Which finite simple groups are unit groups?
Davis, Christopher James; Occhipinti, Tommy

Published in:
Journal of Pure and Applied Algebra

DOI:
10.1016/j.jpaa.2013.08.013

Publication date:
2014

Document version
Early version, also known as pre-print

Citation for published version (APA):
WHICH FINITE SIMPLE GROUPS ARE UNIT GROUPS?

CHRISTOPHER DAVIS AND TOMMY OCCHIPINTI

Abstract. We prove that if $G$ is a finite simple group which is the unit group of a ring, then $G$ is isomorphic to either (a) a cyclic group of order 2; (b) a cyclic group of prime order $2^k - 1$ for some $k$; or (c) a projective special linear group $\text{PSL}_n(F_2)$ for some $n \geq 3$. Moreover, these groups do all occur as unit groups. We deduce this classification from a more general result, which holds for groups $G$ with no non-trivial normal 2-subgroup.

Throughout this paper, rings will be assumed to be unital, but not necessarily commutative, and ring homomorphisms send 1 to 1. The finite groups $G$ of odd order which occur as unit groups of rings were determined in [3]. We will prove similar results for a more general class of groups; the description of this class of groups uses the following.

Definition 1. For a finite group $G$, the $p$-core of $G$ is the largest normal $p$-subgroup of $G$. We denote this subgroup by $O_p(G)$. It is the intersection of all Sylow $p$-subgroups of $G$.

We now state the main result. The authors are most grateful to the anonymous referee for our earlier paper [2], who recognized that one of the results proved in that paper could be strengthened into the following.

Theorem 2. Let $G$ denote a finite group such that $O_2(G) = \{1\}$ and such that $G$ is isomorphic to the unit group of a ring $R$. Then

$$G \cong \text{GL}_{n_1}(F_{2^{k_1}}) \times \cdots \times \text{GL}_{n_r}(F_{2^{k_r}}).$$

Before proving Theorem 2, we record the following corollary.

Corollary 3. The finite simple groups which occur as unit groups of rings are precisely the groups

(a) $\mathbb{Z}/2\mathbb{Z}$,
(b) $\mathbb{Z}/p\mathbb{Z}$ for a Mersenne prime $p = 2^k - 1$,
(c) $\text{PSL}_n(F_2)$ for $n \geq 3$.

Proof. If $G$ is a finite simple group, then either $O_2(G) = \{1\}$ or $O_2(G) = G$. If $O_2(G) = G$, then $G$ is a 2-group, and because we are assuming $G$ is simple, we must have $G \cong \mathbb{Z}/2\mathbb{Z}$, which for instance is isomorphic to the unit group of $\mathbb{Z}$.

Hence assume $G$ is a finite simple group which is isomorphic to the unit group of a ring and further assume $O_2(G) = \{1\}$. By Theorem 2 we know

$$G \cong \text{GL}_{n_1}(F_{2^{k_1}}) \times \cdots \times \text{GL}_{n_r}(F_{2^{k_r}}).$$

These groups all occur as unit groups of the corresponding products of matrix rings, so we are reduced to determining which of them are simple; this forces

$$G \cong \text{GL}_n(F_{2^k}).$$

If $n > 1$ and $k > 1$, then the subgroup of invertible scalar matrices forms a nontrivial normal subgroup. Hence two possibilities remain. If $n = 1$, then $\text{GL}_1(F_{2^k})$ is cyclic of order $2^k - 1$; such a group is simple if and only if its order is prime. If $k = 1$, then $\text{GL}_n(F_2) = \text{PSL}_n(F_2)$. For the case $k = 1, n = 2$, we have $\text{PSL}_2(F_2) \cong S_3$ (see for example [4 Section 3.3.1]); this group is not simple. For the cases $k = 1, n \geq 3$, it is well-known that $\text{PSL}_n(F_2)$ is simple (see for example [4 Section 3.3.2]). This completes the proof.

Date: August 6, 2013.

1The authors also thank Colin Adams, John F. Dillon, Dennis Eichhorn, Noam Elkies, Kiran Kedlaya, Charles Toll, and Ryan Vinroot for many useful discussions.
Remark 4. The simple groups $A_8$ and $PSL_2(\mathbb{F}_7)$ also occur as unit groups. This follows immediately from the exceptional isomorphisms

$$A_8 \cong PSL_4(\mathbb{F}_2) \text{ and } PSL_2(\mathbb{F}_7) \cong PSL_3(\mathbb{F}_2).$$


Having recorded the above consequences of the main result, we now gather the preliminary results used in its proof. We begin with the following observation.

**Lemma 5.** Let $G$ denote a finite group with $O_2(G) = \{1\}$, and let $R$ denote a ring with $R^x \cong G$. Then $R$ has characteristic 2.

**Proof.** The elements 1 and $-1$ are units in $R$ and are in the center of $R$, hence are in the center of $R^x$. By the assumption $O_2(G) = \{1\}$, the center of $G$ cannot contain any elements of order 2. Hence $1 = -1$. □

**Lemma 6.** Keep notation as in Lemma 5, and fix an isomorphism $R^x \cong G$. Because $R$ has characteristic 2, we have a natural map

$$\varphi : F_2[G] \rightarrow R$$

extending the fixed embedding of $G$ into $R$. The image of $\varphi$ is a ring with unit group isomorphic to $G$.

**Proof.** Write $S$ for the image of $\varphi$. On one hand, we have that $S^x \subseteq R^x \cong G$. On the other hand, the induced map $\varphi : G \rightarrow S^x \rightarrow R^x$ is surjective. This shows that the unit group of $S$ is isomorphic to $G$. □

**Lemma 7.** Let $R$ denote a finite ring of characteristic 2. If $J \subseteq R$ is a two-sided ideal such that $J^2 = 0$, then $1 + J$ is a normal elementary abelian 2-subgroup of $R^x$.

**Proof.** Note that for any $j, k \in J$ and $r \in R^x$, we have

- $(1 + j)^2 = 1 + j^2 = 1$;
- $(1 + j)(1 + k) = 1 + j + k + jk = 1 + j + k = (1 + k)(1 + j)$;
- $r(1 + j)r^{-1} = 1 + rjr^{-1} \in 1 + J$.

The first of these calculations shows that $1 + J$ is a subset of $R^x$, and the three calculations together show that it is a normal elementary abelian 2-group. □

We now use these preliminary results to prove our main theorem.

**Proof of Theorem 2.** By Lemma 6, we may assume $R$ is a finite ring (and in particular artinian) and has characteristic 2. Let $J$ denote a two-sided ideal of $R$ such that $J^2 = 0$. By Lemma 7, the set $1 + J$ is a normal 2-subgroup of $R^x$, and so by the assumption $O_2(G) = \{1\}$, we have $J = \{0\}$. Thus the ring $R$ has no non-zero two-sided ideals $J$ with $J^2 = 0$, and hence $R$ has no non-zero two-sided nilpotent ideals. By [1] Theorem 5.4.5], the artinian ring $R$ is semisimple. By Wedderburn’s Theorem [1] Theorem 5.3.4], we have

$$R \cong M_{n_1}(D_1) \times \cdots \times M_{n_r}(D_r)$$

for some $n_1, \ldots, n_r \geq 1$ and some division algebras $D_1, \ldots, D_r$. Our ring $R$ is finite and hence each $D_i$ is finite. By another theorem of Wedderburn [1] Theorem 3.8.6], we have that each $D_i$ is a finite field. Finally, because the ring $R$ has characteristic 2, each field $D_i$ has characteristic 2. This completes the proof. □

**REFERENCES**


University of California, Irvine, Dept of Mathematics, Irvine, CA 92697

Current address: University of Copenhagen, Dept of Mathematical Sciences, Universitetsparken 5, 2100 København Ø, Denmark

E-mail address: davis@math.ku.dk

University of California, Irvine, Dept of Mathematics, Irvine, CA 92697

Current address: Carleton College, Dept of Mathematics, Northfield, MN 55057

E-mail address: tocchipinti@carleton.edu