Term Structure Modeling of SOFR
Evaluating the Importance of Scheduled Jumps
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Abstract

As interest rate benchmarks move from LIBOR to overnight Risk-Free Rates (RFR), it has become increasingly important for models to accurately capture the interest rate dynamics at the overnight tenor. Overnight rates closely track central bank policy rate decisions resulting, in highly discontinuous dynamics around scheduled meeting dates. In this paper, we construct a dynamic term structure model, which accounts for the discontinuous short-rate dynamics. We show that the model is able to jointly fit the overnight US policy rate, SOFR and SOFR futures rates through the recent Fed hiking cycle. Comparing our model with a standard continuous time-homogeneous short-rate model, we find several indications that our model avoids the clear misspecification of the continuous model, in particular with regard to the short-rate dynamics around meeting dates of the Federal Open Market Committee (FOMC). This effect begins to disappear as the term of the rates under consideration is increased, suggesting that diffusive dynamics are a reasonably accurate reflection of the evolution of market expectations embodied in longer-term interest rates.

Keywords: SOFR, Jumps, FOMC, Futures, Options.

JEL Classification: C5, E43, G13.
1 Introduction

Since Vasicek (1977), time-homogeneous affine term structure models have been the main tool when estimating dynamic term structure models. However, the push by regulators during recent years to move from LIBOR to overnight Risk-Free Rates (RFR) as the main benchmarks in fixed income products has increased the importance of accurately modeling rates at the overnight tenor. Overnight rates are highly dependent on central bank policy rates, which predominantly change on dates that are scheduled in advance and known to the market participants. Time-homogeneous and diffusive models ignore these scheduled announcement effects and therefore cannot be expected to accurately capture the dynamics of the policy rate and consequently the overnight benchmark.

Our focus is the Secured Overnight Financing Rate (SOFR), the USD RFR benchmark scheduled to replace USD LIBOR following its June 2023 cessation deadline. Studying the historical changes in SOFR following Federal Open Market Committee (FOMC) meetings, we argue that the primary US policy rate determining the level of SOFR is the Interest on Reserve Balances (IORB). We then construct an affine dynamic term structure model, which jointly models IORB, overnight SOFR and SOFR futures rates. In our framework, the dynamics of IORB are modelled as a time-inhomogeneous pure jump process consisting of both scheduled and unscheduled jumps reflecting the scheduled and unscheduled meetings of the FOMC. modeling IORB as a pure jump process, the filtered IORB short-rate path of the estimated model is consistent with the observed piecewise constant dynamics.

SOFR is computed using aggregate data from the overnight treasury repo market. It is therefore not entirely piecewise constant between FOMC announcements, but also affected by the general funding conditions in the overnight treasury repo market. We model the additional diffusive dynamics of SOFR by considering a stochastic SOFR-IORB spread. Our framework thus models the variation in SOFR futures rates as reflecting changes in expectations about future changes in the underlying policy rate as measured by the IORB as well as a stochastic SOFR-IORB spread.

Previous papers have also studied the effect of central bank policy rate announcements on the term structure. Piazzesi (2005) provides an early attempt to model the Federal
Funds target rate accounting for the FOMC meeting calendar. In her model the target rate follows a pure jump process driven by Poisson processes, with jumps during FOMC meetings triggered by elevated jump intensities. More recently, inspired by the UK LIBOR Transition, Backwell and Hayes (2022) propose and estimate a pure jump multicurve model for GBP LIBOR and SONIA OIS rates. In their model overnight, SONIA is included as a direct observation of the SONIA short-rate state variable, while term rates are modelled using a set of auxiliary state variables determining the distribution of future expected and unexpected jumps. Gellert and Schlögl (2021) put forward a model for SOFR in the Heath-Jarrow-Morton framework in which the target rate is piecewise constant between FOMC meeting dates while forward rates evolve diffusively due to the changing market expectations about future FOMC decisions driven by underlying Wiener processes. Brace et al. (2022) extend the model with stochastic volatility to also fit options on SOFR futures.\footnote{Scheduled or expected jumps are also referred to in the literature as stochastic discontinuities. Recently, Multiple theoretical papers have also studied the modeling of stochastic discontinuities in relation to term structure modeling see e.g. Keller-Ressel et al. (2019), Fontana et al. (2020), and Fontana et al. (2022).} While these studies also include the scheduled central bank announcements, our paper is the first to seek to include data on the overnight policy and benchmark rates, not as a direct observation of a state variable, but as a part of the estimation sample. Furthermore, the model we construct falls within the traditional class of affine term structure models. The tractable nature of this framework allows us to include option data as well as futures data in the estimation sample.

In the empirical section of the paper we compare our jump model to an instance of the standard time-homogeneous continuous affine term structure models. The models are estimated on daily overnight rate, futures rate data using maximum-likelihood estimation in conjunction with the Kalman filter. While the continuous model is able to fit the cross-section of futures rates fairly well, as also shown in Skov and Skovmand (2021), this is no longer the case when overnight rates are introduced in the estimation sample. Studying the individual log-likelihood contributions we find that, unlike the jump model, estimating the continuous model results in significant deterioration of the log-likelihood contributions on observation dates with changes in the overnight policy rate following FOMC announcements. The misspecification also results in the continuous model being unable to jointly fit overnight
and futures rates during the Fed hiking cycle in 2022, whereas the time-inhomogeneous jump model is consistent with both overnight and futures rates. Comparing real-time policy rate forecasts produced by both models we find that the jump model performs better at short horizons (one month) while the forecasts become similar when considering longer forecast periods. Finally, accounting for the FOMC meeting calendar is not only important when modeling overnight rates, but also in contracts with shorter tenors such as the one-month futures or during the accumulation period of contracts set in arrears as is the case in both one- and three-month SOFR futures and the one-month SOFR futures option.

The paper is structured as follows. Section 2 studies the dynamics of SOFR in relation to overnight US policy rates. Next, in section 3 We present the time-inhomogeneous affine jump framework used to jointly model IORB and SOFR. Section 5 details the data and estimation used in the empirical analysis while the results are discussed in Section 6.

2 Policy Rates, FOMC Announcements and SOFR

Two of the primary policy rates set by the Federal Reserve are the Federal Funds Target Range and the Interest Rate on Reserve Balances (IORB). The Federal Funds Target Range indicates an acceptable interval for the Effective Federal Funds Rate (EFFR). The IORB replaces the Interest Rate on Required Reserves (IORR) and Interest Rate on Excess Reserves (IOER), which were discontinued on July 29, 2021.\(^1\) The last time IORR and IOER differed was in 2008 and thus before our data sample. With IORR and IOER identical during our entire data period, we will refer to all three rates during our sample simply as IORB.

Figure 1 plots the Target Range, IORB, and SOFR. The plot suggests a stochastic spread between SOFR and each of the policy rates. Note that, while mostly within the target range, SOFR fixes both below and above IORB. Also, while spikes in SOFR were frequent during the start of our data period, there have been no significant spikes in SOFR since the October 11, 2019 Fed announcement to conduct operations in the overnight repo market.\(^3\) Importantly, we see that SOFR closely tracks the changes in the policy rates set by the Federal Reserve. Since the nineties, changes in the policy rates have historically been announced following the

\(^2\)See https://www.federalreserve.gov/newsevents/pressreleases/bcreg20210602a.htm

\(^3\)See https://www.federalreserve.gov/newsevents/pressreleases/monetary20191011a.htm
eight annual scheduled FOMC meetings, with few exceptions of changes after unscheduled meetings during crises such as the Great Financial Crisis or the beginning of the COVID-19 pandemic. This suggests that correctly modeling the jumps in policy rates around FOMC meetings is crucial when modeling the dynamics of overnight SOFR.

Studying the rate changes following FOMC announcements further, Table I lists the Target Range and IORB changes made by the Fed and the corresponding change in SOFR as well as EFFR on the day following the FOMC announcement during our sample period. The table suggests that the primary policy rate affecting the level of both SOFR and EFFR is the IORB and not the Target Range. E.g., following the scheduled meeting on June 13, 2018 the target range increased by 25 basis points, while IORB only increased by 20 basis points, which caused an increase in SOFR and EFFR by 19 and 20 basis points, respectively. Furthermore, unlike the target range, changes in the IORB are not restricted to the well-known increments in multiples of 25 basis points, but may change by as little as 5 basis...
<table>
<thead>
<tr>
<th>FOMC Date</th>
<th>$\Delta \text{Target Range}$</th>
<th>$\Delta \text{IORB}$</th>
<th>$\Delta \text{SOFR}$</th>
<th>$\Delta \text{EFFR}$</th>
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Table I: Changes in overnight policy and market rates around FOMC announcements. * denotes unscheduled meetings. Dates in red have $\Delta \text{Target Range} \neq \Delta \text{IORB}$. All values are in basis points.

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*For a brief discussion on how the IOER emerged as the key policy rate set by the Fed, see Gellert and Schlögl (2021).
3 Modeling the Joint Dynamics of IORB and SOFR

We fix the filtered probability space \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, Q)\) and specify the dynamics directly under the risk-neutral \(Q\)-measure defined by the continuously compounded savings account numeraire. To reflect the piecewise constant dynamics of IORB we model the IORB-specific short-rate as a pure jump process

\[
dr_t = J_t^D dN_t^D + dJ_t^P
\]

where \(N_t^D\) is defined as a deterministic counting process reflecting the deterministic jump dates. The set of deterministic jump dates \(\{\tau_1, ..., \tau_M\}\) are denoted as \(\mathcal{T}\) and identified as the days following the scheduled FOMC meetings when the potential changes in IORB become effective. Furthermore, \(J_t^P\) is a pure jump process with jump arrival intensity \(\nu_t^P\) and iid jump sizes \(Z_1, Z_2, ...\) capturing unscheduled jumps in the target rate. We assume that the scheduled jumps, \(J_t^D\), have an exponentially affine jump size distribution with parameters determined by affine transformations of the underlying state process \(X_t^j\). The dynamics of \(X_t^j\) are modelled using a standard \(n\)-dimensional time-homogeneous continuous affine process

\[
dX_t^j = \theta^j (K^j - X_t^j)dt + \Sigma^j D(X_t^j)dW_t^j
\]

where \(D(X_t^j)\) is a diagonal matrix with \(D(X_t^j)_{ii} = \sqrt{\alpha_i + \beta_i' X_t^j}\). We note that \(X_t^j\) must be specified such that the parameters of the jump size distribution stay in the space that they are defined on. Thus, in our framework the joint specification of \(J_t^D\) and \(X_t^j\) dictates the term structure of scheduled FOMC meeting jumps. The specification of \(r_t\) implies that it is an affine time-inhomogeneous process. Since there is no diffusion term in the dynamics of \(r_t\) it remains constant between jumps and the time \(t\) risk-neutral expectation of the IORB for any future time point \(u > t\) is given by the current rate and expected sum of future jumps

\[
E_t^Q [r_u] = r_t + \sum_{j=\eta(t)}^{\eta(u)-1} E_t^Q [J_{\tau_j}^D] + E_t^Q \left[ \sum_{i=N_t^P}^{N_t^P-1} Z_i \right]
\]

with \(\eta(t) = \{j \in \mathbb{N}|\tau_{j-1} \leq t < \tau_j\}\) and \(N_t^P\) the amount of realized unscheduled jumps at time \(t\). Having described the dynamics of the IORB, we model the SOFR specific short-rate, denoted \(r_t^s\), as the sum of IORB and an affine stochastic spread process, \(s_t\), such that

\[
r_t^s = r_t + s_t.
\]
Given that spikes in SOFR were only present during the beginning of our sample and seem to have disappeared in recent years, we do not attempt to incorporate spikes in our modeling of the dynamics of SOFR. We refer to Andersen and Bang (2020) for a potential way to treat spikes in SOFR.

### 3.1 Model Specification

To obtain a working specification of our framework, we will assume that the size of scheduled jumps is normally distributed with state-dependent mean and constant variance

\[
J_t^D \sim \mathcal{N}(\gamma^Q + \Gamma^Q X_t^j, \omega^2).
\]

Furthermore, we let \(\gamma^Q = 0\) and \(\Gamma^Q = (1, 0)'\) and specify \(X_t^j = (\xi_t, \theta_t)'\) as a two-factor Gaussian process

\[
\begin{align*}
    d\xi_t &= \kappa^\xi (\theta_t - \xi_t) dt + \sigma^\xi dW_t^\xi, \\
    d\theta_t &= \kappa^\theta (\theta_t - \theta_t) dt + \sigma^\theta \left( \rho dW_t^\xi + \sqrt{1 - \rho^2} dW_t^\theta \right).
\end{align*}
\]

The joint specification of \(J_t^D\) and \(X_t^j\) implies that the mean jump sizes are modelled by \(\xi_t\), which in turn evolves stochastically around the central tendency factor \(\theta_t\). Having specified a Gaussian distribution for the scheduled jumps and ignoring the unscheduled jumps, our model can be seen as a special case of the framework presented in Kim and Wright (2014). However, rather than modeling the target rate specifically, that paper models government bond yields using a diffusive term for the short-rate while considering scheduled jumps on dates with employment report announcements. Next, we assume that the unscheduled jumps \(J_t^P\) arrive at a constant intensity \(\nu^P\) and specify the corresponding jump size distributions \(Z_1, Z_2, \ldots\) as normal with mean \(\mu^P\) and volatility \(\sigma^P\). Lastly, since the spread between SOFR and IORB is both negative and positive during our sample, we model it using a simple one-factor Gaussian process

\[
ds_t = \theta^s (\kappa^s - s_t) dt + \sigma^s dW_t^s.
\]
Defining the joint process $X_t = (r_t, \xi_t, \theta_t, s_t)'$, our model specification constitutes a four-factor model for the joint dynamics of IORB and SOFR.\(^5\) We relate the dynamics under the physical and risk-neutral measures through the likelihood process

$$\frac{dL_t}{L_t} = \Psi (J^D_t, X^j_t) \ dN^D_t + \Lambda' dW^P_t.$$  

(9)

Given the fairly short sample size, we consider a parsimonious risk premium specification providing a close link between the risk-neutral and physical dynamics. First, we specify a completely affine market price of risk specification for the diffusive state variables. Thus, we define the market price of risk as $\Lambda = (\lambda^\xi, \lambda^\theta, \lambda^s)'$ such that the diffusive dynamics under the physical and risk-neutral measures are related through

$$dW^Q_t = dW^P_t + \Lambda dt.$$  

(10)

Second, we consider an essentially affine scheduled jump risk premium extension as in Kim and Wright (2014)

$$\Psi (J^D_t, X^j_t) = \exp \left( -\gamma^D_t + (\gamma^P + \Gamma^P X^j_t) - \frac{1}{2} \gamma^2 \right) - 1$$  

(11)

with $\gamma = \phi + \Phi X^j_t$. This implies that the physical jump distribution is $J^P_t \sim N(\gamma^P + \Gamma^P X^j_t, \omega^2)$ with $\gamma^P = \gamma^Q + \omega \phi$ and $\Gamma^P = \Gamma^Q + \omega \Phi$. We further restrict the scheduled jump risk specification by $\phi = 0$ and $\Phi = (\Phi_1, 0)'$ such that $\gamma^P = 0$ and $\Gamma^P = (\Gamma^P_1, 0)'$. With these restrictions the jump risk premium effectively scales the effect of $\xi_t$ on the mean jump

\(^5\)Another possible choice, which would preserve the affine property of $r_t$, would be to use scaled Poisson distributions for the up and down jumps, i.e., $J^D_t \sim s \cdot (\text{Pois}(\gamma_u + \Gamma_u X^j_t) - \text{Pois}(\gamma_d + \Gamma_d X^j_t))$. The realized jumps in the IORB during our sample presented in Section XX suggest that we would need to set $s = 0.0005$ to be able to capture the realized changes in the IORB during our sample. While this implies a discrete jump size distribution, which may seem more realistic, estimating such a model results in large values of the rate parameters controlling the Poisson distributions. Thus, while on average the sum of these would reflect the scheduled jumps given a sufficiently flexible state process, the simulated paths would result in frequent and large up/down jumps in the target rate dissimilar to the observed path. Specifying Poisson jumps also implies that the rate parameter in each Poisson distribution needs to stay positive, meaning that we cannot model $X^j_t$ using a Gaussian process, which reduces the tractability of the model. Nonetheless, the pricing formulas in Appendix A are easily modified to allow for Poisson jumps with state-dependent parameters driven by independent square-root processes, for example.
size under the physical measure. This allows us to directly evaluate the impact of the jump risk premium by comparing the estimated value of $\Gamma_P^1$ with the fixed value $\Gamma_Q^1 = 1$ under the risk-neutral measure. In order to identify the dynamics of the unscheduled jumps under both measures we do not allow for a risk premium on the unscheduled jumps, see Section 5.3 for further discussion.

### 3.2 A Continuous Specification

In order to compare the proposed jump model, we also consider the class of standard time-homogeneous continuous models. Specifically, we consider the maximally identifiable three-factor Gaussian model from Dai and Singleton (2000) as a continuous model for the dynamics of IORB. The IORB related short-rate is therefore affine in the latent state variables $r_t = \rho_0 + \rho_1 \cdot X_t$, which evolve as

$$dX_t = K^Q (\theta^Q + X_t) \, dt + \Sigma dW^Q_t. \tag{12}$$

As in Kim and Wright (2014) we rotate the specification such that the third variable defines the short-rate. The parameterization then becomes

$$K^Q = \begin{bmatrix} K_{11} & 0 & 0 \\ K_{21} & K_{22} & 0 \\ K_{31} & K_{32} & K_{33} \end{bmatrix}, \quad \theta^Q = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}, \quad \rho = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{13}$$

Where $c$ is a scaling constant, which we fix at 0.01. Similarly to the jump specification, we define the SOFR specific short rate as the sum of the target rate and a stochastic spread process driven by a single-factor Gaussian process $r_t^s = r_t + s_t$. Likewise, we also specify a completely affine market price of risk specification with $\Lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda^s)'$ such that the diffusive dynamics again are related by

$$dW^Q_t = dW^P_t + \Lambda dt. \tag{14}$$

The jump and continuous specifications are thus both four-factor models for the joint dynamics of IORB and SOFR, however, we note that the continuous specification contains an extra diffusive Brownian motion term.
SOFR Futures and Futures Options

Interest rate futures referencing SOFR are traded on the CME. The futures price at expiration is \(100 \left(1 - R(S, T)\right)\) with \(R(S, T)\) denoting the futures rate. The mark-to-market feature implies that the futures price is a martingale under the risk-neutral measure (see e.g. Hunt and Kennedy (2004)). The present value of the futures contract is therefore given by the risk-neutral expectation

\[
F(t; S, T) = \mathbb{E}_t^Q \left[100 \left(1 - R(S, T)\right)\right].
\] (15)

Contracts referencing both a one- and three-month backward-looking rate are available. Denoting the realized SOFR fixings \(R_{t_i}^s\), the one-month futures rate is determined as the arithmetic average during the contract month

\[
R^{1m}(S, T) = \frac{1}{T - S} \sum_{i=1}^{N} \frac{1}{360} R_{t_i}^s,
\] (16)

with \(S \leq t_1, \ldots, t_N \leq T\). The three-month futures rate is given by the compounded average of the overnight fixings

\[
R^{3m}(S, T) = \frac{1}{T - S} \left( \prod_{i=1}^{N} \left(1 + d_i R_{t_i}^s\right) - 1 \right)
\] (17)

with \(d_i\) the number of days until the next fixing multiplied by the day count fraction.

Futures contracts provide multiple advantages when studying the short end of the term structure. Specifically, SOFR futures data contains the earliest record of historical data on SOFR linked derivatives and the most liquidly traded short term SOFR contracts. Furthermore, since futures are exchange-traded products, our data consists of actual traded prices instead of quotes from brokers, as is often the case with data on over-the-counter products such as Overnight Index Swaps (OIS). Finally, spot-starting OIS rates involve compounding of the underlying overnight rate throughout the entire period of the swap and thus the compounding period increases as the maturity of the contract increases. In contrast, the futures rate is a direct measurement of risk-neutral SOFR expectations during the reference period of the contract, which is always either one or three months. This feature becomes increasingly important when we want to be able to properly differentiate between the ability of each model in fitting the term structure of SOFR around FOMC meetings.
Options on SOFR futures are available on both the one- and three-month underlying futures contracts. A key difference between the one- and three-month SOFR option is the expiration date of the option. The option referencing the three-month futures expires on the Friday before the third Wednesday of the contract month when the underlying three-month futures starts to accumulate its rate. The one-month futures option expires at the end of the contract month of the underlying one-month futures contract. The one-month SOFR futures therefore allows one to study the option behaviour as the backward-looking rate fixes, however, as of writing trading is only active in the three-month futures options. Like the futures contract, the three-month SOFR futures option also provides the earliest record of traded SOFR derivatives with optionality. Trading in SOFR futures options has greatly increased during 2022, as the US LIBOR cessation date approaches, and in November 2022 open interest in SOFR futures options has surpassed open interest in Eurodollar futures options.\(^6\) Finally, while the undiscounted SOFR futures contract is a martingale under the risk-neutral measure due to the mark-to-market feature, the SOFR futures option payoff requires discounting. However, since October 2020 SOFR has been the price alignment interest (PAI) at CME and thus SOFR futures options can be accurately priced using a single-curve setup.

5 Data and Estimation

In this section we describe the data set and assumptions made in our model estimation as well as the estimation method.

5.1 FOMC Meeting Data

We obtain the historical and future scheduled FOMC calendar from the Federal Reserve website.\(^7\) Meetings are scheduled approximately a year in advance. Since we only consider


\(^7\)See [https://www.federalreserve.gov/monetarypolicy/fomccalendars.htm](https://www.federalreserve.gov/monetarypolicy/fomccalendars.htm)
maturities up to around one year we use the actual historical scheduled FOMC date.\(^8\) Our jump model does not include any FOMC meeting calendar uncertainty. Instead, we model the target rate jumps following unscheduled meetings as unscheduled jumps. Therefore, we also do not consider any changes or unscheduled meetings in the FOMC meeting calendar. This means that unscheduled meetings such as those that happened on March 3 and 15, 2020 are not a part of our FOMC calendar. Similarly, the meeting on March 18, 2020, which was cancelled, is also not removed from our FOMC calendar data.

5.2 Overnight, Futures and Futures Option Data

First, we collect the overnight SOFR and IORB data used in the estimation from the website of the Federal Reserve Bank of St. Louis. Next, we obtain futures data from Refinitiv Eikon. Our data sample consists of daily data starting in June 2018 (following the introduction of SOFR futures on CME) and ending in February 2023, resulting in a total of 1178 SOFR futures trading days. Since we study the effect of FOMC meetings, we focus on the short end of the term structure. Furthermore, futures contracts close to expiry are usually the most liquidly traded. Therefore, we consider futures contracts with a maturity up to around a year. Futures contracts referencing SOFR are available as both one-month and three-month contracts. We include the seven nearest one-month SOFR futures contracts and five nearest quarterly three-month SOFR futures contracts. In Section 6.6 we extend the sample with recent data on SOFR futures options. Similarly to futures contracts, options on futures close to expiry are also the most liquid, in particular futures options expiring within a year. Based on this, we include the four nearest quarterly SOFR futures options. These have only recently become liquid and our options data sample therefore starts June 1, 2022, consisting of 170 observations, which include options. Since the main focus is estimate SOFR variance

\(^8\)We note that our framework can easily be extended to longer maturities simply by extrapolating the FOMC dates. Scheduled FOMC meeting announcements are always on Wednesdays, with the potential target rate change the day after on Thursday and thus in order to obtain the future scheduled meeting dates for e.g. 2024, one can choose the Wednesdays closest to the equivalently scheduled meeting date in 2023. The exact timing of the meeting dates are far less important for the model estimation when considering meetings further than a year out, and thus extrapolating the dates has little impact.
and not fit the actual smile, for each option maturity we only consider the out-of-the-money call option closest to being at-the-money for each contract maturity. Furthermore, we only consider options with an open interest larger than 100. Finally, we exclude options with a quoted price less than 0.02, which tend to be less accurate due to the minimum price fluctuation of the contracts. This usually impacts the nearest options when the options are close to expiry. SOFR futures options are American options. We therefore adjust option prices for the early exercise premium as in Broadie et al. (2007) and Bikbov and Chernov (2005) by calculating the Black-implied volatility including the early exercise feature using a binomial tree. We then calculate the approximated European option price by inserting the obtained implied volatility in the European version of the Black formula. When calculating both the American Black-implied volatility and the subsequent European option price we need to fix a constant rate for discounting. Here, we use the fitted SOFR curve from the estimation without options to obtain the prevailing SOFR term rate between the observation date and the maturity date of the option. Since we only include OTM options with up to a year of maturity the American feature has little impact on the price of the option. The median size of the American premium measured in implied volatility is 0.4 basis points, while the observed implied volatilities range from 48 to 180 basis points, with a median size of 124 basis points.

5.3 Maximum-Likelihood and the Kalman Filter

We estimate the models using maximum-likelihood in conjunction with the Kalman filter. This involves casting each model in its state space representation consisting of a transition and measurement equation. The method is frequently used in the estimation of dynamic term structure models, we therefore leave out the exact steps of the Kalman filtering process.\(^9\) Focusing on the state transition equation of the time-inhomogeneous short-rate process in the jump model, we note that when estimating the jump model we cannot explicitly identify the individual effect of scheduled and unscheduled jumps on the term structure. However, once realized we know both the size of the jump in IORB and if it was scheduled or not from

\(^9\)See Skov and Skovmand (2021) and Skov and Skovmand (2022) for applications and details on estimating continuous time-homogeneous term structure models on historical SOFR futures data.
the following FOMC announcements. Assuming no risk-premium on the unscheduled jumps, this allows us to pre-estimate the effect of the unscheduled jumps outside the Kalman filtering algorithm from the empirical frequency and jump sizes. Thus, under both the $P$- and $Q$-measure we fix the unscheduled jump intensity at the empirical frequency during our sample $\nu^P = 0.428$ and the unscheduled jump size distributions are likewise fixed at the empirical mean $\mu^P = -0.0075$ and variance $\sigma^P = 1.25e^{-5}$. For the filtering we then update the target rate with its realized unscheduled jump sizes during the unscheduled rate changes in March of 2020. Thus, only considering the scheduled jumps in IORB, we define the state transition equation of the discretized short rate process $r_t$ with $\Delta t = t_i - t_{i-1} = 1/252$ as

$$
\begin{align*}
    r_{t_i} &= \begin{cases} 
    r_{t_{i-1}} + \Gamma^P X^P_{t_{i-1}} + \vartheta_t & t_i \in \mathcal{T} \\
    r_{t_{i-1}} & t_i \notin \mathcal{T}
    \end{cases}
\end{align*}
$$

(18)

where $\vartheta_t \sim \mathcal{N}(0, \omega^2)$. The conditional variance of $r_t$ is therefore $\omega^2$ if $t_i \in \mathcal{T}$ and zero if $t_i \notin \mathcal{T}$. The zero conditional variance of $r_t$ on non-FOMC dates due to the offline estimation of unscheduled jumps is essential during estimation since it implies that the policy rate cannot change during the Kalman filtering process, thus enforcing the piecewise constant pattern of the policy rate between FOMC meeting dates. Similarly to the continuous model specification, the remaining state variables in the jump model are simply time-homogeneous Gaussian diffusions and we obtain the transition equation from the discretized dynamics under the physical measure.

The standard Kalman filter also requires an affine measurement equation. From Appendix A we note that the one-month futures rates are already affine in the state variables, whereas we use log transformations and consider futures and overnight yields to obtain affine expressions for the overnight and three-month rates, i.e.,

$$
\begin{align*}
    y^{O/N}(t,T) &= \frac{1}{T-t} \log \left( 1 + (T-t)R^{O/N}(t,T) \right), \\
    y^{3m}(t;S,T) &= \frac{1}{T-S} \log \left( 1 + (T-S)f^{3m}(t;S,T) \right). 
\end{align*}
$$

(19) (20)

Denoting by $y_{t_i}$ the stacked vector of futures and overnight observations for each observation

---

10Recall from table I that we observe two unscheduled jumps of $-100$ and $-50$ basis points during our sample.
date $t_i$, the measurement equation can be written as

$$y_{t_i} = A_{t_i} + B_{t_i}X_{t_i} + \epsilon_{t_i}. \quad (21)$$

$\epsilon_{t_i}$ is a vector of measurement errors, which we assume to be independent for each observed overnight and futures rate. Furthermore, we assume mean zero Gaussian errors with standard deviation $\sigma^{rates}$ for each observation. The assumption puts a tight requirement for the models to simultaneously fit the entire range of overnight and futures rates.\footnote{Skov and Skovmand (2021) allow for individual variances of each futures observation. This additional flexibility of the measurement errors makes it harder to identify if the models are misspecified.} When including data on futures options, the measurement equation becomes non-affine in the state vector. Instead, we apply the extended Kalman filter, computing the required derivatives using small perturbations in the state vector. We also assume a separate standard deviation $\sigma^{options}$ for the option measurement errors.

### 6 Empirical Results

#### 6.1 Parameter estimates and state variables

Parameter estimates for both the continuous and jump model specifications are presented in Table II. Focusing on the jump model estimates, we first observe a near zero estimate for the mean of the stochastic mean process $\theta$. Jumps in the policy rate are thus expected to mainly occur at the nearest FOMC dates, with the Federal Reserve expected to keep its policy rate fixed in the long run. Furthermore, we find a significant negative correlation between the jump state factors, as indicated by $\rho = -0.94$. A positive shock to the expectation of the nearest FOMC meeting is thus expected to be offset by an almost equivalent decrease in the expectation of future jumps. Finally, while the market price of risk parameter estimates display fairly large standard deviations, we see a sizeable scheduled jump risk premium as implied by $\Gamma_{1}^{P} = 0.61$, which is notably lower than the fixed $\Gamma_{1}^{Q} = 1$.

Figure 2 plots the filtered policy related short-rate factor and SOFR-IORB spread factor for each specification. The figure clearly shows how the continuous short rate specification is not able to reflect the discontinuous path for IORB, whereas the jump specification is fully...
<table>
<thead>
<tr>
<th>Jump</th>
<th>Estimate</th>
<th>Continuous</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\kappa^r$</td>
<td>1.6988 (0.3072)</td>
<td>$\rho_0$</td>
<td>0.0033 (0.0013)</td>
</tr>
<tr>
<td>$\kappa^\theta$</td>
<td>1.7046 (0.3047)</td>
<td>$K_{11}$</td>
<td>1.8398 (2.9764)</td>
</tr>
<tr>
<td>$\kappa^s$</td>
<td>0.0442 (0.0330)</td>
<td>$K_{21}$</td>
<td>2.1303 (0.6243)</td>
</tr>
<tr>
<td>$\theta^\theta$</td>
<td>-0.0003 (0.0000)</td>
<td>$K_{22}$</td>
<td>0.3574 (0.0180)</td>
</tr>
<tr>
<td>$\theta^s$</td>
<td>0.1056 (0.0777)</td>
<td>$K_{31}$</td>
<td>-2.3387 (2.3426)</td>
</tr>
<tr>
<td>$\sigma^\xi$</td>
<td>0.0070 (0.0002)</td>
<td>$K_{32}$</td>
<td>-1.9901 (1.6970)</td>
</tr>
<tr>
<td>$\sigma^\theta$</td>
<td>0.0088 (0.0026)</td>
<td>$K_{33}$</td>
<td>1.8505 (2.9684)</td>
</tr>
<tr>
<td>$\sigma^s$</td>
<td>0.0112 (0.0002)</td>
<td>$\kappa^s$</td>
<td>200.903 (3.7261)</td>
</tr>
<tr>
<td>$\rho$</td>
<td>-0.9447 (0.0138)</td>
<td>$\theta^s$</td>
<td>-0.0001 (0.0000)</td>
</tr>
<tr>
<td>$\omega$</td>
<td>0.0013 (0.0000)</td>
<td>$\sigma_{31}$</td>
<td>-0.0025 (0.0013)</td>
</tr>
<tr>
<td>$\lambda^\xi$</td>
<td>0.4895 (0.8824)</td>
<td>$\sigma_{32}$</td>
<td>-0.0006 (0.0010)</td>
</tr>
<tr>
<td>$\lambda^\theta$</td>
<td>0.8414 (0.5654)</td>
<td>$\sigma_{33}$</td>
<td>0.0036 (0.0006)</td>
</tr>
<tr>
<td>$\lambda^s$</td>
<td>-0.4783 (0.4175)</td>
<td>$\sigma^s$</td>
<td>0.0298 (0.0003)</td>
</tr>
<tr>
<td>$\Gamma_1^P$</td>
<td>0.6088 (0.0481)</td>
<td>$\lambda_1$</td>
<td>-0.2539 (0.8591)</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>$\lambda_2$</td>
<td>0.5479 (0.6389)</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>$\lambda_3$</td>
<td>-0.6841 (0.7330)</td>
</tr>
<tr>
<td>-</td>
<td>-</td>
<td>$\lambda^s$</td>
<td>-1.4819 (1.1976)</td>
</tr>
<tr>
<td>$\sigma^\text{rates} \times 10^4$</td>
<td>2.74 (0.10)</td>
<td>$\sigma^\text{rates} \times 10^4$</td>
<td>3.79 (0.07)</td>
</tr>
</tbody>
</table>

Log-likelihood 107,109  Log-likelihood 102,961

Table II: Full sample parameter estimates for the jump and continuous model specifications. Standard deviations are shown in parenthesis.

consistent with such a path. Furthermore, during the recent Fed hiking cycle, in order to compensate for the continuous short-rate path, the spread factor oscillates around jumps to fit the overnight SOFR fixings.
Figure 2: The plots display the filtered $r_t$ and $s_t$ state variables in both the continuous and jump specification.

6.2 Log-likelihood contributions and FOMC dates

The continuous and jump models are not nested models, hindering the use of standard likelihood ratio tests. However, simply comparing the maximized log-likelihoods for each model in Table II, we note that the jump specification produces a significantly higher overall log-likelihood value, suggesting a better fit to the observed data. In order to further study the cause of the difference in log-likelihood values, consider the daily log-likelihood contributions based on the optimal parameters. The daily log-likelihood contributions throughout the sample period are plotted in Figure 3. During the first two years of the sample we see occasional downward spikes in the daily log-likelihood contributions of both models due to overnight spikes in SOFR, most notably during the SOFR surge of September 2019. As noted in Section 3, we do not attempt to model spikes in SOFR and the negative spikes in log-likelihood values are thus to be expected. Likewise, the large drop in log-likelihoods following the market stress of the COVID-19 Crisis is no surprise. During the subsequent zero interest rate environment both specifications produce steady log-likelihood contributions with slightly higher values obtained by the jump-model. The largest differences are seen during the most recent period in 2022, following the multiple policy rate hikes by the Fed. Figure 4 highlights the log-likelihood values from January 2022 to the end of our sample

Electronic copy available at: https://ssrn.com/abstract=4431839
in February 2023. The plot shows the distinct crashes in the log-likelihood values of the continuous model specification following FOMC meetings, whereas the jump specification correctly captures the scheduled jumps in the overnight rates at FOMC dates and thus displays little to no difference in log-likelihood contributions on these dates. The large deterioration in log-likelihood values clearly reflects the misspecification of the continuous model around scheduled FOMC dates with policy rate changes.

6.3 In-Sample Fit

In order to compare the ability of each of the models to fit the time variation in the cross-section of overnight and futures rates, we compute the fitted overnight and futures rates based on the filtered state variables. The resulting root-mean-square errors (RMSE) are reported in Table III. Based on the full sample RMSEs, we observe an improvement across almost all futures rates when comparing the jump specification to the classical continuous model. Focusing on the overnight rates, the continuous model has a surprisingly good fit to overnight SOFR. This is because of the SOFR spread process, $s_t$, which in the continuous
Figure 4: Daily log-likelihood values from January 2022 to the end of our sample based on the optimal parameter estimates. The vertical dotted lines marks FOMC meeting dates.

The model is estimated with a very strong mean reversion, $\kappa^* = 200.9$, and mean zero such that variations in this process basically only impact the overnight SOFR fixings and enables the model to fit the overnight SOFR fixings even when spikes are present. In the jump model, the SOFR spread accounts for the entire term structure of the spread. Thus, it is not able to fully capture the spikes during the first part of the sample.

We also report RMSEs for subsamples of the full data period. Specifically, we consider the first period of our sample from June 1, 2018 to the end of 2022, and the final part of the sample from January 1, 2022 to February 2, 2023 separately. Studying the RMSEs of the subsamples, we see that the improvement in the jump specification is mainly driven by a significant improvement in the most recent period from January 2022 to February 2023. During this period the Federal Reserve increased its policy rates multiple times following scheduled FOMC meetings. While there was a degree of uncertainty about the size of the policy rate hikes, the hikes were anticipated thus creating a highly discontinuous expected path for the policy rate and in turn SOFR. The jump specification is perfectly capable of fitting these discontinuities and shows no real deterioration in fit during this period. However,
Table III: Comparative RMSEs for the jump and continuous models. "Start" refers to the subsample covering the first period of our sample from June 1, 2018 to the end of 2022, and "End" refers to the final part of the sample from January 1, 2022 to February 2, 2023.

<table>
<thead>
<tr>
<th>Model</th>
<th>Sample</th>
<th>O/N Rates</th>
<th>One-Month Futures</th>
<th>Three-Month Futures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>IORB</td>
<td>SOFR</td>
<td>1st</td>
</tr>
<tr>
<td>Jump</td>
<td>Full</td>
<td>1.0</td>
<td>5.1</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>Start</td>
<td>1.0</td>
<td>5.5</td>
<td>2.2</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>0.7</td>
<td>3.3</td>
<td>1.3</td>
</tr>
<tr>
<td></td>
<td>Full</td>
<td>6.6</td>
<td>1.4</td>
<td>3.9</td>
</tr>
<tr>
<td>Continuous</td>
<td>Start</td>
<td>4.4</td>
<td>1.4</td>
<td>2.7</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>11.1</td>
<td>1.2</td>
<td>6.6</td>
</tr>
<tr>
<td>Jump</td>
<td>Full</td>
<td>-</td>
<td>-</td>
<td>1.8</td>
</tr>
<tr>
<td></td>
<td>Start</td>
<td>-</td>
<td>-</td>
<td>1.7</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>-</td>
<td>-</td>
<td>2.2</td>
</tr>
<tr>
<td>Continuous</td>
<td>Full</td>
<td>-</td>
<td>-</td>
<td>1.6</td>
</tr>
<tr>
<td></td>
<td>Start</td>
<td>-</td>
<td>-</td>
<td>0.9</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>-</td>
<td>-</td>
<td>2.8</td>
</tr>
<tr>
<td>Jump</td>
<td>Full</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Start</td>
<td>-</td>
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<td>-</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Continuous</td>
<td>Full</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>Start</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>End</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

The continuous specification clearly fails to accurately capture the expected short-rate path as seen by the significant increase in RMSEs across all overnight and futures rates.

Table III suggests better fits for the nearest one- and three-month futures contracts compared to the second nearest contract. This is somewhat misleading and a result of the fixing in arrears of the futures rate, which means that the nearest futures contract has already accumulated a part of its underlying overnight SOFR fixings. Therefore, when a large part of these fixings have already accumulated, even a highly misspecified model will provide a
decent fit to the futures rate. To account for this, we adjust for the rate that has already accumulated and re-calculate the RMSEs based on just the part of the futures rate that has not yet fixed.\(^{12}\) Focusing on the last period of the sample starting from January 1, 2022, we find that the RMSE for the nearest one-month futures in the continuous model is 11.2 bps compared to 3.0 bps for the jump model. Similarly, for the three-month contract the RMSE is 11.0 and 3.8 bps for the continuous and jump model, respectively. Figure 5 plots the absolute errors between the observed and model-implied futures rates for the nearest contracts while adjusting for the part of the contract, which has already accumulated. The graphs clearly show the better fit of the jump model, especially for the one-month contract around FOMC meetings.

Backwell and Hayes (2022) find that unscheduled jumps are necessary when fitting the term structure (up to one year) of SONIA. However, in their model the state variables impact the jump distribution for the next meeting only, and all scheduled jumps in the short rate after the upcoming meeting have expectation zero. This implies, that in their model futures contracts starting to accumulate after the next scheduled meeting are determined solely, by unscheduled jumps. Their model setup thus necessitates a more rich model structure for the arrival intensity of unscheduled jumps in order to get a realistic representation of the futures curve. Furthermore, the need to have a significant unscheduled jump component is likely exacerbated by the relatively low frequency of scheduled jumps in the UK. Backwell and Hayes (2022) consider the quarterly meetings of the Bank of England’s Monetary Policy Committee (MPC) as opposed to the eight annual meetings of the FOMC. In contrast, our results thus suggest that a pure jump specification of the short rate can fit the term structure without including stochastic intensity or a state dependent jump size distribution of the unscheduled jumps (possibly because unscheduled jumps appear quite rarely). This

\(^{12}\)Specifically we adjust the observed and modelled rates by removing the accumulated fixings and rescaling the resulting rate for the remaining period to reflect an annual rate. E.g. for the nearest one-month contract with \(S < t < T\) we obtain

\[
\begin{align*}
f_{\text{adjusted}}^{1m}(t; S, T) &= \frac{T - S}{T - t} \left( f_{\text{1m}}^{1m}(t; S, T) - \frac{1}{T - S} \sum_{i=1}^{N_t} \frac{1}{360} R_{t_i} \right) \\
\end{align*}
\]

where \(S \leq t_1, ..., t_{N_t} \leq t\).
Figure 5: Absolute fitting errors for the nearest futures contracts during the recent Fed hiking cycle. The futures rates are adjusted to only reflect the implied futures rate for the remaining reference period of the contract. The vertical lines mark scheduled FOMC dates.

does not imply that unscheduled jumps are not part of the US overnight rate dynamic, as clearly seen from the recent COVID-19 crisis. However, the impact of the unscheduled and scheduled jumps on the term structure cannot be separately identified from futures contracts and the scheduled FOMC meetings are sufficiently frequent to allow for the scheduled jumps to capture the term structure of SOFR futures rates.

Unsurprisingly, and as seen by the RMSEs, correctly modeling the discontinuities becomes increasingly important the shorter the tenor of the considered rate due to the averaging of the overnight fixings present in longer tenors. Next, we therefore investigate the ability of each model to fit subsets of the daily observed rates. We start by removing the overnight rates from the sample, leaving only the one-month and three-month futures rates in the estimation. Subsequently, we also remove the one-month futures contracts, such that only the three-month futures contracts remain. After removing the overnight rates from the sample, we observe close to identical fits in the three-month futures rates between the models. However, the jump specification still significantly outperforms the continuous model in fitting the one-
month futures rate, particularly during the end of the sample. This becomes increasingly important if one estimates a model based on Federal Funds futures, since these contracts only exist with a one-month reference period. It suggests that even when the overnight fixings are not of concern, accurately modeling the stochastic discontinuities of the underlying overnight rate can significantly improve the fit of a model. Lastly, when our sample consists of only three-month SOFR futures rates we see a notable decrease in the RMSEs of the continuous specification, which then outperforms our jump specification. The tenor structure of this subsample is effectively similar to a sample of the Eurodollar futures that three-month SOFR futures are replacing, except for the nearest three-month SOFR futures contract, which due to the accumulation during the contract period differs from the nearest Eurodollar futures contract. Therefore, when the fixings are based on sufficiently long tenors, as is the case with contracts referencing LIBOR or three-month SOFR futures contracts, a continuous specification seems able to fit the term structure even if the underlying overnight rates are highly discontinuous.

6.4 Model Implied Term Rates and CME Term SOFR

As a part of the transition to SOFR, the ARRC expressed the need for a forward-looking term SOFR rate once sufficiently liquid derivative markets referencing SOFR to support such a rate had developed. In May 2021, the ARRC chose CME as the administrator of term SOFR. CME Term SOFR for tenors of 1, 3, and 6 month was accepted by the ARRC.

13This is not to say that the underlying rates are the same: The accumulated overnight rate underlying three-month SOFR futures is quite distinct from the LIBOR term rate underlying Eurodollar futures. For a discussion of this issue, see Backwell et al. (2023).


15Despite the name, Term SOFR does not reflect actual term lending. Rather, it is derived from the futures market. We refer to Backwell et al. (2023), Skov and Skovmand (2022), and Filipović and Trolle (2013) for a discussion on term rates.

16See https://www.newyorkfed.org/medialibrary/Microsites/arrc/files/2021/20210521-ARRC-Press-Release-Term-Rate-RFP.pdf
on July 29, 2021 and later on May 21, 2022 the 12 month tenor was also endorsed\textsuperscript{17}. With over three trillion USD in loans as of February 2023, a model should be able to produce forward-looking SOFR term rates consistent with those published by the CME\textsuperscript{18}. Using the historical record of CME Term SOFR rates since the approval by the ARRC as an out of sample benchmark, we evaluate the ability of each model to produce SOFR term rates consistent with those published by the CME. In order to accurately compare the model-implied term rates we follow the conventions used by CME when calculating Term SOFR. CME Term SOFR Rates are published on all overnight SOFR publishing dates at 5:00 am and based on CME SOFR futures rates from the previous trading day. The tenor starts on the second business day (included) after the publication day calculated using the Following business day convention. Next, the term is calculated for the relevant tenor using the Modified Following convention with an Actual/360 day count convention. The rates are based on the five and twelve nearest one- and three-month futures, respectively\textsuperscript{19}. Thus, denoting by $t$ the calculation date, $S$ the start date and $T$ the end date of the tenor, we calculate the model-implied term SOFR as

$$R^s_{\text{Term}}(t; S, T) = \frac{360}{T - S} \left( \frac{p^s(t, S)}{p^s(t, T)} - 1 \right).$$

(22)

where $p^s(t, T)$ is the SOFR zero coupon bond calculated as $\psi(0, t, T)$ in Appendix A. There are, however, still multiple reasons for the term rates to differ. First, the CME methodology computes futures rates from forward rates and thus ignores the futures convexity correction (see Skov and Skovmand (2021)). Second, CME term SOFR rates are based on discrete compounding whereas we use a continuously compounded approximation. Finally, CME Term SOFR uses aggregated futures prices between 7:00am and 2:00pm CT, while we estimate the models using end-of-day prices. Table IV reports the RMSEs between the model-implied and observed CME Term SOFR rates. As with the futures rates, we note a substantial improve-
Table IV: Out of sample CME Term SOFR RMSEs.

<table>
<thead>
<tr>
<th>Model</th>
<th>One-Month</th>
<th>Three-Month</th>
<th>Six-Month</th>
<th>Twelve-Month</th>
</tr>
</thead>
<tbody>
<tr>
<td>Jump</td>
<td>3.1</td>
<td>2.0</td>
<td>2.0</td>
<td>2.7</td>
</tr>
<tr>
<td>Continuous</td>
<td>9.3</td>
<td>3.7</td>
<td>2.2</td>
<td>2.4</td>
</tr>
</tbody>
</table>

The agreement (relative to the continuous model) in the fit of the one-month term rate. Already at the three-month tenor the improvement decreases notably, while the errors become similar in size at the six- and twelve-month tenors. The agreement between the jump model and the CME term rates is no surprise, as the calculation of CME Term SOFR is based on a step-function for the SOFR forward rates, which are allowed to jump on FOMC meeting dates as in Heitfield and Park (2019). The results further emphasize the need to include scheduled jumps to accurately capture the dynamics of the short end of the SOFR curve if CME Term SOFR continues to gain popularity as a forward-looking term rate benchmark.

### 6.5 Overnight Policy Rate Forecasts

In this section, we investigate the ability of each model specification to forecast future values of the IORB. In order to obtain real time estimates of the model-forecasted IORB fixings, we perform rolling re-estimations starting from January 3, 2022 to the end of our data period in February 2023. Each estimation is therefore based on at least three and a half years of daily data from June 2018 to January 2022. In the jump specification, we calculate the forecasted IORB value under the $\mathbb{P}$-measure as

$$
\mathbb{E}^P_t [r_u] = r_t + \sum_{j=\eta(t)}^{\eta(u)-1} \mathbb{E}^P_t \left[ J^P_{ij} \right] + \mathbb{E}^P_t \left[ \sum_{i=N^P_t-1}^{N^P_t-1} Z_i \right] \\
= r_t + \sum_{j=\eta(t)}^{\eta(u)-1} \left( \gamma^P + \Gamma^P \mathbb{E}^P_t \left[ X_{ij} \right] \right) + (u - t) \nu^P \mu^P
$$

(23)

While the focus of our study is on SOFR and SOFR futures, similar models estimated on EFFR and Federal Funds futures data would allow for much longer data samples. Also, the EFFR-IORB spread is less volatile compared to the SOFR-IORB spread, which might further improve on the accuracy of the forecasted IORB.
Figure 6: The figure compares the one-, three-, and six-month IORB forecasts for both the continuous and jump specification. The plot displays the time $t$ forecasts based on rolling re-estimations as well as the future realized value of the IORB.

with each $\mathbb{E}^P_t \left[ X^j_{\tau_j} \right]$ computed as in Equation (38), but under the $P$-dynamics.

In Figure 6, we plot the IORB forecasts against the actual subsequent realization of the policy rate. Focusing on the one-month IORB forecast, we note that this results in at most one FOMC meeting during the forecasting period. The forecasted IORB value for IORB in the jump specification is therefore still highly discontinuous with the forecasted IORB values jumping whenever a new FOMC meeting enters the forecasting period. When we observe an FOMC meeting during the forecasting period the forecasted IORB changes as the market expectations about the future FOMC decision changes, however, when there is no FOMC meeting during the forecasting period, the only change in the forecasted IORB is due to the very slight changes in the filtered $r_t$ value because of the rolling re-estimations. Also, with no FOMC meetings during the forecasting period the only difference between the current and forecasted value of IORB is due to the impact of the constant intensity, unscheduled jumps. These features of the jump specification result in very different short-term policy forecasts compared to the one-month IORB forecasts in the continuous model, which con-
Table V: IORB Forecast errors summarized as mean and root-mean-squared errors. All values are in basis points.

<table>
<thead>
<tr>
<th>Model</th>
<th>One-Month</th>
<th>Three-Month</th>
<th>Six-Month</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>RMSE</td>
<td>Mean</td>
</tr>
<tr>
<td>Jump</td>
<td>1.2</td>
<td>11.4</td>
<td>-28.0</td>
</tr>
<tr>
<td>Continuous</td>
<td>-6.6</td>
<td>21.4</td>
<td>-35.5</td>
</tr>
<tr>
<td>Random Walk</td>
<td>-34.0</td>
<td>47.4</td>
<td>-114.3</td>
</tr>
</tbody>
</table>

Table V lists summary statistics for the IORB forecast errors for both model specifications. As a reference we also include the static forecasts implied by a random walk model. Comparing the one-month forecast errors, we note that the jump specification greatly outperforms the continuous specification. At a three-month term, however, the relative increase in forecast performance is less evident, while at a six-month term the performance of both models significantly deteriorates, with the continuous version performing slightly better. The sizeable negative three- and six-month mean forecast errors reported are of course a result of the large and frequent policy rate hikes by the Fed during 2022, as also indicated by the even greater negative mean forecasting errors of the random walk model. The results clearly reflect the difficulty in obtaining accurate policy rate estimates beyond the very short term. However, in the short term the forecasting performance can be significantly improved by accurately incorporating the scheduled jumps following FOMC meetings.

### 6.6 Including SOFR Futures Options Data

In this section, we extend the data sample by including data on three-month SOFR futures options. As noted in Section 5, these contracts have only recently started to trade and our study is thus preliminary. Furthermore, neither of the considered models include stochastic
<table>
<thead>
<tr>
<th>Model</th>
<th>Three-Month Option</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1st</td>
</tr>
<tr>
<td>Jump</td>
<td>60.5</td>
</tr>
<tr>
<td>Continuous</td>
<td>16.7</td>
</tr>
</tbody>
</table>

Table VI: Black implied volatility fitting errors summarized as root-mean-squared errors. All values are in basis points.

Volatility and accurately capturing the option dynamics cannot be expected. Instead, the main purpose is to study the impact of scheduled jumps on implied volatilities.

Table 6.6 reports the RMSEs for the implied volatilities. These indicate a substantially better fit by the continuous model specification. Due to the expiration of the option before the three-month accumulation period, the three-month SOFR futures option greatly resembles the Eurodollar futures option it is supposed to replace. Thus, it is an option on a forward-looking three-month rate, the dynamics of which the continuous specification is able to capture. The results are consistent with those in Section 6.3, showing that the continuous specification performs well when a sample of only three-month futures contracts are considered. Furthermore, the additional Brownian motion allows for a more flexible volatility specification in the continuous model. However, based on our previous analysis, one may not expect the continuous model to be able to accurately model the one-month SOFR futures option. This is particularly true for the nearest contract since, as noted in Section 4, the one-month option expire at the end of the accumulation period. The constant volatility continuous model results in a linear decay in implied volatility during the accumulation period. As shown in Figure 7, the jump model, even without stochastic volatility, produces kinks in implied volatility around FOMC dates. Also, implied volatilities are dependent on the existence and timing of an FOMC meeting during the futures reference month. The validity of this behaviour can of course only be verified if these contracts start to trade in the market. The graph based on the estimation on the full data set suggests that the majority

---

An immediate way to include stochastic volatility in the jump model would be to allow for state dependence in the variance of the normally distributed scheduled jumps. However, we note that a part of the volatility also arises from the volatility of the processes determining the mean of the jump size distribution.
of the volatility is not due to the scheduled jumps, but rather the unscheduled jumps and the SOFR-IORB spread process. However, this result is largely due to the spikes observed at the beginning of the sample, which in the estimation result in substantial volatility for the spread process. Consequently, the contribution of scheduled jumps (i.e., FOMC meeting dates) to the time evolution of implied volatility of the one-month options is much more evident when the initial part of the sample is excluded from the model estimation. Arguably, this is a better reflection of market reality going forward, as policy action by the Fed seems to have prevented the recurrence of SOFR spikes in recent years.

7 Conclusion

The ongoing benchmark transition from LIBOR to Risk-Free Rates has made the overnight rate the main building block in fixed income products. Studying the changes in SOFR in relation to US policy rates, we argue that the primary policy rate affecting changes in
the level of SOFR is the IORB. Inspired by this, we develop a dynamic term structure model, which is able to jointly model IORB, SOFR and SOFR futures rates. Comparing our model to a standard time-homogeneous continuous affine term structure model, we find that accounting for scheduled jumps, identified by the FOMC meeting calendar, is necessary to accurately capture shorter-term interest rate dynamics. However, modeling the scheduled jumps becomes less important as the term of the rate is increased. In particular, when only three-month futures rates are considered, the diffusive dynamics of the continuous model are able to describe the time-variation in the cross-section of futures rates.

As the SOFR futures options market matures, an immediate addition for future research would be to extend our framework with stochastic volatility. This would be of particular interest if options on one-month SOFR futures start to trade more actively, as their shorter tenor and expiration date at the end of the underlying futures contract month lead a model incorporating scheduled jumps to predict a behaviour that is quite distinct from the behaviour predicted by a more traditional interest rate term structure model.

### A Affine Pricing of Futures with Scheduled Jumps

Pricing in the standard continuous affine setup is well-known and we refer to Bikbov and Chernov (2005) or Feldhütter et al. (2008) for pricing of Eurodollar futures and futures options and Skov and Skovmand (2022) for a treatment using SOFR futures. When computing the time \( t \) futures rates in the jump setup we will consider the auxiliary unscheduled jump process \( \hat{J}_s^P \) for \( s \geq t \) defined by

\[
d\hat{J}_s^P = dJ_s^P, \quad \hat{J}_t^P = 0.
\]

#### A.1 Overnight Rates

We calculate the model implied overnight IORB as the simple rate

\[
R(t, t + d) = \frac{1}{d} \left( \frac{1}{p(t, t + d)} - 1 \right)
\]
with \( d = \frac{1}{360} \) the day count fraction and \( p(t, t + d) = \mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} r(s)ds} \right] \), which we calculate as
\[
p(t, t + d) = e^{-dr_t} \mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} \sum_{j=\eta(t)}^{\eta(t)-1} J_{t_j}^P + \int_t^{t+d} J_{t}^P du} \right].
\] (26)

When computing the overnight rate, we assume that at the time of observation any scheduled jumps either have already occurred and thus are included in \( r_t \) or are scheduled at least one day ahead and therefore do not impact the overnight rate.\(^{22}\)

Focusing on the unscheduled jumps and recalling that \( \hat{J}_t^P = 0 \) we calculate
\[
\mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} r(s)ds + \int_t^{t+d} \hat{J}_t^P du} \right] = e^{\alpha^J(d)}
\]
where \( \alpha^J(\tau) \) solves the ODE
\[
\frac{\partial \alpha^J(\tau)}{\partial \tau} = \nu^P \left( e^{-\mu^P \tau + \frac{(\sigma^P)^2 \tau^2}{2}} - 1 \right), \quad \alpha^J(0) = 0.
\] (27)

Similarly, the SOFR fixings are defined as
\[
R^s(t, t + d) = \frac{1}{d} \left( \frac{1}{p^s(t, t + d)} - 1 \right)
\] (28)
where \( p^s(t, t + d) \) includes the SOFR-IORB spread process. From the independence of the spread process and unscheduled jumps we have (again assuming that the scheduled jumps do not affect the overnight tenor)
\[
p^s(t, t + d) = \mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} r_u + s_u du} \right] = e^{-dr_t} \mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} J_{t}^P du} \right] \mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} s_u du} \right].
\] (29)

It only remains to compute the contribution from the spread process, which is given by
\[
\mathbb{E}_t^Q \left[ e^{-\int_t^{t+d} s_u du} \right] = e^{\alpha^s(\tau) + \beta^s(\tau)s_t}
\]
where \( \alpha^s(\tau) \) and \( \beta^s(\tau) \) are as in the Vasicek model
\[
\alpha^s(\tau) = -\theta^s \tau + \frac{(\sigma^s)^2 \tau}{2(k^s)^2} + \frac{\theta^s}{k^s} \left( \frac{(\sigma^s)^2}{(k^s)^3} \right) \left( 1 - e^{-k^s \tau} \right) + \frac{\sigma^s (1 - e^{-2k^s \tau})}{4(k^s)^3}.
\] (31)

such that
\[
p^s(t, t + d) = e^{-dr_t} e^{\alpha^J(d) + \alpha^s(\tau) + \beta^s(\tau)s_t}
\] (32)

\(^{22}\)In reality this is always the case, since announced changes to the policy rate are effective at a one-day lag.
A.2 Three-Month Futures

We consider the continuously compounded approximation of the discrete compounding in Equation (17)

\[ f^{\text{sm}}(t; S, T) = \mathbb{E}_t^Q \left[ \frac{1}{T-S} \left( e^{\int_S^T r^*_d du} - 1 \right) \right]. \]  

(33)

By the independence of the scheduled jumps we can write

\[ \mathbb{E}_t^Q \left[ e^{\int_S^T r^*_d du} \right] = e^{(T-S)r_t} \mathbb{E}_t^Q \left[ e^{\int_S^T \sum_{j=q(t)}^{\eta(t)-1} J^D_j du} \right] \mathbb{E}_t^Q \left[ e^{\int_S^T J^P_{t+s} du} \right]. \]  

(34)

For the scheduled jumps we note that the integral can be rewritten as

\[ \int_S^{T-\eta_1^{(u)-1}} \sum_{j=\eta_1(t)}^{\eta(T)-1} J^D_j du = \sum_{j=\eta_1(t)}^{\eta(T)-1} \min(T-S, T-\tau_j) J^D_j. \]  

(35)

Repeated use of iterated expectations then yields

\[
\begin{align*}
\mathbb{E}_t^Q \left[ \sum_{j=\eta_1(t)}^{\eta(T)-1} \min(T-S,T-\tau_j) J^D_j \right] \\
= \mathbb{E}_t^Q \left[ \sum_{j=\eta_1(t)}^{\eta(T)-2} \min(T-S,T-\tau_j) J^D_j \mathbb{E}_{\tau(T)-1}^Q \left[ e^{\min(T-S,T-\tau_{\eta(T)-1}) J^D_{\eta(T)-1}} \right] \right] \\
= \mathbb{E}_t^Q \left[ \sum_{j=\eta_1(t)}^{\eta(T)-2} \min(T-S,T-\tau_j) J^D_j \mathbb{E}_{\tau(T)-1}^Q \left[ e^{\min(T-S,T-\tau_{\eta(T)-1}) (\gamma + \Gamma X^{j}_{\tau(T)-1}) + \frac{\min(T-S,T-\tau_{\eta(T)-1})^2}{2} \omega^2} \right] \right] \\
= \mathbb{E}_t^Q \left[ \sum_{j=\eta_1(t)}^{\eta(T)-3} \min(T-S,T-\tau_j) J^D_j \mathbb{E}_{\tau(T)-2}^Q \left[ e^{\min(T-S,T-\tau_{\eta(T)-1}) (\gamma + \Gamma X^{j}_{\tau(T)-2}) + \frac{\min(T-S,T-\tau_{\eta(T)-1})^2}{2} \omega^2} \right] \right] \\
= \mathbb{E}_t^Q \left[ e^{\min(T-S,T-\tau_{\eta(T)}) (\gamma + \Gamma X^{j}_{\tau(T)}) + \frac{\min(T-S,T-\tau_{\eta(T)})^2}{2} \omega^2} \right] \\
\times \mathbb{E}_{\tau(T)-1}^Q \left[ e^{\min(T-S,T-\tau_{\eta(T)+1}) (\gamma + \Gamma X^{j}_{\tau(T)+1}) + \frac{\min(T-S,T-\tau_{\eta(T)+1})^2}{2} \omega^2} \right] \ldots \\
= e^{\alpha^j(t;S,T)+\beta^j(t;S,T) X^j_t}
\end{align*}
\]  

(36)

Since \( X^j_t \) is Gaussian we have an analytical solution for the Laplace transform and can simply iterate through the expectations using the actual FOMC dates to get \( \alpha^j(t;S,T) \) and
The Gaussian conditional Laplace transform is

\[ \mathbb{E}_t^Q \left[ e^{a + b'X^j_t} \right] = e^a e^{b' \mathbb{E}_t^Q[X^j_t]} + \frac{1}{2} b' \mathbb{V}_t^Q[X^j_t] \]  

(37)

with mean

\[ \mathbb{E}_t^Q \left[ X^j_u \right] = \left( I - e^{-K^j(u-t)} \right) \theta^j + e^{-K^j(u-t)} X^j_t, \]  

(38)

and variance

\[ \mathbb{V}_t^Q \left[ X^j_u \right] = \int_t^u e^{-K^j(\nu-t)} \Sigma^j \Sigma^j \left( e^{-K^j(\nu-t)} \right)' d\nu. \]  

(39)

We calculate the variance using the analytical solution in Fisher and Gilles (1996). The unscheduled jumps and stochastic SOFR spread contribution can be calculated using the tower property and standard methods on affine jump diffusions from Duffie et al. (2000) as

\[ e^{\alpha_{A JD}(T-S) + \beta_{A JD}(T-S) \theta + \beta_{A JD}(S-T) \hat{J} \theta} \]

where \( \beta_{A JD}(\tau) \) and \( \alpha_{A JD}(\tau) \) solve the ODEs

\[ \frac{\partial \beta_{A JD}(\tau)}{\partial \tau} = -K^s \beta_{A JD}(\tau) + 1, \quad \beta_{A JD}(0) = 0, \]  

(40)

\[ \frac{\partial \alpha_{A JD}(\tau)}{\partial \tau} = K^s \theta^s \beta_{A JD}(\tau) + \frac{1}{2} \left( \sigma^s \beta_{A JD}(\tau) \right)^2 + \nu^P \left( e^{\nu^P \tau + \left( \sigma^P \tau \right)^2 / 2} - 1 \right), \quad \alpha_{A JD}(0) = 0. \]  

(41)

Next, we calculate

\[ e^{\alpha_{A JD}(T-S) + \beta_{A JD}(T-S) \theta + \beta_{A JD}(S-T) \hat{J} \theta} \]

where \( \tilde{\beta}_{A JD}(\tau) \) and \( \tilde{\alpha}_{A JD}(\tau) \) solve the ODEs

\[ \frac{\partial \tilde{\beta}_{A JD}(\tau)}{\partial \tau} = -K^s \tilde{\beta}_{A JD}(\tau), \]  

(43)

\[ \frac{\partial \tilde{\alpha}_{A JD}(\tau)}{\partial \tau} = K^s \theta^s \tilde{\beta}_{A JD}(\tau) + \frac{1}{2} \left( \sigma^s \tilde{\beta}_{A JD}(\tau) \right)^2 + \nu^P \left( e^{\nu^P \tau + \left( \sigma^P \tau \right)^2 / 2} - 1 \right) \]  

(44)

with initial conditions \( \tilde{\beta}_{A JD}(0) = \beta_{A JD}(T - S) \) and \( \tilde{\alpha}_{A JD}(0) = \alpha_{A JD}(T - S) \).
A.3 One-Month Futures

Again, we consider the common continuous approximation of the sum in Equation (16)

\[ f^{1m}(t; S, T) = \frac{1}{T - S} \mathbb{E}^Q_t \left[ \int_S^T r_u^s du \right]. \]  

(45)

We treat the scheduled jumps, unscheduled jumps, and stochastic spread separately

\[
\mathbb{E}^Q_t \left[ \int_S^T r_u^s du \right] = (T - S)r_t + \mathbb{E}^Q_t \left[ \int_S^T \sum_{j=\eta(t)}^{\eta(t)-1} J^D_{\tau_j} du \right] \\
+ \mathbb{E}^Q_t \left[ \int_S^T j_u^P du \right] + \mathbb{E}^Q_t \left[ \int_S^T s_u du \right].
\]

Applying Equation (35) and using iterated expectations we get

\[
\mathbb{E}^Q_t \left[ \int_S^T \sum_{j=\eta(t)}^{\eta(t)-1} J^D_{\tau_j} du \right] = \mathbb{E}^Q_t \left[ \sum_{j=\eta(t)}^{\eta(t)-1} \min(T - S, T - \tau_j) J^D_{\tau_j} \right] \\
= \mathbb{E}^Q_t \left[ \sum_{j=\eta(t)}^{\eta(T-1)} \min(T - S, T - \tau_j) \mathbb{E}^Q_{\tau_j} \left[ J^D_{\tau_j} \right] \right] \\
= \mathbb{E}^Q_t \left[ \sum_{j=\eta(t)}^{\eta(T-1)} \min(T - S, T - \tau_j) \left( \gamma + (\Gamma)X^j(\tau_{j-}) \right) \right] \\
= \sum_{j=\eta(t)}^{\eta(T-1)} \min(T - S, T - \tau_j)(\gamma + (\Gamma)\mathbb{E}^Q_t \left[ X^j(\tau_{j-}) \right]).
\]

(46)

Thus we only need to calculate mean of the underlying state variables given by (38). Since the unscheduled jumps are modelled as a compound Poisson process we get

\[
\int_S^T \mathbb{E}^Q_t \left[ j_u^P \right] du = \frac{1}{2} (T - S)^2 \nu^P \mu^P.
\]

(47)

And lastly the continuous spread is calculated as

\[
\int_S^T \mathbb{E}^Q_t \left[ s_u \right] du = (T - S)\theta^s + \frac{e^{-\kappa^s(S-t)} - e^{-\kappa^s(T-t)}}{\kappa^s}(s_t - \theta^s).
\]

(48)

A.4 Three-Month SOFR Futures Options

Having adjusted our options data for the American feature, we consider the time \( t \) value of a European call option on a three-month SOFR futures with reference quarter from time...
S to T. The option expires at time u with u < S. Specifically, for three-month quarterly
SOFR futures options traded on the CME the option expires on the last Friday before the
Wednesday when the reference quarter of the contract begins. Let \( k = 100 - K \) then

\[
\pi^3m(t; u, S, T) = E_t^Q \left[ e^{-\int_t^u r_z^m dz} (F^3m(u; S, T) - K)^+ \right] \\
\approx E_t^Q \left[ e^{-\int_t^u r_z^m dz} (k - 100f^3m(u; S, T))^+ \right] \\
= E_t^Q \left[ e^{-\int_t^u r_z^m dz} \left( k - 100 \left( \frac{1}{T-S} \mathbb{E}_u \left[ e^{\int_u^T r_z^m dz} \right] - \frac{1}{T-S} \right) \right)^+ \right].
\]  (49)

Where the approximation is from the continuously compounded approximation of the futures
rate. Let \( \alpha^3m(u; S, T) = \alpha^j(u; S, T) + \tilde{\alpha}^{AJD}(S-u) \) and \( \beta^3m(u; S, T) = (T-S, \beta^j(u; S, T), \tilde{\beta}^{AJD}(S-u))' \) then we can write (49) as

\[
\frac{100}{T-S} e^{\alpha^3m(u; S, T)} E_t^Q \left[ e^{-\int_t^u r_z^m dz} \left( \hat{k} - e^{\beta^3m(u; S, T)'} X_u \right) 1(\beta^3m(u; S, T) X_u < \log(k)) \right] \\
= \frac{100}{T-S} e^{\alpha^3m(u; S, T)} \left( \hat{k}G_{0, \beta^3m(u; S, T)}(\log(\hat{k})) - \hat{G}_{\beta^3m(u; S, T), \beta^3m(u; S, T)}(\log(\hat{k})) \right)
\]  (50)

with \( \hat{k} = e^{-\alpha^3m(u; S, T)} \left( \frac{k(T-S)}{100} + 1 \right) \) and

\[
G_{a,b}(y) = E_t^Q \left[ e^{-\int_t^u r_z^m dz} e^{a' X_u} 1(b' X_u < y) \right].
\]  (51)

We calculate \( G_{a,b}(y) \) using its Fourier transform

\[
\hat{G}_{a,b}(y) = \int_{\mathbb{R}} e^{iwy} dG_{a,b}(y) \\
= E_t^Q \left[ e^{-\int_t^u r_z^m dz} e^{a' X_u} \right] = \psi(a + iwb, t, u).
\]  (52)

Let \( a = (a^r, a^j, a^{AJD})' \) and \( b = (b^r, b^j, b^{AJD})' \) be the coefficients referring to the short-rate, scheduled jump factors and spread factor, respectively. We then calculate the transform as

\[
\psi(a + iwb, t, u) = E_t^Q \left[ e^{-\int_t^u r_z^m dz} e^{(a + iwb)' X_u} \right] \\
= E_t^Q \left[ e^{-(u-t)r_1 - \int_t^u \sum_{j=\eta(t)}^{\eta(z)-1} \mathcal{J}_{ij} dz - \int_t^u \mathcal{J}_z^P dz + s_z dz} e^{(a^r + iwb^r) r_u + (a^j + iwb^j)' X_u + (a^{AJD + iwb^{AJD}}) s_u} \right] \\
= e^{(a^r + iwb^r - (u-t)r_1)} E_t^Q \left[ e^{\sum_{j=\eta(t)}^{\eta(z)-1} (a^r + iwb^r - (u-t)\tau_j) \mathcal{J}_{ij} (a^j + iwb^j)' X_u} \right] \\
\times E_t^Q \left[ e^{-\int_t^u s_z dz + \mathcal{J}_z^P dz} e^{(a^{AJD + iwb^{AJD}}) s_u + (a^r + iwb^r) \mathcal{J}_u^P} \right].
\]  (53)
The scheduled jumps in (53) can be calculated by repeated use of iterated expectations using a similar approach to (36)

\[
E_t^Q \left[ e^{\sum_{n=1}^{q(n)} (a' + iwb' - (u - \tau_i)) J_n^P e^{(a' + iwb')' X_{s_n}^i}} \right]
\]

\[
= E_t^Q \left[ e^{(a' + iwb' - (u - \tau_{n(t)}))(\gamma + \Gamma X_{t_{n(t)}}^j)} + \frac{(a' + iwb' - (u - \tau_{n(t)}))^2}{2} \right]
\]

\[
\times E_{\tau_{n(t)}}^Q e^{(a' + iwb' - (u - \tau_{n(t)+1}))(\gamma + \Gamma X_{t_{n(t)+1}}^j)} + \frac{(a' + iwb' - (u - \tau_{n(t)+1}))^2}{2} \ldots
\]

\[
\times E_{\tau_{n(T)-2}}^Q e^{(a' + iwb' - (u - \tau_{n(T)-1}))(\gamma + \Gamma X_{t_{n(T)-1}}^j)} + \frac{(a' + iwb' - (u - \tau_{n(T)-1}))^2}{2} \ldots
\]

\[
\] (55)

The term in Equation (54) is a standard time-homogeneous affine jump diffusion, which we calculate as

\[
E_t^Q \left[ e^{- \int_{s_n}^u \psi \tau \beta^\psi d\tau e^{(a' + iwb')s_n + (a' + iwb')\beta^\psi}} \right] = e^{\alpha^\psi(u-t) + \beta^\psi(u-t)s_t}
\] (56)

where \(\alpha^\psi(\tau)\) and \(\beta^\psi(\tau)\) solve the ODEs

\[
\frac{\partial \beta^\psi(\tau)}{\partial \tau} = -\kappa^\psi \beta^\psi(\tau) - 1,
\] (57)

\[
\frac{\partial \alpha^\psi(\tau)}{\partial \tau} = \kappa^\psi \theta^\psi \beta^\psi(\tau) + \frac{1}{2} (\sigma^\psi \beta^\psi(\tau))^2 + \nu^P \left( e^{\mu^P (a' + iwb' - \tau)} + \frac{(a' + iwb')^2}{2} \right)
\] (58)

with initial conditions \(\alpha^\psi(0) = 0\) and \(\beta^\psi(0) = a^{AJD} + iwb^{AJD}\). From the Fourier inversion theorem we then obtain

\[
G_{a,b}(y) = \frac{\psi(a, t, u)}{2} - \frac{1}{\pi} \int_{0}^{\infty} \frac{\psi(a + iwb, t, u) e^{-iyw}}{w} dw.
\] (59)

During estimation we truncate the integral at 5000 and evaluate it with Gauss–Legendre quadrature using 50 points of the integral. We note that due to the time-in-homogeneity we get different \(\alpha^{3m}(u, S, T)\) and \(\beta^{3m}(u, S, T)\) for each futures contract that the options are referencing. However, we only consider options referencing the quarterly three-month SOFR futures contracts with fixed accumulations periods. This results in a manageable set of initial conditions for our estimation. Note that this is very different from something like a caplet where the accumulation period would change for each observation date and thus require new initial values for each observation date in the sample.
A.5 One-Month SOFR Futures Options

We again disregard the American feature of the one-month futures option and consider the time \( t \) value of a European call option on a one-month SOFR futures with reference month from time \( S \) to \( T \). The one-month option expires at the end of the accrual period of the underlying futures contract, i.e. at time \( T \)

\[
\pi^{1m}(t; S, T) = E_t^Q \left[ e^{-\int_t^T r^*_s dz} \left( F^{1m}(T; S, T) - K \right)^+ \right]
\]

\[
\approx E_t^Q \left[ e^{-\int_t^T r^*_s dz} \left( 100(1 - f^{1m}(T; S, T)) - K \right)^+ \right]
\]

\[
= \frac{100}{T - S} E_t^Q \left[ e^{-\int_t^T r^*_s dz} \left( k - \int_S^T r^*_s dz \right)^+ \right]
\]

\[
= \frac{100}{T - S} p^s(t, T) E_t^Q \left[ \left( k - \int_S^T r^*_s dz \right)^+ \right]. \quad (60)
\]

Where \( k = \frac{T - S}{100} (100 - K) \) and \( Q_T \) denotes the \( T \)-forward measure defined by the \( p^s(t, T) \) bond numeraire. The expectation in (60) can then be calculated using Theorem 4 in Filipović et al. (2017) as

\[
E_t^Q \left[ \left( k - \int_S^T r^*_s dz \right)^+ \right] = \frac{1}{\pi} \int_0^\infty \text{Re} \left( \frac{\hat{q}(h + i w)}{(h + i w)^2} \right) dw \quad (61)
\]

with \( \hat{q}(x) = E_t^{Q_T} \left[ e^{x(k - \int_S^T r^*_s dz)} \right] \) where \( x \in \mathbb{C} \) and \( h > 0 \) such that \( \hat{q}(h) < \infty \). In order to compute \( \hat{q}(x) \) we first change back to the risk-neutral measure

\[
\hat{q}(x) = E_t^{Q_T} \left[ e^{x(k - \int_S^T r^*_s dz)} \right] = \frac{1}{p^s(t, T)} E_t^Q \left[ e^{-\int_t^T r^*_s dz} e^{x(k - \int_S^T r^*_s dz)} \right] \quad (62)
\]

Note that the \( p^s(t, T) \) in the denominator cancels out when inserted in (60). Focusing on the nearest futures option with \( t \geq S \) we include the accumulated part of the rate in the modified strike as \( \hat{k} = k - \int_S^T r^*_s dz \) such that

\[
\hat{q}(x) = \frac{e^{x \hat{k}}}{p^s(t, T)} E_t^Q \left[ e^{-(1+x)\int_t^T r^*_s dz} \right]
\]

\[
= e^{-(T-t)(1+x)} r^*_T E_t^Q \left[ e^{\sum_{j=\eta(t)}^{(T-)} \left( -a_{(T)} + (1+x) \right) r^*_j} \right] E_t^Q \left[ e^{-(1+x)\int_t^T r^*_s + \hat{r}^*_T dz} \right]. \quad (63)
\]
Again, we calculate the first expectation as in (36)

\[
\mathbb{E}^Q_t \left[ e \sum_{j=\eta(t)}^{\eta(T)-1} -(u-\tau_j)(1+x)J_{ij} \right] \\
= \mathbb{E}^Q_t \left[ e^{-(u-\tau_{\eta(t)})(1+x)(\gamma+\Gamma' X_{\eta(t)}^j)} + \frac{(u-\tau_{\eta(t)})(1+x))^2}{2} \omega^2 \right] \\
\times \mathbb{E}^Q_{\eta(t)} \left[ e^{-(u-\tau_{\eta(t)+1})(1+x)(\gamma+\Gamma' X_{\eta(t)+1}^j)} + \frac{(u-\tau_{\eta(t)+1})(1+x))^2}{2} \omega^2 \right] \\
\times \mathbb{E}^Q_{\eta(T)-2} \left[ e^{-(u-\tau_{\eta(T)-1})(1+x)(\gamma+\Gamma' X_{\eta(T)-1}^j)} + \frac{(u-\tau_{\eta(T)-1})(1+x))^2}{2} \omega^2 \right] \ldots \right] 
\]

(64)

The second expectation is given by

\[
\mathbb{E}^Q_t \left[ e^{-(1+x) \int_t^T s_r dz} \right] = e^{\alpha q(T-t)+\beta q(T-t)s_t} \tag{65}
\]

where \(\alpha^q(\tau)\) and \(\beta^q(\tau)\) solve the ODEs

\[
\frac{\partial \beta^q(\tau)}{\partial \tau} = -\kappa^q \beta^q(\tau) - (1 + x), \tag{66}
\]

\[
\frac{\partial \alpha^q(\tau)}{\partial \tau} = \kappa^q \theta^q \beta^q(\tau) + \frac{1}{2} \left( \sigma^q \beta^q(\tau) \right)^2 + \nu^P \left( e^{\nu^P(-1+x)\tau} + \frac{(\sigma^q(-1+x)\tau))^2}{2} - 1 \right) \tag{67}
\]

with initial conditions \(\alpha^q(0) = \beta^q(0) = 0\). For longer dated options with \(t < S\), \(\hat{q}(x)\) can be computed using iterated expectations

\[
\hat{q}(x) = \frac{e^{xk}}{p^r(t,T)} E^Q_t \left[ e^{-\int_t^S r_z dz} E^Q_S \left[ e^{x(-1+x) \int_S^T r_z dz} \right] \right]. \tag{68}
\]

References


Electronic copy available at: https://ssrn.com/abstract=4431839