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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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Abstract. In this paper we prove that for any associative ring \( R \), and for any left \( R \)-module \( M \) with finite projective dimension, the Gorenstein injective dimension \( \text{Gid}_R M \) equals the usual injective dimension \( \text{id}_R M \). In particular, if \( \text{Gid}_R R \) is finite, then also \( \text{id}_R R \) is finite, and thus \( R \) is Gorenstein (provided that \( R \) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \( (R, \mathfrak{m}, k) \), the Gorenstein rings are characterized by the condition \( \text{id}_R R < \infty \). From the dual of [10, Proposition (2.27)] \( \text{(10, Proposition 10.2.3) is a special case} \) it follows that the Gorenstein injective dimension \( \text{Gid}_R (\cdot) \) is a refinement of the usual injective dimension \( \text{id}_R (\cdot) \) in the following sense:

For any \( R \)-module \( M \) there is an inequality \( \text{Gid}_R M \leq \text{id}_R M \), and if \( \text{id}_R M < \infty \), then there is an equality \( \text{Gid}_R M = \text{id}_R M \).

Now, since the injective dimension \( \text{id}_R R \) of \( R \) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \( \text{Gid}_R R \) of \( R \) measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring \( R \) with \( \text{Gid}_R R < \infty \) also has \( \text{id}_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

This result is proved by Christensen \( \text{[2, Theorem (6.3.2)] in the case where} \) \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian ring, and let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}^m_R(k, M) \neq 0 \) for some \( m \in \mathbb{N}_0 \) (this happens for example if \( M \neq 0 \) is finitely generated). If either

\( \text{(i) } \text{Gid}_R M < \infty \text{ and } \text{id}_R M < \infty \) or \( \text{(ii) } \text{fd}_R M < \infty \text{ and } \text{Gid}_R M < \infty \),

then \( R \) is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R,m,k)\) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_RM\), \(\text{fd}_RM\), and \(\text{id}_RM\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_RM\), \(\text{Gfd}_RM\), and \(\text{Gid}_RM\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_RM < \infty\), then \(\text{Gid}_RM = \text{id}_RM\). In particular, if \(\text{Gid}_R \subset < \infty\), then also \(\text{id}_R \subset < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_RM \subset \text{id}_RM\) always, it suffices to prove that \(\text{id}_RM \subset \text{Gid}_RM\). Naturally, we may assume that \(\text{Gid}_RM \subset < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_RM = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(I,E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \text{Ker}(E_1 \rightarrow E_0)\). In particular, there exists a short exact sequence \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_RM \subset < \infty\), it follows by [3] Lemma 1.3] that \(\text{Ext}_R^1(M,M') = 0\). Thus \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_RM > 0\). By [10] Theorem (2.15)] there exists an exact sequence \(0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0\) where \(H\) is Gorenstein injective and \(\text{id}_RC = \text{Gid}_RM - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \rightarrow H' \rightarrow I \rightarrow H \rightarrow 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{c}
0 & & 0 \\
0 & \rightarrow & M & \rightarrow & H & \rightarrow & C & \rightarrow & 0 \\
0 & \rightarrow & P & \rightarrow & I & \rightarrow & C & \rightarrow & 0 \\
H' & \rightarrow & H'
\end{array}
\]

Since \( I \) is injective and \( \text{id}_R M \leq G \text{id}_R M \) by the second row. Since \( H' \) is Gorenstein injective and \( \text{pd}_R M < \infty \), it follows (as before) by \([4\text{, Lemma 1.3]}\) that \( \text{Ext}^1_R(M, H') = 0 \). Consequently, the first column \( 0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0 \) splits. Therefore \( P \cong M \oplus H' \), and hence \( \text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M \).

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \( M \) is an \( R \)-module with \( \text{id}_R M < \infty \), then \( \text{Gpd}_R M = \text{pd}_R M \).

Theorem (2.6) below is a “flat version” of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \( \text{LeftFPD}(R) \) of \( R \) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(R) \) of \( R \) is defined similarly.

**Remark 2.4.** When \( R \) is commutative and Noetherian, we have that \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) equals the Krull dimension of \( R \), by \([3\text{, Théorème (3.2.6)}]\) (Seconde partie)].

Furthermore, we will need the following result from \([10\text{, Proposition (3.11)}]\):

**Proposition 2.5.** For any (left) \( R \)-module \( M \) the inequality

\[
\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M
\]

holds. If \( R \) is right coherent, then we have \( \text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M \).

We are now ready to state:

**Theorem 2.6.** For any \( R \)-module \( M \), the following conclusions hold:

(i) Assume that \( \text{LeftFPD}(R) \) is finite. If \( \text{fd}_R M < \infty \), then \( \text{Gid}_R M = \text{id}_R M \).

(ii) Assume that \( R \) is left and right coherent with finite \( \text{RightFPD}(R) \). If \( \text{id}_R M < \infty \), then \( \text{Gfd}_R M = \text{fd}_R M \).

**Proof.** (i) If \( \text{fd}_R M < \infty \), then also \( \text{pd}_R M < \infty \), by \([11\text{, Proposition 6]}\) (since \( \text{LeftFPD}(R) < \infty \)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \( R \) is left coherent, we have that \( \text{fd}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty \), by \([12\text{, Lemma 3.1.4]}\). By assumption, \( \text{RightFPD}(R) < \infty \), and therefore also
pd_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [4] Proposition 6. Now Theorem 2.1 gives that
\[ \text{Gid}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \]
(without assumptions on $R$), and by Proposition 2.5 above, we also get $\text{Gfd}_{R} M = \text{Gid}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$, since $R$ is right coherent. The proof is done.

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [5, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that $R$ is commutative and Noetherian. For an $R$-module $M$, an integer $n$, and a prime ideal $p$ in $R$, we write $\beta^{R}_{n}(p, M)$, respectively, $\mu^{R}_{n}(p, M)$, for the $n$th Betti number, respectively, $n$th Bass number, of $M$ at $p$.

Foxby [5, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an $R$-module $M$ to be the set
\[ \text{supp}_{R} M = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_{0} : \beta^{R}_{n}(p, M) \neq 0 \} \]

Let us mention the most basic results about the small support, all of which can be found in [5] pp. 157 - 159 and [7] Chapter 14:

(a) The small support, $\text{supp}_{R} M$, is contained in the usual (large) support, $\text{Supp}_{R} M$, and $\text{supp}_{R} M = \text{Supp}_{R} M$ if $M$ is finitely generated. Also, if $M \neq 0$, then $\text{supp}_{R} M \neq \emptyset$.

(b) $\text{supp}_{R} M = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_{0} : \mu^{R}_{n}(p, M) \neq 0 \}$.

(c) Assume that $(R, m, k)$ is local. If $M$ is an $R$-module with finite depth, that is,
\[ \text{depth}_{R} M := \inf \{ m \in \mathbb{N}_{0} \mid \text{Ext}^{m}_{R}(k, M) \neq 0 \} < \infty \]

(this happens for example if $M \neq 0$ is finitely generated), then $m \in \text{supp}_{R} M$, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [5, Proposition 2.10] is immediate:

**Theorem 3.2.** Assume that $R$ is commutative and Noetherian. Let $M$ be any $R$-module, and assume that any of the following four conditions is satisfied:

(i) $\text{Gpd}_{R} M < \infty$ and $\text{id}_{R} M < \infty$,

(ii) $\text{pd}_{R} M < \infty$ and $\text{Gid}_{R} M < \infty$,

(iii) $R$ has finite Krull dimension, and $\text{Gfd}_{R} M < \infty$ and $\text{id}_{R} M < \infty$,

(iv) $R$ has finite Krull dimension, and $\text{fd}_{R} M < \infty$ and $\text{Gid}_{R} M < \infty$.

Then $R_{p}$ is a Gorenstein local ring for all $p \in \text{supp}_{R} M$.

**Corollary 3.3.** Assume that $(R, m, k)$ is a commutative local Noetherian ring. If there exists an $R$-module $M$ of finite depth, that is,
\[ \text{depth}_{R} M := \inf \{ m \in \mathbb{N}_{0} \mid \text{Ext}^{m}_{R}(k, M) \neq 0 \} < \infty \]

and which satisfies either

(i) $\text{Gfd}_{R} M < \infty$ and $\text{id}_{R} M < \infty$, or

(ii) $\text{fd}_{R} M < \infty$ and $\text{Gid}_{R} M < \infty$,

then $R$ is Gorenstein.
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References


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