Rings with finite Gorenstein injective dimension

Holm, Henrik Granau

Published in:
Proceedings of the American Mathematical Society

Publication date:
2004

Document version
Peer reviewed version

Citation for published version (APA):
RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

Abstract. In this paper we prove that for any associative ring \( R \), and for any left \( R \)-module \( M \) with finite projective dimension, the Gorenstein injective dimension \( \text{Gid}_R M \) equals the usual injective dimension \( \text{id}_R M \). In particular, if \( \text{Gid}_R R \) is finite, then also \( \text{id}_R R \) is finite, and thus \( R \) is Gorenstein (provided that \( R \) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \((R, \mathfrak{m}, k)\), the \emph{Gorenstein rings} are characterized by the condition \( \text{id}_R R < \infty \). From the dual of \([10, \text{Proposition } (2.27)]\) (\([6, \text{Proposition } 10.2.3]\) is a special case) it follows that the \emph{Gorenstein injective dimension} \( \text{Gid}_R (\cdot) \) is a \emph{refinement} of the usual injective dimension \( \text{id}_R (\cdot) \) in the following sense:

For any \( R \)-module \( M \) there is an inequality \( \text{Gid}_R M \leq \text{id}_R M \), and if \( \text{id}_R M < \infty \), then there is an equality \( \text{Gid}_R M = \text{id}_R M \).

Now, since the injective dimension \( \text{id}_R R \) of \( R \) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \( \text{Gid}_R R \) of \( R \) measure? As a consequence of Theorem \((2.1)\) below, it turns out that:

An associative ring \( R \) with \( \text{Gid}_R R < \infty \) also has \( \text{id}_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

This result is proved by Christensen \([2, \text{Theorem } (6.3.2)]\) in the case where \((R, \mathfrak{m}, k)\) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem \((2.1)\), together with a series of related results. Among these results is Theorem \((3.2)\), which has the nice, and easily stated, Corollary \((3.3)\):

Assume that \((R, \mathfrak{m}, k)\) is a commutative local Noetherian ring, and let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}_R^m(k, M) \neq 0 \) for some \( m \in \mathbb{N}_0 \) (this happens for example if \( M \neq 0 \) is finitely generated). If either

(i) \( \text{Gid}_R M < \infty \) and \( \text{id}_R M < \infty \) or
(ii) \( \text{fd}_R M < \infty \) and \( \text{Gid}_R M < \infty \),

then \( R \) is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: *Gorenstein injective* modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, *Gorenstein projective* modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. *Gorenstein flat* modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—*left* \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_R M\), \(\text{fd}_R M\), and \(\text{id}_R M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_R M\), \(\text{Gfd}_R M\), and \(\text{Gid}_R M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\). In particular, if \(\text{Gid}_R R < \infty\), then also \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_R M \leq \text{id}_R M\) always, it suffices to prove that \(\text{id}_R M \leq \text{Gid}_R M\). Naturally, we may assume that \(\text{Gid}_R M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_R M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \text{Ker}(E_1 \to E_0)\). In particular, there exists a short exact sequence \(0 \to M' \to E \to M \to 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows by [4, Lemma 1.3] that \(\text{Ext}^1_R(M, M') = 0\). Thus \(0 \to M' \to E \to M \to 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_R M > 0\). By [10, Theorem (2.15)] there exists an exact sequence \(0 \to M \to H \to C \to 0\) where \(H\) is Gorenstein injective and \(\text{id}_R C = \text{Gid}_R M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \to H' \to I \to H \to 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \to & M & \to & H & \to & C & \to & 0 \\
0 & \to & P & \to & I & \to & C & \to & 0 \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& H' & \to & H' & & & & & \\
& 0 & \uparrow & 0 & & & & & \\
\end{array}
\]

Since \( I \) is injective and \( \text{id}_R P \leq \text{Gid}_R M \) by the second row. Since \( H' \) is Gorenstein injective and \( \text{pd}_R M < \infty \), it follows (as before) by Lemma 1.3 that \( \text{Ext}^{1}_{R}(M, H') = 0 \). Consequently, the first column \( 0 \to H' \to P \to M \to 0 \) splits. Therefore \( P \cong M \oplus H' \), and hence \( \text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M \).

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \( M \) is an \( R \)-module with \( \text{id}_R M < \infty \), then \( \text{Gpd}_R M = \text{pd}_R M \).

Theorem (2.6) below is a "flat version" of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \( \text{LeftFPD}(R) \) of \( R \) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(R) \) of \( R \) is defined similarly.

**Remark 2.4.** When \( R \) is commutative and Noetherian, we have that \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) equals the Krull dimension of \( R \), by Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from Proposition (3.11):

**Proposition 2.5.** For any (left) \( R \)-module \( M \) the inequality

\[
\text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M
\]

holds. If \( R \) is right coherent, then we have \( \text{Gid}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M \).

We are now ready to state:

**Theorem 2.6.** For any \( R \)-module \( M \), the following conclusions hold:

(i) Assume that \( \text{LeftFPD}(R) \) is finite. If \( \text{fd}_R M < \infty \), then \( \text{Gid}_R M = \text{id}_R M \).

(ii) Assume that \( R \) is left and right coherent with finite \( \text{RightFPD}(R) \). If \( \text{id}_R M < \infty \), then \( \text{Gfd}_R M = \text{fd}_R M \).

**Proof.** (i) If \( \text{fd}_R M < \infty \), then also \( \text{pd}_R M < \infty \), by Proposition 6] (since \( \text{LeftFPD}(R) < \infty \)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \( R \) is left coherent, we have that \( \text{fd}_R \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty \), by Lemma 3.1.4]. By assumption, \( \text{RightFPD}(R) < \infty \), and therefore also...
pd_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) < \infty$, by [11, Proposition 6]. Now Theorem 2.1 gives that $Gid_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = id_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$. It is well known that $fd_RM = id_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (without assumptions on $R$), and by Proposition 2.5 above, we also get $Gfd_RM = Gid_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$, since $R$ is right coherent. The proof is done. \\

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that $R$ is commutative and Noetherian. For an $R$-module $M$, an integer $n$, and a prime ideal $p$ in $R$, we write $\beta_n^R(p, M)$, respectively, $\mu_n^R(p, M)$, for the $n$th Betti number, respectively, $n$th Bass number, of $M$ at $p$.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an $R$-module $M$ to be the set

$$\text{supp}_RM = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_0: \beta_n^R(p, M) \neq 0 \}.$$ 

Let us mention the most basic results about the small support, all of which can be found in [8] pp. 157 – 159 and [7] Chapter 14:

(a) The small support, $\text{supp}_RM$, is contained in the usual (large) support, $\text{Supp}_RM$, and $\text{supp}_RM = \text{Supp}_RM$ if $M$ is finitely generated. Also, if $M \neq 0$, then $\text{supp}_RM \neq 0$.

(b) $\text{supp}_RM = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_0: \mu_n^R(p, M) \neq 0 \}$.

(c) Assume that $(R, m, k)$ is local. If $M$ is an $R$-module with finite depth, that is,

$$\text{depth}_RM := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty$$

(this happens for example if $M \neq 0$ is finitely generated), then $m \in \text{supp}_RM$, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

Theorem 3.2. Assume that $R$ is commutative and Noetherian. Let $M$ be any $R$-module, and assume that any of the following four conditions is satisfied:

(i) $Gpd_RM < \infty$ and $id_RM < \infty$,
(ii) $pd_RM < \infty$ and $Gid_RM < \infty$,
(iii) $R$ has finite Krull dimension, and $Gfd_RM < \infty$ and $id_RM < \infty$,
(iv) $R$ has finite Krull dimension, and $fd_RM < \infty$ and $Gid_RM < \infty$.

Then $R_p$ is a Gorenstein local ring for all $p \in \text{supp}_RM$. \\

Corollary 3.3. Assume that $(R, m, k)$ is a commutative local Noetherian ring. If there exists an $R$-module $M$ of finite depth, that is,

$$\text{depth}_RM := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty,$$

and which satisfies either

(i) $Gfd_RM < \infty$ and $id_RM < \infty$, or
(ii) $fd_RM < \infty$ and $Gid_RM < \infty$,

then $R$ is Gorenstein. 

\square
Acknowledgments

I would like to express my gratitude to my Ph.D. advisor Hans-Bjørn Foxby for his support, and our helpful discussions.

References


