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Holm, Henrik Granau

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GORENSTEIN DERIVED FUNCTORS

HENRIK HOLM

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-, -)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}^n_R(-, -)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $- \otimes_R -$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_RM$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_RM \leq \text{pd}_RM$, with equality $\text{G-dim}_RM = \text{pd}_RM$ when $\text{pd}_RM < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\mathcal{GP}$, $\mathcal{GI}$ and $\mathcal{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_RM$, $\text{Gid}_RM$ and $\text{Gfd}_RM$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.
Now, given our base ring $R$, the usual right derived functors $\text{Ext}^n_R(-,-)$ of $\text{Hom}_R(-,-)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{\mathcal{GP}}(-,-)$ and $\text{Ext}^n_{\mathcal{GI}}(-,-)$ of $\text{Hom}_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{GP}$-resolution $G = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ (please see $2.4$ below for the definition of proper resolutions), we define

$$\text{Ext}^n_{\mathcal{GP}}(M,N) := H^n(\text{Hom}_R(G,N)).$$

From $2.3$ it will follow that $\text{Ext}^n_{\mathcal{GP}}(-,-)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(\mathcal{GP})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $\mathcal{GP}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{\mathcal{GI}}(M',-)$, which is defined on the full subcategory, $\text{RightRes}_M(\mathcal{GI})$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $\mathcal{GI}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{GI}}(M,N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(\mathcal{GP})$ and $N \in \text{RightRes}_M(\mathcal{GI})$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda $[9]$ Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{GI}}(M,N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see $[9]$ Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, $[12]$ Proposition 2.18 (which is restated in this paper as Proposition $2.4$) implies that the category $\text{LeftRes}_M(\mathcal{GP})$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $\mathcal{GP}$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $\mathcal{GI}$-resolution. So $\text{RightRes}_M(\mathcal{GI})$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem $5.6$ in this text proves that the functorial isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M,N) \cong \text{Ext}^n_{\mathcal{GI}}(M,N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems $4.8$ and $4.10$ give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $\mathcal{GP}$-resolutions and proper left $\mathcal{GF}$-resolutions. This has also been proved by Enochs and Jenda $[9]$ Theorem 12.2.2 in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T: \mathcal{C} \rightarrow \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \rightarrow M$, such that $X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ is also a complex, and such that the sequence

$$\text{Hom}_{\mathcal{C}}(X, X^+) = \cdots \rightarrow \text{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \text{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \text{Hom}_{\mathcal{C}}(X, M) \rightarrow 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_C(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_C(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_C(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

Proposition 2.2. Let $f: M \rightarrow M'$ be a morphism in $\mathcal{C}$, and consider the diagram

$$\cdots \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$$

$$\cdots \rightarrow X'_2 \rightarrow X'_1 \rightarrow X'_0 \rightarrow M' \rightarrow 0$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \rightarrow X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

Proof. The proof is an exercise; please see [9, Exercise 8.1.2].

Remark 2.3. A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precover of $M \in \mathcal{C}$ is a morphism $\varphi: X \rightarrow M$, where $X \in \mathcal{X}$, such that the sequence

$$\text{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\text{Hom}_{\mathcal{C}}(X', \varphi)} \text{Hom}_{\mathcal{C}}(X', M) \rightarrow 0$$

is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1})$, $i > 0$, and $X_0 \rightarrow M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \rightarrow \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n^{\text{th}}$ left derived functor

$$L_n^X T: \text{LeftRes}_C(\mathcal{X}) \rightarrow \mathcal{E}$$
of \( T \), with respect to the class \( \mathcal{X} \), by setting \( \text{L}_n^\mathcal{X}T(M) = H_n(T(X)) \), where \( X \) is any proper left \( \mathcal{X} \)-resolution of \( M \in \text{LeftRes}_\mathcal{C}(\mathcal{X}) \). Similarly, the \( n \)th right derived functor

\[
\text{R}_n^\mathcal{X}T : \text{RightRes}_\mathcal{C}(\mathcal{X}) \rightarrow \mathcal{E}
\]

of \( T \) with respect to \( \mathcal{X} \) is defined by \( \text{R}_n^\mathcal{X}T(N) = H_n(T(Y)) \), where \( Y \) is any proper right \( \mathcal{X} \)-resolution of \( N \in \text{RightRes}_\mathcal{C}(\mathcal{X}) \). These constructions are well-defined and functorial in the arguments \( M \) and \( N \) by Proposition 2.2.

The situation where \( T \) is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category \( \mathcal{D} \), together with a full subcategory \( \mathcal{Y} \subseteq \mathcal{D} \) and an additive functor \( F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \) in two variables. We will assume that \( F \) is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of \( F \) is not important, and the definitions and results below can easily be modified to fit the situation where \( F \) is covariant in both variables, say.

For fixed \( M \in \mathcal{C} \) and \( N \in \mathcal{D} \) we can then consider the two right derived functors as in 2.4:

\[
\text{R}_n^\mathcal{X}F(-, N) : \text{LeftRes}_\mathcal{C}(\mathcal{X}) \rightarrow \mathcal{E} \quad \text{and} \quad \text{R}_n^\mathcal{Y}F(M, -) : \text{RightRes}_\mathcal{D}(\mathcal{Y}) \rightarrow \mathcal{E}.
\]

If furthermore \( M \in \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) and \( N \in \text{RightRes}_\mathcal{D}(\mathcal{Y}) \), we can ask for a sufficient condition to ensure that

\[
\text{R}_n^\mathcal{X}F(M, N) \cong \text{R}_n^\mathcal{Y}F(M, N),
\]

functorial in \( M \) and \( N \). Here we wrote \( \text{R}_n^\mathcal{X}F(M, N) \) for the functor \( \text{R}_n^\mathcal{X}F(-, N) \) applied to \( M \). Another, and perhaps better, notation could be

\[
\text{R}_n^\mathcal{X}F(-, N)[M].
\]

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor \( F : \mathcal{C} \times \mathcal{D} \rightarrow \mathcal{E} \) which is contravariant in the first variable and covariant in the second variable, together with the full subcategories \( \mathcal{X} \subseteq \mathcal{C} \) and \( \mathcal{Y} \subseteq \mathcal{D} \). Assume that we have full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \text{LeftRes}_\mathcal{C}(\mathcal{X}) \) and \( \text{RightRes}_\mathcal{D}(\mathcal{Y}) \), respectively, satisfying:

(i) \( \mathcal{X} \subseteq \mathcal{X} \) and \( \mathcal{Y} \subseteq \mathcal{Y} \).

(ii) Every \( M \in \mathcal{X} \) has an augmented proper left \( \mathcal{X} \)-resolution \( \cdots \rightarrow X_i \rightarrow X_0 \rightarrow M \rightarrow 0 \), such that \( 0 \rightarrow F(M, Y) \rightarrow F(X_0, Y) \rightarrow F(X_1, Y) \rightarrow \cdots \) is exact for all \( Y \in \mathcal{Y} \).

(iii) Every \( N \in \mathcal{Y} \) has an augmented proper right \( \mathcal{Y} \)-resolution \( 0 \rightarrow N \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \), such that \( 0 \rightarrow F(X, N) \rightarrow F(X, Y^0) \rightarrow F(X, Y^1) \rightarrow \cdots \) is exact for all \( X \in \mathcal{X} \).

Then we have functorial isomorphisms

\[
\text{R}_n^\mathcal{X}F(M, N) \cong \text{R}_n^\mathcal{Y}F(M, N),
\]

for all \( M \in \mathcal{X} \) and \( N \in \mathcal{Y} \).
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories $X, \tilde{X}, C$ and $Y, \tilde{Y}, D$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

$$P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective (G-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \rightarrow P_{-1})$. Gorenstein injective (G-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat (G-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \rightarrow F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\widehat{GP}, \widehat{GI}$ and $\widehat{GF}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\widehat{GP}$-precovers are always surjective, and $\widehat{GP}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$, together with the categories

$$\mathcal{X} = \widehat{GP}, \quad \tilde{\mathcal{X}} = \widehat{GP} \quad \text{and} \quad \mathcal{Y} = \widehat{GI}, \quad \tilde{\mathcal{Y}} = \widehat{GI}.$$  

In this case we define, in the sense of section 2.4

$$\text{Ext}^n_{\widehat{GP}}(-, N) = R^n_{\widehat{GP}}\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}^n_{\widehat{GI}}(M, -) = R^n_{\widehat{GI}}\text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is a $\widehat{GP}$-precover of $M$ (please see Remark 2.23), and $\text{pd}_R K = \text{Gpd}_R M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $\widehat{GP}$-resolution (that is, there is an inclusion $\widehat{GP} \subseteq \text{LeftRes}_M(\widehat{GP})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_R M < \infty$. Then

$$\text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by
\[ \text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \} \].

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences
\[ S = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0 \]
where $G \rightarrow M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,
\[ S' = 0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0 \]
where $\pi: G' \rightarrow M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of
\[ \text{Hom}_R(G^+, H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \xrightarrow{0} \]
To see this, let $\alpha: K' \rightarrow H$ be any homomorphism. We wish to find $g: G' \rightarrow H$ such that $g \iota = \alpha$. Now pick an exact sequence
\[ 0 \rightarrow \tilde{H} \rightarrow E \xrightarrow{g} H \rightarrow 0 \]
where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \rightarrow E$ with $g \varepsilon = \alpha$:

\[ K' \xrightarrow{\alpha} G' \xrightarrow{\varepsilon} E \]

Next, injectivity of $E$ gives $\tilde{\varepsilon}: G' \rightarrow E$ with $\tilde{\varepsilon} \iota = \varepsilon$. Now $g = g \tilde{\varepsilon}: G' \rightarrow H$ is the desired map. \hfill \Box

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \rightarrow N \rightarrow H^0 \rightarrow H^1 \rightarrow \cdots$ be an augmented proper right $GI$-resolution of $N$ (which exists by the dual of Proposition 3.4). Then $\text{Hom}_R(G, H^+) = 0$ is exact for all Gorenstein projective modules $G$.

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{GT}(M, N),$$

which are functorial in $M$ and $N$. \hfill \Box

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{GT}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

$$\begin{align*}
\text{pd}_R M < \infty & \quad \text{or} \quad M \in \text{LeftRes}_M(GP) \quad \text{and} \quad \text{id}_R N < \infty, \\
\text{id}_R N < \infty & \quad \text{or} \quad N \in \text{RightRes}_M(GT) \quad \text{and} \quad \text{pd}_R M < \infty.
\end{align*}$$

(ii) There are natural isomorphisms $\text{Ext}^n_{GT}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

$$\begin{align*}
\text{id}_R N < \infty & \quad \text{or} \quad N \in \text{RightRes}_M(GT) \quad \text{and} \quad \text{pd}_R M < \infty.
\end{align*}$$

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then $\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$ is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $GP$-resolution of $M$, and thus

$$\text{Ext}^n_{GP}(M, N) = H^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \rightarrow G \rightarrow Q^0 \rightarrow \cdots \rightarrow Q^{m-1} \rightarrow C \rightarrow 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{m+i}_R(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt$^n_R(-, -)$. \hfill \Box
In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}, \mathcal{GP}$, etc.), we write $r\mathcal{C}$, respectively, $\mathcal{C}R$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $R^\mathcal{GP}(-) : \mathcal{M}_R \times R\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}_R$ and $N \in R\mathcal{M}$ we define, in the sense of section 2.4:

$$\text{Tor}^\mathcal{GP}_n(-, N) := L_n^\mathcal{GP}(- \otimes_R N)$$

and

$$\text{Tor}^\mathcal{GP}_n(M, -) := L_n^\mathcal{GP}(M \otimes_R -),$$

together with

$$\text{Tor}^\mathcal{GP}_n(-, N) := L_n^\mathcal{GP}(- \otimes_R N)$$

and

$$\text{Tor}^\mathcal{GP}_n(M, -) := L_n^\mathcal{GP}(M \otimes_R -).$$

The first two $\text{Tor}$s use proper left Gorenstein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein flat resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \widetilde{X}) = (\mathcal{GP}_R, \widetilde{\mathcal{GP}}_R)$ or $(\mathcal{GP}_R, \mathcal{GP}_R)$, together with

$$(Y, \widetilde{Y}) = (r\mathcal{GP}, r\widetilde{\mathcal{GP}})$$

and

$$(Y, \widetilde{Y}) = (r\mathcal{GP}, r\widetilde{\mathcal{GP}}),$$

denotes the covariant-covariant version of Theorem 2.6 instead of the stated contravariant-covariant version. We will need the following classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$$\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.$$ 

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10, Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12, Proposition 3.3], [12, Theorem 3.5] and [12, Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $r\mathcal{GP} \subseteq r\mathcal{GF}$.

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \to M \to F^0 \to F^1 \to \cdots$ consisting of flat left $R$-modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}F$-resolution (that is, there is an inclusion $R\mathcal{G}F \subseteq \text{LeftRes}_{R,M}(R\mathcal{G}F)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}P$-resolution of $M$ (which exists by Proposition 4.4). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) = \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.2 implies that every Gorenstein projective right $R$-module also is Gorenstein flat. □

Lemma 4.7. Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition 4.4 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 5.3 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, 4.5 Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 3.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of $\text{Hom}_R(G^+, H) \cong \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$.

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.2 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii). □

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[
\text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N).
\]

(ii) If $\text{Gpd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then
\[
\text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N).
\]

(iii) If $\text{Gfd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[
\text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N).
\]

(iv) If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then
\[
\text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N).
\]

All the isomorphisms are functorial in $M$ and $N$.

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6.

4.9 (Definition of $\text{gTor}$ and $\text{GTor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_R M < \infty$ and $\text{Gfd}_R N < \infty$, then we write
\[
\text{gTor}_n^R(M, N) := \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_R M < \infty$ and $\text{Gpd}_R N < \infty$, then we write
\[
\text{GTor}_n^R(M, N) := \text{Tor}^G_n(M, N) \cong \text{Tor}^G_n(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_R M$, and for every left $R$-module $N$ with $\text{Gpd}_R N < \infty$, we have isomorphisms:
\[
\text{gTor}_n^R(M, N) \cong \text{GTor}_n^R(M, N)
\]
that are functorial in $M$ and $N$.

Finally we compare $\text{gTor}$ (and hence $\text{GTor}$) with the usual $\text{Tor}$.

Theorem 4.11. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_R M < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_R N < \infty$. If either $\text{fd}_R M < \infty$ or $\text{fd}_R N < \infty$, then there are isomorphisms
\[
\text{gTor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)
\]
that are functorial in $M$ and $N$.

Proof. If $\text{fd}_R M < \infty$, then we also have $\text{pd}_R M < \infty$ by [13] Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 4.3, $P$ is also a proper left $\mathcal{P}_R$-resolution of $M$. Hence, Theorem 4.8(ii) and the definitions give:
\[
\text{gTor}_n^R(M, N) = \text{Tor}^G_n(M, N) = H_n(P \otimes_R N) = \text{Tor}_n^R(M, N),
\]
as desired. 

\[\square\]
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REFERENCES


Matematisk Afdeling, Københavns Universitet, Universitetsparken 5, 2100 København Ø, DK – Danmark
E-mail address: holm@math.ku.dk