Gorenstein derived functions
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Abstract. Over any associative ring \( R \) it is standard to derive \( \text{Hom}_R(\cdot, \cdot) \) using projective resolutions in the first variable, or injective resolutions in the second variable, and doing this, one obtains \( \text{Ext}^n_R(\cdot, \cdot) \) in both cases. We examine the situation where projective and injective modules are replaced by Gorenstein projective and Gorenstein injective ones, respectively. Furthermore, we derive the tensor product \( - \otimes_R - \) using Gorenstein flat modules.

1. Introduction

When \( R \) is a two-sided Noetherian ring, Auslander and Bridger \cite{2} introduced in 1969 the G-dimension, \( \text{G-dim}_R M \), for every finite \( (\text{that is, finitely generated}) \) \( R \)-module \( M \). They proved the inequality \( \text{G-dim}_R M \leq \text{pd}_R M \), with equality \( \text{G-dim}_R M = \text{pd}_R M \) when \( \text{pd}_R M < \infty \), along with a generalized Auslander-Buchsbaum formula (sometimes known as the Auslander-Bridger formula) for the G-dimension.

The (finite) modules with G-dimension zero are called Gorenstein projectives. Over a general ring \( R \), Enochs and Jenda in \cite{6} defined Gorenstein projective modules. Avramov, Buchweitz, Martsinkovsky and Reiten proved that if \( R \) is two-sided Noetherian, and \( G \) is a finite Gorenstein projective module, then the new definition agrees with that of Auslander and Bridger; see the remark following \cite[Theorem 4.2.6]{4}. Using Gorenstein projective modules, one can introduce the Gorenstein projective dimension for arbitrary \( R \)-modules. At this point we need to introduce:

1.1 (Notation). Throughout this paper, we use the following notation:

- \( R \) is an associative ring. All modules are—if not specified otherwise—left \( R \)-modules, and the category of all \( R \)-modules is denoted \( \mathcal{M} \). We use \( \mathcal{A} \) for the category of abelian groups (that is, \( \mathbb{Z} \)-modules).
- We use \( \mathcal{GP}, \mathcal{GI} \) and \( \mathcal{GF} \) for the categories of Gorenstein projective, Gorenstein injective and Gorenstein flat \( R \)-modules; please see \cite{6} and \cite{8}, or Definition 2.7 below.
- Furthermore, for each \( R \)-module \( M \) we write \( \text{Gpd}_R M, \text{Gid}_R M \) and \( \text{Gfd}_R M \) for the Gorenstein projective, Gorenstein injective, and Gorenstein flat dimension of \( M \), respectively.

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Now, given our base ring $R$, the usual right derived functors $\text{Ext}^n_R(-,-)$ of $\text{Hom}_R(-,-)$ are important in homological studies of $R$. The material presented here deals with the Gorenstein right derived functors $\text{Ext}^n_{GP}(-,-)$ and $\text{Ext}^n_{GI}(-,-)$ of $\text{Hom}_R(-,-)$.

More precisely, let $N$ be a fixed $R$-module. For an $R$-module $M$ that has a proper left $GP$-resolution $G = \cdots \to G_1 \to G_0 \to 0$ (please see [2.1] below for the definition of proper resolutions), we define

$$\text{Ext}^n_{GP}(M,N) := \text{H}^n(\text{Hom}_R(G,N)).$$

From [2.1] it will follow that $\text{Ext}^n_{GP}(-,-)$ is a well-defined contravariant functor, defined on the full subcategory, $\text{LeftRes}_M(GP)$, of $\mathcal{M}$, consisting of all $R$-modules that have a proper left $GP$-resolution.

For a fixed $R$-module $M'$ there is a similar definition of the functor $\text{Ext}^n_{GP}(M',-)$, which is defined on the full subcategory, $\text{RightRes}_M(GI)$, of $\mathcal{M}$, consisting of all $R$-modules that which have a proper right $GI$-resolution. Now, the best one could hope for is the existence of isomorphisms,

$$\text{Ext}^n_{GP}(M,N) \cong \text{Ext}^n_{GI}(M,N),$$

which are functorial in each variable $M \in \text{LeftRes}_M(GP)$ and $N \in \text{RightRes}_M(GI)$. The aim of this paper is to show a slightly weaker result.

When $R$ is $n$-Gorenstein (meaning that $R$ is both left and right Noetherian, with self-injective dimension $\leq n$ from both sides), Enochs and Jenda [9] Theorem 12.1.4 have proved the existence of such functorial isomorphisms $\text{Ext}^n_{GP}(M,N) \cong \text{Ext}^n_{GI}(M,N)$ for all $R$-modules $M$ and $N$.

It is important to note that for an $n$-Gorenstein ring $R$, we have $\text{Gpd}_RM < \infty$, $\text{Gid}_RM < \infty$, and also $\text{Gpd}_RM \leq n$ for all $R$-modules $M$; please see [9] Theorems 11.2.1, 11.5.1, 11.7.6]. For any ring $R$, [12] Proposition 2.18 (which is restated in this paper as Proposition 5.1) implies that the category $\text{LeftRes}_M(GP)$ contains all $R$-modules $M$ with $\text{Gpd}_RM < \infty$; that is, every $R$-module with finite G-projective dimension has a proper left $GP$-resolution. Also, every $R$-module with finite G-injective dimension has a proper right $GI$-resolution. So $\text{RightRes}_M(GI)$ contains all $R$-modules $N$ with $\text{Gid}_RN < \infty$.

Theorem [5.6] in this text proves that the functorial isomorphisms $\text{Ext}^n_{GP}(M,N) \cong \text{Ext}^n_{GI}(M,N)$ hold over arbitrary rings $R$, provided that $\text{Gpd}_RM < \infty$ and $\text{Gid}_RN < \infty$. By the remarks above, this result generalizes that of Enochs and Jenda.

Furthermore, Theorems [4.8] and [4.10] give similar results about the Gorenstein left derived of the tensor product $- \otimes_R -$, using proper left $GP$-resolutions and proper left $GI$-resolutions. This has also been proved by Enochs and Jenda [9] Theorem 12.2.2) in the case when $R$ is $n$-Gorenstein.

2. Preliminaries

Let $T : \mathcal{C} \to \mathcal{E}$ be any additive functor between abelian categories. One usually derives $T$ using resolutions consisting of projective or injective objects (if the category $\mathcal{C}$ has enough projectives or injectives). This section is a very brief note on how to derive functors $T$ with resolutions consisting of objects in some subcategory $\mathcal{X} \subseteq \mathcal{C}$. The general discussion presented here will enable us to give very short proofs of the main theorems in the next section.
2.1 (Proper Resolutions). Let $\mathcal{X} \subseteq \mathcal{C}$ be a full subcategory. A proper left $\mathcal{X}$-resolution of $M \in \mathcal{C}$ is a complex $X = \cdots \to X_1 \to X_0 \to 0$ where $X_i \in \mathcal{X}$, together with a morphism $X_0 \to M$, such that $X^+ := \cdots \to X_1 \to X_0 \to M \to 0$ is also a complex, and such that the sequence

$$\text{Hom}_{\mathcal{C}}(X, X^+) = \cdots \to \text{Hom}_{\mathcal{C}}(X, X_1) \to \text{Hom}_{\mathcal{C}}(X, X_0) \to \text{Hom}_{\mathcal{C}}(X, M) \to 0$$

is exact for every $X \in \mathcal{X}$. We sometimes refer to $X^+ = \cdots \to X_1 \to X_0 \to M \to 0$ as an augmented proper left $\mathcal{X}$-resolution. We do not require that $X^+$ itself is exact. Furthermore, we use $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$ to denote the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper left $\mathcal{X}$-resolution. Note that $\mathcal{X}$ is a subcategory of $\text{LeftRes}_{\mathcal{C}}(\mathcal{X})$.

Proper right $\mathcal{X}$-resolutions are defined dually, and the full subcategory of $\mathcal{C}$ consisting of those objects that have a proper right $\mathcal{X}$-resolution is $\text{RightRes}_{\mathcal{C}}(\mathcal{X})$.

The importance of working with proper resolutions comes from the following:

**Proposition 2.2.** Let $f: M \to M'$ be a morphism in $\mathcal{C}$, and consider the diagram

$$
\begin{array}{ccccccccc}
\cdots & \to & X_2 & \to & X_1 & \to & X_0 & \to & M & \to & 0 \\
& & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & \\
& \cdots & \to & X'_2 & \to & X'_1 & \to & X'_0 & \to & M' & \to & 0
\end{array}
$$

where the upper row is a complex with $X_n \in \mathcal{X}$ for all $n \geq 0$, and the lower row is an augmented proper left $\mathcal{X}$-resolution of $M'$. Then the following conclusions hold:

(i) There exist morphisms $f_n: X_n \to X'_n$ for all $n \geq 0$, making the diagram above commutative. The chain map $\{f_n\}_{n \geq 0}$ is called a lift of $f$.

(ii) If $\{f'_n\}_{n \geq 0}$ is another lift of $f$, then the chain maps $\{f_n\}_{n \geq 0}$ and $\{f'_n\}_{n \geq 0}$ are homotopic.

**Proof.** The proof is an exercise; please see [9, Exercise 8.1.2].

**Remark 2.3.** A few comments are in order:

- In our applications, the class $\mathcal{X}$ contains all projectives. Consequently, all the augmented proper left $\mathcal{X}$-resolutions occurring in this paper will be exact. Also, all augmented proper right $\mathcal{Y}$-resolutions will be exact, when $\mathcal{Y}$ is a class of $R$-modules containing all injectives.

- Recall (see [15, Definition 1.2.2]) that an $\mathcal{X}$-precovers of $M \in \mathcal{C}$ is a morphism $\varphi: X \to M$, where $X \in \mathcal{X}$, such that the sequence

$$\text{Hom}_{\mathcal{C}}(X', X) \xrightarrow{\text{Hom}_{\mathcal{C}}(X', \varphi)} \text{Hom}_{\mathcal{C}}(X', M) \to 0$$

is exact for every $X' \in \mathcal{X}$. Hence, in an augmented proper left $\mathcal{X}$-resolution $X^+$ of $M$, the morphisms $X_{i+1} \to \ker(X_i \to X_{i-1})$, $i > 0$, and $X_0 \to M$ are $\mathcal{X}$-precovers.

- What we have called proper $\mathcal{X}$-resolutions, Enochs and Jenda [9, Definition 8.1.2] simply call $\mathcal{X}$-resolutions. We have adopted the terminology proper from [13, Section 4].

2.4 (Derived Functors). Consider an additive functor $T: \mathcal{C} \to \mathcal{E}$ between abelian categories. Let us assume that $T$ is covariant, say. Then (as usual) we can define the $n$th left derived functor

$$L_n^X T: \text{LeftRes}_{\mathcal{C}}(\mathcal{X}) \to \mathcal{E}$$
of $T$, with respect to the class $\mathcal{X}$, by setting $L_n^X T(M) = H_n(T(X))$, where $X$ is any proper left $\mathcal{X}$-resolution of $M \in \text{LeftRes}_C(\mathcal{X})$. Similarly, the $n^{th}$ right derived functor

$$R^+_n T : \text{RightRes}_C(\mathcal{X}) \to \mathcal{E}$$

of $T$ with respect to $\mathcal{X}$ is defined by $R^+_n T(N) = H_n(T(Y))$, where $Y$ is any proper right $\mathcal{X}$-resolution of $N \in \text{RightRes}_C(\mathcal{X})$. These constructions are well-defined and functorial in the arguments $M$ and $N$ by Proposition 2.2.2.

The situation where $T$ is contravariant is handled similarly. We refer to [9, Section 8.2] for a more detailed discussion on this matter.

### 2.5 (Balanced Functors).

Next we consider yet another abelian category $\mathcal{D}$, together with a full subcategory $\mathcal{Y} \subseteq \mathcal{D}$ and an additive functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ in two variables. We will assume that $F$ is contravariant in the first variable, and covariant in the second variable.

Actually, the variance of the variables of $F$ is not important, and the definitions and results below can easily be modified to fit the situation where $F$ is covariant in both variables, say.

For fixed $M \in \mathcal{C}$ and $N \in \mathcal{D}$ we can then consider the two right derived functors as in 2.4:

$$R^+_n F(-, N) : \text{LeftRes}_C(\mathcal{X}) \to \mathcal{E} \quad \text{and} \quad R^+_n F(M, -) : \text{RightRes}_D(\mathcal{Y}) \to \mathcal{E}.$$ 

If furthermore $M \in \text{LeftRes}_C(\mathcal{X})$ and $N \in \text{RightRes}_D(\mathcal{Y})$, we can ask for a sufficient condition to ensure that

$$R^+_n F(M, N) \cong R^+_n F(M, N),$$

functorial in $M$ and $N$. Here we wrote $R^+_n F(M, N)$ for the functor $R^+_n F(-, N)$ applied to $M$. Another, and perhaps better, notation could be

$$R^+_n F(-, N)[M].$$

Enochs and Jenda have in [5] developed a machinery for answering such questions. They operate with the term left/right balanced functor (hence the headline), which we will not define here (but the reader might consult [5, Definition 2.1]). Instead we shall focus on the following result:

**Theorem 2.6.** Consider the functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which is contravariant in the first variable and covariant in the second variable, together with the full subcategories $\mathcal{X} \subseteq \mathcal{C}$ and $\mathcal{Y} \subseteq \mathcal{D}$. Assume that we have full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\text{LeftRes}_C(\mathcal{X})$ and $\text{RightRes}_D(\mathcal{Y})$, respectively, satisfying:

(i) $\mathcal{X} \subseteq \mathcal{X}$ and $\mathcal{Y} \subseteq \mathcal{Y}$.

(ii) Every $M \in \mathcal{X}$ has an augmented proper left $\mathcal{X}$-resolution $\cdots \to X_1 \to X_0 \to M \to 0$, such that $0 \to F(M, Y) \to F(X_0, Y) \to F(X_1, Y) \to \cdots$ is exact for all $Y \in \mathcal{Y}$.

(iii) Every $N \in \mathcal{Y}$ has an augmented proper right $\mathcal{Y}$-resolution $0 \to N \to Y^0 \to Y^1 \to \cdots$, such that $0 \to F(X, N) \to F(X, Y^0) \to F(X, Y^1) \to \cdots$ is exact for all $X \in \mathcal{X}$.

Then we have functorial isomorphisms

$$R^+_n F(M, N) \cong R^+_n F(M, N),$$

for all $M \in \mathcal{X}$ and $N \in \mathcal{Y}$. 
Proof. Please see [5, Proposition 2.3]. That the isomorphisms are functorial follows from the construction. The functoriality becomes more clear if one consults the proof of [9, Proposition 8:2], or the proofs of [14, Theorems 2.7.2 and 2.7.6]. □

In the next paragraphs we apply the results above to special categories $X$, $\bar{X}$, $C$ and $\bar{C}$, $D$, including the categories mentioned in 1.1. For completeness we include a definition of Gorenstein projective, Gorenstein injective and Gorenstein flat modules:

**Definition 2.7.** A complete projective resolution is an exact sequence of projective modules, 
\[ P = \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow P_{-1} \rightarrow \cdots, \]
such that $\text{Hom}_{R}(P, Q)$ is exact for every projective $R$-module $Q$. An $R$-module $M$ is called Gorenstein projective ($G$-projective for short), if there exists a complete projective resolution $P$ with $M \cong \text{Im}(P_0 \rightarrow P_{-1})$. Gorenstein injective ($G$-injective for short) modules are defined dually.

A complete flat resolution is an exact sequence of flat (left) $R$-modules, 
\[ F = \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow F_{-1} \rightarrow \cdots, \]
such that $I \otimes_R F$ is exact for every injective right $R$-module $I$. An $R$-module $M$ is called Gorenstein flat ($G$-flat for short), if there exists a complete flat resolution $F$ with $M \cong \text{Im}(F_0 \rightarrow F_{-1})$.

3. Gorenstein deriving $\text{Hom}_{R}(-,-)$

We now return to categories of modules. We use $\tilde{GP}$, $\tilde{GI}$ and $\tilde{GF}$ to denote the class of $R$-modules with finite Gorenstein projective dimension, finite Gorenstein injective dimension, and finite Gorenstein flat dimension, respectively.

Recall that every projective module is Gorenstein projective. Consequently, $GP$-precovers are always surjective, and $\tilde{GP}$ contains all modules with finite projective dimension.

We now consider the functor $\text{Hom}_{R}(-,-): \mathcal{M} \times \mathcal{M} \rightarrow \mathcal{A}$, together with the categories 
\[ \mathcal{X} = \tilde{GP}, \quad \bar{X} = \tilde{GP}, \quad \mathcal{Y} = \tilde{GI}, \quad \bar{Y} = \tilde{GI}. \]
In this case we define, in the sense of section 2.1 
\[ \text{Ext}_{\tilde{GP}}^{n}(-, N) = R_{\tilde{GP}}^{n} \text{Hom}_{R}(-, N) \quad \text{and} \quad \text{Ext}_{\tilde{GI}}^{n}(M, -) = R_{\tilde{GI}}^{n} \text{Hom}_{R}(M, -), \]
for fixed $R$-modules $M$ and $N$. We wish, of course, to apply Theorem 2.6 to this situation. Note that by [12, Proposition 2.18], we have:

**Proposition 3.1.** If $M$ is an $R$-module with Gpd$_{R}M < \infty$, then there exists a short exact sequence $0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0$, where $G \rightarrow M$ is a $\tilde{GP}$-precover of $M$ (please see Remark 2.3), and pd$_{R}K = \text{Gpd}_{R}M - 1$ (in the case where $M$ is Gorenstein projective, this should be interpreted as $K = 0$).

Consequently, every $R$-module with finite Gorenstein projective dimension has a proper left $GP$-resolution (that is, there is an inclusion $\tilde{GP} \subseteq \text{LeftRes}_{\mathcal{M}}(\tilde{GP})$).

Furthermore, we will need the following from [12, Theorem 2.13]:

**Theorem 3.2.** Let $M$ be any $R$-module with Gpd$_{R}M < \infty$. Then 
\[ \text{Gpd}_{R}M = \sup\{n \geq 0 \mid \text{Ext}_{R}^{n}(M, L) \neq 0 \text{ for some } R\text{-module } L\text{ with pd}_{R}L < \infty\}. \]
Remark 3.3. It may be useful to compare Theorem 3.2 to the classical projective dimension, which for an $R$-module $M$ is given by

$$\text{pd}_R M = \{ n \geq 0 \mid \text{Ext}^n_R(M, L) \neq 0 \text{ for some } R\text{-module } L \}.$$ 

It also follows that if $\text{pd}_R M < \infty$, then every projective resolution of $M$ is actually a proper left $\mathcal{GP}$-resolution of $M$.

Lemma 3.4. Assume that $M$ is an $R$-module with finite Gorenstein projective dimension, and let $G^+ = \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$ be an augmented proper left $\mathcal{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective modules $H$.

Proof. We split the proper resolution $G^+$ into short exact sequences. Hence it suffices to show exactness of $\text{Hom}_R(S, H)$ for all Gorenstein injective modules $H$ and all short exact sequences

$$S = 0 \rightarrow K \rightarrow G \rightarrow M \rightarrow 0,$$

where $G \rightarrow M$ is a $\mathcal{GP}$-precover of some module $M$ with $\text{Gpd}_R M < \infty$ (recall that $\mathcal{GP}$-precovers are always surjective). By Proposition 3.1, there is a special short exact sequence,

$$S' = 0 \rightarrow K' \rightarrow G' \rightarrow M \rightarrow 0,$$

where $\pi: G' \rightarrow M$ is a $\mathcal{GP}$-precover and $\text{pd}_R K' < \infty$.

It is easy to see (as in Proposition 3.2) that the complexes $S$ and $S'$ are homotopy equivalent, and thus so are the complexes $\text{Hom}_R(S, H)$ and $\text{Hom}_R(S', H)$ for every (Gorenstein injective) module $H$. Hence it suffices to show the exactness of $\text{Hom}_R(S', H)$ whenever $H$ is Gorenstein injective.

Now let $H$ be any Gorenstein injective module. We need to prove the exactness of

$$\text{Hom}_R(G', H) \xrightarrow{\text{Hom}_R(\iota, H)} \text{Hom}_R(K', H) \longrightarrow 0.$$

To see this, let $\alpha: K' \rightarrow H$ be any homomorphism. We wish to find $g: G' \rightarrow H$ such that $g\iota = \alpha$. Now pick an exact sequence

$$0 \rightarrow \tilde{H} \rightarrow E \xrightarrow{g} H \rightarrow 0,$$

where $E$ is injective, and $\tilde{H}$ is Gorenstein injective (the sequence in question is just a part of the complete injective resolution that defines $H$). Since $\tilde{H}$ is Gorenstein injective and $\text{pd}_R K' < \infty$, we get $\text{Ext}^1_R(K', \tilde{H}) = 0$ by [7, Lemma 1.3], and thus a lifting $\varepsilon: K' \rightarrow E$ with $g\varepsilon = \alpha$:

Next, injectivity of $E$ gives $\bar{\varepsilon}: G' \rightarrow E$ with $\bar{\varepsilon}\iota = \varepsilon$. Now $g = g\bar{\varepsilon}: G' \rightarrow H$ is the desired map. \qed

With a similar proof we get:
Lemma 3.5. Assume that $N$ is an $R$-module with finite Gorenstein injective dimension, and let $H^+ = 0 \to N \to H^0 \to H^1 \to \cdots$ be an augmented proper right $G\mathcal{T}$-resolution of $N$ (which exists by the dual of Proposition 3.7). Then $\text{Hom}_R(G, H^+)$ is exact for all Gorenstein projective modules $G$. \hfill \square

Comparing Lemmas 3.4 and 3.5 with Theorem 2.6, we obtain:

Theorem 3.6. For all $R$-modules $M$ and $N$ with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$, we have isomorphisms

$$\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G\mathcal{T}}(M, N),$$

which are functorial in $M$ and $N$. \hfill \square

3.7 (Definition of GExt). Let $M$ and $N$ be $R$-modules with $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. Then we write

$$\text{GExt}^n_R(M, N) := \text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_{G\mathcal{T}}(M, N)$$

for the isomorphic abelian groups in Theorem 3.6 above.

Naturally we want to compare GExt with the classical Ext. This is done in:

Theorem 3.8. Let $M$ and $N$ be any $R$-modules. Then the following conclusions hold:

(i) There are natural isomorphisms $\text{Ext}^n_{GP}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(\dag) $\text{pd}_R M < \infty$ or (\ddag) $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N < \infty$.

(ii) There are natural isomorphisms $\text{Ext}^n_{G\mathcal{T}}(M, N) \cong \text{Ext}^n_R(M, N)$ under each of the conditions

(\dag) $\text{id}_R N < \infty$ or (\ddag) $N \in \text{RightRes}_M(G\mathcal{T})$ and $\text{pd}_R M < \infty$.

(iii) Assume that $\text{Gpd}_R M < \infty$ and $\text{Gid}_R N < \infty$. If either $\text{pd}_R M < \infty$ or $\text{id}_R N < \infty$, then $\text{GExt}^n_R(M, N) \cong \text{Ext}^n_R(M, N)$ is functorial in $M$ and $N$.

Proof. (i) Assume that $\text{pd}_R M < \infty$, and pick any projective resolution $P$ of $M$. By Remark 3.3, $P$ is also a proper left $G\mathcal{T}$-resolution of $M$, and thus

$$\text{Ext}^n_{GP}(M, N) = \text{H}^n(\text{Hom}_R(P, N)) = \text{Ext}^n_R(M, N).$$

In the case where $M \in \text{LeftRes}_M(GP)$ and $\text{id}_R N = m < \infty$, we see that Gorenstein projective modules are acyclic for the functor $\text{Hom}_R(-, N)$, that is, $\text{Ext}^i_R(G, N) = 0$ (the usual Ext) for every Gorenstein projective module $G$, and every integer $i > 0$. This is because, if $G$ is a Gorenstein projective module, and $i > 0$ is an integer, then there exists an exact sequence $0 \to G \to Q^0 \to \cdots \to Q^{m-1} \to C \to 0$, where $Q^0, \ldots, Q^{m-1}$ are projective modules. Breaking this exact sequence into short exact ones, and applying $\text{Hom}_R(-, N)$, we get $\text{Ext}^i_R(G, N) \cong \text{Ext}^{m+i}_R(C, N) = 0$, as claimed.

Therefore [11] Chapter III, Proposition 1.2A implies that $\text{Ext}^n_R(-, N)$ can be computed using (proper) left Gorenstein projective resolutions of the argument in the first variable, as desired.

The proof of (ii) is similar. The claim (iii) is a direct consequence of (i) and (ii), together with the Definition 3.7 of GExt$^n_R(-, -)$. \hfill \square
4. Gorenstein deriving \(- \otimes_R -\)

In dealing with the tensor product we need, of course, both left and right \(R\)-modules. Thus the following addition to Notation 1.1 is needed:

If \(C\) is any of the categories in Notation 1.1 (\(\mathcal{M}, \mathcal{GP}, \text{ etc.}\)), we write \(\mathcal{R}C\), respectively, \(\mathcal{C}R\), for the category of left, respectively, right, \(R\)-modules with the property describing the modules in \(C\).

Now we consider the functor \(\mathcal{R} : \mathcal{M} \otimes \mathcal{R} \to \mathcal{A}\). For fixed \(M \in \mathcal{M}\) and \(N \in \mathcal{R}M\) we define, in the sense of section 2.4:

\[
\text{Tor}^{\mathcal{GP}}_n (-, N) := L_n^{\mathcal{GP}} (- \otimes_R N) \quad \text{and} \quad \text{Tor}^{\mathcal{GP}}_n (M, -) := L_n^{\mathcal{GP}} (M \otimes_R -),
\]

together with

\[
\text{Tor}^{\mathcal{GP}}_n (-, N) := L_n^{\mathcal{GP}} (- \otimes_R N) \quad \text{and} \quad \text{Tor}^{\mathcal{GP}}_n (M, -) := L_n^{\mathcal{GP}} (M \otimes_R -).
\]

The first two \(\text{Tor}\)s use proper left Gorestein projective resolutions, and the last two \(\text{Tor}\)s use proper left Gorenstein flat resolutions. In order to compare these different \(\text{Tor}\)s, we wish, of course, to apply (a version of) Theorem 2.6 to different combinations of \((X, e_X) = (\mathcal{GP}R, \widehat{\mathcal{GP}})\) or \((\mathcal{GP}R, \mathcal{GP})\); and

\[(Y, e_Y) = (\mathcal{GP}R, \mathcal{GP}) \quad \text{or} \quad (\mathcal{GP}R, \mathcal{GP}),\]

namely, the covariant-covariant version of Theorem A, instead of the stated contravariant-covariant version. We will need the classical notion:

**Definition 4.1.** The left finitistic projective dimension \(\text{LeftFPD}(R)\) of \(R\) is defined as

\[
\text{LeftFPD}(R) = \sup \{ \text{pd}_R M \mid M \text{ is a left } R\text{-module with } \text{pd}_R M < \infty \}.
\]

The right finitistic projective dimension \(\text{RightFPD}(R)\) of \(R\) is defined similarly.

**Remark 4.2.** When \(R\) is commutative and Noetherian, the dimensions \(\text{LeftFPD}(R)\) and \(\text{RightFPD}(R)\) coincide and are equal to the Krull dimension of \(R\), by [10 Théorème (3.2.6) (Seconde partie)].

We will need the following three results, [12 Proposition 3.3], [12 Theorem 3.5] and [12 Proposition 3.18], respectively:

**Proposition 4.3.** If \(R\) is right coherent with finite \(\text{LeftFPD}(R)\), then every Gorenstein projective left \(R\)-module is also Gorenstein flat. That is, there is an inclusion \(\mathcal{GP} \subseteq \mathcal{GF}\).

**Theorem 4.4.** For any left \(R\)-module \(M\), we consider the following three conditions:

(i) The left \(R\)-module \(M\) is \(G\)-flat.

(ii) The Pontryagin dual \(\text{Hom}_\mathbb{Z}(M, \mathbb{Q}/\mathbb{Z})\) (which is a right \(R\)-module) is \(G\)-injective.

(iii) \(M\) has an augmented proper right resolution \(0 \to M \to F^0 \to F^1 \to \cdots\) consisting of flat left \(R\)-modules, and \(\text{Tor}^R_i (I, M) = 0\) for all injective right \(R\)-modules \(I\), and all \(i > 0\).

The implication (i) \(\Rightarrow\) (ii) always holds. If \(R\) is right coherent, then also (ii) \(\Rightarrow\) (iii) \(\Rightarrow\) (i), and hence all three conditions are equivalent.
Proposition 4.5. Assume that $R$ is right coherent. If $M$ is a left $R$-module with $\text{Gfd}_R M < \infty$, then there exists a short exact sequence $0 \to K \to G \to M \to 0$, where $G \to M$ is an $R\text{GF}$-precover of $M$, and $\text{fd}_R K = \text{Gfd}_R M - 1$ (in the case where $M$ is Gorenstein flat, this should be interpreted as $K = 0$).

In particular, every left $R$-module with finite Gorenstein flat dimension has a proper left $R\text{GF}$-resolution (that is, there is an inclusion $R\text{GF} \subseteq \text{LeftRes}_R M(R\text{GF})$).

Our first result is:

Lemma 4.6. Let $M$ be a left $R$-module with $\text{Gpd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\text{GP}$-resolution of $M$ (which exists by Proposition 3.1). Then the following conclusions hold:

(i) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(ii) If $R$ is left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) By Theorem 4.4 above, the Pontryagin dual $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is a Gorenstein injective left $R$-module. Hence $\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z})$ is exact by Proposition 3.3. Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, $T \otimes_R G^+$ is exact too.

(ii) With the given assumptions on $R$, the dual of Proposition 4.3 implies that every Gorenstein projective right $R$-module is also Gorenstein flat.

Lemma 4.7. Assume that $R$ is right coherent with finite $\text{LeftFPD}(R)$. Let $M$ be a left $R$-module with $\text{Gfd}_R M < \infty$, and let $G^+ = \cdots \to G_1 \to G_0 \to M \to 0$ be an augmented proper left $R\text{GF}$-resolution of $M$ (which exists by Proposition 3.4 since $R$ is right coherent). Then the following conclusions hold:

(i) $\text{Hom}_R(G^+, H)$ is exact for all Gorenstein injective left $R$-modules $H$.

(ii) $T \otimes_R G^+$ is exact for all Gorenstein flat right $R$-modules $T$.

(iii) If $R$ is also left coherent with finite $\text{RightFPD}(R)$, then $T \otimes_R G^+$ is exact for all Gorenstein projective right $R$-modules $T$.

Proof. (i) Since $\text{Gfd}_R M < \infty$ and $R$ is right coherent, Proposition 6.5 gives a special short exact sequence $0 \to K' \to G' \to M \to 0$, where $G' \to M$ is an $R\text{GF}$-precover of $M$, and $\text{fd}_R K' < \infty$. Since $R$ has $\text{LeftFPD}(R) < \infty$, Proposition 6] implies that also $\text{pd}_R K' < \infty$. Now the proof of Lemma 3.4 applies.

(ii) If $T$ is a Gorenstein flat right $R$-module, then the left $R$-module $H = \text{Hom}_Z(T, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective, by (the dual of) Theorem 4.4 above. By the result (i), just proved, we have exactness of

$$\text{Hom}_R(G^+, H) \cong \text{Hom}_Z(T \otimes_R G^+, \mathbb{Q}/\mathbb{Z}).$$

Since $\mathbb{Q}/\mathbb{Z}$ is a faithfully injective $\mathbb{Z}$-module, we also have exactness of $T \otimes_R G^+$, as desired.

(iii) Under the extra assumptions on $R$, the dual of Proposition 6.5 implies that every Gorenstein projective right $R$-module is also Gorenstein flat. Thus (iii) follows from (ii).

Theorem 4.8. Assume that $R$ is both left and right coherent, and that both $\text{LeftFPD}(R)$ and $\text{RightFPD}(R)$ are finite. For every right $R$-module $M$, and every left $R$-module $N$, the following conclusions hold:
(i) If \( \text{Gfdr}_RM < \infty \) and \( \text{Gfdr}_RN < \infty \), then
\[
\text{Tor}^G_{n,R}(M, N) \cong \text{Tor}^G_{n,F}(M, N).
\]
(ii) If \( \text{Gpdr}_RM < \infty \) and \( \text{Gfdr}_RN < \infty \), then
\[
\text{Tor}^G_{n,R}(M, N) \cong \text{Tor}^G_{n,F}(M, N) \cong \text{Tor}^G_{n,F}(M, N).
\]
(iii) If \( \text{Gfdr}_RM < \infty \) and \( \text{Gpdr}_RN < \infty \), then
\[
\text{Tor}^G_{n,R}(M, N) \cong \text{Tor}^G_{n,R}(M, N) \cong \text{Tor}^G_{n,F}(M, N).
\]
(iv) If \( \text{Gpdr}_RM < \infty \) and \( \text{Gpdr}_RN < \infty \), then
\[
\text{Tor}^G_{n,R}(M, N) \cong \text{Tor}^G_{n,F}(M, N) \cong \text{Tor}^G_{n,F}(M, N).
\]
All the isomorphisms are functorial in \( M \) and \( N \).

Proof. Use Lemmas 4.6 and 4.7 as input in the covariant-covariant version of Theorem 4.9.

4.9 (Definition of \( g\text{Tor} \) and \( \text{GTor} \)). Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module, and let \( N \) be a left \( R \)-module. If \( \text{Gfdr}_RM < \infty \) and \( \text{Gfdr}_RN < \infty \), then we write
\[
g\text{Tor}^R_n(M, N) := \text{Tor}^{G,R}_{n}(M, N) \cong \text{Tor}^G_{n,F}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.9(i). If \( \text{Gpdr}_RM < \infty \) and \( \text{Gpdr}_RN < \infty \), then we write
\[
\text{GTor}^R_n(M, N) := \text{Tor}^{G,P}_{n}(M, N) \cong \text{Tor}^G_{n,F}(M, N)
\]
for the isomorphic abelian groups in Theorem 4.9(iv).

We can now reformulate some of the content of Theorem 4.8.

Theorem 4.10. Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. For every right \( R \)-module \( M \) with finite \( \text{Gpdr}_RM \), and for every left \( R \)-module \( N \) with \( \text{Gpdr}_RN < \infty \), we have isomorphisms:
\[
g\text{Tor}^R_n(M, N) \cong \text{GTor}^R_n(M, N)
\]
that are functorial in \( M \) and \( N \).

Finally we compare \( g\text{Tor} \) (and hence \( \text{GTor} \)) with the usual \( \text{Tor} \).

Theorem 4.11. Assume that \( R \) is both left and right coherent, and that both \( \text{LeftFPD}(R) \) and \( \text{RightFPD}(R) \) are finite. Furthermore, let \( M \) be a right \( R \)-module with \( \text{Gfdr}_RM < \infty \), and let \( N \) be a left \( R \)-module with \( \text{Gfdr}_RN < \infty \). If either \( \text{fd}_RM < \infty \) or \( \text{fd}_RN < \infty \), then there are isomorphisms
\[
g\text{Tor}^R_n(M, N) \cong \text{Tor}^R_n(M, N)
\]
that are functorial in \( M \) and \( N \).

Proof. If \( \text{fd}_RM < \infty \), then we also have \( \text{pd}_RM < \infty \) by [13] Proposition 6] (since \( \text{RightFPD}(R) < \infty \)). Let \( P \) be any projective resolution of \( M \). As noted in Remark 4.8, \( P \) is also a proper left \( \mathcal{GP}_R \)-resolution of \( M \). Hence, Theorem 4.8(ii) and the definitions give:
\[
g\text{Tor}^R_n(M, N) = \text{Tor}^{G,R}_{n}(M, N) = H_n(P \otimes_R N) = \text{Tor}^R_n(M, N),
\]
as desired.
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References


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