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MOD $p$ HOMOLOGY OF UNORDERED CONFIGURATION SPACES OF SURFACES

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ABSTRACT. We provide a short proof that the dimensions of the mod $p$ homology groups of the unordered configuration space $B_k(T)$ of $k$ points in a torus are the same as its Betti numbers for $p > 2$ and $k \leq p$. Hence the integral homology has no $p$-power torsion. The same argument works for the punctured genus $g$ surface with $g > 0$, thereby recovering a result of Brantner-Hahn-Knudsen via Lubin-Tate theory.

1. INTRODUCTION

The unordered configuration space of $k$ points in a manifold $M$ is the orbit space

$$B_k(M) := \text{Conf}_k(M)\Sigma_k = \{(x_1, \ldots, x_k) \in M^\times k, x_i \neq x_j \text{ for } i \neq j\}/\Sigma_k.$$ 

The main new result of this paper concerns the odd primary homology of $B_k(T)$, where $T$ is a closed torus.

**Theorem 1.1.** Let $p$ be an odd prime. The dimension of $H_i(B_k(T); \mathbb{F}_p)$ over $\mathbb{F}_p$ is given by the $i$th Betti number $\beta_i(B_k(T))$ for all $i$ and $k \leq p$. Hence the integral homology of $B_k(T)$ has no $p$-power torsion for $k \leq p$.

The study of unordered configuration spaces dates back to as early as Segal [Seg73] and McDuff [McD75]. The rational homology groups of these objects are relatively well understood in cases of interests via classical methods, see for instance [BC88][BCT89][Kri94][Tot96][FT00]. In contrast, the odd primary homology groups of unordered configuration spaces have remained mostly intractable. Classically, the only known cases are the following: when $M = \mathbb{R}^n$ for $1 \leq n \leq \infty$ where $\bigoplus_{k \geq 0} H_*(B_k(M); \mathbb{F}_p)$ is the mod $p$ homology of the free $E_n$-algebra on $\Sigma$ [May72] [CLM76][BMMS88], and when the dimension of $M$ is odd where $\bigoplus_{k \geq 0} H_*(B_k(M); \mathbb{F}_p)$ depends only on the $\mathbb{F}_p$-module $H_*(M; \mathbb{F}_p)$ [BCT89][ML88][BCM93].

More recently, advances in the computation of the homology of unordered configuration spaces are made possible by a result of Knudsen [Knu18]. For any manifold $M$ and spectrum $X$, we can consider the *labeled* configuration spectrum

$$B_k(M; X) := \Sigma\text{Conf}_k(M) \otimes X^\otimes k.$$ 

In particular $\Sigma\text{Conf}_k(M) = B_k(M; \Sigma)$. Denote by $s.\mathcal{L}^p$ the monad associated to the free spectral Lie algebra functor $\text{Free}^{s.\mathcal{L}^p}$. The \(\infty\)-category of spectral Lie algebras is cotensored in Spaces, and $(-)^{M^+}$ denotes the cotensor with the one-point compactification of $M$ in this category. Using the machinery of factorization homology, Knudsen established the following equivalence.

**Theorem 1.2.** [Knu18, Section 3.4] Let $M$ be a parallelizable $n$-manifold and $X$ a spectrum of weight one. Then there is an equivalence of weighted spectra

$$\bigoplus_{k \geq 1} B_k(M; X) \simeq |\text{Bar}_s(id, s.\mathcal{L}^p, \text{Free}^{s.\mathcal{L}^p}(\Sigma^\infty X)^{M^+})|.$$ 

The left hand side is weighted by the index $k$ and right hand side induced by the weight on $X$.

Using the bar spectral sequence with rational coefficients associated to the right hand side of (1), Knudsen [Knu17] provided a general formula for the Betti numbers of unordered configuration spaces. Building on Knudsen’s work, Drummond-Cole and Knudsen [DCK17] produced explicit formulae of the Betti numbers of unordered configuration spaces of surfaces. In [BHK19], Brantner, Hahn, and Knudsen studied Knudsen’s spectral sequence with coefficients in Morava $E$-theory at an odd prime. They computed the weight $p$ part
of the labeled configuration spaces in \( \mathbb{R}^n \) and punctured genus \( g \) surfaces \( \Sigma_{g,1} \) for \( g \geq 1 \) with coefficient in a sphere. By letting the height go to infinity, they deduced that:

**Theorem 1.3.** [BHK19, Theorem 1.10] Let \( p \) be an odd prime. The integral homology of \( B_p(\Sigma_{g,1}) \), \( g \geq 1 \) has no \( p \)-power torsion.

The computation via Morava \( E \)-theory becomes rather convoluted if one would like to apply it to the closed genus one surface \( T \), which is the only parallelizable closed surface. Following a similar approach, the second author studied Knudsen’s spectral sequence with odd primary coefficients in [Zha21] and showed that \( H_*(B_k(M); \mathbb{F}_p) \), \( k = 2,3 \) depends on the cohomology ring \( H^*(M^+; \mathbb{F}_p) \) when \( M \) is an even dimensional parallelizable manifold, which is in contrast to the case when \( M \) is odd dimensional.

In this paper, we build on the work of the second author and attack the computation of \( H_*(B_k(T); \mathbb{F}_p) \) directly. We prove Theorem 1.1 by identifying the higher differentials in the odd primary Knudsen’s spectral sequence

\[
E^2_{s,t}(k) = \pi_* \pi_! (\text{Bar}_*(\text{id}_S, \mathcal{L}_n, \text{Free}^{\mathcal{L}_n} (\Sigma^n S)^T) \otimes \mathbb{F}_p)(k) \Rightarrow H_{s+t}(B_k(T); \mathbb{F}_p)
\]

for \( k \leq p \). The argument we use involves a comparison with the spectral sequence computing \( H_*(B_k(T); \mathbb{Q}) \) studied by Knudsen [Knu17] and simple dimension counting. The exact same argument works for the punctured genus \( g \) surface for \( g > 0 \), thereby providing a more direct proof of Theorem 1.3.

1.1. **Outline.** In Section 2, we review the construction of Chevalley-Eilenberg complex of a shifted graded Lie algebra over a field and the structure of the odd primary homology of spectral Lie algebras. Then we recall previous computations of the \( E^2 \)-page of the odd primary Knudsen’s spectral sequence. Comparing with the computation of the rational homology of \( B_k(M) \) by Knudsen where \( M = T \) or \( \Sigma_{g,1} \), we deduce the main result of the paper Theorem 1.1 in Section 3.

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1.3. **Conventions.** Let \( k \) be a field. A weighted graded \( k \)-module \( M \) is an \( \mathbb{N} \)-indexed collection of \( \mathbb{Z} \)-graded \( k \)-modules \( \{M(w)\}_{w \in \mathbb{N}} \). The weight grading of an element \( x \in M(w) \) is \( w \). Morphisms are weight preserving morphisms of graded \( k \)-modules. Denote by \( \text{Mod}_w \) the category of weighted graded \( k \)-modules. We omit the adjectives weighted, graded from here on. The Day convolution \( \otimes \) makes \( \text{Mod}_w \) a symmetric monoidal category and the Koszul sign rule \( x \otimes y = (-1)^{|x||y|} y \otimes x \) depends only on the internal grading. Denote by \( w_{\ell}(M) \) the weight \( i \) part of the \( k \)-module \( M \).

2. **Preliminaries**

2.1. **The Chevalley-Eilenberg complex.** A shifted Lie algebra \( L \) over \( k \) is a \( k \)-module equipped with a shifted Lie bracket

\[
[-,-] : L_m \otimes L_n \to L_{m+n-1}
\]

that satisfies graded commutativity \( [x,y] = (-1)^{|x||y|}[y,x] \), the graded Jacobi identity

\[
(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0,
\]

and adds weight. When \( p = 3 \) we further require that \( [x,x,x] = 0 \) for all \( x \in L \). Denote by \( \text{Lie}_w^\ast \) the category of shifted weighted graded Lie algebras over \( k \).

**Definition 2.1.** [CE48][May66A] Suppose that \( k \) has characteristics away from two. For a \( \text{Lie}_w^\ast \)-algebra \( L \), let \( L_{\text{even}} \) and \( L_{\text{odd}} \) denote the elements in \( L \) with even and odd degree, respectively. The Chevalley-Eilenberg complex of \( L \) is the chain complex

\[
\text{CE}(L;k) = (\bigwedge^\ast (L_{\text{even}}) \otimes L_{\text{odd}}, \partial),
\]
where $\Gamma^*$ and $\Lambda^*$ are respectively the graded shifted divided power and exterior algebra functor over $k$, and the differential $\partial$ on an element $y_k(x_1) \cdots y_{k-1}(x_i) \cdots y_{k-j}(x_j) \cdots y_m(x_m) \langle y_{i_1}, \ldots, y_{i_n} \rangle$ in $\Gamma^*(L_{\mathrm{even}}) \otimes \Lambda^*(L_{\mathrm{odd}})$ is
\[
\sum_{1 \leq i < j \leq m} y_k(x_1) \cdots y_{k-1}(x_i) \cdots y_{k-1}(x_j) \cdots y_m(x_m) \langle y_{i_1}, \ldots, y_i, y_{i_1}, \ldots, y_{i_n} \rangle
\]
\[
+ \sum_{1 \leq i < j \leq m} (-1)^{i+j} y_k(x_1) \cdots y_m(x_m) \langle y_{i_1}, y_{i_2}, \ldots, y_j, \ldots, y_{i_n} \rangle
\]
\[
+ \frac{1}{2} \sum_{j=1}^m y_k(x_1) \cdots y_{k-j}(x_1) \cdots y_m(x_m) \langle (x_i, x_j), y_1, \ldots, y_{i_n} \rangle
\]
\[
+ \sum_{i=1}^m \sum_{j=1}^n (-1)^{i-1} y_k(x_1) \cdots y_{k-1}(x_i) \cdots y_{k-1}(x_1) \cdots y_m(x_m) \langle y_{i_1}, \ldots, y_{i_n} \rangle.
\]

Note that the differential $\partial$ preserves weights, so $H_*(\gamma_k(CE(L;k))) = \gamma_k(H_*(CE(L;k)))$. The Chevallay-Eilenberg complex is useful in computing the Lie algebra homology.

**Theorem 2.2.** [May66A][Pri70] For $L$ a Lie$_k$-algebra, its Lie$_k$-algebra homology is given by

$$H_*^{\mathrm{Lie}_k}(L) := \pi_* \gamma_k(\mathrm{Bar}_*(\gamma_k, L) \otimes k) \cong H_*(\gamma_k(CE(L;k))).$$

2.2. **Operations on the odd primary homology of spectral Lie algebras.** Building on the work of Arone-Mahowald [AM99] and Johnson [Job95], Ching [Chi05] and Salvatore showed that the Goodwillie derivatives $\{\partial_0(\mathrm{Id})\}_n$ of the identity functor $\mathrm{Id} : \mathrm{Top}_\ast \to \mathrm{Top}_\ast$ form an operad in Spectra that is Koszul dual to the non-unital $\Sigma_\infty$-operad. This is called the spectral Lie operad and we denote it by $s\mathcal{L}$. Algebras over the operad $s\mathcal{L}$ are called spectral Lie algebras, with structure maps $\partial_0(\mathrm{Id}) \otimes_{\Sigma_n} L \to L$ for all $k \geq 1$.

In [Kja18], Kjaer studied the structure of the mod $p$ homology of spectral Lie algebras for $p > 2$ following the approach of Behrens [Beh12] and Antolín-Camarena [AC20] when $p = 2$.

**Proposition 2.3.** [Kja18, Definition 3.2] Let $L$ be a spectral Lie algebra. Then $H_*(L; F_p)$ admits unary operations of weight $p$

$$\beta^p Q^j : H_*(L; F_p) \to H_{*+2(p-1)j-1}(L; F_p), \ x \mapsto \xi_*(\sigma^{-1} \beta^p Q^j(x))$$

for $\xi \in \{0, 1\}$, $j \in \mathbb{Z}$. Here $\xi : \partial_0(\mathrm{Id}) \otimes_{\mathcal{L}_p} L \to L$ is the $p$th structure map of the spectral Lie algebra $L$, $\sigma^{-1}$ the desuspension isomorphism, and $\beta^p \tilde{Q}^j$ a mod $p$ Dyer-Lashof operation.

It follows from the unstability of Dyer-Lashof operations that $\beta^p Q^j(x) = 0$ if $j < \frac{|x|}{2}$. There is also a Lie$_{\mathcal{L}_p}$-algebra structure on $H_*(L; F_p)$, induced by the second structure map

$$\partial_2(\mathrm{Id}) \otimes_{\mathcal{L}_p} L \simeq \partial_2(\mathrm{Id}) \otimes_{\mathcal{L}_p} L \simeq S^{-1} \otimes L \to L.$$

**Proposition 2.4.** [Kja18, Proposition 3.7] For $L$ a spectral Lie algebra, $\beta^p Q^j(x, y) = 0$ for any $x$, $y$, $j$ and $x, y \in H_*(L; F_p)$.

Define a functor $\text{Lie}_{\mathcal{L}_p}^* : \text{Mod}_{F_p} \to \text{Mod}_{F_p}$ as follows. For $M \in \text{Mod}_{F_p}$, let $A$ be an $F_p$-basis for the free shifted Lie algebra $\text{Free}^{\mathcal{L}_p}(M)$. The graded $F_p$-module $\text{Lie}_{\mathcal{L}_p}^*(M)$ has basis

$$\{\beta^p_{j_1} Q^k \cdots \beta^p_{j_{i+1}} Q^k x, \ x \in A, j_i \geq \frac{|x|}{2}, j_i \geq p(j_{i+1} - s_{i+1} + 1)\}.$$
2.3. Odd primary Knudsen’s spectral sequence. The odd primary Knudsen’s spectral sequence was first investigated by the second author in [Zha21]. Using the skeletal filtration of the geometric realization of the bar construction in Theorem 1.2, we obtain Knudsen’s spectral sequence with mod \( p \) coefficients

\[
E^2_{s,t}(k) = \pi_s \pi_t (\text{Bar}_* (\text{id}, s, L, \text{Free}^s\mathcal{L}(\Sigma^n X)^{M^n}) \otimes F_p)(k) \Rightarrow H_{s+t}(B_k(M; X); F_p).
\]

By repeatedly applying Theorem 2.5, we see that the \( E^2 \)-page is the homotopy group of a simplicial \( F_p \)-module \( V_* \) with \( (\text{Lie}_R^s)^{\omega_s}(L) \) as the \( s \)th simplicial level, where

\[
L = \bar{H}_s(\text{Free}^s\mathcal{L}(\Sigma^n X)^{M^n}; F_p) \cong \bar{H}^s(M^n; F_p) \otimes \text{Lie}_R^s(\Sigma^n H_0(X; F_p)).
\]

Then \( L \) has a \( \text{Lie}_p^s \)-structure given by

\[
[y_1 \otimes x_1, y_2 \otimes x_2] = (y_1 \cup y_2) \otimes [x_1, x_2]
\]

[BHK19, Proposition 5.9] and the action of the unary operations is given by

\[
\overline{B^sQj}(y \otimes x) = y \otimes \overline{B^sQj}(x)
\]

if there is no nonzero Steenrod operation on \( H^*(M; F_p) \) other than \( Sq^0 \) [Zha21, Proposition 4.4].

At the time of this work, there is no published result on the relations among the unary operations \( \overline{B^sQj} \). Nonetheless, at weight \( k < p^2 \) it suffices to know how unary operations and \( \text{Lie}_p^s \)-brackets commute, which is established by Proposition 2.4. The face maps on \( \text{Lie}_p^s \)-brackets are simply \( \text{Lie}_p^s \)-algebra structure maps. In [Zha21], the second author computed the \( E^2 \)-page of the spectral sequence (2) in weight \( k \leq p \) in terms of \( \text{Lie}_p^s \)-algebra homology.

**Proposition 2.6.** [Zha21, Proposition 6.5] Let \( M \) be a parallelizable manifold of dimension \( n \) and \( X \) any spectrum. Set

\[
g = \bar{H}^s(M^n; F_p) \otimes \text{Lie}_p^s(\Sigma^n H_0(X; F_p))
\]

with \( \text{Lie}_p^s \)-structure given by

\[
[y_1 \otimes x_1, y_2 \otimes x_2] = (y_1 \cup y_2) \otimes [x_1, x_2].
\]

(1) For \( k < p \), the weight \( k \) part of the spectral sequence (2) has \( E^2 \)-page given by

\[
E^2_{s,t}(k) \cong \text{wt}_s H_{s+t}(\text{CE}(g; F_p)).
\]

(2) For \( p \geq 5 \), the weight \( p \) part of the spectral sequence (2) has \( E^2 \)-page given by

\[
E^2(p)_{s,*} \cong \text{wt}_p H_{s,*}(\text{CE}(g; F_p)) \oplus \bigoplus_{y \in H_\ast, x \in B} F_p \left\{ \overline{B^sQj} y \otimes x, \frac{|x| - |y|}{2} \leq j < \frac{|x|}{2} \right\},
\]

where \( H \) is an \( F_p \)-basis of \( \bar{H}^*(M^n; F_p) \) and \( B \) an \( F_p \)-basis of \( H_0(X; F_p) \).

**Remark 2.7.** [Zha21, Remark 6.6] When \( p = 3 \), there has to be an identity \( \overline{B^sQj}(x) = [x, x, x] \) when \( x \) is a class of degree \( 2j \) in the mod 3 homology of the spectral Lie algebra \( L \). This can be seen by looking at the weight 3 part of the spectral sequence (2) for \( M = \mathbb{R}^n, X = S^{2j} \) and comparing with the weight 3 part of the free \( \mathbb{Z}_3 \)-algebra on the generator \( x \) given in [CLM76, III].

In other words, the mod 3 homology of a spectral Lie algebra should have the structure of an operadic \( \text{Lie}_p^s, \)-algebra, denoted by \( \text{Lie}_p^s \), which does not require \([x, x, x] = 0\). The underlying module of \( \text{Lie}_p^s(M) \) is thus given as follows: let \( B \) be an \( F_3 \)-basis for the free \( \text{Lie}_p^s \)-algebra on \( M \). The graded \( F_3 \)-module \( \text{Lie}_p^s(M) \) has basis the quotient of

\[
\overline{B^sQj_1} \cdots \overline{B^sQj_k} [x, x \in B, j_k \geq \frac{|x|}{2}, j_i \geq 3j_{i+1} - e_{i+1} \forall i}
\]

by the relation \( \overline{B^sQj}(x) = [x, x, x] \) for all \( x \in M \) with even degree.

\footnote{Through private communication, we were informed that Nikolay Konovalov has forthcoming work computing the odd primary relations via Goodwillie calculus.}
3. Mod $p$ homology of $B_k(T)$

Let $M = \Sigma_g$ be a genus $g$ surface with $g \geq 1$. Its integral cohomology ring is

$$H^*(\Sigma_g; \mathbb{Z}) = \begin{cases} \mathbb{Z}\{d\} & * = 0 \\ \mathbb{Z}\{a_i \oplus b_i, \ i = 1, \ldots, g\} & * = 1 \\ \mathbb{Z}\{c\} & * = 2 \\ 0 & \text{otherwise} \end{cases}$$

with cup product given by $a_i \cup b_i = c$ for all $i$, $d \cup y = y$ for all $y \in H^*(\Sigma_g; \mathbb{Z})$, and zero otherwise. Hence $H^*(\Sigma_{g,1}; \mathbb{Z})$ has product structure given by $a_i \cup b_i = c$ for all $i$ and zero otherwise, where $\Sigma_{g,1}$ is the punctured genus $g$ surface.

Using the fact that Knudsen’s spectral sequence with rational coefficients always collapses on the $E^2$-page [Knu18], Drummond-Cole and Knudsen produced explicit formulae for the Betti numbers of $B_k(\Sigma_g)$ and $B_k(\Sigma_{g,1})$ for all $k$ and $g$.

**Notation 3.1.** Denote by $g$ the Lie$_p^2$-algebra

$$g = H^*(\Sigma_g; \mathbb{F}_p) \otimes \text{Lie}_p^2(\mathbb{F}_p \{x_2\})$$

with $x_2$ in internal degree 2 and weight 1, with the Lie$_p^2$-structure is given by $[y \otimes x, y' \otimes x'] = (y \otimes y') \otimes [x, x']$. An $\mathbb{F}_p$-basis for $g$ is $B = \{y \otimes x_2, y \otimes [x_2, x_2], y = a_1, b_1, \ldots, a_g, b_g, c, d\}$. Let $g'$ be Lie$_Q^1$-algebra

$$H^*(\Sigma_g; \mathbb{Q}) \otimes \text{Free}^{\text{Lie}_0}(\mathbb{Q}\{x_2\})$$

with brackets given by the same formula as above. A $\mathbb{Q}$-basis for $g'$ is also $B$.

**Notation 3.2.** Denote by $g_1$ the Lie$_p^3$-algebra

$$g_1 = \tilde{H}^*(\Sigma_{g,1}; \mathbb{F}_p) \otimes \text{Lie}_p^3(\mathbb{F}_p \{x_2\})$$

with $x_2$ in internal degree 2 and weight 1, with the Lie$_p^3$-structure is given by $[y \otimes x, y' \otimes x'] = (y \otimes y') \otimes [x, x']$. An $\mathbb{F}_p$-basis for $g_1$ is $B_1 = \{y \otimes x_2, y \otimes [x_2, x_2], y = a_1, b_1, \ldots, a_g, b_g, c\}$. Let $g'_1$ be Lie$_Q^2$-algebra

$$\tilde{H}^*(\Sigma_{g,1}^+; \mathbb{Q}) \otimes \text{Free}^{\text{Lie}_0}(\mathbb{Q}\{x_2\})$$

with brackets given by the same formula as above. A $\mathbb{Q}$-basis for $g'_1$ is also $B_1$.

**Theorem 3.3.** [Knu17] The $i$th Betti number of $B_k(\Sigma_g)$ and $B_k(\Sigma_{g,1})$ are respectively equal to the dimension over $\mathbb{Q}$ of $\bigoplus_{s+i=1}H_{s,i}(\text{wt}_k(\text{CE}(g'; \mathbb{Q})))$ and $\bigoplus_{s+i=1}H_{s,i}(\text{wt}_k(\text{CE}(g'_1; \mathbb{Q})))$ for all $i$.

Explicit formulae for the Betti numbers $\beta_i(B_k(T))$ were obtained by Drummond-Cole and Knudsen in [DCK17, Corollary 4.5-4.7] and $\beta_i(\Sigma_{g,1}^+)$ in [DCK17, Proposition 3.5].

We will deduce the higher differentials in the spectral sequence

$$E^2_{p,k}(k) = \pi_* \pi_! (\text{Bar} \text{id}_+(\text{Free}^p(\Sigma^p ; \mathbb{Z})) \otimes \mathbb{F}_p)(k) \Rightarrow H_{s+i}(B_k(M); \mathbb{F}_p)$$

where $M = T$ or $\Sigma_{g,1}$ by combining Proposition 2.6 with Theorem 3.3.

**Theorem 3.4** (Theorem 1.1). For $k \leq p$, the dimension of $H_i(B_k(T); \mathbb{F}_p)$ over $\mathbb{F}_p$ is equal to the Betti number $\beta_i(B_k(T))$ for all $i$. Hence the integral homology of $B_k(T)$ has no $p$-power torsion for $k \leq p$.

**Proof.** The isomorphism between the $\mathbb{F}_p$-basis of $g$ and the $\mathbb{Q}$-basis of $g'$ (cf. Notation 3.1) induces compatible isomorphisms between the basis

$$\{\gamma_k(u_1) \cdots \gamma_k(u_m)(v_1, \ldots, v_n), \ u_1, \ldots, u_m, v_1, \ldots, v_m \in B\}$$

on each simplicial level of $\text{CE}(g; \mathbb{F}_p)$ and $\text{CE}(g'; \mathbb{Q})$. Since the integral cohomology of $T$ is torsion-free and $\gamma_p(y \otimes x_2)$ in weight $p$ does not receive differentials, the differentials in the two CE complexes preserve...
the isomorphism in weight $k \leq p$. Furthermore, when $p = 3$ the simplicial objects $\text{Bar}_*(\text{id}, \text{Lie}_c^b, g)$ and $\text{Bar}_*(\text{id}, \text{Lie}_c^{b,op}, g)$ are isomorphic in weight $k < p$. Hence

$$\dim F_p \bigoplus_{s+t = i} \text{wt}_k \text{H}_{s,t}(\text{CE}(g; F_p)) = \dim Q \bigoplus_{s+t = i} \text{wt}_k \text{H}_{s,t}(\text{CE}(g'; Q)) = \beta_i(B_k(T))$$

for all $i$ and $k \leq p$, with the right equality given by Theorem 3.3.

By Proposition 2.6.(1), the $E^2$-page of the weight $k < p$ part of the spectral sequence (3) is

$$E^2_{s,t}(k) \cong \text{wt}_k \text{H}_{s,t}(\text{CE}(g; F_p)).$$

Since $B_k(T)$ is of finite type, the dimension of $H_i(B_k(T); F_p)$ over $F_p$ is at least $\beta_i(B_k(T))$ for all $i$. Therefore no higher differential can happen by Equation (4).

Now we tackle the case $k = p$. First we consider $p > 3$. By Proposition 2.6.(2), the $E^2$-page of the weight $k$ part of the spectral sequence (3) is given by

$$E^2_{s,s}(p) \cong \text{wt}_p \text{H}_{s,s}(\text{CE}(g; F_p)) \oplus F_p \{\overline{Q^0}|c \otimes x_2, \overline{\beta Q^0}|c \otimes x_2\}.$$

Note that the class $\overline{\beta Q^0}|c \otimes x_2 \in E^2_{1,-2}$ has total degree $-1$, so it has to be killed by a class with total degree 0 and simplicial degree at least 3. There is exactly one class of total degree 0 in $\text{wt}_p \text{H}_{s,s}(\text{CE}(g; F_p))$ since $\beta_0(B_p(T)) = 1$, which is the class $\gamma_p(c \otimes x_2) \in E^2_{p-1,1-p}$. Therefore there has to be a $d_{p-2}$-differential $\gamma_p(c \otimes x_2) \mapsto \overline{\beta Q^0}|c \otimes x_2$. Appealing to the inequality

$$\sum_{i} \dim F_i H_i(B_k(T); F_p) \geq \sum_{i} \beta_i(B_k(T)) \overset{4}{=} \dim F_p \bigoplus_{s+t = i} \text{wt}_k \text{H}_{s,t}(\text{CE}(g; F_p)) = \sum_{s,t} \dim F_p E^2_{s,t}(p) - 2,$$

we deduce that no other higher differential can happen.

For $p = 3$, we deduce from Remark 2.7 that the normalized complexes of $\text{Bar}_*(\text{id}, \text{Lie}_c^b, g)$ and $\text{Bar}_*(\text{id}, \text{Lie}_c^{b,op}, g)$ differ in the weight 3 part in that there is a differential $\gamma_3(c \otimes x_2) \mapsto \overline{\beta Q^0}|c \otimes x_2$ in the latter but not in the former. Hence in the spectral sequence (3), the element

$$\gamma_3(c \otimes x_2) = [[c \otimes x_2, c \otimes x_2], c \otimes x_2] \in \text{Lie}_c^b \circ \text{Lie}_c^b \left( H^*(T; \mathbb{F}_3) \cong \text{Lie}_c^b(\mathbb{F}_3\{x_2\}) \right)$$

with the two brackets come from different iterations of $\text{Lie}_c^b$. It is mapped by the $d_1$-differential to

$$[[c \otimes x_2, c \otimes x_2], c \otimes x_2] \in \text{Lie}_c^b \left( H^*(T; \mathbb{F}_3) \cong \text{Lie}_c^b(\mathbb{F}_3\{x_2\}) \right),$$

with both brackets coming from the same iteration of $\text{Lie}_c^b$. At weight 3, there are no more differentials between elements containing a unary operation and brackets. It follows that the $E^2$-page of the weight 3 part of the spectral sequence (3) has a basis given by the union of an $\mathbb{F}_3$-basis of $\text{wt}_3 \text{H}_{s,s}(\text{CE}(g; F_p)) / \mathbb{F}_3\{\gamma_3(c \otimes x_2)\}$ and $\{\overline{Q^0}|c \otimes x_2\}$. Combining with Equation (4), we see that

$$\sum_{i} \beta_i(B_3(T)) = \dim F_3 \bigoplus_{s,t} \text{wt}_3 \text{H}_{s,t}(\text{CE}(g; F_3)) = \sum_{s,t} \dim F_3 E^2_{s,t}(3) \geq \sum_{i} \dim F_i H_i(B_3(T); F_3).$$

Hence no higher differential can happen and equality is achieved.

Therefore the dimensions of $H_i(B_k(T); F_p)$ over $F_p$ agree with the Betti numbers for $k \leq p$. Since $B_k(T)$ is a finite complex, we further deduce that its integral homology has no $p$-power torsion for $k \leq p$. \hfill \Box

**Remark 3.5.** The exact same argument works for the punctured genus $g$ surface by comparing the Lie$^b$-algebra homology groups of $g_1$ and $g'_1$ (Notation 3.2) weight $k \leq p$, thereby providing an elementary proof for Theorem 1.3.

**Remark 3.6.** The $d_{p-2}$-differential $\gamma_p(c \otimes x_2) \mapsto \overline{\beta Q^0}|c \otimes x_2$ should be viewed as a universal differential, in the sense that it occurs in the universal case $M = \lim_{n \to \infty} \mathbb{R}^n$, cf. [Zha21, Proposition 6.6]. Then $\lim_{n \to \infty} \Omega^p \text{Free}^{|\mathbb{S}'|}(\Sigma^n X) \simeq X$ and the spectral sequence (2) becomes

$$E^2_{s,t} = \pi_s \pi_t \text{Bar}_*(\text{id}, s\mathbb{L}', \mathbb{S}' \otimes \mathbb{F}_p) \Rightarrow \pi_{s+t} | \text{Bar}_*(\text{id}, s\mathbb{L}', \mathbb{S}' \otimes \mathbb{F}_p)| \cong \pi_{s+t} (\text{Free}^{|\mathbb{S}'|}(\mathbb{S}')),$$
where $\mathbb{E}_{2p}^\nu \otimes \mathbb{F}_p$ is the non-unital $\mathbb{E}_{2p}$-operad in the category of $\mathbb{F}_p$-module spectra. Heuristically, this is because the bottom non-vanishing mod $p$ Dyer-Lashof operation on a class $\alpha$ of degree 2 in an $\mathbb{E}_{2p}^\nu \otimes \mathbb{F}_p$-algebra is given by $\hat{Q}^\nu(x) = x^{\alpha p}$, so $\gamma_p(x)$ is redundant.

**Remark 3.7.** We expect the first $p$-power-torsion classes in $H_*(B_\Sigma(M); \mathbb{Z})$ for $M = T, \Sigma_{g,1}$ to show up at weight $k = 2p$ in the form of $\hat{Q}^\nu c \otimes [x_2, x_3] \in H_*(B_{2p}(M); \mathbb{F}_p)$ with $\nu = 0, 1$.

**References**


