Rings with finite Gorenstein injective dimension

Holm, Henrik Granau

Published in:
Proceedings of the American Mathematical Society

Publication date:
2004

Document version
Peer reviewed version

Citation for published version (APA):
RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

(Communicated by Bernd Ulrich)

Abstract. In this paper we prove that for any associative ring \( R \), and for any left \( R \)-module \( M \) with finite projective dimension, the Gorenstein injective dimension \( Gid_R M \) equals the usual injective dimension \( id_R M \). In particular, if \( Gid_R R \) is finite, then also \( id_R R \) is finite, and thus \( R \) is Gorenstein (provided that \( R \) is commutative and Noetherian).

1. Introduction

It is well known that among the commutative local Noetherian rings \( (R, \mathfrak{m}, k) \), the Gorenstein rings are characterized by the condition \( id_R R < \infty \). From the dual of [10] Proposition (2.27) ([10] Proposition 10.2.3 is a special case) it follows that the Gorenstein injective dimension \( Gid_R(\_\_\_) \) is a refinement of the usual injective dimension \( id_R(\_\_) \) in the following sense:

For any \( R \)-module \( M \) there is an inequality \( Gid_R M \leq id_R M \), and if \( id_R M < \infty \), then there is an equality \( Gid_R M = id_R M \).

Now, since the injective dimension \( id_R R \) of \( R \) measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension \( Gid_R R \) of \( R \) measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring \( R \) with \( Gid_R R < \infty \) also has \( id_R R < \infty \) (and hence \( R \) is Gorenstein, provided that \( R \) is commutative and Noetherian).

This result is proved by Christensen [2] Theorem (6.3.2)] in the case where \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that \( (R, \mathfrak{m}, k) \) is a commutative local Noetherian ring, and let \( M \) be an \( R \)-module of finite depth, that is, \( \text{Ext}^m_R(k, M) \neq 0 \) for some \( m \in \mathbb{N}_0 \) (this happens for example if \( M \neq 0 \) is finitely generated). If either

\( (i) \ Gid_R M < \infty \) and \( id_R M < \infty \) or \( (ii) \ fd_R M < \infty \) and \( Gid_R M < \infty \),

then \( R \) is Gorenstein.
This corollary is also proved by Christensen [Z, Theorem (6.3.2)] in the case where \((R, \mathfrak{m}, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [10]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [3].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_R M\), \(\text{fd}_R M\), and \(\text{id}_R M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_R M\), \(\text{Gfd}_R M\), and \(\text{Gid}_R M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

**Theorem 2.1.** If \(M\) is an \(R\)-module with \(\text{pd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\). In particular, if \(\text{Gid}_R R < \infty\), then also \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

**Proof.** Since \(\text{Gid}_R M \leq \text{id}_R M\) always, it suffices to prove that \(\text{id}_R M \leq \text{Gid}_R M\). Naturally, we may assume that \(\text{Gid}_R M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_R M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(\mathfrak{m}, E)\) is exact for every injective \(R\)-module \(E\), and such that \(M \cong \ker(E_1 \rightarrow E_0)\). In particular, there exists a short exact sequence \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows by [11, Lemma 1.3] that \(\text{Ext}^1_R(M, M') = 0\). Thus \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_R M > 0\). By [10, Theorem (2.15)] there exists an exact sequence \(0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0\) where \(H\) is Gorenstein injective and \(\text{id}_R C = \text{Gid}_R M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \rightarrow H' \rightarrow I \rightarrow H \rightarrow 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
columns:

\[
\begin{array}{cccc}
0 & 0 & & \\
\downarrow & & & \\
M & H & C & 0 \\
\downarrow & & & \\
0 & I & C & 0 \\
\downarrow & & & \\
H' & H' & & \\
\downarrow & & & \\
0 & 0 & & \\
\end{array}
\]

Since \( I \) is injective and \( \text{id}_R M = G \text{id}_R M \) by the second row. Since \( H' \) is Gorenstein injective and \( \text{pd}_R M < \infty \), it follows (as before) by [4, Lemma 1.3] that \( \text{Ext}^1_R(M, H') = 0 \). Consequently, the first column \( 0 \to H' \to P \to M \to 0 \) splits. Therefore \( P \cong M \oplus H' \), and hence \( \text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M \).

Theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If \( M \) is an \( R \)-module with \( \text{id}_R M < \infty \), then \( \text{Gpd}_R M = \text{pd}_R M \). \qed

Theorem (2.6) below is a "flat version" of the two previous theorems. First recall the following.

**Definition 2.3.** The **left finitistic projective dimension** \( \text{LeftFPD}(R) \) of \( R \) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid \text{M is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(R) \) of \( R \) is defined similarly.

**Remark 2.4.** When \( R \) is commutative and Noetherian, we have that \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) equals the Krull dimension of \( R \), by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) \( R \)-module \( M \) the inequality

\[
\text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M
\]

holds. If \( R \) is right coherent, then we have \( \text{Gid}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M \). \qed

We are now ready to state:

**Theorem 2.6.** For any \( R \)-module \( M \), the following conclusions hold:

(i) Assume that \( \text{LeftFPD}(R) \) is finite. If \( \text{fd}_R M < \infty \), then \( \text{Gid}_R M = \text{id}_R M \).

(ii) Assume that \( R \) is left and right coherent with finite \( \text{RightFPD}(R) \). If \( \text{id}_R M < \infty \), then \( \text{Gfd}_R M = \text{fd}_R M \).

**Proof.** (i) If \( \text{fd}_R M < \infty \), then also \( \text{pd}_R M < \infty \), by [14, Proposition 6] (since \( \text{LeftFPD}(R) < \infty \)). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since \( R \) is left coherent, we have that \( \text{fd}_R \text{Hom}_Z(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty \), by [12, Lemma 3.1.4]. By assumption, \( \text{RightFPD}(R) < \infty \), and therefore also...
GidₚHom₂(M, ℚ/ℤ) = idₚHom₂(M, ℚ/ℤ). It is well known that
fdₚM = idₚHom₂(M, ℚ/ℤ)
(without assumptions on R), and by Proposition [35] above, we also get GfdₚM =
GidₚHom₂(M, ℚ/ℤ), since R is right coherent. The proof is done.

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that R is commutative and Noetherian. For an R-module M, an integer n, and a prime ideal p in R, we write βⁿₚ(M) for the nth Betti number, respectively, βⁿₚ(p, M), for the nth Bass number, of M at p.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an R-module M to be the set

suppₚM := { p ∈ Spec R | ∃n ∈ ℤ₀: βⁿₚ(p, M) ≠ 0 }.

Let us mention the most basic results about the small support, all of which can be found in [8] pp. 157 - 159 and [7] Chapter 14:

(a) The small support, suppₚM, is contained in the usual (large) support, SuppₚM, and suppₚM = SuppₚM if M is finitely generated. Also, if M ≠ 0, then suppₚM ≠ ∅.
(b) suppₚM = { p ∈ Spec R | ∃n ∈ ℤ₀: µⁿₚ(p, M) ≠ 0 }.
(c) Assume that (R, m, k) is local. If M is an R-module with finite depth, that is,

depthₚM := inf { m ∈ ℤ₀ | Extₚᵐ(k, M) ≠ 0 } < ∞

(this happens for example if M ≠ 0 is finitely generated), then m ∈ suppₚM, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

Theorem 3.2. Assume that R is commutative and Noetherian. Let M be any R-module, and assume that any of the following four conditions is satisfied:

(i) GpdₚM < ∞ and idₚM < ∞,
(ii) pdₚM < ∞ and GidₚM < ∞,
(iii) R has finite Krull dimension, and GfdₚM < ∞ and idₚM < ∞,
(iv) R has finite Krull dimension, and fdₚM < ∞ and GidₚM < ∞.

Then Rₚ is a Gorenstein local ring for all p ∈ suppₚM.

Corollary 3.3. Assume that (R, m, k) is a commutative local Noetherian ring. If there exists an R-module M of finite depth, that is,

depthₚM := inf { m ∈ ℤ₀ | Extₚᵐ(k, M) ≠ 0 } < ∞,

and which satisfies either

(i) GfdₚM < ∞ and idₚM < ∞, or
(ii) fdₚM < ∞ and GidₚM < ∞,
then R is Gorenstein.
Acknowledgments

I would like to express my gratitude to my Ph.D. advisor Hans-Bjørn Foxby for his support, and our helpful discussions.

References


Mathematiske Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, Danmark
E-mail address: holm@math.ku.dk