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GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring $R$ it is standard to derive $\text{Hom}_R(-,-)$ using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains $\text{Ext}^n_R(-,-)$ in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product $-\otimes_R-$ using Gorenstein flat modules.

1. Introduction

When $R$ is a two-sided Noetherian ring, Auslander and Bridger [2] introduced in 1969 the G-dimension, $\text{G-dim}_R M$, for every finite (that is, finitely generated) $R$-module $M$. They proved the inequality $\text{G-dim}_R M \leq \text{pd}_R M$, with equality $\text{G-dim}_R M = \text{pd}_R M$ when $\text{pd}_R M < \infty$, along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring $R$, Enochs and Jenda in [6] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if $R$ is two-sided Noetherian, and $G$ is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following [4, Theorem (4.2.6)]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary $R$-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- $R$ is an associative ring. All modules are—if not specified otherwise—left $R$-modules, and the category of all $R$-modules is denoted $\mathcal{M}$. We use $\mathcal{A}$ for the category of abelian groups (that is, $\mathbb{Z}$-modules).
- We use $\text{GP}$, $\text{GI}$ and $\text{GF}$ for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat $R$-modules; please see [6] and [8], or Definition 2.7 below.
- Furthermore, for each $R$-module $M$ we write $\text{Gpd}_R M$, $\text{Gid}_R M$ and $\text{Gfd}_R M$ for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of $M$, respectively.

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The aim of this paper is to show a slightly weaker result. More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $G\mathcal{P}$-resolution $G = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}^n_{G\mathcal{P}}(M, N) := H^n(\text{Hom}_R(G, N)).$$

From [2.4] it will follow that $\text{Ext}^n_{G\mathcal{P}}(\cdot, N)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(G\mathcal{P})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $G\mathcal{P}$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{G\mathcal{I}}(M', \cdot)$, which is defined on the full subcategory, $\text{RightRes}_M(G\mathcal{I})$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper right $G\mathcal{I}$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{G\mathcal{P}}(M, N) \cong \text{Ext}^n_{G\mathcal{I}}(M, N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(G\mathcal{P})$ and $N \in \text{RightRes}_M(G\mathcal{I})$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}^n_{G\mathcal{P}}(M, N) \cong \text{Ext}^n_{G\mathcal{I}}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_R M < \infty$, $\text{Gid}_R M < \infty$, and also $\text{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9] Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12] Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(G\mathcal{P})$ contains all $R$-modules $M$ with $\text{Gpd}_R M < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $G\mathcal{P}$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $G\mathcal{I}$-resolution. So $\text{RightRes}_M(G\mathcal{I})$ contains all $R$-modules $N$ with $\text{Gid}_R N < \infty$.

Theorem 5.6 in this text proves that the functorial isomorphisms $\text{Ext}^n_{G\mathcal{P}}(M, N) \cong \text{Ext}^n_{G\mathcal{I}}(M, N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $\cdot \otimes_R \cdot$, using proper left $G\mathcal{P}$-resolutions and proper left $G\mathcal{I}$-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \rightarrow \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $X^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence

$$\text{Hom}_\mathcal{C}(X, X^+) = \cdots \to \text{Hom}_\mathcal{C}(X, X_1) \to \text{Hom}_\mathcal{C}(X, X_0) \to \text{Hom}_\mathcal{C}(X, M) \to 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_\mathcal{C}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_\mathcal{C}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_\mathcal{C}(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let $f: M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram

$$\cdots \to X_2 \to X_1 \to X_0 \to M \to 0$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \to X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precover of $M \in \mathcal{C}$ is a morphism $\varphi: X \to M$, where $X \in \mathcal{X}$, such that the sequence

$$\text{Hom}_\mathcal{C}(X', X) \xrightarrow{\text{Hom}_\mathcal{C}(X', \varphi)} \text{Hom}_\mathcal{C}(X', M) \to 0$$

is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \to \ker(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor

$$L_n^\mathcal{X} T: \text{LeftRes}_\mathcal{C}(\mathcal{X}) \to \mathcal{E}$$
of $T$, with respect to the class $\mathcal{X}$, by setting $L^X_nT(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_C(\mathcal{X})$. Similarly, the $n^{th}$ right derived functor

$$R^X_nT: \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^X_nT(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_C(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9] Section 8.2 for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4

$$R^X_nF(-,N): \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^Y_nF(M,-): \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.$$ 

If furthermore $M \in \text{LeftRes}_C(\mathcal{X})$ and $N \in \text{RightRes}_D(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^X_nF(M,N) \cong R^Y_nF(M,N),$$

functorial in $M$ and $N$. Here we wrote $R^Y_nF(M,N)$ for the functor $R^X_nF(-,N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^X_nF(-,N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5 Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F: \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_C(\mathcal{X})$ and $\text{RightRes}_D(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \bar{\mathcal{X}}$ and $\mathcal{Y} \subseteq \bar{\mathcal{Y}}$.

(ii) Every $M \in \bar{\mathcal{X}}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M,Y) \to F(X_0,Y) \to F(X_1,Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \bar{\mathcal{Y}}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X,N) \to F(X,Y^0) \to F(X,Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^X_nF(M,N) \cong R^Y_nF(M,N),$$

for all $M \in \bar{\mathcal{X}}$ and $N \in \bar{\mathcal{Y}}$. 

Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories $X$, $\tilde{X}$, $C$ and $\mathcal{Y}$, $\tilde{\mathcal{Y}}$, $D$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules, $$P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots,$$
such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \to P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules, $$F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots,$$
such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \to F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\tilde{G}\mathcal{P}$, $\tilde{G}\mathcal{I}$ and $\tilde{G}\mathcal{F}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $G\mathcal{P}$-precovers are always surjective, and $\tilde{G}\mathcal{P}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \to \mathcal{A}$, together with the categories

$$\mathcal{X} = G\mathcal{P}, \quad \tilde{\mathcal{X}} = \tilde{G}\mathcal{P} \quad \text{and} \quad \mathcal{Y} = G\mathcal{I}, \quad \tilde{\mathcal{Y}} = \tilde{G}\mathcal{I}.$$ 

In this case we define, in the sense of section 2.4

$$\text{Ext}_{G\mathcal{P}}^n(-, N) = R^n_{G\mathcal{P}}\text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{G\mathcal{I}}^n(M, -) = R^n_{G\mathcal{I}}\text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is a $G\mathcal{P}$-precover of $M$ (please see Remark 2.5), and $\text{pd}_R K = \text{Gpd}_R M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $G\mathcal{P}$-resolution (that is, there is an inclusion $G\mathcal{P} \subseteq \text{LeftRes}_M(G\mathcal{P})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_R M < \infty$. Then

$$\text{Gpd}_R M = \sup \{n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an \( R \)-module \( M \) is given by

\[
\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}_R^n(M, L) \neq 0 \text{ for some } R\text{-module } L \}.
\]

It also follows that if \( \text{pd}_R M < \infty \), then every projective resolution of \( M \) is actually a proper left \( \mathcal{GP} \)-resolution of \( M \).

**Lemma 3.4.** Assume that \( M \) is an \( R \)-module with finite Gorenstein projective dimension, and let \( G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0 \) be an augmented proper left \( \mathcal{GP} \)-resolution of \( M \) (which exists by Proposition 3.1). Then \( \text{Hom}_R(G^+, H) \) is exact for all Gorenstein injective modules \( H \).

**Proof.** We split the proper resolution \( G^+ \) into short exact sequences. Hence it suffices to show exactness of \( \text{Hom}_R(S, H) \) for all Gorenstein injective modules \( H \) and all short exact sequences

\[
S = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0,
\]

where \( G \rightarrow M \) is a \( \mathcal{GP} \)-precover of some module \( M \) with \( \text{Gpd}_R M < \infty \) (recall that \( \mathcal{GP} \)-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,

\[
S' = 0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0,
\]

where \( \pi : G' \rightarrow M \) is a \( \mathcal{GP} \)-precover and \( \text{pd}_R K' < \infty \).

It is easy to see (as in Proposition 2.2) that the complexes \( S \) and \( S' \) are homotopy equivalent, and thus so are the complexes \( \text{Hom}_R(S, H) \) and \( \text{Hom}_R(S', H) \) for every (Gorenstein injective) module \( H \). Hence it suffices to show the exactness of \( \text{Hom}_R(S', H) \) whenever \( H \) is Gorenstein injective.

Now let \( H \) be any Gorenstein injective module. We need to prove the exactness of

\[
\frac{\text{Hom}_R(G', H)}{\text{Hom}_R(S', H)} \longrightarrow \frac{\text{Hom}_R(K', H)}{\text{Hom}_R(S', H)}.
\]

To see this, let \( \alpha : K' \rightarrow H \) be any homomorphism. We wish to find \( g : G' \rightarrow H \) such that \( g \alpha = \alpha \). Now pick an exact sequence

\[
0 \longrightarrow \tilde{H} \longrightarrow E \xrightarrow{g} H \longrightarrow 0,
\]

where \( E \) is injective, and \( \tilde{H} \) is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines \( H \)). Since \( \tilde{H} \) is Gorenstein injective and \( \text{pd}_R K' < \infty \), we get \( \text{Ext}_R^n(K', \tilde{H}) = 0 \) by \( \mathbb{R} \) Lemma 1.3], and thus a lifting \( \varepsilon : K' \rightarrow E \) with \( g \varepsilon = \alpha \):

\[
\begin{array}{ccc}
K' & \xrightarrow{\alpha} & G' \\
\downarrow{\varepsilon} & & \downarrow{\varepsilon} \\
E & \xrightarrow{g} & H
\end{array}
\]

Next, injectivity of \( E \) gives \( \tilde{\varepsilon} : G' \rightarrow E \) with \( \tilde{\varepsilon} \alpha = \varepsilon \). Now \( g \tilde{\varepsilon} : G' \rightarrow H \) is the desired map. \( \Box \)

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+=0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{G}$-resolution of $N$ (which exists by the dual of Proposition 3.2). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$. \hfill $\square$

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$, we have isomorphisms

$$\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GZ}}(M, N),$$

which are functorial in $M$ and $N$. \hfill $\square$

3.7 (Definition of $G\text{Ext}$). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. Then we write

$$G\text{Ext}^n_R(M, N) := \text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_{\mathcal{GZ}}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare $G\text{Ext}$ with the classical $\text{Ext}$. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{\mathcal{GP}}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{itemize}
    \item[(i)] $\text{pd}_RM < \infty$ or \quad $M \in \text{LeftRes}_M(\mathcal{GP})$ \quad and \quad $\text{id}_RN < \infty$.
    \item[(ii)] $\text{Ext}^n_{\mathcal{GZ}}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{itemize}
    \item[(i)] $\text{id}_RN < \infty$ \quad or \quad $N \in \text{RightRes}_M(\mathcal{GT})$ \quad and \quad $\text{pd}_RM < \infty$.
    \item[(iii)] Assume that $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. If either $\text{pd}_RM < \infty$ or $\text{id}_RN < \infty$, then $G\text{Ext}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$ is functorial in $M$ and $N$.
\end{itemize}

Proof. (i) Assume that $\text{pd}_RM < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{GP}$-resolution of $M$, and thus

$$\text{Ext}^n_{\mathcal{GP}}(M, N) = \text{H}^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(\mathcal{GP})$ and $\text{id}_RN = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{m+i}_R(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A] implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $G\text{Ext}^n_R(-, -)$. \hfill $\square$
4. Gorenstein deriving \(-\otimes_R-\)

In dealing with the tensor product we need, of course, both left and right \(R\)-modules. Thus the following addition to Notation 1.1 is needed:

If \(\mathcal{C}\) is any of the categories in Notation 1.1 (\(\mathcal{M}, \mathcal{GP}, \text{etc.}\), we write \(R\mathcal{C}\), respectively, \(\mathcal{C}R\), for the category of left, respectively, right, \(R\)-modules with the property describing the modules in \(\mathcal{C}\).

Now we consider the functor \(R\mathcal{M}\mathcal{R}\). For \(x \in \mathcal{M}\) and \(N \in R\mathcal{M}\) we define, in the sense of section 2.4:

\[
\text{Tor}^n_{\mathcal{GP}}(-, N) := L_n \mathcal{GP}(- \otimes_R N) \quad \text{and} \quad \text{Tor}^n_{\mathcal{GP}}(M, -) := L_n \mathcal{GP}(M \otimes_R -),
\]

together with

\[
\text{Tor}^n_{\mathcal{GP}}(N, -) := L_n \mathcal{GP}(N \otimes_R -) \quad \text{and} \quad \text{Tor}^n_{\mathcal{GP}}(-, M) := L_n \mathcal{GP}(\otimes_R M),
\]

The first two \(\text{Tor}\)s use proper left Gorestein projective resolutions, and the last two \(\text{Tor}\)s use proper left Gorestein flat resolutions. In order to compare these different \(\text{Tor}\)s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of \((X, \mathcal{X}) = (\mathcal{GP}, \mathcal{GP})\) or \((\mathcal{GP}, \mathcal{GP})\), and \((Y, \mathcal{Y}) = (\mathcal{GF}, \mathcal{GF})\) or \((\mathcal{GF}, \mathcal{GF})\), namely, the covariant-covariant version of Theorem 2.6, instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[
\text{LeftFPD}(R) = \sup\{\text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty\}.
\]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 4.2.** When \(R\) is commutative and Noetherian, the dimensions \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) coincide and are equal to the Krull dimension of \(R\), by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If \(R\) is right coherent with finite \(\text{LeftFPD}(R)\), then every Goreinstein projective left \(R\)-module is also Goreinstein flat. That is, there is an inclusion \(\mathcal{GP} \subseteq \mathcal{GF}\).

**Theorem 4.4.** For any left \(R\)-module \(M\), we consider the following three conditions:

(i) The left \(R\)-module \(M\) is \(G\)-flat.

(ii) The Pontryagin dual Hom\(\mathbb{Z} \to \mathcal{M}, \mathbb{Q} \to \mathcal{Z}\) (which is a right \(R\)-module) is \(G\)-injective.

(iii) \(M\) has an augmented proper right resolution \(0 \to M \to F^0 \to F^1 \to \cdots\) consisting of flat left \(R\)-modules, and \(\text{Tor}_i^R(I, M) = 0\) for all injective right \(R\)-modules \(I\), and all \(i > 0\).

The implication (i) \(\Rightarrow\) (ii) always holds. If \(R\) is right coherent, then also (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\mathcal{G}F$-resolution (that is, there is an inclusion $R\mathcal{G}F \subseteq \text{LeftRes}_{R,M}(R\mathcal{G}F)$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}P$-resolution of $M$ (which exists by Proposition 3.4). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite RightFPD($R$), then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 1.4 above, the Pontryagin dual $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.4. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 1.3 implies that every Gorenstein projective right $R$-module also is Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite LeftFPD($R$). Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\mathcal{G}F$-resolution of $M$ (which exists by Proposition 4.5 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite RightFPD($R$), then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 4.5 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\mathcal{G}F$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has LeftFPD($R$) < ∞, Proposition 6] implies that $\text{pd}_R K' < \infty$. Now the proof of Lemma 4.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_\mathbb{Z}(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 1.4 above. By the result (i), just proved, we have exactness of

\[ \text{Hom}_R(G^+, H) \cong \text{Hom}_\mathbb{Z}(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}). \]

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 4.5 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both LeftFPD($R$) and RightFPD($R$) are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N).
\]

(ii) If \( \text{Gpd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then
\[
\text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N).
\]

(iii) If \( \text{Gfd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N).
\]

(iv) If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then
\[
\text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N).
\]

All the isomorphisms are functorial in \( M \) and \( N \).

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \( \square \)

4.9 (Definition of \( \text{gTor} \) and \( \text{GTor} \)). Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module, and let \( N \) be a left \( R \)-module. If \( \text{Gfd}_R M < \infty \) and \( \text{Gfd}_R N < \infty \), then we write
\[
\text{gTor}_n^R(M, N) := \text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(i). If \( \text{Gpd}_R M < \infty \) and \( \text{Gpd}_R N < \infty \), then we write
\[
\text{GTor}_n^R(M, N) := \text{Tor}_n^{m,n}(M, N) \cong \text{Tor}_n^{m,n}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. For every right \( R \)-module \( M \) with finite \( \text{Gpd}_R M \), and for every left \( R \)-module \( N \) with \( \text{Gpd}_R N < \infty \), we have isomorphisms:
\[
\text{gTor}_n^R(M, N) \cong \text{GTor}_n^R(M, N)
\]
that are functorial in \( M \) and \( N \).

Finally we compare \( \text{gTor} \) (and hence \( \text{GTor} \)) with the usual \( \text{Tor} \).

Theorem 4.11. Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module with \( \text{Gfd}_R M < \infty \), and let \( N \) be a left \( R \)-module with \( \text{Gfd}_R N < \infty \). If either \( \text{fd}_R M < \infty \) or \( \text{fd}_R N < \infty \), then there are isomorphisms
\[
\text{gTor}_n^R(M, N) \cong \text{Tor}_n^R(M, N)
\]
that are functorial in \( M \) and \( N \).

Proof. If \( \text{fd}_R M < \infty \), then we also have \( \text{pd}_R M < \infty \) by 13 Proposition 6] (since \( \text{RightFPD}(R) \) \( < \infty \)). Let \( P \) be any projective resolution of \( M \). As noted in Remark 4.8 \( \text{P} \) is also a proper left \( \mathcal{GP}_R \)-resolution of \( M \). Hence, Theorem 4.8 ii) and the definitions give:
\[
\text{gTor}_n^R(M, N) = \text{Tor}_n^{m,n}(M, N) = \text{H}_n(P \otimes_R N) = \text{Tor}_n^R(M, N),
\]
as desired. \( \square \)
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