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GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring $R$ it is standard to derive $\operatorname{Hom}_R(-, -)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\operatorname{Ext}^n_R(-, -)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\operatorname{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\operatorname{G-dim}_R M \leq \operatorname{pd}_R M$, with equality $\operatorname{G-dim}_R M = \operatorname{pd}_R M$ when $\operatorname{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension. The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).

- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.

- Furthermore, for each $R$-module $M$ we write $\operatorname{Gpd}_R M$, $\operatorname{Gid}_R M$ and $\operatorname{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}^n_R(-,-)$ of \(\text{Hom}_R(-,-)\) are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{\mathcal{GP}}(-,-)$ and $\text{Ext}^n_{\mathcal{GI}}(-,-)$ of $\text{Hom}_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{GP}$-resolution $G = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}^n_{\mathcal{GP}}(M, N) := H^n(\text{Hom}_R(G, N)).$$

From [2.4] it will follow that $\text{Ext}^n_{\mathcal{GP}}(-, N)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(\mathcal{GP})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{\mathcal{GI}}(M', -)$, which is defined on the full subcategory, $\text{RightRes}_M(\mathcal{GI})$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\mathcal{GP})$ and $N \in \text{RightRes}_M(\mathcal{GI})$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9] Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12] Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(\mathcal{GP})$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\text{RightRes}_M(\mathcal{GI})$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GI}}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GI}$-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T: \mathcal{C} \rightarrow \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \rightarrow M$, such that $X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ is also a complex, and such that the sequence
\[ \text{Hom}_{\mathcal{C}}(X, X^+) = \cdots \rightarrow \text{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \text{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \text{Hom}_{\mathcal{C}}(X, M) \rightarrow 0 \]
is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_C(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_C(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_C(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let $f: M \rightarrow M'$ be a morphism in $\mathcal{C}$, and consider the diagram
\[
\begin{array}{cccccccccc}
\cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & \\
\cdots & \rightarrow & X'_2 & \rightarrow & X'_1 & \rightarrow & X'_0 & \rightarrow & M' & \rightarrow & 0
\end{array}
\]
where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \rightarrow X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precover of $M \in \mathcal{C}$ is a morphism $\varphi: X \rightarrow M$, where $X \in \mathcal{X}$, such that the sequence
\[ \text{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\text{Hom}_{\mathcal{C}}(X', \varphi)} \text{Hom}_{\mathcal{C}}(X', M) \rightarrow 0 \]
is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1})$, $i > 0$, and $X_0 \rightarrow M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \rightarrow \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor
\[ L_n^X T: \text{LeftRes}_C(\mathcal{X}) \rightarrow \mathcal{E} \]
of $T$, with respect to the class $\mathcal{X}$, by setting $L^n_\mathcal{X}T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$. Similarly, the $n^{th}$ right derived functor

$$R^n_\mathcal{X}T : \text{RightRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^n_\mathcal{X}T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_\mathcal{C}(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4

$$R^n_X F(-, N) : \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^n_Y F(M, -) : \text{RightRes}_\mathcal{D}(\mathcal{Y}) \to \mathcal{E}.$$

If furthermore $M \in \text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $N \in \text{RightRes}_\mathcal{D}(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^n_X F(M, N) \cong R^n_Y F(M, N),$$

functorial in $M$ and $N$. Here we wrote $R^n_X F(M, N)$ for the functor $R^n_X F(-, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^n_X F(-, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ and $\text{RightRes}_\mathcal{D}(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \bar{\mathcal{X}}$ and $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$.

(ii) Every $M \in \bar{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \bar{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^n_X F(M, N) \cong R^n_Y F(M, N),$$

for all $M \in \bar{\mathcal{X}}$ and $N \in \bar{\mathcal{Y}}$. 
Proof. Please see [3, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [3, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories \( \mathcal{X}, \mathcal{X}, \mathcal{C} \) and \( \mathcal{Y}, \mathcal{Y}, \mathcal{D} \), including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

\[
P = \cdots \to P_1 \to P_0 \to P_{-1} \to \cdots
\]

such that \( \text{Hom}_R(P, Q) \) is exact for every projective \( R \)-module \( Q \). An \( R \)-module \( M \) is called Gorenstein projective (G-projective for short), if there exists a complete projective resolution \( P \) with \( M \cong \text{Im}(P_0 \to P_{-1}) \). Gorenstein injective (G-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) \( R \)-modules,

\[
F = \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots
\]

such that \( I \otimes_R F \) is exact for every injective right \( R \)-module \( I \). An \( R \)-module \( M \) is called Gorenstein flat (G-flat for short), if there exists a complete flat resolution \( F \) with \( M \cong \text{Im}(F_0 \to F_{-1}) \).

3. **Gorenstein deriving \( \text{Hom}_R(-, -) \)**

We now return to categories of modules. We use \( \mathcal{GP}, \mathcal{GI} \) and \( \mathcal{GF} \) to denote the class of \( R \)-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, \( \mathcal{GP} \)-precovers are always surjective, and \( \mathcal{GP} \) contains all modules with finite projective dimension.

We now consider the functor \( \text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \to \mathcal{A} \), together with the categories

\[
\mathcal{X} = \mathcal{GP}, \quad \mathcal{X} = \mathcal{GP}, \quad \mathcal{Y} = \mathcal{GI}, \quad \mathcal{Y} = \mathcal{GI}.
\]

In this case we define, in the sense of section 2.4

\[
\text{Ext}^n_{\mathcal{GP}}(-, N) = R^n_{\mathcal{GP}}\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}^n_{\mathcal{GI}}(M, -) = R^n_{\mathcal{GI}}\text{Hom}_R(M, -),
\]

for fixed \( R \)-modules \( M \) and \( N \). We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If \( M \) is an \( R \)-module with \( \text{Gpd}_RM < \infty \), then there exists a short exact sequence \( 0 \to K \to G \to M \to 0 \), where \( G \to M \) is a \( \mathcal{GP} \)-precover of \( M \) (please see Remark 2.23), and \( \text{pd}_R K = \text{Gpd}_RM - 1 \) (in the case where \( M \) is Gorenstein projective, this should be interpreted as \( K = 0 \)).

Consequently, every \( R \)-module with finite Gorenstein projective dimension has a proper left \( \mathcal{GP} \)-resolution (that is, there is an inclusion \( \mathcal{GP} \subseteq \text{LeftRes}_M(\mathcal{GP}) \)).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let \( M \) be any \( R \)-module with \( \text{Gpd}_RM < \infty \). Then

\[
\text{Gpd}_RM = \sup\{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R \text{-module } L \text{ with } \text{pd}_RL < \infty\}.
\]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective
dimension, which for an $R$-module $M$ is given by
\[ \text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \}. \]
It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually
a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective
dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper
left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is
exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it
suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$
and all short exact sequences
\[ S = 0 \to K \to G \to M \to 0, \]
where $G \to M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that
$\mathcal{GP}$-precovers are always surjective). By Proposition 3.1 there is a special short
exact sequence,
\[ S' = 0 \to K' \to G' \to M \to 0, \]
where $\pi : G' \to M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homo-
topy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for
every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of
$\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of
\[ \text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \to 0. \]
To see this, let $\alpha : K' \to H$ be any homomorphism. We wish to find $g : G' \to H$
such that $g \iota = \alpha$. Now pick an exact sequence
\[ 0 \to \bar{H} \to E \xrightarrow{g} H \to 0, \]
where $E$ is injective, and $\bar{H}$ is Gorenstein injective (the sequence in question is just
a part of the complete injective resolution that defines $H$). Since $\bar{H}$ is Gorenstein
injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R(K', \bar{H}) = 0$ by [7, Lemma 1.3], and thus a
lifting $\varepsilon : K' \to E$ with $g \varepsilon = \alpha$:

\[ \begin{array}{ccc}
K' & \xrightarrow{\alpha} & G' \\
\downarrow \varepsilon & & \downarrow g \\
H & \xrightarrow{\iota} & E
\end{array} \]

Next, injectivity of $E$ gives $\tilde{\varepsilon} : G' \to E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g = g \tilde{\varepsilon} : G' \to H$ is the
desired map. \qed

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $GI$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$. □

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}_{G_P}^n(M, N) \cong \text{Ext}_{G_T}^n(M, N),$$

which are functorial in $M$ and $N$. □

3.7 (Definition of $G$Ext). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$G\text{Ext}_{R}^n(M, N) := \text{Ext}_{G_P}^n(M, N) \cong \text{Ext}_{G_T}^n(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare $G$Ext with the classical $\text{Ext}$. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}_{G_P}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions

(\begin{itemize}
  \item[(i)] $\text{pd}_R M < \infty$ or $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N < \infty$.
  \item[(ii)] $\text{id}_R N < \infty$ or $N \in \text{RightRes}_M(GI)$ and $\text{pd}_R M < \infty$.
\end{itemize})

(ii) There are natural isomorphisms $\text{Ext}_{G_T}^n(M, N) \cong \text{Ext}_R^n(M, N)$ under each of the conditions

(\begin{itemize}
  \item[(i)] $\text{pd}_R M < \infty$ or $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N < \infty$.
  \item[(iii)] Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then $G\text{Ext}_R^n(M, N) \cong \text{Ext}_R^n(M, N)$ is functorial in $M$ and $N$.
\end{itemize})

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $GP$-resolution of $M$, and thus

$$\text{Ext}_{G_P}^n(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}_R^n(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}_R^n(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}_R^i(G, N) \cong \text{Ext}_R^{m+i}(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A implies that $\text{Ext}_R^n(\cdot, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $G\text{Ext}_R^n(\cdot, \cdot)$. □
4. Gorenstein deriving $- \otimes_R -$ 

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $C$ is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP}$, etc.), we write $rC$, respectively, $C_R$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $C$.

Now we consider the functor $- \otimes_R -$ : $\mathcal{M}_R \times R\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in R\mathcal{M}$ we define, in the sense of section 2.4:

$$\text{Tor}^n_{GP}(R, M; N) := \text{L}^n_{GP}(M \otimes_R N) \quad \text{and} \quad \text{Tor}^n_{GP}(M; -) := \text{L}^n_{GP}(M \otimes_R -),$$

together with

$$\text{Tor}^n_{Fr}(R, M; N) := \text{L}^n_{Fr}(M \otimes_R N) \quad \text{and} \quad \text{Tor}^n_{Fr}(M; -) := \text{L}^n_{Fr}(M \otimes_R -).$$

The first two $\text{Tor}$s use proper left Gorestein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, e X) = (GP_R, f GP_R)$ or $(GF_R, g GF_R)$;

and

$$(Y, e Y) = (n GP, n GF) \quad \text{or} \quad (n GF, n GF),$$

namely, the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$$\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.$$ 

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $n GP \subseteq n GF$.

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.
(ii) The Pontryagin dual $\text{Hom}_{\mathbb{Q}/\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.
(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}^R_i(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Theorem 4.8. Assume that $R$ is both left and right coherent, and that both \text{LeftFPD}(R) and \text{RightFPD}(R) are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\text{GP}$-resolution of $M$ (which exists by Proposition 4.5). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite \text{RightFPD}(R), then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.4. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat. 

Lemma 4.7. Assume that $R$ is right coherent with finite \text{LeftFPD}(R). Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\text{GF}$-resolution of $M$ (which exists by Proposition 4.4 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite \text{RightFPD}(R), then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 4.5 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' = \cdots \to G_1 \to G_0 \to M \to 0$ is an $R\text{GF}$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has \text{LeftFPD}(R) < \infty, \text{Proposition 6}$ implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii). 

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both \text{LeftFPD}(R) and \text{RightFPD}(R) are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
$$\text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N).$$

(ii) If $\text{Gpd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
$$\text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N) \cong \text{Tor}_n^G(M, N).$$

(iii) If $\text{Gfd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
$$\text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N) \cong \text{Tor}_n^G(M, N).$$

(iv) If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
$$\text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N) \cong \text{Tor}_n^G(M, N).$$

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \qed

4.9 (Definition of $g\text{Tor}$ and $\text{GTor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then we write
$$g\text{Tor}_n^R(M, N) := \text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then we write
$$\text{GTor}_n^R(M, N) := \text{Tor}^R_n^G(M, N) \cong \text{Tor}_n^G(M, N)$$
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_R M$, and for every left $R$-module $N$ with $\text{Gpd}_R N < \infty$, we have isomorphisms:
$$g\text{Tor}_n^R(M, N) \cong \text{GTor}_n^R(M, N)$$
that are functorial in $M$ and $N$.

Finally we compare $g\text{Tor}$ (and hence $\text{GTor}$) with the usual Tor.

Theorem 4.11. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_R M < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_R N < \infty$. If either $\text{fd}_R M < \infty$ or $\text{fd}_R N < \infty$, then there are isomorphisms
$$g\text{Tor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)$$
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_R M < \infty$, then we also have $\text{pd}_R M < \infty$ by [13, Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 4.8, $P$ is also a proper left $\mathcal{GP}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
$$g\text{Tor}_n^R(M, N) = \text{Tor}_n^R(M, N) = H_n(P \otimes_R N) = \text{Tor}_n^R(M, N),$$
as desired. \qed
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References


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