RINGS WITH FINITE GORENSTEIN INJECTIVE DIMENSION

HENRIK HOLM

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Abstract. In this paper we prove that for any associative ring $R$, and for any left $R$-module $M$ with finite projective dimension, the Gorenstein injective dimension $\text{id}_{G}R M$ equals the usual injective dimension $\text{id}R M$. In particular, if $\text{id}R R$ is finite, then also $\text{id}R R$ is finite, and thus $R$ is Gorenstein (provided that $R$ is commutative and Noetherian).

1. INTRODUCTION

It is well known that among the commutative local Noetherian rings $(R, \mathfrak{m}, k)$, the Gorenstein rings are characterized by the condition $\text{id}R R < \infty$. From the dual of [10] Proposition (2.27) ([6] Proposition 10.2.3] is a special case) it follows that the Gorenstein injective dimension $\text{id}_{G}R (-)$ is a refinement of the usual injective dimension $\text{id}R (-)$ in the following sense:

For any $R$-module $M$ there is an inequality $\text{id}_{G}R M \leq \text{id}R M$, and if $\text{id}R M < \infty$, then there is an equality $\text{id}_{G}R M = \text{id}R M$.

Now, since the injective dimension $\text{id}R R$ of $R$ measures Gorensteinness, it is only natural to ask what does the Gorenstein injective dimension $\text{id}_{G}R R$ of $R$ measure? As a consequence of Theorem (2.1) below, it turns out that:

An associative ring $R$ with $\text{id}_{G}R R < \infty$ also has $\text{id}R R < \infty$ (and hence $R$ is Gorenstein, provided that $R$ is commutative and Noetherian).

This result is proved by Christensen [2] Theorem (6.3.2)] in the case where $(R, \mathfrak{m}, k)$ is a commutative local Noetherian Cohen-Macaulay ring with a dualizing module. The aim of this paper is to prove Theorem (2.1), together with a series of related results. Among these results is Theorem (3.2), which has the nice, and easily stated, Corollary (3.3):

Assume that $(R, \mathfrak{m}, k)$ is a commutative local Noetherian ring, and let $M$ be an $R$-module of finite depth, that is, $\text{Ext}^m_R(k, M) \neq 0$ for some $m \in \mathbb{N}_0$ (this happens for example if $M \neq 0$ is finitely generated). If either

(i) $\text{Gid}_{R} M < \infty$ and $\text{id}R M < \infty$ or (ii) $\text{fd}_{R} M < \infty$ and $\text{Gid}_{R} M < \infty$,

then $R$ is Gorenstein.
This corollary is also proved by Christensen [2, Theorem (6.3.2)] in the case where \((R, m, k)\) is Cohen-Macaulay with a dualizing module. However, Theorem (3.2) itself (dealing not only with local rings) is a generalization of [8, Proposition 2.10] (in the module case) by Foxby from 1979.

We should briefly mention the history of Gorenstein injective, projective and flat modules: Gorenstein injective modules over an arbitrary associative ring, and the related Gorenstein injective dimension, was introduced and studied by Enochs and Jenda in [3]. The dual concept, Gorenstein projective modules, was already introduced by Auslander and Bridger [1] in 1969, but only for finitely generated modules over a two-sided Noetherian ring. Gorenstein flat modules were also introduced by Enochs and Jenda; please see [5].

1.1. Setup and notation. Let \(R\) be any associative ring with a nonzero multiplicative identity. All modules are—if not specified otherwise—left \(R\)-modules. If \(M\) is any \(R\)-module, we use \(\text{pd}_R M\), \(\text{id}_R M\), and \(\text{id}_R M\) to denote the usual projective, flat, and injective dimension of \(M\), respectively. Furthermore, we write \(\text{Gpd}_R M\), \(\text{Gfd}_R M\), and \(\text{Gid}_R M\) for the Gorenstein projective, Gorenstein flat, and Gorenstein injective dimension of \(M\), respectively.

2. Rings with finite Gorenstein injective dimension

Theorem 2.1. If \(M\) is an \(R\)-module with \(\text{pd}_R M < \infty\), then \(\text{Gid}_R M = \text{id}_R M\). In particular, if \(\text{Gid}_R R < \infty\), then also \(\text{id}_R R < \infty\) (and hence \(R\) is Gorenstein, provided that \(R\) is commutative and Noetherian).

Proof. Since \(\text{Gid}_R M \leq \text{id}_R M\) always, it suffices to prove that \(\text{id}_R M \leq \text{Gid}_R M\). Naturally, we may assume that \(\text{Gid}_R M < \infty\).

First consider the case where \(M\) is Gorenstein injective, that is, \(\text{Gid}_R M = 0\). By definition, \(M\) is a kernel in a complete injective resolution. This means that there exists an exact sequence \(E = \cdots \rightarrow E_1 \rightarrow E_0 \rightarrow E_{-1} \rightarrow \cdots\) of injective \(R\)-modules, such that \(\text{Hom}_R(I, E)\) is exact for every injective \(R\)-module \(I\), and such that \(M \cong \ker(E_1 \rightarrow E_0)\). In particular, there exists a short exact sequence \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\), where \(E\) is injective, and \(M'\) is Gorenstein injective. Since \(M'\) is Gorenstein injective and \(\text{pd}_R M < \infty\), it follows by [14, Lemma 1.3] that \(\text{Ext}^1_R(M, M') = 0\). Thus \(0 \rightarrow M' \rightarrow E \rightarrow M \rightarrow 0\) is split-exact; so \(M\) is a direct summand of the injective module \(E\). Therefore, \(M\) itself is injective.

Next consider the case where \(\text{Gid}_R M > 0\). By [10, Theorem (2.15)] there exists an exact sequence \(0 \rightarrow M \rightarrow H \rightarrow C \rightarrow 0\) where \(H\) is Gorenstein injective and \(\text{id}_R C = \text{Gid}_R M - 1\). As in the previous case, since \(H\) is Gorenstein injective, there exists a short exact sequence \(0 \rightarrow H' \rightarrow I \rightarrow H \rightarrow 0\) where \(I\) is injective and \(H'\) is Gorenstein injective. Now consider the pull-back diagram with exact rows and
Since $I$ is injective and $\text{id}_R M < 1$, we get $\text{id}_R P \leq \text{Gid}_R M$ by the second row. Since $H'$ is Gorenstein injective and $\text{pd}_R M < \infty$, it follows (as before) by [4, Lemma 1.3] that $\text{Ext}_R^1(M, H') = 0$. Consequently, the first column $0 \to H' \to P \to M \to 0$ splits. Therefore $P \cong M \oplus H'$, and hence $\text{id}_R M \leq \text{id}_R P \leq \text{Gid}_R M.$

The theorem above has, of course, a dual counterpart:

**Theorem 2.2.** If $M$ is an $R$-module with $\text{id}_R M < 1$, then $\text{Gpd}_R M = \text{pd}_R M$. □

**Theorem 2.6** below is a \textit{flat version} of the two previous theorems. First recall the following.

**Definition 2.3.** The \textit{left finitistic projective dimension} $\text{LeftFPD}(R)$ of $R$ is defined as

$\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}$.

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 2.4.** When $R$ is commutative and Noetherian, we have that $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ equals the Krull dimension of $R$, by [3, Théorème (3.2.6) (Seconde partie)].

Furthermore, we will need the following result from [10, Proposition (3.11)]:

**Proposition 2.5.** For any (left) $R$-module $M$ the inequality

$\text{Gid}_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{Gfd}_R M$

holds. If $R$ is right coherent, then we have $\text{Gid}_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) = \text{Gfd}_R M$. □

We are now ready to state:

**Theorem 2.6.** For any $R$-module $M$, the following conclusions hold:

(i) Assume that $\text{LeftFPD}(R)$ is finite. If $\text{fd}_R M < \infty$, then $\text{Gid}_R M = \text{id}_R M$.

(ii) Assume that $R$ is left and right coherent with finite $\text{RightFPD}(R)$. If $\text{id}_R M < \infty$, then $\text{Gfd}_R M = \text{fd}_R M$.

**Proof.** (i) If $\text{fd}_R M < \infty$, then also $\text{pd}_R M < \infty$, by [11, Proposition 6] (since $\text{LeftFPD}(R) < \infty$). Hence the desired conclusion follows from Theorem (2.1) above.

(ii) Since $R$ is left coherent, we have that $\text{fd}_R \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \leq \text{id}_R M < \infty$, by [12, Lemma 3.1.4]. By assumption, $\text{RightFPD}(R) < \infty$, and therefore also
pd_r\text{Hom}_Z(M, Q/\mathbb{Z}) < \infty$, by [11] Proposition 6. Now Theorem 2.1 gives that $\text{Gid}_R\text{Hom}_Z(M, Q/\mathbb{Z}) = \text{id}_R\text{Hom}_Z(M, Q/\mathbb{Z})$. It is well known that $\text{fd}_R M = \text{id}_R\text{Hom}_Z(M, Q/\mathbb{Z})$ (without assumptions on $R$), and by Proposition 2.5 above, we also get $\text{Gfd}_R M = \text{Gid}_R\text{Hom}_Z(M, Q/\mathbb{Z})$, since $R$ is right coherent. The proof is done. 

3. A theorem on Gorenstein rings by Foxby

We end this paper by generalizing a theorem [8, Proposition 2.10] on Gorenstein rings by Foxby from 1979. For completeness, we briefly recall:

3.1. The small support. Assume that $R$ is commutative and Noetherian. For an $R$-module $M$, an integer $n$, and a prime ideal $p$ in $R$, we write $\beta^n_R(p, M)$, respectively, $\mu^n_R(p, M)$, for the $n$th Betti number, respectively, $n$th Bass number, of $M$ at $p$.

Foxby [8, Definition p. 157] or [7, (14.8)] defines the small (or homological) support of an $R$-module $M$ to be the set

$$\text{supp}_R M = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_0: \beta^n_R(p, M) \neq 0 \}.$$ 

Let us mention the most basic results about the small support, all of which can be found in [8, pp. 157 - 159] and [7, Chapter 14]:

(a) The small support, $\text{supp}_R M$, is contained in the usual (large) support, $\text{Supp}_R M$, and $\text{supp}_R M = \text{Supp}_R M$ if $M$ is finitely generated. Also, if $M \neq 0$, then $\text{supp}_R M \neq 0$.

(b) $\text{supp}_R M = \{ p \in \text{Spec } R \mid \exists n \in \mathbb{N}_0: \mu^n_R(p, M) \neq 0 \}$.

(c) Assume that $(R, m, k)$ is local. If $M$ is an $R$-module with finite depth, that is,

$$\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty$$

(this happens for example if $M \neq 0$ is finitely generated), then $m \in \text{supp}_R M$, by (b) above.

Now, given these facts about the small support, and the results in the previous section, the following generalization of [8, Proposition 2.10] is immediate:

**Theorem 3.2.** Assume that $R$ is commutative and Noetherian. Let $M$ be any $R$-module, and assume that any of the following four conditions is satisfied:

(i) $\text{Gpd}_R M < \infty$ and $\text{id}_R M < \infty$,

(ii) $\text{pd}_R M < \infty$ and $\text{Gid}_R M < \infty$,

(iii) $R$ has finite Krull dimension, and $\text{Gfd}_R M < \infty$ and $\text{id}_R M < \infty$,

(iv) $R$ has finite Krull dimension, and $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$.

Then $R_p$ is a Gorenstein local ring for all $p \in \text{supp}_R M$. 

**Corollary 3.3.** Assume that $(R, m, k)$ is a commutative local Noetherian ring. If there exists an $R$-module $M$ of finite depth, that is,

$$\text{depth}_R M := \inf \{ m \in \mathbb{N}_0 \mid \text{Ext}_R^m(k, M) \neq 0 \} < \infty,$$

and which satisfies either

(i) $\text{Gfd}_R M < \infty$ and $\text{id}_R M < \infty$, or

(ii) $\text{fd}_R M < \infty$ and $\text{Gid}_R M < \infty$,

then $R$ is Gorenstein. 

\[ \square \]
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References


Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, Danmark

E-mail address: holm@math.ku.dk