Rings with finite Gorenstein injective dimension

Holm, Henrik Granau

Published in:
Proceedings of the American Mathematical Society

Publication date:
2004

Document version
Peer reviewed version

Citation for published version (APA):
RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

(Communicated by Bernd Ulrich)

Abstract. In this paper we prove that for any associative ring \( R \), and for any left \( R \)-module \( M \) with finite projective dimension, the Gorenstein injective dimension \( \text{Gid}_R M \) equals the usual injective dimension \( \text{id}_R M \). In particular, if \( \text{Gid}_R R \) is finite, then also \( \text{id}_R R \) is finite, and thus \( R \) is Gorenstein (provided that \( R \) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \( (R, \mathfrak{m}, k) \), the Gorenstein rings are characterized by the condition \( \text{id}_R R < \infty \). From the dual of \cite{10} Proposition (2.27) \( \cite{10} \) Proposition 10.2.3 is a special case) it follows that the Gorenstein injective dimension \( \text{Gid}_R (\cdot) \) is a refinement of the usual injective dimension \( \text{id}_R (\cdot) \) in the following sense:

For any \( R \)-module \( M \) there is an inequality \( \text{Gid}_R M \leq \text{id}_R M \), and if \( \text{id}_R M < \infty \), then there is an equality \( \text{Gid}_R M = \text{id}_R M \).

Now, since the injective dimension \( \text{id}_R R \) of \( R \) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \( \text{Gid}_R R \) of \( R \) measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring \( R \) with \( \text{Gid}_R R < \infty \) also has \( \text{id}_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

This result is proved by Christensen \cite{2} Theorem (6.3.2)] in the case where \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian ring, and let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}^m_R(k, M) \neq 0 \) for some \( m \in \mathbb{N}_0 \) (this happens for example if \( M \neq 0 \) is finitely generated). If either

(i) \( \text{Gid}_R M < \infty \) and \( \text{id}_R M < \infty \) or
(ii) \( \text{fd}_R M < \infty \) and \( \text{Gid}_R M < \infty \),

then \( R \) is Gorenstein.

Received by the editors January 28, 2003.

2000 Mathematics Subject Classification. Primary 13D02, 13D05, 13D07, 13H10; Secondary 16E05, 16E10, 16E30.

Key words and phrases. Gorenstein dimensions, homological dimensions, Gorenstein rings.

©2003 American Mathematical Society
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \( R \) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \( R \)-modules. If \( M \) is any \( R \)-module, we use \( \text{pd}_R M \), \( \text{fd}_R M \), and \( \text{id}_R M \) to denote the usual projective, flat, and injective dimension of \( M \), respectively. Furthermore, we write \( \text{Gpd}_R M \), \( \text{Gfd}_R M \), and \( \text{Gid}_R M \) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \( M \), respectively.

2. Rings with finite Gorenstein injective dimension

Theorem 2.1. If \( M \) is an \( R \)-module with \( \text{pd}_R M < \infty \), then \( \text{Gid}_R M = \text{id}_R M \). In particular, if \( \text{Gid}_R R < \infty \), then also \( \text{id}_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

Proof. Since \( \text{Gid}_R M \leq \text{id}_R M \) always, it suffices to prove that \( \text{id}_R M < \text{Gid}_R M \). Naturally, we may assume that \( \text{Gid}_R M < \infty \).

First consider the case where \( M \) is Gorenstein injective, that is, \( \text{Gid}_R M = 0 \). By definition, \( M \) is a kernel in a complete injective resolution. This means that there exists an exact sequence \( E = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots \) of injective \( R \)-modules, such that \( \text{Hom}_R(I, E) \) is exact for every injective \( R \)-module \( I \), and such that \( M \cong \ker(E_1 \to E_0) \). In particular, there exists a short exact sequence \( 0 \to M' \to E \to M \to 0 \), where \( E \) is injective, and \( M' \) is Gorenstein injective. Since \( M' \) is Gorenstein injective and \( \text{pd}_R M < \infty \), it follows by [4, Lemma 1.3] that \( \text{Ext}^1_R(M, M') = 0 \). Thus \( 0 \to M' \to E \to M \to 0 \) is split-exact; so \( M \) is a direct summand of the injective module \( E \). Therefore, \( M \) itself is injective.

Next consider the case where \( \text{Gid}_R M > 0 \). By [10, Theorem (2.15)] there exists an exact sequence \( 0 \to M \to H \to C \to 0 \) where \( H \) is Gorenstein injective and \( \text{id}_R C = \text{Gid}_R M - 1 \). As in the previous case, since \( H \) is Gorenstein injective, there exists a short exact sequence \( 0 \to H' \to I \to H \to 0 \) where \( I \) is injective and \( H' \) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{c}
0 & 0 \\
0 & 0 \\
0 & 0 \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \\
\rightarrow & M & \rightarrow H & \rightarrow C & \rightarrow 0 \\
\rightarrow & P & \rightarrow I & \rightarrow C & \rightarrow 0 \\
\rightarrow & H' & \rightarrow H' & & \\
\rightarrow & 0 & & & \\
\end{array}
\]

Since \(I\) is injective and \(id_R C = \text{Gid}_R M - 1\) we get \(id_R P \leq \text{Gid}_R M\) by the second row. Since \(H'\) is Gorenstein injective and \(pd_R M < \infty\), it follows (as before) by [4, Lemma 1.3] that \(\text{Ext}_1^R(M, H') = 0\). Consequently, the first column \(0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0\) splits. Therefore \(P \cong M \oplus H'\), and hence \(id_R M \leq id_R P \leq \text{Gid}_R M\). \(\square\)

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \(M\) is an \(R\)-module with \(id_R M < \infty\), then \(\text{Gpd}_R M = \text{pd}_R M\). \(\square\)

Theorem 2.6 below is a “flat version” of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \} \]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 2.4.** When \(R\) is commutative and Noetherian, we have that \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) equals the Krull dimension of \(R\), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \(R\)-module \(M\) the inequality

\[\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M\]

holds. If \(R\) is right coherent, then we have \(\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M\). \(\square\)

We are now ready to state:

**Theorem 2.6.** For any \(R\)-module \(M\), the following conclusions hold:

(i) Assume that \(\text{LeftFPD}(R)\) is finite. If \(\text{fd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\).

(ii) Assume that \(R\) is left and right coherent with finite \(\text{RightFPD}(R)\). If \(\text{id}_R M < \infty\), then \(\text{Gfd}_R M = \text{fd}_R M\).

**Proof.** (i) If \(\text{fd}_R M < \infty\), then also \(\text{pd}_R M < \infty\), by [11, Proposition 6] (since \(\text{LeftFPD}(R) < \infty\)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \(R\) is left coherent, we have that \(\text{fd}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty\), by [12, Lemma 3.1.4]. By assumption, \(\text{RightFPD}(R) < \infty\), and therefore also...
pd_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by \[11\, \text{Proposition} \, 6\]. Now Theorem \[24\] gives that \( \text{Gid}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \). It is well known that \( \text{fd}_{R} M = \text{id}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \) (without assumptions on \( R \)), and by Proposition \[25\] above, we also get \( \text{Gfd}_{R} M = \text{Gid}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \), since \( R \) is right coherent. The proof is done. \( \square \)

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem \[8, \text{Proposition} \, 2.10\] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that \( R \) is commutative and Noetherian. For an \( R \)-module \( M \), an integer \( n \), and a prime ideal \( p \) in \( R \), we write \( \beta_{n}^{R}(p, M) \), respectively, \( \mu_{n}^{R}(p, M) \), for the \( n \)th Betti number, respectively, \( n \)th Bass number, of \( M \) at \( p \).

Foxby \[8, \text{Definition} \, p. \, 157\] or \[7, (14.8)\] defines the small (or homological) support of an \( R \)-module \( M \) to be the set

\[ \text{supp}_{R} M = \{ \, p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_{0}: \beta_{n}^{R}(p, M) \neq 0 \, \} \]

Let us mention the most basic results about the small support, all of which can be found in \[8, \text{pp.} \, 157 - 159\] and \[7, \text{Chapter} \, 14\]:

(a) The small support, \( \text{supp}_{R} M \), is contained in the usual (large) support, \( \text{Supp}_{R} M \), and \( \text{supp}_{R} M = \text{Supp}_{R} M \) if \( M \) is finitely generated. Also, if \( M \neq 0 \), then \( \text{supp}_{R} M \neq 0 \).

(b) \( \text{supp}_{R} M = \{ \, p \in \text{Spec} \, R \mid \exists n \in \mathbb{N}_{0}: \mu_{n}^{R}(p, M) \neq 0 \, \} \).

(c) Assume that \((R, m, k)\) is local. If \( M \) is an \( R \)-module with finite depth, that is,

\[ \text{depth}_{R} M := \inf \{ \, m \in \mathbb{N}_{0} \mid \text{Ext}_{R}^{m}(k, M) \neq 0 \, \} < \infty \]

(this happens for example if \( M \neq 0 \) is finitely generated), then \( m \in \text{supp}_{R} M \), by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of \[8, \text{Proposition} \, 2.10\] is immediate:

**Theorem 3.2.** Assume that \( R \) is commutative and Noetherian. Let \( M \) be any \( R \)-module, and assume that any of the following four conditions is satisfied:

(i) \( \text{Gpd}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \),

(ii) \( \text{pd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \),

(iii) \( R \) has finite Krull dimension, and \( \text{Gfd}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \),

(iv) \( R \) has finite Krull dimension, and \( \text{fd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \).

Then \( R_{p} \) is a Gorenstein local ring for all \( p \in \text{supp}_{R} M \). \( \square \)

**Corollary 3.3.** Assume that \((R, m, k)\) is a commutative local Noetherian ring. If there exists an \( R \)-module \( M \) of finite depth, that is,

\[ \text{depth}_{R} M := \inf \{ \, m \in \mathbb{N}_{0} \mid \text{Ext}_{R}^{m}(k, M) \neq 0 \, \} < \infty \]

and which satisfies either

(i) \( \text{Gfd}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \), or

(ii) \( \text{fd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \),

then \( R \) is Gorenstein. \( \square \)
Acknowledgments

I would like to express my gratitude to my Ph.D. advisor Hans-Bjørn Foxby for his support, and our helpful discussions.

References


MATEMATISK AFDELING, KØBENHAVNS UNIVERSITET, UNIVERSITETSPARKEN 5, 2100 KØBENHAVN Ø, DANMARK
E-mail address: holm@math.ku.dk