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Subgroups of PL⁺I which do not embed into Thompson’s group F

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Abstract. We will give a general criterion—the existence of an $F$-obstruction—for showing that a subgroup of PL⁺I does not embed into Thompson’s group $F$. An immediate consequence is that Cleary’s “golden ratio” group $F_\tau$ does not embed into $F$, answering a question of Burillo, Nucinkis, and Reeves. Our results also yield a new proof that Stein’s groups $F_{p,q}$ do not embed into $F$, a result first established by Lodha using his theory of coherent actions. We develop the basic theory of $F$-obstructions and show that they exhibit certain rigidity phenomena of independent interest. In the course of establishing the main result of the paper, we prove a dichotomy theorem for subgroups of PL⁺I. In addition to playing a central role in our proof, it is strong enough to imply both Rubin’s reconstruction theorem restricted to the class of subgroups of PL⁺I and also Brin’s ubiquity theorem.

1. Introduction

In this article, we aim to give a partial answer to the following question: When does a group of piecewise linear homeomorphisms of the unit interval fail to embed into Richard Thompson’s group $F$? Thompson’s group $F$ is the subgroup of PL⁺I consisting of those functions whose breakpoints occur at dyadic rationals and whose slopes are powers of 2. We isolate the notion of an $F$-obstruction based on Poincaré’s rotation number and show that subgroups of PL⁺I which contain $F$-obstructions do not embed into $F$.

In the course of proving the main result of the paper, we establish a dichotomy theorem for subgroups of PL⁺I. This result seems likely to be of independent interest as it is already sufficiently powerful to prove both Brin’s ubiquity theorem [3] and the restriction of Rubin’s reconstruction theorem [18, Corollary 3.5 (c)] to the class of subgroups of PL⁺I (see also [17, Theorem 4] and [1, Theorem E16.3] which were precursors to [18]).

1.1. Rotation numbers and $F$-obstructions

Recall that if $\gamma$ is a homeomorphism of the circle $\mathbb{R}/\mathbb{Z}$, then the rotation number of $\gamma$ is defined to be

$$\lim_{n \to \infty} \frac{\gamma^n(x) - x}{n}$$
modulo 1, where \( \tilde{\gamma}: \mathbb{R} \to \mathbb{R} \) is a lift of \( \gamma \) (this limit always exists and, modulo 1, does not depend on \( x \) or the choice of \( \tilde{\gamma} \)). Observe that if \( \gamma \) is a rotation of \( \mathbb{R}/\mathbb{Z} \) by \( r \in (0, 1) \), then we can take \( \tilde{\gamma}(x) = x + r \) and the rotation number of \( \gamma \) is \( r \). In fact, Poincaré showed that if \( \gamma \) is any homeomorphism such that no finite power has a fixed point, \( \gamma \) is semiconjugate to the irrational rotation specified by its rotation number. On the other hand, if \( \gamma^q \) has a fixed point for some \( q \), then we can take \( x \in \mathbb{R} \) and \( \tilde{\gamma} \) such that \( \tilde{\gamma}^p(x) = x + p \) for some \( p \in \mathbb{Z} \) with \( 0 \leq p < q \). It follows that the rotation number is \( p/q \).

If \( f, g \in \text{Homeo}_+I \) and \( s \in I \) are such that

\[
s < f(s) \leq g(s) < f(g(s)) = g(f(s)),
\]

then the rotation number of \( f \) modulo \( g \) at \( s \) is the rotation number of the function on \([s, g(s)]\) defined by \( x \mapsto g^{-m}(f(x)) \), where \( m \) is such that \( s \leq g^{-m}(f(x)) < g(s) \). This map is a homeomorphism of a circle when \([s, g(s)]\) is given a suitable topology.

A pair \((f, g)\) of elements of \( \text{PL}_+I \) is an \( F \)-obstruction if there is \( s \) such that either:

1. \( s < f(s) \leq g(s) < f(g(s)) = g(f(s)) \) and the rotation number of \( f \) modulo \( g \) at \( s \) is irrational;
2. \( s > f(s) \geq g(s) > f(g(s)) = g(f(s)) \) and the rotation number of \( f^{-1} \) modulo \( g^{-1} \) at \( f(g(s)) \) is irrational.

It follows from the work of Ghys and Sergiescu [12] that the standard way of representing \( F \) in \( \text{PL}_+I \) does not contain any \( F \)-obstructions (see Section 3).

The main result of this paper is that the property of being an \( F \)-obstruction is preserved by monomorphisms into \( \text{PL}_+I \).

**Theorem 1.1.** If \((f, g)\) is an \( F \)-obstruction and \( \phi: (f, g) \to \text{PL}_+I \) is a monomorphism, then \((\phi(f), \phi(g))\) is an \( F \)-obstruction. In particular, if \( G \leq \text{PL}_+I \) contains an \( F \)-obstruction, then \( G \) does not embed into \( F \).

### 1.2. A dichotomy for subgroups of \( \text{PL}_+I \)

Theorem 1.1 is first established for \( F \)-obstructions which generate a group with a single orbital—a component of support. The general case is then handled by way of a dichotomy theorem for subgroups of \( \text{PL}_+I \). This dichotomy is strong enough to imply both Brin’s ubiquity theorem [3] and a form of Rubin’s reconstruction theorem [18, Corollary 3.5 (c)] for subgroups of \( \text{PL}_+I \) (see Section 6).

If \( G \leq \text{PL}_+I \), then we say that \( J \) is a resolvable orbital of \( G \) if \( J \) is an orbital of \( G \) and \( \{\text{supt}(g) \cap J \mid g \in G\} \) forms a base for the topology on \( J \). If \( G \leq \text{Homeo}_+I \), a partial function \( \psi: I \to I \) is \( G \)-equivariant if its domain is \( G \)-invariant and for all \( g \in G \) and \( x \in \text{dom}(\psi) \), \( \psi(g(x)) = g(\psi(x)) \). Our dichotomy theorem can now be stated as follows:

**Theorem 1.2.** Suppose that \( G \leq \text{PL}_+I \) and \( J \) is a resolvable orbital of \( G \). If \( K_i \) \((i < n)\) is a sequence of orbitals of \( G \), then exactly one of the following is true:
(1) There is $g \in G$ whose support intersects $J$ but is disjoint from $K_i$ for all $i < n$.

(2) There is $i < n$ and a monotone surjection $\psi : K_i \rightarrow J$ which is $G$-equivariant.

**Remark 1.3.** A more general result has been obtained independently by Brum, Matte Bon, Rivas, and Triestino [6, Corollary 5.17].

### 1.3. Corollaries of Theorem 1.1

Our original motivation for proving Theorem 1.1 is the following corollary, which answers Question 10.2 of [7] (see Section 7 for the definitions of $F_\tau$ and $F_{p,q}$).

**Corollary 1.4.** Cleary’s group $F_\tau$ does not embed into $F$.

Theorem 1.1 also gives a new proof of the following result first established by Lodha using his theory of coherent actions.

**Corollary 1.5 ([15]).** Stein’s groups $F_{p,q}$ do not embed into $F$ if $p$, $q$ are relatively prime natural numbers.

In the next corollary, we view $PL_+ I$ as consisting of functions from $\mathbb{R}$ to $\mathbb{R}$ by defining its elements to be the identity outside of $I$. Here $F^{t \mapsto t - \xi}$ is the set of conjugates of elements of $F$ by $t \mapsto t - \xi$.

**Corollary 1.6.** If $0 < \xi < 1$ is irrational, then $\langle F \cup F^{t \mapsto t - \xi} \rangle$ does not embed into $F$.

In the course of proving Theorem 1.1, we will also establish the following results. An $F$-obstruction is basic if the group it generates has connected support.

**Theorem 1.7.** If two basic $F$-obstructions generate isomorphic groups, then the groups are topologically conjugate via a homeomorphism of their supports.

**Theorem 1.8.** If $(f, g)$ is an $F$-obstruction, then $F$ embeds into $\langle f, g \rangle$.

Theorem 1.8 generalizes a result of Bleak [2, §3.3] which asserts that if $G \leq PL_+ I$ and the left or right group of germs at some point is nondiscrete—or equivalently noncyclic—then $F$ embeds into $G$. Note that this implies the restriction [15, Theorem 1.6] to the class of subgroups of $PL_+ I$.

We conjecture that the converse to Theorem 1.1 holds for finitely generated groups.

**Conjecture 1.9.** If $G \leq PL_+ I$ is finitely generated and does not contain an $F$-obstruction, then $G$ embeds into $F$.

Notice that this conjecture implies every finitely generated subgroup of $PL_+ I$ either contains a copy of $F$ or else embeds into $F$; whether such a dichotomy holds was asked by Matthew Brin. A natural test case is the group $F_{2/3}$ consisting of those elements of $PL_+ I$ having breakpoints in $\mathbb{Z}[1/6]$ and having slopes which are powers of $2/3$. It appears to be unknown both whether this group contains an $F$-obstruction and whether it embeds into $F$, see [8, Question 4.6].
This paper is organized as follows. After recalling some terminology and notation in Section 2 and establishing that $F$ does not contain $F$-obstructions in Section 3, we prove Theorem 1.8 in Section 4. Section 5 contains the proof of Theorem 1.2. In Section 6, we use Theorem 1.2 to complete the proofs of Theorems 1.1 and 1.7 and give new derivations of both Brin’s ubiquity theorem and Rubin’s theorem for sub-groups of PL$^+I$. Finally, the computations needed for Corollaries 1.4–1.6 are presented in Section 7.

2. Preliminaries

Throughout the paper, counting will start at 0 and $i, j, k, l, m, n$ will only be used to denote integers. If $A$ and $B$ are subsets of an ordered set, we will sometimes write $A \leq B$ to indicate that every element of $A$ is less than every element of $B$.

As already mentioned, Homeo$^+_I$ is the collection of orientation preserving homeomorphisms of the unit interval $I := [0, 1]$. Homeo$^+_I$ is a group with the operation of composition. PL$^+_I$ is the subgroup of Homeo$^+_I$ consisting of those elements which are piecewise linear. If $f \in \text{PL}^+_I$, we say that $s$ is a breakpoint of $f$ if the derivative of $f$ at $s$ is undefined. If $s$ is not a breakpoint of $f$, we will refer to $f'(s)$ as the slope of $f$ at $s$.

Thompson’s group $F$ consists of those elements of PL$^+_I$ whose slopes are integer powers of 2 and whose breakpoints are in $\mathbb{Z}[\frac{1}{2}]$. When there is a need to emphasize that we are working with this particular group and not an isomorphic copy, we will refer to it as the standard model of $F$. The reader is referred to [9] for the basic analysis of Thompson’s group $F$ and [4] for background on PL$^+_I$.

Going forward, we will adopt the convention common in the literature that elements of Homeo$^+_I$ act on $I$ from the right. Thus we will write $xf$ for the application of $f \in \text{Homeo}^+_I$ to $x \in I$. If $f \in \text{Homeo}^+_I$, then the support of $f$ is defined to be

$$\text{supt}(f) := \{x \in I \mid xf \neq x\}.$$ 

If $A \subseteq \text{Homeo}^+_I$, then the support of $A$ is defined to be

$$\text{supt}(A) := \{x \in I \mid \exists g \in A \ (xg \neq x)\} = \cup\{\text{supt}(g) \mid g \in A\}.$$ 

Notice that supt $A = \text{supt}(A)$. We will write $\overline{\text{supt}} A$ for the closure of supt $A$. A connected component of the support of $f$ is an orbital of $f$; similarly one defines the orbital of a subgroup of Homeo$^+_I$. If $f$ has a single orbital, we will say that $f$ is a bump. If $f$ is a bump and $sf > s$ for some (equivalently all) $s$ in its support, then we say $f$ is a positive bump; otherwise $f$ is a negative bump.

If $f \in \text{Homeo}^+_I$ and $X \subseteq I$ is a union of orbitals and fixed points of $f$, then $f|_X \in \text{Homeo}^+_I$ is defined by

$$sf|_X := \begin{cases} sf & \text{if } s \in X, \\ s & \text{otherwise.} \end{cases}$$
This map will be referred to as the projection to $X$. If $G \leq \text{Homeo}_+ I$ and $X$ is a union of orbitals and fixed points of $G$, then the projection of $G$ to $X$ is the image of $G$ under the homomorphism $f \mapsto f|_X$; we will sometimes use "the projection of $G$ to $X"$ to refer to the homomorphism itself.

If $f, g$ are elements of a group $G$, define $g^f := f^{-1}gf$ and $[f, g] := f^{-1}g^{-1}fg = f^{-1}g^f = (g^{-1})^f g$. It is easily checked that if $f, g \in \text{PL}_+ I$, then $\text{supt}(f^g) = \text{supt}(f)g$.

If $A$ and $B$ are sets of group elements, we will write $[A, B]$ for $\{[a, b] \mid a \in A$ and $b \in B\}$. The subgroup of $G$ generated by $[G, G]$ is denoted by $G'$. If $G = G'$, then we say that $G$ is perfect. If $G, H \leq \text{PL}_+ I$, we will say that $G$ commutes with $H$ if every element of $G$ commutes with every element of $H$.

We finish this section with some well-known results which will be needed later in the paper.

**Proposition 2.1** (see [9]). If $a$ and $b$ are elements of a group and $[a^b, b^a] = [a^{ba^{-1}}, b^{ab^{-1}}]$ is the identity but $ab \neq ba$, then $\langle a, b \rangle$ is isomorphic to Thompson’s group $F$. In particular, if $s_0 < s_1 < t_0 < t_1$ and $a_0, a_1 \in \text{Homeo}_+ I$ are such that $\text{supt}(a_i) = (s_i, t_i)$ and $t_0a_1 \leq s_1a_0$, then $\langle a_0, a_1 \rangle$ is isomorphic to $F$.

The next theorem is known as Brin’s ubiquity theorem. If $G \leq \text{PL}_+ I$, $J$ is an orbital of $G$ and $g \in G$, we say $g$ approaches the left (right) end of $J$ if the closure of $\text{supt}(g) \cap J$ contains the left (right) endpoint of $J$.

**Theorem 2.2** ([3]). Suppose that $G \leq \text{PL}_+ I$ and there is an orbital $J$ of $G$ such that some element of $G$ approaches one end of $J$ but not the other. Then there is a subgroup of $G$ isomorphic to $F$.

**Lemma 2.3** ([4]). If $G \leq \text{PL}_+ I$ and $a \in G'$, then $\overline{\text{supt}}(a) \subseteq \text{supt} G$.

The next lemma is more or less established in [4] in the course of showing that non-abelian subgroups of $\text{PL}_+ I$ contain infinite rank free abelian groups. We leave the details to the interested reader.

**Lemma 2.4.** If $G$ is a subgroup of $\text{Homeo}_+ I$ and $X \subseteq \text{supt} G$ is compact, then there exists $g \in G$ such that for all nonzero $k \in \mathbb{Z}$, $Xg^k \cap X = \emptyset$.

## 3. $F$ does not contain $F$-obstructions

In this section, we will prove the following proposition.

**Proposition 3.1.** No pair of elements of the standard model of $F$ is an $F$-obstruction.
Proof. Recall that Thompson’s group $T$ consists of all piecewise linear homeomorphisms of $\mathbb{R}/\mathbb{Z}$ which map 0 to a dyadic rational, whose breakpoints are dyadic rationals, and whose slopes are powers of 2. Ghys and Sergiescu [12] have shown that every element of $T$ has a rational rotation number. It therefore suffices to show that if $f, g \in F$ and $s \in I$ with $s < sf \leq sg < sfg = sgf$, then the associated homeomorphism $\gamma$ defined in the introduction is topologically conjugate to an element of $T$.

Let $s, f$ and $g$ be given as above and let $s_0 < s$ be a dyadic rational such that $s < s_0 g$, noting that $sg < s_0 g^2$. By conjugating by an element of $F$ and revising $f, g, s$, and $s_0$ if necessary, we can assume that for some $k$, $s_0 g = s_0 + 2^{-k}$ and $s_0 g < 1 - 2^{-k}$. By further conjugating by an element $h$ of $F$ which satisfies $th = t$ if $t \leq s_0 + 2^{-k}$ and $th = tg^{-1} + 2^{-k}$ if $s_0 + 2^{-k} \leq t \leq s_0 g^2$, we can additionally assume that if $s_0 \leq t \leq s_0 g$, then $tg = t + 2^{-k}$. (This conjugacy argument is essentially the staircase algorithm of [14].)

Repeating this procedure on the interval $[s_0 g, s_0 g^2]$, we can assume without loss of generality that if $s_0 \leq t < s_0 g^2$, then $tg = t + 2^{-k}$.

The homeomorphism $\gamma$ associated to this revised choice of $f, g$ and $s$ is topologically conjugate to the homeomorphism associated to the original choice of $f, g$, and $s$. Moreover, $\gamma$ is a homeomorphism of $\mathbb{R}/2^{-k}\mathbb{Z}$ which maps dyadic rationals to dyadic rationals, whose breakpoints are dyadic rationals, and whose slopes are powers of 2. Clearly, $\gamma$ is topologically conjugate to an element of $T$ and hence by [12], $\gamma$ has a rational rotation number.

\section{4. $F$-obstructions yield copies of $F$}

A key step in proving Theorem 1.1 is to demonstrate that if $f, g \in \text{PL}_+ I$ is a basic $F$-obstruction and $J := \text{supp}(f, g)$, then $J$ is a resolvable orbital of $(f, g)$. When combined with Proposition 2.1, this readily yields many copies of $F$ inside $(f, g)$, establishing Theorem 1.8 as a byproduct. The first step is the following lemma.

**Lemma 4.1.** If $(f, g)$ is an $F$-obstruction, then $f$ and $g$ do not commute.

**Proof.** Let $s$ witness that $(f, g)$ is an $F$-obstruction and let $J$ be the orbital of $(f, g)$ such that $s \in J$. If $f|J$ and $g|J$ commute, then by [5] there must be $h$ such that

$$f|J = hp \quad \text{and} \quad g|J = hg$$

for integers $p$ and $q$. This implies that the rotation number of $f$ modulo $g$ at $s$ is $p/q \in \mathbb{Q}$, which is a contradiction.

For the duration of this section, fix a basic $F$-obstruction $(f, g)$ and fix $s \in I$ which witnesses this. Specifically, set $C := [s, sg)$ and let $\gamma: C \to C$ be defined by

$$x\gamma := \begin{cases} 
xf & \text{if } xf < sg, \\
xfg^{-1} & \text{if } sg \leq xf.
\end{cases}$$
Notice that if \( s \leq x < sg \) and \( sg \leq xf \), then \( s \leq xfg^{-1} < sg \); this last inequality holds since \( xf < sgf = sfg \leq sg^2 \) by our hypothesis. Define a metric \( d \) on \( C \) by

\[
d(x, y) := \min(y - x, sg - y + x - s)
\]

whenever \( x < y \).

With this metric, \( C \) is homeomorphic to a circle and \( \gamma \) is an orientation preserving homeomorphism of \( C \). Our hypothesis is that the rotation number of \( \gamma \) is irrational. Notice that this implies that \( sf \neq sg \) (otherwise this would give a rotation number of 0) and hence \( sf < sg \).

Since \( \gamma \) is piecewise linear, Herman’s variation of Denjoy’s theorem [13] (see also [16]) implies the orbits of \( \gamma \) are dense and moreover that \( \gamma = \alpha^{-1} \theta \alpha \) for some irrational rotation \( \theta \) of \( C \) and some homeomorphism \( \alpha \) of \( C \). Since \( \alpha \) is uniformly continuous, \( \theta \) is an isometry, and \( \gamma^n = \alpha^{-1} \theta^n \alpha \), we can witness the uniform continuity of \( \gamma^n \) independently of \( n \); for every \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for all \( x, y \in C \) and \( n \in \mathbb{Z} \) if \( d(x, y) < \delta \), then \( d(x\gamma^n, y\gamma^n) < \varepsilon \).

Noting that this assertion remains unchanged if we swap the roles of \((x, y)\) and \((xy^n, y\gamma^n)\), we will sometimes employ the contrapositive of this implication: if \( d(x, y) \geq \varepsilon \) and \( n \in \mathbb{Z} \), then \( d(x\gamma^n, y\gamma^n) \geq \delta \). Notice that \( d(x, y) \leq |x - y| \) and \( d(x, y) = |x - y| \) if \( |x - y| \leq (sg - s)/2 \).

Since \( f \neq g \), \( sf \neq sg \), \( g^{-1}f^{-1} \in \text{PL}_+I \) and \( sf = sgf \), there are \( t > s \) and \( c > 0 \) such that \( xfg^{-1}f^{-1} = cx + (1 - c)s \) whenever \( s \leq x \leq t \). If \( c \leq 1 \), then \( xfg \leq xgf \) for all \( x \in [s, t] \) and if \( c \geq 1 \), then \( xfg \geq xgf \) for all \( x \in [s, t] \).

**Lemma 4.2.** There is \( \delta > 0 \) such that for all \( n \geq 0 \) and all \( x < y \) in \( C \) with \( |x - y| < \delta \):

- if \( c \leq 1 \) and \( x\gamma^n < y\gamma^n \), then there is \( h \in \{f, g\} \) such that \( xh = x\gamma^n < y\gamma^n \leq yh \);
- if \( c \geq 1 \) and \( x\gamma^{-n} < y\gamma^{-n} \), then there is \( h \in \{f, g\} \) such that \( xh = x\gamma^{-n} < y\gamma^{-n} \leq yh \).

**Proof.** First observe that \( sf \neq sf \neq gf^{-1} = sgf \neq sf \) and hence if \( s \leq x^* < sf \), then \( x^*g^{-1} \neq sf \neq gf^{-1} = sg \).

**Claim 4.3.** There is \( \delta > 0 \) such that for all \( s \leq x < y < sg \) with \( |x - y| < \delta \) and for all \( n \in \mathbb{Z} \):

- if \( x\gamma^n < y\gamma^n \), then \( d(x\gamma^n, y\gamma^n) = x\gamma^n - y\gamma^n \);
- if \( x\gamma^n > y\gamma^n \), then \( d(x\gamma^n, y\gamma^n) = sg - x\gamma^n + y\gamma^n - s \).

**Proof.** Let \( \delta > 0 \) be such that \( \delta < (sg - s)/6 \) and for all \( s \leq x, y < sg \) and \( n \in \mathbb{Z} \) if \( d(x, y) < \delta \), then \( d(x\gamma^n, y\gamma^n) < (sg - s)/6 \). This implies that whenever \( s \leq x, y < sg \) and \( n \in \mathbb{Z} \) if \( d(x, y) \geq (sg - s)/6 \), then \( d(x\gamma^n, y\gamma^n) \geq \delta \).

Now suppose that \( s \leq x < y < sg \) are given with \( |x - y| < \delta \), and let \( z \) be the midpoint of the longest arc of \( C \) connecting \( x\gamma^n \) and \( y\gamma^n \). Since \( d(x, y) < \delta \), \( d(x\gamma^n, y\gamma^n) < (sg - s)/6 \) and therefore \( \min(d(x\gamma^n, z), d(z, y\gamma^n)) \geq (sg - s)/6 \). It follows that

\[
\min(d(x, z\gamma^{-n}), d(z\gamma^{-n}, y)) \geq \delta.
\]
and therefore either $s \leq zy^{-n} < x$ or $y < zy^{-n} < sg$—i.e., $zy^{-n}$ is not between $x$ and $y$ in the cyclic order on $C$. Since $y$ preserves the cyclic order, $z$ is not between $xy^n$ and $yy^n$ in the cyclic order.

Suppose that $n \in \mathbb{Z}$ and $xy^n < yy^n$. Then either $s \leq z < xy^n$ or $yy^n < z < sg$. In the former case $xy^n - z \geq d(xy^n, z) \geq (sg - s)/6$ and in the latter case $z - yy^n \geq d(yy^n, z) \geq (sg - s)/6$. In either case

$$xy^n - s + sg - yy^n \geq (sg - s)/6 > d(xy^n, yy^n),$$

and hence $d(xy^n, yy^n) = yy^n - xy^n$.

On the other hand, if $n \in \mathbb{Z}$ is such that $xy^n > yy^n$, then $yy^n < z < xy^n$. This implies $(sg - s)/6 \leq xy^n - z \leq xy^n - yy^n$, and therefore $d(xy^n, yy^n) = sg - xy^n + yy^n - s$.

Let $\epsilon > 0$ be such that $\epsilon < sg - sf$ and if $|x^* - y^*| < \epsilon$ and $x^* f < sg \leq y^* f$, then $y^* fg^{-1} < t$. Find $\delta > 0$ satisfying the conclusion of Claim 4.3 and such that additionally if $d(x^*, y^*) < \delta$, then $d(x^*, y^*) < \epsilon$ for all $n \in \mathbb{Z}$.

We will now verify the conclusion of the lemma by induction on $n \geq 0$ under the assumption $c \leq 1$; the case $c \geq 1$ is handled by an analogous computation. If $n = 0$, then we can take $h$ to be the identity and there is nothing to show. Now suppose that $n > 0$, $x < y$ and $xy^n < yy^n$. If $sf \leq xy^n$, then $xy^n = xy^{n-1} f$ and $yy^n = yy^{n-1} f$. By our induction hypothesis, there is $h_0 \in \langle f, g \rangle$ such that $xy^{n-1} = xh_0$ and $yy^{n-1} \leq yh_0$. Since $f$ is order preserving, $yy^{n-1} f \leq yh_0 f$ and since $xy^{n} = xh_0 f$, $h := h_0 f$ satisfies the conclusion of the lemma. Similarly, if $yy^n < sf$, then $xy^n = xy^{n-1} fg^{-1}$ and $yy^n = yy^{n-1} fg^{-1}$ and we can apply our induction hypothesis to find $h_0 \in \langle f, g \rangle$ such that $xh_0 = xy^{n-1}$ and $yy^{n-1} \leq yh_0$. It follows that $h := h_0 fg^{-1}$ satisfies

$$xh = xh_0 fg^{-1} = xy^n < yy^n = yy^{n-1} fg^{-1} \leq yh_0 fg^{-1} = yh.$$

Finally, suppose that $xy^n < sf \leq yy^n$. By choice of $\delta$ and its property asserted in Claim 4.3, this implies that $d(xy^n, yy^n) = yy^n - xy^n$. It follows that $xy^n = xy^{n-1} f g^{-1}$ and $yy^n = yy^{n-1} f$. Observe that $xy^{n-1} > yy^{n-1}$, hence $n > 1$ and $d(xy^{n-1}, yy^{n-1}) = sg - xy^{n-1} + yy^{n-1} - s$. Since $d(xy^{n-1}, yy^{n-1}) < \epsilon$, it follows that $sf < sg - \epsilon < xy^{n-1}$. Thus $xy^n = xy^{n-2} f^2 g^{-1}$ and $yy^n = yy^{n-2} fg^{-1} f$. Observe that $xy^{n-2} < sg f^{-1} < yy^{n-2}$. By the induction hypothesis, there is $h_0 \in \langle f, g \rangle$ such that $xh_0 = xy^{n-2}$ and $yy^{n-2} \leq yh_0$. Define $h = h_0 f^2 g^{-1}$. Since $d(xy^{n-2}, yy^{n-2}) < \epsilon$ by our choice of $\delta$ and since $xy^{n-2} f < sg \leq yy^{n-2} f$, it follows from our choice of $\epsilon$ that $s \leq yy^{n-2} fg^{-1} < t$. Therefore,

$$yy^{n-2} fg^{-1} f g \leq yy^{n-2} fg^{-1} g f = yy^{n-2} f^2.$$

Acting on the right by $g^{-1}$ yields that

$$yy^n = yy^{n-2} fg^{-1} f \leq yy^{n-2} f^2 g^{-1} \leq yh_0 f^2 g^{-1} = yh,$$

and hence $h$ satisfies the conclusion of the lemma.
Proposition 4.4. Suppose that \((f, g)\) is a basic \(F\)-obstruction. There are dense sets \(A, B \subseteq J := \text{supt}(f, g)\) such that if \(a \in A\) and \(b \in B\) with \(a < b\), then there is \(h \in \langle f, g \rangle\) such that \(\text{supt}(h) = (a, b)\). In particular, the support of \(\langle f, g \rangle\) is a resolvable orbital of \(\langle f, g \rangle\).

Proof. We will first show that there is \(A_0 \subseteq [s, sg]\) which is dense in \([s, sg]\) such that if \(a \in A_0\), then \((a, sg)\) is an initial part of the support of some element of \(\langle f, g \rangle\). By Lemma 4.1, the commutator \([f, g]\) is not the identity and by Lemma 2.3, the infimum of its support is in \(J\). Let \(h_0 \in \langle f, g \rangle\) be such that \(p := \inf \text{supt}([f, g]^{h_0})\) is in \((s, sg)\). Such \(h_0\) exists since every \(\langle f, g \rangle\)-orbit of a point in \(J\) intersects \([s, sg]\). Let \(q > p\) be such that \((p, q)\) is an initial part of the support of \([f, g]^{h_0}\). Let \(\delta > 0\) satisfy the conclusions of Lemma 4.2 and Claim 4.3 and moreover satisfy for all \(x < y\) in \(C\):

- if \(d(x, y) \geq q - p\), then for all \(n\), \(d(xy^n, yy^n) \geq \delta\);
- if \(d(x, y) < \delta\) and \(xy^n > yy^n\), then \(xy^{n-1} < yy^{n-1}\).

There are now two cases depending on whether \(c \leq 1\).

If \(c \leq 1\), then define \(A_0 := \{p y^n \mid n \geq 0\}\). By Herman’s variation of Denjoy’s theorem [13] (see also [16]), \(A_0\) is dense in \([s, sg]\).

Claim 4.5. If \(a \in A_0\), then either \((a, a + \delta)\) or \((a, sg)\) is an initial part of the support of some element of \(\langle f, g \rangle\).

Proof. If \(a \in A_0\), let \(n \geq 0\) be such that \(a = p y^n\). If \(a < q y^n\), then by choice of \(\delta\) there is \(h \in \langle f, g \rangle\) such that

\[a = ph = p y^n < a + \delta \leq q y^n \leq qh.\]

Thus, \((a, a + \delta)\) is an initial segment of the support of \([f, g]^{h_0 h}\). If \(p y^n > q y^n\), then by choice of \(\delta\) we have that \(af^{-1} = p y^{n-1} < q y^{n-1}\) and there is \(h\) such that \(p y^{n-1} = ph\) and \(q y^{n-1} \leq qh\). It follows that \((af^{-1}, q y^{n-1}) \subseteq (ph, qh)\) is an initial part of the support of \([f, g]^{h_0 h}\) and hence \((a, sg)\) is an initial part of the support of \([f, g]^{h_0 h f}\).  

Now suppose that \(a \in A_0\) and use the density of \(A_0\) to select a sequence \(a_0 = a < a_1 < \cdots < a_k = sg\) such that if \(i < k\) then \(a_i \in A_0\) and \(a_{i+2} - a_{i} < \delta\) if \(i < k - 1\). For each \(i < k\), let \(h_i \in \langle f, g \rangle\) be such that \((a_i, a_{i+1})\) is an initial part of \(\text{supt}(h_i)\) and \(a_{i+2} \leq a_{i+1} h_i\) if \(0 \leq i \leq k - 2\). If \(h := \prod_{i<k} h_i\), then \((a, a_1) \subseteq \text{supt}(h)\) and \(sg \leq a_1 h\). It follows \(h\) has \((a, sg)\) as an initial part of its support.

If \(tfg \geq tgf\), then we define \(A_0 := \{p y^n \mid n \leq 0\}\) and an analogous argument gives the desired conclusion. Next, using a similar argument construct an analogous dense \(B_0 \subseteq [s, sg]\) such that if \(b \in B_0\), then there is an element of \(\langle f, g \rangle\) whose support has \([s, b]\) as a final segment. If \(a \in A_0\) and \(b \in B_0\) with \(a < b\), then let \(h_0\) and \(h_1\) be such that \((a, sg)\) is an initial segment of the support of \(h_0\) and \([s, b]\) is a final segment of the support of \(h_1\). Furthermore, select \(h_0\) and \(h_1\) such that \(a < bh_0 < ah_1 < b\) and observe that

\[[h_0, h_1] = h_0^{-1} h_1^{-1} h_0 h_1 = h_0^{-1} \cdot (h_0)^{h_1} = (h_1^{-1})^{h_0} \cdot h_1.\]
Since to the left of \( ah_1 \), the product \( h_0^{-1} \cdot (h_0)^{h_1} \) acts as \( h_0^{-1} \) followed by a function which is the identity to the left of \( ah_1 \), \([h_0, h_1]\) is increasing on \((a, ah_1]\) and the identity to the left of \( a \). Similarly, to the right of \( bh_0 < ah_1 \), \([h_0, h_1]\) is \( (h_1^{-1})^0 \cdot h_1 \) acts as \( h_1 \) and so is increasing on \([bh_0, b)\) and is the identity to the right of \( b \). Hence \( \text{supt}(\{h_0, h_1\}) = (a, b) \) and \([h_0, h_1]\) is increasing on \((a, b)\).

Finally, define \( A := A_0(f, g) \) and \( B := B_0(f, g) \) and observe that \( A \) and \( B \) are both dense in \( J \). Let \((x, y) \subseteq J\) be a maximal open interval containing \((s, sg)\) such that if \( a \in A \cap (x, y) \) and \( b \in B \cap (x, y) \) with \( a < b \), then there is an element of \( \langle f, g \rangle \) with support \((a, b)\). It suffices to show that \((x, y) = J\). Suppose for contradiction that this is not true—then either \( x \) or \( y \) are in \( J \).

If \( x \in J \), let \( h \in \{ f^{\pm 1}, g^{\pm 1} \} \) be such that \( xh \in (x, y) \); such \( h \) exists since \((s, sg) \subseteq (x, y)\). Notice that \( x' := xh^{-1} < x \). Let \( x' < a < b < y \) with \( a \in A \) and \( b \in B \). It suffices to show that \((a, b)\) is the support of an element of \( \langle f, g \rangle \) as this will contradict the maximality of \((x, y)\). If \( x < a \), then \((a, b)\) is the support of an element of \( \langle f, g \rangle \) by our choice of \((x, y)\). Similarly, if \( x' < a < b \leq x \), then \( x < ah < bh < y \) and there is \( h_0 \in \{ f, g \} \) with support \((ah, bh)\). It follows that \( h_0^{-1} \) has support \((a, b)\). If \( x' < a \leq x < b \), then \( x < ah \leq xh \). Let \( b' \in B \) be such that \( xh < b' < \min(bh, y) \). Let \( h_0 \in \langle f, g \rangle \) be a positive bump with support \((ah, b')\), noting that \( h_0^{-1} \) has support \((a, b'h^{-1})\). Let \( a' \in A \) be such that \( a' < b'h^{-1} < b \) and let \( h_1 \) be a positive bump with support \((a', b)\). Now \( h_0^{-1} \cdot h_1 \) is a positive bump with support \((a, b)\). This gives the desired contradiction. The case \( y \in J \) is handled by an analogous argument.

Theorem 1.8 is an immediate consequence of Proposition 4.4 and Brin’s ubiquity theorem [3].

5. A dichotomy for subgroups of \( \text{PL}_+ I \)

In this section, we will prove Theorem 1.2.

**Theorem 1.2.** Suppose that \( G \leq \text{PL}_+ I \) and \( J \) is a resolvable orbital of \( G \). If \( K_i \ (i < n) \) is a sequence of orbitals of \( G \), then exactly one of the following is true:

1. There is \( g \in G \) whose support intersects \( J \) but is disjoint from \( K_i \) for all \( i < n \).
2. There is \( i < n \) and a monotone surjection \( \psi: K_i \to J \) which is \( G \)-equivariant.

This theorem will be proved through a series of lemmas. The first gives a criterion for the existence of an equivariant surjection between orbitals of a subgroup of \( \text{Homeo}_+ I \).

**Lemma 5.1.** Suppose that \( J \) is a resolvable orbital of \( G \leq \text{Homeo}_+ I \) and \( K \) is an orbital of \( G \). If there are nonempty open intervals \( U \) and \( V \) such that

1. \( \bar{U} \subseteq J \) and \( \bar{V} \subseteq K \), and
2. for all \( g \in G \), \( Ug \cap U \neq \emptyset \) if and only if \( Vg \cap V \neq \emptyset \),

then there is a \( G \)-equivariant surjection \( \psi: K \to J \) which is monotone.
Proof. Define
\[ V^* := \cup \{Vh \mid h \in G \text{ and } Uh \subseteq U \} \]
and observe that \( V^* \) is an open interval containing \( V \).

Claim 5.2. For all \( g \in G \):

1. \( Ug \cap U \neq \emptyset \) if and only if \( V^*g \cap V^* \neq \emptyset \);
2. \( V^* \subseteq K \);
3. \( Ug \subseteq U \) if and only if \( V^*g \subseteq V^* \);
4. if \( \overline{Ug} \subseteq U \), then \( V^*g \subseteq V^* \).

Proof. Let \( g \in G \). If \( U \cap Ug \neq \emptyset \), then \( \emptyset \neq V \cap Vg \subseteq V^* \cap V^*g \). Next, suppose that \( x \in V^* \cap V^*g \) for some \( g \) and let \( h_0 \) and \( h_1 \) be such that \( Uh_0 \cup Uh_1 \subseteq U \) and \( x \in Vh_0 \cap Vh_1g \). It follows that \( V \cap Vh_1g \neq \emptyset \), which implies \( U \cap Uh_1g \neq \emptyset \) which in turn implies \( Uh_0 \cap Uh_1g \subseteq U \cap Ug \neq \emptyset \). This establishes (1).

Observe that if \( V^* \) contains an endpoint of \( K \), then for any \( g \in G \), \( V^*g \cap V^* \neq \emptyset \). On the other hand, since \( J \) is a resolvable orbital of \( G \) and \( \overline{U} \subseteq J \), there is \( g \in G \) such that \( Ug \cap U = \emptyset \). Thus (2) follows from (1).

We will now prove (3). First suppose that \( Ug \subseteq U \) for some \( g \in G \). If \( y \in V^*g \), let \( h \) be such that \( Uh \subseteq U \) and \( y^{-1}g \in Vh \). Then \( Vhg \subseteq Ug \subseteq U \) and so \( y \in Vhg \subseteq V^* \). Suppose now \( Ug \) is not contained in \( U \). Since \( J \) is a resolvable orbital of \( G \), there is \( h \in G \) such that \( Uh \) intersects \( Ug \) but not \( U \). It follows from (1) that \( V^*h \) intersects \( V^*g \) but not \( V^* \) and hence that \( V^*g \) is not contained in \( V^* \).

Finally, suppose that \( \overline{Ug} \subseteq U \) for some \( g \in G \). Since \( J \) is a resolvable orbital of \( G \), there are \( h_0, h_1 \in G \) such that:

- \( Uh_0 \cap Uh_1 = \emptyset \),
- \( Uh_0 \cup Uh_1 \subseteq U \),
- \( Uh_0 \) and \( Uh_1 \) intersect \( Ug \) but neither are contained in \( Ug \).

It follows from items (1) and (3) that these same conditions hold of \( V^* \) in place of \( U \). This implies that the endpoints of \( V^*g \) are contained in \( V^*h_0 \cup V^*h_1 \) and hence that \( V^*g \subseteq V^* \).

By replacing \( V \) with \( V^* \) if necessary, we can assume that \( V \) has the additional properties of \( V^* \) in Claim 5.2—these will be referred to as the revised hypotheses on \( U \) and \( V \).

Define \( \psi \) to consist of all pairs \( (x, y) \in K \times J \) such that for all \( g \in G \), whenever \( y \in Ug, x \in Vg \). To see that \( \psi \) is a (partial) function, suppose \( y_0 \neq y_1 \) are in \( J \). Since \( J \) is a resolvable orbital of \( G \) there are \( g_i, h_i \in G \) such that \( y_i \in Ug_i \subseteq Vg_i \subseteq Uh_i \) and \( Uh_0 \cap Uh_1 = \emptyset \). By our revised hypotheses, \( Vg_0 \cap Vg_1 = \emptyset \). Hence there is no \( x \) such that \( (x, y_0) \) and \( (x, y_1) \) are in \( \psi \). It also follows immediately from the definition that \( (x, y) \in \psi \) if and only if \( (xg, yg) \in \psi \) and hence \( \psi \) is \( G \)-equivariant.

Next, let us say that two intervals are linked if they intersect and neither is a subset of the other. Observe that for any \( g \in G \), \( U \) and \( Ug \) are linked if and only if \( V \) and \( Vg \) are.
linked. Also, a pair of intervals is linked if and only if each contains an endpoint of the other. If $A$ and $B$ are intervals, then we will write $A \prec_l B$ to mean that the pair $A, B$ is linked and the left endpoint of $A$ is less than the left endpoint of $B$. Clearly, if $A$ and $B$ is a linked pair of intervals, then exactly one of $A \prec_l B$ or $B \prec_l A$.

**Claim 5.3.** Either for all $g \in G$, $U \prec_l Ug$ implies $V \prec_l Vg$ or for all $g \in G$, $U \prec_l Ug$ implies $Vg \prec_l V$.

**Proof.** Observe that if $U \prec_l Ug$, then $Ug^{-1} \prec_l U$. Hence if the claim is false, there are $g_0, g_1 \in G$ such that $Ug_0 \prec_l U \prec_l Ug_1$ and yet $V \prec_l Vg_0, Vg_1$. Since $J$ is a resolvable orbital of $G$, there is $h \in G$ such that $\text{sup}(h) \cap J \subset U$ and $Ug_0 \cap Ug_1 h = \emptyset$. Since $Uh = U, Vh = V$ and because the right endpoint of $V$ is in $Vg_1$, it is also in $Vg_1 h$. In particular, this right endpoint is in both $Vg_0$ and $Vg_1 h$ while $Ug_0$ and $Ug_1 h$ are disjoint, contrary to our hypothesis.

**Claim 5.4.** The function $\psi$ is monotone.

**Proof.** Suppose that for all $g, h \in G$, $Ug \prec_l Uh$ implies $Vg \prec_l Vh$. Let $\psi(x_0) = y_0 < y_1 = \psi(x_1)$ and let $g_i \in G$ be such that $y_i \in Ug_i$ and $Ug_0 \cap Ug_1 = \emptyset$. By resolvability of $G$ on $J$, there is $h$ such that $Uh$ links both $Ug_0$ and $Ug_1$. In particular, $Ug_0 \prec_l Uh \prec_l Ug_1$, which implies $Vg_0 \prec_l Vh \prec_l Vg_1$. Since $x_i \in \overline{Vg_i}$ and $Vg_0 \cap Vg_1 \emptyset$, it follows that $x_0 \leq x_1$; since $\psi$ is a (partial) function we must have $x_0 < x_1$. Similarly, if $Ug \prec_l Uh$ always implies $Vh \prec_l Vg$, then $\psi$ is monotone decreasing.

**Claim 5.5.** The function $\psi$ is a surjection from $K$ onto $J$.

**Proof.** In order to see that $\psi$ is a surjection, let $y \in J$ be given. By assumption,

$$\mathcal{F} := \{ \overline{Vg} \mid g \in G \text{ and } y \in Ug \}$$

is a pairwise intersecting collection of intervals. By our revised hypotheses on $U$ and $V$ and by the resolvability of $G$ on $J$, $\mathcal{F}$ contains elements whose closure is contained in $K$. Thus $\psi^{-1}(y) = \cap \mathcal{F}$ is a nonempty interval.

Now suppose that $x \in K$. Since $\psi$ is a surjection, its domain is nonempty; since $\psi$ is $G$-equivariant, its domain $X$ contains elements both to the left and right of $x$. Set

$$x_0 := \sup\{s \in X \mid s \leq x\}, \quad x_1 := \inf\{s \in X \mid x \leq s\}.$$ 

Notice that since $\psi$ is a monotone surjection, $\psi(x_0) = \psi(x_1)$. Since we have shown $\psi$-preimages of points are intervals, $x \in \text{dom}(\psi)$.

Our strategy for proving Theorem 1.2 will now be to carefully select a subgroup $H \leq G$ whose support has nice properties which will allow us to define intervals $U$ and $V$ as in Lemma 5.1. The first step toward this goal is the following lemma. Recall that a group $G$ is perfect if $G' = G$. 
Lemma 5.6. Suppose that $J$ is a resolvable orbital of $G \leq \text{PL}_+ I$ and $K$ is an orbital of $G$. There is $H \leq G$ such that:

1. $H$ is perfect;
2. $H$ has finitely many orbitals;
3. $\text{suppt} H \cap J$ is a resolvable orbital of $H$ with closure contained in $J$;
4. $\text{suppt} H \cap K$ has closure contained in $K$.

This lemma will itself be proved through a series of lemmas.

Lemma 5.7. If $A, B \leq \text{Homeo}_+ I$ and for each $a \in A$ we have $\text{suppt} (a) \subseteq \text{suppt} B$, then $A' \leq ([A, B], [A, B])$.

Proof. Suppose that $a_0, a_1 \in A$ are arbitrary; it suffices to show that

$$[a_0, a_1] \in [[A, B], [A, B]].$$

Set

$$X := \text{suppt} \{a_0, a_1\}.$$

By Lemma 2.4, there is $b \in B$ such that both $Xb$ and $Xb^{-1}$ are disjoint from $X$. Since $(a_i^{-1})^b$ is supported on $Xb$ and $(a_0^{-1})^{-1}$ is supported on $Xb^{-1}$, these terms commute with each other and with $a_0$ and $a_1$, which are supported on $X$. Thus

$$[[b^{-1}, a_0], [b, a_1]] = [(a_0^{-1})^{-1} a_0, (a_1^{-1})^{-1} a_1] = [a_0, a_1]$$

is in $[[A, B], [A, B]]$ as desired. 

Lemma 5.8. Let $G$ be a subgroup of $\text{PL}_+ I$. If $\text{suppt} G' = \text{suppt} G$, then $\text{suppt} G'' = \text{suppt} G$ and $G''$ is perfect.

Proof. If $a \in G'$, then by Lemma 2.3

$$\text{suppt} (a) \subseteq \text{suppt} G = \text{suppt} G'.$$

Thus we can apply Lemma 5.7 to $A = B = G'$ and obtain that

$$G'' \leq ([G', G'], [G', G']) \leq G'''.$$

Thus $G'' = G'''$ and $G''$ is perfect. To see that $\text{suppt} G'' = \text{suppt} G$, suppose that $x \in \text{suppt} G$. By assumption, there is $g \in G'$ such that $x \in \text{suppt}(g)$. By Lemmas 2.3 and 2.4, there is $f \in G'$ such that the supports of $g$ and $g^f$ are disjoint. It follows that $x g = x[f, g]$ and therefore that $x$ is in the support of $[f, g] \in G''$. 

Lemma 5.9. If $G \leq \text{PL}_+ I$ is perfect and $H \leq G$ is a normal subgroup with $\text{suppt} H = \text{suppt} G$, then $G = H$. 
Proof. Since $H$ is a normal subgroup of $G$, $\langle[G, H], [G, H]\rangle \leq H$. By Lemma 2.3, $\text{supt}(g) \subseteq \text{supt}G = \text{supt}H$ for every $g \in G$. Applying Lemma 5.7 to $A = G$ and $B = H$, we obtain that $G = G' \leq \langle[G, H], [G, H]\rangle \leq H$. \hfill \blacksquare

Lemma 5.10. Let $H_0 \leq \text{PL}_+I$ and $J$ be an orbital of $H_0$ such that $H_0|_J$ is both perfect and the normal closure of a single element. Then there exists a perfect subgroup $H$ of $H_0$ with finitely many orbitals such that $H|_J = H_0|_J$.

Proof. Since $H_0|_J$ is perfect $H''_0|_J = H_0|_J$. Fix $h \in H''_0$ with the normal closure of $h|_J$ in $H_0|_J$ equal to $H_0|_J$. Let $H_1$ be the normal closure of $h$ in $H''_0$ and let $H_2$ be the normal closure of $h$ in $H_1$. Define $\mathcal{Z}$ to be the orbitals of $H_0$ that are also orbitals of $H_2$ and set $U := \cup \mathcal{Z}$. Define $\mathcal{V}$ to be the orbitals of $H_0$ which intersect $\text{supt}H_2$ but are not orbitals of $H_2$ and set $V := \cup \mathcal{V}$.

Claim 5.11. $\text{supt}H_2|_V$ is contained in $\text{supt}H_0$.

Proof. We will first argue that if $L$ is an orbital of $H_1|_V$, then the closure of $L$ is contained in the support of $H_0$. To see this, suppose $L$ is an orbital of $H_1|_V$. If $L$ were an orbital of $H_0$, then it would also be an orbital of $H_0$ since $H_1 \leq H'_0$. Hence Lemma 5.8 would imply that $H''_0|_L$ is perfect and Lemma 5.9 would imply $H''_0|_L = H_1|_L$. Since this in turn would imply $H_2|_L = H_1|_L = H''_0|_L$ and that $L$ is in $\mathcal{Z}$, this is impossible. Thus $L$ is not an orbital of $H_0$.

Let $g \in H_0$ be such that $Lg \not\subseteq L$. Since $Lg$ is contained in the support of $H_0$ and $L$ is an orbital of $H_1$, $Lg$ must be disjoint from $L$, and in particular the closure of $L$ is contained in $\text{supt}H_0$. Now let $X$ be the closure of the union of the orbitals of $H_1|_V$ which intersect $\text{supt}(h)$. We have shown that $X \subseteq \text{supt}H_0$. Since $X$ is $H_1$-invariant, $\text{supt}H_2|_V \subseteq X$.

By Claim 5.11, we can find $g \in H_0$ such that $V \cap \text{supt}H_2 \cap \text{supt}H_2^g = \emptyset$. Define $H := \langle[H_2^g, H_2]\rangle$, noting that $\text{supt}H \subseteq U$. By Lemma 5.8, $H''_0|_U$ is perfect. By Lemma 5.9, $H_1|_U = H''_0|_U$ and therefore $H_2|_U = H''_0|_U$. Since $H''_0$ is a normal subgroup of $H_0$, we have $H''_0^g = H''_0$. Putting this all together, we obtain

$$H = H|_U = \langle[H_2^g|_U, H_2|_U]\rangle = \langle[H''_0^g|_U, H''_0|_U]\rangle$$

$$= \langle[H''_0|_U, H''_0|_U]\rangle = H''_0|_U = H''_0.$$

Thus $H = H''_0|_U$ is perfect and has finitely many components of support. \hfill \blacksquare

Lemma 5.12. Suppose that $A, B \leq \text{PL}_+I$ are perfect groups and $N$ is the normal closure of $B$ in $\langle A \cup B \rangle$. If $S$ denotes the union of the orbitals of $\langle A \cup B \rangle$ which are not orbitals of $N$, then $A|_S$ is contained in $\langle A \cup B \rangle$.

Proof. We will first show that for all $a \in A$ there exists $b \in N$ such that $ab|_S$ is in $\langle A \cup B \rangle$. Define $A_0$ to be the set of all $a \in A$ such that there exists $b \in N$ with $ab|_S$ is in $\langle A \cup B \rangle$. Since $A$ is perfect, it suffices to show that $[A, A] \subseteq A_0$ and that $A_0$ is a group.
Toward showing $[A, A] \subseteq A_0$, let $f, g \in A$, and let us set $X$ equal to the closure of $\text{sup}(f, g) \setminus S$. Since $A$ is perfect, Lemma 2.3 implies $X$ is contained in $\text{sup} N$. By Lemma 2.4, there is $h \in N$ such that $X h \cap X = \emptyset$. Since $f$ and $g^h$ have disjoint supports, $[f, g^h]$ agrees with the identity on $I \setminus S$. In particular, $[f, g^h]_S = [f, g^h]$ is in $(A \cup B)$. We can rewrite $[f, g^h]$ as $[f, g](h^{-1})f g^h (h^{-1})^g h$. Since $(h^{-1})^g f g (h^{-1})^g h$ is in $N$ we have shown that $[f, g]$ is in $A_0$.

It remains to show that $A_0$ is closed under composition and taking inverses. Let $a_0, a_1 \in A_0$ and fix $b_0, b_1 \in N$ with $a_0 b_0|_S$ and $a_1 b_1|_S$ in $(A \cup B)$. Since $a_0 a_1 b_0^a a_1 b_1|_S = (a_0 b_0)|_S (a_1 b_1)|_S$ is in $(A \cup B)$ and $b_0^a b_1$ is in $N$, it follows that $a_0 a_1 \in A_0$. Since $(a_0 b_0)^{-1}|_S = a_0^{-1}(b_0^{-1})^a|_S$ is in $(A \cup B)$ and $(b_0^{-1})^a^{-1}$ is in $N$, it follows that $a_0^{-1} \in A_0$. Since $a_0, a_1 \in A_0$ were arbitrary, $A_0$ is closed under multiplication and taking inverses and hence is a group. Thus $A_0 = A$.

Next we will show that $N|_S$ is contained in $(A \cup B)$. Notice that this is sufficient to complete the proof since if $a \in A$, then for some $b \in N$, $(ab)|_S$ is in $(A \cup B)$ and hence $a|_S = (ab)|_S (b^{-1})|_S$ is in $(A \cup B)$. Since $B$ is perfect, it is generated by $[B, B]$ and hence $N$ is the normal closure of $[B, B]$ in $(A \cup B)$. Thus it suffices to show that if $b_0, b_1 \in B$, then $[b_0, b_1]|_S$ is in $(A \cup B)$. Toward this end, let $b_0, b_1 \in B$ be arbitrary. Observe that any endpoint of a connected component $U$ of $S$ is a limit point of $U \setminus \text{sup} N$. Thus there is a set $X \subseteq S$ which is a finite union of intervals with endpoints in $S \setminus \text{sup} N$ such that $\text{sup}(b_0, b_1) \cap S \subseteq X$.

We now claim that $X$ is contained in the support of $A$. If there were $x \in X$ fixed by every element of $A$, then $x$ is in some component $L$ of the support of $B$. In this case, however, the union of the translates of $L$ by elements of $(A \cup B)$ is an orbital of both $N$ and $(A \cup B)$, contradicting that $L$ is contained in $S$. Thus it must be that $X$ is contained in the support of $A$.

By Lemma 2.4, there is $a \in A$ such that $X a \cap X = \emptyset$. Thus there is $g \in N$ such that $h := (ag)|_S$ is in $(A \cup B)$. We will be finished once we show that $[[h, b_0], b_1] = [b_0, b_1]|_S$. Define $Y := S \setminus X$ and $Z := I \setminus S$ and set $c := (b_0^{-1})^g h$. Observe that $X, Y$, and $Z$ are all invariant under $b_0, b_1$ and $c$. Furthermore, $[h, b_0] = cb_0$ agrees with $b_0$ on $X$ and the identity on $Z$. Since $b_1$ agrees with the identity on $Y$, we have that $[[h, b_0], b_1] = [cb_0, b_1]$ coincides with $[b_0, b_1]$ on $X$ and is the identity elsewhere. Since $[b_0, b_1]$ is the identity on $Y$, it follows that $[[h, b_0], b_1] = [b_0, b_1]|_S$.

Proof of Lemma 5.6. Let $s < t$ be in $J$ and set $U := (s, t)$. Define $H_0 := \{g \in G \mid \text{sup}(g) \cap J \subseteq U\}$. Since $J$ is a resolvable orbital of $G$, $H_0|_J$ is perfect by Lemma 5.8. It is easily checked that $U$ is a resolvable orbital of $H_0$ and hence Lemma 5.9 implies that $H_0|_J$ is the normal closure of a single element. By Lemma 5.10, there is $H \leq H_0$ such that $H$ is perfect, has finitely many orbitals, and $H|_J = H_0|_J$.

If the closure of $\text{sup}(H \cap K)$ is contained in $K$, then $H$ satisfies the conclusion of the lemma. If the closure of $\text{sup}(H \cap K)$ contains an endpoint of $K$, then let $g \in G$ be such that $U g \cap U = \emptyset$. Let $N$ be the normal closure of $H^g$ in $(H \cup H^g)$ and let $S$ be the union of the orbitals of $(H \cup H^g)$ which are not also orbitals of $N$. Observe that $U \subseteq S$
and that the closure of \( S \cap K \) does not contain the endpoints of \( K \). Applying Lemma 5.12 to \( A = H \) and \( B = H^g \), the projection \( H|_S \) is contained in \( \langle H \cup H^g \rangle \leq G \) and satisfies the conclusion of the lemma.

**Proof of Theorem 1.2.** The theorem is proved by induction on \( n \) with the bulk of the proof dedicated to the base case \( n = 1 \). Suppose that \( n = 1 \) and write \( K \) for \( K_0 \). Applying Lemma 5.6, fix a perfect subgroup \( H \leq G \) with finitely many orbitals such that:

- \( U := \text{supt } H \cap J \) is a nonempty interval with closure contained in \( J \);
- the closure of \( \text{supt } H \cap K \) is contained in \( K \);
- \( U \) is a resolvable orbital of \( H \);
- the number of orbitals of \( H \) is minimized.

If \( \text{supt } H \cap K = \emptyset \), then the first alternative of the theorem holds. Suppose now that this is not the case. Define \( V \) to be the leftmost component of \( \text{supt } H \cap K \). By Lemma 5.1, it suffices to show that for all \( g \in G \), \( Ug \cap U \neq \emptyset \) if and only if \( Vg \cap V \neq \emptyset \).

**Claim 5.13.** For all \( g \in G \), \( Ug \cap U \neq \emptyset \) implies \( Vg \cap V \neq \emptyset \).

**Proof.** Suppose first for contradiction that for some \( g \in G \), \( Ug \cap U \neq \emptyset \) but \( Vg \cap V = \emptyset \). By replacing \( g \) with \( g^{-1} \) if necessary, we can assume that \( V \) is disjoint from the support of \( H^g \). Let \( N \) denote the normal closure of \( H^g \) in \( \langle H \cup H^g \rangle \) and let \( S \) be the union of all orbitals of \( \langle H \cup H^g \rangle \) which are not orbitals of \( N \). Observe that since \( U \) is a resolvable orbital of \( H \) and \( U \cap Ug \neq \emptyset \), it follows that \( U \) is disjoint from \( S \). On the other hand, \( V \) is contained in \( S \). Thus applying Lemma 5.12 to \( A = H \) and \( B = H^g \) yields that \( H|_S \) is a perfect subgroup of \( G \). Consequently, if \( R = \text{supt } H \setminus S \), then \( H_0 := H|_R \) is also contained in \( G \). Since \( H_0 \) is also perfect, satisfies \( H_0|_U = H|_U \) and has fewer orbitals than \( H \), we have contradicted our choice of \( H \) to minimize the number of its orbitals.

**Claim 5.14.** For all \( g \in G \), \( Ug \cap U = \emptyset \) implies \( Vg \cap V = \emptyset \).

**Proof.** Suppose first for contradiction that for some \( g \in G \), \( Ug \cap U = \emptyset \) but \( Vg \cap V \neq \emptyset \). As in the previous claim, let \( N \) denote the normal closure of \( H^g \) in \( \langle H \cup H^g \rangle \) and let \( S \) be the union of all orbitals of \( \langle H \cup H^g \rangle \) which are not orbitals of \( N \). This time \( U \) is contained in \( S \) and \( V \) is disjoint from \( S \). Lemma 5.12 implies \( H_0 := H|_S \) is a perfect subgroup of \( G \). Since \( H_0 \) has fewer orbitals than \( H \), is perfect, and satisfies \( H_0|_J = H|_J \), we again contradict our choice of \( H \).

Now suppose that \( n \) is given and that the statement of the theorem is true for \( n \). Let \( K_i \) \( (i < n + 1) \) be orbitals of \( G \). We need to show that if (2) fails, then there is \( g \in G \) whose support intersects \( J \) but not \( K_i \) for any \( i < n + 1 \). By our inductive assumption, there is \( g_0 \in G \) whose support intersects \( J \) but is disjoint from \( K_i \) for \( i < n \). Additionally there is \( g_1 \in G \) whose support intersects \( J \) but is disjoint from \( K_n \). Since \( J \) is a resolvable orbital of \( G \), there is \( h \) such that \( g_0^h \) does not commute with \( g_1 \). It follows that \( g = [g_0^h, g_1] \) is as desired.
6. Consequences of the dichotomy theorem

In this section, we will give proofs of Theorems 1.1 and 1.7 using Theorem 1.2. We will also illustrate the utility of Theorem 1.2 by deriving some known results as corollaries.

Proof of Theorem 1.1. Suppose that \((f, g)\) is an \(F\)-obstruction and \(\phi: \langle f, g \rangle \rightarrow \text{PL}_+ I\) is an embedding. By rescaling and translating if necessary, we can assume that the supports of \(\langle f, g \rangle\) and \(\langle \phi(f), \phi(g) \rangle\) are disjoint. Let \(J\) be an orbital of \(\langle f, g \rangle\) such that for some \(s \in J\), the rotation number of \(f\) modulo \(g\) at \(s\) is irrational. Let \(K_i\) \((i < n)\) list the orbitals of \(\langle \phi(f), \phi(g) \rangle\). Observe that since \(\phi\) is an injection, the first alternative of Theorem 1.2 applied to \(G := \langle f\phi(f), g\phi(g) \rangle\) cannot hold. Therefore, there is \(i < n\) and a \(G\)-equivariant monotone surjection \(\psi: K_i \rightarrow J\); let \(t \in K_i\) be minimal such that \(\psi(t) = s\). It is easily checked that \(t\phi(f)\phi(g) = t\phi(g)\phi(f)\). Since rotation numbers are preserved by semiconjugacy, it follows that the rotation number of \(\phi(f)\) modulo \(\phi(g)\) at \(t\) is irrational and hence that \((\phi(f), \phi(g))\) is an \(F\)-obstruction.

We will now recall the statement and context of Rubin’s reconstruction theorem. Suppose that \(X\) is a locally compact Hausdorff space and \(G\) is a group of homeomorphisms of \(X\). The group \(G\)’s action on \(X\) is locally dense if \(X\) has no isolated points and whenever \(x \in U \subseteq X\) with \(U\) open,

\[
\{xg \mid (g \in G) \text{ and } (\text{supt}(g) \subseteq U)\}
\]

is somewhere dense. It is easily checked that if \(X \subseteq I\) is an interval, then this is equivalent to \(X\) being a resolvable orbital of \(G\). Rubin’s theorem asserts that if \(X\) and \(Y\) are locally compact and if \(G \leq \text{Homeo}X\) and \(H \leq \text{Homeo}Y\) are such that the actions of \(G\) and \(H\) on their underlying spaces are locally dense, then any isomorphism between \(G\) and \(H\) is induced by a unique homeomorphism of \(X\) and \(Y\) (this is Corollary 3.5 (c) of [18]).

We will now show how to derive Rubin’s theorem when \(G\) and \(H\) are subgroups of \(\text{PL}_+ I\). Notice that Theorem 1.7 is an immediate consequence of this result and Proposition 4.4.

Corollary 6.1 ([18]). Suppose that \(G, H \leq \text{PL}_+ I\) are nontrivial and each acts on its support in a locally dense manner. If \(\phi: G \rightarrow H\) is an isomorphism, then there is a unique homeomorphism \(\psi: \text{supt } G \rightarrow \text{supt } H\) such that for all \(x \in \text{supt } G\), \(\psi(xg) = \psi(x)\phi(g)\).

Remark 6.2. Both McCleary [17] and Bieri–Strebel [1] had previously proved similar reconstruction theorems for subgroups of \(\text{PL}_+ I\), although under different dynamical hypotheses.

Proof. First observe that since the action of \(G \leq \text{PL}_+ I\) on its support is locally dense, then the only \(G\)-equivariant maps between orbitals of \(G\) are the identity functions. This in particular implies that \(\psi\) is unique if it exists. To prove existence, suppose \(G, H \leq \text{PL}_+ I\) and \(\phi: G \rightarrow H\) are as in the statement of the corollary. By replacing \(G\) and \(H\) by rescaled translates if necessary, we can assume that the supports of \(G\) and \(H\) are disjoint.
Define $\Gamma := \{g\phi(g) \mid g \in G\}$ and let $J$ be an orbital of $G$. Observe that we are finished once we have shown that there is a $\Gamma$-equivariant homeomorphism between $\text{supt} G$ and $\text{supt} H$. Furthermore, it suffices to show that for each orbital $J$ of $G$, there is a unique orbital $K$ of $H$ for which there is a $\Gamma$-equivariant homeomorphism between $J$ and $K$. The statement with the roles of $G$ and $H$ reversed must also hold and $\psi$ is then obtained by pasting together these local homeomorphisms.

Fix $g_0 \in G$ such that $\text{supt}(g_0)$ is nonempty with closure contained in $J$. Let $K_i$ ($i < n$) list the orbitals of $H$ which intersect $\text{supt}(\phi(g_0))$—there are only finitely many such orbitals since $\phi(g_0) \in \text{PL}_+ I$. Since $J$ is a resolvable orbital of $G$, the only element of $G|_J$ which commutes with every conjugate of $g_0$ is the identity. Apply Theorem 1.2 to the group $\Gamma$ and observe that the first alternative cannot hold since if $g_J$ is not the identity, $g$ fails to commute with $g_0^h$ for some $h \in G$. Since the support of $\phi(g_0^h)$ is contained in $\bigcup_{i < n} K_i$, it must be that $\phi(g)|_{K_i}$ is nontrivial for some $i < n$. Thus there is $i < n$ and a $\Gamma$-equivariant surjection $\theta_i: K_i \to J$. Similarly, there is a $\Gamma$-equivariant monotone surjection $\hat{\theta}$ from some orbital $J'$ of $G$ to $K_i$. By the observation made at the start of the proof, $J' = J$ and $\hat{\theta} = \theta_i^{-1}$. In particular, $\theta: J \to K$ is the desired $\Gamma$-equivariant homeomorphism.

\begin{remark}
It should be noted that Theorem 1.2 is false if we replace $\text{PL}_+ I$ with $\text{Homeo}_+ I$. For example, since $F$ is orderable [10], there is $G \leq \text{Homeo}_+ I$ which is isomorphic to $F$ such that every nonidentity element of $G$ has only isolated fixed points; such $G$ cannot be semiconjugate to the standard copy of $F$. It would be interesting to know if there are broader contexts in which Theorem 1.2 holds.
\end{remark}

Next we will derive Brin’s ubiquity theorem from Theorem 1.2.

\begin{corollary}[\cite{3}]
Suppose $G \leq \text{PL}_+ I$ and $K$ is an orbital of $G$ such that some element of $G$ approaches one end of $K$ but not the other. Then $F$ embeds into $G$.
\end{corollary}

\begin{proof}
By replacing $G$ with a rescaled translate if necessary, we can assume that the support of $G$ is contained in $(1/2, 1)$. Let $a$ and $b$ be the generators for the rescaled standard model of $F$ with support $(0, 1/2)$. Let $K_i$ ($i < n$) list the orbitals of $G$ so that $K_0 = K$. The hypothesis combined with Lemma 2.4 readily yields a pair $f, g \in G$ such that $f|_K$ and $g|_K$ satisfy the same relations as $a$ and $b$.

Define $\Gamma := \langle af, bg \rangle$ and apply Theorem 1.2 to the group $\Gamma$, the distinguished orbital $J := (0, 1/2)$, and the orbitals $K_i$ ($i < n$). There is a subset $X \subseteq \{0, \ldots, n - 1\}$ and $\Gamma$-equivariant monotone surjections $\psi_i: K_i \to J$ for $i \in X$ and $h \in \Gamma$ such that if $i < n$ is not in $X$, then $h|_{K_i}$ is the identity; let $\psi: \bigcup_{i \in X} K_i \to J$ be the common extension of the $\psi_i$’s. Using that $J$ is a resolvable orbital of $\Gamma$ and arguing as in the proof of Proposition 4.4, there is $h_0$ in the normal closure of $h$ in $\Gamma$ such that $h_0|_J$ is a positive bump. Observe that the image of the support of $h_0$ under $\psi$ is the union of $(s_0, t_0) := \text{supt}(h_0) \cap J$ and a finite set $E$. Let $g$ be such that $E g \cap E = \emptyset$ and $s_0 < s_0 g < t_0 < t_0 g$. It is now easily checked that for some $m > 0$, $a := h_0^m$ and $b := (h_0^g)^{-m}$ are as in Proposition 2.1.
\end{proof}
Remark 6.5. While we used Brin’s ubiquity theorem to prove Theorem 1.8, it is not required for the proof of Theorem 1.2. Even so, the purpose of deriving Corollaries 6.1 and 6.4 from Theorem 1.2 is not to give new proofs of these facts but rather to demonstrate the ways in which Theorem 1.2 can be used and the utility that resides in it.

7. Some examples

In this section, we will prove Corollaries 1.4–1.6. Recall that Cleary’s group $F_\tau$ is the subgroup of $\text{PL}_+ I$ consisting of those elements whose singularities are in $\mathbb{Z}[\tau]$ and whose slopes are powers of $\tau$, where $\tau$ is the solution to $\tau^2 = \tau + 1$ with $\tau > 1$ [11]. If $1 < p < q$ are relatively prime integers, then Stein’s group $F_{p,q}$ is the subgroup of $\text{PL}_+ I$ consisting of those elements whose singularities are in $\mathbb{Z}[\frac{1}{p}, \frac{1}{q}]$ and whose slopes are the product of a power of $p$ and a power of $q$ [19].

The following observations will allow us to show that Cleary’s and Stein’s groups contain $F_\tau$-obstructions.

Observation 7.1. Suppose that $f, g \in \text{Homeo}_+ I$ and for some $s$ and $0 < \xi < \eta$, $xf = x + \xi$ and $xg = x + \eta$ whenever $s \leq x \leq s + \eta$. Then the rotation number of $f$ modulo $g$ at $s$ is defined and equals $\xi / \eta$.

Observation 7.2. Suppose that $f, g \in \text{Homeo}_+ I$ and for some $s_0 < s_1$ and $1 < a < b$, $xf = a(x - s_0) + s_0$ and $xg = b(x - s_0) + s_0$ whenever $s_0 \leq x \leq s_1$. If $s \in (s_0, s_1)$ is such that $sg \leq s_1$, then the rotation number of $f$ modulo $g$ at $s$ is defined and equals $\log_b(a)$.

The second observation is a consequence of the first by conjugating $f$ and $g$ by $\log_b \frac{x - s_0}{s}$. If $1 < p < q$ are relatively prime integers, then $\log_q(p)$ is irrational. Since $F_{p,q}$ contains elements which have slope $p$ and $q$ near 0, Corollary 1.5 follows from Observation 7.2 and Theorem 1.1.

We now turn to Cleary’s group $F_\tau$. Define $f, g \in F_\tau$ by

$$xf := \begin{cases} \tau x & \text{if } 0 \leq x \leq \tau^{-3}, \\ x + \tau^{-2} - \tau^{-3} & \text{if } \tau^{-3} \leq x \leq \tau^{-1}, \\ x\tau^{-1} + \tau^{-2} & \text{if } \tau^{-1} \leq x \leq 1, \end{cases}$$

$$xg := \begin{cases} \tau^2 x & \text{if } 0 \leq x \leq \tau^{-4}, \\ x + \tau^{-2} - \tau^{-4} & \text{if } \tau^{-4} \leq x \leq \tau^{-1}, \\ x\tau^{-2} + \tau^{-1} & \text{if } \tau^{-1} \leq x \leq 1. \end{cases}$$

If we set $s := \tau^{-3}$, then

$$sf = \tau^{-2} < \tau^{-2} + \tau^{-3} - \tau^{-4} = \tau^{-1} - \tau^{-4} = sg < \tau^{-1}.$$
It follows from Observation 7.1 that the rotation number of $f$ modulo $g$ at $s$ is defined and equals
\[
\frac{\tau - 3 - \tau - 4}{\tau - 2 - \tau - 4} = \frac{\tau^2 - \tau}{\tau^2 - 1} = \tau^{-1}.
\]
Since $\tau^{-1}$ is irrational, $(f, g)$ is an $F$-obstruction.

Finally, we wish to show that the group generated by $F \cup F^{t \mapsto t - \xi}$ contains an $F$-obstruction whenever $0 < \xi < 1$. Recall that $F$ is the subgroup of $\text{PL}_+ I$ consisting of those elements whose singularities occur at dyadic rationals and whose slopes are powers of 2. Let $\xi$ be given and let $n$ be such that $2^{-n} < \xi < 1 - 2^{-n+2}$. Observe that the following functions $f$, $g_0$, and $g_1$ are in either $F$ or $F^{t \mapsto t - \xi}$:

\[
xf := \begin{cases}
2x + \xi & \text{if } -\xi \leq x \leq 2^{-n} - \xi, \\
x + 2^{-n} & \text{if } 2^{-n} - \xi \leq x \leq 1 - 2^{-n+1} - \xi, \\
2^{-1}(x + 1 - \xi) & \text{if } 1 - 2^{-n+1} - \xi \leq x \leq 1 - \xi, \\
x & \text{otherwise},
\end{cases}
\]

\[
xg_0 := \begin{cases}
2x & \text{if } 0 \leq x \leq 2^{-1} - 2^{-n-1}, \\
x + 2^{-1} - 2^{-n-1} & \text{if } 2^{-1} - 2^{-n-1} \leq x \leq 2^{-1}, \\
2^{-n}x + 1 - 2^{-n} & \text{if } 2^{-1} \leq x \leq 1, \\
x & \text{otherwise},
\end{cases}
\]

\[
xg_1 := \begin{cases}
2^n(x + \xi) - \xi & \text{if } -\xi \leq x \leq 2^{-n-1} - \xi, \\
x + 2^{-1} - 2^{-n-1} & \text{if } 2^{-n-1} - \xi \leq x \leq 2^{-n} - \xi, \\
2^{-1}(x + \xi) + 2^{-1} - \xi & \text{if } 2^{-n} - \xi \leq x \leq 1 - \xi, \\
x & \text{otherwise}.
\end{cases}
\]

Set $g := g_0g_1$. Observe that by our choice of $n$, if $0 \leq x \leq (1 - \xi)/2$, then

\[
xf = x + 2^{-n}, \quad xg = x + \frac{1 - \xi}{2}.
\]

Since $0f = 2^{-n} < (1 - \xi)/2 = 0g$, it follows from Observation 7.2 that the rotation number of $f$ modulo $g$ at 0 is defined and equals $2^{-n+1}/(1 - \xi)$, which is irrational. Hence $(f, g)$ is an $F$-obstruction.

**Remark 7.3.** We do not know if $\bigcup_{q \in \mathbb{Q}} F^{t \mapsto t - q}$ embeds into $F$. We conjecture it does not. Note that if

\[
G \leq \left\{ \bigcup_{q \in \mathbb{Q}} F^{t \mapsto t - q} \right\}
\]

is finitely generated, then $G$ is conjugate to a subgroup of $F$. Specifically, if $X \subseteq \frac{1}{n} \mathbb{Z}$, then $\bigcup_{q \in X} F^{t \mapsto t - q}$ is conjugate to a subgroup of the real line model of $F$ via the map $t \mapsto nt$. 

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