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MOD \( p \) HOMOLOGY OF UNORDERED CONFIGURATION SPACES OF SURFACES

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ABSTRACT. We provide a short proof that the dimensions of the mod \( p \) homology groups of the unordered configuration space \( B_k(T) \) of \( k \) points in a torus are the same as its Betti numbers for \( p > 2 \) and \( k \leq p \). Hence the integral homology has no \( p \)-power torsion. The same argument works for the punctured genus \( g \) surface with \( g > 0 \), thereby recovering a result of Brantner-Hahn-Knudsen via Labin-Tate theory.

1. INTRODUCTION

The unordered configuration space of \( k \) points in a manifold \( M \) is the orbit space

\[
B_k(M) := \text{Conf}_k(M)\Sigma_k = \{(x_1, \ldots, x_k) \in M^{\times k}, x_i \neq x_j \text{ for } i \neq j\}/\Sigma_k.
\]

The main new result of this paper concerns the odd primary homology of \( B_k(T) \), where \( T \) is a closed torus.

Theorem 1.1. Let \( p \) be an odd prime. The dimension of \( H_i(B_k(T); \mathbb{F}_p) \) over \( \mathbb{F}_p \) is given by the \( i \)-th Betti number \( \beta_i(B_k(T)) \) for all \( i \) and \( k \leq p \). Hence the integral homology of \( B_k(T) \) has no \( p \)-power torsion for \( k \leq p \).

The study of unordered configuration spaces dates back to as early as Segal [Seg73] and McDuff [McD75]. The rational homology groups of these objects are relatively well understood in cases of interests via classical methods, see for instance [BCM88][BCT89][Kri94][Tot96][FT00]. In contrast, the odd primary homology groups of unordered configuration spaces have remained mostly intractable. Classically, the only known cases are the following: when \( M = \mathbb{R}^n \) for \( 1 \leq n \leq \infty \) where \( \bigoplus_{k \geq 0} H_*(B_k(M); \mathbb{F}_p) \) is the mod \( p \) homology of the free \( E_* \) algebra on \( \mathbb{S} \) [May72] [CLM76][BMMS88], and when the dimension of \( M \) is odd where \( \bigoplus_{k \geq 0} H_*(B_k(M); \mathbb{F}_p) \) depends only on the \( \mathbb{F}_p \) module \( H_*(M; \mathbb{F}_p) \) [BCT89][ML88][BCM93].

More recently, advances in the computation of the homology of unordered configuration spaces are made possible by a result of Knudsen [Knu18]. For any manifold \( M \) and spectrum \( X \), we can consider the labeled configuration spectrum

\[
B_k(M; X) := \Sigma_\infty \text{Conf}_k(M) \otimes X^{\otimes k}.
\]

In particular \( \Sigma_\infty B_k(M) = B_k(M; \mathbb{S}) \). Denote by \( s.L \) the monad associated to the free spectral Lie algebra functor \( \text{Free}^s.L \). The \( \infty \)-category of spectral Lie algebras is cotensored in Spaces, and \( (-)^{M^+} \) denotes the cotensor with the one-point compactification of \( M \) in this category. Using the machinery of factorization homology, Knudsen established the following equivalence.

Theorem 1.2. [Knu18, Section 3.4] Let \( M \) be a parallelizable \( n \)-manifold and \( X \) a spectrum of weight one. Then there is an equivalence of weighted spectra

\[
\bigoplus_{k \geq 1} B_k(M; X) \simeq \text{Bar}_s(\text{id}, s.L, \text{Free}^s.L(\Sigma^nX)^{M^+})/.
\]

The left hand side is weighted by the index \( k \) and right hand side induced by the weight on \( X \).

Using the bar spectral sequence with rational coefficients associated to the right hand side of (1), Knudsen [Knu17] provided a general formula for the Betti numbers of unordered configuration spaces. Building on Knudsen’s work, Drummond-Cole and Knudsen [DCK17] produced explicit formulae of the Betti numbers of unordered configuration spaces of surfaces. In [BHK19], Brantner, Hahn, and Knudsen studied Knudsen’s spectral sequence with coefficients in Morava \( E \)-theory at an odd prime. They computed the weight \( p \) part
of the labeled configuration spaces in $\mathbb{R}^n$ and punctured genus $g$ surfaces $\Sigma_{g,1}$ for $g \geq 1$ with coefficient in a sphere. By letting the height go to infinity, they deduced that:

**Theorem 1.3.** [BHK19, Theorem 1.10] Let $p$ be an odd prime. The integral homology of $B_p(\Sigma_{g,1})$, $g \geq 1$ has no $p$-power torsion.

The computation via Morava $E$-theory becomes rather convoluted if one would like to apply it to the closed genus one surface $T$, which is the only parallelizable closed surface. Following a similar approach, the second author studied Knudsen’s spectral sequence with odd primary coefficients in [Zha21] and showed that $H_* (B_2(M); \mathbb{F}_p)$, $k = 2, 3$ depends on the cohomology ring $H^*(M^\times; \mathbb{F}_p)$ when $M$ is an even dimensional parallelizable manifold, which is in contrast to the case when $M$ is odd dimensional.

In this paper, we build on the work of the second author and attack the computation of $H_*(B_k(T); \mathbb{F}_p)$ directly. We prove Theorem 1.1 by identifying the higher differentials in the odd primary Knudsen’s spectral sequence

$$E^2_{s,t}(k) = \pi_* \pi^* (\text{Bar}_{*} (\text{id} \cdot s \mathcal{L}, \text{Free}^s \mathcal{L} (\Sigma^s)T^+)) \otimes \mathbb{F}_p)(k) \Rightarrow H_{s+t}(B_k(T); \mathbb{F}_p)$$

for $k \leq p$. The argument we use involves a comparison with the spectral sequence computing $H_*(B_2(T); \mathbb{Q})$ studied by Knudsen [Knu17] and simple dimension counting. The exact same argument works for the punctured genus $g$ surface for $g > 0$, thereby providing a more direct proof of Theorem 1.3.

1. **Outline.** In Section 2, we review the construction of Chevalley-Eilenberg complex of a shifted graded Lie algebra over a field and the structure of the odd primary homology of spectral Lie algebras. Then we recall previous computations of the $E^2$-page of the odd primary Knudsen’s spectral sequence. Comparing with the computation of the rational homology of $B_2(M)$ by Knudsen where $M = T$ or $\Sigma_{g,1}$, we deduce the main result of the paper Theorem 1.1 in Section 3.

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1.3. **Conventions.** Let $k$ be a field. A weighted graded $k$-module $M$ is an $\mathbb{N}$-indexed collection of $\mathbb{Z}$-graded $k$-modules $\{M(w)\}_{w \in \mathbb{N}}$. The weight grading of an element $x \in M(w)$ is $w$. Morphisms are weight preserving morphisms of graded $k$-modules. Denote by $\text{Mod}_k$ the category of weighted graded $k$-modules. We omit the adjectives weighted, graded from here on. The Day convolution $\otimes$ makes $\text{Mod}_k$ a symmetric monoidal category and the Koszul sign rule $x \otimes y = (-1)^{|x||y|} y \otimes x$ depends only on the internal grading. Denote by $\text{wt}_i(M)$ the weight $i$ part of the $k$-module $M$.

2. **Preliminaries**

2.1. **The Chevalley-Eilenberg complex.** A shifted Lie algebra $L$ over $k$ is a $k$-module equipped with a shifted Lie bracket

$$[-,-]: L_m \otimes L_n \to L_{m+n-1}$$

that satisfies graded commutativity $[x,y] = (-1)^{|x||y|} [y,x]$, the graded Jacobi identity

$$(-1)^{|x||z|} [x,[y,z]] + (-1)^{|y||z|} [y,[x,z]] + (-1)^{|z||x|} [z,[x,y]] = 0,$$

and adds weight. When $p = 3$ we further require that $[[x,x],x] = 0$ for all $x \in L$. Denote by $\text{Lie}_k^*$ the category of shifted weighted graded Lie algebras over $k$.

**Definition 2.1.** [CE48][May66A] Suppose that $k$ has characteristics away from two. For a Lie_k^*-algebra $L$, let $L_{\text{even}}$ and $L_{\text{odd}}$ denote the elements in $L$ with even and odd degree, respectively. The **Chevalley-Eilenberg complex** of $L$ is the chain complex

$$\text{CE}(L;k) = (\Gamma^*(L_{\text{even}}) \otimes \Lambda^*(L_{\text{odd}}), \partial),$$
where $\Gamma^\bullet$ and $\Lambda^\bullet$ are respectively the graded divided power and exterior algebra functors over $k$, and the differential $\partial$ on an element $\gamma_1(x_1) \cdots \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_n(x_n)$ in $\Gamma^*(L_{\even}) \otimes \Lambda^*(L_{\odd})$ is

$$\sum_{1 \leq i < j \leq m} \gamma_i(x_i) \cdots \gamma_{i-1}(x_i) \cdots \gamma_j(x_j) \cdots \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_n(x_n)$$

$$+ \sum_{1 \leq i < j \leq n} (-1)^{i+j-1} \gamma_i(x_i) \cdots \gamma_{i-1}(x_i) \gamma_{i+1}(x_{i+1}) \cdots \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_n(x_n)$$

$$+ \frac{1}{2} \sum_{j=1}^m \gamma_1(x_1) \cdots \gamma_{j-1}(x_j) \cdots \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_n(x_n)$$

$$+ \sum_{i=1}^m \sum_{j=1}^n (-1)^{i+j-1} \gamma_i(x_i) \cdots \gamma_j(x_j) \gamma_{i+1}(x_{i+1}) \cdots \gamma_m(x_m) \gamma_{m+1}(x_{m+1}) \gamma_{m+2}(x_{m+2}) \cdots \gamma_n(x_n).$$

Note that the differential $\partial$ preserves weights, so $H_{e_{\ast}}(w_\ast(CE(L;k))) = w_\ast(H_{\ast,e}(CE(L;k)))$. The Chevellay-Eilenberg complex is useful in computing the Lie $\kappa$-algebra homology.

**Theorem 2.2.** [May66A][Pri70] For $L$ a Lie $\kappa$-algebra, its Lie $\kappa$-algebra homology is given by

$$H_{e_{\ast}}(L) := \pi_{e_{\ast}}(Bar_{\ast}(id, Lie_{\kappa}, L) \otimes k) \cong H_{e_{\ast}}(CE(L;k)).$$

2.2. **Operations on the odd primary homology of spectral Lie algebras.** Building on the work of Arone-Mahowald [AM99] and Johnson [Job95], Ching [Chi05] and Salvatore showed that the Goodwillie derivatives $\{\partial_{\ast}(Id)\}_n$ of the identity functor $\text{Id} : \text{Top}_{\ast} \to \text{Top}_{\ast}$ form an operad in Spectra that is Koszul dual to the non-unit $\mathbb{E}_{\ast}$-operad. This is called the spectral Lie operad and we denote it by $s\mathcal{L}$. Algebras over the operad $s\mathcal{L}$ are called spectral Lie algebras, with structure maps $\partial_3(\text{Id}) \otimes_{\mathbb{H}_{\Sigma}} L_{\Sigma} \to L$ for all $k \geq 1$.

In [Kja18], Kjaer studied the structure of the mod $p$ homology of spectral Lie algebras for $p > 2$ following the approach of Behrens [Beh12] and Antolín-Camarena [AC20] when $p = 2$.

**Proposition 2.3.** [Kja18, Definition 3.2] Let $L$ be a spectral Lie algebra. Then $H_*(L;F_p)$ admits unary operations of weight $p$

$$\partial^FQ^j : H_*(L;F_p) \to H_{*+2(p-1)i-1}(L;F_p), \ x \mapsto \xi_* (\sigma^{-1} \beta^FQ^j(x))$$

for $\varepsilon \in \{0, 1\}$, $j \in \mathbb{Z}$. Here $\xi_* : \partial_{\ast}(\text{Id}) \otimes_{\mathbb{E}_{\ast}} L_{\Sigma} \to L$ is the $p$th structure map of the spectral Lie algebra $L$, $\sigma^{-1}$ the desuspension isomorphism, and $\beta^FQ^j$ a mod $p$ Dyer-Lashof operation.

It follows from the unstability of Dyer-Lashof operations that $\partial^FQ^j(x) = 0$ if $j < |\frac{x}{2}|$. There is also a Lie $\kappa_p$-algebra structure on $H_*(L;F_p)$, induced by the second structure map

$$\partial_3(\text{Id}) \otimes_{\mathbb{H}_{\Sigma}} L_{\Sigma} \cong \partial_3(\text{Id}) \otimes_{\mathbb{H}_{\Sigma}} L_{\Sigma} \cong S^{-1} \otimes_{\mathbb{H}_{\Sigma}} L_{\Sigma} \to L.$$

**Proposition 2.4.** [Kja18, Proposition 3.7] For $L$ a spectral Lie algebra, $\partial^FQ^j(x,y) = 0$ for any $\varepsilon, j$ and $x, y \in H_*(L;F_p)$.

Define a functor $\text{Lie}_{\kappa_p} : \text{Mod}_{\kappa_p} \to \text{Mod}_{\kappa_p}$ as follows. For $M \in \text{Mod}_{\kappa_p}$, let $A$ be an $F_p$-basis for the free shifted Lie algebra $\text{Free}^\kappa_{\kappa_p}(M)$. The graded $F_p$-module $\text{Lie}_{\kappa_p}^\kappa(M)$ has basis

$$\{\beta^1_{e_i}Q^1 \cdots \beta^1_{e_i}Q^k | x, \ x \in A, j_i \geq |\frac{x}{2} |, j_i \geq p j_{i+1} - e_{i+1} \in \mathbb{Z} \}.$$

The Lie $\kappa_p$-structure on $\text{Free}^\kappa_{\kappa_p}$ can be extended to that on $\text{Lie}_{\kappa_p}^\kappa(M)$ via Proposition 2.4.

**Theorem 2.5.** [Kja18, Theorem 5.2] For $X$ a spectrum, there is an isomorphism of Lie $\kappa_p$-algebras

$$\text{Lie}_{\kappa_p}(H_*(X;F_p)) \to H_*(\text{Free}^\kappa_{\kappa_p}(X);F_p).$$
2.3. Odd primary Knudsen’s spectral sequence. The odd primary Knudsen’s spectral sequence was first investigated by the second author in [Zha21]. Using the skeletal filtration of the geometric realization of the bar construction in Theorem 1.2, we obtain Knudsen’s spectral sequence with mod $p$ coefficients

$$E^2_{s,t}(k) = \pi_s \pi_t \left( \text{Bar}\left( \text{id}, s, \mathcal{L}', \text{Free}^{s,\mathcal{L}}(\Sigma^n X)^{M^+}; \mathbb{F}_p \right) \right) \Rightarrow H_{s+t}(B_k(M; X); \mathbb{F}_p).$$

By repeatedly applying Theorem 2.5, we see that the $E^2$-page is the homotopy group of a simplicial $\mathbb{F}_p$-module $V_*$ with $(\text{Lie}_R^s)^{\otimes s}(L)$ as the $s$th simplicial level, where

$$L = H_*(\text{Free}^{s,\mathcal{L}}(\Sigma^n X)^{M^+}; \mathbb{F}_p) \cong \tilde{H}^*(M^+; \mathbb{F}_p) \otimes \text{Lie}_R^s(\Sigma^n H_*; \mathbb{F}_p)).$$

Then $L$ has a $\text{Lie}_R^s$-structure given by

$$[y_1 \otimes x_1, y_2 \otimes x_2] = (y_1 \cup y_2) \otimes [x_1, x_2]$$

[BHK19, Proposition 5.9] and the action of the unary operations is given by

$$\overline{B^s Q^j}(y \otimes x) = y \otimes \overline{B^s Q^j}(x)$$

if there is no nonzero Steenrod operation on $H^*(M; \mathbb{F}_p)$ other than $Sq^0$ [Zha21, Proposition 4.4].

At the time of this work, there is no published result on the relations among the unary operations $\overline{B^s Q^j}$. Nonetheless, at weight $k < p^2$ it suffices to know how unary operations and $\text{Lie}_R^s$-brackets commute, which is established by Proposition 2.4. The face maps on $\text{Lie}_R^s$-brackets are simply $\text{Lie}_R^s$-algebra structure maps. In [Zha21], the second author computed the $E^2$-page of the spectral sequence (2) in weight $k \leq p$ in terms of $\text{Lie}_R^s$-algebra homology.

Proposition 2.6. [Zha21, Proposition 6.5] Let $M$ be a parallelizable manifold of dimension $n$ and $X$ any spectrum. Set

$$g = \tilde{H}^*(M^+; \mathbb{F}_p) \otimes \text{Lie}_R^s(\Sigma^n H_*; \mathbb{F}_p))$$

with $\text{Lie}_R^s$-structure given by $[y_1 \otimes x_1, y_2 \otimes x_2] = (y_1 \cup y_2) \otimes [x_1, x_2].$

1. For $k < p$, the weight $k$ part of the spectral sequence (2) has $E^2$-page given by

$$E^2_{s,t}(k) \cong \text{wt}_k H_{s+t}(\text{CE}(g; \mathbb{F}_p)).$$

2. For $p \geq 5$, the weight $p$ part of the spectral sequence (2) has $E^2$-page given by

$$E^2_{p,s} \cong \text{wt}_p H_{s+t}(\text{CE}(g; \mathbb{F}_p)) \oplus \bigoplus_{y \in H_*; x \in B} \mathbb{F}_p \left( \overline{B^s Q^j}(y \otimes x, \frac{|x| - |y|}{2} \leq j < \frac{|x|}{2} \right),$$

where $H$ is an $\mathbb{F}_p$-basis of $\tilde{H}^*(M^+; \mathbb{F}_p)$ and $B$ an $\mathbb{F}_p$-basis of $H_*(X; \mathbb{F}_p)$.

Remark 2.7. [Zha21, Remark 6.6] When $p = 3$, there has to be an identity $\overline{B^s Q^j}(x) = [x, x], x]$ when $x$ is a class of degree $2j$ in the mod 3 homology of a spectral Lie algebra $L$. This can be seen by comparing the weight 3 part of the spectral sequence (2) for $M = \mathbb{R}^n$, $X = S^{2j}$ and comparing with the weight 3 part of the free $E_n$-algebra on the generator $x$ given in [CLM76, III].

In other words, the mod 3 homology of a spectral Lie algebra should have the structure of an operadic $\text{Lie}_R^s$-algebra, denoted by $\text{Lie}_R^{s,op}$, which does not require $[x, x], x]$ = 0. The underlying module of $\text{Lie}_R^s(M)$ is thus given as follows: let $B$ be an $\mathbb{F}_3$-basis for the free $\text{Lie}_R^{s,op}$-algebra on $M$. The graded $\mathbb{F}_3$-module $\text{Lie}_R^s(M)$ has basis the quotient of

$$\{ \overline{B^s Q^j} \cdots \overline{B^s Q^j} x, x \in B, j_k \geq \frac{|x|}{2}, j_i \geq 3 j_{i+1} - e_{i+1} |v| \}$$

by the relation $\overline{B^s Q^j}(x) = [[x, x], x]$ for all $x \in M$ with even degree.

\footnote{Through private communication, we were informed that Nikolay Konovalov has forthcoming work computing the odd primary relations via Goodwillie calculus.}
3. MOD $p$ HOMOLOGY OF $B_k(T)$

Let $M = \Sigma_g$ be a genus $g$ surface with $g \geq 1$. Its integral cohomology ring is

$$H^*(\Sigma_g; \mathbb{Z}) = \begin{cases} 
\mathbb{Z}\{d\} & * = 0 \\
\mathbb{Z}\{a_i \oplus b_i, i = 1, \ldots, g\} & * = 1 \\
\mathbb{Z}\{c\} & * = 2 \\
0 & \text{otherwise}
\end{cases}$$

with cup product given by $a_i \cup b_i = c$ for all $i$, $d \cup y = y$ for all $y \in H^*(\Sigma_g; \mathbb{Z})$, and zero otherwise. Hence $H^*(\Sigma_g; \mathbb{Z})$ has product structure given by $a_i \cup b_i = c$ for all $i$ and zero otherwise, where $\Sigma_{g,1}$ is the punctured genus $g$ surface.

Using the fact that Knudsen’s spectral sequence with rational coefficients always collapses on the $E^2$-page [Knu18], Drummond-Cole and Knudsen produced explicit formulae for the Betti numbers of $B_k(\Sigma_g)$ and $B_k(\Sigma_{g,1})$ for all $k$ and $g$.

**Notation 3.1.** Denote by $g$ the Lie$_p^q$-algebra

$$g = H^*(\Sigma_g; F_p) \otimes \text{Lie}_p^q(F_p\{x_2\})$$

with $x_2$ in internal degree 2 and weight 1, with the Lie$_p^q$-structure is given by $[y \otimes x, y' \otimes x'] = (y \otimes y') \otimes [x, x']$.

An $F_p$-basis for $g$ is $B_1 = \{y \otimes x_2, y \otimes [x_2, x_2], y = a_1, b_1, \ldots, a_g, b_g, c \}$.

Let $g'$ be Lie$_Q^q$-algebra

$$H^*(\Sigma_g; Q) \otimes \text{Free}^\text{Lie}_Q^q(Q\{x_2\})$$

with brackets given by the same formula as above. A $Q$-basis for $g'$ is also $B$.  

**Notation 3.2.** Denote by $g_1$ the Lie$_p^q$-algebra

$$g_1 = \overline{H}^*(\Sigma_{g,1}; F_p) \otimes \text{Lie}_p^q(F_p\{x_2\})$$

with $x_2$ in internal degree 2 and weight 1, with the Lie$_p^q$-structure is given by $[y \otimes x, y' \otimes x'] = (y \otimes y') \otimes [x, x']$.

An $F_p$-basis for $g_1$ is $B_1 = \{y \otimes x_2, y \otimes [x_2, x_2], y = a_1, b_1, \ldots, a_g, b_g, c \}$.

Let $g'_1$ be Lie$_Q^q$-algebra

$$\overline{H}^*(\Sigma_{g,1}; Q) \otimes \text{Free}^\text{Lie}_Q^q(Q\{x_2\})$$

with brackets given by the same formula as above. A $Q$-basis for $g'_1$ is also $B_1$.

**Theorem 3.3.** [Knu17] The $i$th Betti number of $B_k(\Sigma_g)$ and $B_k(\Sigma_{g,1})$ are respectively equal to the dimension over $Q$ of $\bigoplus_{i+s-t} H_{i+s}(\text{wt}_k(CE(g'; Q)))$ and $\bigoplus_{i+s-t} H_{i+s}(\text{wt}_k(CE(g'_1; Q)))$ for all $i$.

Explicit formulæ for the Betti numbers $\beta_i(B_k(T))$ were obtained by Drummond-Cole and Knudsen in [DCK17, Corollary 4.5-4.7] and $\beta_i(B_k(\Sigma_{g,1}))$ in [DCK17, Proposition 3.5].

We will deduce the higher differentials in the spectral sequence

$$E^2_{ij}(k) = \pi_i \pi_j (\text{Bar} \{id, s \Sigma^s \cdot \text{Free}^{\Sigma^s}(\Sigma^s)^{M^s} \otimes F_p\}(k) \Rightarrow H_{i+s}(B_k(M); F_p))$$

where $M = T$ or $\Sigma_{g,1}$; combining Proposition 2.6 with Theorem 3.3.

**Theorem 3.4** (Theorem 1.1). For $k \leq p$, the dimension of $H_i(B_k(T); F_p)$ over $F_p$ is equal to the Betti number $\beta_i(B_k(T))$ for all $i$. Hence the integral homology of $B_k(T)$ has no $p$-power torsion for $k \leq p$.

**Proof.** The isomorphism between the $F_p$-basis of $g$ and the $Q$-basis of $g'$ (cf. Notation 3.1) induces compatible isomorphisms between the basis

$$\{\gamma_k(u_1) \cdots \gamma_m(u_m)(v_1, \ldots, v_n), u_1, \ldots, u_m, v_1, \ldots, v_n \in B\}$$

on each simplicial level of $CE(g; F_p)$ and $CE(g'; Q)$. Since the integral cohomology of $T$ is torsion-free and $\gamma_p(y \otimes x_2)$ in weight $p$ does not receive differentials, the differentials in the two CE complexes preserve
the isomorphism in weight $k \leq p$. Furthermore, when $p = 3$ the simplicial objects $\text{Bar}_* (\text{id}, \text{Lie}_g^3)$ and $\text{Bar}_* (\text{id}, \text{Lie}_{g'}^{g',3})$ are isomorphic in weight $k < p$. Hence
\begin{equation}
\dim_{\mathbb{F}} \bigoplus_{s+t=i} \text{wt}_k H_{s,t}(\text{CE}(\mathbb{g}; \mathbb{F}_p)) = \dim_{\mathbb{Q}} \bigoplus_{s+t=i} \text{wt}_k H_{s,t}(\text{CE}(\mathbb{g'}; \mathbb{Q})) = \beta_i(B_k(T))
\end{equation}
for all $i$ and $k \leq p$, with the right equality given by Theorem 3.3.

By Proposition 2.6.(1), the $E^2$-page of the weight $k < p$ part of the spectral sequence (3) is
$$E^2_{s,t}(k) \cong \text{wt}_k H_{s,t}(\text{CE}(\mathbb{g}; \mathbb{F}_p)).$$
Since $B_k(T)$ is of finite type, the dimension of $H_i(B_k(T); \mathbb{F}_p)$ over $\mathbb{F}_p$ is at least $\beta_i(B_k(T))$ for all $i$. Therefore no higher differential can happen by Equation (4).

Now we tackle the case $k = p$. First we consider $p > 3$. By Proposition 2.6.(2), the $E^2$-page of the weight $k$ part of the spectral sequence (3) is given by
$$E^2_{v,s}(p) \cong \text{wt}_p H_{v,s}(\text{CE}(\mathbb{g}; \mathbb{F}_p)) \oplus \mathbb{F}_p \{ \overline{Q}^p | c \otimes x_2 \oplus \beta \overline{Q}^p | c \otimes x_2 \}.$$
Note that the class $\overline{Q}^p | c \otimes x_2 \oplus \beta \overline{Q}^p | c \otimes x_2$ has total degree $-1$, so it has to be killed by a class with total degree 0 and simplicial degree at least 3. There is exactly one class of total degree 0 in $\text{wt}_p H_{v,s}(\text{CE}(\mathbb{g}; \mathbb{F}_p))$ since $\beta_0(B_p(T)) = 1$, which is the class $\gamma_p(c \otimes x_2) \in E^2_{p-1,1-p}$. Therefore there has to be a $d_{p-2}$-differential $\gamma_p(c \otimes x_2) \mapsto \overline{Q}^p | c \otimes x_2$. Appealing to the inequality
$$\sum_i \dim_{\mathbb{F}} H_i(B_k(T); \mathbb{F}_p) + \beta_i(B_k(T)) \leq \sum_i \dim_{\mathbb{F}} \text{wt}_k H_{s,t}(\text{CE}(\mathbb{g}; \mathbb{F}_p)) = \sum_i \dim_{\mathbb{F}} E^2_{i,s}(p) - 2,$$
we deduce that no other higher differential can happen.

For $p = 3$, we deduce from Remark 2.7 that the normalized complexes of $\text{Bar}_* (\text{id}, \text{Lie}_g^3)$ and $\text{Bar}_* (\text{id}, \text{Lie}_{g'}^{g',3})$ differ in the weight 3 part in that there is a differential $\gamma_3(c \otimes x_2) \mapsto \overline{Q}^3 | c \otimes x_2$ in the latter but not in the former. Hence in the spectral sequence (3), the element
$$\gamma_3(c \otimes x_2) = [[[c \otimes x_2, c \otimes x_2], c \otimes x_2] \in \text{Lie}_g^3 \circ \text{Lie}_g^3 \left( H^*(T; \mathbb{F}_3) \otimes \text{Lie}_g^3 (\mathbb{F}_3 \{ x_2 \}) \right)$$
with the two brackets come from different iterations of $\text{Lie}_g^3$ is mapped by the $d_1$-differential to
$$[[[c \otimes x_2, c \otimes x_2], c \otimes x_2] \in \text{Lie}_g^3 \left( H^*(T; \mathbb{F}_3) \otimes \text{Lie}_g^3 (\mathbb{F}_3 \{ x_2 \}) \right)$$
with both brackets coming from the same iteration of $\text{Lie}_g^3$. At weight 3, there are no more differentials between elements containing a unary operation and brackets. It follows that the $E^2$-page of the weight 3 part of the spectral sequence (3) has a basis given by the union of an $\mathbb{F}_3$-basis of $\text{wt}_3 H_{v,s}(\text{CE}(\mathbb{g}; \mathbb{F}_3)) / \mathbb{F}_3 \{ \gamma_3(c \otimes x_2) \}$ and $\{ \overline{Q}^3 | c \otimes x_2 \}$. Combining with Equation (4), we see that
$$\sum_i \beta_i(B_3(T)) = \dim_{\mathbb{F}} \bigoplus_{s,t} \text{wt}_3 H_{s,t}(\text{CE}(\mathbb{g}; \mathbb{F}_3)) = \sum_i \dim_{\mathbb{F}} E^2_{i,s}(3) \geq \sum_i \dim_{\mathbb{F}} H_i(B_3(T); \mathbb{F}_3).$$
Hence no higher differential can happen and equality is achieved.

Therefore the dimensions of $H_i(B_k(T); \mathbb{F}_p)$ over $\mathbb{F}_p$ agree with the Betti numbers for $k \leq p$. Since $B_k(T)$ is a finite complex, we further deduce that its integral homology has no $p$-power torsion for $k \leq p$. □

**Remark 3.5.** The exact same argument works for the punctured genus $g$ surface by comparing the Lie $^g$-algebra homology groups of $g_1$ and $g'_1$ (Notation 3.2) weight $k \leq p$, thereby providing an elementary proof for Theorem 1.3.

**Remark 3.6.** The $d_{p-2}$-differential $\gamma_p(c \otimes x_2) \mapsto \overline{Q}^p | c \otimes x_2$ should be viewed as a universal differential, in the sense that it occurs in the universal case $M = \lim_{\to \infty} \mathbb{R}^n$, cf. [Zha21, Proposition 6.6]. Then $\lim_{\to \infty} \Omega^p \text{Free}^\mathbb{R}(\Sigma^k X) \simeq X$ and the spectral sequence (2) becomes
$$E^2_{i,s} = \pi_s \pi_t \text{Bar}_* (\text{id}, \text{id}, S' \otimes \mathbb{F}_p) \Rightarrow \pi_{s+t} \text{Bar}_* (\text{id}, \text{id}, S' \otimes \mathbb{F}_p) \cong \pi_{s+t} (\text{Free}^\mathbb{R} S' \otimes \mathbb{F}_p),$$
where $\mathbb{E}_\infty \otimes \mathbb{F}_p$ is the non-unital $\mathbb{E}_\infty$-operad in the category of $\mathbb{F}_p$-module spectra. Heuristically, this is because the bottom non-vanishing mod $p$ Dyer-Lashof operation on a class $x$ of degree $2j$ in an $\mathbb{E}_\infty \otimes \mathbb{F}_p$ algebra is given by $Q^j(x) = x^{\otimes p}$, so $\gamma_p(x)$ is redundant.

Remark 3.7. We expect the first $p$-power-torsion classes in $H_r(B_2(M); \mathbb{Z})$ for $M = T, \Sigma_{g,1}$ to show up at weight $k = 2p$ in the form of $\beta^p Q^j [x_2, x_2] \in H_r(B_2(M); \mathbb{F}_p)$ with $\epsilon = 0, 1$.

REFERENCES


