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PAPER

Dynamic control of the Bose–Einstein-like condensation transition in scalar active matter

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Abstract
The dynamics of a generic class of scalar active matter exhibiting a diffusivity edge is studied in a confining potential where the amplitude is governed by a time-dependent protocol. For such non-equilibrium systems, the diffusion coefficient vanishes when the single-particle density field reaches a critical threshold, inducing a condensation transition that is formally akin to Bose–Einstein condensation. We show that this transition arises even for systems that do not reach a steady state, leading to condensation in finite time. Since the transition can be induced for a fixed effective temperature by evolving the system, we effectively show that the temporal coordinate constitutes an alternative control parameter to tune the transition characteristics. For a constant-amplitude protocol, our generalised thermodynamics reduces in the steady-state limit to earlier results. Lastly, we show numerically that for periodic modulation of the potential amplitude, the condensation transition is reentrant.

1. Introduction
The exploration of self-organising systems through intrinsic microscopic activity has led to the discovery of remarkable phenomena within the realm of active matter [1], which is a relatively novel class of nonequilibrium systems composed of numerous (interacting) autonomous agents that exhibit persistent motion. Such systems are continuously out of thermodynamic equilibrium due to the local energy consumption of its individual constituents and consequently break detailed balance at a local level, which can lead to collective effects such as motility-induced phase separation [2, 3] or non-Gaussian anomalous diffusion [4]. Active matter presents an interesting avenue for new bio-inspired artificial technology, ranging from the design of nano- and micromachines, to understanding synthetic active tissues [1].

The key to understanding the rich phenomenology of active systems lies in the formulation of minimal models that capture the essential physics [1, 5]. Recently, a class of such active systems, distinguished by a diffusivity edge inducing condensation at steady states, has emerged as a generic mean field description for density-dependent diffusivity, in which some mean field dynamical features are represented [6–9]. This diffusivity edge, characterised by a critical density threshold beyond which particle mobility vanishes, introduces a unique and intriguing facet into the dynamics of scalar active matter.

The concept of a diffusivity edge was introduced phenomenologically in [6], where confining the active system in a harmonic potential led to a condensation transition at the ground state of the potential, where particles aggregate, exhibiting striking similarities with Bose–Einstein condensation (BEC) [10, 11], manifested in the behaviour of the thermodynamic potentials and susceptibilities. The condensation transition persisted even when collective activation at intermediate particle densities was introduced in the diffusivity profile.

This BEC-like condensation transition was shown through Brownian dynamics simulation to occur in a model of magnetic microswimmers confined in a quasi-one-dimensional channel [8, 12], where the microswimmers were focused along the axial direction of the channel by an effective harmonic potential that was externally controlled by a combination of nonequilibrium activity, external driving, and magnetic
interactions. Such a system exhibits three types of dynamical behaviour: focusing, clustering and condensation.

The notion of a diffusivity edge was subsequently extended to periodic potentials in arbitrary dimensions [7], where co-existing point-like condensates formed at all potential minima. The shallowness of the potentials with respect to the effective thermal energy then quantitatively affects the transition, leading to a non-universality in the values of the scaling exponents. In the high energy barrier limit, the condensation transition was found to be similar to the one originally predicted for harmonic confinement [6], but differed in the shallow potential limit, where the effective transition temperature diverges when the average density in a single period of the potential approaches the critical density.

Introducing an external driving force into the periodically confined system was shown to lead to qualitatively distinct stationary regimes depending on the amplitude of the driving force with respect to the potential barriers [9]. For small external forces, the condensation transition was shown to be qualitatively similar to the transition induced by harmonic confinement [6], while large forces lead to a reentrant transition, depending on the average density, where the condensation transition is accompanied by a low-temperature evaporation transition. The reentrant regime was found to be characterised by a depinning transition from a point-wise to a spatially extended condensate occupying a finite volume, in contrast with the point-like condensates found for harmonic confinement. It was shown that exactly at the critical value for the external forcing the potential becomes monotonous and is no longer confining, leading to a non-condensed gas phase even in the limit where the effective temperature is zero.

Motivated by the plethora of steady-state phenomena observed in the aforementioned systems, we now shift focus to the dynamical aspects of active matter systems with a diffusivity edge. The paper is structured as follows. In section 2, we introduce our time-dependent harmonically confined active system, and in section 3 we characterise the associated condensation transition for two protocol choices. Section 4 then develops a fully time-dependent generalised thermodynamic framework, showing the similarities and differences with Bose–Einstein condensation. Subsequently, in section 5, we briefly comment on periodic protocols and give a numerical example. In section 6, we conclude and give an outlook on future studies.

2. Scalar active matter with diffusivity edge

We consider a mean-field description of non-equilibrium matter that is described by an effective single-particle real scalar density field $\rho(r, t)$ in $d$ dimensions. While the results in this work exhibit formal similarities with Bose–Einstein condensation, it should be stressed that the system is purely classical and pertains to generic systems exhibiting a diffusivity edge.

The dynamics conserves the number of particles $N = \int d^d r \rho(r, t)$ for all times $t \geq 0$. It satisfies the conservation law $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$, where the flux is defined as $\mathbf{J} = -D(\rho) \nabla \rho + \mathbf{v}$. The drift velocity $\mathbf{v}$ is a result of the external potential $U(r, t)$, since $\mathbf{v} = M(\rho)(-\nabla U)$, and the diffusion coefficient $D(\rho)$ and mobility $M(\rho)$ are generally functions of the density. In the single-particle limit, their ratio defines a tuning parameter $T_{\text{eff}}$ that is equivalent to an effective temperature [6], and which is a measure for the density fluctuations at the particle level,

$$k_B T_{\text{eff}} = \lim_{\rho \to \rho_c} \frac{D(\rho)}{M(\rho)}. \quad (1)$$

For non-equilibrium systems the fluctuation–dissipation theorem (FDT) is broken, which we implement here by requiring that the single-particle limit of the system is different from the finite-density regime, i.e. we set

$$\frac{D(\rho)}{M(\rho)} \neq \lim_{\rho \to \rho_c} \frac{D(\rho)}{M(\rho)}. \quad (2)$$

Specifically, for large densities we assume the existence of a step function diffusivity edge in the system, i.e. $D(\rho) = M_k k_B T_{\text{eff}} \Theta(\rho_c - \rho)$, where $\rho_c$ is the critical density above which the diffusivity becomes identically zero. Other choices for the diffusion profile involving, e.g. inhibition or activation are also possible, see, e.g. [6]. Note that one can, in fact, also break the FDT by introducing a mobility edge instead of a diffusivity edge. However, in this work, we will only consider the latter. Recently, the fluctuation–dissipation relation was shown to hold for a Bose–Einstein condensate of photons, confirming the thermal nature of the condensate in a quantum gas of light [13]. For such systems, which are true quantum BECs, a notion similar to a diffusivity edge can be found in disordered systems in the presence of a random potential [14, 15], where a critical energy barrier separates extended and Anderson localised states. One must, however, take care not to confuse the quantum and classical notions of diffusion; when for quantum BECs the diffusion coefficient...
reaches a value on the order of $h/m$ the classical picture of diffusion breaks down completely since a single-article description is impossible [15].

We consider the full dynamics of the system by solving the conservation equation, with the spherically symmetric harmonic potential $U(r,t) = \gamma(t)r^2 + \varepsilon(t)r$, which allows for the individual tuning of both the stiffness and the position of the ground state by means of the functions $\gamma(t)$ and $\varepsilon(t)$, respectively. We assume here that $\gamma(t) \geq 0$, such that the potential $U(r,t)$ remains confining for all times $t > 0$. Consequently, the density is a decreasing function of $|r|$, and the maximum density is located at the minimum of the potential at $r_0$ where $U(r_0, t) = 0$, which we henceforth call the ground state. The conservation equation is supplemented with the symmetric $d$-dimensional Gaussian initial condition with standard deviation $\sigma$,

$$\psi(r) = \rho(r, 0) = N e^{-\frac{|r|^2}{2\sigma^2}}. \tag{3}$$

Note that $\psi$ is a real scalar density and not a wavefunction, as would be the case for quantum systems. The general smooth solution remains Gaussian for the non-condensed regime, with a time-dependent variance and amplitude, i.e. $\rho(r, t) = \rho_0(t) e^{\frac{(r-g(t))^2}{2\sigma^2(t)}}$, with $\rho_0(t) = \rho(0, t)$. The function $f(t, T_{\text{eff}})$, representing a time-dependent variance, is given by

$$f(t, T_{\text{eff}}) = \sigma^2 e^{-4H(t, 0)} + 2k_B T_{\text{eff}} M_t \int_0^t d\tau e^{-4H(t, \tau)}, \tag{4}$$

with $H(t, s) = M_t \int_0^s d\tau \gamma(\tau)$. For $t = 0$, it becomes equal to $f(0, T_{\text{eff}}) = \sigma^2$, as required. The function $g(t)$ is given by $g(t) = M_t \int_0^t \delta s e^{-2H(t, s)}$ and represents the time-dependent position of the maximum density or, alternatively, the position of the ground state, i.e. $r_0 = g(t)$. Details of the derivation are given in appendix .

In this work, we set $\varepsilon(t) = 0$, i.e. the location of the harmonic trap remains stationary, such that henceforth also $r_0 = 0$.

When the critical threshold density $\rho_c$ is reached, the system develops a singularity at the potential ground state at $r = 0$, since the normalisation condition cannot be fulfilled. Hence, the complete density profile is given by

$$\rho(r, t) = \begin{cases} \rho_0(t) e^{-\frac{|r|^2}{2\sigma^2(t)}} & \rho_0(t) < \rho_c, \\ N_0 \delta(r) + \rho_c e^{-\frac{|r|^2}{2\sigma^2(t)}} & \rho_0(t) \geq \rho_c, \end{cases} \tag{5}$$

where the ground state density is given by $\rho_0(t) = N/(2\pi f(t, T_{\text{eff}}))^\frac{d}{2}$. At condensation, the ground state density reaches the threshold $\rho_0(T_c) = \rho_c$, which corresponds to $f(t, T_{\text{eff}}) = f(t, T_c)$. Note that for general potentials $U(r)$ the condensate need not be formed solely at the position of the local minimum, but can instead occupy a finite width [9].

At this point, we want to stress that condensation can be achieved either by tuning the effective temperature $T_{\text{eff}}$ similar to previous research [6, 7, 9], or by evolving the system in time, with a suitable (time-dependent) potential. Hence, there exists both a critical temperature $T_c$ and a critical time $t_c$ to condensation.

In this work, we will consider power-law protocols, i.e. $\gamma(t) = \alpha t^\beta$, specifically with the choices $\beta = 0$ and $\beta = 1$, which we will refer to as the constant or the linear protocol, respectively, and denote them wherever necessary with the corresponding $\beta$ superscripts. All calculations that follow are straightforward to extend to other choices of $\gamma(t)$. For the choice $\beta = 0, 1$, the system is always localised at the ground state. By tuning the parameters $\alpha$ and $\beta$, one can access different regimes of anomalous diffusion [15], which offers a means to represent, e.g. systems with memory [16].

In figure 1, a projection of the normalised density profiles $\rho/\rho$ on a one-dimensional subspace is shown for increasing times, as a function of the coordinate $r$ in the positive real half space. Here, setting $r = 0$ in equation (3) yields $\rho \equiv \rho_0(t = 0) = N/\lambda^d$, i.e. the initial occupation of the ground state at $r = 0$, with $\lambda = \sqrt{2\pi \sigma^2}$ the average initial system size. When the condensation threshold at $\rho = \rho_c$ is reached, a singularity forms at the ground state, shown by a thicker vertical line. When not stated otherwise, we will fix the parameters $\alpha = 1/2, \sigma = 1, \rho/\rho_c = 2/3, M_t = 1$, and we work in three spatial dimensions, i.e. $d = 3$.

Experimental verification of our time-dependent model can possibly be achieved, e.g. in the previously studied model for magnetic microswimmers that are confined in a microfluidic channel and experience a Poiseuille flow [8, 12]. A condensed phase arises when such microswimmers are aligned along the axial direction, as a result of an effective harmonic confinement with stiffness $k = k_0 T_{\text{eff}}/(D_{\text{eff}}\gamma)$. This stiffness can be dynamically tuned by changing the external magnetic field or the flow speed, which is then formally similar to the protocols considered in this work.
3. The condensation transition

In general, for any $\gamma(t)$, the ratio $\bar{\rho}/\rho_c$ is a fixed quantity and can be used to find the $(T_{\text{eff}}, t)$ phase diagram for the condensation transition, since $\rho_0(t_c, T_c) = \rho_c$ at condensation. From the expressions for $f(t, T_{\text{eff}})$, i.e. equation (4), and $\rho_0(t, T_{\text{eff}})$, we find after some calculation the following relation between the critical temperature $T_c$ and time to condensation $t_c$, for a fixed initial density ratio:

$$k_B T_c = \frac{\sigma^2}{2M_s} \left[ \frac{(\bar{\rho}/\rho_c)^{\frac{3}{2}} - e^{4H(t_c, 0)} - 1}{G(t_c)} \right],$$

with $G(t) = \int_0^t ds \exp 4H(s, 0)$.

For both protocols, the phase diagram is shown in figure 2 for a fixed initial density ratio $\bar{\rho}/\rho_c = 2/3$, and the critical temperatures are given respectively by

$$T_c^{(0)}(t) = \frac{2\alpha\sigma^2}{k_B} \left( \frac{(\bar{\rho}/\rho_c)^{\frac{3}{2}} - e^{-4M_s\alpha t}}{1 - e^{-4M_s\alpha t}} \right), \quad T_c^{(1)}(t) = \sqrt{\frac{2\alpha}{M_s\pi k_B}} \left( \frac{(\bar{\rho}/\rho_c)^{\frac{3}{2}} - e^{-2M_s\alpha t}}{e^{-2M_s\alpha t} \text{erfi}(\sqrt{2M_s\alpha t})} \right).$$

For long times, the system confined by the potential with the constant protocol reaches a steady state, and it is only possible to reach condensation in finite time at a temperature strictly below $T_c^{(0)} = (\alpha/k_B\pi)(N/\rho_c)^{2/4}$. If we choose $\alpha = k/2$, with $k$ the stiffness of the harmonic trap, the critical temperature in the steady state becomes equal to the previously derived value $T_c^{(0)} = (k/2\pi k_B)(N/\rho_c)^{2/4}$ [6]. For a linear protocol, however, the critical temperature does not saturate to a constant value in the steady state, but is asymptotically linear in time, with slope $T_c^{(0)}$. It can be shown that for a general choice of the protocol power $\beta$, the $\beta$th derivative with respect to time of the asymptotic critical temperature yields $T_c^{(0)}$ in the high-temperature limit. Note that choosing $\bar{\rho} = \rho_c$ for the constant protocol, i.e. initialising the system in the condensed state, equation (6) becomes $k_B T_c = 2\alpha\sigma^2$, independent of time.

Since for the constant protocol $T_c^{(0)}(t)$ is an increasing function of time that converges at $T_c^{(0)}$, the critical temperature is always bounded from above by $T_c^{(0)}$, and vanishes at a particular time $t_0$ at $T_{\text{eff}} = 0$, which is the lower bound for the critical time to condensation; even at zero effective temperature there is a finite amount of time that needs to pass before the density is high enough to induce a condensation transition. This time can be calculated by setting the right-hand-side of equation (6) to zero at $t_c = 0$ and solving for $t_0$. Note that to continue, we will require that $t_0 > 0$, i.e. the system is always initialised in the non-condensed phase. This condition corresponds to the requirement that $\rho_0(t = 0) < \rho_c$, such that initial conditions need to be chosen carefully while preparing the system.

The denominator in equation (6) is nonzero for every $t_0 > 0$, so we can eliminate it and solve for $t_0$ in the numerator only, which yields the following simple equation

$$H(t_0, 0) = -\frac{1}{2d} \ln \left( \frac{\bar{\rho}}{\rho_c} \right).$$

Since this expression for $t_0$ and equation (6) are general for every choice of $\gamma(t)$ (under the condition that the integral that constitutes $H(t, 0)$ converges), the time-dependence of the condensation transition is a universal property of systems exhibiting a diffusivity edge. If the protocol $\gamma(t)$, the initial average density $\bar{\rho}$ and
condensation threshold $\rho_c$ are known, the time to condensation at a fixed effective temperature can be tuned exactly.

The normalisation condition is given by the spatial integral of equation (5), and in the condensed regime, this becomes

$$N = N_c + N \frac{\rho_c}{\rho_0(t)},$$  \hspace{1cm} (9)

such that the condensate fraction $\phi_c = N_c/N$ is given by

$$\phi_c(t, T_{\text{eff}}) = 1 - \frac{\rho_c}{\rho_0(t, T_{\text{eff}})} = 1 - \left( \frac{f(t, T_{\text{eff}})}{f(t, T_c)} \right)^{\frac{1}{2}},$$  \hspace{1cm} (10)

which is remarkably similar to the equivalent expression for the condensation transition of the ideal Bose gas [11]. The function $f(t, T_{\text{eff}})$ given by

$$f^{(0)}(t, T_{\text{eff}}) = \sigma^2 e^{-4 M_{c} \alpha t} + \frac{k_B T_{\text{eff}} (1 - e^{-4 M_{c} \alpha t})}{2 \alpha},$$  \hspace{1cm} (11)

for the constant protocol, and by

$$f^{(1)}(t, T_{\text{eff}}) = e^{-2 M_{c} \alpha t} \left( \sqrt{2 \pi M_{c} \alpha} k_B T_{\text{eff}} M_{e} \text{erfi} \left( \sqrt{2 M_{c} \alpha t} \right) + 2 \alpha \sigma^2 \right),$$  \hspace{1cm} (12)

for the linear protocol.

The condensate fraction for both the constant and linear protocols converges to $\phi_c = 1 - (T_{\text{eff}}/T_c)^{\frac{1}{2}}$ with $T_{\text{eff}} < T_c$ for long times. In figure 3, we show $\phi_c$ as a function of temperature $T_{\text{eff}}$ and time $t$. From panels (b) and (d), it can be seen that for $T_{\text{eff}} > 0$, the condensate fraction converges to a value $\phi_c^{\infty} = 1 - (T_{\text{eff}}/T_c)^{\frac{1}{2}}$, for the constant protocol. For the linear protocol, however, the number fraction always converges to unity in the long-time limit, since $T_c \to \infty$. Due to the increasing potential stiffness, particles in the smooth regions are effectively 'pushed' into the condensate at $t = 0$ at later times, experiencing an increased effective driving force towards the ground state.

For a constant protocol with $T_{\text{eff}} > T_c$, the relaxation to the steady state $\rho_{\lambda}(r) = \lim_{t \to \infty} \rho(r, t)$ is expected to be exponential, i.e., $\rho(r, t) \propto \rho_{\lambda}(r) + \sum_i a_i(r) e^{-t/\tau_i}$, where the $a_i(r)$ are unknown coefficients. The Laplace transform of this ansatz results in

$$\hat{\rho}(r, s) = \int_0^\infty dt \rho(r, t) e^{-st} = \frac{\rho_{\lambda}(r)}{s} + \sum_i \frac{a_i(r)}{s + \tau_i}.$$  \hspace{1cm} (13)

The different timescales in the system now correspond to simple poles of $\hat{\rho}(r, s)$ in the negative $s$ half-space, where the characteristic inverse timescale $\tau_i^{-1}$ of the system is given by the last negative pole. For

![Figure 2](image-url) Temperature-time phase diagrams for the condensation transition for (a) the constant and (b) the linear protocols. The condensed phase is represented by the shaded regions. For the constant protocol, the steady-state transition temperature is given by $T_{\text{eff}} = T_{c}^{\infty}$, while for the linear protocol it does not exist, but the transition line is asymptotically linear, $T_c \sim t T_c^{\infty}$ for long times. The zero-temperature transition times $t_0$ are indicated by arrows.
simply, we consider only the maximum value of the density $\rho_0(t)$ for the constant protocol. Laplace transforming leads to

$$\hat{\rho}_0(s) = \frac{\rho_0(0)}{s} \frac{d}{2} F_1 \left( \frac{d}{2} ; \frac{s}{4\alpha M_s} , 1 + \frac{s}{4\alpha M_s} , 1 - \frac{2\alpha \sigma^2}{k_B T_{\text{eff}}} \right),$$

where $2 F_1 (a,b;c;z)$ is the hypergeometric function. Finding the relaxation timescales $\tau_i$ of the system therefore amounts to finding the singularities of (14). The hypergeometric function is defined as a power series for $|z| < 1$, i.e.

$$2 F_1 (a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where $(q)_n = q(q+1) \ldots (q+n-1)$ is the rising Pochhammer symbol. The singularities at $s = s^*$ are therefore given by the condition $\epsilon = -k$, $k \in \mathbb{N}$, which can be rewritten in terms of $s^*$, i.e. $s^* = 4\alpha M_s (k + 1)$. The characteristic timescale is then given by $\tau^{-1} = 4\alpha M_s$, which for $\alpha = k/2$ reduces to $\tau^{-1} = 2kM_s$. This result of course could also have been found by inspecting the first term in equation (4), leading us to conjecture that the characteristic timescale is set by $1/4M_s \gamma$. Note that the characteristic relaxation timescale is set only by the single-particle mobility and the trap stiffness, and not by, e.g. the effective temperature. For infinite systems, only one scaling regime is possible, while for finite systems, a size-dependent diffusive to ballistic crossover can be identified [18].

In the condensed regime, we study the relaxation of the condensate fraction $\phi_c$ by Laplace transforming equation (10). This yields

$$\hat{\phi}_c(s,T_{\text{eff}}) = \frac{1}{s} \left[ 1 - \left( \frac{T_{\text{eff}}}{T_c} \right)^{\frac{d}{2}} 2 F_1 \left( \frac{d}{2} ; \frac{s}{4\alpha M_s} , 1 + \frac{s}{4\alpha M_s} , 1 - \frac{2\alpha \sigma^2}{k_B T_{\text{eff}}} \right) \right],$$

such that the characteristic relaxation timescale is once again given by $\tau^{-1} = 4\alpha M_s$. Note that we assume that the system always equilibrates faster than the potential stiffness is modulated, i.e. $\tau < 1/|\gamma|$. When $|\gamma|$ is finite and the stiffness is changed faster than the characteristic relaxation time, the system is in nonequilibrium and the effective volume of the system, set by the variance of the density profile, is larger than the equilibrium value.

We now turn to the development of a time-dependent generalised thermodynamics for the system.
4. Generalised thermodynamics

The average potential energy \( \langle U(t, T_{\text{eff}}) \rangle \) is given by the expectation value of the potential, i.e.

\[
\langle U(t, T_{\text{eff}}) \rangle = \int d^d r U(r,t) \rho(r,t) = dN \gamma(t) f(t, T_{\text{eff}}) \left\{ \begin{array}{ll} \frac{1}{T_{\text{eff}}} & T_{\text{eff}} > T_c, \\ \frac{1}{T_{\text{eff}}} \frac{T_{\text{eff}}}{T_c}^{1/2} & T_{\text{eff}} \leq T_c. \end{array} \right.
\]  

(17)

In the long-time limit, the energy converges to

\[
\lim_{t \to \infty} \langle U(t, T_{\text{eff}}) \rangle = \frac{d}{2} N k_b T_{\text{eff}} \left\{ \begin{array}{ll} \frac{1}{T_{\text{eff}}} & T_{\text{eff}} > T_c, \\ \frac{1}{T_{\text{eff}}} \frac{T_{\text{eff}}}{T_c}^{1/2} & T_{\text{eff}} \leq T_c. \end{array} \right.
\]  

(18)

where one needs to take into account that the critical temperature is not necessarily finite in this limit, as is the case for, e.g. the linear protocol, where \( T_{\infty}^c \) and consequently also the energy diverge. For short times, however, the energy is a polynomial of the time \( t \), with the same exponent as the protocol \( \gamma(t) \), i.e. the energy is asymptotically

\[
\langle U(t, T_{\text{eff}}) \rangle \sim (dN \sigma^2) t^\beta \left\{ \begin{array}{ll} 1 & T_{\text{eff}} > T_c, \\ \frac{2}{\beta} e^{-2dH(t,0)} & T_{\text{eff}} \leq T_c. \end{array} \right.
\]  

(19)

For long times, asymptotic analysis reveals that the average potential energy is asymptotically equivalent to a constant for the constant protocol, and to \( t^{-d/2} \) for the linear protocol.

At \( T_{\text{eff}} = 0 \), the internal energy reduces to

\[
\langle U(t, T_{\text{eff}} = 0) \rangle = dN \sigma^2 \gamma(t) e^{-4H(t,0)} \left\{ \begin{array}{ll} 1 & t < t_c, \\ \frac{2}{\beta} e^{-2dH(t,0)} & t \geq t_c. \end{array} \right.
\]  

(20)

In figure 4, we show the average potential energy as a function of effective temperature and time, in panels (a), (b) and (c), (d) for the constant and linear protocols, respectively.
The heat capacity is then given by the derivative of the internal energy with respect to the effective temperature,

$$ C(t, T_{\text{eff}}) = dN \gamma(t) \frac{\partial f}{\partial T_{\text{eff}}} \left\{ \begin{array}{ll} 1 & T_{\text{eff}} > T_c, \\ (1 + \frac{t}{2}) \frac{\rho_c}{\rho_0} & T_{\text{eff}} \leq T_c. \end{array} \right. $$

For $T_{\text{eff}} > T_c$, it can be seen that the heat capacity is independent of the effective temperature, regardless of the shape of $\gamma(t)$, since $\partial f/\partial T_{\text{eff}}$ is independent of $T_{\text{eff}}$. For short times, $C$ is asymptotically equivalent to a power law of time, with exponent $1 + \beta$, i.e. $C/Nk_B \sim t^{1+\beta}$. For long times, however, the heat capacity is once again asymptotically equivalent to a constant for the constant protocol, and to $t^{-d/2}$ for the linear protocol, similar to the asymptotics of the energy. The heat capacity as a function of time and effective temperature is shown in figure 5.

The jump $\Delta C = C(\rho = \rho_c^+) - C(\rho = \rho_c^-)$ in the heat capacity for a fixed time $t$ is given by

$$ \frac{\Delta C}{Nk_B} = d^2 \gamma(t) M_s t e^{-4H(t, s)} \int_0^t ds e^{-4H(t, s)}, $$

which, for the constant and linear protocols, are given respectively by

$$ \frac{\Delta C^{(0)}}{Nk_B} = \frac{d^2}{4} \left( 1 - e^{-4M_s \alpha t} \right), $$

$$ \frac{\Delta C^{(1)}}{Nk_B} = \frac{d^2}{2} \sqrt{\frac{M_s \alpha \pi}{2}} t e^{-2M_s \alpha t} \text{erfi}(\sqrt{2M_s \alpha t}), $$

where the superscripts denote the power of $t$ in the protocol ((0) for constant, (1) for linear). Both expressions converge to $d^2/4$ in the long-time limit. The jump in the heat capacity is shown as a function of time in figure 6. For the constant protocol, $\Delta C$ is an increasing function of time, while for the linear protocol, it reaches a maximum at a finite time before converging to the long-time value.

To compute the bulk pressure, we use a mechanical definition [19], which is defined as the average force per unit surface area exerted by particles on a potential $U$ confining the system within a volume $V$, i.e.,

$$ P_{\text{mech}} = \int_0^\infty dU \rho(U), $$

where $\rho$ is the density of the system.
which is identical to the thermodynamic pressure calculated by means of the Helmholtz free energy, $P_{\text{mech}} = P_{\text{therm}} = P$ [6, 7]. Writing the density as a function of the time-dependent potential and integrating, the pressure reads

$$P(t, T_{\text{eff}}) = 2\gamma(t)f(t, T_{\text{eff}}) \begin{cases} \rho_0 & T_{\text{eff}} > T_c \\ \rho_c & T_{\text{eff}} \leq T_c \end{cases}$$

$$= 2N\gamma(t)f(t, T_{\text{eff}}) \begin{cases} V^{-1} & V > V_c \\ V^{-1} & V \leq V_c \end{cases}$$  (25)

where $V = N/\rho_0 = (N - N_c)/\rho_c$ is the effective volume of the system, with $V_c = (2\pi \sigma^2)^{\frac{d}{2}}(\rho_c/\rho_c)$. For high effective temperature, the pressure goes as $P \sim T_{\text{eff}}^{-\frac{d}{2}}$, while for low $T_{\text{eff}}$, it behaves asymptotically as a constant that depends on time, i.e. $P \sim 2\alpha \sigma^2 \rho_c \exp\left[-4M \alpha t^{1+\beta}\right]$. For short times, the pressure is asymptotically a power of $t$, following the exponent $\beta$ of the protocol. The pressure as a function of $T_{\text{eff}}$ and $t$ is shown in figure 7.

The second line in equation (25) shows that the pressure decouples from the average volume $V$ when below the critical volume $V_c$, indicating that the isothermal incompressibility $\kappa^{-1} = -V\left(\frac{\partial^2 F}{\partial V^2}\right)_{T_{\text{eff}}}$ diverges at condensation. The divergence of the isothermal compressibility at criticality is indicative of ideal Bose gas behaviour [20], and has recently been observed in optical quantum gases [21] at room temperature.

Eliminating now the effective temperature from the first line of equation (25), we find that the critical point $P_c$ for the pressure isotherms is given by $P_c = \frac{\gamma(t)}{N^{\frac{d}{2}}/\rho_c^{\frac{d}{2}}}$. Alternatively, for the constant protocol, one can eliminate the time $t$ from the pressure, to find the critical point for the pressure isostemal line, yielding $P_c = T_c/\rho_c$. While this can formally be done numerically for any type of protocol, analytical progress is unfeasible even for the linear protocol.

Finally, the chemical potential $\mu$ can be defined as the conjugate variable to the particle number $N$, and is given as a derivative of the Helmholtz free energy

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T_{\text{eff}}, V}.$$

After some calculation, we find that $\mu$ is given by

$$\mu = d\gamma(t)f(t, T_{\text{eff}}) \begin{cases} -2k_B T_{\text{eff}} \ln\left(\frac{\tilde{Z}_V}{T_c}\right) \left(\frac{\rho_c}{\pi^2 + 2M M_{\text{eff}} T_{\text{eff}}^\gamma(t)}\right) & T_{\text{eff}} > T_c \\ 0 & T_{\text{eff}} \leq T_c \end{cases}$$

(27)

For the constant protocol, the expression for the chemical potential in the non-condensed regime becomes $\mu(T_{\text{eff}} > T_c) = -\frac{d}{2}k_B T_{\text{eff}} \ln(T/T_c)(1 - \exp(-4M_{\text{eff}} t))$. In the steady-state regime, this reduces to known results [7].

Equations (10), (21), (25) and (27) highlight a remarkable similarity to the thermodynamics of Bose–Einstein condensation, and all thermodynamic observables reduce to the steady-state results obtained
Figure 7. Pressure $P/\rho_c k_B T_{\text{eff}}$ for the constant (a)–(b) and linear (c)–(d) protocols, as a function of the effective temperature $T_{\text{eff}}$ (a), (c) and time $t$ (b), (d). The dashed lines indicate the limiting behaviour. Asymptotics are shown by indicating the slope.

in [6] when the protocol is chosen to be constant with $a = k/2$. The dynamical behaviour of our model is moreover akin to the recently derived model for the condensation of a dilute Bose gas subjected to evaporative cooling [22, 23]. Conversely, the discontinuous jump in the heat capacity is a phenomenon that only occurs for dimensions higher than four [11], and it is a consequence of the discrete jump in diffusivity at the condensation threshold $\rho_c$, indicating that the equivalence between our system and BEC is not exact.

5. Periodic protocols

We now turn briefly to the study of periodic protocols, where the amplitude of the harmonic potential is cyclically modulated with a frequency $\omega$. Since it is generally not feasible to analytically solve the integrals involved in equation (4) and associated quantities, we will resort to numerical methods. We discuss only the condensation transition, and refer to future research for the thermodynamic analysis.

To proceed, we choose the protocol as $\gamma(t) = \cos(\omega t + \varphi)$, with $\omega = 2\pi/\tau$ the frequency of the modulation with period $\tau$, and $\varphi$ the phase. We subsequently solve equation (6) to find the critical temperature as a function of the critical moment to condensation. The resulting phase diagram is shown in figure 8. As a consequence of the periodic modulation, the condensation is now reentrant as a function of time, for a fixed value of the effective temperature. In the inset, the number fraction is drawn, showing that the number of particles in the condensate is now a non-monotonous function of time.

Note that at $T_{\text{eff}} = 0$, the reentrant behaviour is periodic in time, and the number fraction reaches the global maximum in every condensed phase. Taking the derivative of the number fraction at $T_{\text{eff}} = 0$ with respect to time and setting the result equal to zero leads to the condition

$$\frac{\partial f}{\partial t}(t, T_{\text{eff}} = 0) = -4\sigma \frac{\partial H(t, 0)}{\partial t} e^{-4\Omega(t, 0)} = 0.$$  \hspace{1cm} (28)$$

Solving $\partial H(t, 0)/\partial t = 0$ for $t$ and checking the second derivative shows that the maxima are located at $t_{\text{max}} = \tau/4 + j\tau, j \in \mathbb{N}$. Inserting $t_{\text{max}}$ back into equation (10) at zero effective temperature shows that the maximum number fraction $\phi_{c,\text{max}}$ is given by

$$\phi_{c,\text{max}} = 1 - \frac{\rho_c}{\bar{\rho}} e^{-\frac{4\mu}{\sigma^2}}.$$  \hspace{1cm} (29)$$

which is fully determined by the initial system parameters.
6. Conclusions and outlook

We have studied the dynamics of a non-equilibrium system of particles subjected to both a diffusivity edge and time-dependent harmonic confinement, in arbitrary dimensions. In addition to the temperature-dependent results reported in earlier findings [6, 7, 9], we have revealed the finite-time emergence of the condensation transition, which can be tuned by both the effective temperature and the time. For constant protocols, and in the steady-state limit, the behaviour and thermodynamics of the active system is equivalent to that of the ideal Bose gas in free space, while the dynamics are closely related to exactly solvable models for evaporative cooling of such gases [22, 23].

Moreover, we have shown that for any choice of protocol, the time to condensation and the critical temperature are related through equation (6). This relation, even at zero effective temperature, emphasises the universal nature of the time-dependence in systems with a diffusivity edge, governed by the interplay of protocol, initial density, and condensation threshold. This universal property enables precise tuning of the time to condensation when these parameters are known. This relation was then extended to a system with periodic modulation of the potential amplitude, leading to a reentrant condensation transition as a function of time. Extending the current dynamic approach to the BEC-like condensation transition in periodically confined systems would enable experimental modulation of both spatial and temporal periodicity.

It would be an interesting avenue of future research to study active systems with a diffusivity edge beyond the mean field approach employed here. Since, e.g. the divergence of the isothermal incompressibility is a result of the existence of off-diagonal-long-range order in regular BEC, the current mean field approach is insufficient to capture density correlations leading to this divergence. As such, fluctuations become key and one possible application of this research could be in the field of stochastic thermodynamics, where work can be extracted [24, 25].

Finally, note that since $\varepsilon(t)$ can be removed from the smooth solution for $\rho < \rho_c$ by shifting to a moving frame with time-dependent translation $\mathbf{r} \rightarrow \mathbf{r} + M_s \int_0^t ds \varepsilon(s) e^{-2H(t,s)}$, the results for the non-condensed phase hold true when $\varepsilon \neq 0$, although it is not guaranteed that this symmetry holds for the condensed phase where the diffusion vanishes. This can possibly lead to interesting extensions of the research to dragged condensates.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

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Appendix. Derivation of the time-dependent density profile

To find the general time-dependent density profile for the smooth ($\rho < \rho_c$) region, we follow closely the derivation in [26]. Starting from the conservation equation with the single-particle diffusion and mobility coefficients,

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (D_r \nabla \rho - M_r \rho (-\nabla U)),$$

(A1)

with the general time-dependent potential $U(r, t) = \gamma(t) r^2 + \varepsilon(t) r$, we can rewrite equation (A1) as follows

$$\frac{\partial \rho}{\partial t} = D_r \nabla^2 \rho + M_r \varepsilon(t) \nabla \rho + 2 \gamma(t) M_r \rho \nabla (\rho \nabla r).$$

(A2)

Henceforth, we will suppress the time dependence of $\gamma(t)$ and $\varepsilon(t)$ to ease notation. Taking the $d$-dimensional Fourier transform $\tilde{f} \equiv \hat{F} \{ f(r) \} = \int_{\mathbb{R}^d} d^d r \langle f(r) \rangle \exp(-i \mathbf{k} \cdot \mathbf{r})$ of equation (A2) yields the following equation for the time derivative of the density

$$\frac{\partial \tilde{\rho}}{\partial t} = -D_r |\mathbf{k}|^2 + M_r \varepsilon \mathbf{k} + 2M_r \gamma \mathbf{k} \nabla \tilde{\rho}.$$  

(A3)

We seek a solution to equation (A3) of the form

$$\tilde{\rho}(\mathbf{k}, t) = \prod_{n=1}^{\infty} e^{b_n(t)|\mathbf{k}|^2}.$$  

(A4)

Substituting (A4) into (A3) gives the following

$$\sum_{n=1}^{\infty} \left( b_n'(t) + 2\gamma M_r n b_n(t) \right) |\mathbf{k}|^n + i M_r \varepsilon \mathbf{k} + D_r |\mathbf{k}|^2 = 0,$$

(A5)

where a prime indicates the time derivative, such that the coefficients $b_n$ are given by

$$b_1(t) = b_{1,0} e^{-2H(t,0)} - i M_r \int_0^t ds \varepsilon(s) e^{-2H(t,s)},$$

$$b_2(t) = b_{2,0} e^{-4H(t,0)} - D_r \int_0^t ds e^{-4H(t,s)},$$

$$b_n(t) = b_{n,0} e^{-2nH(t,0)} \quad \text{for } n \geq 3$$

(A6)

with $H(t,s) = M_r \int_s^t d\tau \gamma(\tau)$. Substituting the coefficients back into the solution (A4)

$$\tilde{\rho}(\mathbf{k}, t) = \exp \left\{ \left( b_{1,0} e^{-2H(t,0)} - i M_r \int_0^t ds \varepsilon(s) e^{-2H(t,s)} \right) \mathbf{k} \right\} \times \exp \left\{ \left( b_{2,0} e^{-4H(t,0)} - D_r \int_0^t ds e^{-4H(t,s)} \right) |\mathbf{k}|^2 \right\} \prod_{n=3}^{\infty} \exp \left\{ b_{n,0} e^{-2nH(t,0)} |\mathbf{k}|^n \right\},$$

(A7)

and inserting the Fourier transform of the initial condition (3), i.e.

$$\tilde{\psi}(\mathbf{k}) = \hat{F} \{ \psi(r) \} = \frac{\exp(\sqrt{\tau} |\mathbf{k}|^2)}{\sqrt{2\pi}},$$

(A8)

in equation (A7) fixes the values of the unknown constants to be $b_{1,0} = b_{3,0} = b_{4,0} = \ldots = 0$, and $b_{2,0} = -\sigma^2/2$. Inserting the $b_{n,0}$ back into equation (A7) gives

$$\tilde{\rho}(\mathbf{k}, t) = \exp \left\{ -i M_r \mathbf{k} \int_0^t ds \varepsilon(s) e^{-2H(t,s)} \right\} \exp \left\{ -\left( \frac{\sigma^2}{2} e^{-4H(t,0)} + D_r \int_0^t ds e^{-4H(t,s)} \right) |\mathbf{k}|^2 \right\}. $$

(A9)

Finally, taking the inverse Fourier transform $f \equiv \hat{F}^{-1} \{ \tilde{f}(\mathbf{k}) \} = (2\pi)^{-d} \int_{\mathbb{R}^d} d^d k \tilde{f}(\mathbf{k}) \exp \{ i \mathbf{k} \cdot \mathbf{r} \}$ of the solution (A9), we arrive at

$$\rho(r, t) = (2\pi f(t, T))^{-\frac{d}{2}} \exp \left\{ -\frac{(r - g(t))^2}{2f(t, T)} \right\},$$

(A10)

with $f(t, T) = \sigma^2 e^{-4H(t,0)} + 2k_B T \varepsilon M_r \int_0^t d\tau \exp e^{-4H(t,\tau)}$, which is equation (4) in the main text, and $g(t) = M_r \int_0^t ds \varepsilon(s) e^{-2H(t,s)}$. 


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