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Portfolio Selection With Exploration of New Investment Opportunities

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Portfolio selection with exploration
of new investment opportunities

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Abstract

We introduce a model for portfolio selection with an extendable investment universe where an agent with mean-variance preferences faces a trade-off between exploiting existing and exploring for new investment opportunities. When the agent chooses to explore, a new risky asset is discovered and the agent subsequently invests in the extended universe. We first provide conditions for wellposedness and characterize the optimal amount devoted to exploration. Our model shows that incremental exploration does not pay off, that it is increasingly worthwhile to explore in worse market environments, and that investment performance measured by the Sharpe ratio is increasing in the initial wealth of the agent indicating that richer agents can make better use of new investment opportunities. We further generalize our model and verify the robustness of the main findings with regards to various modeling assumptions.

Keywords: portfolio selection; mean-variance optimization; exploration vs exploitation; investment universe; alternative investments

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1 Introduction

The vast majority of papers on portfolio selection start with a sentence of the form ‘The investment universe consists of one risk-free and $N$ risky assets’ or an equivalent statement. They then proceed by further specifying the prices of these assets in terms of their return distributions or as stochastic processes, describe the preferences of the agent, and solve a stochastic optimization problem to determine the optimal investment strategy.

In this paper, we question and remove one implicit assumption of the above setup, namely that the number of $N + 1$ available investment opportunities is fixed and that the agent has to allocate all of his/her wealth within this specified investment universe. In practice, even if $N$ is large, it is arguably not the case that the $N + 1$ assets present in the specified investment universe cover every conceivable investment opportunity for the agent. Rather, in our view, the correct interpretation of the above setup is that these are the investment opportunities which are salient to the agent and well understood in terms of their relevant distributional properties.

In reality, investors are in a constant search for opportunities and yield, especially since the advent of zero-interest-rate and negative-interest-rate policies that have reduced the yield that investors were accustomed to obtain in fixed income assets. The new investment environment created in response to the Great Financial Crisis of 2008 has pushed investors more and more to contemplate alternative investment vehicles. For instance, global annual Venture Capital investment has grown at 17% CAGR over the past decade.\(^1\) The Web 2.0 movement is fueling the “fourth industrial revolution”, with strong surges in innovations and market share battles between existing and newly created companies. This environment is leading standard institutional investors to consider investments in alternative assets and ventures that they would not have dared touching before the collapse of fixed income yields. Moreover, the exploding debt of nations since 2008 and its recent on-going surge due to policy responses to Covid-19 together with a growing sense of urgency to address the sustainability of the Human-Earth system is making clearer the need for massive investments in R&D from both public and private sectors to foster technological innovations that are becoming new investments vehicles. We are thus witnessing a progressing blurring of investment practices tending to combine holding shares

of established public companies and participating in all types of venture capital initiatives, including the recent SPAC boom.

Therefore, there is a need to formalize theoretically this new investment landscape by combining the more standard optimization process of exploiting an existing universe with the novel (from a portfolio optimization perspective) exploration of new investment opportunities. We thus propose to extend the setup with a fixed investment universe by giving the agent the option to explore for new investment opportunities. If this option is exercised, the agent chooses to devote a part of his/her wealth for exploration. This amount represents the costs associated with exploration, for example for hiring a team of analysts or acquiring information. Exploration results in the discovery of a new asset, whose distributional properties depend on the amount devoted to exploration in a way that the asset becomes more attractive to the agent when a larger amount is devoted to exploration. After discovery of the new asset, the agent then distributes the remainder of his/her wealth among the assets of the extended investment universe in order to optimize his/her preferences over the resulting terminal wealth. The agent thus faces the general and ubiquitous trade-off between exploration and exploitation studied in various research domains, e.g., organization science (March, 1991; Uotila et al., 2009), neuroscience (Daw et al., 2006), reinforcement learning (Sutton and Barto, 2018), and cross-disciplinary studies (Berger-Tal et al., 2014; Sornette et al., 2019). In our model, exploration refers to the discovery of new investment opportunities while exploitation corresponds to investing in the existing investment universe. Wang and Zhou (2020) consider a reinforcement learning approach for mean-variance portfolio selection where the space of strategies is relaxed to distributions of controls and where there is a trade-off between exploration and exploitation induced by adding an entropy-regularized reward function to the objective which encourages exploitation. However, they do not consider the possibility to expand the investment universe and it is not transparent how the agent benefits from exploration other than receiving a higher expected entropy-regularized reward. To the best of our knowledge, the present paper is thus the first to explicitly model a trade-off between exploration and exploitation for portfolio selection where exploration results in a tangible benefit for the agent through the discovery of a new investment opportunity. Our goal is to develop a formal model which provides guidance to individual investors, fund managers, and policy makers on whether to invest in exploration...
and on how to balance a trade-off between exploration in new and exploitation of existing investment opportunities.

In order to introduce new concepts in a familiar framework, we deliberately keep the mathematical model as simple as possible. Specifically, in this paper, we start with a single-period model for an agent with mean-variance preferences. This setup is analytically tractable and still rich enough to allow for a number of economic interpretations and discussion of novel model predictions. Various extensions will be discussed in Section 5, where we relax some of the assumptions in our main setting and further provide a multi-period formulation.

Mean-variance preferences have been introduced in the seminal work of Markowitz (1952) and thereafter studied in numerous papers. For the single-period setting, Markowitz (1959) gives an overview of the new approach in monograph form, Tobin (1958) derived the mutual fund theorem, Lintner (1965); Mossin (1966); Sharpe (1964) independently obtained the capital asset pricing model, and Merton (1972) provided an explicit expression for the efficient frontier. For the dynamic setting, the time-inconsistency of the mean-variance objective significantly complicated the analysis. This challenge was overcome with the embedding technique of Li and Ng (2000) and Zhou and Li (2000), who obtained the pre-committed optimal solution explicitly, which led to a second wave of research on dynamic mean-variance optimization (Bielecki et al., 2005; Chiu and Wong, 2011, 2014; Cui et al., 2014, 2012; Gao and Li, 2013; Han and Wong, 2021; Li and Zhou, 2006; Li et al., 2002; Lim, 2004; Lim and Zhou, 2002; Xia, 2005; Zhou and Yin, 2003). A second approach to time-inconsistent optimization problems is to formulate an intrapersonal game against future versions of oneself and to determine a time-consistent equilibrium strategy (Basak and Chabakauri, 2010; Björk and Murgoci, 2014; Björk et al., 2014; Czichowsky, 2013; Dai et al., 2021; He and Jiang, 2019a,b; Hu et al., 2012, 2017; Wang and Forsyth, 2011). With the exception of the stream of the literature where the agent is not required to invest all their wealth (Ehrbar, 1990), respectively take a cash-flow stream out of the market (Bäuerle and Grether, 2015; Cui et al., 2012; Dang and Forsyth, 2016), the agent is typically required to distribute all available wealth among the available assets. Most closely related to our paper are the studies of van Nieuwerburgh and Veldkamp (2010) on information acquisition, where the investor allocates a given quantity of information capacity among a fixed number of assets and subsequently chooses an optimal portfolio, and of Way et al. (2019) on
investment in technologies, where the amount invested lowers the per unit cost of a technology and thus indirectly influences its return distribution. However, to the best of our knowledge, no previous work considered the possibility of extending the investment universe and discovering new assets through exploration.

To determine the optimal amount $\kappa$ devoted to exploration and optimal investment strategy, we first solve a mean-variance optimization problem with a fixed amount devoted to exploration. This problem is of standard form, but its solution and optimal value provide valuable insight for the more general problem where the amount $\kappa$ devoted to exploration is a decision variable. As a corollary, we obtain that incremental exploration does lead to worse investment performance unless the marginal Sharpe ratio of the newly discovered asset at $\kappa = 0$ is infinite. This shows that one has to be willing to devote a substantial amount for exploration in order to be able to reap its benefits. We then impose the assumption of reasonable asymptotic elasticity on the Sharpe ratio of newly discovered asset, which is analogous to the same condition on the utility function for utility maximization in incomplete markets (Kramkov and Schachermayer, 1999). We show that, under this assumption, the mean-variance optimization problem with exploration is well-posed and the optimal amount devoted to exploration is characterized as a solution to an equation.

We then analyze the relationship between the existing investment universe and the optimal amount $\kappa$ devoted to exploration. The quality of the existing investment universe can be summarized by a single associated constant which corresponds to the square of the Sharpe ratio achieved by the optimal investment strategy when there is no option for exploration. We show that the optimal amount devoted to exploration decreases when the quality of the existing investment universe increases. Furthermore, unless the marginal Sharpe ratio of the newly discovered asset at $\kappa = 0$ is infinite, there always exists an investment universe such that it is optimal not to exercise the option to explore for new investment opportunities but instead invest the whole wealth in the existing universe. There is no need to explore for new investment opportunities when the existing universe is good enough. We discuss the difficulties to test this prediction empirically due to the implicit assumption of constant characteristics of newly discovered asset and leave an empirical investigation of this hypothesis as a challenging direction for future research. Nonetheless, our analysis does have policy implications because it
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highlights the potential benefits of exploration in an environment where traditional asset classes have unfavorable characteristics. This indicates that monetary and fiscal policies inflating prices of traditional assets would benefit from being complemented by policies of innovation and exploration.

Next, we investigate how the optimal amount devoted to exploration and the resulting investment performance depend on the initial wealth of the agent. We first obtain the intuitive result that the optimal amount devoted to exploration is increasing in the initial wealth of the agent. More interestingly, we find that the investment performance as measured by the Sharpe ratio is increasing in the initial wealth as well. This is different from the classical mean-variance setting with a fixed number of assets, where the Sharpe ratio of the optimal investment does not depend on the initial wealth for the agent. Intuitively, a wealthier agent can devote a larger amount for exploring new investment opportunities and thus discovers a new asset which is performing better in terms of its relevant distributional characteristics. The opportunity to invest in this newly discovered asset then outweighs the larger amount the agent initially had to devote to exploration since otherwise he/she would not have made this investment in the first place. While the empirical evidence on size and performance is mixed or even opposite to this prediction for various reasons at the level of individual funds (Chen et al., 2004; Pástor et al., 2015; Phillips et al., 2018; Pollet and Wilson, 2008; Yan, 2008), evidence on the performance of asset managers sitting above the fund level, such as family of funds (Chen et al., 2004; Pollet and Wilson, 2008), university endowments (Lerner et al., 2008), non-profit endowment funds (Lo et al., 2020), and pension funds (Dyck and Pomorski, 2016), do support the prediction that wealthier agents are able to achieve better performance and link this finding with investments in alternative assets.

Finally, we propose an explicit functional parametrization for the expected return and standard deviation of the newly discovered asset and illustrate the obtained results in numerical examples. The chosen parametrization leads to realistic investment behavior for a broad range of existing investment universes and initial wealth levels simultaneously and thus supports the plausibility of our model.

Our base model, and the analysis thereof, rests on some simplifying assumptions. Namely, we suppose that there is zero correlation between the existing investment universe and the
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newly discovered asset, and that there is a single asset to be explored. These assumptions are primarily made to obtain a more parsimonious model. We further assume that the functional mapping the amount devoted for exploration to the distributional characteristics of the newly discovered asset is deterministic. In other words, we suppose that the agent knows ex ante the distributional characteristics of the newly discovered asset as a function of the amount devoted for exploration. There is no uncertainty in our base model. In the final part of the paper, we relax those assumptions and consider possible extensions of our model. Allowing for correlation between the newly discovered asset and the assets of the existing universe is a technical extension and the results conform with those of the base model. If the newly discovered asset is more strongly correlated with some assets of the existing universe than with others, the integration of the newly discovered asset into the investment universe will lead to a rebalancing of the portfolio that would be optimal without an option for exploration as one could expect. Our qualitative results also remain robust when the agent can explore for more than one new asset. One interesting new observation is that, when economies of scale for the cost of information acquisition occur, it is possible that there are two local optima for the number of newly discovered assets: A small local optimum if the agent focuses on reaping low hanging fruits and a larger number reflecting the attempt to benefit from economies of scale. The case where there is uncertainty about the distributional properties of the newly asset is conceptually interesting due to the dynamic inconsistency of the variance, but our main findings remain again robust and this extension even completely reduces to the base model when there is a concurrent decision about how much to devote for exploration and how to invest the remaining wealth in available assets. We finally examine how the model could be extended to a dynamic setting and investigate a simple two-period model in greater detail. Our initial results on this point show that devoting resources to discover new investment opportunities is mostly beneficial at the beginning of the investment horizon and, subsequently, when (and if) wealth decreases below a certain value. On the contrary, when the agent is already close to reach his/her objective at maturity, investing additional funds for the discovery of new assets is not optimal.

The remainder of this paper is organized as follows. We review the classical single-period mean-variance optimization model and propose an extension by giving the agent the option to
explore for new investment opportunities in Section 2. Section 3 contains the main theoretical results on the well-posedness of the problem (Subsection 3.1), the optimal amount devoted to exploration as a function of the existing investment universe (Subsection 3.2), and the initial wealth (Subsection 3.3). The numerical examples related to the main setting are in Section 4. In Section 5, we investigate extensions of our model to account for correlation between the new asset and assets in the initial investment universe (Subsection 5.1), exploration of multiple new assets (Subsection 5.2), uncertainty (Subsection 5.3), and dynamic exploration (Subsection 5.4). Section 6 concludes. All proofs and additional results are placed in the appendices.

2 A model of portfolio selection with an extendable investment universe

We start with reviewing the classical mean-variance model for portfolio selection of Markowitz (1952), and then provide an extension of this model that allows for the exploration in new investment opportunities.

In standard models for portfolio selection, an agent considers a fixed investment universe consisting of a risk-free bond providing a deterministic return $r_f$ and $N$ risky stocks with random returns $r = (r_1, \ldots, r_N)'$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote an investment universe by $\mathcal{U}$. Under mean-variance preferences, knowledge about the mean and covariance structure of the risky assets is sufficient for portfolio analysis. We suppose that $r \in L^2(\mathbb{P})$ and let

$$\bar{r} := \mathbb{E}[r], \quad \Sigma := \mathbb{E} [(r - \bar{r})(r - \bar{r})']$$

denote the vector of expected risky returns and the covariance matrix of the risky returns respectively. We suppose that $\Sigma$ is positive definite and that $\bar{r} \neq r_f 1_N$, where $1_N \in \mathbb{R}^n$ denotes the vector with all components equal to one. Those assumptions are made to exclude mathematically degenerate situations. In practice, one typically has $\bar{r} > r_f 1_N \geq 1_N$.

An agent with initial wealth $x_0$ seeks to select the portfolio of minimal risk, where risk is measured by the variance, among all those portfolios that on average achieve a target expected return $\mu \geq r_f$. The agent chooses an investment strategy described by the amount invested
in the risk-free asset $u_0$ and the amounts invested in the $N$ risky assets $\mathbf{u} \in \mathbb{R}^N$, where $u_i$, $i = 1, \ldots, N$ denotes the absolute amount invested in the $i^{th}$ risky asset. The objective of the agent is to solve the classical mean-variance optimization problem

$$
\min_{\mathbf{u} \in \mathbb{R}^N, u_0 \in \mathbb{R}} \quad \mathbf{u}' \Sigma \mathbf{u}
\text{ s.t. } \quad \mathbf{u}' \bar{\mathbf{r}} + u_0 r_f = \mu x_0, \quad (\text{MV}(\mu, x_0))
\quad \mathbf{u}' \mathbf{1}_N + u_0 = x_0.
$$

The first constraint in (MV($\mu, x_0$)) corresponds to the requirement that the portfolio on average has to achieve the target expected return while the second is a budget constraint. We recall the following constants associated with the investment universe $\mathcal{U}$,

$$
A = \mathbf{1}_N ' \Sigma^{-1} \bar{\mathbf{r}}, \quad B = \bar{\mathbf{r}} ' \Sigma^{-1} \bar{\mathbf{r}}, \quad C = \mathbf{1}_N ' \Sigma^{-1} \mathbf{1}_N, \quad D = r_f^2 C - 2 r_f A + B,
$$

which play a role in the explicit derivation of the optimal value and optimal solution to the classical mean-variance optimization problem (MV($\mu, x_0$)). The optimal value of (MV($\mu, x_0$)) is given by

$$
\sigma_{MV}^2 = \frac{(\mu - r_f)^2}{D} x_0^2.
$$

The constant $D$ is thus of particular importance and indicates how good the investment universe $\mathcal{U}$ is from the perspective of a mean-variance investor since it corresponds to the square of the Sharpe ratio achieved by the optimal investment strategy

$$
\sqrt{D} = \frac{\mu - r_f}{\sigma_{MV}} x_0.
$$

In practice, the $N + 1$ assets considered in (MV($\mu, x_0$)) do not constitute the totality of available investment opportunities. Rather, the correct interpretation of the above investment problem is that the $N + 1$ assets are the investment opportunities which are salient to the agent and are well understood in terms of their relevant distributional properties. We herein extend the above model by giving the agent the option to devote a part of his/her wealth for exploring new investment opportunities. We emphasize that this observation is not specific to the single-period mean-variance investment problem considered herein, but applies to the vast majority of the literature on portfolio selection. In particular, a framework with an extendable
investment universe can be studied in a great number of other specifications of the financial market and for preferences of the investor other than mean-variance.

In our model of mean-variance optimization with exploration, the agent has the option to devote an amount $\kappa \geq 0$ for exploring new investment opportunities. If this option is exercised and the agent devotes $\kappa > 0$ for exploration, he/she discovers a new asset that extends the previously available investment universe. The newly discovered asset achieves a random return $r_e(\kappa)$ which depends on the amount $\kappa$ devoted to exploration. We assume that the expected return $\bar{r}_e(\kappa)$ and standard deviation $\Sigma_e(\kappa)$ of the newly discovered asset are known to the agent as a function of $\kappa$, that $\bar{r}_e(0) = r_f$, $\bar{r}_e(\kappa)$ is increasing and continuously differentiable, and that $\Sigma_e(\kappa) > 0$ for any $\kappa \geq 0$, $\Sigma_e$ is decreasing and continuously differentiable. Assuming that $\bar{r}_e(0) = r_f$ and $\Sigma_e(0) > 0$ constitutes a ‘no-free-lunch’ condition: One can not discover a desirable new asset by devoting nothing to exploration. On the other hand, the assumption that $\bar{r}_e$ is increasing and $\Sigma_e$ decreasing reflects the intuition that devoting a larger amount for exploration leads to a newly discovered asset that is more attractive from the perspective of a mean-variance agent. The above setting implicitly assumes that the distributional characteristics of the newly discovered asset as a function of the amount devoted for exploration are deterministic and known ex ante by the agent. We will relax this assumption and discuss the case of stochastic distributional characteristics in Subsection 5.3. We further assume that the newly discovered asset is uncorrelated with assets of the original investment universe, i.e., that $\text{Cov}(r_e(\kappa), r_i) = 0$ for $i = 1, \ldots, N$. This assumption is primarily made in order to obtain a parsimonious model which allows for transparent economic interpretations. The assumption is realistic if the newly discovered asset is an alternative investment, but less so when it adds another stock to an original universe consisting of stocks. We will remove this assumption and consider the case where the newly discovered asset is correlated with the assets of the existing universe in Subsection 5.1. To summarize, what differentiates the newly discovered asset from those in the existing investment universe is that its discovery and the knowledge about its distributional properties are costly. On the other hand, the assets in the existing investment universe are readily available and their distributional properties are well understood from the outset.

The objective of the agent is to concurrently determine whether to exercise the option to
explore for new investment opportunities and, if the option is exercised, to specify an optimal amount devoted to exploration and an optimal allocation in the extended investment universe. While, if the option is not exercised, the agent’s objective is to determine an optimal allocation in the existing investment universe. In the case where the agent does not exercise the option to explore for new investment opportunities, he/she would simply solve the classical mean-variance optimization problem \( \text{MV}(\mu, x_0) \). However, if the option to explore is exercised, the agent solves the mean-variance optimization problem with exploration

\[
\inf_{\kappa \geq 0} \left( \min_{u \in \mathbb{R}^N, v \in \mathbb{R}, u_0 \in \mathbb{R}} \left\{ u^T \Sigma u + \Sigma^2_e(\kappa)v^2 \right\} \right), \quad (\text{MVE}(\mu, x_0))
\]

where \( v \in \mathbb{R} \) denotes the amount the agent invests in the newly discovered asset. Note in particular the adjusted budget constraint, where the amount devoted to exploration \( \kappa \) is deducted from the initial wealth \( x_0 \) at the stage of the portfolio optimization problem. While decisions are made concurrently, \( (\text{MVE}(\mu, x_0)) \) is a nested minimization problem which can be solved in a sequential manner. This will significantly simplify the analysis.

**Remark 1.** Our model for portfolio selection with exploration shares some principles with the framework of real options. The main difference is that, in our case, the decision to explore for new investment opportunities is not based on option-like payoffs, for the agent is allowed to control the amount to be invested in the new (tradable) asset. In this regard, it is more customary for standard real option problems to assume that investors must decide whether to undertake a one-off investment. Without attempting to be exhaustive, we refer to Brennan and Schwartz (1985); Dixit (1989); Grenadier and Malenko (2010); Henderson (2007); Leippold and Stromberg (2017); McDonald and Siegel (1986) for a few seminal and more recent works on real options.
3 The optimal amount devoted to exploration, investment strategy, and investment performance

In this section, we first solve a mean-variance optimization problem with a fixed amount devoted to exploration and provide conditions under which (MVE(\(\mu, x_0\))) is well-posed and admits an optimal strategy in Subsection 3.1. We then investigate how the optimal amount devoted to exploration and investment performance depend on the existing investment universe in Subsection 3.2, and on the initial wealth of the agent in Subsection 3.3.

3.1 Well-posedness and the asymptotic elasticity of the Sharpe ratio of the newly discovered asset

For a fixed \(\kappa \geq 0\), we first consider the inner optimization problem of (MVE(\(\mu, x_0\))), which we call the mean-variance optimization problem with a fixed amount devoted to exploration

\[
\min_{u \in \mathbb{R}^N, v \in \mathbb{R}, u_0 \in \mathbb{R}} \quad u' \Sigma u + \kappa^2 v^2 \\
\text{s.t.} \quad u' \bar{r} + v \bar{r}_e(\kappa) + u_0 r_f = \mu x_0, \\
\quad u' 1_N + v + u_0 = x_0 - \kappa. 
\]

(MVEF(\(\mu, x_0; \kappa\)))

We denote the optimal value of (MVEF(\(\mu, x_0; \kappa\))) by \(\sigma^2_{MVEF}(\kappa)\). The following proposition presents the optimal strategy and value for (MVEF(\(\mu, x_0; \kappa\))). The method for solving (MVEF(\(\mu, x_0; \kappa\))) is standard and well-known.

**Proposition 1.** Consider an investment universe \(\mathcal{U}\) with associated \(A, B, C, D\) given in (1) and let

\[
A_e(\kappa) = A + \frac{\bar{r}_e(\kappa)}{\Sigma^2_e(\kappa)}, \quad B_e(\kappa) = B + \frac{\bar{r}_e(\kappa)^2}{\Sigma^2_e(\kappa)}, \\
C_e(\kappa) = C + \frac{1}{\Sigma^2_e(\kappa)}, \quad D_e(\kappa) = D + \frac{(\bar{r}_e(\kappa) - r_f)^2}{\Sigma^2_e(\kappa)}. 
\]

(3)
The problem \((\text{MVEF}(\mu, x_0; \kappa))\) has the unique solution

\[
\begin{align*}
\mathbf{u} &= \frac{\mu x_0 - r_f(x_0 - \kappa)}{D_e(\kappa)} \Sigma^{-1}_{e}(\bar{\mathbf{r}}_e - r_f \mathbf{1}_N), \\
v &= \frac{\mu x_0 - r_f(x_0 - \kappa) \bar{\mathbf{r}}_e(\kappa) - r_f}{D_e(\kappa)} \\
u_0 &= x_0 - \kappa - \frac{\mu x_0 - r_f(x_0 - \kappa)}{D_e(\kappa)} (A_e(\kappa) - r_f \Sigma e_c(\kappa))
\end{align*}
\]  

(4)

and the optimal value is given by

\[
\sigma^2_{\text{MVEF}}(\kappa) = \frac{(\mu x_0 - r_f(x_0 - \kappa))^2}{D_e(\kappa)}. 
\]

(5)

We note from (3) and (4) that the optimal value of \((\text{MVEF}(\mu, x_0; \kappa))\) depends only on the distributional characteristics of the newly discovered asset through the Sharpe-ratio of the newly discovered asset, which we denote by 

\[
S_e(\kappa) = \frac{\bar{\mathbf{r}}_e(\kappa) - r_f}{\Sigma e(\kappa)}, \quad \kappa \geq 0.
\]

(6)

Therefore, we conclude that the optimal amount devoted to exploration only depends on \(S_e(\kappa)\) and not on the separate values of \(\bar{\mathbf{r}}_e(\kappa)\) and \(\Sigma e(\kappa)\). Interestingly, the same conclusion holds true for the optimal investment in the risky assets of the original investment universe, \(\mathbf{u}\), given in (4), which is decreasing in the Sharpe ratio of the newly discovered asset and thus in the amount devoted to exploration. However, the optimal investment in the newly discovered asset, \(\mathbf{v}\), and in the risk-free asset, \(u_0\), do depend on the separate values of the mean and standard deviation of the newly discovered asset and its Sharpe ratio therefore is not a sufficient statistic for optimal investment. It is also noteworthy that, while the cumulative amount invested in the risky assets of the original universe is decreasing in the amount devoted to exploration, the distribution of the investment among the risky assets of the original investment universe does not depend on the amount devoted to exploration and is always proportional to \(\Sigma^{-1}_{e}(\bar{\mathbf{r}} - r_f \mathbf{1}_N)\). In particular, a version of the mutual fund separation theorem still holds in our model.

**Remark 2.** While we make the assumption that the newly discovered asset is uncorrelated with assets of the original investment universe primarily to obtain a parsimonious model, the above version of the mutual fund separation theorem depends crucially on this assumption. If the newly discovered asset were correlated with the assets of the existing investment universe, and
stronger so with some than with others, then the optimal distribution of the investment among the risky assets of the original investment universe would depend on the amount devoted to exploration; cf. Subsection 5.1.

The following corollary shows that the case of not exercising the option to explore for new investment opportunities is equivalent to exercising the option and then not devoting any amount for exploration. This in particular implies that it is sufficient to consider \( (\text{MVE}(\mu, x_0)) \).

**Corollary 1.** The classical mean-variance problem without exploration \( (\text{MV}(\mu, x_0)) \) is equivalent to the mean-variance optimization problem with a fixed amount devoted to exploration equal to zero \( (\text{MVEF}(\mu, x_0; \kappa = 0)) \) in the sense that both the optimal strategy and optimal value coincide.

**Remark 3.** The reduction of the problem obtained in Corollary 1 would not hold if there doesn’t exist a risk-free asset. Indeed, without a risk-free asset, there are financial markets in terms of \( \bar{\mathbf{r}} \) and \( \Sigma \) where a lower variance can be achieved by throwing money away, see Cui et al. (2015).

Next, we obtain a further corollary showing that there exists an \( \epsilon > 0 \), such that devoting an amount \( \kappa \in (0, \epsilon) \) for exploration leads to a worse result than not devoting any amount for exploration as long as the marginal Sharpe ratio \( S_e(\kappa) \) of the newly discovered asset is finite at zero.

**Corollary 2.** (i) Suppose that \( \kappa_0 > 0 \) is such that \( S'_e(\kappa_0) = 0 \) and let \( \mu > r_f \). There exists an \( \epsilon > 0 \) such that \( \sigma^2_{\text{MVEF}}(\kappa) > \sigma^2_{\text{MVEF}}(\kappa_0) \) for any \( \kappa \in (\kappa_0, \kappa_0 + \epsilon) \).

(ii) Suppose that \( S'_e(0) < \infty \) and let \( \mu > r_f \). There exists an \( \epsilon > 0 \) such that \( \sigma^2_{\text{MVEF}}(\kappa) > \sigma^2_{\text{MVEF}}(0) \) for any \( \kappa \in (0, \epsilon) \).

Corollary 2 shows the intuitive result that it is suboptimal to marginally increase the amount devoted to exploration at a point where the marginal Sharpe ratio of the newly discovered asset is zero. The second statement of the corollary is more interesting and surprising: Unless the marginal Sharpe ratio of the newly discovered asset at \( \kappa = 0 \) is infinite, incremental exploration does not pay off. One must put a significant amount at risk in order to harvest the potential benefits of exploring for new investment opportunities. Note that this is not due to the fact that incremental exploration would not result in a desirable asset. As long as the expected
return of the newly discovered asset is strictly increasing as a function of the amount devoted for exploration in a neighborhood of zero, the agent would invest in the newly discovered asset if it were available for free. It is the cost required to discover and understand the new asset that makes incremental exploration not worthwhile. While we are not aware of empirical evidence that would directly support or confute the prediction that incremental exploration does not pay off, Brown et al. (2019) document that large institutional investors commit a significant amount of resources into acquisition of private information before investing in hedge funds when publicly available information does not already lead to precise estimates of return characteristics.

While the classical mean-variance optimization problem $\text{(MV}(\mu, x_0))$ is always well-posed, this is not necessarily the case for the mean-variance problem with exploration $\text{(MVE}(\mu, x_0))$. We therefore make the following additional assumption on the Sharpe ratio of the newly discovered asset as a function of the cost devoted to exploration which, as we will see, assures that $\text{(MVE}(\mu, x_0))$ is well-posed.

**Assumption 1.** The Sharpe ratio of the newly discovered asset has asymptotic elasticity smaller than one, i.e.,

$$\limsup_{\kappa \to \infty} \frac{\kappa S_e'(\kappa)}{S_e(\kappa)} < 1.$$  

Assumption 1 is essentially saying that the Sharpe ratio of the newly discovered asset grows asymptotically slower than linearly as a function of the amount devoted to exploration. The same condition is required for the utility function for utility maximization in incomplete markets, see Kramkov and Schachermayer (1999).

**Theorem 1.** Under Assumption 1, the mean-variance optimization problem allowing for exploration $\text{(MVE}(\mu, x_0))$ is well-posed and there is an optimal cost for exploration and optimal investment strategy. In particular, both the optimal value of and optimal amount for exploration and optimal investment strategy strategy for $\text{(MVE}(\mu, x_0))$ are finite. Furthermore, the optimal amount devoted to exploration $\kappa^*$ is either zero, or satisfies

$$r_f D + r_f S_e(\kappa^*)^2 = S_e(\kappa^*) S_e'(\kappa^*) \left[(\mu - r_f) x_0 + r_f \kappa^*\right]. \quad (7)$$

We suppose that Assumption 1 holds for the remainder of this paper.
3.2 Dependence of the optimal amount devoted to exploration on the existing investment universe

In this subsection, we investigate how the optimal amount devoted to exploration depends on the existing investment universe $\mathcal{U}$.

We first show that, unless the marginal Sharpe ratio $S_e(\kappa)$ of the newly discovered asset is infinite at $\kappa = 0$, it is optimal not to explore for new investment opportunities when the existing investment universe is sufficiently attractive, and that such a sufficiently attractive investment universe exists for any initial wealth and target expected return for the agent.

**Proposition 2.** Suppose that $S'_e(0) < \infty$. Then there is an investment universe $\mathcal{U}$ with associated $D$ given in (1) and (2) such that it is optimal not to exercise the option to explore for new investment opportunities in any investment universe $\tilde{\mathcal{D}}$ with associated $\tilde{D} \geq D$.

The following result extends the above proposition and shows that, the better the existing investment universe the investor starts out with, the smaller the optimal amount devoted to exploration.

**Proposition 3.** Consider two investment universes $\mathcal{U}$ and $\tilde{\mathcal{U}}$ with associated $D$ and $\tilde{D}$ respectively satisfying $\tilde{D} > D$, given in (1) and (2). Let $\kappa^*$ and $\tilde{\kappa}^*$ denote the optimal amounts devoted to exploration in each universe. It then holds that $\tilde{\kappa}^* \leq \kappa^*$.

When an investment universe $\mathcal{U}$ consists of $N$ risky assets and $\tilde{\mathcal{U}}$ is obtained from $\mathcal{U}$ by adding $m$ additional risky assets, then $\tilde{D} \geq D$. Propositions 2 and 3 thus indicate that it is suboptimal for the agent to explore for new investment opportunities if the existing investment universe is already large enough and that, in general, the larger the existing investment universe, the smaller the optimal amount devoted to exploring for an additional asset.

Propositions 2 and 3 indicate that it is worthwhile to increase the amount devoted to exploration if the market conditions of the existing investment universe are unfavorable relative to the properties of newly discovered assets. Such conditions could for example exist when asset prices of traditional investments are inflated due to monetary and fiscal policy so that investors expect lower future returns. At the level of policy making, our model thus indicates that such policies could benefit from being complemented or even replaced by an innovation policy to explore new horizons for inventions and innovations.
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We end this subsection with a word of caution on how to interpret Propositions 2 and 3. Our results do not imply that one should devote a larger amount for exploration of new investment opportunities in a bear market, and a smaller amount in a bull market. Indeed, the results obtained in this section are based on the implicit assumption that the expected return and variance of the newly discovered asset as functions of the amount devoted to exploration are fixed. However, it is likely the case that, keeping the amount devoted to exploration constant, new investment opportunities are worse in a bear market, and better in a bull market. This discussion shows that it will be difficult to test this prediction of our model using empirical data. For doing so, one would have to compare environments where the attributes in terms of risk and return offered by traditional investments differ markedly, but alternative investments would offer similar risk and return characteristics. A comparison of investment behavior under such alternative environments has not yet been investigated in the literature to the best of our knowledge and empirically testing the hypothesis of this subsection thus remains an open problem for empirical research.

3.3 Optimal investment performance as a function of the initial wealth: The rich perform better

In this subsection, we turn our attention to the analysis of the investment behavior and performance as a function of the initial wealth of the agent.

The following proposition shows that it is optimal to devote a larger amount for exploration when the initial wealth of the agent is larger. This result is intuitive and shows that the current model leads to reasonable investment behavior.

Proposition 4. Consider two agents with wealth \(x_0\) and \(\tilde{x}_0\) respectively, such that \(x_0 < \tilde{x}_0\). Let \(\kappa^*\) and \(\tilde{\kappa}^*\) denote the optimal amounts devoted to exploration for these two agents with their respective levels of initial wealth. It then holds that \(\kappa^* \leq \tilde{\kappa}^*\).

In the classical mean-variance optimization problem \((MV(\mu, x_0))\), the optimal investment performance measured by the Sharpe ratio does not depend on the initial wealth \(x_0\) of the agent but only on the investment universe \(\mathcal{U}\) through the associated constant \(D\) given in (1) and (2). The following Theorem shows that, when the agent is given the option to explore for new
investment opportunities, the Sharpe ratio of the optimal investment behavior is increasing in the initial wealth, conforming with the empirical evidence. We denote the Sharpe ratio achieved in $(\text{MVE}(\mu, x_0))$ by choosing an optimal amount devoted to exploration and optimally investing by $S^*_{\text{MVE}}(\mu, x_0)$.

**Theorem 2.** Consider two agents with wealth $x_0$ and $\tilde{x}_0$ respectively, such that $x_0 < \tilde{x}_0$. The Sharpe ratio achieved by the richer agent is larger than the Sharpe ratio achieved by the poorer agent, $S^*_{\text{MVE}}(\mu, \tilde{x}_0) \geq S^*_{\text{MVE}}(\mu, x_0)$. Furthermore, if the Sharpe ratio $S_e(\kappa)$ of the newly discovered asset is strictly increasing as a function of the amount $\kappa$ devoted to exploration, then the inequality is strict if and only if $\tilde{\kappa}^* > 0$.

The empirical evidence on the effect of scale on performance in the fund industry is mixed, and does not immediately confirm the prediction of Theorem 2. Chen et al. (2004) document an inverse relation between fund size and fund performance and suggest that these diseconomies of scale are related to liquidity concerns. These findings are confirmed in Yan (2008), who presents strong evidence for adverse effects of scale on performance among funds with low liquidity, but no significant relation between fund size and fund performance among the funds in the three most liquid quintiles of the funds considered therein. Pollet and Wilson (2008) finds that individual funds grow by increasing their investment in existing assets rather than increasing the numbers of assets in their portfolio. This indicates that our model might not apply at the individual fund level, and, therefore, that individual fund performance is not a good quantity to study the predictions of our model. Later, Pástor et al. (2015) and Phillips et al. (2018) do not find any significant relationship between fund size and fund performance at the fund level when accounting for endogeneity of fund size and performance.

To summarize, on the fund level, the empirical evidence on the relationship between fund size and performance does not confirm the prediction of Theorem 2 and in some studies point in the opposite direction. However, there are manifold reasons for negative impacts of size on performance at the fund level which are not captured in our model. We therefore consider asset managers sitting above the fund level, such as family of funds, pension funds, or nonprofit endowments. These agents have additional degrees of freedom, often consider individual funds as risky assets, and invest in alternative assets for which the process of exploring new investment options applies and thus makes our model relevant.
At the level of family of funds, such as Black Rock, Fidelity, or Vanguard, Chen et al. (2004) find that the size of the family is positively related to the performance. Pollet and Wilson (2008) further document that fund family growth corresponds to a growth of the number of funds instead of increased allocations in each individual funds and seems to be associated with new investment ideas. This indicates that our model is more applicable at the level of family of funds, where its implications in terms of both strategy and performance coincide with empirical observations. Lerner et al. (2008) find that investment returns achieved by larger university endowments outperform those of smaller endowments and document that, among other factors, the better performance of larger endowments is linked to a higher proportion of wealth invested in alternative assets. Similarly, Lo et al. (2020) find that larger funds significantly outperform smaller funds based on tax-return data from all public nonprofit endowment funds in the United States over the 2009-2017 period. They further show that some of this difference is driven by asset allocation decisions. The results of Dahiya and Yermack (2020) that four-factor alphas are lower for endowment of large nonprofit funds than those of smaller funds would point in the opposite direction but is based on the analysis of a much smaller number of funds than in Lo et al. (2020). For pension funds, Dyck and Pomorski (2016) find that investment performance is increasing in the scale of the funds. While some of the difference in performance can be attributed to other factors such as greater use of internal management and governance, a substantial factor behind the better performance of larger pension funds comes from the fact that they devote a larger amount to alternative investments such as private equity or real estate.

There is thus substantial empirical evidence showing that investment performance improves with wealth for various classes of asset managers above the fund level and that the increased performance is partially linked to asset allocation decisions and investments in alternative assets. This evidence is consistent with the predictions of our model. Potentially, portfolio selection with exploration of new investment opportunities could thus also provide a new perspective on our understanding of the recent increase in inequality throughout most of the developed world. Richer agents can make better use of the opportunities offered by exploration, which in turn leads to better investment performance. As we have seen in the previous subsection, this is of particular relevance in the current investment environment, where traditional asset prices are
4 Numerical examples

We consider a numerical example where the expected return and covariance matrix of the existing risky assets are as in Example 1 of Cui et al. (2015), which in turn is based on the example in Chapter 7 of Sharpe et al. (1995) and the example on page 176 in Markowitz (1959). The original investment universe \( \mathcal{U} \) consists of three risky assets with expected return and covariance structure given by

\[
\bar{r} = \begin{pmatrix} 1.162 \\ 1.246 \\ 1.228 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{pmatrix}
\] (8)

and a risk-free asset providing a return \( r_f = 1.03 \). Under the above statistics for the original investment universe, we have \( D = 1.6282 \).

**Remark 4.** The original investment universe \( \mathcal{U} \) only affects the optimal amount devoted to exploration and resulting investment performance through the associated constant \( D \) given in (1) and (2). The qualitative results regarding the optimal amount devoted to exploration and resulting performance of this example also cover existing investment universes with a much larger number of risky assets. Based on this observation, we will only vary \( D \) when illustrating the effect of \( \mathcal{U} \) on optimal investment and performance.

We consider the following parametrization for the expected return and standard deviation of the newly discovered asset,

\[
\bar{r}_e(\kappa) = r_f + \alpha \arctan\left(\frac{\kappa}{\beta}\right) \quad \text{and} \quad \Sigma_e(\kappa) = s_0 + \frac{1}{(s_1 + \kappa)^2}, \quad \kappa \geq 0,
\] (9)

for parameters \( \alpha, s_0 \geq 0 \) and \( \beta, s_1 > 0 \). Straightforward computation shows that the resulting Sharpe ratio of the newly discovered asset has asymptotic elasticity equal to zero for any value of the parameters \( (\bar{r}_e(\kappa) \to r_f + \alpha \text{ for large } \kappa) \). In particular, parameterization (9) satisfies Assumption 1 and thus leads to a well-posed investment problem according to Theorem 1. Furthermore, the marginal Sharpe ratio of the newly discovered asset is finite (equal to \( \alpha/\beta \)) at \( \kappa = 0 \) and strictly positive everywhere such that all our results apply.
We will use the following parameter values as benchmark values for our analysis: $\alpha = 0.3$, $\beta = 100'000$, $\gamma = 0.5$, $s_0 = 0.25$, and $s_1 = 100'000$. Given these values, an agent managing an initial wealth of $x_0 = 10'000'000$ USD and target expected return $\mu = 1.2$ would devote an amount $\kappa^* = 283'372$ USD for exploration. This roughly corresponds to the cost of a full-time position for a seasoned investment professional. The optimal investment strategies (in USD) are given by

$$u = \begin{pmatrix} 904'171 \\ 783'743 \\ 2'870'480 \end{pmatrix}, \quad v = 3'071'691, \quad u_0 = 2'086'543,$$

corresponding to a diversified portfolio of the risky assets of the existing investment universe, the newly discovered asset, and the risk-free asset.

In Figure 1, we plot the optimal variance which can be achieved in the mean-variance problem with a fixed amount devoted to exploration ($\text{MVEF}(\mu, x_0; \kappa)$) as a function of the amount devoted to exploration $\kappa \geq 0$. The figure in particular illustrates the result obtained in Corollary 2 that a small amount devoted to exploration leads to a worse investment performance compared with not exploring for new assets at all. One has to commit a sufficient amount in order to reap the benefits of exploration.

Next, Figure 2 illustrates how the optimal amount devoted to exploration and achieved investment performance depends on the initial wealth in different existing investment universes. The existing investment universes are characterized by the associated constant $D \in \{0.5, 1, 1.6282, 2, 2.5\}$, where $D = 1.6282$ is the benchmark value corresponding to the market described in (8). The absolute amount devoted to exploration is shown in Subfigure 2a, while Subfigure 2b shows the amount devoted to exploration as a fraction of the initial wealth. Similarly, Subfigure 2c shows the amount invested in the newly discovered asset and Subfigure 2d shows the investment in the newly discovered asset as a fraction of the initial wealth.

These figures highlight that it is suboptimal to explore new investment opportunities when the initial wealth is below a certain amount. For the benchmark parameters, it only pays to explore for new investment opportunities when the initial wealth exceeds about 2'500'000 USD. Thus, only those who are already sufficiently wealthy can benefit from exploration. Once this threshold is reached, the agent strictly increases his/her amount devoted to exploration and
Figure 1: Objective value of \((MVEF(\mu, x_0; \kappa))\) as a function of the amount devoted to exploration \(\kappa\)

Notes. The figure shows the objective value of the mean-variance optimization problem with a fixed amount devoted to exploration \((MVEF(\mu, x_0; \kappa))\) for the market with return statistics specified in (8) and a newly discovered asset with returns characterized by (9). The initial wealth is \(x_0 = 10^8,000\) USD and the target expected returns is \(\mu = 1.2\).

investment in the newly discovered asset with increasing wealth. However, as a fraction of wealth, both the amount devoted to exploration and the investment in the newly discovered asset are decreasing in wealth once the threshold wealth for exercising the option to explore is reached. This decrease in the fraction invested in the newly discovered asset can be understood as follows: An agent with a larger initial wealth devotes a larger amount for exploration and therefore obtains a newly discovered asset with a larger expected return. If the investment allocations would remain constant as a fraction of initial wealth, he/she would thus achieve a higher expected return than the target expected return \(\mu\). The richer agent can thus afford to put a larger share of his/her wealth into the risk-free asset and thus reduces the fraction invested in all the risky assets, in particular also in the newly discovered one, with the result of decreasing
Figure 2: Amount devoted to exploration and investment in the newly discovered asset as a function of the initial wealth

Notes. The figure shows the optimal amount devoted to exploration, respectively the optimal investment in the newly discovered asset as a function of the initial wealth and the constant $D$ (square of Sharpe ratio) given in (1) and (2) associated with the existing investment universe in semi-logarithmic plots. The existing investment universe is specified in (8) and the newly discovered asset has returns characterized by (9).

the variance of his/her portfolio. We further observe that the threshold initial wealth above which exploration becomes economical is increasing, while the amount devoted to exploration and the investment in the newly discovered asset are decreasing in the constant $D$ associated with the existing investment universe. This is intuitive and consistent with Proposition 3.

Remark 5. The numerical results obtained in this section show that our framework leads to
reasonable predictions on the optimal behavior for the parametrization chosen in (9) for a broad range of specifications for the existing investment universe and initial wealth levels.

The investment performance, measured by the Sharpe ratio, as a function of the initial wealth is shown in Figure 3. The figure confirms the finding of Theorem 2 that a framework allowing for exploration of new investment opportunities can explain that agents with larger wealth achieve better investment performance.

Figure 3: Sharpe ratio as a function of the initial wealth

Notes. The figure shows the optimal Sharpe ratio of the investment as a function of the initial wealth and the constant $D$ (square of Sharpe ratio) associated with the existing investment universe in semi-logarithmic plots. The existing investment universe is specified in (8) and the newly discovered asset has returns characterized by (9).
5 Extensions

In this section, we relax some of our assumptions and discuss possible extensions of the base model. In Subsection 5.1 we remove the assumption that the newly discovered assets are uncorrelated with assets in the initial investment universe. Next, we allow for exploration of multiple assets in Subsection 5.2. Two model formulations with uncertainty about the distributional characteristics of the newly discovered asset are then examined in Subsection 5.3. Last, in Subsection 5.4 we introduce a dynamic framework for exploration.

5.1 Correlation between the new asset and the existing universe

Our basic assumptions remain unchanged, except that now the newly discovered asset can be correlated with one or more assets in the existing investment universe, i.e., we do not set a priori $\text{Cov}(r_e(\kappa), r_i) = 0$ for $i = 1, \ldots, N$. Let $\tilde{\mathbf{r}} = (r_1, \ldots, r_{N+1}(\kappa))'$ be the extended vector of random returns, with $r_{N+1}(\kappa) = r_e(\kappa)$. We denote by $\tilde{r}_j := \mathbb{E}[r_j]$, for $j = 1, \ldots, N$, the expected return of the $j^{th}$ risky asset, and by $\tilde{r}_e(\kappa) := \mathbb{E}[r_e(\kappa)]$ the expected return of the new asset. Also, let $\tilde{\Sigma}(\kappa) = [\sigma_{i,j}]$ be the extended $(N+1) \times (N+1)$ covariance matrix, where we set $\sigma_{i,N+1}(\kappa) = \text{Cov}(r_e(\kappa), r_i)$, for $i = 1, \ldots, N$, and $\sigma_{N+1,N+1}(\kappa) = \Sigma_e^2(\kappa)$. We assume that $\tilde{\Sigma}(\kappa)$ is positive definite for any $\kappa \geq 0$.

The agent then solves the following revised mean-variance optimization problem with exploration:

$$\inf_{\kappa \geq 0} \left\{ \begin{array}{l} \min_{\mathbf{u} \in \mathbb{R}^N, v \in \mathbb{R}, u_0 \in \mathbb{R}} \left( \sum_{i=1}^N \sum_{j=1}^N \sigma_{i,j} u_i u_j + \sigma_{N+1,N+1}(\kappa) v^2 \ight. \\
+ 2 \sum_{i=1}^N \sigma_{N+1,i}(\kappa) u_i v \\
\left. \right) \\
\text{s.t.} \\
\sum_{i=1}^N u_i \tilde{r}_i + v \tilde{r}_{N+1}(\kappa) + u_0 r_f = \mu x_0, \\
\sum_{i=1}^N u_i + v + u_0 = x_0 - \kappa. \end{array} \right\} \quad (\text{MVE}_\rho(\mu, x_0))$$

While being surely more realistic, the model formulation accounting for correlation between the new asset and the existing investment opportunities leads to a less transparent solution; see Appendix B for details. Nonetheless, we can identify two scenarios. When the new asset has positive correlation with the initial investment universe, the main insights of our model remain valid, supporting the basic intuition provided by the simplified case without correlation between
the discovered asset and the pre-existing assets. On the other hand, different conclusions might be drawn if the newly discovered asset is assumed to have negative correlation with some assets in the initial investment universe. For instance, in this case there exist situations in which the agent would find optimal to reduce the variance of the extended portfolio by investing even a small amount for information acquisition on a new asset with low return and negative correlation with the initial assets (thus providing a counterexample to Corollary 2(ii)). Asset classes with these characteristics are certainly not common, so we leave this prediction to an empirical investigation.

The next theorem provides conditions for the well-posedness of problem \( \text{MVE}_\rho(\mu, x_0) \). This result parallels Theorem 1.

**Theorem 3.** Consider \( \tilde{D}_e(\kappa) \) as given in (35), and assume that

\[
\limsup_{\kappa \to \infty} \frac{\kappa \tilde{D}_e'(\kappa)}{\tilde{D}_e(\kappa)} < 1. \tag{10}
\]

The problem \( \text{MVE}_\rho(\mu, x_0) \) is well-posed and there is an optimal cost for exploration and optimal investment strategy. In particular, both the optimal value of and optimal amount for exploration and optimal investment strategy for \( \text{MVE}_\rho(\mu, x_0) \) are finite. Furthermore, the optimal amount devoted to exploration \( \kappa^* \) is either zero, or satisfies

\[
r_f \tilde{D}_e(\kappa^*) = \frac{1}{2} \tilde{D}_e'(\kappa^*) [ (\mu - r_f) x_0 + r_f \kappa^* ] . \tag{11}
\]

For the well-posedness of problem \( \text{MVE}_\rho(\mu, x_0) \), the assumption in (10) now requires that the performance (from the perspective of a mean-variance investor) measured by \( \tilde{D}_e(\kappa) \) of the extended portfolio grows asymptotically slower than linearly as a function of the amount devoted to exploration. This condition is intuitive and generalizes Assumption 1.

### 5.2 Exploration of multiple assets

We relax the assumption that the agent can only explore a single new asset by allowing for concurrent exploration of an arbitrary number \( M \in \mathbb{N}_0 \) of new assets, where, notably, \( M \) is itself a decision variable. The random returns of the new assets are given by \( r^M_e = (r_{e,1}, \ldots, r_{e,M})' \), with \( r_{e,l} \) denoting the random return of the \( l^{th} \) new asset, for \( l = 1, \ldots, M \). We define by \( \bar{r}_e^M = \).
The expected returns and by $\Sigma^M_e := \mathbb{E} \left[ (\bar{r}^M_e - \bar{r}^M_{e,l})(\bar{r}^M_{e,l} - \bar{r}^M_{e,l})' \right]$ the covariance matrix of the newly discovered assets.

In this setting, exploring each new asset comes at a cost $\kappa_l > 0$, for $l = 1, \ldots, M$, which is exogenously given. The vector of such costs is denoted as $\kappa = (\kappa_1, \ldots, \kappa_M)'$. For simplicity, we go back to the assumption of newly discovered assets being independent of the assets in the initial investment universe, as well as of one another. We will keep this assumption for the remainder of the section.

The agent then chooses an investment strategy described by the amounts $u_0 \in \mathbb{R}$ and $u \in \mathbb{R}^N$ invested in the initial investment universe, the number $M \in \mathbb{N}_0$ of new assets he/she wants to explore, and the amount $v \in \mathbb{R}^M$ allocated in the $M$ newly discovered assets, where $v_l$ denotes the amount invested in the $l^{th}$ new asset, for $l = 1, \ldots, M$.

This scenario corresponds to solving the following mean-variance optimization problem with exploration of multiple assets:

$$\inf_{M \in \mathbb{N}_0} \left( \min_{u \in \mathbb{R}^N, v \in \mathbb{R}^M, u_0 \in \mathbb{R}} \begin{pmatrix} u' \Sigma u + v' \Sigma^M_e v \\ \text{s.t.} \quad u' \bar{r} + v' \bar{r}^M_{e,l} + u_0 r_f = \mu x_0, \\ u'1_N + v'1_M + u_0 = x_0 - \kappa'1_M. \end{pmatrix} \right), \quad (MVE_M(\mu, x_0))$$

As usual, we can proceed by first considering the inner optimization problem of $(MVE_M(\mu, x_0))$ for a fixed $M \in \mathbb{N}_0$. The proof of the next proposition follows the same arguments developed for the proof of Proposition 1, and is therefore omitted.

**Proposition 5.** Let

$$A_M = 1'_N \Sigma^{-1} \bar{r} + \sum_{l=1}^M \frac{\bar{r}_{e,l}}{\Sigma^2_{e,l}}, \quad B_M = \bar{r}' \Sigma^{-1} \bar{r} + \sum_{l=1}^M \frac{\bar{r}_{e,l}^2}{\Sigma^2_{e,l}},$$

$$C_M = 1'_N \Sigma^{-1} 1_N + \sum_{l=1}^M \frac{1}{\Sigma^2_{e,l}}, \quad D_M = r_f^2 C_M - 2r_f A_M + B_M + \sum_{l=1}^M \frac{(\bar{r}_{e,l} - r_f)^2}{\Sigma^2_{e,l}}.$$

The problem $(MVE_M(\mu, x_0))$ has the unique solution

$$u = \frac{\mu x_0 - r_f (x_0 - \kappa'1_M)}{D_M} \Sigma^{-1} (\bar{r} - r_f 1_N),$$

$$u_l = \frac{\mu x_0 - r_f (x_0 - \kappa'1_M)}{D_M} \frac{\bar{r}_{e,l} - r_f}{\Sigma^2_{e,l}}, \quad l = 1, \ldots, M,$$

$$u_0 = x_0 - \kappa'1_M - \frac{\mu x_0 - r_f (x_0 - \kappa'1_M)}{D_M} (A_M - r_f C_M)$$

(12)
and the optimal value is given by

$$\sigma_{MVE_M}^2(M) = \frac{(\mu x_0 - r_f(x_0 - \kappa))^2}{D_M}.$$  \hspace{1cm} (13)

Plugging (13) into (MVE_M(\mu, x_0)), we obtain

$$\min_{M \in \mathbb{N}_0} \left( \frac{(\mu x_0 - r_f(x_0 - \kappa))^2}{D_M} \right).$$  \hspace{1cm} (14)

The minimization in (14) now involves a discrete decision variable \( M \in \mathbb{N}_0 \). Such mixed-integer programming problems are notoriously difficult to solve analytically. Clearly, we can characterize a local condition for the minimum by requiring \( M \) to be such that

$$\sigma_{MVE_M}^2(M+1) \leq \sigma_{MVE_M}^2(M) \leq \sigma_{MVE_M}^2(M-1).$$

We illustrate the above model with an example.

**Example 1.** We retrieve the same setting of the numerical example in Section 4. The original investment universe consists of three risky assets with expected return and covariance matrix given in (8), and a risk-free asset providing a return \( r_f = 1.03 \). We also set the initial wealth to \( x_0 = 10'000'000 \) USD and the target expected return to \( \mu = 1.3 \).

The statistical properties of the newly discovered assets are chosen according to the recursion

$$\bar{r}_{e,1} = 1.25, \quad \Sigma_{e,1} = 0.3,$$

$$\bar{r}_{e,l} = \max\left\{1, \bar{r}_{e,l-1} - \frac{0.1}{l^2}\right\}, \quad \Sigma_{e,l} = \Sigma_{e,l-1} - \frac{0.1}{l^2}, \quad j = 2, 3, \ldots$$  \hspace{1cm} (15)

Note that the recursive expressions in (15) yield a sequence of assets with decreasing Sharpe ratios, i.e., \( \frac{\bar{r}_{e,j}-r_f}{\Sigma_{e,j}} > \frac{\bar{r}_{e,j+1}-r_f}{\Sigma_{e,j+1}} \), for \( j \in \mathbb{N} \). This property reflects the intuition that the agent discovers “low hanging fruits” first and that it becomes increasingly difficult to find new profitable opportunities.

The cost of exploration remains to be defined. For this, we will distinguish two cases: in the first case, the cost is equal for each new asset and given by \( k_l = 300'000 \) USD, for \( l = 1, \ldots, M \). In the second case, we assume the presence of economies of scale and set \( k_l = \max\{100'000, 300'000 - (l - 1)10'000\} \) USD, for \( l = 1, \ldots, M \).

For reference, the optimal investment strategies are provided in Appendix C. In Figure 4, we show the optimal variance of problem (MVE_M(\mu, x_0)) as a function of the number of new assets. In Subfigure 4a the cost for exploration is assumed to be fixed, while in Subfigure 4b we assume that economies of scale occur during the process of exploration.
Figure 4: Objective value of $(\text{MVE}_M(\mu, x_0))$ as a function of the number of new assets.

(a) No economies of scale

(b) Economies of scale

Notes. The figure shows the objective value for the mean-variance problem with exploration of many assets under the assumption of fixed or decreasing cost for exploration. The initial investment universe is described in (8) and the newly discovered assets are characterized by (15). The initial wealth is $x_0 = 10'000'000$ USD and the target expected return $\mu = 1.3$.

Although we provided here only an illustrative example, Figure 4 hints already at the following insights. In the absence of economies of scale for the cost of exploration, Subfigure 4a shows that it is optimal to focus only on a small number of new investment opportunities. On the other hand, when economies of scale arise (Subfigure 4b) the agent also attains a second local optimum by pooling a larger number of new investments.\textsuperscript{2} The latter scenario can be interpreted in terms of the increase in expertise or positive externalities within a company when multiple new opportunities are undertaken; such benefits might reduce the cost of exploration and information acquisition for further projects, thus justifying investments which would not be appealing from the initial perspective.\textsuperscript{3}

\textsuperscript{2}We observed this phenomenon under a certain range of parameters. In several other cases, for instance when the economies of scale are too slow or the target expected return is significantly lower, the agent finds him/herself in a similar situation as in Subfigure 4a without economies of scale, where the optimum is attained only for a small number of new assets.

\textsuperscript{3}This could be the case, for instance, of a company that wants to invest in asset classes for which it lacks internal expertise. The company could then find optimal to either hire a new analyst to focus on a small number of bets (say, venture capital), or hire an entire team of analysts dedicated to build a diversified sub-portfolio of alternative investments.
We leave a detailed study of the robustness of these conclusions in different contexts and under more general assumptions as an interesting and challenging avenue for future research.

5.3 Uncertainty: concurrent vs sequential optimization

In the model formulations discussed thus far, it is assumed that the distributional properties of the newly discovered assets are known as functions of the amount devoted to information acquisition. Herein, we consider a setting where the agent chooses the investment for information acquisition without knowing ex-ante the true value of the expected return of the new asset, which is denoted by $\bar{r}_e$. For simplicity, we assume that variance $\Sigma^2_e > 0$ is known and limit our analysis on a binary outcome for the exploration: success or failure. If exploration is successful, the new asset yields a positive expected return $\bar{r}_e > r_f$; if exploration is not successful, the new asset yields an expected return $\bar{r}_e = r_f$, thus resulting in an unworthy investment.\(^4\)

Remark 6. Non-binary outcomes for the mean return of the new asset could be studied, e.g., by assuming $\bar{r}_e$ to be drawn with cumulative distribution function $F_{\bar{r}_e}(\kappa)$ depending on the amount devoted for exploration $\kappa \geq 0$. One could further generalize the setting by also allowing for the variance $\Sigma^2_e$ to be random and drawn according to another cumulative distribution function $F_{\Sigma^2_e}(\kappa)$, $\kappa \geq 0$.

The outcome of the exploration is defined through a random variable $S$, taking values in $\{0, 1\}$. We denote by $p(\kappa) = P[S = 1 | \kappa]$ the probability of success. A key assumption of this model is that the probability of successful exploration depends increasingly on the amount devoted to information acquisition $\kappa$, that is, the agent can positively affect his/her chances of success by investing a larger amount for exploration.\(^5\) Different to our main setting, however,

\(^4\)Since the variance is assumed to be strictly positive, if the new asset yields an expected return that is equal to the risk-free rate, it is easy to see that a mean-variance agent would choose not to invest in this asset. Further, note that in principle one could even suppose that $\bar{r}_e < r_f$. We rule out this possibility throughout the paper, in order to avoid that the agent decides to invest for information acquisition on a new asset with the aim to short-sell it. Although this is not an unrealistic scenario, in practice it might result in too risky a strategy to carry out.

\(^5\)To specify the probability of success, a possible choice is to use the Gompertz function $p(\kappa) = e^{-e^{\alpha - \beta \kappa}}$, for some constants $\alpha$ and $\beta$. Originally applied by Gompertz (1825) to model mortality rates, Gompertz functions
this investment does not translate into a higher ex-post (after observation of the outcome of \( S \)) mean return \( \bar{r}_e \), which is now assumed to be fixed.

The present setting can be dealt with by using two distinct approaches, which we label as concurrent and sequential optimization. Both methods might be relevant for particular applications.

Under the concurrent formulation, the agent concurrently decides on the amount to explore and the investment strategy. Mathematically, it is easy to observe that this problem boils down to solving our initial optimization problem \((\text{MVE}(\mu, x_0))\) where we set \( \bar{r}_e(\kappa) = \bar{r}_e p(\kappa) + r_f(1 - p(\kappa)) \), i.e., the expected return of the newly discovered asset is taken as the weighted average between the two scenarios, with weights given by the probabilities of success or failure; see, for instance, Section 5 in Décamps et al. (2005), in which the authors compare an investment timing problem under uncertain and average mean return. Furthermore, note that this equivalence holds without additional complications also for general settings as described in Remark 6. Indeed, the latter observation suggests that our main theoretical results are robust with respect to uncertainty about the distributional properties of the new asset.

The sequential formulation, on the other hand, is more delicate. In this case the agent decides first how much to devote to exploration, observes the characteristics of the newly discovered asset, and only then commits to an amount to be invested in the new asset. A conceptual as well as technical difficulty arises in light of the fact that the mean-variance objective is not dynamically consistent, that is, plans of action that are optimal in the present may no longer be optimal at future times (Strotz (1955)). In particular, such time-inconsistency entails that Bellman’s principle of optimality does not hold in this case, thus complicating the analysis. Conceptually, one has to decide on whether to consider an agent who is able to pre-commit or a sophisticated agent not able to pre-commit and then solve for an equilibrium solution. For brevity, in this subsection we will only outline the results for the pre-committed strategy and defer to further studies the treatment of the sophisticated agent. We refer to Björk and Murgoci (2014) and He and Zhou (2021) for reviews of the literature on time-inconsistent are commonly used to describe variables whose growth rate is slow at the beginning and at the end of a given interval (Winsor (1932)). For instance, fixing \( \alpha = 3, \beta = 10^{-5}, p(\kappa) \) would yield a vanishingly small probability of success if \( \kappa = 0 \) and a probability of success of 1/2 if the agent spends 336’651 USD in information acquisition.
problems in discrete- and continuous-time settings respectively.

The initial investment universe is represented by a risk-free and, for simplicity, a single risky asset \( N = 1 \) with excess return \( R := r - r_f \). Similarly, we consider a newly explorable asset with excess return \( R_e := r_e - r_f \). The expected values of the excess returns are denoted by \( \bar{R} \) for the existing asset and \( \bar{R}_e \) for the new asset, while the standard deviations are denoted by \( \Sigma \) and \( \Sigma_e \), respectively.

For a given initial wealth \( x_0 > 0 \), the agent’s wealth \( x_t \), for \( t = 1, 2 \), is determined by

\[
\begin{align*}
  x_1 &= x_0 - \kappa, \\
  x_2 &= r_f x_1 + uR + I_{\{S=1\}}vR_e,
\end{align*}
\]

(16)

where \( I_{\{S=1\}} \) is an indicator function taking value 1 if \( S = 1 \) and 0 otherwise. We emphasize that the time variable \( t \in \{0, 1, 2\} \) does not denote a multiperiod investment problem as we will study in the next Subsection 5.4, but only indicates that the two decision variables are now to be determined sequentially: At step \( t = 0 \), the agent determines the amount devoted for exploration \( \kappa \geq 0 \); at step \( t = 1 \), the agent observes the characteristics of the newly discovered asset and decides on the investment strategy \( u \) and \( v \); at the final step \( t = 2 \), the agent observes the outcome of the risky returns.

In a compact form, the \textit{mean-variance optimization problem with sequential exploration} can be stated as follows:

\[
\begin{align*}
  \inf_{\kappa \geq 0, u \in \mathbb{R}, v \in \mathbb{R}} & \quad \text{Var} \left[ x_2 \right], \\
\text{s.t.} & \quad \mathbb{E} \left[ x_2 \right] = \mu x_0, \\
& \quad x_1 = x_0 - \kappa, \\
& \quad x_2 = r_f x_1 + uR + I_{\{S=1\}}vR_e.
\end{align*}
\]

(MVSE(\( \mu, x_0 \)))

To remedy the issue of time-inconsistency of this problem, we apply the well-known embedding technique of Li and Ng (2000) and Zhou and Li (2000), which consists in reformulating a dynamic mean-variance optimization by means of an equivalent (time-consistent) quadratic objective; see Appendix A.10.

The next proposition provides the optimal strategies and value for problem \( \text{MVSE}(\mu, x_0) \).
Proposition 6. The optimal investment strategies for problem \((\text{MVSE}(\mu, x_0))\) are given by

\[
\begin{align*}
    u &= -\left( r_f(x_0 - \kappa) - (\mu x_0 - \lambda) \right) \frac{\bar{R} \Sigma^2_{\epsilon} \Sigma}{(\Sigma^2 + \bar{R}^2)(\Sigma^2 + \Sigma^2_{\epsilon} + \mathbf{1}_{\{S=1\}} \bar{R}^2_{\epsilon}) - \mathbf{1}_{\{S=1\}} \bar{R}^2 \bar{R}^2_{\epsilon}} , \\
    v &= -\mathbf{1}_{\{S=1\}}(r_f(x_0 - \kappa) - (\mu x_0 - \lambda)) \frac{\bar{R} \Sigma^2_{\epsilon}}{(\Sigma^2 + \bar{R}^2)(\Sigma^2_{\epsilon} + \bar{R}^2_{\epsilon}) - \bar{R}^2 \bar{R}^2_{\epsilon}} ,
\end{align*}
\]

(17)

and the optimal value is

\[
\sigma^2_{\text{MVSE}}(\kappa) = (r_f(x_0 - \kappa) - (\mu x_0 - \lambda))^2 \left( \frac{\Sigma^2 \Sigma^2_{\epsilon}}{(\Sigma^2 + \bar{R}^2)(\Sigma^2_{\epsilon} + \bar{R}^2_{\epsilon}) - \bar{R}^2 \bar{R}^2_{\epsilon}} p(\kappa) \right) + \frac{\Sigma^2}{\Sigma^2 + \bar{R}^2} (1 - p(\kappa)) - \lambda^2 .
\]

(18)

Moreover, the optimal amount invested for exploration \(\kappa\) is either 0, or satisfies

\[
2 \left( (\Sigma^2 + \bar{R}^2) \Sigma^2_{\epsilon} + p(\kappa) \Sigma^2 \right) + (r_f(x_0 - \kappa) - (\mu x_0 - \lambda)) p'(\kappa) \Sigma^2 = 0,
\]

(19)

and \(\lambda\) is chosen so that \(E[x_2^*] = \mu x_0\), where \(x_2^*\) denotes the value of \(x_2\) under the optimal strategies.

Proposition 6 shows that, qualitatively, the optimal policies for the sequential formulation correspond to those for the static case. A subtle difference lies in the interpretation of the amount devoted for exploration \(\kappa\). In this case, given the distributional properties of a new asset, the agent chooses \(\kappa\) in order to increase the probability that exploration is successful. In our starting model, on the other hand, \(\kappa\) is chosen so that a new asset with better distributional properties is discovered.

5.4 Multi-period setting

In the previous subsection, we described a sequential, quasi-static formulation in which the agent has only a single opportunity for exploration and portfolio selection. We conclude this section by introducing a multi-period setting where the processes of information acquisition and investment in new assets are dynamic. We will first outline a general framework for a finite time horizon \(T \geq 1\) and then turn our attention to a two-period example.

Some notation is borrowed and adapted from Subsection 5.3. For simplicity, we consider again an initial investment universe with one risk-free and one risky asset. The risk-free rate is
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set constantly equal to \( r_f \) and the random excess return of the risky asset between time \( t \) and \( t + 1 \) is given by \( R_t = r_t - r_f \), where \( r_t \in L^2(\mathbb{P}) \) is the random total return between time \( t \) and \( t + 1 \). The expected value and standard deviation of the excess return at time \( t \) are given by \( \bar{R}_t \) and \( \Sigma_t \), respectively, and we denote by \( u_t \) the absolute amount of the agent’s wealth invested in the risky asset at the beginning of the \( t^{th} \) time period.

Additionally, a sequence of \( M \geq 1 \) new assets are discovered at times \( \tau_1, \ldots, \tau_M \). Without loss of generality, we set \( \tau_{j-1} \leq \tau_j \leq T - 1 \), for \( j = 2, \ldots, M \). These exploration times are supposed to be fixed and known in advance by the agent. To give some perspective, this setting can be used to model a scenario in which private companies announce their initial public offerings, thus providing new investment opportunities to portfolio managers. These new opportunities, however, typically require additional costs for their return characteristics (costs and risks) to be well understood that some agents might not be willing to bear. To the best of our knowledge, and surprisingly, this problem does not appear to have been studied in the literature. When the new investment opportunities are discovered, the agent will choose an optimal (lump sum) amount \( \kappa_{e,j}^{\tau_j} \geq 0 \), for \( j = 1, \ldots, M \), to devote for information acquisition on the \( j^{th} \) new asset, and \( v_{e,j}^t \) to be allocated in the same asset at each time period \( t \geq \tau_j \). The random total returns of the new assets between time \( t \) and \( t + 1 \) are expressed by \( r_{e,j,t} \in L^2(\mathbb{P}) \), and their distributional properties are assumed to depend on the information acquired at the time of exploration \( \tau_j \). More precisely, the expected value and standard deviation of the new assets between time \( t \) and \( t + 1 \) are defined by \( \bar{r}_{e,j,t} (\kappa_{e,j}^{\tau_j}) \) and \( \Sigma_{e,j,t} (\kappa_{e,j}^{\tau_j}) \), for \( j = 1, \ldots, M \). In turn, we define the mean excess return by \( \bar{R}_{e,j,t} (\kappa_{e,j}^{\tau_j}) \). Consistent with assumptions in Section 2, a “no-free-lunch” condition imposing \( \bar{R}_{e,j,t}(0) = 0 \) and \( \Sigma_{e,j,t}(0) > 0 \), for \( t = 0, 1, \ldots, T - 1 \) and \( j = 1, \ldots, M \), is in place.

Remark 7. For the sake of parsimony, in this formulation we suppose that information can be acquired only at the time \( \tau_j \) of discovery of a new asset. Under this assumption, if agents do not exercise the option to explore the new asset, they cannot revise their choice later on during the investment horizon. An extension of this model could be to assume a Bayesian setting where the information acquisition process extends over time and the distributional properties of a new asset at time \( t \) depend on both observations and information acquired up to that time. In this case, agents would be allowed to update their beliefs and jump onto the new opportunity at later
stages (after $\tau_j$).

Another alternative, but conceptually different, model formulation could be to assume that exploration happens at stopping times which are discretionary (that is, decision variables) for the agent. Although seemingly more general, and certainly an interesting control/stopping problem, this situation poses in practice less constraints on the agent, as he/she can now choose to explore at any time during the investment horizon instead of only at a fixed number of exogenously given times.

Our probability space $(\Omega, \mathcal{G}, \mathbb{P})$ is endowed with a filtration $\mathcal{G}_t$, for $t = 0, 1, \ldots, T - 1$, generated by the excess returns $R_0, R_1, \ldots, R_{t-1}$ of the asset in the initial investment universe and, after exploration, by the excess returns $R_{e_j, \tau_j}, \ldots, R_{e_j, t-1}$ of the newly discovered assets; see Appendix D for details on the construction of $(\mathcal{G}_t)_{t=0}^{T-1}$. We will assume that trading strategies $u = (u_t)_{t=0}^{T-1}$, $\kappa^e_{\tau_j}, v^e_j = (v^e_j)_{t=\tau_j}^{T-1}$, for $j = 1, \ldots, M$, are adapted to $(\mathcal{G}_t)_{t=0}^{T-1}$.

For a given initial wealth $x_0 > 0$, the wealth process $(x_t)_{t=0}^T$ is now determined by

$$x_{t+1} = r_f x_t + u_t R_t + \sum_{j=1}^M \left( 1_{\{t \geq \tau_j\}} v^e_{t,j} R_{e_j,t}(\kappa^e_{\tau_j}) - 1_{\{t = \tau_j\}} r_f \kappa^e_{\tau_j} \right), \quad t = 0, 1, \ldots, T - 1. \tag{5.3}$$

Owing to the discussion on the dynamic inconsistency of the mean-variance objective in Subsection 5.3, we can directly state the following mean-variance portfolio selection problem with dynamic exploration:

$$\inf_{u, \kappa^e, v^e} \mathbb{E} \left[ (x_T - (\mu x_0 - \lambda))^2 \right], \quad (\text{MVDE}(\mu, x_0))$$

s.t. $x_{t+1} = r_f x_t + u_t R_t + \sum_{j=1}^M \left( 1_{\{t \geq \tau_j\}} v^e_{t,j} R_{e_j,t}(\kappa^e_{\tau_j}) - 1_{\{t = \tau_j\}} r_f \kappa^e_{\tau_j} \right), \quad t = 0, 1, \ldots, T - 1$

$$u \in \mathbb{R}^T, \quad \kappa^e_{\tau_j} \geq 0, \quad v^e_j \in \mathbb{R}^{T-\tau_j}, \quad j = 1, \ldots, M,$$

where $\kappa = (\kappa^e_{\tau_1}, \ldots, \kappa^e_{\tau_M})$ and $v^e = (v^e_1, \ldots, v^e_M)$, and $\lambda$ is a Lagrange multiplier chosen so that $\mathbb{E} [x_T^\tau] = \mu x_0$, with $x_T^\tau$ denoting the value of $x_T$ under the optimal strategies.

\[6\] We point out here a slight inconsistency in the notation. Differently from the single-period setting, where we use $u_0$ to denote the investment in the risk-free asset, for the multi-period setting $u_0$ denotes the investment in the risky asset at time 0.

\[7\] For the dynamic formulation of the wealth process we directly employed the self-financing condition and replaced the investment in the risk-free asset at time $t$ by $x_t - u_t - \sum_{j=1}^M (1_{\{t \geq \tau_j\}} v^e_{t,j} + 1_{\{t = \tau_j\}} \kappa^e_{\tau_j}).$
In the next example, we highlight some peculiar features of problem \((\text{MVDE}(\mu,x_0))\) by examining a two-period setting \((T = 2)\).

**Example 2.** Imagine that a new asset is discovered at each time period: that is, fix \(\tau_1 = 0\) and \(\tau_2 = 1\). The mean and standard deviation of the new assets are specified according to the same parametrization used in Section 4, i.e,

\[
\tilde{R}_{e_1,t}(\kappa_{0}^{e_1}) = \alpha_t \arctan \left( \frac{\kappa_{0}^{e_1}}{\beta} \right), \quad \Sigma_{e_1,t}(\kappa_{0}^{e_1}) = s_0^t + \frac{1}{(s_1^t + \kappa_{0}^{e_1})^2}, \quad t = 1, 2, \kappa_{0}^{e_1} \geq 0,
\]

\[
\tilde{R}_{e_2,1}(\kappa_{1}^{e_2}) = \nu \arctan \left( \frac{\kappa_{1}^{e_2}}{\eta} \right), \quad \Sigma_{e_2,1}(\kappa_{1}^{e_2}) = q_0 + \frac{1}{(q_1 + \kappa_{1}^{e_2})^2}, \quad \kappa_{1}^{e_2} \geq 0,
\]

for some constants \(\alpha_1, \alpha_2, \beta, s_0^0, s_0^1, s_1^0, s_1^1, \gamma, \nu, \eta, q_0, q_1, \zeta\); specifics on the parameters are given in Appendix E.

In order to guarantee a straightforward interpretation of the results, we further assume that asset returns are distributed according to a binomial model (Cox and Ross (1976); Cox et al. (1979)). Namely, we suppose that \(r_{t}, \text{ for } t = 1, 2\), takes either the value \(1 + \delta_t\), \(\delta_t > 0\), with probability \(p_t \in (0, 1)\) or the value \(1 - \delta_t\) with probability \(1 - p_t\). Equivalent assumptions are made for \(r_{e_1,t}, \text{ with } t = 1, 2\), and \(r_{e_2,1}\). \(^8\)

Under this framework, the mean-variance problem with exploration \((\text{MVDE}(\mu,x_0))\) reads as

\[
\inf_{u, \kappa, v^e} \mathbb{E} \left[ (x_2 - (\mu x_0 - \lambda))^2 \right],
\]

s.t.

\[
x_1 = r_f(x_0 - \kappa_{0}^{e_1}) + u_0 R_0 + v_{0}^{e_1} R_{e_1,0}(\kappa_{0}^{e_1}),
\]

\[
x_2 = r_f(x_1 - \kappa_{1}^{e_2}) + u_1 R_1 + v_{1}^{e_1} R_{e_1,1}(\kappa_{0}^{e_1}) + v_{1}^{e_2} R_{e_2,1}(\kappa_{1}^{e_2}),
\]

\(u \in \mathbb{R}^2, \kappa_{0}^{e_1} \geq 0, v_{0}^{e_1} \in \mathbb{R}^2, \kappa_{1}^{e_2} \geq 0, v_{1}^{e_2} \in \mathbb{R}\).

We sketch the procedure to solve problem \((21)\) in Appendix E, and summarize the qualitative results in the following proposition.

**Proposition 7.** Consider the context of Example 2. Then:

\(^8\)Note that the choice of a binomial model is not inconsistent with the specification in \((20)\). Indeed, one can easily find the parameters of a symmetric binomial distribution that are consistent with desired mean and variance. Additionally, we emphasize that the analysis and solution of this problem do not depend on the choice of the return distribution, which herein is made purely to provide a simple scenario-based interpretation of the results.
(1) The optimal strategy at time 0 is uniquely characterized by the agent exercising the option to invest in the first newly discovered asset.

(2) The optimal strategy at time 1 is contingent on the performance of the extended portfolio between time 0 and time 1, which defines four cases:

(i) $R_0 > 0$ and $R_{e1,0} > 0$;

(ii) $R_0 < 0$ and $R_{e1,0} > 0$;

(iii) $R_0 > 0$ and $R_{e1,0} < 0$;

(iv) $R_0 < 0$ and $R_{e1,0} < 0$.

In case (i), the agent does not exercise the option to invest in the second newly discovered asset.\textsuperscript{9} In cases (ii), (iii) and (iv), the agent invests in the second newly discovered asset, and both the investment for information acquisition and the wealth allocated in this asset increase with respect to the wealth drop after the first time period.

The initial results provided in Proposition 7 suggest that, for a mean-variance agent, investing on exploration of new assets is optimal at the early stage of the investment horizon and, subsequently, in cases of losses by the present portfolio of assets (possibly reducing wealth below a certain threshold). Oppositely, under positive past performance, devoting resources to the discovery of new assets is suboptimal.

Further, the amount devoted to exploration of new assets increases as the performance of the extended portfolio declines. In this regard, it is in fact well-known that mean-variance agents invest more when the difference between their present and target wealth is larger, thus becoming more risk-seeking in worse scenarios. This feature, whose desirability certainly depends on the context, has nonetheless received extensive empirical support; see, for instance, Andrade and Iyer (2009); Langer and Weber (2008); Page et al. (2014); Smith et al. (2009); Zhang and Semmler (2009).

\textsuperscript{9} In this case, the decision to not re-explore at time 1 does not imply that the second new asset is generally unworthy. Rather, given the characteristics of the currently available assets, one should interpret this decision as being dependent on the wealth level at that time and the likelihood to reach the objective at maturity.
6 Conclusions

We extended the classical mean-variance optimization problem by allowing the agent to devote an amount of his/her initial wealth for exploration of new investment opportunities. The novel mean-variance optimization problem with exploration is well-posed under the assumption of reasonable asymptotic elasticity of the Sharpe ratio of the newly discovered asset. We found that the optimal amount devoted to exploration can be determined by solving an associated equation and decreases when the quality of the existing investment universe increases. In contrast to the classical mean-variance setting where the investment performance is not affected by the initial wealth, richer agents can make better use of the option to explore for new investment opportunities and thus achieve a better investment performance in our model consistent with the empirical evidence for asset managers above the fund level. An explicit parametrization and numerical examples show that the model leads to reasonable investment behavior across a broad range of specifications for the existing investment universe and initial wealth levels.

For this paper, we have deliberately kept the base model as simple as possible in order to introduce new concepts in a familiar framework. We have subsequently shown that the new insights, predictions, and policy implications obtained herein are robust under more general conditions. Nonetheless, there remain several meaningful ways to extend the model beyond the single-period mean-variance framework. In particular, we have preliminarily discussed a multi-period setting which is notably flexible and can capture, for instance, portfolio selections problems when new stocks are listed through initial public offerings. We conjecture that a complete study of the latter problem, especially in combination with some of the features mentioned below, paves the way to novel perspectives. From a mathematical point of view, a continuous-time version of the problem is also worth careful consideration.

Further directions for future research are pointed out next. Firstly, portfolio optimization problems with an option to explore for new investment opportunities can be studied under preferences other than mean-variance objective, such as expected utility or behavioral preferences. While we expect our main qualitative results to hold under an expected utility formulation, behavioral features such as loss aversion or probability distortion might lead to new insights. Relatedly, uncertainty aversion over the distributional properties of the new asset could be
integrated in the framework through a robust worst-case formulation (Garlappi et al. (2006); Goldfarb and Iyengar (2003)), \( \alpha \)-maxmin preferences (Ghirardato et al. (2004)), or smooth ambiguity (Klibanoff et al., 2005). Finally, it would be interesting to study exploration from the perspective of general equilibrium theory. This would allow us to investigate how new investment opportunities are priced and what impact is to be expected on the prices of existing assets.

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Appendices

A Proofs

A.1 Proof of Proposition 1

Introducing the Lagrangian multipliers \( \lambda_1 \) and \( \lambda_2 \) for the expected wealth and budget constraint in \((\text{MVEF}(\mu, x_0; \kappa))\) respectively, the unconstrained problem becomes

\[
\min_{u \in \mathbb{R}^N, v \in \mathbb{R}, u_0 \in \mathbb{R}} \left( \frac{1}{2} (u' \Sigma u + \Sigma^2_{\kappa}(\kappa)v^2) - \lambda_1 (u' \bar{r} + v \bar{r}_{\kappa}(\kappa) + u_0 r_f - \mu x_0) - \lambda_2 (u' 1_N + v + u_0 - (x_0 - \kappa)) \right),
\]
where we added an auxiliary multiplier of 1/2 to the original objective. The Karush-Kuhn-Tucker conditions of the above program are

\[
\begin{align*}
\Sigma u - \lambda_1 \bar{r} - \lambda_2 1_N &= 0, \\
\Sigma_e^2(\kappa) v - \lambda_1 \bar{r}_e(\kappa) - \lambda_2 &= 0, \\
-\lambda_1 r_f - \lambda_2 &= 0, \\
u' \bar{r} + v \bar{r}_e(\kappa) + u_0 r_f &= \mu x_0, \\
u' 1_N + v + u_0 &= x_0 - \kappa.
\end{align*}
\]

From (24) we obtain that \( \lambda_2 = -r_f \lambda_1 \). Using (22) and (23), this yields that \( u = \lambda_1 \Sigma^{-1} (\bar{r} - r_f 1_N) \) and \( v = \lambda_1 \frac{\bar{r}_e(\kappa) - r_f}{\Sigma_e^2(\kappa)} \). From (26) we then get that

\[
u_0 = x_0 - \kappa - \lambda_1 (\bar{r} - r_f 1_N)' \Sigma^{-1} 1_N - \lambda_1 \frac{\bar{r}_e(\kappa) - r_f}{\Sigma_e^2(\kappa)} = x_0 - \kappa - \lambda_1 (A_e(\kappa) - r_f C_e).
\]

Next, (25) becomes

\[
\begin{align*}
\mu x_0 &= \lambda_1 (\bar{r} - r_f 1_N)' \Sigma^{-1} \bar{r} + \lambda_1 \frac{\bar{r}_e(\kappa) - r_f \bar{r}_e(\kappa) + r_f (x_0 - \kappa - \lambda_1 (A_e(\kappa) - r_f C_e(\kappa)))}{\Sigma_e^2(\kappa)} \\
&= \lambda_1 \left( B - r_f A + \frac{\bar{r}_e(\kappa)^2 - r_f \bar{r}_e(\kappa)}{\Sigma_e^2(\kappa)} - r_f A_e(\kappa) + r_f^2 C_e(\kappa) \right) + r_f (x_0 - \kappa)
\end{align*}
\]

and thus \( \lambda_1 = \frac{\mu x_0 - r_f (x_0 - \kappa)}{\Sigma_e^2(\kappa)} \). Replacing \( \lambda_1 \) with this expression in the earlier equalities for \( u, v, \) and \( u_0 \) yields (4). The optimal value is given by

\[
\frac{(\mu x_0 - r_f (x_0 - \kappa))^2}{D_e(\kappa)^2} (\bar{r} - r_f 1_N)' \Sigma^{-1} (\bar{r} - r_f 1_N) + \frac{(\mu x_0 - r_f (x_0 - \kappa))^2}{D_e(\kappa)^2} \frac{(\Sigma^2(\kappa))}{D_e(\kappa)^2} \frac{(\bar{r}_e(\kappa) - r_f)^2}{\Sigma_e^2(\kappa)},
\]

which easily simplifies to (5).

\[\square\]

### A.2 Proof of Corollary 1

Using that \( \bar{r}_e(0) = 0 \) and \( \Sigma_e(0) > 0 \), this follows immediately from (4) and (5).

\[\square\]

### A.3 Proof of Corollary 2

Taking the derivative of \( \sigma^2_{MVEF}(\kappa) \) given in (5) yields

\[
\frac{d}{d\kappa} \sigma^2_{MVEF}(\kappa) = \frac{2r_f (\mu x_0 - r_f (x_0 - \kappa)) (D + S_e(\kappa)^2) - 2S_e(\kappa) S'_e(\kappa) (\mu x_0 - r_f (x_0 - \kappa))^2}{D + S_e(\kappa)^2}, \tag{27}
\]
Since $S_e(0) = 0$ and $S'_e(0) < \infty$ we thus have that

$$\frac{d}{d\kappa} \sigma^2_{MVEF}(0) = \frac{2r_f (\mu x_0 - r_fx_0)}{D} > 0.$$  

Using continuity, we easily conclude (ii). The first claim (i) follows immediately from (27) and continuity.

A.4 Proof of Theorem 1

Recall the derivative of $\sigma^2_{MVEF}(\kappa)$ derived in (27). Since $\mu > r_f$, $\frac{d}{d\kappa} \sigma^2_{MVEF}(\kappa)$ is positive if and only if

$$r_fD + r_f S_e(\kappa)^2 > S_e(\kappa)S'_e(\kappa) (\mu x_0 - r_fx_0) + r_f\kappa S_e(\kappa)S'_e(\kappa),$$

and zero if and only if equality holds in (28). Due to Assumption 1, there exist $\epsilon > 0$ and $K' > 0$ such that $S_e(\kappa) > (1 + \epsilon)\kappa S'_e(\kappa)$ for any $\kappa > K'$. Take $K = \max \left\{ K', \frac{\mu - r_f}{\epsilon r_f} x_0 \right\}$. For $\kappa > K$ we have

$$r_fD + r_f S_e(\kappa)^2 > r_f S_e(\kappa)(1 + \epsilon)\kappa S'_e(\kappa) > S_e(\kappa)S'_e(\kappa)(\mu - r_f)x_0 + r_f\kappa S_e(\kappa)S'_e(\kappa),$$

where we used that $\kappa > K'$ for the first and that $\kappa > \frac{\mu - r_f}{\epsilon r_f} x_0$ for the second inequality. Therefore, $\frac{d}{d\kappa} \sigma^2_{MVEF}(\kappa) > 0$ for any $\kappa > K'$. We can thus optimize $\sigma^2_{MVEF}(\kappa)$ over the compact interval $[0, K]$. Due to continuity of $\sigma^2_{MVEF}$, the optimum exists and is finite. If the optimal solution is not zero, it must satisfy the first-order condition (28) with equality which is (7).

A.5 Proof of Proposition 2

Let $f : [0, \infty) \to \mathbb{R}$ be defined by

$$f(\kappa) = r_f S_e(\kappa)^2 - S_e(\kappa)S'_e(\kappa)(\mu - r_f)x_0 + r_f\kappa S_e(\kappa)S'_e(\kappa).$$

Because $S'_e(0) < \infty$, we have $f(0) = 0$ and the proof of Theorem 1 shows that Assumption 1 implies that there exists a $K > 0$ such that $f(\kappa) > 0$ for $\kappa > K$. Therefore, $\inf_{\kappa > 0} f(\kappa) > -\infty$ and there thus exists an investment universe $\mathcal{U}$ and associated $D > 0$ given in (1) and (2) such that (7) does not have a solution. It is clear that the same conclusion holds in any investment universe $\tilde{D}$ with associated $\tilde{D} \geq D$. 

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A.6 Proof or Proposition 3

For this proof, we denote the optimal variance of \( \text{MVEF}(\mu, x_0; \kappa) \) by \( \sigma^2_{\text{MVEF}}(\kappa; D) \) in order to make the dependence on the existing investment universe \( \mathcal{U} \) characterized by \( D \) explicit.

We first show the following claim. Let \( \kappa_1 < \kappa_2 \) such that \( \sigma^2_{\text{MVEF}}(\kappa_1; D) \leq \sigma^2_{\text{MVEF}}(\kappa_2; D) \). Then it also holds that \( \sigma^2_{\text{MVEF}}(\kappa_1; \tilde{D}) < \sigma^2_{\text{MVEF}}(\kappa_2; \tilde{D}) \).

Indeed, fix \( \kappa_1 \leq \kappa_2 \) with \( \sigma^2_{\text{MVEF}}(\kappa_1; D) \leq \sigma^2_{\text{MVEF}}(\kappa_2; D) \) and note that

\[
\frac{1}{\sigma^2_{\text{MVEF}}(\kappa_1; \tilde{D})} - \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_1; D)} = \frac{\tilde{D} - D}{\left( (\mu - r_f)x_0 + r_f\kappa_1 \right)^2} > \frac{\tilde{D} - D}{\left( (\mu - r_f)x_0 + r_f\kappa_2 \right)^2} = \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_2; \tilde{D})} - \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_2; D)}.
\]

Therefore,

\[
\frac{1}{\sigma^2_{\text{MVEF}}(\kappa_1; \tilde{D})} > \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_2; \tilde{D})} + \left( \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_1; D)} - \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_2; D)} \right) \geq \frac{1}{\sigma^2_{\text{MVEF}}(\kappa_2; D)},
\]

which proves the claim.

From the claim, we immediately obtain that, for \( \kappa' > \kappa^* \), \( \sigma^2_{\text{MVEF}}(\kappa^*; \tilde{D}) < \sigma^2_{\text{MVEF}}(\kappa'; \tilde{D}) \) and thus that \( \tilde{\kappa}^* \leq \kappa^* \).

\[\square\]

A.7 Proof of Proposition 4

For this proof, we denote the optimal variance of \( \text{MVEF}(\mu, x_0; \kappa) \) by \( \sigma^2_{\text{MVEF}}(\kappa; x_0) \) in order to make the dependence on the initial wealth explicit.

Similar to the proof of Proposition 3, we start with the following claim. Let \( \kappa_1 < \kappa_2 \) be such that \( \sigma^2_{\text{MVEF}}(\kappa_1; x_0) \geq \sigma^2_{\text{MVEF}}(\kappa_2; x_0) \). Then it also holds that \( \sigma^2_{\text{MVEF}}(\kappa_1; \tilde{x}_0) > \sigma^2_{\text{MVEF}}(\kappa_2; \tilde{x}_0) \).

To see this, we differentiate \( \sigma^2_{\text{MVEF}}(\kappa; x) \) with respect to \( x \) to obtain

\[
\frac{d}{dx}\sigma^2_{\text{MVEF}}(\kappa_1, x)|_{x=x_0} = 2(\mu - r_f)\frac{(\mu - r_f)x_0 + r_f\kappa_1}{D + S_e(\kappa_1)^2}
\]

\[
= 2\frac{\mu - r_f}{\sqrt{D + S_e(\kappa_1)^2}}\sigma_{\text{MVEF}}(\kappa_1, x_0)
\]

\[
> 2\frac{\mu - r_f}{\sqrt{D + S_e(\kappa_2)^2}}\sigma_{\text{MVEF}}(\kappa_2, x_0)
\]

\[
= \frac{d}{dx}\sigma^2_{\text{MVEF}}(\kappa_2, x)|_{x=x_0},
\]

(29)
From the above, we obtain that \( \frac{d}{dx} \sigma_{MVEF}^2(\kappa_1, x) > \frac{d}{dx} \sigma_{MVEF}^2(\kappa_2, x) \) for any \( x \geq x_0 \). If this were not the case, we let \( \bar{x} = \sup \{ x \geq x_0 \mid \frac{d}{dx} \sigma_{MVEF}^2(\kappa_1, x) \geq \frac{d}{dx} \sigma_{MVEF}^2(\kappa_2, x) \} \) and by continuity have \( x_0 < \bar{x} < \infty \). However,

\[
\begin{align*}
\sigma_{MVEF}^2(\kappa_1, \bar{x}) &- \sigma_{MVEF}^2(\kappa_1, x_0) = \int_{x_0}^{\bar{x}} \frac{d}{dx} \sigma_{MVEF}^2(\kappa_1, x) dx \\
&> \int_{x_0}^{2} \frac{d}{dx} \sigma_{MVEF}^2(\kappa_2, x) dx \\
&= \sigma_{MVEF}^2(\kappa_2, \bar{x}) - \sigma_{MVEF}^2(\kappa_2, x_0),
\end{align*}
\]

and we therefore have that

\[
\sigma_{MVEF}^2(\kappa_1, \bar{x}) > \sigma_{MVEF}^2(\kappa_2, \bar{x}) + \sigma_{MVEF}^2(\kappa_1, x_0) - \sigma_{MVEF}^2(\kappa_2, x_0) \geq \sigma_{MVEF}^2(\kappa_2, \bar{x}).
\]

With exactly the same argument as in (29), we would thus obtain that \( \frac{d}{dx} \sigma_{MVEF}^2(\kappa_2, \bar{x}) \) in contradiction to the definition of \( \bar{x} \). We conclude that \( \frac{d}{dx} \sigma_{MVEF}^2(\kappa_1, x) > \frac{d}{dx} \sigma_{MVEF}^2(\kappa_2, x) \) for any \( x \geq x_0 \). Repeating the argument in (30) with \( \bar{x} \) replaced with \( \bar{x}_0 \) we immediately obtain the claim.

We next prove the proposition. If \( \kappa^* = 0 \), the statement is trivial, we thus suppose that \( \kappa^* > 0 \). For \( \kappa' < \kappa^* \), the claim implies that \( \sigma_{MVEF}^2(\kappa', \bar{x}_0) > \sigma_{MVEF}^2(\kappa^*, \tilde{x}_0) \) and therefore that \( \tilde{x}_0 > \kappa^* \).

### A.8 Proof of Theorem 2

The optimal costs devoted to exploration, \( \kappa^* \) and \( \tilde{\kappa}^* \), for \( (MVE(\mu, x_0)) \) and \( (MVE(\mu, \tilde{x}_0)) \) exist according to Theorem 1. If \( \tilde{\kappa}^* = 0 \), then \( \kappa^* = 0 \) by Proposition 4 and the two Sharpe ratios coincide and are given by \( \sqrt{D} \). If \( \tilde{\kappa}^* > 0 \) but \( \kappa^* = 0 \), then clearly \( S_{MVE}(\mu, \tilde{x}_0) > S_{MVE}(\mu, x_0) \).

We thus suppose that \( \kappa^* > 0 \). According to Proposition 1, the optimal Sharpe ratio which can be achieved in \( (MVE(\mu, x_0)) \) is given by

\[
S_{MVE}^*(\mu, x_0) = \frac{(\mu - r_f)x_0}{(\mu - r_f)x_0 + r_f \kappa^*} \sqrt{D + S_e(\kappa^*)^2}.
\]

Considering \( \tilde{\kappa} = \frac{\kappa^*}{\kappa^*} \tilde{x}_0 > \kappa^* \), which is not necessarily optimal for \( (MVE(\mu, \tilde{x}_0)) \), we obtain that

\[
S_{MVE}^*(\mu, \tilde{x}_0) \geq \frac{(\mu - r_f)x_0}{(\mu - r_f)x_0 + r_f \tilde{\kappa} x_0} \sqrt{D + S_e(\kappa^*/\tilde{x}_0)^2} \geq S_{MVE}^*(\mu, x_0)
\]

and the last inequality is strict if \( S_e(\kappa) \) is strictly increasing. \( \square \)
A.9 Proof of Theorem 3

The proof contains similar arguments as the proof of Theorem 1. Taking the derivative of \( \sigma_{MVEF}^2(\kappa) \) from (37) yields

\[
\frac{d}{d\kappa} \sigma_{MVEF}^2(\kappa) = \frac{2r_f (\mu x_0 - r_f (x_0 - \kappa)) \tilde{D}_e(\kappa) - \tilde{D}_e'(\kappa) (\mu x_0 - r_f (x_0 - \kappa))^2}{(\tilde{D}_e(\kappa))^2}. 
\]

Since \( \mu > r_f \), \( \frac{d}{d\kappa} \sigma_{MVEF}^2(\kappa) \) is positive if and only if

\[
2r_f \tilde{D}_e(\kappa) > (\mu x_0 - r_f (x_0 - \kappa)) \tilde{D}_e'(\kappa) + r_f \kappa \tilde{D}_e'(\kappa)
\]

and zero if and only if equality holds in (31). Due the assumption in (10), there exist \( \epsilon > 0 \) and \( K' > 0 \) such that \( \tilde{D}_e(\kappa) > (1 + \epsilon) \kappa \tilde{D}_e'(\kappa) \) for any \( \kappa > K' \). Take \( K > \max \{ K', \frac{\mu - r_f}{(1 + 2\epsilon) r_f} x_0 \} \).

For \( \kappa > K \) we have

\[
2r_f \tilde{D}_e(\kappa) > 2r_f (1 + \epsilon) \kappa \tilde{D}_e'(\kappa) > \tilde{D}_e'(\kappa) (\mu - r_f) x_0 + r_f \kappa \tilde{D}_e'(\kappa),
\]

where we used that \( \kappa > K' \) for the first and that \( \kappa > \frac{\mu - r_f}{(1 + 2\epsilon) r_f} x_0 \) for the second inequality. Therefore, \( \frac{d}{d\kappa} \sigma_{MVEF}^2(\kappa) > 0 \) for any \( \kappa > K' \). We can thus optimize \( \sigma_{MVEF}^2(\kappa) \) over the compact interval \([0, K]\). Due to continuity of \( \sigma_{MVEF}^2(\kappa) \), the optimum exists and is finite. If the optimal solution is not zero, it must satisfy the first-order condition (31) with equality which is (11).

A.10 Proof of Proposition 6

By applying the embedding technique in Li and Ng (2000), we obtain the following reformulation of \( (MVSE(\mu, x_0)) \):

\[
\inf_{\kappa \geq 0, u \in \mathbb{R}, v \in \mathbb{R}} \mathbb{E} \left[ (x_2 - (\mu x_0 - \lambda))^2 \right],
\]

s.t. \( x_1 = x_0 - \kappa \),

\[
x_2 = r_f x_1 + uR + 1_{\{S_1 = 1\}} v R_e,
\]

where the constraint on the target expected wealth has been incorporated by means of a Lagrange multiplier \( \lambda \); namely, \( \lambda \) is chosen so that \( \mathbb{E} [x_2^*] = \mu x_0 \), where \( x_2^* \) denotes the value of \( x_2 \) under the optimal strategies.
The auxiliary problem (32) is now well-suited to be solved via dynamic programming (DP), and note that the solution to this problem will yield automatically the solution to problem (MVSE(\(\mu, x_0\))), in the sense that the optimal strategies coincide.

The DP algorithm starts at \(t = 1\). For given state variables \(x_1\) and \(\kappa\), the agent solves
\[
\inf_{u \in \mathbb{R}, v \in \mathbb{R}} \mathbb{E} \left[ (x_2 - (\mu x_0 - \lambda))^2 \right| x_1, \kappa] =: J_{MVSE}^1(x_1, \kappa) \\
= \inf_{u \in \mathbb{R}, v \in \mathbb{R}} \mathbb{E} \left[ (r_f x_1 + uR + 1_{\{S=1\}}vR_e - (\mu x_0 - \lambda))^2 \right| x_1, \kappa],
\]
From standard first-order optimality conditions, the optimal policies can be computed as
\[
u = -1_{\{S=1\}}(r_f x_1 - (\mu x_0 - \lambda)) \left( \frac{\bar{R} \Sigma_e^2}{(\Sigma^2 + \bar{R}^2)(\Sigma_e^2 + 1_{\{S=1\}} \bar{R}^2 e) - 1_{\{S=1\}} \bar{R}^2 e} \right).
\]
Replacing (17) into the objective function \(J_{MVSE}^1(x_1, \kappa)\), we obtain
\[
J_{MVSE}^1(x_1, \kappa) = (r_f x_1 - (\mu x_0 - \lambda))^2 \left( \frac{\Sigma^2 \Sigma_e^2}{(\Sigma^2 + \bar{R}^2)(\Sigma_e^2 + 1_{\{S=1\}} \bar{R}^2 e) - 1_{\{S=1\}} \bar{R}^2 e} \right).
\]
Next, proceeding backwards, the problem at time \(t = 0\) is given by
\[
\inf_{\kappa \geq 0} \mathbb{E} \left[ J_{MVSE}^1(x_1, \kappa) \right] =: J_{MVSE}^0(\kappa) \\
= \inf_{\kappa \geq 0} (r_f (x_0 - \kappa) - (\mu x_0 - \lambda))^2 \left( \frac{\Sigma^2 \Sigma_e^2}{(\Sigma^2 + \bar{R}^2)(\Sigma_e^2 + 1_{\{S=1\}} \bar{R}^2 e) - 1_{\{S=1\}} \bar{R}^2 e} p(\kappa) + \frac{\Sigma^2}{\Sigma^2 + \bar{R}^2} (1 - p(\kappa)) \right).
\]
The remaining step for the proof is to compute the optimal value for problem \((\text{MVSE}(\mu, x_0))\). To do this, observe that under the optimal strategies the objective function in (32) is equal to \(\text{var}[x_2^n] + \lambda^2\), thus easily giving (18).

\[ \Box \]

\section*{B Optimal strategies and value for problem \((\text{MVE}_\rho(\mu, x_0))\)}

\textbf{Proposition B.8.} For a fixed \(\kappa \geq 0\), we denote by \((\text{MVE}_\rho(\mu, x_0; \kappa))\) the inner optimization problem in \((\text{MVE}_\rho(\mu, x_0))\). Also, let

\[ \tilde{D}_e(\kappa) = \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j}^{-1}(\kappa) (\bar{r}_i - r_f) (\bar{r}_j - r_f) + 2 \sum_{i=1}^{N} \sigma_{i,j}^{-1}(\kappa) (\bar{r}_i - r_f) (\bar{e}(\kappa) - r_f) \]

\[ + \sigma_{N+1,N+1}^{-1}(\kappa) (\bar{e}(\kappa) - r_f)^2, \tag{35} \]

where \(\sigma_{i,j}^{-1}(\kappa)\), for \(i, j = 1, \ldots, N + 1\), are defined as the elements of the inverse of the extended covariance matrix, i.e., \(\tilde{\Sigma}^{-1}(\kappa) = [\sigma_{i,j}^{-1}(\kappa)]\).

The problem \((\text{MVE}_\rho(\mu, x_0; \kappa))\) has the unique solution

\[ u_i = \frac{\mu x_0 - r_f (x_0 - \kappa)}{\tilde{D}_e(\kappa)} \left( \sum_{j=1}^{N} \sigma_{i,j}^{-1}(\kappa) (\bar{r}_j - r_f) + \sigma_{i,N+1,j}^{-1}(\kappa) (\bar{e}(\kappa) - r_f) \right), \quad i = 1, \ldots, N, \]

\[ v = \frac{\mu x_0 - r_f (x_0 - \kappa)}{\tilde{D}_e(\kappa)} \left( \sum_{j=1}^{N} \sigma_{N+1,j}^{-1}(\kappa) (\bar{r}_j - r_f) + \sigma_{N+1,j}^{-1}(\kappa) (\bar{e}(\kappa) - r_f) \right), \]

\[ u_0 = x_0 - \kappa - \frac{\mu x_0 - r_f (x_0 - \kappa)}{\tilde{D}_e(\kappa)} \left( \sum_{i=1}^{N} \sum_{j=1}^{N} \sigma_{i,j}^{-1}(\kappa) (\bar{r}_j - r_f) + \sum_{j=1}^{N} \sigma_{N+1,j}^{-1}(\kappa) (\bar{e}(\kappa) - r_f) \right. \]

\[ + \sum_{j=1}^{N} \sigma_{N+1,j}^{-1}(\kappa) (\bar{r}_j - r_f) + \sigma_{N+1,N+1}^{-1}(\kappa) (\bar{e}(\kappa) - r_f) \right) \]

and the optimal value is given by

\[ \sigma_{\text{MVE}_\rho}^2(\kappa) = \frac{(\mu x_0 - r_f (x_0 - \kappa))^2}{\tilde{D}_e(\kappa)}. \tag{37} \]

\textbf{Proof.} The proof follows with minor modifications of the proof of Proposition 1. \(\Box\)

\section*{C Optimal strategies for Example 1}

We provide below the strategies corresponding to the optima (highlighted in red) in Figure 4. All values are in USD.
Case without economies of scale (Subfigure 4a).
Number of new assets: $M = 2$

$$
u = \begin{pmatrix} 2'135'695 \\ 1'851'240 \\ 6'780'214 \end{pmatrix}, \quad v = \begin{pmatrix} 3'039'119 \\ 3'205'802 \end{pmatrix}, \quad u_0 = -7'612'072.$$

Case with economies of scale (Subfigure 4b).
(i) Number of new assets: $M = 2$

$$
u = \begin{pmatrix} 2'129'065 \\ 1'845'493 \\ 6'759'166 \end{pmatrix}, \quad v = \begin{pmatrix} 3'029'685 \\ 3'195'851 \end{pmatrix}, \quad u_0 = -7'549'262.$$

(ii) Number of new assets: $M = 38$

$$
u = \begin{pmatrix} 800'505 \\ 693'885 \\ 2'541'373 \end{pmatrix}, \quad v = \begin{pmatrix} 1'139'128 \\ 1'201'605 \\ : \\ 1'299'390 \\ 1'299'578 \end{pmatrix}, \quad u_0 = -48'625'126.$$

D An enlarged filtration for exploration

We denote by $\mathcal{F}_t^{(j)}$, for $t = 0, 1, \ldots, T - 1$ and $j = 1, \ldots, M$, the filtration generated by the excess returns of the $j^{th}$ new asset (which, we recall, are observable only after time $\tau_j$):

$$\mathcal{F}_t^{(j)} = \begin{cases} \sigma(R_{e,\tau_j}, \ldots, R_{e,t-1}) , & \text{if } \tau_j + 1 \leq t \leq T - 1, \\ \{\emptyset\} , & \text{if } 0 \leq t \leq \tau_j. \end{cases}$$

Thus, setting $\mathcal{G}^{(0)} := \mathcal{F}$, the enlarged filtration $\mathcal{G}^{(j)} = \left(\mathcal{G}_t^{(j)}\right)_{0 \leq t \leq T}$ is given by

$$\mathcal{G}_t^{(j)} := \mathcal{F}_t^{(j)} \lor \mathcal{G}_t^{(j-1)}, \quad 0 \leq t \leq T, \; j = 1, \ldots, M,$$

where $\mathcal{F}_t^{(j)} \lor \mathcal{G}_t^{(j-1)}$ denotes the smallest $\sigma$-algebra containing $\mathcal{F}_t^{(j)}$ and $\mathcal{G}_t^{(j-1)}$. $\mathcal{G}_0$ is the trivial $\sigma$-algebra over $\Omega$. 
Sketch of the solution of problem (21)

As usual, the dynamic programming algorithm starts from period \( T - 1 = 1 \). The agent solves

\[
\inf_{u_t, e_t^1, e_t^2, \kappa_t^1} \mathbb{E} \left[ (x_2 - (\mu x_0 - \lambda))^2 \mid x_1, \kappa_0^1 \right] =: J_1(x_1, \kappa_0^1)
\]

\[
= \inf_{u_t, e_t^1, e_t^2, \kappa_t^1} \mathbb{E} \left[ (r_f(x_1 - \kappa_1^2) + u_t R_t + v_t^1 R_{e_t, 1}(\kappa_0^1) + v_t^2 R_{e_t, 2}(\kappa_1^2) - (\mu x_0 - \lambda))^2 \mid x_1, \kappa_0^1 \right],
\]

such that \( u_t \in \mathbb{R}, v_t^1 \in \mathbb{R}, v_t^2 \in \mathbb{R} \) and \( \kappa_1^2 \geq 0 \). Let

\[
D_e(\kappa_0^1, \kappa_1^2) = \sum_1^2 \sum_1^2 (\kappa_0^1) \sum_2^2 (\kappa_1^2) e + \tilde{R}_{e_1, 1}(\kappa_0^1) \sum_1^2 \sum_1^2 (\kappa_1^2)
\]

\[
\tilde{R}_{e_2, 1}(\kappa_1^2) \sum_2^2 \sum_2^2 (\kappa_0^1).
\]

For a fixed \( \kappa_1^2 \geq 0 \), we obtain the following optimal strategies:

\[
u_t^1 = -(r_f(x_1 - \kappa_1^2) - (\mu x_0 - \lambda)) \left( \frac{\tilde{R}_{e_1, 1}(\kappa_0^1) \sum_2^2 (\kappa_1^2)}{D_e(\kappa_0^1, \kappa_1^2)} \right),
\]

\[
u_t^2 = -(r_f(x_1 - \kappa_1^2) - (\mu x_0 - \lambda)) \left( \frac{\tilde{R}_{e_2, 1}(\kappa_1^2) \sum_2^2 (\kappa_0^1)}{D_e(\kappa_0^1, \kappa_1^2)} \right).
\]

Replacing (38) into \( J_1(x_1, \kappa_0^1) \) yields

\[
J_1(x_1, \kappa_0^1, \kappa_1^2) = (r_f(x_1 - \kappa_1^2) - (\mu x_0 - \lambda))^2 \left( \frac{\sum_1^2 \sum_2^2 (\kappa_0^1) \sum_2^2 (\kappa_1^2)}{D_e(\kappa_0^1, \kappa_1^2)} \right),
\]

Similarly to the problems studied thus far, we can solve for \( \kappa_1^2 \) by differentiating \( J_1(x_1, \kappa_0^1, \kappa_1^2) \) (with respect to \( \kappa_1^2 \)) and setting the derivative to 0.

Next, proceeding backwards, the problem to solve at time 0 is

\[
\inf_{u_0, v_0^1, \kappa_0^1} \mathbb{E} \left[ J_1(x_1, \kappa_0^1) \right],
\]

such that \( u_0 \in \mathbb{R}, v_0^1 \in \mathbb{R} \) and \( \kappa_0^1 \geq 0 \). At this point, it is important to observe that, by our assumption about the dependence on \( \kappa \) of the distributional properties of the new assets, the function in (39) is not anymore quadratic in the wealth process. This feature differentiates our model from standard mean-variance optimization problems, where the quadratic form of the objective is typically preserved over time. Unfortunately, however, this does not allow us to obtain closed form expressions for the optimal strategies at time 0 (and, more generally, at times \( t \leq T - 2 \)), for which we thus rely on a numerical approach.
Parameters of the model.
\[
x_0 = 10'000'000 \text{ USD}, \quad \mu = 1.4, \quad r_f = 1.03, \quad r_0 = r_1 = 1.162, \quad \Sigma^2_0 = \Sigma^2_1 = 0.0146,
\]
\[
\alpha_1 = \alpha_2 = 0.3, \quad \beta = 100'000, \quad s^0_0 = s^0_1 = 0.25, \quad s^1_0 = s^1_1 = 100'000, \quad \gamma = 0.5,
\]
\[
\nu = 0.28, \quad \eta = 100'000, \quad q_0 = 0.24, \quad q_1 = 100'000, \quad \zeta = 0.5.
\]

Optimal strategies.
At time 0, the numerical values of the optimal strategies (in USD) are given by
\[
u_0 = 6'461'538, \quad v^c_0 = 5'384'615, \quad \kappa^c_0 = 538'461.
\]
At time 1, we distinguish four cases:

(i) \( R_0 > 0 \) and \( R_{e1,0} > 0 \) : \( u_1 = 269'354, \quad v^c_1 = 196'734, \quad v^c_2 = 0, \quad \kappa^c_1 = 0 \);

(ii) \( R_0 < 0 \) and \( R_{e1,0} > 0 \) : \( u_1 = 4'093'485, \quad v^c_1 = 2'989'859, \quad v^c_2 = 2'570'624, \quad \kappa^c_1 = 245'915 \);

(iii) \( R_0 > 0 \) and \( R_{e1,0} < 0 \) : \( u_1 = 7'754'530, \quad v^c_1 = 5'663'867, \quad v^c_2 = 5'388'414, \quad \kappa^c_1 = 371'882 \);

(iv) \( R_0 < 0 \) and \( R_{e1,0} < 0 \) : \( u_1 = 10'963'206, \quad v^c_1 = 8'007'467, \quad v^c_2 = 7'897'924, \quad \kappa^c_1 = 455'926 \).

In case (i), during the first time period the existing portfolio (including the first newly explored asset) experiences positive returns. In this case, wealth at \( t = 1 \) is \( x_1 = 13'452'978 \) USD, and the agent does not explore again in the second new asset (\( v^c_2 = \kappa^c_2 = 0 \)).

In cases (ii) and (iii), the two assets provide one positive and one negative return. In case (ii) (respectively, (iii)), wealth at \( t = 1 \) is \( x_1 = 10'841'237 \) USD (respectively, \( x_1 = 7'938'763 \) USD). In these cases, the agent re-explores for a second new asset and invests higher amounts as the difference with the target wealth (which is set at \( x_2 = 14'000'000 \) USD) becomes larger.

In case (iv), both assets yield negative returns and wealth drops at \( x_1 = 5'327'023 \) USD. In this case, the agent re-explores for a second new asset and tries to compensate for the substantial loss by investing with high leverage in the risky assets.

Under the above optimal strategies, the optimal variance is attained at \( \sim 5'132 \times 10^8 \). As a benchmark, the heuristic \( 1/N \) rule, together with a 5% investment for exploration at each time period, yields a variance of \( \sim 1'826 \times 10^9 \) with expected terminal wealth of \( 13'752'910 \) USD. To this extent, the model of exploration vs exploitation as discussed in this paper can thus lead to significant improvements of the mean-variance objective.
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