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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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Abstract. In this paper we prove that for any associative ring \(R\), and for any left \(R\)-module \(M\) with finite projective dimension, the Gorenstein injective dimension \(\text{Gid}_R M\) equals the usual injective dimension \(\text{id}_R M\). In particular, if \(\text{Gid}_R R\) is finite, then also \(\text{id}_R R\) is finite, and thus \(R\) is Gorenstein (provided that \(R\) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \((R, \mathfrak{m}, k)\), the Gorenstein rings are characterized by the condition \(\text{id}_R R < \infty\). From the dual of [10, Proposition (2.27)] (13 Proposition 10.2.3] is a special case) it follows that the Gorenstein injective dimension \(\text{Gid}_R (\cdot)\) is a refinement of the usual injective dimension \(\text{id}_R (\cdot)\) in the following sense:

For any \(R\)-module \(M\) there is an inequality \(\text{Gid}_R M \leq \text{id}_R M\), and if \(\text{id}_R M < \infty\), then there is an equality \(\text{Gid}_R M = \text{id}_R M\).

Now, since the injective dimension \(\text{id}_R R\) of \(R\) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \(\text{Gid}_R R\) of \(R\) measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring \(R\) with \(\text{Gid}_R R < \infty\) also has \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

This result is proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that \((R, \mathfrak{m}, k)\) is a commutative local Noetherian ring, and let \(M\) be an \(R\)-module of finite depth, that is, \(\text{Ext}^m_R(k, M) \neq 0\) for some \(m \in \mathbb{N}_0\) (this happens for example if \(M \neq 0\) is finitely generated). If either

(i) \(\text{Gid}_R M < \infty\) and \(\text{id}_R M < \infty\) or
(ii) \(\text{fd}_R M < \infty\) and \(\text{Gid}_R M < \infty\),

then \(R\) is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, m, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem \((3.2)\) itself (dealing not only with local rings) is a generalization of \([8, \text{Proposition 2.10}]\) (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in \([3]\). The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger \([1]\) in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see \([5]\).

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_R M\), \(\text{fd}_R M\), and \(\text{id}_R M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_R M\), \(\text{Gfd}_R M\), and \(\text{Gid}_R M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\). In particular, if \(\text{Gid}_R R < \infty\), then also \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_R M \leq \text{id}_R M\) always, it suffices to prove that \(\text{id}_R M \leq \text{Gid}_R M\). Naturally, we may assume that \(\text{Gid}_R M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_R M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \text{Ker}(E_1 \to E_0)\). In particular, there exists a short exact sequence \(0 \to M' \to E \to M \to 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows by \([11, \text{Lemma 1.3}]\) that \(\text{Ext}^1_R(M, M') = 0\). Thus \(0 \to M' \to E \to M \to 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_R M > 0\). By \([10, \text{Theorem (2.15)}]\) there exists an exact sequence \(0 \to M \to H \to C \to 0\) where \(H\) is Gorenstein injective and \(\text{id}_R C = \text{Gid}_R M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \to H' \to I \to H \to 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & & \\
0 & M & H \\
\downarrow & & \\
0 & P & I \\
\downarrow & & \\
0 & C & 0 \\
\end{array}
\]

Since \(I\) is injective and \(\text{id}_R M < 1\) we get \(\text{id}_R P \leq \text{Gid}_R M\) by the second row. Since \(H'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows (as before) by [4, Lemma 1.3] that \(\text{Ext}_R^1(M, H') = 0\). Consequently, the first column \(0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0\) splits. Therefore \(P \cong M \oplus H'\), and hence \(\text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M\).

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \(M\) is an \(R\)-module with \(\text{id}_R M < \infty\), then \(\text{Gpd}_R M = \text{pd}_R M\).

Theorem (2.6) below is a "flat version" of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 2.4.** When \(R\) is commutative and Noetherian, we have that \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) equals the Krull dimension of \(R\), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \(R\)-module \(M\) the inequality

\[
\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M
\]

holds. If \(R\) is right coherent, then we have \(\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M\).

We are now ready to state:

**Theorem 2.6.** For any \(R\)-module \(M\), the following conclusions hold:

(i) Assume that \(\text{LeftFPD}(R)\) is finite. If \(\text{fd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\).

(ii) Assume that \(R\) is left and right coherent with finite \(\text{RightFPD}(R)\). If \(\text{id}_R M < \infty\), then \(\text{Gfd}_R M = \text{fd}_R M\).

**Proof.** (i) If \(\text{fd}_R M < \infty\), then also \(\text{pd}_R M < \infty\), by [11, Proposition 6] (since \(\text{LeftFPD}(R) < \infty\)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \(R\) is left coherent, we have that \(\text{fd}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty\), by [12, Lemma 3.1.4]. By assumption, \(\text{RightFPD}(R) < \infty\), and therefore also
pd_R\, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [11] Proposition 6]. Now Theorem 2.1 gives that 
\text{Gid}_R\, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_R\, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}). It is well known that 
\text{fd}_R M = \text{id}_R\, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) 
(without assumptions on \(R\)), and by Proposition 2.5 above, we also get \text{Gfd}_R M = \text{Gid}_R\, \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$, since \(R\) is right coherent. The proof is done. \(\square\)

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that \(R\) is commutative and Noetherian. For an \(R\)-module \(M\), an integer \(n\), and a prime ideal \(p\) in \(R\), we write \(\beta_n^R(p, M)\), respectively, \(\mu_n^R(p, M)\), for the \(n\)th Betti number, respectively, \(n\)th Bass number, of \(M\) at \(p\).

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an \(R\)-module \(M\) to be the set 
\[ \text{supp}_R M = \{ p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_0 : \beta_n^R(p, M) \neq 0 \}. \]

Let us mention the most basic results about the small support, all of which can be found in [8] pp. 157 - 159 and [7] Chapter 14:

(a) The small support, \(\text{supp}_R M\), is contained in the usual (large) support, \(\text{Supp}_R M\), and \(\text{supp}_R M = \text{Supp}_R M\) if \(M\) is finitely generated. Also, if \(M \neq 0\), then \(\text{supp}_R M \neq 0\).

(b) \(\text{supp}_R M = \{ p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_0 : \mu_n^R(p, M) \neq 0 \}\).

(c) Assume that \((R, m, k)\) is local. If \(M\) is an \(R\)-module with finite depth, that is, \(\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty\) (this happens for example if \(M \neq 0\) is finitely generated), then \(m \in \text{supp}_R M\), by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

**Theorem 3.2.** Assume that \(R\) is commutative and Noetherian. Let \(M\) be any \(R\)-module, and assume that any of the following four conditions is satisfied:

(i) \(\text{Gpd}_R M < \infty\) and \(\text{id}_R M < \infty\),

(ii) \(\text{pd}_R M < \infty\) and \(\text{Gid}_R M < \infty\),

(iii) \(R\) has finite Krull dimension, and \(\text{Gfd}_R M < \infty\) and \(\text{id}_R M < \infty\),

(iv) \(R\) has finite Krull dimension, and \(\text{fd}_R M < \infty\) and \(\text{Gid}_R M < \infty\).

Then \(R_p\) is a Gorenstein local ring for all \(p \in \text{supp}_R M\). \(\square\)

**Corollary 3.3.** Assume that \((R, m, k)\) is a commutative local Noetherian ring. If there exists an \(R\)-module \(M\) of finite depth, that is, \(\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty\),

and which satisfies either

(i) \(\text{Gfd}_R M < \infty\) and \(\text{id}_R M < \infty\), or

(ii) \(\text{fd}_R M < \infty\) and \(\text{Gid}_R M < \infty\),

then \(R\) is Gorenstein. \(\square\)
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References


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