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RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

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Abstract. In this paper we prove that for any associative ring $R$, and for any left $R$-module $M$ with finite projective dimension, the Gorenstein injective dimension $\text{Gid}_R M$ equals the usual injective dimension $\text{id}_R M$. In particular, if $\text{Gid}_R R$ is finite, then also $\text{id}_R R$ is finite, and thus $R$ is Gorenstein (provided that $R$ is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings $(R; \mathfrak{m}; k)$, the Gorenstein rings are characterized by the condition $\text{id}_R R < \infty$. From the dual of [10] Proposition (2.27) ([10] Proposition 10.2.3] is a special case) it follows that the Gorenstein injective dimension $\text{Gid}_R (-)$ is a refinement of the usual injective dimension $\text{id}_R (-)$ in the following sense:

For any $R$-module $M$ there is an inequality $\text{Gid}_R M \leq \text{id}_R M$, and if $\text{id}_R M < \infty$, then there is an equality $\text{Gid}_R M = \text{id}_R M$.

Now, since the injective dimension $\text{id}_R R$ of $R$ measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $\text{Gid}_R R$ of $R$ measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring $R$ with $\text{Gid}_R R < \infty$ also has $\text{id}_R R < \infty$ (and hence $R$ is Gorenstein, provided that $R$ is commutative and Noetherian).

This result is proved by Christensen [2] Theorem (6.3.2)] in the case where $(R; \mathfrak{m}; k)$ is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that $(R; \mathfrak{m}; k)$ is a commutative local Noetherian ring, and let $M$ be an $R$-module of finite depth, that is, $\text{Ext}_R^m(k; M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

(i) $\text{Gid}_R M < \infty$ and $\text{id}_R M < \infty$ or
(ii) $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$,

then $R$ is Gorenstein.
This corollary is also proved by Christensen \cite{Christensen2} Theorem (6.3.2)] in the case where \((R, m, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem \cite{Christensen2} 6.3.2 itself (dealing not only with local rings) is a generalization of \cite{Foxby} Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in \cite{Enochs}. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger \cite{Auslander} in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see \cite{Enochs2}.

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_{\!R}M\), \(\text{fd}_{\!R}M\), and \(\text{id}_{\!R}M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_{\!R}M\), \(\text{Gfd}_{\!R}M\), and \(\text{Gid}_{\!R}M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_{\!R}M < \infty\), then \(\text{Gid}_{\!R}M = \text{id}_{\!R}M\). In particular, if \(\text{Gid}_{\!R}R < \infty\), then also \(\text{id}_{\!R}R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_{\!R}M \leq \text{id}_{\!R}M\) always, it suffices to prove that \(\text{id}_{\!R}M \leq \text{Gid}_{\!R}M\). Naturally, we may assume that \(\text{Gid}_{\!R}M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_{\!R}M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \to E_1 \to E_0 \to E_{-1} \to \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_{\!R}(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \ker(E_1 \to E_0)\). In particular, there exists a short exact sequence \(0 \to M' \to E \to M \to 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_{\!R}M < \infty\), it follows by \cite{Enochs} Lemma 1.3] that \(\text{Ext}_{\!R}^1(M, M') = 0\). Thus \(0 \to M' \to E \to M \to 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_{\!R}M > 0\). By \cite{Enochs} Theorem (2.15)] there exists an exact sequence \(0 \to M \to H \to C \to 0\) where \(H\) is Gorenstein injective and \(\text{id}_{\!R}C = \text{Gid}_{\!R}M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \to H' \to I \to H \to 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{c}
\begin{array}{ccc}
0 & 0 & \\
0 & M & H \\
0 & P & I \\
H' & H' & 0
\end{array}
\end{array}
\]

Since \(I\) is injective and \(\text{id}_R M < 1\) we get \(\text{id}_R P \leq \text{Gid}_R M\) by the second row. Since \(H'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows (as before) by [4, Lemma 1.3] that \(\text{Ext}_R^1(M, H') = 0\). Consequently, the first column \(0 \rightarrow H' \rightarrow P \rightarrow M \rightarrow 0\) splits. Therefore \(P \cong M \oplus H'\), and hence \(\text{id}_R M < 1\).

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \(M\) is an \(R\)-module with \(\text{id}_R M < 1\), then \(\text{Gpd}_R M = \text{pd}_R M\).

**Theorem (2.6)** below is a "flat version" of the two previous theorems. First recall the following.

**Definition 2.3.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 2.4.** When \(R\) is commutative and Noetherian, we have that \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) equals the Krull dimension of \(R\), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \(R\)-module \(M\) the inequality

\[
\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M
\]

holds. If \(R\) is right coherent, then we have \(\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M\).

We are now ready to state:

**Theorem 2.6.** For any \(R\)-module \(M\), the following conclusions hold:

(i) Assume that \(\text{LeftFPD}(R)\) is finite. If \(\text{fd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\).

(ii) Assume that \(R\) is left and right coherent with finite \(\text{RightFPD}(R)\). If \(\text{id}_R M < \infty\), then \(\text{Gfd}_R M = \text{fd}_R M\).

**Proof.** (i) If \(\text{fd}_R M < \infty\), then also \(\text{pd}_R M < \infty\), by [11, Proposition 6] (since \(\text{LeftFPD}(R) < \infty\)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \(R\) is left coherent, we have that \(\text{fd}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty\), by [12, Lemma 3.1.4]. By assumption, \(\text{RightFPD}(R) < \infty\), and therefore also...
pd_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) < \infty \text{, by } \[11 \text{ Proposition } 6\]. Now Theorem \[15\] gives that 
Gid_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) = \text{id}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \text{. It is well known that } 
fd_{R} M = \text{id}_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \text{ (without assumptions on } R) \text{, and by Proposition \[15\] above, we also get } Gfd_{R} M = Gid_{R} \text{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \text{, since } R \text{ is right coherent. The proof is done.} \square

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem \[8 \text{ Proposition } 2.10\] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that \( R \) is commutative and Noetherian. For an \( R \)-module \( M \), an integer \( n \), and a prime ideal \( p \) in \( R \), we write \( \beta_{n}^{R}(p, M) \), respectively, \( \mu_{n}^{R}(p, M) \), for the \( n \)th Betti number, respectively, \( n \)th Bass number, of \( M \) at \( p \).

Foxby \[8 \text{ Definition p. } 157\] or \[7 \text{ (14.8)] defines the small (or homological) support of an \( R \)-module \( M \) to be the set

\[
\text{supp}_{R} M = \{ \ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_{0} : \beta_{n}^{R}(p, M) \neq 0 \ \}.
\]

Let us mention the most basic results about the small support, all of which can be found in \[8 \text{ pp. } 157 - 159\] and \[7 \text{ Chapter } 14\]:

(a) The small support, \( \text{supp}_{R} M \), is contained in the usual (large) support, \( \text{Supp}_{R} M \), and \( \text{supp}_{R} M = \text{Supp}_{R} M \) if \( M \) is finitely generated. Also, if \( M \neq 0 \), then \( \text{supp}_{R} M \neq 0 \).

(b) \( \text{supp}_{R} M = \{ \ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_{0} : \mu_{n}^{R}(p, M) \neq 0 \ \} \).

(c) Assume that \((R, m, k)\) is local. If \( M \) is an \( R \)-module with finite depth, that is,

\[
\text{depth}_{R} M := \inf \{ \ m \in \mathbb{N}_{0} : \text{Ext}_{M}^{m}(k, M) \neq 0 \ \} < \infty
\]

(this happens for example if \( M \neq 0 \) is finitely generated), then \( m \in \text{supp}_{R} M \), by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of \[8 \text{ Proposition } 2.10\] is immediate:

**Theorem 3.2.** Assume that \( R \) is commutative and Noetherian. Let \( M \) be any \( R \)-module, and assume that any of the following four conditions is satisfied:

(i) \( \text{Gpd}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \),

(ii) \( \text{pd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \),

(iii) \( R \) has finite Krull dimension, and \( \text{Gfd}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \),

(iv) \( R \) has finite Krull dimension, and \( \text{fd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \).

Then \( R_{p} \) is a Gorenstein local ring for all \( p \in \text{supp}_{R} M \). \( \square \)

**Corollary 3.3.** Assume that \((R, m, k)\) is a commutative local Noetherian ring. If there exists an \( R \)-module \( M \) of finite depth, that is,

\[
\text{depth}_{R} M := \inf \{ \ m \in \mathbb{N}_{0} : \text{Ext}_{M}^{m}(k, M) \neq 0 \ \} < \infty
\]

and which satisfies either

(i) \( \text{Gid}_{R} M < \infty \) and \( \text{id}_{R} M < \infty \), or

(ii) \( \text{fd}_{R} M < \infty \) and \( \text{Gid}_{R} M < \infty \),

then \( R \) is Gorenstein. \( \square \)
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References


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