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Robust optimal asset-liability management under square-root factor processes and model ambiguity: A BSDE approach

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Abstract
This paper studies robust optimal asset-liability management problems for an ambiguity-averse manager in a possibly non-Markovian environment with stochastic investment opportunities. The manager has access to one risk-free asset and one risky asset in a financial market. The market price of risk relies on a stochastic factor process satisfying an affine-form, square-root, Markovian model, whereas the risky asset’s return rate and volatility are potentially given by general non-Markovian, unbounded stochastic processes. This financial framework includes, but is not limited to, the constant elasticity of variance (CEV) model, the family of 4/2 stochastic volatility models, and some path-dependent non-Markovian models, as exceptional cases. As opposed to most of the papers using the Hamilton-Jacobi-Bellman-Issacs (HJBI) equation to deal with model ambiguity in the Markovian cases, we address the non-Markovian case by proposing a backward stochastic differential equation (BSDE) approach. By solving the associated BSDEs explicitly, we derive, in closed form, the robust optimal controls and robust optimal value functions for power and exponential utility, respectively. In addition, analytical solutions to some particular cases of our model are provided. Finally, the effects of model ambiguity and market parameters on the robust optimal investment strategies are illustrated under the CEV model and 4/2 model with numerical examples.

Keywords — Ambiguity aversion, asset-liability management, non-Markovian model, square-root factor process, backward stochastic differential equation

1 Introduction
Asset-liability management (ALM) is one of the important concerns not only for financial institutions, such as pension funds, banks, and insurance companies but also for individual investors

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who coordinate the existing and future assets and liabilities to earn an adequate return. Based on Markowitz [34]'s mean-variance criterion, Sharpe and Tint [46] first investigated the ALM problem in a single-period setting, and Leippold et al. [28] extended the results to a multi-period setting. By applying the linear-quadratic control theory, Chiu and Li [13] and Xie et al. [55] considered the continuous-time mean-variance ALM problems with uncontrollable liabilities described by a geometric Brownian motion and a drifted Brownian motion, respectively. Chen et al. [8] and Chen and Yang [9] further extended the results of Chiu and Li [13] and Leippold et al. [28] to the case with Markovian regime-switching markets. Chiu and Wong [14] studied an ALM problem with asset correlation driven by a multivariate Wishart process. Under the framework of expected utility maximization, Liang and Ma [31] considered the ALM problems under power and exponential utility with mortality and salary risks, and the optimal approximation investment strategies were derived. Pan and Xiao [41, 42] studied an ALM problem with inflation risks and liquidity constraints, respectively. For other relevant works on ALM problems, readers may refer to Zeng and Li [60], Chang [6], Pan and Xiao [43], Peng and Chen [44], and references therein.

The motivation for this paper is three-fold. First, most of the above-mentioned literature on the ALM problems assumes that the volatility of risky asset’s price is a constant or deterministic function, which violates the well-documented evidence to support the existence of stochastic (local) volatility, mainly referred to French et al. [22], Heston [25], Cox [17], Lewis [29], and Grasselli [24]. In the last decade, some papers have emerged that investigated the optimal ALM problems with various stochastic investment opportunities. For example, Zhang and Chen [61] studied a mean-variance ALM problem under the constant elasticity of variance (CEV) model with multiple risky assets. Li et al. [30] considered the derivative-based optimal investment strategy for a mean-variance ALM problem under the Heston model. Zhang [63] stepped further by incorporating the Cox-Ingersoll-Ross (CIR) interest rate and the family of 4/2 model (Grasselli [24]) into an ALM problem with derivative trading. Sun et al. [50] studied a mean-variance ALM problem with a reinsurance option in a complete market under an affine diffusion equation. Besides the mean-variance criterion, Pan et al. [40] considered an ALM problem for the exponential utility function under the Heston model. Zhang [64] investigated an ALM problem in an incomplete market setting with an affine diffusion factor process for the hyperbolic absolute risk aversion utility function.

Second, most of the literature mentioned above on ALM problems assumes that the asset-liability manager knows exactly the true probability measure. In many situations, however, economic agents are skeptical about the true model, because, for instance, as shown by Merton [36] and Cochrane [15], the drift parameters are difficult to estimate with precision. In addition, experimental evidence from Ellsberg [19] and Bossaerts et al. [4] demonstrate that economic agents display not only risk aversion but also ambiguity aversion. In this sense, it is plausible to incorporate model ambiguity into portfolio choice problems. In the pioneering work of Anderson et al. [1], a robust control approach was proposed to address model ambiguity in continuous-time stochastic control problems, where the agent regards a particular probability measure as a reference measure and considers a set of alternative probability measures which are close to the reference measure in terms of relative entropy. Maenhout [32] refined the robust control approach by proposing the homothetic robustness, and Uppal and Wang [51] extended the analysis of Maenhout [32] by allowing different levels of ambiguity aversion about the state variables. Maenhout [33] considered a robust portfolio selection problem with a mean-reverting expected stock return. Flor and Larsen [21] studied a robust investment problem in a setting with stochastic interest rates. Munk and Rubtsov [37] extended the work of Flor and Larsen [21] by incorporating an unobservable inflation rate. Escobar et al. [20] considered a robust investment problem with derivatives trading under the Heston model. Zeng et al. [59] analyzed a robust derivative-based pension investment problem with
stochastic income and volatility. Recently, Cheng and Escobar [12] investigated robust investment under the state-of-the-art 4/2 model. In the field of ALM, Yuan and Mi [57] considered a robust investment problem for maximizing the minimal expected utility of terminal wealth and minimizing the maximal cumulative deviation, respectively. Chen et al. [10] studied a robust ALM problem in a regime-switching market. As the literature on robust investment problems is abundant, the above review is not exhaustive. Other works considering robust investment problems under various scenarios include Yi et al. [56], Zheng et al. [65], Wang and Li [52], Wang et al. [53], Chang et al.[7], Baltas et al. [2], Wei et al. [54], to name but only a few.

Third, although robust investment problems have been extensively studied over the last decade, one common feature shared by most of the existing works is that the exogenous parameter processes are assumed to be only constants or Markovian diffusion processes. In the Markovian case, such problems can be studied by using the dynamic programming principle and solving the corresponding Hamilton-Jacobi-Bellman-Issacs (HJBI) equations (see, for example, Mataramvura and Øksendal [35]). These methods, however, cannot be applied directly to the non-Markovian setting because the dynamic programming principle no longer works. To handle the non-Markovian case, Øksendal and Sulem [38] studied an optimal investment problem under model ambiguity by proposing a backward stochastic differential equation (BSDE) approach, where the performance functional (value function) is written as the solution of an associated controlled BSDE and the comparison theorem for BSDEs plays a key role. But this approach is strongly linked to the exponential utility function. Øksendal and Sulem [39] extended the analysis of Øksendal and Sulem [38] for general utility functions by developing a forward-backward stochastic differential equation (FBSDE) approach. Following the methodology of Øksendal and Sulem [39], Peng et al. [45] considered an optimal investment-consumption and reinsurance problem under model ambiguity.

In this paper, we investigate a robust optimal ALM problem under model ambiguity in the presence of stochastic volatility. The risk- and ambiguity-averse manager has access to a financial market consisting of one risk-free asset (money account) and one risky asset (stock) and is subject to an uncontrollable random liability. Unlike most of the preceding literature on robust decision problems, it is not a prerequisite to assume that the risky asset’s return rate and volatility are specifically Markovian processes as they may depend on past values. Inspired by Shen and Zeng [47] and Zhang [62], we only suppose that the market price of risk relies on an affine-form, square-root, Markovian process, which includes, but is not limited to, the Black-Scholes model, CEV model, Heston model, 3/2 model, 4/2 model, and some non-Markovian models, as exceptional cases (see Example 2.1-2.3). In the spirit of Maenhout [32] and Uppal and Wang [51], the manager is assumed to have different levels of ambiguity aversion about the risky asset price and volatility and aims to maximize the terminal surplus under the worst-case scenario for power and exponential utility, respectively. Given the potentially non-Markovian setting, the HJBI equation approach does not work, and a BSDE approach is disentangled. Different from Øksendal and Sulem [38, 39], where the value function is written as the value at time zero of the solution to a controlled FBSDE and the comparison theorem for solutions of BSDEs is applied, we propose to construct a stochastic process hinging upon any admissible control, and such that its value at time zero does not depend on any admissible control and its terminal value equals the utility of the terminal surplus penalized by model ambiguity. The proposed stochastic process is shown to be either sub-martingale or super-martingale for any admissible control, and even martingale for a particular control under the reference measure, which then leads to the associated uncontrolled BSDEs. By solving the BSDEs explicitly, we derive the analytical expressions for the robust optimal controls and robust optimal value functions for the above two utility maximization problems. Furthermore, several special cases of our model are discussed and the corresponding results are provided in closed form.  

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Finally, the economic effects of model ambiguity and model parameters on the behavior of robust optimal investment strategies are analyzed by giving numerical examples. To sum up, we think that this paper has three main contributions:

1. In the literature on the ALM problems, the model ambiguity and stochastic volatility are simultaneously taken into consideration in a potentially non-Markovian modeling framework for the very first time, whereas in Yuan and Mi [57] and Chen et al. [10], only Markovian cases were investigated and stochastic volatility was not taken into account; Zhang and Chen [65], Li et al. [30], Pan et al. [40], Sun et al. [50], and Zhang [63, 64] considered the presence of stochastic volatility but not model uncertainty.

2. At the mathematical level, compared with the literature on the robust investment problems considering the Markovian models and using the HJBI equation approach, such as Yi et al. [56], Flor and Larsen [21], Escobar et al. [20], Zheng et al. [65], Wang and Li [52], Wang et al. [53], Cheng and Escobar [12], Chang et al. [7], Baltas et al. [2], and Wei et al. [54], a novel BSDE approach, which has distinct differences with the FBSDE approach proposed in Øksendal and Sulem [38, 39], is disentangled to deal with the non-Markovian setting.

3. A general class of stochastic volatility models is considered for modeling the risky asset’s price and volatility, embracing the CEV model, Heston model, 3/2 model, 4/2 model, and some path-dependent models, as particular cases. Furthermore, closed-form expressions for the robust optimal controls and robust optimal value functions are derived for the power and exponential utility functions, and explicit solutions to some special cases of our model are recovered, such as Gao [23], Zheng et al. [65], Sun et al. [49], Cheng and Escobar [12, 11], and Zhang [62].

The remainder of this paper is organized as follows. In Section 2, we formulate the model and establish the robust optimal ALM problems for the power and exponential utility functions. Section 3 and 4 derive the robust optimal solutions to the power and exponential utility cases, respectively. Section 5 discusses the effects of model ambiguity and model parameters on the robust optimal investment strategies with numerical analysis. Section 6 concludes our work. All proofs are given in the Appendix.

2 General formulation

In this paper, we consider the optimal ALM problems for an asset-liability manager with ambiguity aversion under the expected utility maximization framework. We assume that assets can be traded continuously, infinite short-selling and leverage are allowed, and no transaction costs or taxes are involved. Let $T > 0$ be a fixed constant describing the decision-making horizon and $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions on which are defined two one-dimensional, mutually independent Brownian motions $\{W_{1,t}\}_{t \in [0,T]}$ and $\{W_{2,t}\}_{t \in [0,T]}$. The filtration $\mathbb{F} := \{\mathcal{F}_t\}_{t \in [0,T]}$ is assumed to be generated by the two Brownian motions, $\mathbb{P}$ stands for a real-world probability measure, and $\mathbb{E}^\mathbb{P} [\cdot]$ denotes the expectation associated with measure $\mathbb{P}$. In what follows, we introduce several spaces on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:

- $\mathcal{L}^{2, loc}_{2, \mathbb{F}}(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t \in [0,T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{P} \left( \int_0^T |f_t|^2 \, dt < \infty \right) = 1$;
- $\mathcal{L}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t \in [0, T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}^\mathbb{P} \left[ \int_0^T |f_t|^2 \, dt \right] < \infty$;
- $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted processes $\{f_t\}_{t \in [0, T]}$ with $\mathbb{P}$-a.s. continuous sample paths such that $\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0, T]} |f_t|^{2p} \right] < \infty$, $p = 1, 2$;
- $\mathcal{S}_{\mathbb{F}, \mathbb{P}}^\infty(0, T; \mathbb{R})$: the space of all real-valued, $\mathbb{F}$-adapted uniformly bounded processes with $\mathbb{P}$-a.s. continuous sample paths.

2.1 Financial market and random liability

Assume that the financial market consists of one risk-free asset (money account) and one risky asset (stock). The price process $\{B_t\}_{t \in [0, T]}$ of the risk-free asset evolves according to

$$
 dB_t = r B_t \, dt, \quad B_0 = 1,
$$

where the constant $r \in \mathbb{R} \setminus \{0\} = \mathbb{R}_0$ is the risk-free interest rate. The price process $\{S_t\}_{t \in [0, T]}$ of the risky asset is described by the following stochastic differential equation (SDE):

$$
 dS_t = \mu_t S_t \, dt + \sigma_t S_t \, dW_{1,t}, \quad S_0 = s_0 \in \mathbb{R}^+,
$$

(2.1)

where $\mu_t$ and $\sigma_t > 0$ are two potentially unbounded and non-Markovian $\mathbb{F}$-adapted stochastic processes describing the risky asset’s return rate and volatility at time $t$, respectively. Assume that the market price of volatility risk is related to an affine form, square-root factor process $\{\alpha_t\}_{t \in [0, T]}$ as follows:

$$
 \frac{\mu_t - r}{\sigma_t} = \lambda \sqrt{\alpha_t}, \quad \lambda \in \mathbb{R}_0,
$$

(2.2)

where the dynamics of $\alpha_t$ is given by

$$
 d\alpha_t = \kappa (\theta - \alpha_t) \, dt + \sqrt{\alpha_t} \left( \rho_1 \, dW_{1,t} + \rho_2 \, dW_{2,t} \right), \quad \alpha_0 \in \mathbb{R}^+.
$$

(2.3)

with the speed of mean reversion $\kappa$, long-run level $\theta$, and volatility $\sqrt{\rho_1^2 + \rho_2^2}$. In line with Chapter 6.3 in Jeanblanc et al. [26], we assume that the constants $\kappa, \theta \in \mathbb{R}$ satisfy $\kappa \theta \in \mathbb{R}^+$ to ensure the process $\alpha_t \geq 0$ for all $t \in [0, T]$, $\mathbb{P}$ almost surely, while no specific conditions are imposed on the constants $\rho_1, \rho_2 \in \mathbb{R}$. Notice that we do not impose the Feller condition for strict positivity of $\alpha_t$, i.e., $2\kappa \theta \geq \rho_1^2 + \rho_2^2$ in our case.

The above financial modelling framework (2.1)-(2.3) was studied in Shen and Zeng [47] and Zhang [62] in the context of solving a mean-variance investment-reinsurance problem and a defined contribution pension investment problem with stochastic income and inflation risks, respectively. It is also worth mentioning that this modelling framework is general embracing not only a wide class of stochastic (local) volatility models, such as the CEV model, Heston model, 3/2 model, and 4/2 model (see Examples 2.1 and 2.2) but also some non-Markovian models (Example 2.3), as exceptional cases.

Example 2.1 (CEV model). If $\mu_t = \mu$ and $\sigma_t = \sigma S_t^{\beta}$, where $\mu \in \mathbb{R}^+$, $\sigma \in \mathbb{R}^+$, and $\beta \leq -\frac{1}{2}$ such that $\mu \neq r$, then the risky asset price $S_t$ is given by the CEV model:

$$
 dS_t = S_t \left( \mu \, dt + \sigma S_t^{\beta} \, dW_{1,t} \right), \quad S_0 = s_0 \in \mathbb{R}^+,
$$

(2.4)
where $\beta$ is called the elasticity parameter. By setting $\alpha_t = S_t^{-2\beta}$, $\kappa = 2\beta\mu$, $\theta = (\beta + \frac{1}{2})\frac{\sigma^2}{\rho}$, $\rho_1 = -2\beta\sigma$, $\rho_2 = 0$ and $\lambda = \frac{\mu - r}{\sigma}$, we have
\[
d\alpha_t = 2\beta\mu \left[ \left( \beta + \frac{1}{2} \right) \frac{\sigma^2}{\rho} - S_t^{-2\beta} \right] dt - 2\beta\sigma S_t^{-\beta} dW_{1,t}
= \kappa(\theta - \alpha_t) dt + \sqrt{\alpha_t} (\rho_1 dW_{1,t} + \rho_2 dW_{2,t}).
\]
For the particular case when $\beta = 0$, the condition $\kappa\theta \geq 0$ is still met and the CEV model degenerates to the Black-Scholes model.

**Example 2.2** (The family of 4/2 models). If $\mu_t = r + \lambda(c_1\alpha_t + c_2)$, $\sigma_t = c_1\sqrt{\alpha_t} + \frac{c_2}{\sqrt{\alpha_t}}$, $V_t = \alpha_t$, $\kappa \in \mathbb{R}^+$, $\theta \in \mathbb{R}^+$, $\rho_1 = \sigma_v\rho$ and $\rho_2 = \sigma_v\sqrt{1 - \rho^2}$, where $c_1 \geq 0$, $c_2 \geq 0$, $\sigma_v \in \mathbb{R}^+$, and $\rho \in [-1, 1]$, then the risky asset price process $S_t$ is governed by the family of 4/2 stochastic volatility models (Grasselli [24]):
\[
\begin{align*}
    dS_t &= S_t \left[ (r + \lambda(c_1V_t + c_2)) dt + \left(c_1\sqrt{V_t} + \frac{c_2}{\sqrt{V_t}} \right) dW_{1,t} \right], \quad S_0 = s_0 \in \mathbb{R}^+, \\
    dV_t &= \kappa(\theta - V_t) dt + \sigma_v\sqrt{V_t} \left( \rho dW_{1,t} + \sqrt{1 - \rho^2} dW_{2,t} \right), \quad V_0 = v_0 = \alpha_0 \in \mathbb{R}^+,
\end{align*}
\]
where $V_t$ is the variance driver process with mean-reversion rate $\kappa$, long-run mean $\theta$, volatility $\sigma_v$, and correlation coefficient between the risky asset price and its variance driver $\rho$. For the 4/2 model (2.5), we impose the Feller condition, i.e., $2\kappa\theta \geq \sigma_v^2$ to keep the process $V_t$ strictly positive for $t \in [0, T]$, $\mathbb{P}$ almost surely.

**Remark 1.** The 4/2 model (2.5) is featured by two embedded parsimonious models, the Heston model (Heston [25]) and 3/2 model (Lewis [29]) via the constants $c_1$ and $c_2$. Particularly, the case $(c_1, c_2) = (1, 0)$ corresponds to the Heston model, while the specification $(c_1, c_2) = (0, 1)$ is known as the 3/2 model.

**Example 2.3** (A path-dependent stochastic volatility model). If $\mu_t = r + \lambda\sqrt{\alpha_t}\sigma(\alpha_{[0,t]})$ and $\sigma_t = \hat{\sigma}(\alpha_{[0,t]})$ for some functional $\hat{\sigma} : C([0, t]; \mathbb{R}) \to \mathbb{R}^+$, where $\alpha_{[0,t]} := (\alpha_s)_{s \in [0, t]}$ is the restriction of $\alpha \in C([0, T]; \mathbb{R})$ to $C([0, t]; \mathbb{R})$, i.e., the space of real-valued, continuous functions defined on $[0, t]$. In this case, the risky asset price process $S_t$ is given by the following path-dependent stochastic volatility model:
\[
\begin{align*}
    dS_t &= S_t \left[ (r + \lambda\sqrt{\alpha_t}\hat{\sigma}(\alpha_{[0,t]})) dt + \hat{\sigma}(\alpha_{[0,t]}) dW_{1,t} \right], \quad S_0 = s_0 \in \mathbb{R}^+, \\
    d\alpha_t &= \kappa(\theta - \alpha_t) dt + \sqrt{\alpha_t} (\rho_1 dW_{1,t} + \rho_2 dW_{2,t}), \quad \alpha_0 \in \mathbb{R}^+.
\end{align*}
\]
Due to the path-dependence of the return rate and volatility of the risky asset price, the model (2.6) is a special case of the non-Markovian stochastic volatility models. For more details on (2.6), readers may consult Siu [48].

Consider an asset-liability manager who is subject to an uncontrollable liability commitment with an initial value $L_0$. Similar to Zhang and Chen [61] and Sun et al. [50], we assume that the liability process $L_t$ is driven by the following SDE:
\[
dL_t = L_t [\mu_t dt + \sigma_t (\lambda\alpha_t dt + \sqrt{\alpha_t} dW_{1,t})], \quad L_0 = l_0 \in \mathbb{R}^+, \quad (2.7)
\]
where $\mu_t \in \mathbb{R}$ is the drift coefficient and the constant $\sigma_t \in \mathbb{R}$ is a volatility scale factor measuring how the risk source of the risky asset affects the random liability. In the following subsection, we will formulate the robust optimal ALM problems from the point of view of the manager.
2.2 Ambiguity and optimization problem

In the traditional framework of the ALM problem, the asset-liability manager is assumed to be ambiguity-neutral and completely convinced by the dynamics of the available risky asset price, factor process, and random liability under the real-world probability measure $\mathbb{P}$. However, the fact is that the manager may not know exactly the true model in many cases, for example, due to parameter uncertainty, and thus, any particular probability measure used to describe the model may lead to potential model misspecification. For this reason, it is desirable to take model uncertainty into account for an ambiguity-averse manager when he makes investment decisions. To incorporate model ambiguity, we assume that the ambiguity-averse manager’s knowledge of ambiguity is characterized by the measure $\mathbb{P}$, which is referred to as the reference measure. The ambiguity-averse manager is skeptical about the reference measure $\mathbb{P}$ and only regards it as an approximation to the truly real-world measure. Therefore, he considers some adverse alternative measures to seek robust optimal investment strategies. In line with Anderson et al. [1], the alternative measures are assumed to be equivalent to, i.e., mutually absolutely continuous with the reference measure $\mathbb{P}$, and we denote by $\mathcal{Q}$ such a class of alternative measures $\mathcal{Q}$, i.e., $\mathcal{Q} := \{\mathbb{Q} \mid \mathbb{Q} \sim \mathbb{P}\}$. More specifically, for each $\mathbb{Q} \in \mathcal{Q}$, there is a two-dimensional $\mathbb{F}$-adapted process $\phi = (\phi_1, \phi_2) := (\{\phi_{1,t}\}_{t \in [0,T]}, \{\phi_{2,t}\}_{t \in [0,T]})$, which can be referred as the probability distortion process, such that the following Radon-Nikodym derivative process $\varphi^\phi$:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} := \varphi^\phi = \exp \left\{ \int_0^t \phi_{1,s} dW_{1,s} + \int_0^t \phi_{2,s} dW_{2,s} - \frac{1}{2} \int_0^t (\phi_{1,s}^2 + \phi_{2,s}^2) \, ds \right\}$$

(2.8)

is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale. For this, we shall only consider the distortion process $\phi$ satisfying the following Novikov’s condition:

$$\mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{1}{2} \int_0^T (\phi_{1,t}^2 + \phi_{2,t}^2) \, dt \right\} \right] < +\infty,$$

(2.9)

and denote by $\hat{\Phi}$ the space of all distortion process $\phi$ such that (2.9) holds. According to Girsanov’s theorem, the dynamics of the standard Brownian motions $W^\mathbb{Q}_{1,t}$ and $W^\mathbb{Q}_{2,t}$ under the alternative measure $\mathbb{Q} \in \mathcal{Q}$ are given by

$$dW^\mathbb{Q}_{1,t} = dW_{1,t} - \phi_{1,t} \, dt, \quad dW^\mathbb{Q}_{2,t} = dW_{2,t} - \phi_{2,t} \, dt.$$

Suppose that the asset-liability manager has an initial wealth $x_0 \in \mathbb{R}^+$. Denote by $\pi_t$ the proportion of wealth invested in the risky asset at time $t$, then the process $\pi := \{\pi_t\}_{t \in [0,T]}$ represents the investment strategy. Let $X^\pi := \{X^\pi_t\}_{t \in [0,T]}$ be the wealth process associated with strategy $\pi$. Under a self-financing condition, the dynamics of $X^\pi_t$ is then given by

$$dX^\pi_t = \left(1 - \pi_t\right)X^\pi_t \frac{dB_t}{B_t} + \pi_t X^\pi_t \frac{dS_t}{S_t}$$

$$= \left[r + (\mu_t - r)\pi_t\right]X^\pi_t \, dt + \sigma_t \pi_t X^\pi_t \, dW_{1,t}$$

$$= \left[r + (\mu_t - r + \sigma_t \phi_{1,t})\pi_t\right]X^\pi_t \, dt + \sigma_t \pi_t X^\pi_t \, dW^\mathbb{Q}_{1,t}, \quad X^\pi_0 = x_0.$$

(2.10)

In this paper, we will subsequently consider two utility maximization problems when the risk preferences of the ambiguity-averse manager are characterized by a power utility function $U(x) = x^\gamma/\gamma$ with the relative risk aversion $\gamma \in \mathbb{R}^-$ and an exponential utility function $U_2(x) = -e^{-qx}/q$ with the absolute risk aversion $q \in \mathbb{R}^+$. To this end, we give below the formal definitions of the admissible strategies for these two utility maximization problems, respectively.
Definition 2.4 (Admissible strategy for power utility). A control \((\pi, \phi)\) is said to be admissible if the following conditions are satisfied:

1. \(\pi\) is \(F\)-adapted and \(\phi \in \Phi\);

2. for any initial data \((x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+\) such that \(x_0 + \dot{G}_{1,0} l_0 \in \mathbb{R}^+\), the associated asset process \((2.10)\) admits a pathwise unique solution such that \(X_t^\pi + \dot{G}_{1,t} L_t > 0\), \(P\) almost surely, for all \(t \in [0, T]\), where \(G_{1,t}\) is given by \((3.14)\) below;

3. either the family of random variables

\[
\left\{ \left( \Phi_{\tau_n \wedge T} \left( Y_{1, \tau_n \wedge T} \frac{X_{\tau_n \wedge T}^\pi + \dot{G}_{1, \tau_n \wedge T} L_{\tau_n \wedge T}}{\gamma} \right) + \int_0^{\tau_n \wedge T} \frac{\phi_{1,t}^2}{2 \psi_{1,t}} + \frac{\phi_{2,t}^2}{2 \psi_{2,t}} \, dt \right) \right\}_{n \in \mathbb{N}}
\]

is uniformly integrable under \(P\) measure for any sequence of \(F\)-stopping times \(\{\tau_n\}_{n \in \mathbb{N}}\) such that \(\tau_n \uparrow +\infty\), where \(\hat{\phi} = (\hat{\phi}_1, \hat{\phi}_2) \in \Phi\) is given in \((3.17)\) with \(X_t^\pi\) and \(\pi_t\), respectively, and \(\psi_{1,t}, \psi_{2,t}\), and \(Y_{1,t}\) are given by \((2.12)\) and \((3.6)\) below, or the family of random variables

\[
\left\{ \left( \Phi_{\tau_n \wedge T} \left( Y_{1, \tau_n \wedge T} \frac{X_{\tau_n \wedge T}^\pi + \dot{G}_{1, \tau_n \wedge T} L_{\tau_n \wedge T}}{\gamma} \right) + \int_0^{\tau_n \wedge T} \frac{\phi_{1,t}^2}{2 \psi_{1,t}} + \frac{\phi_{2,t}^2}{2 \psi_{2,t}} \, dt \right) \right\}_{n \in \mathbb{N}}
\]

is uniformly integrable under \(P\) measure for any sequence of \(F\)-stopping times \(\{\tau_n\}_{n \in \mathbb{N}}\) such that \(\tau_n \uparrow +\infty\), where \(\hat{\pi}_t = \pi_t^*\) is given in \((3.17)\) and \(\hat{\psi}_{1,t}\) and \(\hat{\psi}_{2,t}\) are given in \((2.12)\) with \(X_t^\pi\) replaced by \(X_t^\pi\).

The set of all admissible controls is denoted by \(\Pi_p \otimes \Phi\).

Remark 2. The technical condition 3 in Definition 2.4 implies that both the controls \((\pi, \hat{\phi})\) and \((\hat{\pi}, \phi)\) are immediately admissible whenever there exists at least one control \((\pi, \phi) \in \Pi_p \otimes \Phi\). For the sake of tractability, we suppose that the set of admissible controls is not empty throughout the rest of the paper.

For the power utility case, the ambiguity-averse manager aims to seek a robust investment strategy \(\pi\) to maximize the expected utility from the terminal surplus \(X_T^\pi - L_T\) under the worst-case alternative measure. Inspired by Maenhout [32], the robust optimal ALM problem for the ambiguity-averse manager is formulated as

\[
\sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} J_p(\pi, \phi) := \sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} \mathbb{E}^\mathbb{Q} \left[ \frac{(X_T^\pi - L_T)^\gamma}{\gamma} + \int_0^T \left( \frac{\phi_{1,t}^2}{2 \psi_{1,t}} + \frac{\phi_{2,t}^2}{2 \psi_{2,t}} \right) dt \right], \tag{2.11}
\]

where \(J_p(\pi, \phi)\) denotes the value function associated with admissible control \((\pi, \phi)\), the minimization over \(\phi \in \Phi\) reflects the asset-liability manager’s aversion to ambiguity, and \(\psi_{1,t}\) and \(\psi_{2,t}\) are two \(\mathbb{R}^+\)-valued, \(\mathbb{F}\)-adapted stochastic processes capturing the level of ambiguity aversion with respect to model misspecification. The larger the levels of ambiguity \(\psi_{1,t}\) and \(\psi_{2,t}\) are, the more skeptical the manager is about the reference measure \(P\), and the smaller the penalty for a given deviation from the reference measure is. For the extreme case where \(\psi_{1,t} = \psi_{2,t} = +\infty\), the integral term within \((2.11)\) vanishes and the manager considers all alternative measures equally. For the other extreme case where \(\psi_{1,t} = \psi_{2,t} = 0\), i.e., the manager is completely confident that the reference measure \(P\) is the true measure, any alternative measure deviating from the reference measure \(P\).
where the set of admissible controls is not empty, i.e., there exists at least a control \( \bar{\pi} \). Similar to the above power utility case, in the rest of the paper, we suppose that the ALM problem (2.11) reduces to the traditional ALM problem without model ambiguity, i.e., \( \sup_{\pi \in \Pi} \mathbb{E}^F \left[ \frac{(X^\pi_T - L_T)}{\gamma} \right] \).

For analytical tractability, we assume \( \psi_{1,t} \) and \( \psi_{2,t} \) are state-dependent. Similar to the existing works, such as Maenhout [32], Escobar et al. [20], and Wang and Li [52], we set

\[
\psi_{i,t} = \beta_i \frac{Y_{1,t}(X^\pi_t + G_{1,t})^\gamma}{\gamma}, \quad i = 1, 2,
\]

where \( G_{1,t} \) and \( Y_{1,t} \) are given by (3.5) and (3.6), respectively, and positive constants \( \beta_i \in \mathbb{R}^+ \), \( i = 1, 2 \), are called the ambiguity-aversion parameters. In particular, \( \beta_1 \) can be interpreted as the level of ambiguity about the risky asset dynamics, while \( \beta_2 \) represents the ambiguity aversion about the stochastic factor process.

**Definition 2.5** (Admissible strategy for exponential utility). A control \( (\pi, \phi) \) is said to be admissible if the following conditions are met:

1. \( \pi \) is \( F \)-adapted and \( \phi \in \Phi \);

2. for any initial data \( (x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+ \), the associated asset process (2.10) admits a pathwise unique solution;

3. either the family of random variables

\[
\left\{ \varphi_{\tau_n \wedge T}^\phi \left( - \frac{e^{-q(X^\pi_{\tau_n \wedge T} Y_{2,\tau_n \wedge T} + G_{2,\tau_n \wedge T})}}{q} + \int_0^{\tau_n \wedge T} \frac{\phi_{1,t}^2}{2 \eta_{1,t}} + \frac{\phi_{2,t}^2}{2 \eta_{2,t}} dt \right) \right\}_{n \in \mathbb{N}}
\]

is uniformly integrable under \( \mathbb{P} \) measure for any sequence of \( F \)-stopping times \( \{\tau_n\}_{n \in \mathbb{N}} \) such that \( \tau_n \uparrow +\infty \), where \( \phi = (\phi_1, \phi_2) \in \Phi \) is given by (4.17) with \( X^\pi_t \) and \( \pi_t^* \) replaced by \( X^\pi_t \) and \( \pi_t \), respectively, and \( \eta_{1,t}, \eta_{2,t}, Y_{2,t}, \) and \( G_{2,t} \) are given by (2.14), (4.4), and (4.7) below, or the family of

\[
\left\{ \varphi_{\tau_n \wedge T}^\phi \left( - \frac{e^{-q(X^\pi_{\tau_n \wedge T} Y_{2,\tau_n \wedge T} + G_{2,\tau_n \wedge T})}}{q} + \int_0^{\tau_n \wedge T} \frac{\phi_{1,t}^2}{2 \eta_{1,t}} + \frac{\phi_{2,t}^2}{2 \eta_{2,t}} \right) \right\}_{n \in \mathbb{N}}
\]

is uniformly integrable under \( \mathbb{P} \) measure for any sequence of \( F \)-stopping times \( \{\tau_n\}_{n \in \mathbb{N}} \) such that \( \tau_n \uparrow +\infty \), where \( \pi_t = \pi_t^* \) is given in (4.17) and \( \eta_{1,t} \) and \( \eta_{2,t} \) are given in (2.14) with \( X^\pi_t \) replaced by \( X^\pi_t \).

Denote by \( \Pi_e \otimes \Phi \) the set of all admissible controls.

**Remark 3.** Similar to the above power utility case, in the rest of the paper, we suppose that the set of admissible controls is not empty, i.e., there exists at least a control \( (\pi, \phi) \in \Pi_e \otimes \Phi \). As a result, both controls \( (\hat{\pi}, \hat{\phi}) \) and \( (\pi, \phi) \) are admissible as well based on condition 3 in Definition 2.5.

The robust optimal ALM problem under the exponential utility case is formally written as follows:

\[
\sup_{\pi \in \Pi_e} \inf_{\phi \in \Phi} J_e(\pi, \phi) := \sup_{\pi \in \Pi_e} \inf_{\phi \in \Phi} \mathbb{E}^F \left[ - \frac{e^{-q(X^\pi_T - L_T)}}{q} + \int_0^T \left( \frac{\phi_{1,t}^2}{2 \eta_{1,t}} + \frac{\phi_{2,t}^2}{2 \eta_{2,t}} \right) dt \right],
\]

(2.13)
where \( J_\epsilon(\pi, \phi) \) denotes the value function associated with admissible control \((\pi, \phi)\), and the two \( \mathbb{R}^+ \)-valued, \( \mathbb{F} \)-adapted stochastic processes \( \eta_{i,t} \) and \( \eta_{2,t} \) characterize the level of ambiguity aversion with respect to model ambiguity. Again, for the sake of tractability, we assume that \( \eta_{i,t}, \ i = 1, 2 \) are state-dependent. More specifically, we make the following assumption on \( \eta_{i,t}, \ i = 1, 2 \):

\[
\eta_{i,t} = \frac{\beta_i}{e^{-q(X_i^t Y_{2,t} + G_{2,t})}}, \ i = 1, 2, \tag{2.14}
\]

where \( Y_{2,t} \) and \( G_{2,t} \) are given by (4.4) and (4.7), respectively, and positive constants \( \beta_i \in \mathbb{R}^+ \), \( i = 1, 2 \), denote the ambiguity-aversion parameters. For an ambiguity-neutral manager, the robust optimization problem (2.13) degenerates to finding an admissible investment strategy such that \( \sup_{\pi \in \Pi} \mathbb{E}^P \left[ -e^{-q(X_i^T - L_T)} \right] \) is attained.

**Remark 4.** Given the possibly non-Markovian structures of the market model, the dynamic programming problem (refer to Mataramvura and Øksendal) is not applicable in our case. We, therefore, solve the above two utility maximization problems (2.11) and (2.13) in Section 3 and 4 by proposing a novel BSDE approach. This distinguishes our paper from the published works considering robust investment problems in the Markovian settings and using the HJBI approach; see, for example, Yi et al. [56], Flor and Larsen [21], Munk and Rubstov [37], Escobar et al. [20], Zheng et al. [65], Zeng et al. [59], Wang and Li [52], Chen et al. [10], Wang et al. [53], Cheng and Escobar [12], Chang et al. [7], Baltas et al. [2], Wei et al. [54], and etc.

### 3 Optimal investment strategies for the power utility case

This section is dedicated to deriving the robust optimal investment strategies for the power utility maximization problem (2.11) by using a BSDE approach. To this end, we introduce two continuous \((\mathbb{F}, \mathbb{P})\)-semi-martingales \( Y_{1,t} \) and \( G_{1,t} \) with the following canonical decomposition:

\[
dY_{1,t} = P_{1,t} \, dt + Z_{1,t} \, dW_{1,t} + Z_{2,t} \, dW_{2,t}
\]

and

\[
dG_{1,t} = H_{1,t} \, dt + \Lambda_{1,t} \, dW_{1,t} + \Lambda_{2,t} \, dW_{2,t},
\]

where \( P_{1,t} \) and \( H_{1,t} \) are two undetermined \( \mathbb{F} \)-adapted processes, and \( Z_{1,t}, Z_{2,t}, \Lambda_{1,t} \), and \( \Lambda_{2,t} \) lie in \( L_{\mathbb{F}, \mathbb{P}}^2(0, T; \mathbb{R}) \). Applying Itô’s formula to

\[
\phi_t^\phi \left( Y_{1,t} \left( X_i^T + G_{1,t} \right)^\gamma \right) \geq \int_0^t \phi_{1,s}^2 + \phi_{2,s}^2 \, ds
\]

and using the method of completion of squares, we have

\[
\phi_t^\phi \left( Y_{1,t} \phi_{1,t} + Z_{1,t} \left( X_i^T + G_{1,t} \right) \right) + Y_{1,t} \left( X_i^T + G_{1,t} \right)^\gamma - (X_i^T \pi_t \sigma_t + \Lambda_{1,t}) + \left( \int_0^t \phi_{1,s}^2 \right)_{2 \psi_{1,s}}
\]

\[
+ \phi_{2,t}^2 \, ds \right) \phi_{2,t} \, dW_{2,t} + \phi_t^\phi \left( Y_{1,t} \phi_{2,t} + Z_{2,t} \right) \left( X_i^T + G_{1,t} \right)^\gamma + Y_{1,t} \left( X_i^T + G_{1,t} \right)^\gamma - \Lambda_{2,t} + \left( \int_0^t \phi_{2,s}^2 \right)_{2 \psi_{2,s}}.
\]
We expect that by introducing the two continuous semi-martingales \( Y_{1,t} \) and \( G_{1,t} \), the stochastic process \( \varphi_{t}^{\phi} \left( \frac{X_{t}^{\pi} + G_{1,t}}{\gamma} \right) + \int_{0}^{t} \frac{\phi_{1,s}^{\Phi}}{2\psi_{2,s}^{\Phi}} + \frac{\phi_{2,s}^{\Phi}}{2\psi_{2,s}^{\Phi}} \, ds \) is a local \((\mathbb{F}, \mathbb{P})\)-martingale under an admissible control \((\pi^{*}, \phi^{*}) \in \Pi_{p} \otimes \Phi\), a local \((\mathbb{F}, \mathbb{P})\)-super-martingale for \((\pi, \phi) \in \Pi_{p} \otimes \Phi\), and a local \((\mathbb{F}, \mathbb{P})\)-sub-martingale for \((\hat{\pi}, \hat{\phi}) \in \Pi_{p} \otimes \Phi\), respectively. For this, we can determine the process \( P_{1,t} \) and \( H_{1,t} \) by formally letting the last two terms on the right-hand side of (3.1) be zero. Inspired by this result, we propose the following backward stochastic Riccati equation (BSRE) of \((Y_{1,t}, Z_{1,t}, Z_{2,t})\):

\[
\left\{ \begin{array}{l}
\frac{dY_{1,t}}{dt} = \left[ -r_{1,t} \left( \frac{\gamma}{\gamma - 1} - \frac{\lambda^{2} \alpha_{t}}{\alpha_{t}} \right) + \frac{\gamma}{\gamma - 1} \lambda \sqrt{\alpha_{t}} Z_{1,t} \right] \frac{Y_{1,t}}{\gamma - 1} + \frac{\gamma}{\gamma - 1} \lambda \sqrt{\alpha_{t}} Z_{1,t} \\
+ \frac{1}{2} \left( \beta_{1} + \frac{(\gamma - \beta_{1})^{2}}{\gamma - 1} \right) \frac{Z_{1,t}^{2}}{Y_{1,t}} + \beta_{2} \frac{Z_{2,t}^{2}}{Y_{1,t}} \right] dt + Z_{1,t} dW_{1,t} + Z_{2,t} dW_{2,t},
\end{array} \right.
\]  

(3.2)

and the linear BSDE of \((G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})\):

\[
\left\{ \begin{array}{l}
\frac{dG_{1,t}}{dt} = \left( r_{1,t} G_{1,t} + \Lambda_{1,t} \Lambda_{2,t} - \frac{Z_{2,t}^{2}}{Y_{1,t}} \right) dt + \Lambda_{1,t} dW_{1,t} + \Lambda_{2,t} dW_{2,t},
\end{array} \right.
\]  

(3.3)

Moreover, by separating the dependence of BSDE (3.3) on the liability value \( L_{T} \) and applying Itô’s formula, we can decompose BSDE of \((G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})\) into the following linear BSDE of \((\bar{G}_{1,t}, \bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t})\):

\[
\left\{ \begin{array}{l}
\frac{d\bar{G}_{1,t}}{dt} = \left( r_{1,t} \bar{G}_{1,t} + (\lambda - \sigma_{t}) \sqrt{\alpha_{t}} \bar{\Lambda}_{1,t} - \frac{Z_{2,t}^{2}}{Y_{1,t}} \bar{\Lambda}_{2,t} \right) dt + \bar{\Lambda}_{1,t} dW_{1,t} + \bar{\Lambda}_{2,t} dW_{2,t},
\end{array} \right.
\]  

(3.4)

and the solutions \((G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t})\) and \((\bar{G}_{1,t}, \bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t})\) are related via the following linear formulation:

\[
(G_{1,t}, \Lambda_{1,t}, \Lambda_{2,t}) = (G_{1,t} L_{T}, \bar{\Lambda}_{1,t} + \bar{G}_{1,t} \sigma_{t} \sqrt{\alpha_{t}} L_{t}, \bar{\Lambda}_{2,t} L_{t}).
\]  

(3.5)
Throughout this section, by a solution to BSRE (3.2), we mean a triplet of stochastic processes $(Y_{1,t}, Z_{1,t}, Z_{2,t}) \in \mathcal{S}_{F,P}^\infty(0,T;\mathbb{R}^+) \otimes \mathcal{L}^2_{F,P}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{F,P}(0,T;\mathbb{R})$ and satisfies (3.2). Similarly, the solution to linear BSDE (3.4) is a triplet of stochastic process $(\bar{G}_{1,t}, \bar{\Lambda}_{1,t}, \bar{\Lambda}_{2,t}) \in \mathcal{S}_{F,P}^\infty(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{F,P}(0,T;\mathbb{R}) \otimes \mathcal{L}^2_{F,P}(0,T;\mathbb{R})$.

**Remark 5.** Note that the generator of BSRE (3.2) depends on the market price of risk rather than the return rate and volatility of the risky asset price, which implies that the solvability of BSRE (3.2) is completely determined by the square-root factor process $\alpha_t$ (2.3), and it is, therefore, unnecessary to specify the return rate $\mu_t$ and volatility $\sigma_t$ as Markovian processes. However, due to the unboundedness of $\alpha_t$, the established theory of BSDEs (see, for example, El Karoui et al. [18], Bender an Kohlmann [3], Kobylanski [27], Briand and Hu [5]) cannot be applied to (3.2) directly. Similar to Shen and Zeng [47] and Zhang [62], we first propose one explicit solution to (3.2) by trial and verify its uniqueness by using Girsanov’s measure change technique and the standard results of quadratic BSDE with bounded terminal condition (Kobylanski [27]).

In this section, we impose the following assumption on the model parameters. This guarantees that the factor process $\alpha_t$ preserves the affine-form, square-root structure under an equivalent probability measure $\mathbb{P}$ which is well-defined in the proof of Theorem 3.5.

**Assumption 3.1.** $\kappa + \rho_1 \lambda \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \neq 0$.

**Proposition 3.2.** One solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ to BSRE (3.2) is given by

$$Y_{1,t} = \exp \{ f_1(t) + g_1(t)\alpha_t \},$$

(3.6)

and

$$(Z_{1,t}, Z_{2,t}) = (\rho_1 g_1(t) \sqrt{\alpha_t Y_{1,t}}, \rho_2 g_1(t) \sqrt{\alpha_t Y_{1,t}}),$$

(3.7)

where functions $f_1(t)$ and $g_1(t)$ solve the following ordinary differential equations (ODEs):

$$\frac{dg_1(t)}{dt} = \left( \frac{\gamma - \beta_1}{2\gamma(\gamma - 1 - \beta_1)} \rho_2^2 - \frac{\gamma - \beta_1}{2\gamma} \rho_2 \right) g_1^2(t) + \left( \kappa + \frac{(\gamma - \beta_1)\lambda \rho_1}{\gamma - 1 - \beta_1} \right) g_1(t) + \frac{\gamma \lambda^2}{2(\gamma - 1 - \beta_1)},$$

(3.8)

and

$$\frac{df_1(t)}{dt} = -\kappa \theta g_1(t) - r \gamma,$$

(3.9)

with boundary conditions $f_1(T) = g_1(T) = 0$. Moreover, the closed-form solutions to ODEs (3.8) and (3.9) are given by

$$g_1(t) = \frac{n_{g_1^+} n_{g_1^-} (1 - e^{\sqrt{\Delta g_1} (T-t)})}{n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta g_1} (T-t)}},$$

(3.10)

and

$$f_1(t) = \left( r \gamma + \kappa \theta n_{g_1^-} \right) (T - t) + \frac{\kappa \theta (n_{g_1^+} - n_{g_1^-})}{\sqrt{\Delta g_1}} \log \left( \frac{n_{g_1^+} - n_{g_1^-}}{n_{g_1^+} - n_{g_1^-} e^{\sqrt{\Delta g_1} (T-t)}} \right),$$

(3.11)
where $\Delta_{g_1}, n_{g_1^+},$ and $n_{g_1^-}$ are given by
\[
\begin{align*}
\Delta_{g_1} &= \left(\kappa + \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}\right)^2 - \frac{\lambda^2}{\gamma - 1 - \beta_1} \left(\frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}\rho_1^2 - (\gamma - \beta_2)\rho_2^2\right), \\
n_{g_1^+} &= \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}\rho_1^2 - \frac{\gamma - \beta_2}{\gamma} \rho_2^2, \\
n_{g_1^-} &= \frac{\gamma - \beta_1}{\gamma(\gamma - 1 - \beta_1)}\rho_1^2 - \frac{\gamma - \beta_2}{\gamma} \rho_2^2.
\end{align*}
\] (3.12)

Proof. See Appendix A.1. □

Remark 6. From (3.10), we have $\frac{dg_1(t)}{dt} = \frac{2\gamma\lambda^2\Delta_{g_1}e^{\sqrt{\Delta_{g_1}}(t-t)}}{(\gamma-1-\beta_1)\left(\frac{\gamma-\beta_1}{\gamma-1-\beta_1}\rho_1^2 - \frac{\gamma-\beta_2}{\gamma}\rho_2^2\right)^2\left(n_{g_1^+} - n_{g_1^-}\right)e^{\sqrt{\Delta_{g_1}}(t-t)}} > 0$
due to $\gamma \in \mathbb{R}^-$. In other words, function $g_1(t)$ is strictly increasing over $[0,T]$, and thus, we have $g_1(t) \in [g_1(0), 0]$ and $f_1(t) \leq \tau g(T-t)$.

Proposition 3.3. The solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ proposed in Proposition 3.2 lies in $S_{F, P}^\infty(0, T; \mathbb{R}^+) \otimes L_{F, P}^2(0, T; \mathbb{R}) \otimes L_{P, P}^2(0, T; \mathbb{R})$. More precisely, $Y_t \leq e^{\tau g(T-t)}$, for $t \in [0, T]$, $\mathbb{P}$ almost surely.

Proof. See Appendix A.2. □

Before verifying that the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in Proposition 3.2 is the unique solution to BSRE (3.2) in the space $S_{F, P}^\infty(0, T; \mathbb{R}^+) \otimes L_{F, P}^2(0, T; \mathbb{R}) \otimes L_{P, P}^2(0, T; \mathbb{R})$, we present the following auxiliary result on the stochastic exponential process of the square-root factor process $\alpha_t$ (refer to Lemma A1 in Shen and Zeng [47]).

Lemma 3.4 (Bona-fide martingale property). If $m_1(t)$ and $m_2(t)$ are two bounded functions over $[0, T]$, the following stochastic exponential process

\[
\exp \left\{ \int_0^t m_1(s)\sqrt{\alpha_s} \, dW_{1,s} + \int_0^t m_2(s)\sqrt{\alpha_s} \, dW_{2,s} - \frac{1}{2} \int_0^t (m_1^2(s) + m_2^2(s)) \alpha_s \, ds \right\}
\]

is an $(\mathbb{F}, \mathbb{P})$-martingale.

Theorem 3.5. Suppose that Assumption 3.1 holds true. The solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in (3.6) and (3.7) is the unique solution to BSRE (3.2).

Proof. See Appendix A.3. □

After deriving the closed-form expression of the unique solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ to BSRE (3.3), the linear BSDE (3.4) of $(\tilde{G}_{1,t}, \tilde{A}_{1,t}, \tilde{A}_{2,t})$ can be reformulated as follows:
\[
\begin{align*}
\frac{d\tilde{G}_{1,t}}{dt} &= \left((r - \rho_1)\tilde{G}_{1,t} + (\lambda - \sigma_t)\sqrt{\alpha_t}\tilde{A}_{1,t} - \rho_2 g_1(t)\sqrt{\alpha_t}\tilde{A}_{2,t}\right) \\
- \tilde{A}_{1,t} dW_{1,t} + \tilde{A}_{2,t} dW_{2,t},
\end{align*}
\] (3.13)

Proposition 3.6. The unique solution to linear BSDE (3.13) is given by
\[
\tilde{G}_{1,t} = -e^{(r-\mu)(t-T)},
\] (3.14)
and
\[
(\tilde{A}_{1,t}, \tilde{A}_{2,t}) = (0, 0).
\] (3.15)
Proof. See Appendix A.4.

Remark 7. It is crucial to identify that the second control component $\Lambda_{2,t}$ of the solution to linear BSDE (3.3) is zero from Proposition 3.6 and the relationship between $\Lambda_{2,t}$ and $\bar{\Lambda}_{2,t}$ given in (3.5) above, which, in turn, allows us to remove the fourth and fifth drift terms on the right-hand side of (3.1).

After solving the associated BSDEs explicitly and deriving their uniqueness results, now we are ready to state our first main result.

Theorem 3.7. For any initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + G_{1,0}l_0 \in \mathbb{R}^+$, suppose that Assumption 3.1 and the following conditions hold

$$
\max\{k_0, k_1, k_2\} \leq \frac{\kappa^2}{\rho_1^2 + \rho_2^2} \tag{3.16}
$$

with $k_0, k_1, \text{ and } k_2$ given by

$$
\begin{aligned}
k_0 &= \sup_{t \in [0,T]} (120 + 32\sqrt{14}) \left[ \frac{(\beta_1 \rho_1 g_1(t) + \lambda \gamma \beta_1)^2}{\gamma^2(\gamma - 1 - \beta_1)^2} + \frac{\beta_2^2 \rho_2^2 g_2(t)}{\gamma^2} \right], \\
k_1 &= \sup_{t \in [0,T]} (2 + \sqrt{2})(8(\gamma - \beta_1) \rho_1 g_1(t) + 8\gamma \lambda)^2, \\
k_2 &= \sup_{t \in [0,T]} \frac{32(\gamma - \beta_1) \lambda \rho_1 g_1(t) + 32\lambda^2 \gamma}{\beta_1 + 1 - \gamma} + (128\gamma^2 - 16\gamma) \frac{(2 - \beta_1) \rho_1 g_1(t) + \lambda)^2}{(\beta_1 + 1 - \gamma)^2}.
\end{aligned}
$$

Then, for the following control $(\pi^*, \phi^*)$

$$
\begin{aligned}
\pi^*_t &= \frac{1}{X^*_t \sigma_t} \left[ X^*_t + \bar{G}_{1,t} L_t \left( \frac{\gamma - \beta_1}{Y_{1,t}} Z_{1,t} + \lambda \sqrt{\alpha_t} \right) - \sigma_t \sqrt{\alpha_t} \bar{G}_{1,t} L_t \right], \\
\phi^*_1 &= - \frac{\beta_1}{\gamma} Z_{1,t} + \frac{\beta_1 (X^*_t \pi^*_1 \sigma_t + \sigma_t \sqrt{\alpha_t} \bar{G}_{1,t} L_t)}{X^*_t + \bar{G}_{1,t} L_t} = \frac{\beta_1}{\gamma} Z_{1,t} + \frac{\lambda \beta_1}{\gamma - 1 - \beta_1} \sqrt{\alpha_t}, \\
\phi^*_2 &= - \frac{\beta_2}{\gamma} Z_{2,t},
\end{aligned}
$$

(3.17)

where $X^*_t$ is the asset process associated with $\pi^*_t$, and the closed-form expressions for $Y_{1,t}, Z_{1,t}, Z_{2,t}$ and $\bar{G}_{1,t}$ are given by (3.6), (3.7), and (3.14), respectively, we have

(i) the distortion process $\phi^* \in \Phi$, i.e., the Radon-Nikodym derivative process $\varphi^*_i$ is a uniformly integrable $(\mathbb{F}, \mathbb{P})$-martingale, and $X^*_t + \bar{G}_{1,t} L_t > 0$, $\mathbb{P}$ almost surely, for all $t \in [0, T]$;

(ii) $\varphi^*_i Y_{1,t} (X^*_t + \bar{G}_{1,t} L_t)^\gamma \in S_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}^+)$;

(iii) $\varphi^*_i \psi^* \left( \int_0^t \frac{(\phi^*_1)^2}{2 \psi^*_1} + \frac{(\phi^*_2)^2}{2 \psi^*_2} ds \right) \in S^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}^+)$, where $\psi^*_i = \frac{\beta_i}{Y_{1,t} (X^*_t + \bar{G}_{1,t} L_t)^\gamma}$, for $i = 1, 2$.

In the affirmative, the control $(\pi^*, \phi^*) \in \Pi_p \otimes \Phi$ is the optimal control of the robust ALM problem (2.11), and the optimal value function is given by

$$
J_p(\pi^*, \phi^*) = Y_{1,0} \frac{(x_0 + \bar{G}_{1,0} l_0)^\gamma}{\gamma}. \tag{3.18}
$$

Proof. See Appendix A.5.
Remark 8. The feasibility of the technical condition (3.16) is guaranteed by the monotonicity of function \( g_1(t) \), and in essence, this sufficient condition is imposed to show that \((\pi^*, \phi^*)\) is a saddle point of the value function \( J_p(\pi, \phi) \) for the robust control problem (2.11). More specifically, it proves that (i)-(iii) in Theorem 3.7 hold for the control \((\pi^*, \phi^*)\) and verifies that \((\pi^*, \phi^*) \in \Pi_p \otimes \Phi\) by confirming conditions 1-3 in Definition 2.4 above.

Remark 9. Note that although robust investment problems using BSDE approaches have been considered in the literature (see, for example, Øksendal and Sulem [38, 39]), our BSDE approach is different from that in [38, 39], where the value function is written as the solution of an associated controlled BSDE and the comparison theorem for solutions of BSDEs is applied to show that the optimal control is a saddle point of the value function. Recalling (3.1) and the proof of Theorem 3.7 above, we propose to construct a stochastic process depending on any admissible control \((\pi, \phi)\) instead, and such that its terminal value equals the sum of the utility of the terminal surplus and the penalty term within the expectation of (2.11), which then leads to two uncontrolled BSDEs (3.2)-(3.3). By showing that the proposed stochastic process is a local \((\mathbb{F}, \mathbb{P})\)-martingale under \((\pi^*, \phi^*) \in \Pi_p \otimes \Phi\), a local \((\mathbb{F}, \mathbb{P})\)-super-martingale for \((\pi, \phi) \in \Pi_p \otimes \Phi\), and a local \((\mathbb{F}, \mathbb{P})\)-sub-martingale for \((\tilde{\pi}, \tilde{\phi}) \in \Pi_p \otimes \Phi\), respectively, and using some localization techniques, we confirm that the optimal control \((\pi^*, \phi^*)\) is a saddle point of the value function \( J_p(\pi, \phi) \).

Remark 10. To our knowledge, the results provided in Theorem 3.7 are not reported in the existing literature. If we further set \( b_0 = \mu_1 = \sigma_1 = 0 \) in Theorem 3.7, then we derive the explicit solutions to the robust portfolio selection problem under power utility and square-root factor processes. If we specify \( \beta_1 = \beta_2 = 0 \), the analytical solutions to the optimal ALM problem under power utility and square-root factor processes are provided. In other words, the benefits of Theorem 3.7 are two-fold.

The next three corollaries provide the explicit results for three particular cases of our model, the CEV model (2.4), the family of 4/2 models (2.5), and non-Markovian stochastic volatility model (2.6), respectively.

Corollary 3.8 (CEV model). If the risky asset price \( S_t \) follows the CEV model (2.4) with any initial data \((x_0, s_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+\) such that \( x_0 + G_{1,0}l_0 \in \mathbb{R}^+\), and suppose that \( \mu - r(\gamma - \beta_1) \neq 0 \) and the following conditions hold

\[
\max\left\{ \tilde{k}_0, \tilde{k}_1, \tilde{k}_2 \right\} \leq \frac{\mu^2}{\sigma^2},
\]

with \( \tilde{k}_0, \tilde{k}_1, \text{ and } \tilde{k}_2 \) given by

\[
\left\{
\begin{align*}
\tilde{k}_0 &= \sup_{t \in [0, T]} (120 + 32\sqrt{14}) \frac{(\gamma \frac{\mu-r}{\sigma} - 2\beta \sigma \bar{g}_1(t))^2 \beta_1^2}{\gamma^2 (\gamma - 1 - \beta_1)^2}, \\
\tilde{k}_1 &= \sup_{t \in [0, T]} (2 + \sqrt{2}) \frac{(16(\gamma - 1 - \beta_1)\beta \sigma \bar{g}_1(t))^2}{(\beta_1 + 1 - \gamma)^2}, \\
\tilde{k}_2 &= \sup_{t \in [0, T]} 32 \lambda^2 (4(\gamma - 1 - \beta_1)(\mu - r)\beta \bar{g}_1(t)) + (128 \gamma^2 - 16 \gamma) \frac{(\mu-r)^2 - 2\beta \sigma^2 \beta_1 \bar{g}_1(t)^2}{(\beta_1 + 1 - \gamma)^2},
\end{align*}
\right.
\]

then, the optimal control and optimal value function of the robust ALM problem (2.11) are, respec-
tively, given by

\[
\begin{align*}
\pi_t^* &= \left(\mu - r\right)\left(X_t^* + G_{1,t}L_t\right) - \frac{2\left(\gamma - \beta_1\right)\beta\sigma g_1(t)\left(X_t^* + G_{1,t}L_t\right) + \left(\beta_1 + 1 - \gamma\right)\gamma\sigma_t G_{1,t}L_t}{X_t^*\sigma^2 S_t^{2\beta}}/\left(\beta_1 + 1 - \gamma\right), \\
\phi_{1,t}^* &= \frac{\beta_1}{\left(\gamma - 1 - \beta_1\right)S_t^\beta} \left(\frac{\mu - r}{\sigma} - \frac{2\beta\sigma g_1(t)}{\gamma}\right), \\
\phi_{2,t}^* &= 0,
\end{align*}
\]

and

\[
J_p(\pi^*, \phi^*) = \left(\frac{x_0 - G_{1,0}l_0}{\gamma}\right) \exp\left\{f_1(0) + g_1(0)s_0^{-2\beta}\right\},
\]

where \(G_{1,t}\) is given by (3.14), and functions \(f_1(t)\) and \(g_1(t)\) are given by

\[
f_1(t) = r\gamma + (2\beta^2 + \beta)\sigma^2 n_{g_1^+}(T - t) + \frac{(2\beta^2 + \beta)\sigma^2 (n_{g_1^+} - n_{g_1^-})}{\sqrt{\Delta g_1}} \log \left(\frac{n_{g_1^+} - n_{g_1^-}}{n_{g_1^+} - n_{g_1^-}e^{\sqrt{\Delta g_1}(T-t)}}\right),
\]

and

\[
g_1(t) = \frac{n_{g_1^+} - n_{g_1^-}}{n_{g_1^+} - n_{g_1^-}e^{\sqrt{\Delta g_1}(T-t)}}.
\]

with \(\Delta g_1, n_{g_1^+},\) and \(n_{g_1^-}\) given by

\[
\begin{align*}
\Delta g_1 &= 4\beta^2 \left[\left(\frac{\mu - (\gamma - \beta_1)(\mu - r)}{\gamma - 1 - \beta_1}\right)^2 - \frac{\gamma - \beta_1}{(\gamma - 1 - \beta_1)^2}(\mu - r)^2\right], \\
n_{g_1^+} &= \frac{2\beta \left(\frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}(\mu - r) - \mu\right) + \sqrt{\Delta g_1}}{\frac{\gamma - \beta_1}{\gamma(\gamma - 1 - \beta_1)} 4\beta^2\sigma^2}, \\
n_{g_1^-} &= \frac{2\beta \left(\frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}(\mu - r) - \mu\right) - \sqrt{\Delta g_1}}{\frac{\gamma - \beta_1}{\gamma(\gamma - 1 - \beta_1)} 4\beta^2\sigma^2}.
\end{align*}
\]

Proof. Substituting the parameters specified in Example 2.1 into (3.16)-(3.18) leads to the results immediately. \(\square\)

**Remark 11.** If we set \(l_0 = \mu_1 = \sigma_1 = 0\) in Corollary 3.8, we obtain the closed-form solutions to the robust portfolio selection problem under the CEV model and power utility. If we plug \(\beta_1 = 0\) into Corollary 3.8 instead, then we provide the explicit solutions to the optimal ALM problem under the CEV model and power utility. To our knowledge, these results are not reported in the existing literature.

**Corollary 3.9** (The family of 4/2 models). If the risky asset price process \(S_t\) and the variance driver process \(V_t\) are governed by the family of 4/2 models (2.5) with any initial data \((x_0, v_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+\) such that \(x_0 + G_{1,0}l_0 \in \mathbb{R}^+\), and suppose that \(\kappa + \sigma_0\lambda\frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \neq 0\) and the following conditions hold

\[
\max\{\bar{k}_0, \bar{k}_1, \bar{k}_2\} \leq \frac{\kappa^2}{\sigma_v^2},
\]

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with \( \tilde{k}_0, \tilde{k}_1, \) and \( \tilde{k}_2 \) given by

\[
\begin{align*}
\tilde{k}_0 &= \sup_{t \in [0, T]} (120 + 32\sqrt{14}) \left[ \frac{(\beta_1 \sigma_v \rho g_1(t) + \lambda \gamma \beta_1)^2}{\gamma^2(\gamma - 1 - \beta_1)^2} + \frac{\beta_2^2 \sigma_v^2 (1 - \rho^2) g_1^2(t)}{\gamma^2} \right], \\
\tilde{k}_1 &= \sup_{t \in [0, T]} (2 + \sqrt{2}) \frac{(8(\gamma - \beta_1) \sigma_v \rho g_1(t) + 8\gamma \lambda)^2}{(\beta_1 + 1 - \gamma)^2}, \\
\tilde{k}_2 &= \sup_{t \in [0, T]} \frac{32(\gamma - \beta_1) \lambda \sigma_v \rho g_1(t) + 32\gamma^2}{\beta_1 + 1 - \gamma} + (128\gamma^2 - 16\gamma) \frac{(\gamma - \beta_1) \sigma_v \rho g_1(t) + \lambda)^2}{(\beta_1 + 1 - \gamma)^2},
\end{align*}
\]

then, the optimal control and optimal value function of the robust ALM problem (2.11) are, respectively, given by

\[
\begin{align*}
\pi^* &= \frac{V_t}{X_t^*(c_1 V_t + c_2)} \left[ \frac{X_t^* + \tilde{G}_{1,t} L_t}{\beta_1 + 1 - \gamma} \left( \frac{\gamma - \beta_1}{\gamma} \sigma_v \rho g_1(t) + \lambda \right) - \sigma_t \tilde{G}_{1,t} L_t \right], \\
\phi_1^* &= \left[ \frac{\beta_1}{\gamma(\gamma - 1 - \beta_1)} \sigma_v \rho g_1(t) + \frac{\lambda \beta_1}{\gamma - 1 - \beta_1} \right] \sqrt{V_t}, \\
\phi_2^* &= -\frac{\beta_2}{\gamma} \sigma_v \sqrt{1 - \rho^2 g_1(t)} \sqrt{V_t},
\end{align*}
\]

and

\[
J_p(\pi^*, \phi^*) = \frac{(x_0 - \tilde{G}_{1,0} l_0)^\gamma}{\gamma} \exp \left\{ \tilde{f}_1(0) + \tilde{g}_1(0) v_0 \right\},
\]

where \( \tilde{G}_{1,t} \) is given by (3.14), and functions \( \tilde{f}_1(t) \) and \( \tilde{g}_1(t) \) are given by

\[
\tilde{f}_1(t) = \left( r \gamma + \kappa \theta n_{g_1^+} \right) (T - t) + \frac{\kappa \theta \left( n_{g_1^-} - n_{g_1^+} \right)}{\sqrt{\Delta_{g_1}}} \log \left( \frac{n_{g_1^-} - n_{g_1^+}}{n_{g_1^-} - n_{g_1^+} e^{\sqrt{\Delta_{g_1}}(T - t)}} \right)
\]

and

\[
\tilde{g}_1(t) = \frac{n_{g_1^-} n_{g_1^+}}{n_{g_1^-} - n_{g_1^+} e^{\sqrt{\Delta_{g_1}}(T - t)}}
\]

with \( \Delta_{g_1}, n_{g_1^+}, \) and \( n_{g_1^-} \) given by

\[
\begin{align*}
\Delta_{g_1} &= \left( \kappa + \frac{(\gamma - \beta_1) \lambda \sigma_v \rho}{\gamma - 1 - \beta_1} \right)^2 - \frac{\kappa^2 \sigma_v^2}{\gamma - 1 - \beta_1} \left( \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \rho^2 - (\gamma - \beta_2)(1 - \rho^2) \right), \\
n_{g_1^+} &= -\frac{\left( \kappa + \frac{(\gamma - \beta_1) \lambda \sigma_v \rho}{\gamma - 1 - \beta_1} \right) + \sqrt{\Delta_{g_1}}}{\sigma_v^2 \left( \frac{\gamma - \beta_1}{\gamma(\gamma - 1 - \beta_1)} \rho^2 - \frac{\gamma - \beta_2}{\gamma(1 - \beta_1)}(1 - \rho^2) \right)}, \\
n_{g_1^-} &= -\frac{\left( \kappa + \frac{(\gamma - \beta_1) \lambda \sigma_v \rho}{\gamma - 1 - \beta_1} \right) - \sqrt{\Delta_{g_1}}}{\sigma_v^2 \left( \frac{\gamma - \beta_1}{\gamma(\gamma - 1 - \beta_1)} \rho^2 - \frac{\gamma - \beta_2}{\gamma(1 - \beta_1)}(1 - \rho^2) \right)}.
\end{align*}
\]

Proof. Plugging the specified parameters of the 4/2 model (2.5) into (3.16)-(3.18) yields the above results.

\[\square\]

**Remark 12.** Setting either \((c_1, c_2) = (1, 0)\) or \((c_1, c_2) = (0, 1)\) in the 4/2 model (2.5), Corollary 3.9 provides the explicit expressions for the optimal controls and optimal value functions of the robust ALM problem (2.11) under the Heston model and 3/2 model, respectively, and neither of them is considered in the published works.
Remark 13. It is worth mentioning that Cheng and Escobar [12] recently solved the robust portfolio selection problem under the 4/2 model in a complete market. Corollary 3.9 extends the results of Cheng and Escobar [12] to the case with random liabilities in an incomplete market setting. Moreover, if we ignore model ambiguity by imposing $\beta_1 = \beta_2 = 0$ in Corollary 3.9, it can be verified that our result generalizes that of Cheng and Escobar [11] to the case with random liabilities.

Corollary 3.10 (Non-Markovian path-dependent model). If the risky asset price process $S_t$ and its volatility driver process are governed by the path-dependent stochastic volatility model (2.6) with initial data $(x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+$ such that $x_0 + \bar{G}_{1,0}l_0 \in \mathbb{R}^+$, and suppose that Assumption 3.1 and condition (3.16) hold true, then the optimal control and optimal value function of the robust ALM problem (2.11) are, respectively, given by

$$
\begin{align*}
\pi_t^* &= \frac{1}{X_t^* \hat{\sigma}(\alpha_{[0,t]})} \left[ X_t^* + \bar{G}_{1,t}L_t \left( \frac{\gamma - \beta_1 Z_{1,t}}{\gamma Y_{1,t}} + \lambda \sqrt{\alpha_t} \right) - \sigma_t \sqrt{\alpha_t} \bar{G}_{1,t}L_t \right], \\
\phi_{1,t}^* &= \frac{\beta_1}{\gamma(\gamma - 1 - \beta_1)} Y_{1,t} + \frac{\lambda \beta_1}{\gamma - 1 - \beta_1} \sqrt{\alpha_t}, \\
\phi_{2,t}^* &= -\frac{\beta_2}{\gamma} Z_{2,t},
\end{align*}
$$

and

$$
J_p(\pi^*, \phi^*) = Y_{1,0} \frac{(x_0 + \bar{G}_{1,0}l_0)^\gamma}{\gamma},
$$

where the closed-form expressions for $Y_{1,t}, Z_{1,t}, Z_{2,t}$ and $\bar{G}_{1,t}$ are given by (3.6), (3.7), and (3.14), respectively.

Proof. Replacing $\sigma_t$ in Theorem 3.7 by $\hat{\sigma}(\alpha_{[0,t]})$ leads to the above results immediately. \qed

4 Optimal investment strategies for the exponential utility case

In this section, we investigate the robust optimization problem under exponential utility (2.13) by using a BSDE approach. Similar to the previous section, to find the BSDEs associated with problem (2.13), we introduce the following continuous $(\mathbb{F}, \mathbb{P})$-semi-martingales $Y_{2,t}$ and $G_{2,t}$ with canonical decomposition as follows:

$$
dY_{2,t} = P_{2,t} dt + M_{1,t} dW_{1,t} + M_{2,t} dW_{2,t},
$$

and

$$
dG_{2,t} = H_{2,t} dt + \Gamma_{1,t} dW_{1,t} + \Gamma_{2,t} dW_{2,t},
$$

where $P_{2,t}$ and $H_{2,t}$ are two undetermined $\mathbb{F}$-adapted processes, and $M_{1,t}, M_{2,t}, \Gamma_{1,t},$ and $\Gamma_{2,t}$ belong to $\mathcal{L}_{2,loc}^2(0, T; \mathbb{R})$. An application of Itô’s formula to $\varphi_\phi^\phi \left( -e^{-q(Y_{2,t} + G_{2,t})} + \int_0^t \left( \frac{\phi_{1,s}^2}{2q_{1,s}} + \frac{\phi_{2,s}^2}{2q_{2,s}} \right) ds \right)$

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leads to
\[
d\varphi^\phi_t \left( -\frac{e^{-q(X_t^\pi Y_{2,t} + G_{2,t})}}{q} + \int_0^t \left( \frac{\phi^2_{1,s}}{2\eta_{1,s}} + \frac{\phi^2_{2,s}}{2\eta_{2,s}} \right) ds \right)
= \varphi^\phi_t \left[ e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} \left( (\sigma_1 \pi_t Y_{2,t} + M_{1,t})X_t^\pi + \Gamma_{1,t} \right) + \phi_1,t \left( \int_0^t \left( \frac{\phi^2_{1,s}}{2\eta_{1,s}} + \frac{\phi^2_{2,s}}{2\eta_{2,s}} \right) ds \right)
- e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} \right] dW_{1,t} + \varphi^\phi_t \left[ e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} (M_{2,t} X_t^\pi + \Gamma_{2,t}) + \left( \int_0^t \left( \frac{\phi^2_{1,s}}{2\eta_{1,s}} + \frac{\phi^2_{2,s}}{2\eta_{2,s}} \right) ds \right)
- e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} \right] \phi_2,t \right] dW_{2,t} + \frac{\varphi^\phi_t}{2\eta_{2,t}} \left[ \phi_1,t + e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} (\sigma_1 \pi_t Y_{2,t} + M_{1,t})X_t^\pi
+ \Gamma_{1,t} \right] \eta_{1,t} \right]^2 dt
+ \frac{\varphi^\phi_t}{2\eta_{2,t}} \left[ \phi_2,t + e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} (M_{2,t} X_t^\pi + \Gamma_{2,t}) \eta_{2,t} \right]^2 dt
- \frac{q + \beta_1}{2} \varphi^\phi_t e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} \left[ (\sigma_1 \pi_t Y_{2,t} + M_{1,t})X_t^\pi + \Gamma_{1,t} - \frac{1}{q + \beta_1} \left( \lambda \sqrt{\alpha_t} + M_{1,t} Y_{2,t} \right) \right]^2 dt
- \frac{q + \beta_2}{2} \varphi^\phi_t e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} M_{2,t}^2 (X_t^\pi)^2 dt
- (q + \beta_2) \varphi^\phi_t e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} M_{2,t} X_t^\pi \Gamma_{2,t} dt
+ \varphi^\phi_t e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} \left[ H_{2,t} + \frac{\left( \lambda \sqrt{\alpha_t} + M_{1,t} Y_{2,t} \right)^2}{2(q + \beta_1)} - \frac{q + \beta_2}{2} \Gamma_{2,t} - \left( \lambda \sqrt{\alpha_t} + M_{1,t} Y_{2,t} \right) \Gamma_{1,t} \right] dt
+ \varphi^\phi_t e^{-q(X_t^\pi Y_{2,t} + G_{2,t})} X_t^\pi \left[ r_{Y_{2,t}} + P_{2,t} - \left( \lambda \sqrt{\alpha_t} + M_{1,t} Y_{2,t} \right) M_{1,t} \right] dt.
\]

(4.1)

We expect that \( \varphi^\phi_t \left( -\frac{e^{-q(X_t^\pi Y_{2,t} + G_{2,t})}}{q} + \int_0^t \left( \frac{\phi^2_{1,s}}{2\eta_{1,s}} + \frac{\phi^2_{2,s}}{2\eta_{2,s}} \right) ds \right) \) is a local \((\mathbb{F}, \mathbb{P})\)-martingale for an admissible control \((\pi^*, \phi^*) \in \Pi_c \otimes \Phi\), a local \((\mathbb{F}, \mathbb{P})\)-super-martingale for \((\pi, \tilde{\phi}) \in \Pi_c \otimes \Phi\), and a local \((\mathbb{F}, \mathbb{P})\)-sub-martingale for \((\tilde{\pi}, \phi) \in \Pi_c \otimes \Phi\), respectively. Inspired by this, the stochastic processes \(P_{2,t}\) and \(H_{2,t}\) can be determined by simply letting the last two terms on the right-hand side of (4.1) be zero. In other words, we find the following BSRE of \((Y_{2,t}, M_{1,t}, M_{2,t})\):

\[
\begin{cases}
dY_{2,t} = -rY_{2,t} \left( \lambda \sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) dt + M_{1,t} dW_{1,t} + M_{2,t} dW_{2,t}, \\
Y_{2,\tau} = 1,
\end{cases}
\]

(4.2)

and quadratic BSDE of \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\):

\[
\begin{cases}
dG_{2,t} = \left[ \left( \lambda \sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right) \Gamma_{1,t} + \frac{q + \beta_2}{2} \Gamma_{2,t} - \frac{1}{2(q + \beta_1)} \left( \lambda \sqrt{\alpha_t} + \frac{M_{1,t}}{Y_{2,t}} \right)^2 \right] dt \\
+ \Gamma_{1,t} dW_{1,t} + \Gamma_{2,t} dW_{2,t}, \\
G_{2,\tau} = -L_{\tau}.
\end{cases}
\]

(4.3)

Here, a solution to BSRE (4.2) is a triplet of \(\mathbb{F}\)-adapted processes \((Y_{2,t}, M_{1,t}, M_{2,t}) \in S^\infty_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}^+) \otimes \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R})\); a solution to quadratic BSDE (4.3) is a triplet of \(\mathbb{F}\)-adapted processes \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t}) \in \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}) \otimes \mathcal{L}^2_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R})\).
Similar to Section 3, we make the following assumptions on the model parameters to ensure that $\alpha_t$ is an affine-form, square-root factor process under the well-defined probability measure $\tilde{P}$ in Theorem 4.4.

**Assumption 4.1.** $\kappa + \lambda \rho_1 \neq 0$.

**Proposition 4.2.** The unique solution to BSRE (4.2) is given by

$$Y_{2,t} = e^{r(T-t)},$$

and

$$(M_{1,t}, M_{2,t}) = (0, 0).$$

**Proof.** See Appendix A.6.

After solving BSRE (4.2) explicitly, we can simplify quadratic BSDE (4.3) to the following form:

$$\begin{cases}
    dG_{2,t} = \left( \lambda \sqrt{\alpha_t} \Gamma_{1,t} + \frac{q + \beta_2}{2} \Gamma_{2,t}^2 - \frac{1}{2(q + \beta_1)} \lambda^2 \alpha_t \right) dt + \Gamma_{1,t} dW_{1,t} + \Gamma_{2,t} dW_{2,t}, \\
    G_{2,T} = -L_T.
\end{cases}$$

**Proposition 4.3.** One solution $(G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})$ to quadratic BSDE (4.6) is given by

$$G_{2,t} = f_2(t) + g_2(t) \alpha_t + h_2(t) L_t$$

and

$$(\Gamma_{1,t}, \Gamma_{2,t}) = \left((\rho_1 g_2(t) + \sigma_1 h_2(t) L_t) \sqrt{\alpha_t}, \rho_2 g_2(t) \sqrt{\alpha_t}\right),$$

where functions $f_2(t), g_2(t),$ and $h_2(t)$ solve the following ODEs:

$$\frac{dg_2(t)}{dt} = \frac{q + \beta_2}{2} \rho_2^2 g_2(t) + (\kappa + \lambda \rho_1) g_2(t) - \frac{1}{2(q + \beta_1)} \lambda^2, \ g_2(T) = 0,$$

$$\frac{df_2(t)}{dt} = -\kappa g_2(t), \ f_2(T) = 0,$$

and

$$\frac{dh_2(t)}{dt} = -\mu h_2(t), \ h_2(T) = -1.$$

Furthermore, the closed-form expressions for $g_2(t), f_2(t),$ and $h_2(t)$ are given by

$$g_2(t) = \begin{cases}
    -\frac{\lambda^2}{2(q + \beta_1)} (t - T), & \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda \rho_1 = 0; \\
    -\frac{\lambda^2}{2(q + \beta_1)(\kappa + \lambda \rho_1)} \left( e^{(\kappa + \lambda \rho_1)(t - T)} - 1 \right), & \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda \rho_1 \neq 0; \\
    \frac{g_2^+ - g_2^-}{g_2^+ - g_2^-} e^{\lambda \sqrt{\Delta g_2}(T-t)}, & \text{if } \rho_2 \neq 0,
\end{cases}$$

where $g_2^+, g_2^-$ are defined as in (4.12).

Electronic copy available at: https://ssrn.com/abstract=4182575
Theorem 4.5. For any initial data and the following conditions hold

\[ f_2(t) = \begin{cases} \frac{\kappa \theta \lambda^2}{4(q + \beta_1)}(t - T)^2, & \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda \rho_1 = 0; \\ -\frac{\lambda^2 \kappa \theta}{2(q + \beta_1)(\kappa + \lambda \rho_1)} \left(1 - e^{(\kappa + \lambda \rho_1)(t - T)} \right) \left(1 - \frac{t - T}{\kappa + \lambda \rho_1} \right), & \text{if } \rho_2 = 0 \text{ and } \kappa + \lambda \rho_1 \neq 0; \\ \kappa \theta n_g(T - t) + \frac{\kappa \theta (n_g^+ - n_g^-)}{\sqrt{\Delta g_2}} \log \left(\frac{n_g^+ - n_g^-}{n_g^+ - n_g^- e^{\sqrt{\Delta g_2}(T - t)}}\right), & \text{if } \rho_2 \neq 0, \end{cases} \tag{4.13} \]

and

\[ h_2(t) = -e^{\mu(t - T)}, \tag{4.14} \]

where \( \Delta g_2, n_g^+, \) and \( n_g^- \) are given by

\[ \Delta g_2 = (\kappa + \lambda \rho_1)^2 + \frac{q + \beta_2}{q + \beta_1^2} \lambda^2, \quad n_g^+ = \frac{-(\kappa + \lambda \rho_1) + \sqrt{\Delta g_2}}{(q + \beta_2) \rho_2^2}, \quad n_g^- = \frac{-(\kappa + \lambda \rho_1) - \sqrt{\Delta g_2}}{(q + \beta_2) \rho_2^2}. \tag{4.15} \]

Proof. See Appendix A.7.

In the next theorem, we show that the candidate solution \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\) presented in Proposition 4.3 lies in \( L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \otimes L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \otimes L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \) and is the unique solution to quadratic BSDE (4.6).

Theorem 4.4. Suppose that Assumption 4.1 holds true. The solution \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\) given by (4.7) and (4.8) is the unique solution to BSDE (4.6) and belongs to \( L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \otimes L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \otimes L^2_{\mathbb{P},1}(0, T; \mathbb{R}) \).

Proof. See Appendix A.8.

Having derived the closed-form solutions to BSDEs (4.2) and (4.3), we are ready to state our second main result below.

Theorem 4.5. For any initial data \((x_0, \alpha_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+\), suppose that Assumption 4.1 and the following conditions hold

\[ \max \{b_0, b_1, b_2\} \leq \frac{\kappa^2}{\rho_1^2 + \rho_2^2}, \tag{4.16} \]

with \( b_0, b_1, \) and \( b_2 \) given by

\[
\begin{align*}
  b_0 & = \sup_{t \in [0, T]} \left( 120 + 32 \sqrt{14} \right) \frac{\beta_1^2 \lambda^2}{(q + \beta_1)^2} + \frac{\beta_2^2 \rho_2^2 g_2^2(t)}{(q + \beta_2)^2}, \\
  b_1 & = \sup_{t \in [0, T]} \frac{64(2 + \sqrt{2})}{(q + \beta_1)^2} \left( \frac{\lambda^2}{(q + \beta_1)^2} + \rho_2^2 g_2^2(t) \right) q^2, \\
  b_2 & = \sup_{t \in [0, T]} \frac{128q^2 \lambda^2}{(q + \beta_1)^2} - \frac{16q \lambda^2}{q + \beta_1} + (112q^2 - 16q \beta_2) \rho_2^2 g_2^2(t).
\end{align*}
\]
Then, for the following control \((\pi^*, \phi^*)\)

\[
\begin{align*}
\pi_t^* &= \frac{\lambda}{q+\beta_1} \sqrt{\alpha_t} - \Gamma_{1,t}, \\
\phi_{1,t}^* &= -\beta_1 (\sigma_t \pi_{1,t} Y_{2,t} X_t^* + \Gamma_{1,t}) = -\frac{\beta_1}{q+\beta_1} \lambda \sqrt{\alpha_t}, \\
\phi_{2,t}^* &= -\beta_2 \Gamma_{2,t},
\end{align*}
\]

(4.17)

where \(X_t^*\) is the asset process associated with \(\pi_t^*\), and the closed-form expressions for \(Y_{2,t}, \Gamma_{1,t},\) and \(\Gamma_{2,t}\) are given by (4.4) and (4.8), respectively, we have

(i) the distortion process \(\phi^* \in \Phi\), i.e., the Radon-Nikodym derivative process \(\varphi^*_t\) is a uniformly integrable \((\mathbb{F}, \mathbb{P})\)-martingale;

(ii) \(-\varphi_t^* e^{-q(Y_{2,t}+\Gamma_{2,t})} \in \mathcal{S}^{2,2}_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R});

(iii) \(\varphi_t^* \left( \int_0^t (\phi_{1,s}^*)^2 + (\phi_{2,s}^*)^2 \, ds \right) \in \mathcal{S}^{2,2}_{\mathbb{F}, \mathbb{P}}(0, T; \mathbb{R}), \) where \(\eta_{i,t}^* = e^{-q(X_{1,t} Y_{2,t} + \Gamma_{2,t})},\) for \(i = 1, 2\).

In the affirmative, the control \((\pi^*, \phi^*) \in \Pi_t \otimes \Phi\) is the optimal control of the robust ALM problem (2.13), and the optimal value function is given by

\[
J_c(\pi^*, \phi^*) = -\frac{e^{-q(x_0 Y_{2,0} + G_{2,0})}}{q},
\]

(4.18)

where the explicit expression for \(G_{2,t}\) is given by (4.7).

Proof. See Appendix A.9.

Remark 14. To our knowledge, the results shown in Theorem 4.5 are not reported in the existing literature. If we further consider the special case without model ambiguity by setting \(\beta_1 = \beta_2 = 0\), we obtain the explicit expressions for the optimal investment strategy and optimal value function of the ALM problem under exponential utility and square-root factor process. If we plug \(l_0 = \mu_t = \sigma_t = 0\) into Theorem 4.5, the analytical solutions to the robust optimal portfolio selection problems under exponential utility are derived.

In the next three corollaries, we present the explicit expressions for the robust optimal controls and robust optimal value functions under the CEV model, 4/2 model, and path-dependent stochastic volatility model in Example 2.1-2.3, respectively.

Corollary 4.6 (CEV model). If the risky asset price \(S_t\) follows the CEV model (2.4) with any initial data \((x_0, s_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+\), and suppose that the following conditions hold

\[
\max \left\{ \tilde{b}_0, \tilde{b}_1 \right\} \leq \frac{\mu^2}{\sigma^2}
\]

with \(\tilde{b}_0\) and \(\tilde{b}_1\) given by

\[
\begin{align*}
\tilde{b}_0 &= (120 + 32\sqrt{14}) \frac{(\mu - r)^2 \beta_1^2}{\sigma^2 (q + \beta_1)^2}, \\
\tilde{b}_1 &= 64 (2 + \sqrt{2}) \frac{(\mu - r)^2 q^2}{\sigma^2 (q + \beta_1)^2},
\end{align*}
\]

then, the optimal control and optimal value function of the robust ALM problem (2.13) are, respectively, given by

\[
(\pi_{1,t}^*, \phi_{1,t}^*, \phi_{2,t}^*) = \left( \frac{\mu - r}{\sigma (q + \beta_1)} + 2 \beta_1 \tilde{g}_2(t) - \sigma_1 h_2(t) L_t}{X_t Y_{2,t} S_t^{2/3}}, -\frac{\beta_1}{q + \beta_1} \frac{\mu - r}{\sigma S_t^\beta}, 0 \right),
\]

(4.19)
and

\[ J_e(\pi^*, \phi^*) = -\frac{e^{-q(x_0Y_{2,0} + f_2(0) + \tilde{g}_2(0)S_0^\beta + h_2(0)t_0)}}{q}, \]

where \( Y_{2,t} \) and \( h_2(t) \) are given by (4.4) and (4.14), respectively, and functions \( \tilde{f}_2(t) \) and \( \tilde{g}_2(t) \) are as follows:

\[ \tilde{f}_2(t) = -\frac{(\mu - r)^2 (\beta + \frac{1}{2})}{2(q + \beta_1)r} \left( 1 - \frac{e^{2\beta r(t-T)}}{2\beta r} + t - T \right) \]

and

\[ \tilde{g}_2(t) = \frac{(\mu - r)^2}{4\beta r(q + \beta_1)^2} \left( 1 - e^{2\beta r(t-T)} \right). \]

**Proof.** Substituting the specified parameters of the CEV model (2.4) in Example 2.1 yields the above results. In addition, it is easy to see that Assumption 4.1 always holds for the CEV model since \( 2\beta r \neq 0 \).

\[ \square \]

**Remark 15.** If we ignore random liabilities by imposing \( l_0 = \mu_t = \sigma_t = 0 \), then the optimal value function and optimal control for the robust portfolio selection problem under the CEV model and exponential utility are provided, which are the identical to that presented in Theorem 3.4 in Zheng et al. [65] when no reinsurance is involved. If we further ignore model ambiguity by setting \( \beta_1 = 0 \), the optimal investment strategy recovers the results provided in Sun et al. [50] and Proposition 2 in Gao [23].

**Corollary 4.7** (The family of 4/2 model). If the risky asset price process \( S_t \) and the variance driver process \( V_t \) follow the family of 4/2 models (2.5) with any initial data \( (x_0, v_0, l_0) \in \mathbb{R}^+ \otimes \mathbb{R}^+ \otimes \mathbb{R}^+ \), and suppose that \( \kappa + \sigma_v \rho \lambda \neq 0 \) and the following conditions hold

\[ \max \{ b_0, b_1, b_2 \} \leq \frac{\kappa^2}{\sigma_v^2} \]

with \( b_0, b_1, b_2 \) given by

\[
\begin{align*}
\hat{b}_0 &= \sup_{t \in [0,T]} (120 + 32\sqrt{14}) \left( \frac{\beta_1^2 \lambda^2}{(q + \beta_1)^2} + \beta_2^2 \sigma_v^2 (1 - \rho^2) \tilde{g}_2^2(t) \right), \\
\hat{b}_1 &= \sup_{t \in [0,T]} 64(2 + \sqrt{2}) \left( \frac{\lambda^2}{(q + \beta_1)^2} + \sigma_v^2 (1 - \rho^2) \tilde{g}_2^2(t) \right) q^2, \\
\hat{b}_2 &= \sup_{t \in [0,T]} \frac{128q^2 \lambda^2}{(q + \beta_1)^2} - \frac{16q^2}{q + \beta_1} + (112q^2 - 16q \beta_2) \sigma_v^2 (1 - \rho^2) \tilde{g}_2^2(t),
\end{align*}
\]

then, the optimal control and optimal value function of the robust ALM problem (2.13) are, respectively, given by

\[ (\pi^*_t, \phi^*_1, t, \phi^*_2, t) = \left( \frac{V_t \left( \frac{\lambda}{q + \beta_1} - \sigma_v \rho \tilde{g}_2(t) - \sigma_1 h_2(t)L_t \right)}{(c_1 V_t + c_2) X_t Y_{2, t}}, -\frac{\beta_1 \lambda}{q + \beta_1} \sqrt{V_t}, -\beta_2 \sigma_v \sqrt{1 - \rho^2} \tilde{g}_2(t) \sqrt{V_t} \right), \]

and

\[ J_e(\pi^*, \phi^*) = -\frac{e^{-q(x_0Y_{2,0} + f_2(0) + \tilde{g}_2(0)v_0 + h_2(0)l_0)}}{q}. \]
where $Y_{2,t}$ and $h_{2}(t)$ are given by (4.4) and (4.14), respectively, and functions $\bar{f}_{2}(t)$ and $\bar{g}_{2}(t)$ are given as follows:

$$\bar{f}_{2}(t) = \begin{cases} -\frac{\lambda^{2}\kappa\theta}{2(q+\beta_{1})(\kappa + \lambda\sigma_{t}\rho)} \left( 1 - e^{(\kappa + \lambda\sigma_{t}\rho)(t-T)} \right) \left( \kappa + \lambda\sigma_{t}\rho \right) + t - T \right), & \text{if } \rho = \pm 1; \\
\kappa\theta n_{2}^{-} (T-t) + \frac{\kappa(\bar{n}_{2}^{-} - n_{2}^{+})}{\sqrt{\Delta_{g_{2}}}} \log \left( \frac{n_{2}^{+} - \bar{n}_{2}^{-}}{n_{2}^{+} - n_{2}^{-} e^{\sqrt{\Delta_{g_{2}}}(T-t)}} \right), & \text{if } \rho \neq \pm 1, \end{cases}$$

and

$$\bar{g}_{2}(t) = \begin{cases} -\frac{\lambda^{2}}{2(q+\beta_{1})(\kappa + \lambda\sigma_{t}\rho)} \left( e^{(\kappa + \lambda\sigma_{t}\rho)(t-T)} - 1 \right), & \text{if } \rho = \pm 1; \\
n_{2}^{+} n_{2}^{-} \left( 1 - e^{\sqrt{\Delta_{g_{2}}}(T-t)} \right), & \text{if } \rho \neq \pm 1, \end{cases}$$

with $\Delta_{g_{2}}, n_{2}^{+},$ and $n_{2}^{-}$ given by

$$\Delta_{g_{2}} = (\kappa + \lambda\sigma_{t}\rho)^{2} + \frac{q + \beta_{2}}{q + \beta_{1}} \sigma_{t}^{2}(1-\rho^{2}) \lambda^{2}, \quad n_{2}^{+} = \frac{-(\kappa + \lambda\sigma_{t}\rho) + \sqrt{\Delta_{g_{2}}}}{(q + \beta_{2})\sigma_{t}^{2}(1-\rho^{2})}, \quad n_{2}^{-} = \frac{-(\kappa + \lambda\sigma_{t}\rho) - \sqrt{\Delta_{g_{2}}}}{(q + \beta_{2})\sigma_{t}^{2}(1-\rho^{2})}.$$

Proof. Substituting the specified parameters of the 4/2 model (2.5) in Example 2.2 into Theorem 4.5 by the specification $\hat{\pi}_{t}$ of the path-dependent stochastic volatility model (2.6) with any initial data $(x_{0},\alpha_{0},l_{0}) \in \mathbb{R}^{+} \otimes \mathbb{R}^{+} \otimes \mathbb{R}^{+}$, and suppose that condition (4.16) holds true, then optimal control and optimal value function of the robust ALM problem 2.13 are, respectively, given by

$$\left( \pi_{t}^{*}, \phi_{1,t}^{*}, \phi_{2,t}^{*} \right) = \left( \frac{\lambda}{q+\beta_{1}} \sqrt{\alpha_{t}} \Gamma_{1,t} - \frac{\beta_{1}}{q+\beta_{1}} \lambda \sqrt{\alpha_{t}} - \beta_{2}\Gamma_{2,t} \right),$$

and

$$J_{e}(\pi^{*},\phi^{*}) = \frac{e^{-q(x_{0}Y_{2,0}+G_{2,0})}}{q},$$

where the closed-form expressions for $Y_{2,t}, G_{2,t}, \Gamma_{1,t},$ and $\Gamma_{2,t}$ are given by (4.4), (4.7), and (4.8), respectively.

Proof. Replacing $\sigma_{t}$ in Theorem 4.5 by the specification $\hat{\sigma}(\alpha_{[0,t]})$ of the path-dependent model (2.6) in Example 2.3 yields the above results immediately. \hfill $\square$
5 Numerical Analysis

In this section, we devote ourselves to showing the effects of model parameters on the behavior of the robust optimal investment strategy by giving numerical examples. In the following numerical illustrations, we are mainly concerned about the exponential utility case under the CEV model (2.4) and 4/2 stochastic volatility model (2.5) since these two models are extensively studied in the literature in recent years and the power utility case can be conducted in a similar manner. Throughout this section, unless otherwise specified, the fundamental values of the model parameters are given as follows: \( r = 0.02, \mu_l = 0.01, \sigma_l = 0.2, x_0 = 1, l_0 = 0.5, T = 0.1, \beta_1 = 1.5, \beta_2 = 1, q = 2; \) in the 4/2 model, \( \kappa = 7.3479, \theta = 0.0328, \sigma_v = 0.6612, \rho = -0.7689, \lambda = 2.9428, c_1 = 0.9051, c_2 = 0.023, v_0 = 0.04, \) mainly referred to Cheng and Escobar [12]; in the CEV model, \( \mu = 0.05, \sigma = 0.25, \beta = -0.7, s_0 = 0.5. \) For simplicity but without loss of generality, we focus on the analysis at time \( t = 0 \) and vary the value of one parameter with others fixed at each time. The range allowed for the parameters is the possibility that the conditions in Corollary 4.6-4.7 are respectively met.

5.1 Effects of parameters in the 4/2 model on the robust investment strategy

In this subsection, we are interested in the effects of some model parameters in the 4/2 stochastic volatility model (2.5) on the robust optimal investment strategy \( \pi^* \) given in Corollary 4.7.

Figure 1 displays the effects of the ambiguity aversion parameters \( \beta_1 \) and \( \beta_2 \) and the risk aversion coefficient \( q \) on the robust optimal investment strategy \( \pi^* \). From Figure 1(a), we find that \( \pi^* \) decreases with respect to \( \beta_1 \) and \( q \). Along with the growth of \( \beta_1 \), the asset-liability manager is more ambiguity-averse about the risky asset dynamics. Hence, the manager is willing to reduce the investment proportion in the risky asset. With the increase of \( q \), the manager becomes more risk-averse and tends to accept a lower risk for the investment. So, less wealth will be invested in the risky asset. For a similar reason, as the ambiguity aversion parameter \( \beta_2 \) becomes larger, the manager is more ambiguity-averse about the risky asset variance driver process. Therefore, the investment proportion in the risky asset is reduced, which is consistent with the results shown in Figure 1(b).

Figure 1: Effects of the ambiguity parameters \( \beta_1 \) and \( \beta_2 \) and the risk aversion coefficient \( q \) on the robust optimal investment strategy \( \pi^* \) under the 4/2 model (2.5)

Figure 2 contributes to the evolution of the robust optimal investment strategy \( \pi^* \) with respect to the random liability parameters \( \mu_l, \sigma_l \), the slope of the market price of volatility risk \( \lambda \), the cor-
relation between the risky asset price and instantaneous variance driver $\rho$, and the mean-reversion rate and volatility of the variance driver process $\kappa$ and $\sigma_v$. From Figure 2(a), we observe that $\pi^*$ increases slightly with respect to $\mu_l$. When $\mu_l$ is growing, the appreciation rate of the random liability becomes larger. In this case, the manager is willing to increase the investment proportion in the risky asset to obtain a higher terminal surplus. Figure 2(a) also indicates that the investment proportion in the risky asset increases along with the volatility scale factor $\sigma_l$. In fact, as $\sigma_l$ increases, the volatility of the random liability caused by the risky asset becomes larger. As a result, the manager tends to invest more wealth into the risky asset as a hedging instrument to reduce the volatility risk of the random liability to an acceptable level. In addition, when $\rho$ decreases from 0.9 to −0.9, the investment proportion in the risky asset increases as revealed by Figure 2(b). This is due to the fact that decreasing $\rho$ leads the hedge demand $-\sigma_v \rho g_2(t)$ $(g_2(t) > 0)$ to increase. Figure 2(b) also shows that the investment proportion in the risky asset increases with respect to $\lambda$. Since $\lambda$ depicts the slope of the market price of volatility risk, a larger value of $\lambda$ implies that the manager could obtain higher returns by investing in the risky asset. Finally, Figure 2(c) illustrates that $\pi^*$ decreases along with $\kappa$ but increases along with $\sigma_v$. Indeed, since $\kappa$ stands for the mean-reversion rate of the variance driver process, the variance driver process moves faster towards the constant long-run level $\theta$ as $\kappa$ increases. Along with the growth of $\kappa$, the volatility risk of the risky asset becomes smaller, and hence the investment proportion in the risky asset is reduced. Conversely, as the volatility coefficient $\sigma_v$ increases, the instantaneous variance of the risky asset price fluctuates more dramatically and the manager faces a higher volatility risk. As a result, a larger investment proportion in the risky asset is necessary to hedge against the volatility risk.

![Figure 2](https://ssrn.com/abstract=4182575)

**5.2 Effects of parameters in the CEV model on the robust investment strategy**

This subsection focuses on the effects of some model parameters in the CEV model (2.4) on the robust optimal investment strategy $\pi^*$ given in Corollary 4.6. More specifically, Figure 3 describes how the robust optimal investment strategy $\pi^*$ evolves with respect to $\beta_1$, $q$, $\mu$, $r$, and $\beta$.

Similar to the results shown in Figure 1(a) under the 4/2 model, we can observe from Figure 3(a) that under the CEV model, the robust optimal investment strategy $\pi^*$ has negative relationships with both the ambiguity aversion parameter $\beta_1$ and risk aversion coefficient $q$. In other words, the manager tends to put less wealth into the risky asset when he/she becomes either more ambiguity-averse or more risk-averse. Figure 3(b) shows that the robust optimal investment strategy $\pi^*$ is positively correlated with the parameter $\mu$ under the CEV model. Along with the growth of $\mu$, the manager can earn a higher risk premium from the risky asset as $\mu$ stands for the expected
return rate of the risky asset. In this case, the manager is willing to increase the proportion of wealth invested in the risky asset to derive a higher terminal surplus. From Figure 3(b), we also find that $\pi^*$ decreases with respect to $\sigma$, which can be interpreted by the fact that $\sigma$ characterizes the risky asset’s local volatility, and when $\sigma$ becomes larger, the risky asset displays greater local volatility. Therefore, the manager has the motivation to decrease the amount of wealth invested in the risky asset to avoid amplified volatility risk. Finally, from Figure 3(c), we observe that the robust optimal investment strategy $\pi^*$ increases when the elasticity parameter $\beta$ increases from $-1$ to $-0.7$. This can be explained by the economic implication of $\beta$; the negativeness of $\beta$ indicates the existence of the leverage effect, and when $\beta$ becomes less negative, the volatility risk turns out to be less significant, and thus, the manager would increase the investment in the risky asset. Figure 3(c) also reveals that the optimal proportion of wealth invested into the risky asset $\pi^*$ has a negative relationship with the risk-free interest rate $r$. Varying $r$ from 0.01 to 0.05, the expected rate return of the risk-free asset becomes higher. Hence, the manager would invest more in the risk-free asset and less in the risky asset to reduce the overall risk.

Figure 3: Effects of parameters $\beta_1, q, \sigma, \mu, r$ and $\beta$ on the robust optimal investment strategy $\pi^*$ under the CEV model (2.4)

6 Conclusion

In this paper, we investigate robust ALM problems for a manager with both risk and ambiguity aversion in the presence of stochastic volatility. The manager is subject to random liabilities and has access to a financial market consisting of one risk-free asset and one risky asset, where the market price of risk follows an affine-form, square-root, Markovian model, while the return rate and volatility are possibly non-Markovian, unbounded stochastic processes. The modeling framework embraces the CEV model, the family of the state-of-the-art 4/2 models, and some non-Markovian models, as particular cases. The manager is allowed to have different levels of ambiguity about the risky asset price and volatility and aims to seek a robust optimal investment strategy against the worst-case measure among the class of alternative measures equivalent to the reference measure. In the non-Markovian case, the dynamic programming principle along with the HJBI equation approach no longer works, and thus, a novel BSDE approach is proposed. To find the associated BSDEs, we propose to construct a stochastic process depending on any admissible control, and such that its value at time zero does not rely on any admissible control and its terminal value equals the utility of the terminal surplus penalized by model ambiguity. By solving the BSDEs explicitly, we derive, in closed form, the robust optimal controls and robust optimal value functions for the power and exponential utility functions, respectively. Moreover, analytical solutions to some special cases of our model are obtained. Finally, the economic impacts of model ambiguity and model parameters on the robust optimal investment strategy are analyzed with numerical
examples, from which we find that (1) the levels of ambiguity aversion about the risky asset’s price and volatility both reduce the robust optimal investment proportion in the risky asset; (2) the robust optimal investment strategy is more sensitive to the level of ambiguity about the risky asset dynamics. As far as we know, this paper is the first to address the ALM problems in the presence of model ambiguity as well as stochastic volatility, and more importantly, there is no existing literature using the above BSDE approach to study robust decision problems in the non-Markovian setting. So, this study is meaningful from both theoretical and practical perspectives.

Built on the current study, some promising directions for future research might be followed. For instance, (1) this paper investigates the robust ALM problems within the expected utility maximization framework. One may consider other non-utility criteria, such as the mean-variance criterion. (2) In addition to model ambiguity, the manager may also face partial information. (3) It may also be of interest to apply the proposed BSDE approach to address robust pension investment or investment-consumption problems in non-Markovian cases.

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A.1 Proof of Proposition 3.2

Proof. Conjecture that the first component $Y_{1,t}$ of the solution to BSRE (3.2) has the following exponential-affine form:

$$Y_{1,t} = \exp \left\{ f_1(t) + g_1(t)\alpha_t \right\},$$

where $f_1(t)$ and $g_1(t)$ are two differentiable functions which shall be determined later with terminal conditions $f_1(T) = g_1(T) = 0$. An application of Itô’s formula to $Y_{1,t}$ then leads to

$$dY_{1,t} = Y_{1,t} \left[ \frac{df_1(t)}{dt} + \frac{dg_1(t)}{dt}\alpha_t + \kappa(\theta - \alpha_t)g_1(t) + \frac{1}{2} \left( \rho_1^2 + \rho_2^2 \right) g_1^2(t)\alpha_t \right] dt$$

$$+ \rho_1 g_1(t)\sqrt{\alpha_t} Y_{1,t} dW_{1,t} + \rho_2 g_1(t)\sqrt{\alpha_t} Y_{1,t} dW_{2,t}.$$  \hspace{1cm} (A.1)

Match the diffusion coefficients in (A.1) with BSRE (3.2) by letting $Z_{1,t} = \rho_1 g_1(t)\sqrt{\alpha_t} Y_{1,t}$ and $Z_{2,t} = \rho_2 g_1(t)\sqrt{\alpha_t} Y_{1,t}$. We can rewrite the generator of BSRE (3.2) as follows:

$$Y_{1,t} \left[ \frac{\gamma \lambda^2}{2(\gamma - 1 - \beta)} + \frac{(\gamma - \beta)\lambda \rho_1 g_1(t)}{\gamma - 1 - \beta} + \frac{\rho_2^2 g_1^2(t)}{2\gamma} \left( \beta_1 + \frac{(\gamma - \beta)^2}{\gamma - 1 - \beta} \right) + \frac{\beta_2 \rho_2^2 g_1^2(t)}{2\gamma} \right] \alpha_t - r\gamma. \hspace{1cm} (A.2)$$

Comparing (A.2) with the drift coefficient of (A.1) leads to the ODEs (3.8) and (3.9) governing $g_1(t)$ and $f_1(t)$, respectively.

Furthermore, denote by $\Delta_{g_1} = \left( \kappa + \frac{(\gamma - \beta_1)\lambda \rho_1}{\gamma - 1 - \beta} \right)^2 - \frac{\lambda^2}{\gamma - 1 - \beta} \left( \frac{\gamma - \beta_1}{\gamma - 1 - \beta} \rho_1^2 - (\gamma - \beta_2)\rho_2^2 \right) > 0$. We can reformulate the Riccati ODE (3.8) as follows:

$$\frac{dg_1(t)}{dt} = \left( \frac{\gamma - \beta_1}{2\gamma(\gamma - 1 - \beta)} \rho_1^2 - \frac{\gamma - \beta_2}{2\gamma} \rho_2^2 \right) \left( g_1(t) - n_{g_1} \right) \left( g_1(t) - n_{\bar{g}_1} \right),$$

where $n_{g_1}$ and $n_{\bar{g}_1}$ are given in (3.12). By using the separation method and some simple calculations, the closed-form expression of $g_1(t)$ is given in (3.10). Finally, noticing the boundary condition that $f_1(T) = 0$ and substituting $g_1(t)$ back into ODE (3.9), we have the closed-form expression of $f_1(t)$ given in (3.11). \hfill \qed
A.2 Proof of Proposition 3.3

Proof. The uniformly boundedness of the process \( Y_{1,t} \) follows immediately from the negativeness of function \( g_1(t) \) and the positiveness of the square-root factor process \( \alpha_t \), for \( t \in [0,T] \). More precisely, we have \( \mathbb{P} \) almost surely,

\[
Y_{1,t} = \exp \left\{ f_1(t) + g_1(t)\alpha_t \right\} \leq \exp \left\{ \int_t^T \kappa \theta g_1(s) \, ds + r\gamma(T - t) \right\} \leq \exp \{ r\gamma(T - t) \} < +\infty.
\]

As a result, we find from (3.7) that

\[
\mathbb{E}^\mathbb{P} \left[ \int_0^T Z_{1,t}^2 \, dt \right] \leq C_i \int_0^T \mathbb{E}^\mathbb{P} [\alpha_t] \, dt = C_i \int_0^T \left( \alpha_0 e^{-\kappa t} + \kappa \theta \int_0^t e^{-\kappa(t-s)} \, ds \right) \, dt < +\infty,
\]

where \( C_i = \rho_i^2 g_1^2(0) e^{2|\gamma|T} \), for \( i = 1, 2 \). This completes the proof. \( \square \)

A.3 Proof of Theorem 3.5

Proof. First of all, it follows from Lemma 3.4 that the probability measure \( \tilde{\mathbb{P}} \) defined by

\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \exp \left\{ -\int_0^T \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \lambda \sqrt{\alpha_t} \, dW_{1,t} - \frac{1}{2} \int_0^t \left( \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \right)^2 \lambda^2 \alpha_t \, dt \right\}
\]

is equivalent to the reference measure \( \mathbb{P} \). By Girsanov’s theorem, the following processes \( \tilde{W}_{1,t} \) and \( \tilde{W}_{2,t} \)

\[
\tilde{W}_{1,t} = \int_0^t \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \lambda \sqrt{\alpha_s} \, ds + W_{1,t} \quad \text{and} \quad \tilde{W}_{2,t} = W_{2,t}
\]

are two standard Brownian motions under \( \tilde{\mathbb{P}} \). In addition, the \( \tilde{\mathbb{P}} \)-dynamics of the factor process \( \alpha_t \) is given by

\[
d\alpha_t = \left( \kappa + \rho_1 \lambda \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1} \right) \left( \frac{\kappa \theta}{\kappa + \rho_1 \lambda \frac{\gamma - \beta_1}{\gamma - 1 - \beta_1}} - \alpha_t \right) \, dt + \sqrt{\alpha_t} \left( \rho_1 \, d\tilde{W}_{1,t} + \rho_2 \, d\tilde{W}_{2,t} \right),
\]

which preserves the affine-form, square-root structure under Assumption 3.1. Moreover, the BSRE (3.2) can be rewritten under \( \tilde{\mathbb{P}} \) measure as follows:

\[
\begin{align*}
\begin{cases}
 dY_{1,t} = & \left[ \left( -r\gamma + \frac{\gamma}{2(\gamma - 1 - \beta_1)} \lambda^2 \alpha_t \right) Y_{1,t} + \frac{1}{2\gamma} \left( \beta_1 + \frac{(\gamma - \beta_1)^2}{\gamma - 1 - \beta_1} \right) Z_{1,t}^2 Y_{1,t} + \frac{\beta_2}{2\gamma} Z_{2,t}^2 Y_{1,t} \right] \, dt \\
 & + Z_{1,t} \, d\tilde{W}_{1,t} + Z_{2,t} \, d\tilde{W}_{2,t}, \\
 Y_{1,0} = & 1, \\
 Y_{1,t} > & 0, \text{ for all } t \in [0,T). 
\end{cases}
\end{align*}
\]

Denote by \( (\tilde{Y}_{1,t}, \tilde{Z}_{1,t}, \tilde{Z}_{2,t}) \) another solution to BSRE (3.2), which is different from the proposed one given in Proposition 3.2. Define the following difference process:

\[
(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t}) = \left( \log(Y_{1,t}) - \log(\tilde{Y}_{1,t}), \frac{Z_{1,t}}{Y_{1,t}} - \frac{\tilde{Z}_{1,t}}{\tilde{Y}_{1,t}}, \frac{Z_{2,t}}{Y_{1,t}} - \frac{\tilde{Z}_{2,t}}{\tilde{Y}_{1,t}} \right).
\]
Apply Itô’s formula to $\Delta \log(Y_{1,t})$ yields the following BSDE of $(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t})$:

$$
\begin{cases}
    d\Delta \log(Y_{1,t}) = \frac{1}{2} \left[ \left( \beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma} - 1 \right) \left( \frac{Z_{1,t}^2}{Y_{1,t}^2} - \frac{Z_{1,t}^2}{Y_{1,t}^2} \right) + \left( \frac{\beta_2}{\gamma} - 1 \right) \left( \frac{Z_{2,t}^2}{Y_{1,t}^2} - \frac{Z_{2,t}^2}{Y_{1,t}^2} \right) \right] dt \\
    + \Delta Z_{1,t} d\tilde{W}_{1,t} + \Delta Z_{2,t} d\tilde{W}_{2,t}, \\
    \Delta \log(Y_{1,T}) = 0.
\end{cases}
$$

(A.4)

Furthermore, since $\alpha_t$ is still an affine-form, square-root factor process under $\tilde{\mathbb{P}}$ measure, we can define the following equivalent probability measure $\tilde{\mathbb{P}}$ on $\mathcal{F}_T$ as a result of Lemma 3.4 and the explicit expression of the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in Proposition 3.2:

$$
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \big|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \left( \beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma} - 1 \right) \rho_1 g_1(t) \sqrt{\alpha_t} d\tilde{W}_{1,t} - \int_0^T \left( \frac{\beta_2}{\gamma} - 1 \right) \rho_2 g_1(t) \sqrt{\alpha_t} d\tilde{W}_{2,t} \right. \\
- \frac{1}{2} \left[ \left( \beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma} - 1 \right) \rho_1^2 + \left( \frac{\beta_2}{\gamma} - 1 \right)^2 \rho_2^2 \right] g_1^2(t) \alpha_t dt \left. \right\}.
$$

So, by Girsanov’s theorem, the following processes $\tilde{W}_{1,t}$ and $\tilde{W}_{2,t}$:

$$
\tilde{W}_{1,t} = \int_0^t \left( \beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma} - 1 \right) \frac{Z_{1,s}}{Y_{1,s}} ds + \tilde{W}_{1,t} \quad \text{and} \quad \tilde{W}_{2,t} = \int_0^t \left( \frac{\beta_2}{\gamma} - 1 \right) \frac{Z_{2,s}}{Y_{1,s}} ds + \tilde{W}_{2,t}
$$

(A.5)

are two standard Brownian motions under $\tilde{\mathbb{P}}$ measure. Therefore, it follows from (A.4) and (A.5) that $(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t})$ solves the following quadratic BSDE under $\tilde{\mathbb{P}}$ measure:

$$
\begin{cases}
    d\Delta \log(Y_{1,t}) = - \frac{1}{2} \left[ \left( \beta_1 + \frac{(\gamma-\beta_1)^2}{\gamma} - 1 \right) \Delta Z_{1,t}^2 + \left( \frac{\beta_2}{\gamma} - 1 \right) \Delta Z_{2,t}^2 \right] dt \\
    + \Delta Z_{1,t} d\tilde{W}_{1,t} + \Delta Z_{2,t} d\tilde{W}_{2,t}, \\
    \Delta \log(Y_{1,T}) = 0.
\end{cases}
$$

This quadratic BSDE clearly satisfies all the regularity conditions in Kobylanski [27]. Thus, by Theorem 2.3 and 2.6 in Kobylanski [27], we can conclude that the above quadratic BSDE admits a unique solution which is $(\Delta \log(Y_{1,t}), \Delta Z_{1,t}, \Delta Z_{2,t}) = (0, 0, 0)$. In other words, the proposed solution $(Y_{1,t}, Z_{1,t}, Z_{2,t})$ given in (3.6) and (3.7) must be the unique solution to BSRE (3.2) in the space $\mathcal{S}_{F,T}^\infty(0, T; \mathbb{R}^+) \times \mathcal{L}_F^2(0, T; \mathbb{R}) \times \mathcal{L}_F^2(0, T; \mathbb{R})$. This completes the proof. \hfill \Box
A.4 Proof of Proposition 3.6

Proof. By Lemma 3.4, the following Radon-Nikodym derivative
\[
\frac{d\tilde{P}}{dP}\bigg|_{\mathcal{F}_T} = \exp\left\{ -\int_0^T (\lambda - \sigma_t)\sqrt{\alpha_t} \, dW_{1,t} + \int_0^T \rho_2 g_1(t)\sqrt{\alpha_t} \, dW_{2,t} - \frac{1}{2} \int_0^T ((\lambda - \sigma_t)^2 + \rho_2^2 g_1^2(t)) \alpha_t \, dt \right\}
\]
is well-defined, and thus, the probability measure \(\tilde{P}\) is also well-defined and equivalent to \(P\). By Girsanov’s theorem, the following two processes
\[
\hat{W}_{1,t} = \int_0^t (\lambda - \sigma_t)\sqrt{\alpha_s} \, ds + W_{1,t} \quad \text{and} \quad \hat{W}_{2,t} = -\int_0^t \rho_2 g_1(s)\sqrt{\alpha_s} \, ds + W_{2,t}
\]
are two standard Brownian motions under \(\tilde{P}\). So, linear BSDE (3.13) can be reformulated under \(\tilde{P}\) measure as follows:
\[
\begin{aligned}
\{d\hat{G}_{1,t} = (r - \mu_t)\hat{G}_{1,t} \, dt + \hat{\Lambda}_{1,t} \, d\hat{W}_{1,t} + \hat{\Lambda}_{2,t} \, d\hat{W}_{2,t}, \\
\hat{G}_{1,T} = -1, \quad \text{(A.6)}
\end{aligned}
\]
which is linear BSDE with standard data (refer to El Karoui et al. [18]) and has deterministic coefficients in the generator. Then by Theorem 2.1 and Proposition 2.2 in El Karoui et al. [18], we notice that (3.14) and (3.15) form the unique solution to (A.6) and to (3.13) as well. This completes the proof. \(\square\)

A.5 Proof of Theorem 3.7

Proof. In the first place, we show that \(\phi^* = \left\{ \{\phi_{1,t}^*\}_{t \in [0,T]}, \{\phi_{2,t}^*\}_{t \in [0,T]} \right\}\) given in (3.17) lies in \(\Phi\), i.e.,
\[
\mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} \int_0^T ((\phi_{1,t}^*)^2 + (\phi_{2,t}^*)^2) \, dt \right\} \right] < +\infty.
\]
Indeed, recalling from Remark 6 that \(g_1(t)\) is bounded by \([g_1(0), 0]\) and using the Laplace transform of an integrated square-root diffusion process (see, for example, Theorem 5.1 in Zeng and Taksar [58]) and condition (3.16), we have
\[
\mathbb{E}^P \left[ \exp \left\{ \frac{1}{2} \int_0^T ((\phi_{1,t}^*)^2 + (\phi_{2,t}^*)^2) \, dt \right\} \right] \leq \mathbb{E}^P \left[ \exp \left\{ \frac{k_0}{2} \int_0^T \alpha_t \, dt \right\} \right] < +\infty,
\]
where the constant \(k_0\) is given in (3.16). Substituting \(\pi_t^*\) into \(X_t^*\) and applying Itô’ formula to \(X_t^* + G_{1,t} L_t\) under the reference measure \(P\), we find that
\[
\frac{d(X_t^* + G_{1,t} L_t)}{X_t^* + G_{1,t} L_t} = \left[ r + \frac{\gamma - \beta_1}{\beta_1 + 1 - \gamma} \lambda \rho_1 g_1(t) + \frac{\lambda^2}{\beta_1 + 1 - \gamma} \alpha_t \right] \, dt + \frac{\gamma - \beta_1}{\beta_1 + 1 - \gamma} \lambda \rho_1 g_1(t) + \frac{\lambda^2}{2(\beta_1 + 1 - \gamma)^2} \alpha_t \sqrt{\alpha_t} \, dW_{1,t}. \quad \text{(A.7)}
\]
Solving the linear SDE (A.7) explicitly, we have
\[
X_t^* + G_{1,t} L_t = (x_0 + G_{1,0} l_0) \exp \left\{ \int_0^t \left[ r + \left( \frac{\gamma - \beta_1}{\beta_1 + 1 - \gamma} \lambda \rho_1 g_1(s) + \frac{\lambda^2}{2(\beta_1 + 1 - \gamma)^2} \alpha_s \right) \alpha_s \right] ds \right. \\
+ \int_0^t \frac{\gamma - \beta_1}{\beta_1 + 1 - \gamma} \lambda \rho_1 g_1(s) + \frac{\lambda}{2(\beta_1 + 1 - \gamma)^2} \alpha_s dW_{1,s} \left. \right\} > 0,
\]
\(\text{(A.8)}\)
for the initial data \((x_0, \alpha_0, l_0)\) such that \(x_0 + \hat{G}_{1,0}l_0 \in \mathbb{R}^+\). Then, it follows from (A.8) and Proposition 3.3 that

\[
\left| Y_{1,t} \left( \frac{X_t^* + \hat{G}_{1,t}L_t}{\gamma} \right)^8 \right| \\
\leq c \exp \left\{ 8 \gamma \int_0^t \frac{\gamma - \beta_1}{\beta_1 + 1 - \gamma} \rho_1 g_1(s) \sqrt{\alpha_s} dW_{1,s} - 32 \gamma^2 \int_0^t \frac{\gamma - \beta_1}{(\beta_1 + 1 - \gamma)^2} \alpha_s ds \right\} \\
\times \exp \left\{ \int_0^t \frac{(8 - \beta_1) \alpha_1 g_1(s) + 8 \lambda^2 \gamma}{\beta_1 + 1 - \gamma} + (32 \gamma^2 - 4 \gamma) \frac{(\gamma - \beta_1) \rho_1 g_1(s) + \lambda^2}{\beta_1 + 1 - \gamma} \alpha_s ds \right\},
\]

(A.9)

where \(c\) is a positive constant. We observe that \(K_{1,t}\) is the stochastic exponential process of continuous \((\mathcal{F}, \mathbb{P})\)-local martingale \(\int_0^t \frac{(8 - \beta_1) \alpha_1 g_1(s) + 8 \lambda^2 \gamma}{\beta_1 + 1 - \gamma} \sqrt{\alpha_s} dW_{1,s}\), and thus, it follows from Theorem 15.4.6 in Cohen and Elliott [16], Theorem 5.1 in Zeng and Taksar [58] and condition (3.16) that

\[
\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} K_{1,t}^2 \right] \leq 2 \left\{ \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{2 + \sqrt{2}}{2} \int_0^T (8 - \beta_1) \alpha_1 g_1(t) + 8 \lambda^2 \gamma}{\beta_1 + 1 - \gamma} \alpha_t dt \right\} \right\}^{\frac{1}{2} \frac{\sqrt{2 + \sqrt{2}} - 1}{2 + \sqrt{2}}}
\]

\[
\leq 2 \left\{ \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{k_1}{2} \int_0^T \alpha_t dt \right\} \right]\right\}^{\frac{1}{2} \frac{\sqrt{2 + \sqrt{2}} - 1}{2 + \sqrt{2}}} < +\infty,
\]

(A.10)

where the constant \(k_1\) is given in (3.16). Similarly, by using Theorem 5.1 in Zeng and Taksar [58] and condition 3.16, we have

\[
\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} \left| Y_{1,t} \left( \frac{X_t^* + \hat{G}_{1,t}L_t}{\gamma} \right)^8 \right| \right] < +\infty.
\]

(A.11)

where the constant \(k_2\) is given by (3.16). Then, it follows from Hölder’s inequality and (A.9)-(A.11) that

\[
\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} Y_{1,t} \left( \frac{X_t^* + \hat{G}_{1,t}L_t}{\gamma} \right)^8 \right] < +\infty.
\]

(A.12)

Hence, as a result of Theorem 15.4.6 in Cohen and Elliott [16], Theorem 5.1 in Zeng and Taksar [58], condition (3.16), the explicit expressions for \(\phi_{1,t}^*\) and \(\phi_{2,t}^*\) given in (3.17) and Hölder’s inequality, we obtain

\[
\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} \left| \phi_{1,t}^* Y_{1,t} \frac{(X_t^* + \hat{G}_{1,t}L_t)^\gamma}{\gamma} \right|^4 \right] \\
\leq \left\{ \mathbb{E}^\mathbb{P} \left[ \left| \phi_{1,t}^* \right|^8 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} \left| Y_{1,t} \frac{(X_t^* + \hat{G}_{1,t}L_t)^\gamma}{\gamma} \right|^8 \right] \right\}^{\frac{1}{2}} \\
\leq \left\{ \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{k_0}{2} \int_0^T \alpha_t dt \right\} \right] \right\}^{\frac{1}{2} \frac{\sqrt{120 + 32 \sqrt{8} \gamma^2 - 1}}{240 + 64 \sqrt{8} \gamma^2}} < +\infty,
\]

(A.13)
More importantly, we find that
\[
\mathbb{E}^\pi \left[ \sup_{t \in [0,T]} |\varphi_t^{\sigma^*} \left( \int_0^t \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} ds \right) |^2 \right] \\
\leq c \left\{ \mathbb{E}^\pi \left[ \sup_{t \in [0,T]} |\varphi_t^{\sigma^*} |^4 \right] \right\}^{\frac{3}{4}} \left\{ \mathbb{E}^\pi \left[ \left( \int_0^T \frac{\phi_{1,t}^*} {2\psi_{1,t}^*} + \frac{(\phi_{2,t}^*)^2} {2\psi_{2,t}^*} dt \right)^4 \right] \right\}^{\frac{1}{4}} \left\{ \mathbb{E}^\pi \left[ \left| Y_{1,t}(X_t^* + G_{1,t}L_t) \right|^8 \right] \right\}^{\frac{1}{8}} \left\{ \mathbb{E}^\pi \left[ \exp \left( \int_0^T \alpha_t dt \right) \right] \right\}^{\frac{1}{4}} < +\infty,
\]
where \( c \) is a positive constant which differs between lines, the first inequality follows from Hölder’s inequality and the positiveness of \( \psi_{1,t}^* \) and \( \psi_{2,t}^* \) due to \( X_t^* + G_{1,t}L_t > 0 \) as shown in (A.8), the second inequality makes use of the explicit expressions of \( \phi_{1,t}^* \) and \( \phi_{2,t}^* \) given in (3.17), the third inequality follows from the simple algebraic result that \( x^2 \leq ax^2, \ x \in \mathbb{R}^+ \) for some constant \( a \in \mathbb{R}^+ \), and the last strict inequality is due to (A.12)-(A.13), Theorem 5.1 in Zeng and Taksar [58], and condition (3.16).

Based on (A.13)-(A.14), for any sequence of stopping times \( \{\tau_n\}_{n \in \mathbb{N}} \) such that \( \tau_n \uparrow +\infty \) as \( n \to +\infty \), we know that
\[
\sup_{\tau_n \land T} \mathbb{E}^\pi \left[ \left| \varphi_{\tau_n \land T}^{\sigma^*} Y_{1,\tau_n \land T} \left( \frac{X_{\tau_n \land T}^* + G_{1,\tau_n \land T}L_{\tau_n \land T}}{\gamma} \right)^{\gamma} \right| \right] < +\infty,
\]
and
\[
\sup_{\tau_n \land T} \mathbb{E}^\pi \left[ \left| \varphi_{\tau_n \land T}^{\sigma^*} \left( \int_{\tau_n \land T}^{\tau_n \land T} \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} ds \right) \right| \right] < +\infty.
\]
Then, \( \{ \varphi_{\tau_n \land T}^{\sigma^*} Y_{1,\tau_n \land T} \left( \frac{X_{\tau_n \land T}^* + G_{1,\tau_n \land T}L_{\tau_n \land T}}{\gamma} \right)^{\gamma} \}_{n \in \mathbb{N}} \) and \( \{ \varphi_{\tau_n \land T}^{\sigma^*} \left( \int_{\tau_n \land T}^{\tau_n \land T} \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} ds \right) \}_{n \in \mathbb{N}} \) are two uniformly integrable families under the reference measure \( \mathbb{P} \) since both functions \( t_1(x) = x^4 \) and \( t_2(x) = x^2 \) are test functions of uniform integrability (see, for example, Proposition 11.7 in Zitkovic[66]). Thus, we can conclude from the above results that the control \( (\pi^*, \phi^*) \in \Pi_\pi \otimes \Phi \) is the optimal control of the robust ALM problem (2.11). In fact, plugging \( (\pi^*, \phi_{1,t}^*, \phi_{2,t}^*) \) into (3.1) leads to
\[
d\varphi_t^{\phi^*} \left( Y_{1,t}(X_t^* + G_{1,t}) \right) \gamma + \int_0^t \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} ds \\
= \varphi_t^{\phi^*} \left( Y_{1,t}\phi_{1,t}^* + Z_{1,t} \right) \left( \frac{X_t^* + G_{1,t}}{\gamma} \right)^{\gamma} + Y_{1,t}(X_t^* + G_{1,t})^{\gamma-1}(X_t^* \pi_t^{\sigma_t} L_{\tau_1} + \Lambda_{1,t}) + \int_0^t \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} ds \left( \phi_{2,s}^* \right)^2 \\
+ \int_0^t \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} ds \left( \phi_{2,s}^* \right)^2 dW_{1,t} + \varphi_t^{\phi^*} \left( Y_{1,t}\phi_{1,t}^* + Z_{2,t} \right) \frac{(X_t^* + G_{1,t})}{\gamma} + Y_{1,t}(X_t^* + G_{1,t})^{\gamma-1}\Lambda_{2,t} \\
+ \int_0^t \frac{\phi_{1,s}^*}{2\psi_{1,s}^*} ds \left( \phi_{2,s}^* \right)^2 \\
+ \int_0^t \frac{\phi_{1,s}^*}{2\psi_{2,s}^*} ds \left( \phi_{2,s}^* \right)^2 dW_{2,t}.
\]
Due to the path-wise continuity of the stochastic integrals on the right-hand side of (A.15), we see that they are \((\mathcal{F}, \mathbb{P})\)-local martingales. Therefore, there exists a localizing sequence \(\{\tau_n\}_{n \in \mathbb{N}}\) such that \(\tau_n \uparrow +\infty\) as \(n \to +\infty\) and when stopped by such a sequence, the aforementioned local martingales are true \((\mathcal{F}, \mathbb{P})\)-martingales. Then, integrating both sides of (A.15) from 0 to \(\tau_n \wedge T\) and taking expectations, we have

\[
\mathbb{E}^\mathbb{P}\left[ \varphi_{\tau_n \wedge T}^\phi \left( Y_{1, \tau_n \wedge T} \frac{(X_{\tau_n \wedge T}^* + G_{1, \tau_n \wedge T})^\gamma}{\gamma} + \int_0^{\tau_n \wedge T} \frac{(\phi_{1,s}^*)^2}{2\psi_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\psi_{2,s}^*} \, ds \right) \right] = Y_{1,0} \frac{(x_0 + G_{1,0} l_0)^\gamma}{\gamma}. \quad (A.16)
\]

As we have shown that the term in the expectation on the left-hand side of (A.16) is uniformly integrable, by using the equivalence between \(\mathcal{L}^1\) convergence and uniformly integrability and sending \(n \to +\infty\), we have from (A.16)

\[
J_p(\pi, \phi) = \mathbb{E}^{\mathbb{Q}^\phi}\left[ \frac{(X_{T}^* - L_T)^\gamma}{\gamma} + \int_0^T \frac{(\phi_{1,t}^*)^2}{2\psi_{1,t}^*} + \frac{(\phi_{2,t}^*)^2}{2\psi_{2,t}^*} \, dt \right] = \mathbb{E}^\mathbb{P}\left[ \varphi_T^\phi \left( \frac{(X_{T}^* - L_T)^\gamma}{\gamma} + \int_0^T \frac{(\phi_{1,t}^*)^2}{2\psi_{1,t}^*} + \frac{(\phi_{2,t}^*)^2}{2\psi_{2,t}^*} \, dt \right) \right] = Y_{1,0} \frac{(x_0 + G_{1,0} l_0)^\gamma}{\gamma},
\]

where \(\mathbb{Q}^\phi\) stands for the probability measure corresponding to the Radon-Nikodym derivative \(\varphi_T^\phi\).

In addition, on one hand, for the admissible strategy \((\pi, \phi) \in \Pi_p \otimes \Phi\), by using some similar localization techniques, it follows from the first part of condition 3 in Definition 2.4 and (3.1) that

\[
J_p(\pi, \phi) = \frac{\gamma - 1 - \beta_1}{2} \mathbb{E}^\mathbb{P}\left[ \int_0^T \varphi_t^\phi Y_{1,t}(X_t^\pi + G_{1,t} l_0)^{-\gamma} \left[ X_t^\pi \pi_t \sigma_t + \Lambda_1,t + \frac{X_t^\pi + G_{1,t} l_0}{\gamma - 1 - \beta_1} \left( \frac{\gamma - \beta_1}{\gamma} Z_{1,t} \right) Y_{1,t} \right] \, dt \right] + Y_{1,0} \frac{(x_0 + G_{1,0} l_0)^\gamma}{\gamma} \leq J_p(\pi, \phi),
\]

which implies that

\[
\inf_{\phi \in \Phi} \sup_{\pi \in \Pi_p} J_p(\pi, \phi) \leq \sup_{\pi \in \Pi_p} J_p(\pi, \phi) \leq J_p(\pi^*, \phi^*). \quad (A.18)
\]

On the other hand, for the admissible strategy \((\hat{\pi}, \hat{\phi}) \in \Pi_p \otimes \Phi\), from the second part of condition 3 in Definition 2.4 and (3.1), we also find that

\[
J_p(\hat{\pi}, \hat{\phi}) = \mathbb{E}^\mathbb{P}\left[ \int_0^T \varphi_t^\hat{\phi} \left[ \phi_{1,t} + \frac{(X_t^\hat{\pi} + G_{1,t} l_0)^\gamma}{\gamma} Z_{1,t} + Y_{1,t} \left( X_t^\hat{\pi} + G_{1,t} l_0 \right)^{\gamma-1} \left( \frac{\gamma - \beta_1}{\gamma} Z_{1,t} \right) \right] \, dt \right] + \mathbb{E}^\mathbb{P}\left[ \int_0^T \varphi_t^\hat{\phi} \left[ \phi_{2,t} + \frac{(X_t^\hat{\pi} + G_{1,t} l_0)^\gamma}{\gamma} Z_{2,t} + Y_{1,t} (X_t^\hat{\pi} + G_{1,t} l_0)^{\gamma-1} \Lambda_{2,t} \right] \, dt \right] + Y_{1,0} \frac{(x_0 + G_{1,0} l_0)^\gamma}{\gamma} \geq J_p(\pi^*, \phi^*). \]

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This result indicates that
\[ J_p(\pi^*, \phi^*) \leq \inf_{\phi \in \Phi} J_p(\hat{\pi}, \phi) \leq \sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} J_p(\pi, \phi). \quad (A.19) \]

Since we always have \( \inf(\sup) \geq \sup(\inf) \), we must have equality everywhere in (A.18)-(A.19). This proves that \( \sup_{\pi \in \Pi_p} \inf_{\phi \in \Phi} J_p(\pi, \phi) = J_p(\pi^*, \phi^*) = Y_{1, 0}^{(x_0 + G_1, a_0)} \) and \((\pi^*, \phi^*) \in \Pi_p \otimes \Phi \) is the optimal control of the robust ALM problem (2.11).

### A.6 Proof of Proposition 4.2

**Proof.** From Lemma 3.4 we know that the probability measure \( \tilde{\mathbb{P}} \) defined by
\[
\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} \big |_{\mathcal{F}_T} = \exp \left\{ - \int_0^T \sqrt{\alpha_t} dW_{1, t} - \frac{1}{2} \int_0^T \lambda^2 \alpha_t \, dt \right\}
\]
is equivalent to the reference measure \( \mathbb{P} \). Then, the following processes \( \tilde{W}_{1, t} \) and \( \tilde{W}_{2, t} \)
\[
\tilde{W}_{1, t} = \int_0^t \sqrt{\alpha_s} \, ds + W_{1, t} \quad \text{and} \quad \tilde{W}_{2, t} = W_{2, t}
\]
are two standard Brownian motions under \( \tilde{\mathbb{P}} \) due to Girsanov’s theorem. Applying Itô’s formula to \( \log(Y_{2, t}) \) under \( \tilde{\mathbb{P}} \) measure, we have the following quadratic BSDE of \( \left( \log(Y_{2, t}), \frac{M_{1, t}}{Y_{2, t}}, \frac{M_{2, t}}{Y_{2, t}} \right) \):
\[
\begin{cases}
  d \log(Y_{2, t}) = \left[ -r + \frac{1}{2} \left( \frac{M_{1, t}}{Y_{2, t}} \right)^2 - \frac{1}{2} \left( \frac{M_{2, t}}{Y_{2, t}} \right)^2 \right] \, dt + \frac{M_{1, t}}{Y_{2, t}} \, dW_{1, t} + \frac{M_{2, t}}{Y_{2, t}} \, dW_{2, t}, \\
  \log(Y_{2, T}) = 0.
\end{cases}
\quad (A.20)
\]

Clearly, quadratic BSDE (A.20) satisfies all the regularity conditions in Kobylanski [27]. By Theorem 2.3 and 2.6 in Kobylanski [27], we can conclude that quadratic BSDE (A.20) admits a unique solution. Hence, BSRE (4.2) admits a unique solution as well. Moreover, it is straightforward to verify that (4.4) and (4.5) form the unique solution to BSRE (4.2). This completes the proof. \( \Box \)

### A.7 Proof of Proposition 4.3

**Proof.** We conjecture that the first component \( G_{2, t} \) of the solution to quadratic BSDE (4.6) has the following affine form:
\[
G_{2, t} = f_2(t) + g_2(t) \alpha_t + h_2(t) L_t,
\]
where \( f_2(t), g_2(t), \) and \( h_2(t) \) are three undetermined differentiable functions with terminal conditions \( f_2(T) = g_2(T) = 0 \) and \( h_2(T) = -1 \). Using Itô’s formula to \( G_{2, t} \), we derive
\[
dG_{2, t} = \left[ \frac{df_2(t)}{dt} + \kappa \theta g_2(t) \right] \alpha_t + \left( \frac{dg_2(t)}{dt} - \kappa g_2(t) \right) \alpha_t + \left( \mu_1 + \lambda \sigma_1 \alpha_t \right) h_2(t) \frac{dh_2(t)}{dt} \frac{dt}{\sqrt{\alpha_t}} \right] \, dt + \left( \rho_{1} g_2(t) + \sigma h_2(t) L_t \right) \sqrt{\alpha_t} \, dW_{1, t} + \rho_{2} g_2(t) \sqrt{\alpha_t} \, dW_{2, t}.
\]

Let \( \Gamma_{1, t} = \left( \rho_{1} g_2(t) + \sigma h_2(t) L_t \right) \sqrt{\alpha_t} \) and \( \Gamma_{2, t} = \rho_{2} g_2(t) \sqrt{\alpha_t} \) and substitute them into quadratic BSDE (4.6). Then, the generator of (4.6) can be rewritten as follows:
\[
\left( \lambda \rho_{1} g_2(t) + \frac{q + \beta_2}{2} \rho_{2} g_2^2(t) - \frac{1}{2(q + \beta_1)} \lambda^2 \right) \alpha_t + \lambda \sigma h_2(t) \alpha_t L_t.
\]

(A.22)
Comparing \((A.22)\) and the drift coefficient of \((A.21)\) and separating the dependence on \(\alpha_t, L_t\), and \(\alpha_t L_t\), we obtain the ODEs \((4.9)\)–\((4.11)\).

Moreover, when \(\rho_2 \neq 0\), we denote by \(\Delta g_2 = (\kappa + \lambda \rho_1)^2 + \frac{\rho_2}{\delta^2 + \beta_1} \rho_2^2 \lambda^2 > 0\) and rewrite the Riccati ODE \((4.9)\) as follows:

\[
\frac{d g_2(t)}{dt} = \frac{q + \beta_2}{2} \rho_2^2 \left( g_2(t) - n_{g_2} \right) \left( g_2(t) - n_{g_2} \right),
\]

where \(n_{g_2}^+\) and \(n_{g_2}^-\) are given by \((4.15)\). After some tedious calculations, we derive the closed-form expression of \(g_2(t)\) given in \((4.12)\). When \(\rho_2 = 0\) and \(\kappa + \lambda \rho_1 \neq 0\), the Riccati ODE \((4.9)\) degenerates to the following linear ODE:

\[
\frac{d g_2(t)}{dt} = (\kappa + \lambda \rho_1) g_2(t) - \frac{1}{2(q + \beta_1)} \lambda^2,
\]

and we immediately find that

\[
g_2(t) = -\frac{\lambda^2}{2(q + \beta_1)(\kappa + \lambda \rho_1)} \left( e^{(\kappa + \lambda \rho_1)(t-T)} - 1 \right).
\]

For the case when \(\rho_2 = 0\) and \(\kappa + \lambda \rho_1 = 0\), we have from \((4.9)\) that \(g_2(t) = -\frac{\lambda^2}{2(q + \beta_1)}(t - T)\). Substituting \((4.12)\) into the ODE \((4.10)\) gives the closed-form expressions of \(f_2(t)\) in \((4.13)\). Finally, by a simple calculation, the explicit solution \(h_2(t)\) to ODE \((4.11)\) is given by \((4.14)\).

\[
\Box
\]

**A.8 Proof of Theorem 4.4**

*Proof.* In the first part of the proof, we show that the proposed solution \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\) lies in the space \(L^2_{F,P}(0,T; \mathbb{R}) \otimes L^2_{F,P}(0,T; \mathbb{R}) \otimes L^2_{\Gamma,F,P}(0,T; \mathbb{R})\). For this, from the \(P\)-dynamics of the random liabilities \((2.7)\) we observe that

\[
\mathbb{E}^P \left[ \int_0^T L_t^4 \, dt \right] = l_0^4 \int_0^T e^{4\mu t_l} \mathbb{E}^P \left[ \exp \left\{ (4\lambda \sigma_1 - 2\sigma_2^2) \int_0^t \alpha_s \, ds + 4\sigma_l \int_0^t \sqrt{\alpha_s} \, dW_{1,s} \right\} \right] \, dt
\]

\[\leq l_0^4 \int_0^T e^{4\mu t_l} \left\{ \mathbb{E}^P \left[ \exp \left\{ (8\lambda \sigma_l + 28\sigma_1^2) \int_0^t \alpha_s \, ds \right\} \right] \right\}^\frac{1}{2} \, dt,
\]

where the inequality follows from the Hölder’s inequality and the fact that the stochastic exponential process \(\exp \left\{ 8\sigma_l \int_0^t \sqrt{\alpha_s} \, dW_{1,s} - 32\sigma_1^2 \int_0^t \alpha_s \, ds \right\}\) is an \((\mathcal{F},\mathbb{P})\)-martingale by Lemma 3.4. To calculate the term \(\mathbb{E}^P \left[ \exp \left\{ (8\lambda \sigma_l + 28\sigma_1^2) \int_0^t \alpha_s \, ds \right\} \right]\), let \(\mathbb{E}^F \left[ \cdot | \mathcal{F}_u \right] \) be the conditional expectation under \(\mathbb{P}\) given \(\mathcal{F}_u\), for \(u \leq t\). By using the Markovian structure of the process \(\alpha_t\), we have

\[
\mathbb{E}^P \left[ \exp \left\{ (8\lambda \sigma_l + 28\sigma_1^2) \int_0^t \alpha_s \, ds \right\} \bigg| \mathcal{F}_u \right] = F(\alpha_u, u), \text{ for } u \leq t,
\]

where \(F : \mathbb{R}^+ \otimes [0,t] \rightarrow \mathbb{R}^+\) is an unknown function. By Feynman-Kac theorem, we know that the function \(F\) is governed by the following partial differential equation (PDE):

\[
\begin{cases}
\frac{\partial F}{\partial t}(x,u) + \kappa(\theta - x) \frac{\partial F}{\partial x}(x,u) + \frac{1}{2} (\rho_1^2 + \rho_2^2) x \frac{\partial^2 F}{\partial x^2}(x,u) + (8\lambda \sigma_l + 28\sigma_1^2)xF(x,u) = 0, \\
F(x,t) = 1.
\end{cases}
\]
Furthermore, it can be shown that $F(x, u) = \exp \{M(u; t)x + N(u; t)\}$, for $u \in [0, t]$, where $M(u; t)$ and $N(u; t)$ satisfy the following ODEs:

$$\frac{dM(u; t)}{du} = -\frac{\rho_1^2 + \rho_2^2}{2} M^2(u; t) + \kappa M(u; t) - \left(8\lambda \sigma_1 + 28\sigma_1^2\right), \quad M(t; t) = 0,$$

and

$$\frac{dN(u; t)}{du} = -\kappa \theta M(u; t), \quad N(t; t) = 0.$$

As in Proposition 4.3, we can show the closed-form expressions for $M(u; t)$ and $N(u; t)$ as follows:

$$M(u; t) = \begin{cases} n_M^+ n_M^- \left(1 - e^{\Delta_M (t-u)}\right), & \text{if } \Delta_M > 0; \\ \frac{n_M^+ - n_M^- e^{\Delta_M (t-u)}}{(\rho_1^2 + \rho_2^2)(t-u)n_M^2}, & \text{if } \Delta_M = 0; \\ \frac{\sqrt{-\Delta_M}}{(\rho_1^2 + \rho_2^2) \tan \left(\arctan \left(\frac{\kappa}{\sqrt{-\Delta_M}}\right) - \frac{\sqrt{-\Delta_M}}{2} (t-u)\right)}, & \text{if } \Delta_M < 0, \end{cases}$$

(A.25)

and

$$N(u; t) = \int_u^t \kappa \theta M(s; t) \, ds$$

(A.26)

where $\Delta_M, n_M^+, n_M^-$, and $n_M$ are given by

$$\Delta_M = \kappa^2 - 2(\rho_1^2 + \rho_2^2)(8\lambda \sigma_1 + 28\sigma_1^2), \quad n_M = \frac{\kappa}{\rho_1^2 + \rho_2^2}, \quad n_M^+ = -\frac{\kappa + \sqrt{\Delta_M}}{-\rho_1^2 - \rho_2^2}, \quad n_M^- = -\frac{\kappa - \sqrt{\Delta_M}}{-\rho_1^2 - \rho_2^2}.$$ 

Then, combining (A.23)-(A.26) and using the law of total expectation, we derive

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T L_t^4 \, dt \right] \leq l_0^4 \int_0^T \exp \left\{ 4\mu t + \frac{M(0; t)}{2} \alpha_0 + \frac{N(0; t)}{2} \right\} \, dt < +\infty.$$

Additionally, note that for all $t \in [0, T]$, the second moment of the square-root factor process $\alpha_t$ is explicitly given by

$$\mathbb{E}^\mathbb{P} \left[ \alpha_t^2 \right] = (\alpha_0 e^{-\kappa t} + \theta \left(1 - e^{-\kappa t}\right))^2 + \alpha_0 \left(\frac{\rho_1^2 + \rho_2^2}{\kappa} \left(e^{-\kappa t} - e^{-2\kappa t}\right) + \theta \left(\frac{\rho_1^2 + \rho_2^2}{\kappa} \left(1 - e^{-\kappa t}\right)\right)^2. $$

Therefore, from the explicit expressions for $G_{2,t}, \Gamma_{1,t}$ and $\Gamma_{2,t}$ given in (4.7)-(4.8) we derive

$$\mathbb{E}^\mathbb{P} \left[ \int_0^T G_{2,t}^2 + \Gamma_{1,t}^2 + \Gamma_{2,t}^2 \, dt \right] \leq c \left[ 1 + \int_0^T \mathbb{E}^\mathbb{P} \left[ \alpha_t^2 \right] \, dt + \mathbb{E}^\mathbb{P} \left[ \int_0^T L_t^4 \, dt \right] \right] < +\infty,$$

where $c$ is a positive constant. This finishes the first part of the proof.

Next, we show that the proposed solution given by (4.7)-(4.8) is the unique solution to quadratic BSDE (4.6). To this end, note from Lemma 3.4 that the probability measure $\tilde{\mathbb{P}}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ via the following Radon-Nikodym derivative:

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_T} = \exp \left\{- \int_0^T \lambda \sqrt{\alpha_t} \, dW_{1,t} - \frac{1}{2} \int_0^T \lambda^2 \alpha_t \, dt \right\}.$$
Then, the following stochastic process \( \hat{W}_{1,t} \) and \( \hat{W}_{2,t} \) are standard Brownian motions under \( \hat{P} \) measure:
\[
\hat{W}_{1,t} = \int_0^t \lambda \sqrt{\alpha_s} \, ds + W_{1,t} \quad \text{and} \quad \hat{W}_{2,t} = W_{2,t}
\]
by Girsanov’s theorem and the \( \hat{P} \)-dynamics of the square-root factor process \( \alpha_t \) is given by
\[
da\alpha_t = (\kappa + \lambda \rho_1) \left( \frac{\kappa \theta}{\kappa + \lambda \rho_1} - \alpha_t \right) \, dt + \sqrt{\alpha_t} \left( \rho_1 \, d\hat{W}_{1,t} + \rho_2 \, d\hat{W}_{2,t} \right),
\]
which preserves the affine-form, square-root structure under Assumption 4.1. Reformulate BSDE (4.6) of \((G_{2,t}, \Gamma_{1,t}, \Gamma_{2,t})\) under \( \hat{P} \) as follows:
\[
\begin{align*}
dG_{2,t} &= \left( \frac{q + \beta_2}{2} \Gamma_{2,t}^2 - \frac{1}{2(q + \beta_1)} \lambda^2 \alpha_t \right) \, dt + \Gamma_{1,t} \, d\hat{W}_{1,t} + \Gamma_{2,t} \, d\hat{W}_{2,t}, \\
G_{2,T} &= L_T,
\end{align*}
\]
and suppose that there exists another solution \((\hat{G}_{2,t}, \hat{\Gamma}_{1,t}, \hat{\Gamma}_{2,t})\) to (A.27), which is different from the proposed solution given in Proposition 4.3. Then, the difference process \((\Delta G_{2,t}, \Delta \Gamma_{1,t}, \Delta \Gamma_{2,t}) := (G_{2,t} - \hat{G}_{2,t}, \Gamma_{1,t} - \hat{\Gamma}_{1,t}, \Gamma_{2,t} - \hat{\Gamma}_{2,t})\) must solve the following BSDE:
\[
\begin{align*}
d\Delta G_{2,t} &= \frac{q + \beta_2}{2} \left( \Gamma_{2,t}^2 - \hat{\Gamma}_{2,t}^2 \right) \, dt + \Delta \Gamma_{1,t} \, d\hat{W}_{1,t} + \Delta \Gamma_{2,t} \, d\hat{W}_{2,t}, \\
\Delta G_{2,T} &= 0.
\end{align*}
\]
Furthermore, we notice from the explicit expression for \( \Gamma_{2,t} \) given in (4.8) and Lemma 3.4 that the following probability measure \( \tilde{P} \) is well-defined and equivalent to \( \hat{P} \) on \( \mathcal{F}_T \):
\[
\frac{d\tilde{P}}{d\hat{P}} \bigg|_{\mathcal{F}_T} = \exp \left\{ - \int_0^T (q + \beta_2) \rho_2 g_2(t) \sqrt{\alpha_t} \, d\hat{W}_{2,t} - \int_0^T \frac{(q + \beta_2)^2 \rho_2^2 g_2^2(t)}{2} \alpha_t \, dt \right\}
\]
\[
= \exp \left\{ - \int_0^T (q + \beta_2) \Gamma_{2,t} \, d\hat{W}_{2,t} - \frac{(q + \beta_2)^2}{2} \int_0^T \Gamma_{2,t}^2 \, dt \right\},
\]
and \( \hat{W}_{2,t} = \int_0^t (q + \beta_2) \Gamma_{2,s} \, ds + \hat{W}_{2,t} \) and \( \hat{W}_{1,t} = \hat{W}_{1,t} \) are two standard Brownian motions under \( \tilde{P} \). Hence, BSDE (A.28) can be rewritten under \( \tilde{P} \):
\[
\begin{align*}
d\Delta G_{2,t} &= - \frac{q + \beta_2}{2} \Delta \Gamma_{2,t}^2 \, dt + \Delta \Gamma_{1,t} \, d\hat{W}_{1,t} + \Delta \Gamma_{2,t} \, d\hat{W}_{2,t}, \\
\Delta G_{2,T} &= 0,
\end{align*}
\]
which is a quadratic BSDE satisfying all the regularity conditions in Kobylanski [27], and we can conclude that \((\Delta G_{2,t}, \Delta \Gamma_{1,t}, \Delta \Gamma_{2,t}) = (0, 0, 0)\) is the unique solution by Theorem 2.3 and 2.6 in Kobylanski [27]. This proves that the proposed solution given in Proposition 4.3 is the unique solution to BSDE (4.6).

**A.9 Proof of Theorem 4.5**

**Proof.** First of all, by Theorem 5.1 in Zeng and Taksar [58], it follows from the explicit expressions for \( \phi^+ \) and \( \Gamma_{2,t} \) given in (4.17) and (4.8) that
\[
\mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{1}{2} \int_0^T (\phi_{1,t}^2 + (\phi_{2,t}^2) \, dt \right\} \right] = \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{1}{2} \int_0^T \left( \frac{\beta_1^2 \lambda^2}{(q + \beta_1)^2} + \beta_2^2 \rho_2^2 g_2(t) \right) \alpha_t \, dt \right\} \right]
\]
\[
\leq \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \frac{b_0}{2} \int_0^T \alpha_t \, dt \right\} \right] < +\infty,
\]
Electronic copy available at: https://ssrn.com/abstract=4182575
the constant $b_0$ is given in (4.16). This shows that $\phi^* \in \Phi$. Plugging $\pi_t^*$ into $X_t^*$ and using Itô’s formula to $X_t^* Y_{2,t} + G_{2,t}$ under the reference measure $\mathbb{P}$, we derive

$$X_t^* Y_{2,t} + G_{2,t} = x_0 Y_{2,0} + G_{2,0} + \int_0^t \left( \frac{\lambda^2}{2(q + \beta_1)} + \frac{q + \beta_2}{2} \rho_2 g_2(s) \right) \alpha_s \, ds + \int_0^t \frac{\lambda}{q + \beta_1} \sqrt{\alpha_s} \, dW_{1,s}$$

$$+ \int_0^t \rho_2 g_2(s) \sqrt{\alpha_s} \, dW_{2,s},$$

from which it follows that

$$e^{-8q(X_t^* Y_{2,t} + G_{2,t})}$$

$$= \mathcal{E}_P \left\{ \sup_{t \in [0,T]} 8q \alpha_s \, dW_{1,s} - \int_0^t 8q \rho_2 g_2(s) \sqrt{\alpha_s} \, dW_{2,s} - 32q^2 \int_0^t \left( \frac{\lambda^2}{(q + \beta_1)^2} + \rho_2 g_2(s) \right) \alpha_s \, ds \right\}$$

$$\mathcal{E}_P \left\{ \sup_{t \in [0,T]} 8q \alpha_s \, dW_{1,s} - \int_0^t 8q \rho_2 g_2(s) \sqrt{\alpha_s} \, dW_{2,s} - 32q^2 \int_0^t \left( \frac{\lambda^2}{(q + \beta_1)^2} + \rho_2 g_2(s) \right) \alpha_s \, ds \right\}$$

where $c$ is a positive constant. We notice that $K_{3,t}$ is the stochastic exponential process of continuous $(\mathbb{F}, \mathbb{P})$-local martingale $\int_0^t \frac{-8qX}{q + \beta_1} \sqrt{\alpha_s} \, dW_{1,s} - \int_0^t 8q \rho_2 g_2(s) \sqrt{\alpha_s} \, dW_{2,s}$. Then, applying the Hölder’s inequality, Theorem 5.1 in Zeng and Taksar [58] and Theorem 15.4.6 in Cohen and Elliott [16], we derive

$$\mathbb{E}^P \left[ \sup_{t \in [0,T]} e^{-8q(X_t^* Y_{2,t} + G_{2,t})} \right] \leq c \left\{ \mathbb{E}^P \left[ \sup_{t \in [0,T]} K_{3,t}^2 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}^P \left[ \sup_{t \in [0,T]} K_{1,t}^2 \right] \right\}^{\frac{1}{2}}$$

$$\leq c \left\{ \mathbb{E}^P \left[ \exp \left\{ \frac{b_1}{2} \int_0^T \alpha_t \, dt \right\} \right] \right\}^{\frac{\sqrt{2} + \sqrt{2} - 1}{4 + \sqrt{2}}} \left\{ \mathbb{E}^P \left[ \exp \left\{ \frac{b_2}{2} \int_0^T \alpha_t \, dt \right\} \right] \right\}^{\frac{1}{2}}$$

where the constant $c$ might differ between lines, and $b_1$ and $b_2$ are given in (4.16). Thus, using the explicit expressions for $\phi^*_{1,t}$ and $\phi^*_{2,t}$ given in (4.17) and applying Theorem 15.4.6 in Cohen and Elliott [16] and Theorem 5.1 in Zeng and Taksar [58] again, we find from (A.29) that

$$\mathbb{E}^P \left[ \sup_{t \in [0,T]} \left| \phi_{1,t}^* e^{-q(X_t^* Y_{2,t} + G_{2,t})} \right| \right] \leq \frac{1}{q^8} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} |\phi_{1,t}^*|^8 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E}^P \left[ \sup_{t \in [0,T]} e^{-8q(X_t^* Y_{2,t} + G_{2,t})} \right] \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{q^8} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} |\phi_{1,t}^*|^8 \right] \right\}^{\frac{1}{2}} \left\{ \mathbb{E} \left[ \sup_{t \in [0,T]} e^{-8q(X_t^* Y_{2,t} + G_{2,t})} \right] \right\}^{\frac{1}{2}}$$

$$\leq \frac{1}{q^8} \sqrt{8} \left\{ \mathbb{E}^P \left[ \exp \left\{ \frac{b_0}{2} \int_0^T \alpha_t \, dt \right\} \right] \right\}^{\frac{\sqrt{120} + \sqrt{32}}{240 + 64 \sqrt{14}}} \left\{ \mathbb{E}^P \left[ \sup_{t \in [0,T]} e^{-8q(X_t^* Y_{2,t} + G_{2,t})} \right] \right\}^{\frac{1}{2}} < + \infty.$$
Additionally, it is easy to be shown from condition (4.16), the Hölder’s inequality, (A.29)-(A.30), and the trivial algebraic result that
\[ x^8 \leq ae^x, \quad x \in \mathbb{R}^+ \text{ for some constant } a \in \mathbb{R}^+ \] that
\[
\mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} \left| \varphi_t^\phi \left( \int_0^t \frac{(\phi_{1,s}^*)^2}{2\eta_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\eta_{2,s}^*} \, ds \right) \right|^2 \right] 
\leq c \left\{ \mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} |\varphi_t^\phi|^8 \right] \right\}^{\frac{1}{4}} \left\{ \mathbb{E}^\mathbb{P} \left[ \sup_{t \in [0,T]} e^{-q(X_t^*,Y_t^*,G_t^*)} \right] \right\}^{\frac{1}{4}} \left\{ \mathbb{E}^\mathbb{P} \left[ \exp \left\{ \int_0^T \alpha_t \, dt \right\} \right] \right\}^{\frac{1}{4}} < +\infty,
\]
(A.31)
where \( c \) is a positive constant. The above results verify (i)-(iii) in the statement of Theorem 4.5, and thus, it follows from (A.30)-(A.31) immediately that the following two families of random variables
\[
\left\{ \frac{\varphi_{\tau_n \wedge T}^\phi - e^{-q(X_{\tau_n \wedge T}^*,Y_{\tau_n \wedge T}^*,G_{\tau_n \wedge T})}}{\sqrt{q}} \right\}_{n \in \mathbb{N}} \text{ and } \left\{ \varphi_{\tau_n \wedge T}^\phi \left( \int_0^{\tau_n \wedge T} \frac{(\phi_{1,s}^*)^2}{2\eta_{1,s}^*} + \frac{(\phi_{2,s}^*)^2}{2\eta_{2,s}^*} \, ds \right) \right\}_{n \in \mathbb{N}}
\]
are uniformly integrable under the reference measure \( \mathbb{P} \), where \( \{\tau_n\}_{n \in \mathbb{N}} \) is an arbitrary sequence of stopping times such that \( \tau_n \uparrow +\infty \) as \( n \to +\infty \). Hence, we can conclude that the control \((\pi^*,\phi^*) \in \Pi_e \otimes \Phi \). The proof for the optimality of the admissible control \((\pi^*,\phi^*) \) given in (4.17) is similar to Theorem 3.7, so we omit it here. \( \square \)