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GORENSTEIN DERIVED FUNCTORS

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Abstract. Over any associative ring \( R \) it is standard to derive \( \text{Hom}_R(-,-) \) using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains \( \text{Ext}^n_R(-,-) \) in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product \(- \otimes_R -\) using Gorenstein flat modules.

1. Introduction

When \( R \) is a two-sided Noetherian ring, Auslander and Bridger \[2\] introduced in 1969 the G-dimension, \( \text{G-dim}_R M \), for every finite (that is, finitely generated) \( R \)-module \( M \). They proved the inequality \( \text{G-dim}_R M \leq \text{pd}_R M \), with equality \( \text{G-dim}_R M = \text{pd}_R M \) when \( \text{pd}_R M < \infty \), along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring \( R \), Enochs and Jenda in \[6\] defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if \( R \) is two-sided Noetherian, and \( G \) is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following \[4, \text{Theorem (4.2.6)}\]. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary \( R \)-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- \( R \) is an associative ring. All modules are—if not specified otherwise—left \( R \)-modules, and the category of all \( R \)-modules is denoted \( \mathcal{M} \). We use \( \mathcal{A} \) for the category of abelian groups (that is, \( \mathbb{Z} \)-modules).
- We use \( \mathcal{GP}, \mathcal{GI} \) and \( \mathcal{GF} \) for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat \( R \)-modules; please see \[6\] and \[8\], or Definition \[2.7\] below.
- Furthermore, for each \( R \)-module \( M \) we write \( \text{Gpd}_R M, \text{Gid}_R M \) and \( \text{Gfd}_R M \) for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of \( M \), respectively.

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Now, given our base ring $R$, the usual right derived functors $\mathrm{Ext}^n_R(-,-)$ of $\mathrm{Hom}_R(-,-)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\mathrm{Ext}^n_{\mathcal{G}}(-,-)$ and $\mathrm{Ext}^n_{\mathcal{G}'}(-,-)$ of $\mathrm{Hom}_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $\mathcal{G}'$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.4] below for the definition of proper resolutions), we define

$$\mathrm{Ext}^n_{\mathcal{G}'}(M, N) := H^n(\mathrm{Hom}_R(G, N)).$$

From [2.4] it will follow that $\mathrm{Ext}^n_{\mathcal{G}'}(-,-)$ is a well-defined contravariant functor, defined on the full subcategory, $\mathrm{LeftRes}_M(\mathcal{G}')$, of $M$, consisting of all $R$-modules that have a proper left $\mathcal{G}'$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\mathrm{Ext}^n_{\mathcal{G}'}(M', -)$, which is defined on the full subcategory, $\mathrm{RightRes}_M(\mathcal{G}')$, of $M$, consisting of all $R$-modules that which have a proper right $\mathcal{G}'$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\mathrm{Ext}^n_{\mathcal{G}'}(M, N) \cong \mathrm{Ext}^n_{\mathcal{G}'}(M, N),$$

which are functorial in each variable $M \in \mathrm{LeftRes}_M(\mathcal{G}')$ and $N \in \mathrm{RightRes}_M(\mathcal{G}')$.

The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9, Theorem 12.1.4] have proved the existence of such functorial isomorphisms $\mathrm{Ext}^n_{\mathcal{G}'}(M, N) \cong \mathrm{Ext}^n_{\mathcal{G}'}(M, N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\mathrm{Gpd}_R M < \infty$, $\mathrm{Gid}_R M < \infty$, and also $\mathrm{Gpd}_R M < \infty$ for all $R$-modules $M$; please see [9, Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12, Proposition 2.18] (which is restated in this paper as Proposition 5.1) implies that the category $\mathrm{LeftRes}_M(\mathcal{G}')$ contains all $R$-modules $M$ with $\mathrm{Gpd}_R M < \infty$; that is, every $R$-module with finite $G$-projective dimension has a proper left $\mathcal{G}'$-resolution. Also, every $R$-module with finite $G$-injective dimension has a proper right $\mathcal{G}'$-resolution. So $\mathrm{RightRes}_M(\mathcal{G}')$ contains all $R$-modules $N$ with $\mathrm{Gid}_R N < \infty$.

Theorem 3.6 in this text proves that the functorial isomorphisms $\mathrm{Ext}^n_{\mathcal{G}'}(M, N) \cong \mathrm{Ext}^n_{\mathcal{G}'}(M, N)$ hold over arbitrary rings $R$, provided that $\mathrm{Gpd}_R M < \infty$ and $\mathrm{Gid}_R N < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems 4.8 and 4.10 give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $\mathcal{G}'$-resolutions and proper left $\mathcal{G}'$-resolutions. This has also been proved by Enochs and Jenda [9, Theorem 12.2.2] in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \rightarrow M$, such that $X^+ := \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ is also a complex, and such that the sequence
\[
\text{Hom}_{\mathcal{C}}(X, X^+) = \cdots \rightarrow \text{Hom}_{\mathcal{C}}(X, X_1) \rightarrow \text{Hom}_{\mathcal{C}}(X, X_0) \rightarrow \text{Hom}_{\mathcal{C}}(X, M) \rightarrow 0
\]
is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_{\mathcal{C}}(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let $f : M \rightarrow M'$ be a morphism in $\mathcal{C}$, and consider the diagram
\[
\begin{array}{cccccccc}
\cdots & \rightarrow & X_2 & \rightarrow & X_1 & \rightarrow & X_0 & \rightarrow & M & \rightarrow & 0 \\
\downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\cdots & \rightarrow & X'_2 & \rightarrow & X'_1 & \rightarrow & X'_0 & \rightarrow & M' & \rightarrow & 0
\end{array}
\]
where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n : X_n \rightarrow X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

*Proof.* The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.
- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precover of $M \in \mathcal{C}$ is a morphism $\varphi : X \rightarrow M$, where $X \in \mathcal{X}$, such that the sequence
\[
\text{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\text{Hom}_{\mathcal{C}}(X', \varphi)} \text{Hom}_{\mathcal{C}}(X', M) \rightarrow 0
\]
is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \rightarrow \text{Ker}(X_i \rightarrow X_{i-1})$, $i > 0$, and $X_0 \rightarrow M$ are $\mathcal{X}$-precovers.
- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [3, Section 4].

2.4 (Derived Functors). Consider an additive functor $T : \mathcal{C} \rightarrow \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor
\[
L_n^\mathcal{X}T : \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \rightarrow \mathcal{E}
\]
of \( T \), with respect to the class \( \mathcal{X} \), by setting \( L_n^X T(M) = H_n(T(X)) \), where \( X \) is any proper left \( \mathcal{X} \)-resolution of \( M \in \text{LeftRes}_C(\mathcal{X}) \). Similarly, the \( n \)th right derived functor

\[
R^n_{\mathcal{X}} T : \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}
\]

of \( T \) with respect to \( \mathcal{X} \) is defined by \( R^n_{\mathcal{X}} T(N) = H_n(T(Y)) \), where \( Y \) is any proper right \( \mathcal{X} \)-resolution of \( N \in \text{RightRes}_C(\mathcal{X}) \). These constructions are well-defined and functorial in the arguments \( M \) and \( N \) by Proposition 2.2.

The situation where \( T \) is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

2.5 (Balanced Functors). Next we consider yet another abelian category \( \mathcal{D} \), together with a full subcategory \( \mathcal{Y} \subseteq \mathcal{D} \) and an additive functor \( F : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \) in two variables. We will assume that \( F \) is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of \( F \) is not important, and the definitions and results below can easily be modified to fit the situation where \( F \) is covariant in both variables, say.

For fixed \( M \in \mathcal{C} \) and \( N \in \mathcal{D} \) we can then consider the two right derived functors as in 2.4:

\[
R^n_{\mathcal{X}} F(-, N) : \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^n_{\mathcal{Y}} F(M, -) : \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.
\]

If furthermore \( M \in \text{LeftRes}_C(\mathcal{X}) \) and \( N \in \text{RightRes}_D(\mathcal{Y}) \), we can ask for a sufficient condition to ensure that

\[
R^n_{\mathcal{X}} F(M, N) \cong R^n_{\mathcal{Y}} F(M, N),
\]

functorial in \( M \) and \( N \). Here we wrote \( R^n_{\mathcal{X}} F(M, N) \) for the functor \( R^n_{\mathcal{X}} F(-, N) \) applied to \( M \). Another, and perhaps better, notation could be

\[
R^n_{\mathcal{X}} F(-, N)[M].
\]

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [9, Definition 2.1]). Instead we shall focus on the following result:

\textbf{Theorem 2.6.} Consider the functor \( F : \mathcal{C} \times \mathcal{D} \to \mathcal{E} \) which is contravariant in the first variable and covariant in the second variable, together with the full subcategories \( \mathcal{X} \subseteq \mathcal{C} \) and \( \mathcal{Y} \subseteq \mathcal{D} \). Assume that we have full subcategories \( \mathcal{X} \) and \( \mathcal{Y} \) of \( \text{LeftRes}_C(\mathcal{X}) \) and \( \text{RightRes}_D(\mathcal{Y}) \), respectively, satisfying:

\begin{enumerate}[(i)]
\item \( \mathcal{X} \subseteq \overline{\mathcal{X}} \) and \( \mathcal{Y} \subseteq \overline{\mathcal{Y}} \).
\item Every \( M \in \overline{\mathcal{X}} \) has an augmented proper left \( \mathcal{X} \)-resolution \( \cdots \rightarrow X_1 \rightarrow X_0 \rightarrow M \rightarrow 0 \), such that \( 0 \rightarrow F(M, Y) \rightarrow F(X_0, Y) \rightarrow F(X_1, Y) \rightarrow \cdots \) is exact for all \( Y \in \mathcal{Y} \).
\item Every \( N \in \overline{\mathcal{Y}} \) has an augmented proper right \( \mathcal{Y} \)-resolution \( 0 \rightarrow N \rightarrow Y^0 \rightarrow Y^1 \rightarrow \cdots \), such that \( 0 \rightarrow F(X, N) \rightarrow F(X, Y^0) \rightarrow F(X, Y^1) \rightarrow \cdots \) is exact for all \( X \in \mathcal{X} \).
\end{enumerate}

Then we have functorial isomorphisms

\[
R^n_{\mathcal{X}} F(M, N) \cong R^n_{\mathcal{Y}} F(M, N),
\]

for all \( M \in \overline{\mathcal{X}} \) and \( N \in \overline{\mathcal{Y}} \).
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8.2.14], or the proofs of [14] Theorems 2.7.2 and 2.7.6. □

In the next paragraphs we apply the results above to special categories $\mathcal{X}$, $\tilde{\mathcal{X}}$, $\mathcal{C}$ and $\mathcal{Y}$, $\tilde{\mathcal{Y}}$, $\mathcal{D}$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules,

$$P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots,$$

such that $\text{Hom}_R(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \rightarrow P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules,

$$F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots,$$

such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \rightarrow F_{-1})$.

3. Gorenstein deriving $\text{Hom}_R(-, -)$

We now return to categories of modules. We use $\tilde{\mathcal{GP}}, \tilde{\mathcal{GI}}$ and $\tilde{\mathcal{GF}}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $\mathcal{GP}$-precovers are always surjective, and $\tilde{\mathcal{GP}}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_R(-, -) : \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$, together with the categories

$$\mathcal{X} = \mathcal{GP}, \quad \tilde{\mathcal{X}} = \tilde{\mathcal{GP}} \quad \text{and} \quad \mathcal{Y} = \mathcal{GI}, \quad \tilde{\mathcal{Y}} = \tilde{\mathcal{GI}}.$$

In this case we define, in the sense of section 2.4

$$\text{Ext}_{\tilde{\mathcal{GP}}}^n(-, N) = \text{R}^n_{\tilde{\mathcal{GP}}} \text{Hom}_R(-, N) \quad \text{and} \quad \text{Ext}_{\tilde{\mathcal{GI}}}^n(M, -) = \text{R}^n_{\tilde{\mathcal{GI}}} \text{Hom}_R(M, -),$$

for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.3 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with $\text{Gpd}_R M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is a $\mathcal{GP}$-precover of $M$ (please see Remark 2.3), and $\text{pd}_R K = \text{Gpd}_R M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $\mathcal{GP}$-resolution (that is, there is an inclusion $\tilde{\mathcal{GP}} \subseteq \text{LeftRes}_M(\mathcal{GP})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with $\text{Gpd}_R M < \infty$. Then

$$\text{Gpd}_R M = \sup\{n \geq 0 \mid \text{Ext}_{\tilde{\mathcal{GP}}}^n(M, L) \neq 0 \text{ for some } R\text{-module } L \text{ with } \text{pd}_R L < \infty\}.$$
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by
\[ \text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \}. \]
It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $GP$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $GP$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences
\[ S = 0 \to K \to G \to M \to 0, \]
where $G \to M$ is a $GP$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $GP$-precovers are always surjective). By Proposition 3.1 there is a special short exact sequence,
\[ S' = 0 \to K' \to G' \to M \to 0, \]
where $\pi: G' \to M$ is a $GP$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 2.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of
\[ \text{Hom}_R(G', H) \to \text{Hom}_R(K', H) \to 0. \]
To see this, let $\alpha: K' \to H$ be any homomorphism. We wish to find $g: G' \to H$ such that $g\pi = \alpha$. Now pick an exact sequence
\[ 0 \to \tilde{H} \to E \xrightarrow{g} H \to 0, \]
where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \to E$ with $g\varepsilon = \alpha$:

\[
\begin{array}{c}
K' \xrightarrow{\alpha} G' \\
\downarrow \quad \downarrow \varepsilon \\
H \xrightarrow{\xi} E
\end{array}
\]

Next, injectivity of $E$ gives $\tilde{\varepsilon}: G' \to E$ with $\tilde{\varepsilon}\pi = \varepsilon$. Now $g\tilde{\varepsilon}: G' \to H$ is the desired map. \qed

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $\mathcal{G}I$-resolution of $N$ (which exists by the dual of Proposition 3.7). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$. \hfill $\Box$

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}I}(M, N),$$

which are functorial in $M$ and $N$. \hfill $\Box$

3.7 (Definition of $\text{GExt}$). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_{\mathcal{G}I}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare $\text{GExt}$ with the classical $\text{Ext}$. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{\mathcal{G}P}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{align*}
& (i) \; \text{pd}_R M < \infty \quad \text{or} \quad (i) \; M \in \text{LeftRes}_M(\mathcal{G}P) \quad \text{and} \quad \text{id}_R N < \infty.
& (i) \; \text{pd}_R M < \infty \quad \text{or} \quad (i) \; M \in \text{LeftRes}_M(\mathcal{G}P) \quad \text{and} \quad \text{id}_R N < \infty.
\end{align*}

(ii) There are natural isomorphisms $\text{Ext}^n_{\mathcal{G}I}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

\begin{align*}
& (i) \; \text{id}_R N < \infty \quad \text{or} \quad (i) \; N \in \text{RightRes}_M(\mathcal{G}I) \quad \text{and} \quad \text{pd}_R M < \infty.
& (ii) \; \text{id}_R N < \infty \quad \text{or} \quad (ii) \; N \in \text{RightRes}_M(\mathcal{G}I) \quad \text{and} \quad \text{pd}_R M < \infty.
\end{align*}

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then

$$\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$$

is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $\mathcal{G}P$-resolution of $M$, and thus

$$\text{Ext}^n_{\mathcal{G}P}(M, N) = \text{H}^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(\mathcal{G}P)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$.

This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{i+m+1}_R(C, N) = 0$, as claimed.

Therefore [II Chapter III, Proposition 1.2A] implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of $\text{GExt}^n_R(-, -)$. \hfill $\Box$
4. Gorenstein deriving $- \otimes_R -$ 

In dealing with the tensor product we need, of course, both left and right $R$-modules. Thus the following addition to Notation 1.1 is needed:

If $\mathcal{C}$ is any of the categories in Notation 1.1 ($\mathcal{M}$, $\mathcal{GP}$, etc.), we write $\mathcal{R}\mathcal{C}$, respectively, $\mathcal{C}\mathcal{R}$, for the category of left, respectively, right, $R$-modules with the property describing the modules in $\mathcal{C}$.

Now we consider the functor $\mathcal{R}$:

$\mathcal{M} \times \mathcal{R}\mathcal{M} \to \mathcal{A}$. For fixed $M \in \mathcal{M}$ and $N \in \mathcal{R}\mathcal{M}$ we define, in the sense of section 2.4:

$\text{Tor}_{n}^{\mathcal{GP}}(-, N) := L_{n}^{\mathcal{GP}}(- \otimes_R N)$ and $\text{Tor}_{n}^{\mathcal{NP}}(M, -) := L_{n}^{\mathcal{GP}}(M \otimes_R -)$,

together with $\text{Tor}_{n}^{\mathcal{GF}}(-, N) := L_{n}^{\mathcal{GP}}(- \otimes_R N)$ and $\text{Tor}_{n}^{\mathcal{NF}}(M, -) := L_{n}^{\mathcal{GP}}(M \otimes_R -)$.

The first two $\text{Tor}$s use proper left Gorenstein projective resolutions, and the last two $\text{Tor}$s use proper left Gorenstein at resolutions. In order to compare these different $\text{Tor}$s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of $(X, \tilde{X}) = (\mathcal{GP}_R, \tilde{\mathcal{GP}}_R)$ or $(\mathcal{GP}_R, \tilde{\mathcal{GP}}_R)$, and $\text{Tor}_{n}^{\mathcal{GP}}(-, N)$ and $\text{Tor}_{n}^{\mathcal{GP}}(M, -)$.

The classical notion:

**Definition 4.1.** The left finitistic projective dimension $\text{LeftFPD}(R)$ of $R$ is defined as

$\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}$.

The right finitistic projective dimension $\text{RightFPD}(R)$ of $R$ is defined similarly.

**Remark 4.2.** When $R$ is commutative and Noetherian, the dimensions $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ coincide and are equal to the Krull dimension of $R$, by [10 Theorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If $R$ is right coherent with finite $\text{LeftFPD}(R)$, then every Gorenstein projective left $R$-module is also Gorenstein flat. That is, there is an inclusion $\mathcal{GP} \subseteq \mathcal{GF}$.

**Theorem 4.4.** For any left $R$-module $M$, we consider the following three conditions:

(i) The left $R$-module $M$ is $G$-flat.

(ii) The Pontryagin dual $\text{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$ (which is a right $R$-module) is $G$-injective.

(iii) $M$ has an augmented proper right resolution $0 \rightarrow M \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots$ consisting of flat left $R$-modules, and $\text{Tor}_i^R(I, M) = 0$ for all injective right $R$-modules $I$, and all $i > 0$.

The implication (i) $\Rightarrow$ (ii) always holds. If $R$ is right coherent, then also (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that \( R \) is right coherent. If \( M \) is a left \( R \)-module with \( \text{Gfd}_R M < \infty \), then there exists a short exact sequence \( 0 \to K \to G \to M \to 0 \), where \( G \to M \) is an \( R\overline{GF} \)-precover of \( M \), and \( \text{fd}_R K = \text{Gfd}_R M - 1 \) (in the case where \( M \) is Gorenstein flat, this should be interpreted as \( K = 0 \)).

In particular, every left \( R \)-module with finite Gorenstein flat dimension has a proper left \( R\overline{GF} \)-resolution (that is, there is an inclusion \( R\overline{GF} \subseteq \text{LeftRes}_R(M(R\overline{GF})) \)).

Our first result is:

Lemma 4.6. Let \( M \) be a left \( R \)-module with \( \text{Gpd}_R M < \infty \), and let \( G^+ = \cdots \to G_1 \to G_0 \to M \to 0 \) be an augmented proper left \( R\overline{GP} \)-resolution of \( M \) (which exists by Proposition 4.4). Then the following conclusions hold:

(i) \( T \otimes_R G^+ \) is exact for all Gorenstein flat right \( R \)-modules \( T \).
(ii) If \( R \) is left coherent with finite \( \text{RightFPD}(R) \), then \( T \otimes_R G^+ \) is exact for all Gorenstein projective right \( R \)-modules \( T \).

Proof. (i) By Theorem 4.4 above, the Pontryagin dual \( H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z}) \) is a Gorenstein injective left \( R \)-module. Hence \( \text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}) \) is exact by Proposition 3.3. Since \( \mathbb{Q}/\mathbb{Z} \) is a faithfully injective \( \mathbb{Z} \)-module, \( T \otimes_R G^+ \) is exact too.

(ii) With the given assumptions on \( R \), the dual of Proposition 4.3 implies that every Gorenstein projective right \( R \)-module also is Gorenstein flat. \( \square \)

Lemma 4.7. Assume that \( R \) is right coherent with finite \( \text{LeftFPD}(R) \). Let \( M \) be a left \( R \)-module with \( \text{Gfd}_R M < \infty \), and let \( G^+ = \cdots \to G_1 \to G_0 \to M \to 0 \) be an augmented proper left \( R\overline{GF} \)-resolution of \( M \) (which exists by Proposition 4.4 since \( R \) is right coherent). Then the following conclusions hold:

(i) \( \text{Hom}_R(G^+, H) \) is exact for all Gorenstein injective left \( R \)-modules \( H \).
(ii) \( T \otimes_R G^+ \) is exact for all Gorenstein flat right \( R \)-modules \( T \).
(iii) If \( R \) is also left coherent with finite \( \text{RightFPD}(R) \), then \( T \otimes_R G^+ \) is exact for all Gorenstein projective right \( R \)-modules \( T \).

Proof. (i) Since \( \text{Gfd}_R M < \infty \) and \( R \) is right coherent, Proposition 4.4 gives a special short exact sequence \( 0 \to K' \to G' \to M \to 0 \), where \( G' \to M \) is an \( R\overline{GF} \)-precover of \( M \), and \( \text{fd}_R K' < \infty \). Since \( R \) has \( \text{LeftFPD}(R) < \infty \), Proposition 6] implies that also \( \text{pd}_R K' < \infty \). Now the proof of Lemma 4.4 applies.

(ii) If \( T \) is a Gorenstein flat right \( R \)-module, then the left \( R \)-module \( H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z}) \) is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

\[ \text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}) .\]

Since \( \mathbb{Q}/\mathbb{Z} \) is a faithfully injective \( \mathbb{Z} \)-module, we also have exactness of \( T \otimes_R G^+ \), as desired.

(iii) Under the extra assumptions on \( R \), the dual of Proposition 4.3 implies that every Gorenstein projective right \( R \)-module is also Gorenstein flat. Thus (iii) follows from (ii). \( \square \)

Theorem 4.8. Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. For every right \( R \)-module \( M \), and every left \( R \)-module \( N \), the following conclusions hold:
(i) If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
\[ \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N). \]

(ii) If $\text{Gpd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then
\[ \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N). \]

(iii) If $\text{Gfd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
\[ \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N). \]

(iv) If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then
\[ \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N). \]

All the isomorphisms are functorial in $M$ and $N$.

**Proof.** Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 2.6. \qed

4.9 (Definition of $\mathcal{G}\text{Tor}$ and $\mathcal{G}\text{Tor}$). Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module, and let $N$ be a left $R$-module. If $\text{Gfd}_RM < \infty$ and $\text{Gfd}_RN < \infty$, then we write
\[ \mathcal{G}\text{Tor}^R_n(M, N) := \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(i). If $\text{Gpd}_RM < \infty$ and $\text{Gpd}_RN < \infty$, then we write
\[ \mathcal{G}\text{Tor}^R_n(M, N) := \text{Tor}^{\mathcal{G}_n}_n(M, N) \cong \text{Tor}^{\mathcal{G}_n}_n(M, N) \]
for the isomorphic abelian groups in Theorem 4.8(iv).

We can now reformulate some of the content of Theorem 4.8.

**Theorem 4.10.** Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$ with finite $\text{Gpd}_RM$, and for every left $R$-module $N$ with $\text{Gpd}_RN < \infty$, we have isomorphisms:
\[ \mathcal{G}\text{Tor}^R_n(M, N) \cong \mathcal{G}\text{Tor}^R_n(M, N) \]
that are functorial in $M$ and $N$.

Finally we compare $\mathcal{G}\text{Tor}$ (and hence $\mathcal{G}\text{Tor}$) with the usual Tor.

**Theorem 4.11.** Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. Furthermore, let $M$ be a right $R$-module with $\text{Gfd}_RM < \infty$, and let $N$ be a left $R$-module with $\text{Gfd}_RN < \infty$. If either $\text{fd}_RM < \infty$ or $\text{fd}_RN < \infty$, then there are isomorphisms
\[ \mathcal{G}\text{Tor}^R_n(M, N) \cong \text{Tor}^R_n(M, N) \]
that are functorial in $M$ and $N$.

**Proof.** If $\text{fd}_RM < \infty$, then we also have $\text{pd}_RM < \infty$ by [13] Proposition 6] (since $\text{RightFPD}(R) < \infty$). Let $P$ be any projective resolution of $M$. As noted in Remark 3.3, $P$ is also a proper left $\mathcal{G}\text{P}_R$-resolution of $M$. Hence, Theorem 4.8 ii) and the definitions give:
\[ \mathcal{G}\text{Tor}^R_n(M, N) = \text{Tor}^{\mathcal{G}_n}_n(M, N) = \text{Hom}_R(P \otimes_R N) = \text{Tor}^R_n(M, N), \]
as desired. \qed
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