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Compactness and weak-star continuity of derivations on weighted convolution algebras

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ABSTRACT

Let ω be a continuous weight on $\mathbb{R}^+$ and let $L^1(\omega)$ be the corresponding convolution algebra. By results of Grønbæk and Bade & Dales the continuous derivations from $L^1(\omega)$ to its dual space $L^\infty(1/\omega)$ are exactly the maps of the form

$$(D_\varphi f)(t) = \int_0^\infty f(s)\,\frac{s}{t+s}\,\varphi(t+s)\,ds \quad (t \in \mathbb{R}^+ \text{ and } f \in L^1(\omega))$$

for some $\varphi \in L^\infty(1/\omega)$. Also, every D_φ has a unique extension to a continuous derivation $\overline{D}_\varphi : M(\omega) \to L^\infty(1/\omega)$ from the corresponding measure algebra. We show that a certain condition on φ implies that $\overline{D}_\varphi$ is weak-star continuous. The condition holds for instance if $\varphi \in L_0^\infty(1/\omega)$. We also provide examples of functions φ for which $\overline{D}_\varphi$ is not weak-star continuous. Similarly, we show that D_φ and $\overline{D}_\varphi$ are compact under certain conditions on φ. For instance this holds if $\varphi \in C_0(1/\omega)$ with $\varphi(0) = 0$. Finally, we give various examples of functions φ for which D_φ and $\overline{D}_\varphi$ are not compact.

1. Introduction

Traditionally the study of derivations from a Banach algebra to its Banach modules has mainly focused on the existence of such derivations. In some recent papers by Choi and Heath the aim has instead been to characterise the derivations from a concrete Banach algebra to its dual space, and then to use this characterisation to study various properties of the derivations: Every bounded derivation from $l^1(\mathbb{Z}^+)$ to its dual space $l^\infty(\mathbb{Z}^+)$ is of the form

$$D_\psi(\delta_0) = 0 \quad \text{and} \quad D_\psi(\delta_j)(\delta_k) = \frac{j}{j+k}\,\psi_{j+k} \quad (j, k \in \mathbb{Z}^+,\ j \neq 0)$$

for some $\psi \in l^\infty(\mathbb{Z}^+)$. It was shown in [11] that D_ψ is compact if and only if $\psi \in c_0(\mathbb{Z}^+)$. Moreover, the weakly compact derivations from $l^1(\mathbb{Z}^+)$ to its dual space are described in [3] in terms of the so-called translation-finite sets. Finally, the compact derivations from the disc algebra to its dual space are characterised in [4]. In this paper we continue this line of thinking and consider properties of derivations from weighted convolution algebras $L^1(\omega)$ on $\mathbb{R}^+$ to their dual spaces.

Let $L^1(\mathbb{R}^+)$ be the Banach space of (equivalence classes of) integrable functions f on $\mathbb{R}^+ = [0, \infty)$ with the norm $\|f\| = \int_0^\infty |f(t)|\,dt$. Throughout this paper ω will be a continuous weight function on $\mathbb{R}^+$, that is, a positive and continuous function on $\mathbb{R}^+$ satisfying $\omega(0) = 1$ and $\omega(t+s) \le \omega(t)\omega(s)$ for all $t, s \in \mathbb{R}^+$. We then define $L^1(\omega)$ as the weighted space of functions f on $\mathbb{R}^+$ for which $f\omega \in L^1(\mathbb{R}^+)$ with the inherited norm

$$\|f\| = \int_0^\infty |f(t)|\omega(t)\,dt.$$

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With the usual convolution product

$$(f * g)(t) = \int_0^t f(s)g(t-s)\,ds \quad \text{for } t \in \mathbb{R}^+ \text{ and } f, g \in L^1(\omega),$$

it is well known that $L^1(\omega)$ is a commutative Banach algebra. Similarly, the space $M(\omega)$ of locally finite, complex Borel measures μ on $\mathbb{R}^+$ for which

$$\|\mu\| = \int_{\mathbb{R}^+} \omega(t)\,d|\mu|(t) < \infty$$

is a Banach algebra under convolution and contains $L^1(\omega)$ as a closed ideal. Also, $M(\omega)$ can be identified as the multiplier algebra of $L^1(\omega)$ and this induces a strong topology on $M(\omega)$ by identifying a measure with the corresponding convolution operator.

Moreover, we let $L^\infty(1/\omega)$ denote the Banach space of measurable functions φ on $\mathbb{R}^+$ for which φ/ω is essentially bounded with the norm $\|\varphi\| = \operatorname{ess\,sup}_{t\in\mathbb{R}^+} |\varphi(t)|/\omega(t)$. It is well known that the duality

$$\langle f, \varphi\rangle = \int_0^\infty f(t)\varphi(t)\,dt \quad \text{for } f \in L^1(\omega) \text{ and } \varphi \in L^\infty(1/\omega)$$

identifies $L^\infty(1/\omega)$ isometrically isomorphically with the dual space of $L^1(\omega)$.

We denote by $C_b(1/\omega)$ the closed subspace of $L^\infty(1/\omega)$ of continuous functions in $L^\infty(1/\omega)$, and by $C_0(1/\omega)$ the closed subspace of $C_b(1/\omega)$ of functions $h \in C_b(1/\omega)$ for which h/ω vanishes at infinity. Then $M(\omega)$ is isometrically isomorphic to the dual space of $C_0(1/\omega)$ with the duality being defined by

$$\langle h, \mu\rangle = \int_{\mathbb{R}^+} h(t)\,d\mu(t) \quad \text{for } h \in C_0(1/\omega) \text{ and } \mu \in M(\omega).$$

We will need yet another closed subspace of $L^\infty(1/\omega)$. For $\varphi \in L^\infty(\mathbb{R}^+)$ we say that $\varphi(t) \to 0$ as $t \to \infty$ if

$$\operatorname*{ess\,sup}_{t\ge T} |\varphi(t)| \to 0 \quad \text{as } T \to \infty.$$

We then define $L_0^\infty(1/\omega)$ to be the closed subspace of $L^\infty(1/\omega)$ of those $\varphi \in L^\infty(1/\omega)$ for which $\varphi(t)/\omega(t) \to 0$ as $t \to \infty$.

Recall that the dual space $L^\infty(1/\omega) = L^1(\omega)^*$ becomes a Banach $L^1(\omega)$-module via the action

$$\langle f, g\cdot\varphi\rangle = \langle f * g, \varphi\rangle \quad \text{for } f, g \in L^1(\omega) \text{ and } \varphi \in L^\infty(1/\omega).$$

An easy calculation shows that the module action can be expressed as

$$(g\cdot\varphi)(t) = \int_0^\infty g(s)\varphi(t+s)\,ds \quad \text{for } t \in \mathbb{R}^+,\ g \in L^1(\omega) \text{ and } \varphi \in L^\infty(1/\omega).$$

In particular it follows that $C_0(1/\omega)$ is a Banach $L^1(\omega)$-submodule of $L^\infty(1/\omega)$. Also, if we consider $M(\omega) = C_0(1/\omega)^*$ as a dual Banach $L^1(\omega)$-module, then

$$\begin{aligned}
\langle h, g\cdot\mu\rangle &= \langle g\cdot h, \mu\rangle = \int_{\mathbb{R}^+}\int_0^\infty g(s)h(t+s)\,ds\,d\mu(t)\\
&= \int_{\mathbb{R}^+}\int_t^\infty g(r-t)h(r)\,dr\,d\mu(t)\\
&= \int_0^\infty \int_{[0,r]} g(r-t)\,d\mu(t)\,h(r)\,dr = \langle h, g*\mu\rangle
\end{aligned}$$

for $g \in L^1(\omega)$, $h \in C_0(1/\omega)$ and $\mu \in M(\omega)$. The dual module action $g\cdot\mu$ of $g \in L^1(\omega)$ on $\mu \in M(\omega)$ thus coincides with the usual convolution product $g * \mu$. (One may also have wished for the product $g\cdot\varphi$ for $g \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$ to coincide with the usual convolution product of g and φ. This could be obtained by choosing instead to identify $L^1(\omega)^*$ with the space $L^\infty(\mathbb{R}^-, \omega(-t))$ on $\mathbb{R}^-$ with the duality $\langle f, \varphi\rangle = \int_0^\infty f(t)\varphi(-t)\,dt$ for $f \in L^1(\omega)$ and $\varphi \in L^\infty(\mathbb{R}^-, \omega(-t))$. This approach is taken, for instance, in [5]. We prefer instead (as in [9]) to represent all our spaces on $\mathbb{R}^+$, and pay the price of the form of the product $g\cdot\varphi$.)

We now turn to derivations from $L^1(\omega)$ to its dual space $L^\infty(1/\omega)$. Recall that a linear map $D : L^1(\omega) \to L^\infty(1/\omega)$ is called a derivation if

$$D(f * g) = f\cdot Dg + g\cdot Df \quad \text{for } f, g \in L^1(\omega).$$

The following result was first proved by Grønbæk [9, Theorem 3.7] with φ satisfying the (formally weaker) condition

$$\sup_{s\in\mathbb{R}^+} \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \frac{s|\varphi(t+s)|}{(t+s)\omega(t)\omega(s)} < \infty.$$

Bade and Dales ([1, Theorem 2.5] or [5, Proposition 5.6.26]) then strengthened Grønbæk's result by showing that this condition is equivalent to $\varphi \in L^\infty(1/\omega)$. For $s \in \mathbb{R}^+$ we denote by δ_s the unit point measure at s.

Theorem 1.1 (*Grønbæk and Bade and Dales*)**.** *Let $\varphi \in L^\infty(1/\omega)$. Then*

$$(D_\varphi f)(t) = \int_0^\infty f(s)\,\frac{s}{t+s}\,\varphi(t+s)\,ds \quad \text{for } t \in \mathbb{R}^+ \text{ and } f \in L^1(\omega)$$

defines a continuous derivation from $L^1(\omega)$ to $L^\infty(1/\omega)$. Moreover, D_φ has a unique extension to a continuous derivation $\overline{D}_\varphi : M(\omega) \to L^\infty(1/\omega)$. Also, $\overline{D}_\varphi$ is continuous when $M(\omega)$ is equipped with its strong topology and $L^\infty(1/\omega)$ with its weak-star topology, and

$$(\overline{D}_\varphi \delta_s)(t) = \frac{s}{t+s}\,\varphi(t+s) \quad \text{for } t, s \in \mathbb{R}^+.$$

Conversely, every continuous derivation from $L^1(\omega)$ to $L^\infty(1/\omega)$ equals D_φ for some $\varphi \in L^\infty(1/\omega)$.

We mention that if we let X be the densely defined operator on $L^1(\omega)$ given by $(Xf)(t) = tf(t)$ for $t \in \mathbb{R}^+$ and suitable $f \in L^1(\omega)$, and similarly on $L^\infty(1/\omega)$, then we have $D_\varphi f = (Xf)\cdot(X^{-1}\varphi)$ for f and φ in dense subsets of $L^1(\omega)$ resp. $L^\infty(1/\omega)$.

In this paper we study various properties of the derivations $\overline{D}_\varphi$. In Section 2 we consider weak-star continuity, and in Section 3 we rely on some of the results from Section 2 to prove various range and continuity properties of the derivations. Finally, some of these results are used in Section 4, where compactness of the derivations is investigated. We remark that most of the results in this paper also are of interest in the unweighted case where $\omega \equiv 1$.

2. Weak-star continuity

In this section we will study weak-star continuity of the derivations $\overline{D}_\varphi : M(\omega) \to L^\infty(1/\omega)$. This is inspired by a similar result for homomorphisms between weighted algebras due to Grabiner: Let ω_1 and ω_2 be weights, and let $\Phi : L^1(\omega_1) \to L^1(\omega_2)$ be a non-zero continuous homomorphism. Then Φ has a unique extension to a continuous homomorphism $\widetilde{\Phi} : M(\omega_1) \to M(\omega_2)$ [7, Theorems 3.4] and this extension is automatically weak-star continuous [8, Theorem 1.1]. Also, similar results hold for homomorphisms from $L^1(\omega)$ into some other commutative Banach algebras [13]. We also mention that it easily can be seen that a bounded derivation D_ψ from $l^1(\mathbb{Z}^+)$ to its dual space is weak-star continuous if and only if $\psi \in c_0(\mathbb{Z}^+)$.

For $f, g \in L^1(\omega)$ and $\varphi \in L^\infty(1/\omega)$ it follows from Fubini's theorem that

$$\begin{aligned}\langle f, D_\varphi g\rangle &= \int_0^\infty f(s) \int_0^\infty \frac{t}{t+s}\,\varphi(t+s) g(t)\,dt\,ds \\ &= \int_0^\infty \left(\int_0^\infty f(s)\,\frac{t}{t+s}\,\varphi(t+s)\,ds\right) g(t)\,dt. \qquad (1)\end{aligned}$$

This leads us to the following definition. Let

$$(T_\varphi f)(t) = \int_0^\infty f(s)\,\frac{t}{t+s}\,\varphi(t+s)\,ds = \langle f, \overline{D}_\varphi \delta_t\rangle$$

for $f \in L^1(\omega)$, $\varphi \in L^\infty(1/\omega)$ and $t \in \mathbb{R}^+$.

Proposition 2.1. *Let $\varphi \in L^\infty(1/\omega)$.*

(a) *$s \mapsto \overline{D}_\varphi \delta_s$ is weak-star continuous in $L^\infty(1/\omega)$ for $s \in \mathbb{R}^+$.*
(b) *T_φ is a continuous linear operator $T_\varphi : L^1(\omega) \to C_b(1/\omega)$.*

Proof. (a): Translation is continuous in $L^1(\omega)$ [5, Lemma 4.7.6], so translation is weak-star continuous in $L^\infty(1/\omega)$. Hence $s \mapsto \overline{D}_\varphi \delta_s$ is weak-star continuous in $L^\infty(1/\omega)$ for $s > 0$. Also, for $f \in L^1(\omega)$ we have

$$\begin{aligned}|\langle f, \overline{D}_\varphi \delta_s\rangle| &= \left|\int_0^\infty f(t)\frac{s}{t+s}\,\varphi(t+s)\,dt\right| \\ &\le \|\varphi\|\omega(s)\int_0^\infty |f(t)|\omega(t)\frac{s}{t+s}\,dt \to 0\end{aligned}$$

as $s \to 0$ by Lebesgue's dominated convergence theorem, so $\overline{D}_\varphi \delta_s \to 0 = \overline{D}_\varphi \delta_0$ weak-star in $L^\infty(1/\omega)$ as $s \to 0$.

(b): Let $f \in L^1(\omega)$. The estimate

$$|(T_\varphi f)(t)| \le \|\varphi\| \int_0^\infty |f(s)|\omega(t+s)\,ds \le \|\varphi\|\cdot\|f\|\omega(t) \quad \text{for } t \in \mathbb{R}^+$$

shows that T_φ defines a continuous linear operator $T_\varphi : L^1(\omega) \to L^\infty(1/\omega)$. Moreover, it follows from (a) that T_φ maps into $C_b(1/\omega)$. $\square$

We will need the next couple of results. For $n \in \mathbb{N}$ let $e_n = n \cdot 1_{[0,1/n]}$. It is well known that (e_n) is a bounded approximate identity for $L^1(\omega)$. Also, $\langle h, \mu\rangle = \int_{\mathbb{R}^+} h(t)\,d\mu(t)$ is well-defined for $h \in C_b(1/\omega)$ and $\mu \in M(\omega)$.

Lemma 2.2.

(a) *Let $h \in C_0(1/\omega)$. Then $e_n \cdot h \to h$ in $C_0(1/\omega)$ as $n \to \infty$.*
(b) *Let $h \in C_b(1/\omega)$ and $\mu \in M(\omega)$. Then $\langle e_n \cdot h, \mu\rangle \to \langle h, \mu\rangle$ as $n \to \infty$.*

Proof. (a): For $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ we have

$$(e_n \cdot h - h)(t) = \int_0^{1/n} n(h(t+s) - h(t))ds.$$

Given $\varepsilon > 0$, we choose $T \in \mathbb{R}^+$ such that $|h(t)|/\omega(t) < \varepsilon$ for $t \geq T$. For all $n \in \mathbb{N}$ we then have $|(e_n \cdot h - h)(t)|/\omega(t) < (1+\sup_{s\in[0,1]} \omega(s))\varepsilon$ for $t \geq T$. Since h is uniformly continuous on $[0, T+1]$ we can choose $N \in \mathbb{N}$ such that $|h(t+s)-h(t)| < \varepsilon \cdot \inf_{t\leq T} \omega(t)$ for all $0 \leq t \leq T$ and $s \leq \frac{1}{N}$. Hence

$$\frac{|(e_n \cdot h - h)(t)|}{\omega(t)} \leq \sup_{s\leq 1/N} \frac{|h(t+s) - h(t)|}{\omega(t)} < \varepsilon$$

for all $0 \leq t \leq T$ and $n \geq N$. This finishes the proof.

(b): Given $\varepsilon > 0$, we choose $T \in \mathbb{R}^+$ such that $|\mu \cdot \omega|([T, \infty)) < \varepsilon$. We have

$$\begin{aligned}|\langle e_n \cdot h - h, \mu\rangle| &\leq \int_{[0,T)} |(e_n \cdot h - h)(t)|\, d|\mu|(t) + \int_{[T,\infty)} |(e_n \cdot h - h)(t)|\, d|\mu|(t) \\ &\leq \|\mu\| \sup_{0\leq t\leq T} \frac{|(e_n \cdot h - h)(t)|}{\omega(t)} + \|e_n \cdot h - h\| \cdot |\mu \cdot \omega|([T, \infty)).\end{aligned}$$

The second term is bounded by $C\varepsilon$ and it follows from the proof of part (a) that there exists $N \in \mathbb{N}$ such that the same holds for the first term for $n \geq N$. □

Corollary 2.3. *Let $\mu \in M(\omega)$. Then $e_n * \mu \to \mu$ strongly in $M(\omega)$ and weak star in $C_b(1/\omega)^*$ (and in particular weak-star in $M(\omega)$) as $n \to \infty$.*

Proof. For $f \in L^1(\omega)$ we have $e_n * \mu * f \to \mu * f$ in $L^1(\omega)$ as $n \to \infty$ (since (e_n) is a bounded approximate identity for $L^1(\omega)$). Hence $e_n * \mu \to \mu$ strongly in $M(\omega)$ as $n \to \infty$. Moreover, for $h \in C_b(1/\omega)$ we have $\langle h, e_n * \mu\rangle = \langle e_n \cdot h, \mu\rangle \to \langle h, \mu\rangle$ as $n \to \infty$ by Lemma 2.2(b). Hence $e_n * \mu \to \mu$ weak-star in $C_b(1/\omega)^*$ as $n \to \infty$. □

The adjoint of a continuous linear operator is weak-star continuous. The following result shows that the converse also is true for the operators $\overline{D}_\varphi$.

Proposition 2.4. *For $\varphi \in L^\infty(1/\omega)$ the following conditions are equivalent:*

(a) *$\overline{D}_\varphi$ is weak-star continuous.*
(b) *$\overline{D}_\varphi \delta_t/\omega(t) \to 0$ weak-star in $L^\infty(1/\omega)$ as $t \to \infty$.*
(c) *$\operatorname{ran} T_\varphi \subseteq C_0(1/\omega)$.*
(d) *$\operatorname{ran} T_\varphi \subseteq C_0(1/\omega)$ and $T_\varphi^* = \overline{D}_\varphi$.*

Proof.

(a)⇒(b): This follows because $\delta_t/\omega(t) \to 0$ weak-star in $M(\omega)$ as $t \to \infty$.

(b)⇒(c): This follows from Proposition 2.1(b) because $(T_\varphi f)(t)/\omega(t) = \langle f, \overline{D}_\varphi \delta_t/\omega(t)\rangle$ for $f \in L^1(\omega)$ and $t \in \mathbb{R}^+$.

(c)⇒(d): For $f, g \in L^1(\omega)$ it follows from (1) that

$$\langle f, D_\varphi g\rangle = \langle T_\varphi f, g\rangle$$

since $T_\varphi f \in C_0(1/\omega)$ and $g \in L^1(\omega) \subseteq M(\omega)$. Hence $T_\varphi^* = D_\varphi$ on $L^1(\omega)$. Let $\mu \in M(\omega)$. By Corollary 2.3 we have $e_n * \mu \to \mu$ strongly in $M(\omega)$ as $n \to \infty$, so it follows from Theorem 1.1 that $D_\varphi(e_n * \mu) \to \overline{D}_\varphi(\mu)$ weak-star in $L^\infty(1/\omega)$ as $n \to \infty$. Moreover, for $f \in L^1(\omega)$ we have $e_n \cdot T_\varphi f \to T_\varphi f$ in $C_0(1/\omega)$ as $n \to \infty$ by Lemma 2.2(a) and thus

$$\langle f, T_\varphi^*(e_n * \mu)\rangle = \langle T_\varphi f, e_n * \mu\rangle = \langle e_n \cdot T_\varphi f, \mu\rangle \to \langle T_\varphi f, \mu\rangle = \langle f, T_\varphi^* \mu\rangle$$

as $n \to \infty$. Hence $D_\varphi(e_n * \mu) = T_\varphi^*(e_n * \mu) \to T_\varphi^* \mu$ weak-star in $L^\infty(1/\omega)$ as $n \to \infty$. It follows that $T_\varphi^* \mu = \overline{D}_\varphi \mu$ and thus $T_\varphi^* = \overline{D}_\varphi$ on $M(\omega)$.

[Alternatively, a direct but lengthy calculation shows that T_φ^* is a derivation. Since $T_\varphi^* = D_\varphi$ on $L^1(\omega)$, it follows from the uniqueness of the extension from Theorem 1.1 that $T_\varphi^* = D_\varphi$ on $M(\omega)$.]

(d)⇒(a): Is clear. □

We will now show that a certain relatively easily verified condition on φ ensures that the equivalent conditions in Proposition 2.4 hold. The idea is that if φ/ω is not bounded away from zero on large sets, then the definition of $T_\varphi f$ can be used to show that $(T_\varphi f)(t)/\omega(t) \to 0$ as $t \to \infty$ for $f \in L^1(\omega)$. For $\varphi \in L^\infty(1/\omega)$ and $t, \varepsilon \in \mathbb{R}^+$ we let

$$U_{t,\varepsilon} = \{s \in [t, t+1] : |\varphi(s)|/\omega(s) \geq \varepsilon\}$$

(defined except for a set of measure zero). Also, we denote Lebesgue's measure on $\mathbb{R}^+$ by m.

Theorem 2.5. *Let $\varphi \in L^\infty(1/\omega)$ and assume that $m(U_{t,\varepsilon}) \to 0$ as $t \to \infty$ for every $\varepsilon > 0$. Then $\overline{D}_\varphi$ is weak-star continuous (and consequently the other equivalent conditions in Proposition 2.4 also hold).*

We will need the following lemma.

Lemma 2.6. *Let $f \in L^1[0, 1]$ and let (U_n) be a sequence of measurable sets in $[0, 1]$ with $m(U_n) \to 0$ as $n \to \infty$. Then $\int_{U_n} f(t)\, dt \to 0$ as $n \to \infty$.*

Proof. It is sufficient to prove that every subsequence (U_{n_k}) of (U_n) has a subsequence $(U_{n_{k_j}})$ with $\int_{U_{n_{k_j}}} f(t)\, dt \to 0$ as $j \to \infty$. We may therefore assume that $\sum_{n=1}^\infty m(U_n) < \infty$. Then $m(\cup_{j=n}^\infty U_j) \leq \sum_{j=n}^\infty m(U_j) \to 0$ as $n \to \infty$. Let $f_n = f \cdot 1_{U_n}$ for $n \in \mathbb{N}$. It then follows from [10, Theorem A, p. 91] that $f_n \to 0$ a.e. Consequently $\int_{U_n} f(t)\, dt = \int_0^1 f_n(t)\, dt \to 0$ as $n \to \infty$ by Lebesgue's dominated convergence theorem. □

Proof of Theorem 2.5. By Proposition 2.4 we only need to prove that $\operatorname{ran} T_\varphi \subseteq C_0(1/\omega)$. (A similar proof can be given to show directly that condition (b) in Proposition 2.4 holds.) We first let $f \in L^1(\omega)$ with $\operatorname{supp} f \subseteq [0, 1]$. Then

$$|(T_\varphi f)(t)| \leq \int_0^1 |f(s)\varphi(t+s)|\, ds = \int_t^{t+1} |f(r-t)\varphi(r)|\, dr$$

for $t \in \mathbb{R}^+$. Let $\varepsilon > 0$. Then

$$\begin{aligned}\int_{[t,t+1]\setminus U_{t,\varepsilon}} |f(r-t)\varphi(r)|\, dr &\leq \varepsilon \int_t^{t+1} |f(r-t)|\omega(r)\, dr \\ &\leq \varepsilon \int_t^{t+1} |f(r-t)|\omega(r-t)\, dr\, \omega(t) = \varepsilon \|f\| \omega(t)\end{aligned}$$

for $t \in \mathbb{R}^+$. Moreover,

$$\int_{U_{t,\varepsilon}} |f(r-t)\varphi(r)|\, dr = \int_{U_{t,\varepsilon}-t} |f(s)\varphi(t+s)|\, ds \leq \|\varphi\|\omega(t) \int_{U_{t,\varepsilon}-t} |f(s)|\omega(s)\, ds$$

for $t \in \mathbb{R}^+$. It follows from Lemma 2.6 that there exists $T \in \mathbb{R}^+$ such that

$$\int_{U_{t,\varepsilon}-t} |f(s)|\omega(s)\, ds < \varepsilon$$

for $t \geq T$. Hence there is a constant C such that $|(T_\varphi f)(t)| \leq C\varepsilon\omega(t)$ for $t \geq T$, so we conclude that $T_\varphi f \in C_0(1/\omega)$.

Next, we let $f \in L^1(\omega)$ with $\operatorname{supp} f \subseteq [n, n+1]$ for some $n \in \mathbb{N}$. Then $f = \delta_n * g$ for some $g \in L^1(\omega)$ with $\operatorname{supp} g \subseteq [0, 1]$. Also,

$$(T_\varphi f)(t) = \int_n^{n+1} f(s)\, \frac{t}{t+s}\, \varphi(t+s)\, ds = \int_0^1 g(r)\, \frac{t}{t+r+n}\, \varphi(t+r+n)\, dr,$$

so

$$|(T_\varphi f)(t)| \leq \int_0^1 |g(r)|\, \frac{t+n}{t+r+n} |\varphi(t+r+n)|\, dr = (T_{|\varphi|}|g|)(t+n)$$

for $t \in \mathbb{R}^+$. Applying the first part of the proof (the definition of $U_{t,\varepsilon}$ only depends on $|\varphi|$) we get

$$\frac{|(T_\varphi f)(t)|}{\omega(t)} \leq \frac{|(T_{|\varphi|}|g|)(t+n)|}{\omega(t+n)} \cdot \omega(n) \to 0$$

as $t \to \infty$, so $T_\varphi f \in C_0(1/\omega)$. Consequently, $T_\varphi f \in C_0(1/\omega)$ for every $f \in L^1(\omega)$ with compact support and thus for all $f \in L^1(\omega)$. □

Corollary 2.7. *Let $\varphi \in L_0^\infty(1/\omega)$. Then $\overline{D}_\varphi$ is weak-star continuous and $\overline{D}_\varphi = T_\varphi^*$.*

Proof. Let $\varepsilon > 0$. There exists $T \in \mathbb{R}^+$ such that $U_{t,\varepsilon}$ is of measure zero for $t \geq T$. The result thus follows from Theorem 2.5. $\square$

The following corollary shows a class of functions $\varphi \notin L_0^\infty(1/\omega)$ for which $\overline{D}_\varphi$ is weak-star continuous.

Corollary 2.8. *Let (α_n) be a sequence with $0 < \alpha_n < 1$ for $n \in \mathbb{N}$ and $\alpha_n \to 0$ for $n \to \infty$. Define $\varphi \in L^\infty(1/\omega)$ by the weak-star convergent series $\varphi = \sum_{n=1}^\infty 1_{[n,n+\alpha_n]} \cdot \omega$. Then $\varphi \notin L_0^\infty(1/\omega)$, but $\overline{D}_\varphi$ is weak-star continuous.*

Proof. Let $0 < \varepsilon < 1$. For $t \in \mathbb{R}^+$ we let $n = [t]$. Then

$$U_{t,\varepsilon} \subseteq [n, n+\alpha_n] \cup [n+1, n+1+\alpha_{n+1}],$$

so $m(U_{t,\varepsilon}) \leq \alpha_n + \alpha_{n+1} \to 0$ as $t \to \infty$. The result thus follows from Theorem 2.5. $\square$

We do not know whether the condition in Theorem 2.5 also is necessary for $\overline{D}_\varphi$ to be weak-star continuous, but we finish this section by giving a partial result in this direction.

Proposition 2.9. *Suppose that there exists a positive constant C such that $\int_x^{x+1} \omega(y)\,dy \geq C\omega(x)$ for all $x \in \mathbb{R}^+$. Let (a_n) be a sequence in $\mathbb{R}^+$ with $a_{n+1} \geq a_n + 1$ for $n \in \mathbb{N}$. Define $\varphi \in L^\infty(1/\omega)$ by the weak-star convergent series $\varphi = \sum_{n=1}^\infty 1_{[a_n,a_n+1]} \cdot \omega$. Then $\overline{D}_\varphi$ is not weak-star continuous.*

Proof. For $n \geq 2$ (so that $a_n \geq 1$) we have

$$\left\langle 1_{[0,1]}, \frac{\overline{D}_\varphi \delta_{a_n}}{\omega(a_n)} \right\rangle = \int_0^1 \frac{a_n}{t+a_n} \cdot \frac{\varphi(t+a_n)}{\omega(a_n)}\,dt = \int_{a_n}^{a_n+1} \frac{a_n}{s} \cdot \frac{\omega(s)}{\omega(a_n)}\,ds \geq \frac{C}{2}.$$

Hence $\overline{D}_\varphi \delta_{a_n}/\omega(a_n)$ does not tend to 0 weak-star in $L^\infty(1/\omega)$ as $n \to \infty$. Since $\delta_{a_n}/\omega(a_n) \to 0$ weak-star in $M(\omega)$ as $n \to \infty$, this shows that $\overline{D}_\varphi$ is not weak-star continuous. $\square$

Corollary 2.10. *Suppose that there exists a positive constant C such that $\int_x^{x+1} \omega(y)\,dy \geq C\omega(x)$ for all $x \in \mathbb{R}^+$. Then $\overline{D}_\omega$ is not weak-star continuous.*

We remark that Corollaries 2.7 and 2.8 and Proposition 2.9 can be combined to yield a wider class of functions φ for which $\overline{D}_\varphi$ is not weak-star continuous. Namely, if $\varphi = \varphi_1 + \varphi_2$ with $\overline{D}_{\varphi_1}$ weak-star continuous and $\overline{D}_{\varphi_2}$ not weak-star continuous, then $\overline{D}_\varphi$ is not weak-star continuous.

We finish the section by showing that $\overline{D}_\varphi \mu$ can be represented as a weak-star Bochner integral for $\varphi \in L^\infty(1/\omega)$ and $\mu \in M(\omega)$. Heuristically we can think of $\overline{D}_\varphi \mu$ as

$$(\overline{D}_\varphi \mu)(t) = \int_{\mathbb{R}^+} \frac{s}{t+s}\varphi(t+s)\,d\mu(s) = \int_{\mathbb{R}^+} (\overline{D}_\varphi \delta_s)(t)\,d\mu(s) \quad \text{for } t \in \mathbb{R}^+,$$

although the integrals need not be defined. Inspired by this, we will say that

$$\overline{D}_\varphi \mu = \int_{\mathbb{R}^+} \overline{D}_\varphi \delta_s\,d\mu(s)$$

as a weak-star Bochner integral in $L^\infty(1/\omega)$ if

$$\langle f, \overline{D}_\varphi \mu\rangle = \int_{\mathbb{R}^+} \langle f, \overline{D}_\varphi \delta_s\rangle\,d\mu(s) \quad \text{for } f \in L^1(\omega).$$

We remark that the function $(T_\varphi f)(s) = \langle f, \overline{D}_\varphi \delta_s\rangle$ belongs to $C_b(1/\omega)$ by Proposition 2.1(b) for $f \in L^1(\omega)$. Hence

$$\int_{\mathbb{R}^+} \langle f, \overline{D}_\varphi \delta_s\rangle\,d\mu(s) = \langle T_\varphi f, \mu\rangle$$

is well-defined for $f \in L^1(\omega)$ and $\mu \in M(\omega)$. Also, it follows from the proof of [5, Theorem 5.6.24] that

$$D_\varphi g = \int_0^\infty g(s)\overline{D}_\varphi \delta_s\,ds$$

as a weak-star integral in $L^\infty(1/\omega)$ for every $\varphi \in L^\infty(1/\omega)$ and $g \in L^1(\omega)$. We will show that this result can be extended to $\overline{D}_\varphi \mu$. (Moreover, in the proof of Proposition 4.5 we will see that if $\varphi \in C_b(1/\omega)$ with $\varphi(0) = 0$, then $D_\varphi g = \int_0^\infty g(s)\overline{D}_\varphi \delta_s\,ds$ actually exists as a "proper" Bochner integral for $g \in L^1(\omega)$.)

Proposition 2.11. *Let $\varphi \in L^\infty(1/\omega)$ and $\mu \in M(\omega)$. Then*

$$\overline{D}_\varphi \mu = \int_{\mathbb{R}^+} \overline{D}_\varphi \delta_s \, d\mu(s)$$

as a weak-star Bochner integral in $L^\infty(1/\omega)$.

Proof. Let (e_n) be the bounded approximate identity from Lemma 2.2. By Corollary 2.3 we have $e_n * \mu \to \mu$ strongly in $M(\omega)$ as $n \to \infty$, so it follows from Theorem 1.1 that $D_\varphi(e_n * \mu) \to \overline{D}_\varphi(\mu)$ weak-star in $L^\infty(1/\omega)$ as $n \to \infty$. Since $e_n * \mu \in L^1(\omega)$ we have

$$D_\varphi(e_n * \mu) = \int_0^\infty (e_n * \mu)(s) \overline{D}_\varphi \delta_s \, ds$$

as a weak-star integral in $L^\infty(1/\omega)$ for $n \in \mathbb{N}$. We thus have

$$\begin{aligned}\langle f, \overline{D}_\varphi \mu \rangle &= \lim_{n\to\infty} \langle f, D_\varphi(e_n * \mu)\rangle \\ &= \lim_{n\to\infty} \int_0^\infty \langle f, \overline{D}_\varphi \delta_s \rangle (e_n * \mu)(s)\, ds \\ &= \lim_{n\to\infty} \langle T_\varphi f, e_n * \mu \rangle = \langle T_\varphi f, \mu \rangle = \int_{\mathbb{R}^+} \langle f, \overline{D}_\varphi \delta_s \rangle \, d\mu(s),\end{aligned}$$

where we have used Corollary 2.3. $\square$

3. Range and continuity properties

For $\varphi \in L^\infty(1/\omega)$ we saw in (the proof of) Proposition 2.1(a) that $\overline{D}_\varphi \delta_s \to 0$ weak-star in $L^\infty(1/\omega)$ as $s \to 0$. Similarly, one of the equivalent conditions in Proposition 2.4 is that $\overline{D}_\varphi \delta_s/\omega(s) \to 0$ weak-star in $L^\infty(1/\omega)$ as $s \to \infty$. We will now see that by strengthening the conditions on φ, we can obtain norm convergence in both cases. For $\varphi \in L^\infty(1/\omega)$ we say that $\varphi(t) \to 0$ as $t \to 0$ if $\operatorname{ess\,sup}_{t \le T} |\varphi(t)| \to 0$ as $T \to 0$.

Proposition 3.1. *Let $\varphi \in L^\infty(1/\omega)$.*

(a) *$\varphi(s) \to 0$ as $s \to 0$ if and only if $\overline{D}_\varphi \delta_s \to 0$ in $L^\infty(1/\omega)$ as $s \to 0$.*

(b) *$\varphi \in L_0^\infty(1/\omega)$ if and only if $\overline{D}_\varphi \delta_s/\omega(s) \to 0$ in $L^\infty(1/\omega)$ as $s \to \infty$.*

Proof. (a): Assume that $\varphi(s) \to 0$ as $s \to 0$. We have

$$\|\overline{D}_\varphi \delta_s\| = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \frac{s}{t+s} \cdot \frac{|\varphi(t+s)|}{\omega(t)}.$$

Given $\varepsilon > 0$ we choose $0 < S < 1$ such that $\operatorname{ess\,sup}_{s \le S} |\varphi(s)| < \varepsilon$. For $s \le S/2$ we then have

$$\operatorname*{ess\,sup}_{t\le S/2} \frac{s}{t+s} \cdot \frac{|\varphi(t+s)|}{\omega(t)} \le \frac{\varepsilon}{\inf\limits_{t\in[0,1]} \omega(t)}.$$

Also, since $|\varphi(t+s)| \le \|\varphi\| \omega(t)\omega(s)$ for $t, s \in \mathbb{R}^+$, we have

$$\operatorname*{ess\,sup}_{t\ge S/2} \frac{s}{t+s} \cdot \frac{|\varphi(t+s)|}{\omega(t)} \le \frac{2s}{S} \|\varphi\| \omega(s),$$

and it follows that $\overline{D}_\varphi \delta_s \to 0$ in $L^\infty(1/\omega)$ as $s \to 0$.

Conversely, assume that $\overline{D}_\varphi \delta_s \to 0$ in $L^\infty(1/\omega)$ as $s \to 0$. Given $\varepsilon > 0$ we choose $0 < S < 1$ such that $\|\overline{D}_\varphi \delta_s\| < \varepsilon$ for $s \le S$. For $s \le S$ we then have

$$\varepsilon > \|\overline{D}_\varphi \delta_s\| \ge \frac{\operatorname*{ess\,sup}\limits_{t\le s} |\varphi(t+s)|}{2 \sup\limits_{t\in[0,1]} \omega(t)} = C \operatorname*{ess\,sup}_{s\le t\le 2s} |\varphi(t)|.$$

Hence $\operatorname{ess\,sup}_{0\le t\le 2s} |\varphi(t)| \le \varepsilon/C$, so $\varphi(s) \to 0$ as $s \to 0$.

(b): Assume that $\varphi \in L_0^\infty(1/\omega)$. Then

$$\frac{\|\overline{D}_\varphi \delta_s\|}{\omega(s)} = \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \frac{s}{t+s} \cdot \frac{|\varphi(t+s)|}{\omega(t)\omega(s)} \le \operatorname*{ess\,sup}_{t\in\mathbb{R}^+} \frac{|\varphi(t+s)|}{\omega(t+s)} = \operatorname*{ess\,sup}_{t\ge s} \frac{|\varphi(t)|}{\omega(t)} \to 0$$

as $s \to \infty$.

Conversely, assume that $\varphi \notin L_0^\infty(1/\omega)$. Then there exists $\varepsilon > 0$ such that

$$\operatorname*{ess\,sup}_{t\ge T} |\varphi(t)|/\omega(t) \ge \varepsilon$$

for every $T \in \mathbb{R}^+$. Let $k \in \mathbb{N}$. There exists a measurable set $U_k \subseteq [k, \infty)$ with $m(U_k) > 0$ such that $|\varphi(t)|/\omega(t) \ge \varepsilon$ a.e. on U_k. The metric density of U_k at a point $s \in U_k$, that is $\lim_{r\to 0} m(U_k \cap (s-r, s+r))/(2r)$, exists and equals 1 for almost every $s \in U_k$ [14, 7.12]. Let $s_k \in U_k$ be such a point and let $V_{kr} = U_k \cap [s_k, s_k + r)$ for $r > 0$. Then $m(V_{kr}) > 0$ for every $r > 0$. Hence

$$\|\overline{D}_\varphi \delta_{s_k}\| \ge \operatorname*{ess\,sup}_{t\in[0,1]} \frac{s_k}{t+s_k} \cdot \frac{|\varphi(t+s_k)|}{\omega(t)} \ge \frac{1}{2 \inf_{t\in[0,1]} \omega(t)} \operatorname*{ess\,sup}_{t\in[0,1]} |\varphi(t+s_k)| \ge C \operatorname*{ess\,sup}_{t\in V_{kr}} |\varphi(t)|$$

for some constant $C > 0$ and every $0 < r < 1$. Since $V_{kr} \subseteq U_k$ we thus have

$$\|\overline{D}_\varphi \delta_{s_k}\| \ge C\varepsilon \sup_{t\in V_{kr}} \omega(t)$$

for every $0 < r < 1$. Letting $r \to 0$ we thus obtain

$$\|\overline{D}_\varphi \delta_{s_k}\| \ge C\varepsilon\omega(s_k).$$

Since $s_k \ge k$ this shows that we do not have $\overline{D}_\varphi \delta_s/\omega(s) \to 0$ in $L^\infty(1/\omega)$ as $s \to \infty$. □

We now aim to prove that $\operatorname{ran} D_\varphi \subseteq C_0(1/\omega)$ for any $\varphi \in L^\infty(1/\omega)$, and that $\operatorname{ran} \overline{D}_\varphi \subseteq C_0(1/\omega)$ if $\varphi \in C_0(1/\omega)$ with $\varphi(0) = 0$. Let $\varphi \in L^\infty(1/\omega)$. Since $(D_\varphi f)(t) = \int_0^\infty \frac{s}{t+s} f(s)\varphi(t+s)\, ds$ for $f \in L^1(\omega)$ and $t \in \mathbb{R}^+$, we choose to define

$$\psi_t(s) = (\overline{D}_\varphi \delta_s)(t) = \frac{s}{t+s}\varphi(t+s) \quad \text{for } t, s \in \mathbb{R}^+.$$

Then we can express $D_\varphi f$ by

$$(D_\varphi f)(t) = \langle f, \psi_t \rangle \quad \text{for } f \in L^1(\omega) \text{ and } t \in \mathbb{R}^+$$

(once we have verified that $\psi_t \in L^\infty(1/\omega)$ for $t \in \mathbb{R}^+$). We begin by establishing some properties of ψ_t.

Lemma 3.2.

(a) *Let $\varphi \in L^\infty(1/\omega)$. For $t \in \mathbb{R}^+$ we have $\psi_t \in L^\infty(1/\omega)$ with $\|\psi_t\| \le \|\varphi\|\omega(t)$. Moreover, (ψ_t) is weak-star continuous in $L^\infty(1/\omega)$ for $t \in \mathbb{R}^+$.*

(b) *Let $\varphi \in C_0(1/\omega)$ with $\varphi(0) = 0$. For $t \in \mathbb{R}^+$ we have $\psi_t \in C_0(1/\omega)$. Moreover, (ψ_t) is continuous in $C_0(1/\omega)$ for $t \in \mathbb{R}^+$ and $\psi_t/\omega(t) \to 0$ in $C_0(1/\omega)$ as $t \to \infty$.*

Proof. (a): Let $t \in \mathbb{R}^+$. We have

$$\frac{|\psi_t(s)|}{\omega(s)} \le \frac{|\varphi(t+s)|}{\omega(t+s)}\omega(t) \le \|\varphi\|\omega(t) \quad \text{for all } s \in \mathbb{R}^+,$$

so $\psi_t \in L^\infty(1/\omega)$ with $\|\psi_t\| \le \|\varphi\|\omega(t)$. Also, translation is weak-star continuous in $L^\infty(1/\omega)$, so (ψ_t) is weak-star continuous in $L^\infty(1/\omega)$ for $t > 0$. Also, for $f \in L^1(\omega)$ we have

$$\langle f, \psi_t - \psi_0 \rangle = \int_0^\infty f(s)\left(\frac{s}{t+s}\varphi(t+s) - \varphi(s)\right) ds = \int_0^\infty f(s)(\varphi(t+s) - \varphi(s))\, ds - \int_0^\infty f(s)\frac{t}{t+s}\varphi(t+s)\, ds.$$

As $t \to 0$ the first term tends to 0 since translation is weak-star continuous in $L^\infty(1/\omega)$, whereas the second tends to 0 by Lebesgue's dominated convergence theorem since $|\varphi(t+s)| \le \|\varphi\|\omega(t)\omega(s)$ for $t, s \in \mathbb{R}^+$. Hence (ψ_t) is also weak-star continuous in $L^\infty(1/\omega)$ at $t = 0$.

(b): Clearly $\psi_t \in C_0(1/\omega)$ for $t \in \mathbb{R}^+$, and since translation is continuous in $C_0(1/\omega)$, it follows that (ψ_t) is continuous in $C_0(1/\omega)$ for $t > 0$. We will now prove that (ψ_t) is also continuous in $C_0(1/\omega)$ at $t = 0$. For $t \in \mathbb{R}^+$ we have

$$\|\psi_t - \psi_0\| = \operatorname*{ess\,sup}_{s\in\mathbb{R}^+} \frac{\left|\frac{s}{t+s}\varphi(t+s) - \varphi(s)\right|}{\omega(s)}.$$

Given $\varepsilon > 0$, we choose $S_1, S_2 > 0$ such that $|\varphi(s)|/\omega(s) \le \varepsilon$ if $s \le S_1$ or $s \ge S_2$. For $t \in \mathbb{R}^+$ we then have

$$\begin{aligned}\operatorname*{ess\,sup}_{s\ge S_2} \frac{\left|\frac{s}{t+s}\varphi(t+s) - \varphi(s)\right|}{\omega(s)} &\le \operatorname*{ess\,sup}_{s\ge S_2}\left(\frac{\omega(t+s)}{\omega(s)} + 1\right)\varepsilon \\ &\le (\omega(t)+1)\varepsilon.\end{aligned}$$

Also, for $t \in \mathbb{R}^+$ we have

$$\begin{aligned}\operatorname*{ess\,sup}_{s\le S_2} \frac{\left|\frac{s}{t+s}\varphi(t+s) - \varphi(s)\right|}{\omega(s)} &\le \operatorname*{ess\,sup}_{s\le S_2} \frac{s}{t+s}\cdot\frac{|\varphi(t+s)-\varphi(s)|}{\omega(s)} \\ &\quad + \operatorname*{ess\,sup}_{s\le S_1} \frac{t}{t+s}\cdot\frac{|\varphi(s)|}{\omega(s)} + \operatorname*{ess\,sup}_{S_1\le s\le S_2} \frac{t}{t+s}\cdot\frac{|\varphi(s)|}{\omega(s)}.\end{aligned}$$

The first term tends to 0 as $t \to 0$ since translation is continuous in $C_0(1/\omega)$; whereas the second is bounded by ϵ and the third by Ct for some constant C. Together these estimates show that $\psi_t \to \psi_0$ in $C_0(1/\omega)$ as $t \to 0$ as required. Moreover,

$$\begin{aligned}\frac{\|\psi_t\|}{\omega(t)} &\le \operatorname*{ess\,sup}_{s\in\mathbb{R}^+} \frac{|\varphi(t+s)|}{\omega(t)\omega(s)} \le \operatorname*{ess\,sup}_{s\in\mathbb{R}^+} \frac{|\varphi(t+s)|}{\omega(t+s)} \\ &= \operatorname*{ess\,sup}_{s\ge t} \frac{|\varphi(s)|}{\omega(s)} \to 0\end{aligned}$$

as $t \to \infty$. □

Corollary 3.3.

(a) *Let $\varphi \in L^\infty(1/\omega)$. Then $(D_\varphi f)(t) = \langle f, \psi_t\rangle$ for $f \in L^1(\omega)$ and $t \in \mathbb{R}^+$. Moreover,* ran $D_\varphi \subseteq C_0(1/\omega)$.

(b) *Let $\varphi \in C_0(1/\omega)$ with $\varphi(0) = 0$. Then $(\overline{D}_\varphi \mu)(t) = \langle \psi_t, \mu\rangle$ for $\mu \in M(\omega)$ and $t \in \mathbb{R}^+$. Moreover,* ran $\overline{D}_\varphi \subseteq C_0(1/\omega)$.

Proof. (a): Let $f \in L^1(\omega)$. It follows from Lemma 3.2(a) that $(D_\varphi f)(t) = \langle f, \psi_t\rangle$ for $t \in \mathbb{R}^+$. From this it follows that $D_\varphi f$ is continuous on $\mathbb{R}^+$ and that $|(D_\varphi f)(t)| \le \|f\|\cdot\|\psi_t\| \le \|f\|\cdot\|\varphi\|\omega(t)$. Hence $D_\varphi f \in C_b(1/\omega)$. Let $\varepsilon > 0$ and choose $S > 0$ such that $\int_S^\infty |f(s)|\omega(s)\,ds < \varepsilon$. For $t \ge S/\varepsilon$ we then have

$$\begin{aligned}\frac{|(D_\varphi f)(t)|}{\omega(t)} &\le \|\varphi\|\left(\int_0^S |f(s)|\,\frac{s}{t+s}\,\frac{\omega(t+s)}{\omega(t)}\,ds + \int_S^\infty |f(s)|\,\frac{\omega(t+s)}{\omega(t)}\,ds\right) \\ &\le \|\varphi\|\left(\int_0^S \varepsilon|f(s)|\omega(s)\,ds + \int_S^\infty |f(s)|\omega(s)\,ds\right) \le C\varepsilon\end{aligned}$$

for some constant C, so we conclude that $D_\varphi f \in C_0(1/\omega)$.

(b): Let $\mu \in M(\omega)$ and define

$$k(t) = \int_{\mathbb{R}^+} \frac{s}{t+s}\,\varphi(t+s)\,d\mu(s) = \langle \psi_t, \mu\rangle \quad \text{for } t \in \mathbb{R}^+.$$

Then $k \in C_0(1/\omega)$ by Lemma 3.2(b). By Corollary 2.7 we have $\overline{D}_\varphi = T_\varphi^*$, so for $f \in L^1(\omega)$ we have

$$\begin{aligned}\langle f, \overline{D}_\varphi\mu\rangle &= \langle T_\varphi f, \mu\rangle = \int_{\mathbb{R}^+} (T_\varphi f)(s)\,d\mu(s) \\ &= \int_{\mathbb{R}^+}\int_0^\infty f(t)\,\frac{s}{t+s}\,\varphi(t+s)\,dt\,d\mu(s) \\ &= \int_0^\infty \int_{\mathbb{R}^+} \frac{s}{t+s}\,\varphi(t+s)\,d\mu(s)\,f(t)\,dt \\ &= \int_0^\infty k(t)f(t)\,dt = \langle f, k\rangle.\end{aligned}$$

Hence $\overline{D}_\varphi\mu = k \in C_0(1/\omega)$ and the conclusions follow. □

We finish the section with the following result, which will be used in the next section.

Proposition 3.4. *Let $\varphi \in C_b(1/\omega)$ with $\varphi(0) = 0$. Then $(\overline{D}_\varphi \delta_s)$ is continuous in $C_0(1/\omega)$ for $s \in \mathbb{R}^+$.*

Proof. Clearly $\overline{D}_\varphi \delta_s \in C_0(1/\omega)$ for $s \in \mathbb{R}^+$. Let $s_0 > 0$. For $x > -s_0$ and $t \in \mathbb{R}^+$ we have

$$\begin{aligned}(\overline{D}_\varphi \delta_{s_0+x})(t) &= \frac{s_0 + x}{t + s_0 + x}\,\varphi(t + s_0 + x) \\ &= \frac{s_0 + x}{s_0}\,(\overline{D}_\varphi \delta_{s_0})(t + x) = \frac{s_0 + x}{s_0}\,(\delta_{-x} * \overline{D}_\varphi \delta_{s_0})(t).\end{aligned}$$

Since translation is continuous in $C_0(1/\omega)$ we have $\delta_{-x} * \overline{D}_\varphi \delta_{s_0} \to \overline{D}_\varphi \delta_{s_0}$ and thus $\overline{D}_\varphi \delta_{s_0+x} \to \overline{D}_\varphi \delta_{s_0}$ in $C_0(1/\omega)$ as $x \to 0$. Hence $(\overline{D}_\varphi \delta_s)$ is continuous in $C_0(1/\omega)$ at s_0. Finally, by Proposition 3.1(a) we have $\overline{D}_\varphi \delta_s \to \overline{D}_\varphi \delta_0 = 0$ in $C_0(1/\omega)$ as $s \to 0$, so $(\overline{D}_\varphi \delta_s)$ is also continuous in $C_0(1/\omega)$ at $s = 0$. □

4. Compactness

In this section we study compactness of the operators D_φ and $\overline{D}_\varphi$. The main result of the section is Theorem 4.4, which states that for $\varphi \in C_0(1/\omega)$ the operator $\overline{D}_\varphi$ is compact if and only if $\varphi(0) = 0$. We start with some results which show why the condition $\varphi(0) = 0$ as well as the continuity of φ seem to be close to necessary.

Proposition 4.1. *Let $\varphi \in L^\infty(1/\omega)$ and assume that $\overline{D}_\varphi$ is compact. Then $\varphi(s) \to 0$ as $s \to 0$.*

Proof. It follows from Proposition 2.1(a) that $\overline{D}_\varphi \delta_s \to 0$ weak-star in $L^\infty(1/\omega)$ as $s \to 0$. Since $\overline{D}_\varphi$ is compact, we then also have $\overline{D}_\varphi \delta_s \to 0$ in norm in $L^\infty(1/\omega)$ as $s \to 0$. Hence $\varphi(s) \to 0$ as $s \to 0$ by Proposition 3.1(a). □

Theorem 4.2. *Let $\varphi \in L^\infty(1/\omega)$ be real-valued and assume that there exist $t_0 \geq \delta > 0$ such that*

$$\operatorname*{ess\,inf}_{t \in (t_0 - \delta, t_0)} \varphi(t) > \operatorname*{ess\,sup}_{t \in (t_0, t_0 + \delta)} \varphi(t).$$

Then D_φ and $\overline{D}_\varphi$ are not compact.

Proof. Choose $\alpha \in \mathbb{R}$ and $\varepsilon > 0$ such that

$$\varphi(t) \geq \alpha + \varepsilon \text{ a.e. on } (t_0 - \delta, t_0) \quad \text{and} \quad \varphi(t) \leq \alpha - \varepsilon \text{ a.e. on } (t_0, t_0 + \delta).$$

Assume that D_φ is compact and let $f_n = n \cdot 1_{[t_0 - 1/n, t_0]}$ for $n \in \mathbb{N}$ with $1/n \leq t_0$. Then $(D_\varphi f_n)$ has a norm-convergent subsequence $(D_\varphi f_{n_k})$ with limit $h \in L^\infty(1/\omega)$. For $n \in \mathbb{N}$ with $1/n \leq t_0$ and $t \in \mathbb{R}^+$ we have

$$(D_\varphi f_n)(t) = n \int_{t_0 - 1/n}^{t_0} \frac{s}{t + s}\,\varphi(t + s)\,ds,$$

so $(D_\varphi f_n)(t) \leq \alpha - \varepsilon$ for $1/n \leq t \leq \delta$. Hence $h(t) \leq \alpha - \varepsilon$ a.e. on $[0, \delta]$. Let $n \in \mathbb{N}$ with $1/n \leq \delta$ $(\leq t_0)$ and let $t \leq 1/n$. Then

$$\begin{aligned}(D_\varphi f_n)(t) &= n \int_{t_0 - 1/n}^{t_0 - t} \frac{s}{t + s}\,\varphi(t + s)\,ds + n \int_{t_0 - t}^{t_0} \frac{s}{t + s}\,\varphi(t + s)\,ds \\ &= A_n(t) + B_n(t)\end{aligned}$$

with obvious notation. For $t_0 - 1/n \leq s \leq t_0 - t$ we have $t_0 - \delta \leq t_0 - 1/n \leq t + s \leq t_0$, so

$$A_n(t) \geq n\left(\frac{1}{n} - t\right) \frac{t_0 - 1/n}{t + t_0 - 1/n}(\alpha + \varepsilon) \geq (1 - nt)\,\frac{t_0 - 1/n}{t_0}(\alpha + \varepsilon).$$

Hence there exists $N_1 \in \mathbb{N}$ such that $A_n(t) \geq \alpha + \varepsilon/2$ for $n \geq N_1$ and $t \leq 1/n^2$. Also,

$$|B_n(t)| \leq nt\|\varphi\| \sup_{s \leq t_0 + \delta} \omega(s),$$

so there exists $N_2 \in \mathbb{N}$ such that $|B_n(t)| \leq \varepsilon/2$ for $n \geq N_2$ and $t \leq 1/n^2$. Hence $(D_\varphi f_n)(t) \geq \alpha$ for $n \geq \max\{N_1, N_2\}$ and $t \leq 1/n^2$. Hence $\|D_\varphi f_n - h\| \geq \varepsilon$ for $n \geq \max\{N_1, N_2\}$ which contradicts $D_\varphi f_{n_k} \to h$ in $L^\infty(1/\omega)$ a $k \to \infty$. Hence D_φ is not compact and as a consequence $\overline{D}_\varphi$ is not compact either. □

The following corollary implies that simple functions like $\varphi = 1_{[0,1]}$ do not generate compact derivations.

Corollary 4.3. *Let $\varphi \in L^\infty(1/\omega)$ be real-valued and assume that there exists $t_0 > 0$ such that the limits*

$$\lim_{t\to(t_0)_-} \varphi(t) \quad \text{and} \quad \lim_{t\to(t_0)_+} \varphi(t)$$

exist and are different. Then D_φ and $\overline{D}_\varphi$ are not compact.

Proof. The result follows directly from Theorem 4.2 if $\lim_{t\to(t_0)_-} \varphi(t) > \lim_{t\to(t_0)_+} \varphi(t)$. If the opposite inequality holds, then the result follows by considering $-\varphi$. $\square$

Because of the results above, we will focus on $\varphi \in C_b(1/\omega)$ with $\varphi(0) = 0$ in the rest of the paper. For $\varphi \in C_0(1/\omega)$ we have the following characterisation of compact $\overline{D}_\varphi$.

Theorem 4.4. *Let $\varphi \in C_0(1/\omega)$. Then $\overline{D}_\varphi$ is compact if and only if $\varphi(0) = 0$.*

Proof. If $\overline{D}_\varphi$ is compact, then $\varphi(0) = 0$ by Proposition 4.1.

Conversely, assume that $\varphi(0) = 0$ and let (μ_n) be a bounded sequence in $M(\omega)$. Since the weak-star topologies on bounded subsets of $M(\omega)$ and $L^\infty(1/\omega)$ are metrisable, we may by passing to subsequences assume that there exist $\mu \in M(\omega)$ and $h \in L^\infty(1/\omega)$ such that

$$\mu_n \to \mu \text{ weak-star in } M(\omega) \quad \text{and} \quad \overline{D}_\varphi\mu_n \to h \text{ weak-star in } L^\infty(1/\omega)$$

as $n \to \infty$. By Corollary 2.7 we have $\overline{D}_\varphi = T_\varphi^*$. For $f \in L^1(\omega)$ we thus have

$$\langle f, h\rangle = \lim_{n\to\infty} \langle f, \overline{D}_\varphi\mu_n\rangle = \lim_{n\to\infty} \langle T_\varphi f, \mu_n\rangle = \langle T_\varphi f, \mu\rangle = \langle f, \overline{D}_\varphi\mu\rangle,$$

so we deduce that $h = \overline{D}_\varphi\mu$. By Corollary 3.3(b) we have

$$(\overline{D}_\varphi\mu_n - h)(t) = (\overline{D}_\varphi(\mu_n - \mu))(t) = \langle\psi_t, \mu_n - \mu\rangle.$$

Also, (ψ_t) is continuous in $C_0(1/\omega)$ and $\psi_t/\omega(t) \to 0$ in $C_0(1/\omega)$ as $t \to \infty$ by Lemma 3.2(b), so $\{\psi_t/\omega(t) : t \in \mathbb{R}^+\}$ is totally bounded in $C_0(1/\omega)$. Let $\varepsilon > 0$. There exist $t_1, \ldots, t_M \in \mathbb{R}^+$ such that for every $t \in \mathbb{R}^+$ there exists $m \in \{1, \ldots, M\}$ with $\|\psi_t/\omega(t) - \psi_{t_m}/\omega(t_m)\| < \varepsilon$. Choose $N \in \mathbb{N}$ such that $|\langle\psi_{t_m}/\omega(t_m), \mu_n - \mu\rangle| < \varepsilon$ for $m = 1, \ldots, M$ and $n \geq N$. For $t \in \mathbb{R}^+$ and $n \geq N$ we thus have

$$\begin{aligned} \frac{|(\overline{D}_\varphi\mu_n - h)(t)|}{\omega(t)} &= \left|\left\langle \frac{\psi_t}{\omega(t)}, \mu_n - \mu\right\rangle\right| \\ &\leq \left|\left\langle \frac{\psi_{t_m}}{\omega(t_m)}, \mu_n - \mu\right\rangle\right| + \left|\left\langle \frac{\psi_t}{\omega(t)} - \frac{\psi_{t_m}}{\omega(t_m)}, \mu_n - \mu\right\rangle\right| \\ &< (1 + \sup_{n\in\mathbb{N}} \|\mu_n - \mu\|)\varepsilon. \end{aligned}$$

Hence $\overline{D}_\varphi\mu_n \to h$ in $C_0(1/\omega)$ as $n \to \infty$, and we conclude that $\overline{D}_\varphi$ is compact. $\square$

The next few results will show the existence of $\varphi \notin C_0(1/\omega)$ for which D_φ is compact. We do not know whether the approach can be extended to show that $\overline{D}_\varphi$ is compact, and more generally we do not know whether $\overline{D}_\varphi$ is necessarily compact if D_φ is compact. The idea in the following result is to represent the derivation D_φ by Bochner integrals and then use a pre-compactness argument.

Proposition 4.5. *Let $\varphi \in C_b(1/\omega)$. Assume that $\varphi(0) = 0$ and $\overline{D}_\varphi\delta_s/\omega(s)$ has a limit in $L^\infty(1/\omega)$ as $s \to \infty$. Then D_φ is compact.*

Proof. Let $f \in L^1(\omega)$. We observe that

$$(D_\varphi f)(t) = \int_0^\infty f(s)(\overline{D}_\varphi\delta_s)(t)\,ds \quad \text{for } t \in \mathbb{R}^+.$$

Moreover, $(\overline{D}_\varphi\delta_s)$ is continuous in $C_0(1/\omega)$ for $s \in \mathbb{R}^+$ by Proposition 3.4, so

$$D_\varphi f = \int_0^\infty f(s)\overline{D}_\varphi\delta_s\,ds = \int_0^\infty f(s)\omega(s)\,\frac{\overline{D}_\varphi\delta_s}{\omega(s)}\,ds$$

exists as a Bochner integral (see [12, Theorem 3.7.4]). Also, the map $s \mapsto \overline{D}_\varphi\delta_s/\omega(s)$ extends to a continuous map from the one point compactification $[0, \infty]$ of $\mathbb{R}^+$ into $C_0(1/\omega)$, so we deduce that $\{\overline{D}_\varphi\delta_s/\omega(s) : s \in \mathbb{R}^+\} \cup \{\lim_{s\to\infty} \overline{D}_\varphi\delta_s/\omega(s)\}$ is compact in $C_0(1/\omega)$. It thus follows from [6, VI.8.11] (see also [2, Proof of Theorem 2.2]) that D_φ is compact. $\square$

For $\varphi \in C_0(1/\omega)$ with $\varphi(0) = 0$ we have $\overline{D}_\varphi \delta_s/\omega(s) \to 0$ in $C_0(1/\omega)$ as $s \to \infty$ by Proposition 3.1(b), so D_φ is compact by Proposition 4.5. We can therefore recapture the conclusion about D_φ from Theorem 4.4.

We will now use Proposition 4.5 to obtain concrete examples of functions $\varphi \notin C_0(1/\omega)$ which generate compact derivations D_φ.

Proposition 4.6. *Assume that $\omega(s) \geq 1$ for every $s \in \mathbb{R}^+$, $\omega(s) \to \infty$ as $s \to \infty$ and*

$$\sup_{t\in\mathbb{R}^+} \frac{|\omega(t+s) - \omega(s)|}{\omega(t)\omega(s)} \to 0 \quad \text{as } s \to \infty.$$

Let $\varphi = \omega - 1$. Then D_φ is compact, whereas $\varphi \notin C_0(1/\omega)$.

Remark. If we let $\rho_s(t) = \omega(t+s) - \omega(s)$ for $t, s \in \mathbb{R}^+$, then the last assumption can be restated as $\rho_s/\omega(s) \to 0$ in $L^\infty(1/\omega)$ as $s \to \infty$.

Proof. By Proposition 4.5 the result will follow if we can prove that $\overline{D}_\varphi \delta_s/\omega(s) \to 1$ in $L^\infty(1/\omega)$ as $s \to \infty$. We have

$$\frac{\overline{D}_\varphi \delta_s}{\omega(s)} - 1 = \frac{\frac{s}{t+s}(\omega(t+s) - 1)}{\omega(s)} - 1 = \frac{s\omega(t+s) - s - (t+s)\omega(s)}{(t+s)\omega(s)},$$

so

$$\begin{aligned} \left\| \frac{\overline{D}_\varphi \delta_s}{\omega(s)} - 1 \right\| &= \sup_{t\in\mathbb{R}^+} \frac{|s(\omega(t+s) - \omega(s)) - s - t\omega(s)|}{(t+s)\omega(t)\omega(s)} \\ &\leq \sup_{t\in\mathbb{R}^+} \frac{|\omega(t+s) - \omega(s)|}{\omega(t)\omega(s)} + \frac{1}{\omega(s)} + \sup_{t\in\mathbb{R}^+} \frac{t}{(t+s)\omega(t)}. \end{aligned} \tag{2}$$

Given $\varepsilon > 0$ we choose $T > 0$ such that $\omega(t) \geq 1/\varepsilon$ for $t \geq T$. Then

$$\sup_{t\geq T} \frac{t}{(t+s)\omega(t)} \leq \varepsilon$$

for every $s \in \mathbb{R}^+$. Also, for $s \geq T/\varepsilon$ we have

$$\sup_{t\leq T} \frac{t}{(t+s)\omega(t)} \leq \frac{T}{T + T/\varepsilon} < \varepsilon.$$

Hence the third term in (2) tends to 0 as $s \to \infty$. Moreover, the first and second term tend to 0 as $s \to \infty$ by assumption, so we conclude that $\overline{D}_\varphi \delta_s/\omega(s) \to 1$ in $L^\infty(1/\omega)$ as $s \to \infty$. $\square$

Corollary 4.7. *Let $\alpha > 0$, $\omega(t) = (1+t)^\alpha$ $(t \in \mathbb{R}^+)$ and let $\varphi = \omega - 1$. Then D_φ is compact, whereas $\varphi \notin C_0(1/\omega)$.*

Proof. We clearly have $\varphi \notin C_0(1/\omega)$. First, assume that $\alpha \geq 1$. Then

$$\begin{aligned} 0 &\leq \omega(t+s) - \omega(s) = (1+t+s)^\alpha - (1+s)^\alpha \\ &= \alpha \int_s^{t+s} (1+x)^{\alpha-1}\, dx \leq \alpha t(1+t+s)^{\alpha-1} \end{aligned}$$

for $t, s \in \mathbb{R}^+$, so

$$\begin{aligned} \sup_{t\in\mathbb{R}^+} \frac{|\omega(t+s) - \omega(s)|}{\omega(t)} &\leq \frac{\alpha t(1+t+s)^{\alpha-1}}{(1+t)^\alpha} \leq \alpha \left(\frac{1+t+s}{1+t} \right)^{\alpha-1} \\ &\leq \alpha(1+s)^{\alpha-1} \end{aligned}$$

for $s \in \mathbb{R}^+$.

Next, assume that $\alpha < 1$. Then the function $s \mapsto (1+t+s)^\alpha - (1+s)^\alpha$ is decreasing on $\mathbb{R}^+$ for every $t \in \mathbb{R}^+$, so we deduce that

$$\omega(t+s) - \omega(s) = (1+t+s)^\alpha - (1+s)^\alpha \leq (1+t)^\alpha - 1 < \omega(t)$$

for $t, s \in \mathbb{R}^+$. Consequently,

$$\sup_{t\in\mathbb{R}^+} \frac{|\omega(t+s) - \omega(s)|}{\omega(t)} \leq 1$$

for $s \in \mathbb{R}^+$.

In both cases the result thus follows from Proposition 4.6. $\square$

We finish the paper by showing that the condition $\varphi(t)/\omega(t) \to 0$ as $t \to \infty$ from Theorem 4.4 cannot in general be relaxed to $\varphi(t)/\omega(t) \to \alpha$ as $t \to \infty$ for some $\alpha \in \mathbb{C}$. (For $\varphi \in L^\infty(\mathbb{R}^+)$ we say that $\varphi(t) \to \alpha \in \mathbb{C}$ as $t \to \infty$ if $\varphi(t) - \alpha \to 0$ as $t \to \infty$, that is, if $\operatorname{ess\,sup}_{t\geq T}|\varphi(t) - \alpha| \to 0$ as $T \to \infty$.)

Proposition 4.8. *Let $\varphi \in L^\infty(\mathbb{R}^+)$ and assume that $\varphi(t) \to \alpha$ as $t \to \infty$ for some $\alpha \neq 0$. Then $D_\varphi : L^1(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ and $\overline{D}_\varphi : M(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ are not compact.*

Proof. Let $\tilde{\varphi} = \varphi - \alpha$, so that $\tilde{\varphi}(t) \to 0$ as $t \to \infty$. Let $f_n = 1_{[n,n+1]}$ for $n \in \mathbb{N}$. For $t \in \mathbb{R}^+$ we have

$$|(D_{\tilde{\varphi}}f_n)(t)| \leq \int_n^{n+1} |\tilde{\varphi}(t+s)|\,ds \leq \operatorname*{ess\,sup}_{s\geq n} |\tilde{\varphi}(s)|,$$

so we deduce that $D_{\tilde{\varphi}}f_n \to 0$ in $L^\infty(\mathbb{R}^+)$ as $n \to \infty$. Moreover,

$$(D_\alpha f_n)(t) - \alpha = \alpha \int_n^{n+1} \frac{s}{t+s}\,ds - \alpha = -\alpha \int_n^{n+1} \frac{t}{t+s}\,ds$$

for $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$, so $\|D_\alpha f_n - \alpha\| = |\alpha|$ for $n \in \mathbb{N}$. Since $D_\varphi = D_{\tilde{\varphi}} + D_\alpha$, we thus have $\|D_\varphi f_n - \alpha\| \to |\alpha|$ as $n \to \infty$. On the other hand, for $n \in \mathbb{N}$ and $t \in \mathbb{R}^+$ we have

$$|(D_\alpha f_n)(t) - \alpha| \leq |\alpha|\frac{t}{t+n},$$

so it follows from Lebesgue's dominated convergence theorem that $D_\alpha f_n \to \alpha$ weak-star in $L^\infty(\mathbb{R}^+)$ and thus $D_\varphi f_n \to \alpha$ weak-star in $L^\infty(\mathbb{R}^+)$ as $n \to \infty$. We thus deduce that $(D_\varphi f_n)$ has no cluster point as $n \to \infty$. Hence D_φ and $\overline{D}_\varphi$ are not compact. $\square$

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