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Published in:
Reversible Computation - 16th International Conference, RC 2024, Proceedings

DOI:
10.1007/978-3-031-62076-8_9

Publication date:
2024

Document version
Early version, also known as pre-print

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Citation for published version (APA):
Jeopardy: An Invertible Functional Programming Language

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Abstract

Algorithms are ways of mapping problems to solutions. An algorithm is invertible precisely when this mapping is injective, such that the initial problem can be uniquely inferred from its solution.

While invertible algorithms can be described in general-purpose languages, no guarantees are generally made by such languages as regards invertibility, so ensuring invertibility requires additional (and often non-trivial) proof. On the other hand, while reversible programming languages guarantee that their programs are invertible by restricting the permissible operations to those which are locally invertible, writing programs in the reversible style can be cumbersome, and may differ significantly from conventional implementations even when the implemented algorithm is, in fact, invertible.

In this paper we introduce Jeopardy, a functional programming language that guarantees program invertibility without imposing local reversibility. In particular, Jeopardy allows the limited use of uninvertible – and even nondeterministic! – operations, provided that they are used in a way that can be statically determined to be invertible. To this end, we outline an implicitly available arguments analysis and three further approaches that can give a partial static guarantee to the (generally difficult) problem of guaranteeing invertibility.

1 Introduction

Reversible programming languages guarantee program invertibility by enforcing a strict syntactic discipline: programs are comprised only of parts which are themselves immediately invertible (locally invertible), and these parts can only be combined in ways which preserve invertibility. In this way, the global problem of ensuring the invertibility of an entire program is reduced to a local problem of ensuring the invertibility of its parts.

However, writing algorithms in the reversible style can be cumbersome: in a certain sense, it corresponds to requiring that programmers provide machine

∗Supported by DFF–International Postdoctoral Fellowship 0131-00025B.
checkable proofs that their algorithms are invertible. As such, writing programs in these languages requires some experience, and can in some cases be notoriously hard. To mitigate the problem, this work investigates a more relaxed approach to reversible language design that requires only (global) invertibility.

We present the language Jeopardy; a functional language bearing syntactic resemblance to you garden variety functional programming language and which exhibits the expected semantics for programs running in the conventional direction. However, in order to support program inversion for a particular class of morally reversible programs that fail the syntactic condition of reversibility, we also seek to extend the semantics of Jeopardy to be relational in a conservative way. For example, consider the following program and its (manually implemented) inverse:

\[
\text{swap } p = \text{swap}^{-\text{inverse}} (b, a) = \\
\quad \text{let } a = \text{first } p \text{ in } \text{let } p = (\text{invert second}) \ b \text{ in } \\
\quad \text{let } b = \text{second } p \text{ in } \text{let } p = (\text{invert first}) \ a \text{ in } \\
\quad (b, a). p.
\]

Not all parts of \text{swap} are invertible: for instance, \text{first} is not invertible at all. Nonetheless, \text{swap} clearly describes an invertible algorithm, as all of the information needed to reconstruct its input is contained in its output.

To strengthen our intuition about why \text{swap} is invertible, let us inspect the possible ways of implementing \text{first} or \text{second}. Both have to throw some information away, like so:

\[
\text{first } (a, _) = a. \quad \text{first}^{-\text{inverse}} a = (a, _). \\
\text{second } (_, b) = b. \quad \text{second}^{-\text{inverse}} b = (_, b).
\]

Because of this explicit deletion of data, \text{first} and \text{second} are not information preserving transformations and it is this data loss that makes \text{first} and \text{second} non-invertible. However, when considered together as above, we see that the "open" part of their inverse function outputs (i.e., the underscore _ which can be thought of as a unification variable) can always be unified with a "closed" term (not containing such) from the other function.

Though there is a great deal of overlap between reversibility and invertibility, the notion of reversibility, as known from reversible computations, is philosophically distinct from the current work. In particular, we do not ask that programs are implemented only using small locally invertible parts, and can as such be cleanly mapped to a reversible low-level abstract machine or reversible hardware.

**Related work** Program inversion \[18, 3\] concerns the automatic synthesis of the inverse to a given program (if such an inverse exists), while inverse execution seeks to interpret inverse programs from forward programs directly. Like compilation and interpretation, the two are connected by a Futamura projection \[1, 4\]. Reversible programming \[2, 26, 10, 24, 8, 11, 23\], program inversion, and inverse execution have seen applications in areas as diverse as debugging \[5, 15\], high-performance simulation \[21, 20\], quantum computing \[7, 6\]), and robotics \[22, 16\], and is intimately connected reversible model of computation \[14, 2, 9\].
Structure  In the following section (Section ??) we introduce Jeopardy and its syntax. Section 2 will detail the reversible semantics, which includes rules for both forward and backward interpretation, while Section 3 suggests various strategies for conservatively relaxing the reversible semantics. Finally in Section 4 we conclude on what we have learned thus far.

An implementation of Jeopardy can be found at [12].

Jeopardy is a minimalistic first order functional language with user-definable algebraic data types and inverse function invocation. The latter is invoked by the special keyword `invert`. The syntax for algebraic datatype declaration differs slightly from the norm, in that a sum of products has to be declared using the keyword `data` rather than denoted directly in the program using the symbol `+`. This may seem odd to some theoretic computer scientists, but is common notation among programmers. The full grammar can be found in Figure 1.

To clarify, a pattern `p` is either a variable, or a constructor applied to (possibly 0) other patterns. A value `v` is a pattern that does not contain any variables. A program `Δ`, is a list of mutually recursive function and datatype definitions, followed by a main function declaration. Functions are described by a name `f`, an input pattern, two type annotations (one for input and one for output), and a term `t` describing the functions body. A term is either a pattern, an application, or a case statement that branches execution. Application is special, because the operator may be a function symbol, denoting conventional application, or “`invert`” of a function symbol, calling its inverse function from the corresponding inverted program.

Running a program in the conventional direction corresponds to calling the declared main function on a value provided by the caller in an empty context. Similarly, running a program backwards corresponds to calling the main function’s inverse on said value. Since function application is a term, reasoning

$\begin{align*}
x & \in \text{Name} \quad \text{(Well-formed variable names).} \\
c & \in \text{Name} \quad \text{(Well-formed constructor names).} \\
\tau & \in \text{Name} \quad \text{(Well-formed datatype names).} \\
f & \in \text{Name} \quad \text{(Well-formed function names).} \\
p & ::= [c \, p_i] \mid x \quad \text{(Patterns).} \\
v & ::= [c \, v_i] \quad \text{(Values).} \\
\Delta ::= f \ (p : \tau_p) : \tau_l = t \ . \ \Delta \quad \text{(Function definition).} \\
| \text{data } \tau = [c \, \tau_j] \ . \ \Delta \quad \text{(Data type definition).} \\
| \text{main } g . \quad \text{(Main function declaration).} \\
g ::= f \mid (\text{invert } g) \quad \text{(Function).} \\
t ::= p \quad \text{(Patterns in terms).} \\
| g \ p \quad \text{(Function application).} \\
| \text{case } t : \tau \ of \ p_i \to t_i \quad \text{(Case statement).}
\end{align*}$
about programs is reasoning about terms; as such, we will focus on terms from here on out. The syntax of terms has been designed to be small in order to make reasoning easier, at the cost of making programs harder to read and write. In the interest of writing intuitive program examples, we will use a couple of derived syntactic connectives that depend on the program $\Delta$ in which they are written, shown in Figure 2.

1.1 Examples

In order to motivate the need for an invertible functional programming language, the following section compares programs written in the reversible style, to show that Jeopardy programs can get much closer to a conventional way of writing these programs. Suppose that we declared two datatypes $\text{nat}$ and $\text{pair}$ as follows:

\[
\text{data nat} = \text{[zero]} \ [\text{suc nat}]. \\
\text{data pair} = \text{[pair nat nat]}.
\]

It is not uncommon to want to be able to add a pair of numbers, so we write an algorithm to do just that:

\[
\begin{align*}
\text{add} (\text{[zero]}, n) &= n \\
\text{add} (\text{[suc k]}, n) &= \text{add} (k, \text{[suc n]}).
\end{align*}
\]

Our algorithm $\text{add}$ is not invertible, because it does not describe a bijective function. To be precise, $\text{add}$ is not injective, since

\[
\text{add} (\text{[suc [zero]], [zero]}) = \text{add} (\text{[zero]}, \text{[suc [zero]]})
\]

even though

\[
(\text{[suc [zero]], [zero]}) \neq (\text{[zero]}, \text{[suc [zero]]})
\]

However, since $\text{add}$ is linearly typeable, the corresponding RFun and CoreFun programs will evaluate to a runtime error (as they should) upon calling $\text{add}$ with an $m$ that differs from $\text{[zero]}$, as the result will not be syntactically orthogonal to the variable pattern $n$.

Now, suppose the caller of $\text{add}$ happens to know the value of $m$ in advance; meaning that $m$ will be available in the future from the perspective of the inverse program. Then the case to use for inverse interpretation suddenly become
unambiguous: either \( m \) is \([\text{zero}]\) and we can unambiguously choose the first case, or \( m \) is \([\text{suc k}]\), the caller now knows \( k \), and can deterministically uncall \( \text{add} \) recursively in the second case. Therefore, the corresponding Jeopardy program will allow the programmer to call \( \text{add} \) in a context that knows about \( m \) as exemplified below:

\[
\begin{align*}
\text{fibber} \ (m, n) &= (\text{add} \ (m, n), m). \\
\text{fib-pair} \ [\text{zero}] &= ([\text{suc} \ [\text{zero}]], [\text{suc} \ [\text{zero}]]).
\end{align*}
\]

\[
\text{fib-pair} \ [\text{suc} \ k] = \text{fibber} \ (\text{fib-pair} \ k).
\]

\[
\text{fib} \ n = (n, (\text{first} \ (\text{fib-pair} \ n))).
\]

main \( \text{fib} \).

Here, the function \( \text{fib} \) computes the pair containing \( n \) and the \( n \)'th Fibonacci number. Notice that it will never be possible to write an invertible implementation of the Fibonacci function that does not include something extra in its output, since the first couple of outputs has to be \([\text{suc} \ [\text{zero}]])\) for two different inputs. The helper function \( \text{fib-pair} \) becomes deterministic by the same "trick" as explained for \( \text{add} \). Additionally, notice that even though the \( n \) is part of the output of \( \text{fib} \), it is insufficient to uncompute \( \text{fib} \) by projecting out the input argument, since we agreed already, that \( \text{first} \) is not invertible. So, either we have to check that the second part of the output indeed does compute from \( n \) in the conventional direction, or we need to infer a unique environment in which the output was computed as we will discuss in Section 2.

Before moving on to doing so, we will further our intuition about what can be decided about invertible programs by considering the example of implementing \( \text{map} \) which has been specialised\(^1\) to apply a specific function \( f \):

\[
\begin{align*}
\text{data list} &= [\text{nil}] \ [\text{cons} \ \text{nat list}]. \\
\text{data pair} &= [\text{pair} \ \text{list list}].
\end{align*}
\]

\[
\begin{align*}
\text{reverse} \ ([], ys) &= ys. \\
\text{reverse} \ (y : xs, ys) &= \text{reverse} \ (xs, y : ys).
\end{align*}
\]

\[
f \ldots = \ldots
\]

\[
\begin{align*}
\text{map-f-iter} \ ([], ys) &= \text{reverse} \ (ys, []). \\
\text{map-f-iter} \ (x : xs, ys) &= \text{map-f} \ (xs, f \ x : ys).
\end{align*}
\]

\[
\text{map-f} \ xs = \text{map-f-iter} \ (xs, []).
\]

main \( \text{map-f} \).

This definition of \( \text{map-f} \) is clearly invertible: As \( \text{map-f-iter} \) and \( \text{reverse} \) both move elements from the first to the second component of their input, as such, the first component always becomes smaller, so both algorithms terminate in their respective first cases. Moreover their second component is always initially

\(^1\)The problem of extending an invertible (or even a reversible) programming language with (real) higher order functions will be worthy of its own paper.
empty. Consequently, uncalling \texttt{map-f} will have to uncall \texttt{map-f-iter} (and hence also \texttt{reverse}) with an empty list as the second component.

However, for an interpreter to make this conclusion requires a non-trivial program analysis which cannot necessarily be performed inductively over the syntax of the program in general. In this particular case, the core difference between \texttt{fibber} and \texttt{map-f}, is that the context in which \texttt{fibber} calls \texttt{add}, provides information about the variable that \texttt{add} is branching on, making reverse pattern matching deterministic. \texttt{map-f-iter} on the other hand, is branching on \texttt{xs}, about which \texttt{map-f} does not syntactically provide any information about. However, the cases do become syntactically orthogonal when elaborating on the variable that \texttt{map-f} does provide information about:

\begin{align*}
\texttt{map-f-iter} ([], ys) &= \texttt{reverse} (ys, []). \\
\texttt{map-f-iter} (x : xs, y : ys) &= \texttt{map-f} (xs, f x : y : ys). \\
\texttt{map-f-iter} (x : xs, []) &= \texttt{map-f} (xs, f x : []).
\end{align*}

Additionally, unrolling \texttt{map-f-iter} yielded a particularly interesting result, because it suggests two unique entry and exit conditions for a Janus style loop:

Suggesting a way of obtaining tail-recursion optimisation for functional reversible programs that look like tail recursive functions in a conventional programming style. Just like reversible higher-order functions, this has been left as future work.

## 2 Reversible semantics

Aside from the special keyword \texttt{invert}, which invokes inverse interpretation of functions, Jeopardy is purposefully limited to garden variety syntactic constructs. It is desired that these constructs mean the conventional thing when interpreted in the conventional direction, and that more exotic constructions such as RFun’s \texttt{rlet}-statement, can be derived by combining \texttt{invert} with the case-statement and function application. It is likewise desired that such constructions mean the expected thing when inversely interpreted. In this section we present a reversible operational semantics for Jeopardy, in which all function definitions must be locally invertible.

The goal of this exercise is to ensure that invertible algorithms written in the reversible style, are invertible in the same sense as that of corresponding programs, written in languages that force programmers to formulate their algorithms in this way – In Section 3 we proceed to explaining how the reversible semantics can be relaxed to only require global invertibility. That is, to require a programs main function to be invertible, but not necessarily any other functions.
This reversible semantics is inspired by those of RFun and CoreFun, and the main difference is a separation of concerns. For instance, the judgement rules of RFun can be read in two different ways. First, when read in the conventional direction, a term is evaluated to a value in an environment. Second, in the other direction, a resulting value is used to search a term for the unique environment in which that term would have yielded that particular result. In this regard, the semantics of Jeopardy programs are operationalised by four mutually recursive judgements; Figures 3 and 4 show the judgements for interpretation, with and against the conventional direction, while figures 5 and 6 describe an algorithm for inferring unique environments under which linear terms (and inverted linear terms) evaluated to their canonical forms. Likewise, type checking has been factored out into four mutually recursive judgements. Figures 7 and 8 about linear typing (and inverse linear typing), and Figures 9 and 10 about inferring the unique environment in which a linear term is typeable.

The motivation for producing explicit operational semantics for interpretation in both directions, is to enable more fine grained program analysis, such as the ones outlined in Section 3. The remainder of this section will cover the meaning of each judgement of the reversible semantics, and towards the end of this section, we provide the meta theoretic properties that this semantics guarantee local invertibility on functions. To start somewhere, programs conventionally run from “top to bottom”. The corresponding judgement rules are therefore denoted by a downwards pointing arrow, and the form is \[ \Delta \Gamma \vdash t \downarrow v \], where \( \Delta \) is a copy of the program text, \( \Gamma \) is a mapping between variable names and values, and the judgement reads “in \( \Delta \), \( \Gamma \) stands witness that \( t \) evaluates to \( v \) in the conventional direction”.

1. The rules \( \downarrow \text{Variable}, \downarrow \text{Constructor} \) and \( \downarrow \text{Application} \) are the usual rules for looking up variables and applying first order functions. \( \text{unify/2} \) is the most general unifier as usual as well.

2. The \( \downarrow \text{Cases} \) rule says that when the selector term \( t \) evaluates to a value \( v_i \), and \( v_i \) unifies with the \( i \)th pattern \( p_i \), and the \( i \)th term \( t_i \) evaluates to some value \( v \) under the bindings from the unification, then the whole term evaluates to \( v \). The side condition \( \psi \) is an abbreviation for the bidirectional first-match policy, namely that we require \( p_j \) not to unify with \( v_i \) whenever \( j \neq i \) holds, and that \( \Delta[\llbracket t_j \downarrow v_i \rrbracket] \not\rightarrow \Gamma \) should not hold for all such \( j \) either.

3. The \( \uparrow \text{Inversion} \) rule invokes inverse interpretation for function application, which can be seen in Figure 4.

Since the rules for inverse interpretation run in the opposite direction of the conventional one, their names have been annotated with an arrow pointing upwards. Furthermore, since the keyword \texttt{invert} can only appear in an application, there are only two rules.

4. The \( \uparrow \text{Inversion} \) rule says that inverting inverse interpretation is to resume computation in the conventional direction; Not much to see here.

5. Finally, in the \( \uparrow \text{Application} \) rule, the looming problem of inverse interpretation of functional programs emerges from hiding. The rule says that if the argument to an inverse function evaluated to some value \( v' \), and the
\[ \Delta \Gamma \vdash t \downarrow v \quad (\text{for } t \text{ closed under } \Gamma) \]

\[
\vcenter{\begin{array}{ll}
\text{Variable :} & \Delta \Gamma \vdash x \downarrow v \quad (\Gamma(x) = v) \\
\text{Constructor :} & \Delta \Gamma \vdash [c \ p_i] \downarrow [c \ v_i]
\end{array}}
\]

\[
\vcenter{\begin{array}{ll}
\text{Cases :} & \Delta \Gamma \vdash t \downarrow v_i \\
& \Delta \Gamma \vdash \text{case } t : \tau \text{ of } p_i \rightarrow t_i \downarrow v \\
\end{array}}
\]

\[
\vcenter{\begin{array}{ll}
\text{Application :} & \Delta \Gamma \vdash p \downarrow v' \quad \Delta(\text{unify}(v_i, p_i)) \vdash t_i \downarrow v \\
& \Delta \Gamma \vdash f(p' : \cdot) : t' \quad \Delta \Gamma \vdash \text{inversion } g p \downarrow v \\
\end{array}}
\]

Figure 3: Interpretation in the conventional direction. The side condition \((\psi)\) denotes the bidirectional first match policy as detailed in bullet point \([2]\).

\[ \Delta \Gamma \vdash t \uparrow v \quad (\text{for } t \text{ closed under } \Gamma) \]

\[
\vcenter{\begin{array}{ll}
\text{↑Application :} & \Delta \Gamma \vdash p \downarrow v' \quad \Delta(\text{unify}(v', p')) \vdash t' \downarrow v \\
& \Delta \Gamma \vdash [f(p' : \cdot) : t'] \quad \Delta \Gamma \vdash \text{inversion } g p \downarrow v \\
\end{array}}
\]

Figure 4: Inverse interpretation for linear programs.

body of the corresponding function in the source program evaluated to \(v'\) in the conventional direction because of a unique environment \(\Gamma'\), then the result of the inverse function is the arguments for the function in the source program as evaluated under \(\Gamma''\).

We can hide the problem of searching for \(\Gamma'\) in the rules inverse interpretation. But in the interest of separating concerns, we have separate rules about searching for context. The form is \(\Delta[t \downarrow v] \rightsquigarrow \Gamma\) and it reads “in \(\Delta\), the linear term \(t\) evaluated to \(v\) because of the unique environment \(\Gamma'\)”, and the details can be found in Figure \([5]\).

6. The \(\rightarrow\)Variable rule says that if \(x\) was a linear term, and it evaluated to \(v\),

\[ \Delta[t \downarrow v] \rightsquigarrow \Gamma \quad (\text{for linear terms } t) \]

\[
\vcenter{\begin{array}{ll}
\rightarrow\text{Variable :} & \Delta[x \downarrow v] \rightsquigarrow \{x \mapsto v\} \\
\rightarrow\text{Constructor :} & \Delta[p_i \downarrow v_i] \rightsquigarrow \Gamma_i \\
\rightarrow\text{Cases :} & \Delta[t_i \downarrow v_i] \rightsquigarrow \Gamma_i \\
& \Delta \Gamma_i \vdash p_i \downarrow v \\
& \Delta[t \downarrow v] \rightsquigarrow \Gamma \\
\rightarrow\text{Application :} & \Delta[v' \downarrow v'] \rightsquigarrow \Gamma' \\
& \Delta(\text{unify}(v', p')) \vdash t' \downarrow v \\
& \Delta \Gamma' \vdash p' \downarrow v' \\
& \Delta \Gamma' \vdash [f(p' : \cdot) : t'] \rightsquigarrow \Gamma \\
\rightarrow\text{Inversion :} & \Delta \Gamma \vdash \text{inversion } g p \downarrow v \\
\end{array}}
\]

Figure 5: Environment inference for linear programs. Again, the side condition \((\psi)\) denotes the bidirectional first match policy as noted in bullet point \([5]\).
\[ \Delta[t \mapsto v] \rightsquigarrow \Gamma \] (for linear terms \( t \))

\[ \begin{align*}
\text{Application: } & \Delta[p' \downarrow v] \rightsquigarrow \Gamma' \quad \Delta \Gamma' \vdash t' \downarrow v' \quad \Delta[p \downarrow v'] \rightsquigarrow \Gamma \\
\Delta[f \mapsto p'] &= \Delta'[f \mapsto p'] \\
\Delta'[g \mapsto p \downarrow v] &= \Delta[\text{invert } g] p \downarrow v' \rightsquigarrow \Gamma
\end{align*} \]

\[ \begin{align*}
\text{Inversion: } & \Delta[\text{invert } g] p \downarrow v' \rightsquigarrow \Gamma \\
& \Delta[p \mapsto v] \rightsquigarrow \Gamma
\end{align*} \]

Figure 6: Inverse environment inference.

\[ \Delta \Sigma \vdash t : \tau \]

\[ \begin{align*}
\text{Variable: } & \Delta \{ x \mapsto \tau \} \vdash x : \tau \\
\text{Constructor: } & \Delta \Sigma_i \vdash p_i : \tau_i \\
\text{Cases: } & \Delta \Sigma \vdash t : \tau \\
& \Delta[p_i] : [c \pi_1] \circ \Sigma_i \vdash \Sigma_i \circ \Sigma_j \vdash t_i : \tau_i' \\
& \Delta \Sigma \vdash \text{data } \tau = [c \pi_1] \circ \Sigma_i \circ \Sigma_j \\
& \text{case } t : \tau \text{ of } p_i \mapsto t_i : \tau_i' \\
\text{Application: } & \Delta \Sigma \vdash p' : \tau_p \\
& \Delta[p] : \tau_p \circ \Sigma_p \vdash \Sigma_p \vdash t : \tau_i \\
& \Delta[f \mapsto p'] : \tau_i = f \Sigma_i \vdash f p' : \tau_i \\
\text{Inversion: } & \Delta \Sigma \vdash g p : \tau \\
& \Delta \Sigma \vdash \text{invert } g p \downarrow v' \rightsquigarrow \Gamma
\end{align*} \]

Figure 7: Linear typing.

it must have been because of the unique environment, containing a single binding \( x \mapsto v \).

7. The \( \rightarrow \)Constructor rule says that the unique environment under which a constructor evaluated to a value is the composition of the unique (and disjoint by linearity) environments under which its parts evaluated.

8. The \( \rightarrow \)Cases rule, still requires the bidirectional first match policy \( \psi \). See Cases (bullet point (2)).

9. The \( \rightarrow \)Application and \( \rightarrow \)Inversion rules can be found in Figure 6.

Because the rules for environment inference require linearity, we have given typing rules that are usual for linear typing: [25, 10]. The main judgement form is \( \Delta \Sigma \vdash t : \tau \) and it reads, “in the program \( \Delta \), the term \( t \) has type \( \tau \) under \( \Sigma \)”, where \( \Sigma \) is a mapping between variable names and type names. Moreover, just like the rules for interpretation, typing has a typing environment inference algorithm with the form \( \Delta[t : \tau \downarrow \Sigma] \) which reads “In the program \( \Delta \), we know that the linear term \( t \) has type \( \tau \) because of the unique typing environment \( \Sigma \)”.

As mentioned in Section ??, running a program corresponds to applying its main function to a value provided by the caller in an empty context. So, the desirable property for programs to have, is that this application yields a unique result, and that calling the inverted program on the result will yield said provided input. This property has been summarised in Theorems 1 and 2 and the nifty Corollary 2.1.

**Theorem 1.** If \( t \) is a linear term then \( \Delta \Gamma \vdash t \downarrow v \) if and only if \( \Delta[t \downarrow v] \rightsquigarrow \Gamma \).
Figure 8: Inverse linear typing.

Figure 9: Typing environment inference.

Figure 10: Inverse typing environment inference.
Proof outline. By induction on the derivations $\mathcal{D}$ of $\Delta \Gamma \vdash t \downarrow v$ and $\mathcal{C}$ of $\Delta[t \downarrow v] \rightsquigarrow \Gamma$ respectively. Here we give the case for function application:

- Suppose $t$ is a function applied in the conventional direction. Then $t$ looks like $f \, p$, and $\mathcal{D}$ is a derivation of $\Delta \Gamma \vdash f \, p \downarrow v$, and so it must have used the $\downarrow$ Application rule. As such, $\mathcal{D}$ must be constructed from a derivation $\mathcal{D}_1$ of $\Delta \Gamma \vdash p \downarrow v'$ and another derivation $\mathcal{D}_2$ of $\Delta(\text{unify}(v', p')) \vdash t' \downarrow v$ where $p'$ and $t'$ are the argument pattern and function body of $f$ as defined in $\Delta$. Furthermore, by the definition of $\text{unify}(v', p')$ and the fact that $v'$ is a value (and thereby variable-free): If $p'$ is a variable, then $\Delta(\text{unify}(v', p')) \vdash p' \downarrow v'$ holds by the $\downarrow$ Variable rule, and otherwise, $p'$ is a constructor and we can use the $\downarrow$ Constructor rule to obtain the same proof. In either case, we can construct a derivation $\mathcal{D}_3$ of $\Delta(\text{unify}(v', p')) \vdash p' \downarrow v'$.

Now, by the induction hypothesis on $\mathcal{D}_1$, we get a derivation $\mathcal{C}_1$ of $\Delta[p \downarrow v'] \rightsquigarrow \Gamma$, and by the induction hypothesis on $\mathcal{D}_2$, we get a derivation $\mathcal{C}_2$ of $\Delta[t' \downarrow v] \rightsquigarrow \text{unify}(v', p')$. And finally, we can apply the $\rightsquigarrow$ Application rule to $\mathcal{C}_2, \mathcal{D}_3$ and $\mathcal{C}_1$ respectively we obtain a derivation $\mathcal{C}$ of $\Delta[f \, p \downarrow v] \rightsquigarrow \Gamma$.

Conversely, we can throw away $\mathcal{D}_3$ and use the induction hypothesis on $\mathcal{C}_1$ and $\mathcal{C}_2$ to reconstruct $\mathcal{D}_1$ and $\mathcal{D}_2$, which we may use reconstruct $\mathcal{D}$ and we are done.

The difficult bit is to show uniqueness of evaluation for $\downarrow$ Cases, by unfolding the meaning of the bi-directional first match policy $\psi$ specified in bullet point 2.

\[ \text{Theorem 2. Let } \Delta[f(p : \tau_p) : \tau_t] = t, \text{ be a program in which a function } f \text{ has been declared, and consider two values } v \text{ and } w \text{ such that } \Delta \emptyset \vdash v : \tau_p \text{ and } \Delta \emptyset \vdash w : \tau_t \text{ holds. Then } \Delta \emptyset \vdash f \downarrow \downarrow v \text{ if and only if } \Delta \emptyset \vdash f \downarrow \uparrow v. \]

Proof. Suppose $\Delta \emptyset \vdash f \downarrow \downarrow v$, then the derivation must have used the $\downarrow$ Application rule. Consequently, we get a derivation derivation of $\Delta(\text{unify}(v, p)) \vdash t \downarrow w$. Now, by Theorem 1, we get another derivation of $\Delta[t \downarrow w] \rightsquigarrow \text{unify}(v, p)$, and from the definition of the most general unifier, we derive $\text{unify}(v, p) \vdash p \downarrow v$. Since $w$ is a value, clearly $\Delta \emptyset \vdash w \downarrow w$. So, we can apply the $\uparrow$ Application rule to show that $\Delta \emptyset \vdash f \downarrow v$. The converse proof is similar.

From this theorem and the $\downarrow$ Invert rule follows that inversion is well-behaved:

\[ \text{Corollary 2.1. Let } \Delta \text{ be a program, and } f \text{ an arbitrary function defined in } \Delta. \text{ Then } \Delta \Gamma \vdash f \downarrow \downarrow v \text{ if and only if } \Delta \Gamma \vdash (\text{invert } f) \downarrow \downarrow v. \]

3 Implementing Invertible Semantics

So far we have investigated several competing ideas for implementing the invertible semantics of Jeopardy. We have seen in Section 1.3 that in some cases, it is sufficient to extend the bi-directional first match policy of RFun and CoreFun to include information that is implicitly provided by the caller. We have
developed a program analysis, based on the available expressions analysis specified in Nielsen, Nelson, and Hankin [19], called implicitly available arguments analysis which was presented at NIK 2022 [13]. Based on this analysis, one can extend the judgements for evaluating programs with a statically available environment that contains bindings provided by the caller, and the side condition in Figures 3 and 5 will be an extended notion of orthogonality that is allowed to look at the pattern in each case as well. Additionally, available implicit arguments analysis leads the way for a number of program transformations that compile away certain branching constructions in which branching symmetry is not locally decidable (e.g., by syntactic orthogonality). The information provided by this particular analysis can also be used to prune the search space of algorithms, such as the one found in Figure 5; we also conjecture that online partial evaluation can be used to eliminate branches from a case-statement that do not agree with the implicitly provided arguments in the opposite direction of interpretation. An extended type system could also use available expressions to generate a set of constraints in order to conservatively verify that the program complies with this extended notion of term-pattern orthogonality, and that $\psi$ does not need to be checked at runtime in such cases.

The caveat of this kind of analysis is that it is syntax directed, and as seen with the map-f example in Section 1.1, this is not always sufficient. Several strategies are available to us to extend this:

- We can generalise the program transformation employed for map-f-iter in Section 1.1 that generalises the case where the caller provides information about something that the callee does not explicitly branch over.
- We allow free existential variables (unification variables) that we know we can find by unification some time in the future (before it is needed). That is, functions behave like Horn clauses which are known (by a separate program analysis) to succeed exactly once, that guarantees all existential variables are bound (i.e., constrained to be equal to a ground term) by the time they return.
- Instead of a syntax directed analysis, we can instead produce a graph structure in the style of [17]. Using such an analysis, we can enforce that invertible functions terminate in a unique case for each possible constructor in the data type definition for its input. One could even construct a list of graphs from elaborated Jeopardy programs that give names to the parts of the program that are not bound. These graphs could then be used as a more expressive alternative to syntax in further analyses and transformations.

4 Conclusion

The study of invertible computation has, historically, proven useful in understanding energy and entropy preservation, and in understanding information preserving transformations and transmission. However, there is still something to be learned about programs that are invertible, in particular regarding how to make invertible programming less syntactically restrictive, and how to implement this in a reasonably efficient way.
Since program invertibility is undecidable in general, all an invertibility analysis can hope to achieve is a reasonable approximation. In other words, any static analysis will split the expressible programs into three groups: those which are found to be invertible, those which are found to not be invertible, and those for which the analysis can provide no definite answer.

Clearly, the goal of our work is to make this latter class of programs as small as possible. However, since RFun and CoreFun are both R-Turing complete languages, we cannot hope to achieve a more computationally powerful language, though we can hope to make an invertible language that is more concise, familiar, and user friendly by enabling the expression of algorithms in a style that is much closer to that of conventional functional programming languages.

References


