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Holm, Henrik Granau

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GORENSTEIN DERIVED FUNCTORS

HENRIK HOLM

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(\cdot, \cdot)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}^n_R(\cdot, \cdot)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger $\cite{2}$ introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in $\cite{6}$ defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following $\cite{4}$, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see $\cite{6}$ and $\cite{8}$, or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}^n_R(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{\mathcal{GP}}(\cdot, \cdot)$ and $\text{Ext}^n_{\mathcal{GI}}(\cdot, \cdot)$ of $\text{Hom}_R(\cdot, \cdot)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{GP}$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see $[2,1]$ below for the definition of proper resolutions), we define

$$\text{Ext}^n_{\mathcal{GP}}(M, N) := \text{Hom}_R(G, N).$$

From $[2,4]$ it will follow that $\text{Ext}^n_{\mathcal{GP}}(\cdot, N)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(\mathcal{GP})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{\mathcal{GI}}(M', \cdot)$, which is defined on the full subcategory, $\text{RightRes}_M(\mathcal{GI})$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\mathcal{GP})$ and $N \in \text{RightRes}_M(\mathcal{GI})$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12, Proposition 2.18] (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(\mathcal{GP})$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\text{RightRes}_M(\mathcal{GI})$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\otimes_R \cdot$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GI}$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T: \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let \( \mathcal{X} \subseteq \mathcal{C} \) be a full subcategory. A proper left \( \mathcal{X} \)-resolution of \( M \in \mathcal{C} \) is a complex \( X = \cdots \to X_1 \to X_0 \to 0 \) where \( X_i \in \mathcal{X} \), together with a morphism \( X_0 \to M \), such that \( X^+ := \cdots \to X_1 \to X_0 \to M \to 0 \) is also a complex, and such that the sequence

\[
\text{Hom}_{\mathcal{C}}(X, X^+) = \cdots \to \text{Hom}_{\mathcal{C}}(X, X_1) \to \text{Hom}_{\mathcal{C}}(X, X_0) \to \text{Hom}_{\mathcal{C}}(X, M) \to 0
\]

is exact for every \( X \in \mathcal{X} \). We sometimes refer to \( X^+ = \cdots \to X_1 \to X_0 \to M \to 0 \) as an augmented proper left \( \mathcal{X} \)-resolution. We do not require that \( X^+ \) itself is exact. Furthermore, we use \( \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \) to denote the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper left \( \mathcal{X} \)-resolution. Note that \( \mathcal{X} \) is a subcategory of \( \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \).

Proper right \( \mathcal{X} \)-resolutions are defined dually, and the full subcategory of \( \mathcal{C} \) consisting of those objects that have a proper right \( \mathcal{X} \)-resolution is \( \text{RightRes}_{\mathcal{C}}(\mathcal{X}) \).

The importance of working with proper resolutions comes from the following:

Proposition 2.2. Let \( f: M \to M' \) be a morphism in \( \mathcal{C} \), and consider the diagram

\[
\begin{array}{ccccccc}
\cdots & \longrightarrow & X_2 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow{f_2} & & \downarrow{f_1} & & \downarrow{f_0} & & \downarrow{f} & & & \\
\cdots & \longrightarrow & X'_2 & \longrightarrow & X'_1 & \longrightarrow & X'_0 & \longrightarrow & M' & \longrightarrow & 0
\end{array}
\]

where the upper row is a complex with \( X_n \in \mathcal{X} \) for all \( n \geq 0 \), and the lower row is an augmented proper left \( \mathcal{X} \)-resolution of \( M' \). Then the following conclusions hold:

(i) There exist morphisms \( f_n: X_n \to X'_n \) for all \( n \geq 0 \), making the diagram above commutative. The chain map \( \{f_n\}_{n \geq 0} \) is called a lift of \( f \).

(ii) If \( \{f'_n\}_{n \geq 0} \) is another lift of \( f \), then the chain maps \( \{f_n\}_{n \geq 0} \) and \( \{f'_n\}_{n \geq 0} \) are homotopic.

Proof. The proof is an exercise; please see [9, Exercise 8.1.2].

Remark 2.3. A few comments are in order:

- In our applications, the class \( \mathcal{X} \) contains all projectives. Consequently, all the augmented proper left \( \mathcal{X} \)-resolutions occurring in this paper will be exact. Also, all augmented proper right \( \mathcal{Y} \)-resolutions will be exact, when \( \mathcal{Y} \) is a class of \( R \)-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an \( \mathcal{X} \)-precover of \( M \in \mathcal{C} \) is a morphism \( \varphi: X \to M \), where \( X \in \mathcal{X} \), such that the sequence

\[
\text{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\text{Hom}_{\mathcal{C}}(X', \varphi)} \text{Hom}_{\mathcal{C}}(X', M) \xrightarrow{} 0
\]

is exact for every \( X' \in \mathcal{X} \). Hence, in an augmented proper left \( \mathcal{X} \)-resolution \( X^+ \) of \( M \), the morphisms \( X_{i+1} \to \text{Ker}(X_i \to X_{i-1}) \), \( i > 0 \), and \( X_0 \to M \) are \( \mathcal{X} \)-precovers.

- What we have called proper \( \mathcal{X} \)-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call \( \mathcal{X} \)-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor \( T: \mathcal{C} \to \mathcal{E} \) between abelian categories. Let us assume that \( T \) is covariant, say. Then (as usual) we can define the \( n \)th left derived functor

\[
L_n^X T: \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}
\]
of $T$, with respect to the class $\mathcal{X}$, by setting $L^\mathcal{X}_n T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$. Similarly, the $n^{\text{th}}$ right derived functor

$$R^\mathcal{X}_n T : \text{RightRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^\mathcal{X}_n T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_\mathcal{C}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

### 2.5 (Balanced Functors)

Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4.

$$R^\mathcal{X}_n F(-, N) : \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^\mathcal{Y}_n F(M, -) : \text{RightRes}_\mathcal{D}(\mathcal{Y}) \to \mathcal{E}.$$ If furthermore $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $N \in \text{RightRes}_\mathcal{D}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^\mathcal{X}_n F(M, N) \cong R^\mathcal{Y}_n F(M, N),$$

functorial in $M$ and $N$. Here we wrote $R^\mathcal{X}_n F(M, N)$ for the functor $R^\mathcal{X}_n F(-, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^\mathcal{X}_n F(-, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $\text{RightRes}_\mathcal{D}(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \overline{\mathcal{X}}$ and $\mathcal{Y} \subseteq \overline{\mathcal{Y}}$.

(ii) Every $M \in \overline{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \overline{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^\mathcal{X}_n F(M, N) \cong R^\mathcal{Y}_n F(M, N),$$

for all $M \in \overline{\mathcal{X}}$ and $N \in \overline{\mathcal{Y}}$. 

Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows 
from the construction. The functoriality becomes more clear if one consults the 
proof of [5, Proposition 8.2.14], or the proofs of [14 Theorems 2.7.2 and 2.7.6]. \(\Box\)

In the next paragraphs we apply the results above to special categories 
\(X, \tilde{X}, C\) and \(Y, \tilde{Y}, D\), including the categories mentioned in 1.1. For completeness we 
include a definition of Gorenstein projective, Gorenstein injective and Gorenstein 
flat modules:

**Definition 2.7.** A *complete projective resolution* is an exact sequence of projective 
modules,

\[ P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots, \]

such that \(\text{Hom}_R(P, Q)\) is exact for every projective \(R\)-module \(Q\). An \(R\)-module \(M\) 
is called *Gorenstein projective* (*G-projective for short*), if there exists a complete 
projective resolution \(P\) with \(M \cong \text{Im}(P_0 \to P_{-1})\). *Gorenstein injective* (*G-injective 
for short*) modules are defined dually.

3. Gorenstein deriving \(\text{Hom}_R(-, -)\)

We now return to categories of *modules*. We use \(\widehat{GP}, \widehat{GI}\) and \(\widehat{GF}\) to denote the 
class of \(R\)-modules with finite Gorenstein projective dimension, finite Gorenstein 
injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, \(G\)-
precovers are always surjective, and \(\widehat{GP}\) contains all modules with finite projective 
dimension.

We now consider the functor \(\text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \to \mathcal{A}\), together with the 
categories

\[ \mathcal{X} = \mathcal{GP}, \quad \tilde{\mathcal{X}} = \widehat{GP} \quad \text{and} \quad \mathcal{Y} = \mathcal{GI}, \quad \tilde{\mathcal{Y}} = \widehat{GI}. \]

In this case we define, in the sense of section 2.1

\[
\text{Ext}^n_{\hat{GP}}(-, N) = R^n_{\hat{GP}}\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}^n_{\hat{GI}}(M, -) = R^n_{\hat{GI}}\text{Hom}_R(M, -),
\]

for fixed \(R\)-modules \(M\) and \(N\). We wish, of course, to apply Theorem 2.3 to this 
situation. Note that by [12 Proposition 2.18], we have:

**Proposition 3.1.** If \(M\) is an \(R\)-module with \(\text{Gpd}_R M < \infty\), then there exists a 
short exact sequence \(0 \to K \to G \to M \to 0\), where \(G \to M\) is a \(G\)-precover of 
\(M\) (please see Remark [2.2]), and \(\text{pd}_R K = \text{Gpd}_R M - 1\) (in the case where \(M\) 
is Gorenstein projective, this should be interpreted as \(K = 0\)).

Consequently, every \(R\)-module with finite Gorenstein projective dimension has a 
proper left \(G\)-resolution (that is, there is an inclusion \(\widehat{GP} \subseteq \text{LeftRes}_M(\mathcal{GP})\)).

Furthermore, we will need the following from [12 Theorem 2.13]:

**Theorem 3.2.** Let \(M\) be any \(R\)-module with \(\text{Gpd}_R M < \infty\). Then

\[ \text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}. \]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by

$$\text{pd}_RM = \{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L\}.$$ 

It also follows that if $\text{pd}_RM < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences

$$S = 0 \to K \to G \to M \to 0,$$ 

where $G \to M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_RM < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,

$$S' = 0 \to K' \to G' \to M \to 0,$$ 

where $\pi: G' \to M$ is a $\mathcal{GP}$-precover and $\text{pd}_RK' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \xrightarrow{} 0.$$ 

To see this, let $\alpha: K' \to H$ be any homomorphism. We wish to find $g: G' \to H$ such that $g \iota = \alpha$. Now pick an exact sequence

$$0 \to \tilde{H} \xrightarrow{g} E \xrightarrow{g} H \xrightarrow{} 0,$$ 

where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_RK' < \infty$, we get $\text{Ext}^1_R(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \to E$ with $g\varepsilon = \alpha$:

$$\begin{array}{c}
K' \xrightarrow{\iota} G' \\
| \alpha \downarrow \varepsilon \\
H \xrightarrow{g} E \\
\end{array}$$

Next, injectivity of $E$ gives $\tilde{\varepsilon}: G' \to E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g\tilde{\varepsilon}: G' \to H$ is the desired map.

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \rightarrow N \rightarrow H^0 \rightarrow H^1 \rightarrow \cdots$ be an augmented proper right $G$-resolution of $N$ (which exists by the dual of Proposition 3.27). Then $\text{Hom}_R(G, \mathcal{H}^+)$ is exact for all Gorenstein projective modules $G$. \hfill \Box

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}T}(M, N),$$

which are functorial in $M$ and $N$. \hfill \Box

3.7 (Definition of $G\text{Ext}$). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}T}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare $G\text{Ext}$ with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_G(M, N)$ under each of the conditions

- $\text{id}_R M < \infty$ or $M \in \text{LeftRes}_M(\mathcal{G}P)$ and $\text{id}_R N < \infty$.
- $\text{id}_R N < \infty$ or $N \in \text{RightRes}_M(\mathcal{G}T)$ and $\text{pd}_R M < \infty$.

(ii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then $\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$ is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{G}P$-resolution of $M$, and thus

$$\text{Ext}^n_{\mathcal{G}P}(M, N) = \text{Ext}^n_R(P, N) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(\mathcal{G}P)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \rightarrow G \rightarrow Q^0 \rightarrow \cdots \rightarrow Q^{m-1} \rightarrow C \rightarrow 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{m+i}_R(C, N) = 0$, as claimed.

Therefore [11 Chapter III, Proposition 1.2A] implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $G\text{Ext}^n_R(-, -)$. \hfill \Box
4. Gorenstein deriving — ⊗_R —

In dealing with the tensor product we need, of course, both left and right \( R \)-modules. Thus the following addition to Notation 1.1 is needed:

If \( \mathcal{C} \) is any of the categories in Notation 1.1 (\( \mathcal{M}, \mathcal{GP}, \) etc.), we write \( \mathcal{R}\mathcal{C} \), respectively, \( \mathcal{C}\mathcal{R} \), for the category of left, respectively, right, \( R \)-modules with the property describing the modules in \( \mathcal{C} \).

Now we consider the functor \( - \otimes_R - : \mathcal{M}_R \times_R \mathcal{M} \to \mathcal{A} \). For fixed \( M \in \mathcal{M}_R \) and \( N \in R \mathcal{M} \) we define, in the sense of section 2.4:

\[
\text{Tor}^R_{\mathcal{GP}}(\mathcal{M}, -) := L_{\mathcal{GP}}\mathcal{M} \otimes_R - \quad \text{and} \quad \text{Tor}^R_{\mathcal{GP}}(M, -) := L_{\mathcal{GP}}^R(M \otimes_R -),
\]

and

\[
\text{Tor}^R_{\mathcal{GF}}(\mathcal{M}, -) := L_{\mathcal{GF}}\mathcal{M} \otimes_R - \quad \text{and} \quad \text{Tor}^R_{\mathcal{GF}}(M, -) := L_{\mathcal{GF}}^R(M \otimes_R -).
\]

The first two \( \text{Tor} \)'s use proper left Gorestein projective resolutions, and the last two \( \text{Tor} \)'s use proper left Gorestein flat resolutions. In order to compare these different \( \text{Tor} \)'s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of \( (X, -) = (\mathcal{GP}, \mathcal{GP}) \) and \( (Y, -) = (\mathcal{GP}, \mathcal{GP}) \), namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension \( \text{LeftFPD}(R) \) of \( R \) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \( \text{RightFPD}(R) \) of \( R \) is defined similarly.

**Remark 4.2.** When \( R \) is commutative and Noetherian, the dimensions \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) coincide and are equal to the Krull dimension of \( R \), by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If \( R \) is right coherent with finite \( \text{LeftFPD}(R) \), then every Goreinstein projective left \( R \)-module is also Goreinstein flat. That is, there is an inclusion \( \mathcal{GP} \subseteq \mathcal{GF} \).

**Theorem 4.4.** For any left \( R \)-module \( M \), we consider the following three conditions:

(i) The left \( R \)-module \( M \) is \( \text{G-flat} \).

(ii) The Pontryagin dual \( \text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z}) \) (which is a right \( R \)-module) is \( \text{G-injective} \).

(iii) \( M \) has an augmented proper right resolution \( 0 \to M \to F_0 \to F_1 \to \cdots \) consisting of flat left \( R \)-modules, and \( \text{Tor}^R_i(I, M) = 0 \) for all injective right \( R \)-modules \( I \), and all \( i > 0 \).

The implication (i) \( \Rightarrow \) (ii) always holds. If \( R \) is right coherent, then also (ii) \( \Rightarrow \) (iii) \( \Rightarrow \) (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}F$-resolution (that is, there is an inclusion $R\mathcal{G}F \subseteq \text{LeftRes}_R M(R\mathcal{G}F)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition 4.3). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $R\text{FPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.3 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite $R\text{FPD}(R)$. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition 4.3 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $R\text{FPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 4.3 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $R\text{FPD}(R) < \infty$, Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$.

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $R\text{FPD}(R)$ and $R\text{FPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:

\[ ... \]
(i) If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\text{Tor}^{\mathcal{F}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N).
\]
(ii) If \( \text{Gpd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\text{Tor}^{\mathcal{G}_R}_n(M, N) \cong \text{Tor}^{\mathcal{F}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N).
\]
(iii) If \( \text{Gfd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\text{Tor}^{\mathcal{G}_F}_n(M, N) \cong \text{Tor}^{\mathcal{G}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N).
\]
(iv) If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\text{Tor}^{\mathcal{G}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N) \cong \text{Tor}^{\mathcal{G}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N).
\]
All the isomorphisms are functorial in \( M \) and \( N \).

**Proof.** Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 4.6.

---

**4.9 (Definition of \( g\text{Tor} \) and \( G\text{Tor} \)).** Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module, and let \( N \) be a left \( R \)-module. If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then we write
\[
g\text{Tor}^R_n(M, N) := \text{Tor}^{\mathcal{F}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(i). If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then we write
\[
G\text{Tor}^R_n(M, N) := \text{Tor}^{\mathcal{G}_R}_n(M, N) \cong \text{Tor}^{\mathcal{G}_F}_n(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. For every right \( R \)-module \( M \) with finite \( \text{Gpd}_R M \), and for every left \( R \)-module \( N \) with \( \text{Gpd}_R N < \infty \), we have isomorphisms:
\[
g\text{Tor}^R_n(M, N) \cong G\text{Tor}^R_n(M, N)
\]
that are functorial in \( M \) and \( N \).

Finally we compare \( g\text{Tor} \) (and hence \( G\text{Tor} \)) with the usual \( \text{Tor} \).

**Theorem 4.11.** Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module with \( \text{Gfd}_R M < \infty \), and let \( N \) be a left \( R \)-module with \( \text{Gfd}_R N < \infty \). If either \( \text{fd}_R M < \infty \) or \( \text{fd}_R N < \infty \), then there are isomorphisms
\[
g\text{Tor}^R_n(M, N) \cong \text{Tor}^R_n(M, N)
\]
that are functorial in \( M \) and \( N \).

**Proof.** If \( \text{fd}_R M < \infty \), then we also have \( \text{pd}_R M < \infty \) by [13, Proposition 6] (since \( \text{RightFPD}(R) < \infty \)). Let \( P \) be any projective resolution of \( M \). As noted in Remark 4.3, \( P \) is also a proper left \( \mathcal{GP}_R \)-resolution of \( M \). Hence, Theorem 4.8(ii) and the definitions give:
\[
g\text{Tor}^R_n(M, N) = \text{Tor}^{\mathcal{G}_R}_n(M, N) = H_n(P \otimes_R N) = \text{Tor}^R_n(M, N),
\]
as desired. \(\square\)
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References


Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK–Danmark
E-mail address: holm@math.ku.dk