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PARTIAL DEGENERATION OF TENSORS

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Abstract. Tensors are often studied by introducing preorders such as restriction and degeneration: the former describes transformations of the tensors by local linear maps on its tensor factors; the latter describes transformations where the local linear maps may vary along a curve, and the resulting tensor is expressed as a limit along this curve. In this work we introduce and study partial degeneration, a special version of degeneration where one of the local linear maps is constant whereas the others vary along a curve. Motivated by algebraic complexity, quantum entanglement and tensor networks, we present constructions based on matrix multiplication tensors and find examples by making a connection to the theory of prehomogenous tensor spaces. We highlight the subtleties of this new notion by showing obstruction and classification results for the unit tensor. To this end, we study the notion of aided rank, a natural generalization of tensor rank. The existence of partial degenerations gives strong upper bounds on the aided rank of the tensor, which in turn allows one to turn degenerations into restrictions. In particular, we present several examples, based on the W-tensor and the Coppersmith-Winograd tensors, where lower bounds on aided rank provide obstructions to the existence of certain partial degenerations.

1. Introduction

Restriction and degeneration are preorders on the set of tensors that capture many important concepts in classical algebraic geometry, complexity theory, combinatorics, entanglement theory and the study of tensor networks. In this work, we introduce the notion of partial degeneration, a special version of degeneration defining a preorder which is intermediate between restriction and degeneration.

One key contribution of this work is to show that restriction, partial degeneration and degeneration are mutually inequivalent notions. To show a separation between the notions of restriction and partial degeneration we present a number of constructions based on tensors motivated from algebraic complexity theory and tensor networks. We also draw a connection to the theory of prehomogenous tensor spaces which allows us to derive further examples manifesting this separation. To show a separation between partial degenerations and degenerations, we prove a no-go result for the unit tensor which moreover allows us to classify certain families of partial degenerations.

Moreover, we introduce the notion of aided restriction, which is performed on a version of the tensor augmented via an aiding matrix. This raises the question on how large the rank of such an aiding matrix should be in order to allow certain restrictions. We study upper and lower bounds, highlighting the role of degenerations and partial degeneration.

In the remainder of this introduction we describe in more detail the main contributions of this work and briefly outline future directions and open questions.

1.1. Background. Let $U_1, U_2, U_3$ be complex finite-dimensional vectors spaces and write $e_1, e_2, \ldots, e_{u_i}$ for a fixed basis of the space $U_i$. Tensors in $U_1 \otimes U_2 \otimes U_3$ can be understood as resources by introducing the notions of restriction and degeneration: For tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$, we say that $T$ restricts to $S$, and write $T \geq S$, if there exist linear maps $A_i : U_i \to V_i$ for $i = 1, 2, 3$ such that

$$S = (A_1 \otimes A_2 \otimes A_3)T.$$
we say that $T$ degenerates to $S$, and write $T \succeq S$, if $S$ is a limit of restrictions of $T$. It is a classical result that $T \succeq S$ if and only if there are linear maps $A_i(\epsilon) : U_i \to V_i$ depending polynomially on $\epsilon$, that is with entries in the polynomial ring $\mathbb{C}[\epsilon]$, such that

$$(A_1(\epsilon) \otimes A_2(\epsilon) \otimes A_3(\epsilon))T = \epsilon^d S + \epsilon^{d+1} S_1 + \cdots + \epsilon^{d+e} S_e$$

for some natural numbers $d, e$ and some tensors $S_1, \ldots, S_e$. The integers $d$ and $e$ are called approximation degree and error degree, respectively; write $T \succ ^e_d S$ when it is useful to keep track of these integers.

One can define analogous notions for any number $k$ of tensor factors. It is known that for $k = 2$, the notions of restriction and degeneration are equivalent whereas for $k = 3$ they differ. This phenomenon already appears in [Syl52] and it occurs already when the tensor factors are two dimensional. Write $\langle r \rangle = e_1 \otimes e_1 \otimes e_1 + \cdots + e_r \otimes e_r \otimes e_r$ for the $r$-th unit tensor in $\mathbb{C}^r \otimes \mathbb{C}^r \otimes \mathbb{C}^r$. Define the $W$-tensor in $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ to be $W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes e_2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1$. It is a classical result that $(2) \succeq W$ but $(2) \not\succeq W$.

Understanding tensors in this resource theoretic way has led to advances in various fields in mathematics, physics and computer science:

- The study of degenerations has lead to advances in solving combinatorial problems, in particular the sunflower problem and cap sets [EG17, Tao16]. For example, in [CFTZ22], the study of combinatorial degenerations has lead to advances in the problem of finding large corner free sets.
- In quantum information theory, the study of restrictions and degenerations led to an improved understanding of the entanglement of multi-partite quantum states [DVC00, CD07].
- Tensor network representations of many body quantum states are a special case of tensor restrictions. Studying degenerations of tensors can lead to more efficient tensor network representations of quantum states [CLVW20, CGFW21].
- In algebraic complexity theory, tensor restriction and degeneration play an important role in the study of the asymptotic complexity of bilinear maps. In particular, starting from [BCLR79, Bin80], essentially all upper bounds on the exponent of the matrix multiplication rely on degenerations of suitable tensors to the matrix multiplication tensor [Str87, CWS87, AW21]. A refined understanding of these notions, even in small cases, can lead to further improvements [BCS97, Bla13, CGLV22].

In all of these cases, the notions of restriction and degeneration are used to compare tensors: For example, in complexity theory, $T \succeq S$ or $T \succeq S$ reflects the fact that the bilinear map corresponding to $T$ is harder to compute than the one corresponding to $S$; in quantum physics, it expresses the fact that the quantum state described by $T$ is more entangled than the one described by $S$.

1.2. Partial degeneration. In Section 3 of this work we introduce and study the intermediate notion of partial degeneration: For tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$, we say that $T$ partially degenerates to $S$, and write $T \succ ^e_{d,S} S$, if $T$ degenerates to $S$ where the degeneration map $A_1(\epsilon) = A_1$ can be chosen constant in $\epsilon$. Analogous notions can be defined by assuming that $A_2(\epsilon)$ or $A_3(\epsilon)$ are constant. For simplicity, we will always assume that this map is the first one. Hence, we have $T \succ ^e_{d,S} S$ if and only if there are linear maps $A_1, A_2(\epsilon)$ and $A_3(\epsilon)$, with $A_2(\epsilon)$ and $A_3(\epsilon)$ depending polynomially in $\epsilon$ such that $\lim_{\epsilon \to 0} \frac{1}{\epsilon^d} (A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon)) T = S$ for some $d$. As in the case of degeneration, we sometimes write $T \succ ^e_{d,S} S$ to keep track of the approximation and error degrees. We point out that only allowing one of the three maps depending on $\epsilon$ provides a notion of degeneration which is equivalent to restriction, see Remark 3.2. A priori, it is unclear whether the notion of partial degeneration is indeed non-trivial, or whether one might always reduce a degeneration to a partial degeneration or a partial degeneration to a restriction. We will show this is not the case. We point out that an example of partial degeneration has been
known since [Str87], and it was used to achieve a breakthrough result in the study of the complexity of matrix multiplication: the tensor

\[
\text{Str}_q = \sum_{i=1}^{q-1} e_i \otimes e_q \otimes e_i + e_i \otimes e_i \otimes e_q \in \mathbb{C}^{q-1} \otimes \mathbb{C}^q \otimes \mathbb{C}^q
\]

has tensor rank equal to \(2q - 2\) but it is a partial degeneration from \(q\).

In this work, we study for the first time partial degenerations in-depth. In Section 3, we construct various families of examples of honest partial degenerations. We also study the question under which circumstances partial degenerations cannot exist and provide a no-go result for certain partial degenerations of the unit tensor.

In Section 3.3, we construct a family of partial degenerations of the matrix multiplication tensor. Let \(\langle m,n,p \rangle\) be the matrix multiplication tensor associated to the bilinear map multiplying an \(m \times n\) matrix with a \(n \times p\) matrix. To construct a family of partial degenerations of the matrix multiplication tensor \(\langle 2,2,2 \rangle\), the challenge is to show that these are not actually restrictions of \(\langle 2,2,2 \rangle\). To see this, we resort to notion of tensor compressibility in the sense of [LM18].

In Section 3.4, we study the notion of partial degeneration in the setting of prehomogenous tensor spaces where we can find many more examples of honest partial degenerations. We say that the action of a linear algebraic group \(G\) on a vector space \(V\) is prehomogenous if there is an element \(v \in V\) whose orbit under the group is dense in \(V\), equivalently in the Zariski or Euclidean topology. Consider the action of \(\text{GL}(U_2) \times \text{GL}(U_3)\) on \(U_1 \otimes U_2 \otimes U_3\); if this action is prehomogeneous, with the tensor \(T\) having a dense orbit, then every tensor in \(U_1 \otimes U_2 \otimes U_3\) is a partial degeneration of \(T\). Prehomogenous tensor spaces of this form have been studied in [SK77, Ven19] and are well-understood, and the prehomogeneity of the action is determined by simple arithmetic relations among the dimensions of the tensor factors. In Section 3.4, for every instance where the space \(U_1 \otimes U_2 \otimes U_3\) is prehomogenous under the action of \(\text{GL}(U_2) \otimes \text{GL}(U_3)\), we provide an example of a tensor that cannot be a restriction of a tensor with dense orbit. We emphasize that while it is well-understood under which conditions \(U_1 \otimes U_2 \otimes U_3\) is prehomogenous, there are in general no closed formulas for elements having dense orbit. If \(\dim(U_1) = 2\), that is the case of matrix pencils, explicit elements with closed orbit are known, see, e.g., [Gan59, Ch. XIII] and [Pok86]. In Section 3.4, we use these examples to provide explicit partial degenerations for matrix pencils.

In Section 3.5, we study situations in which partial degenerations cannot occur. More precisely, we show that every partial degeneration of the unit tensor \(\langle r \rangle\) to a concise tensor \(T \in U_1 \otimes U_2 \otimes U_3\) with \(\dim(U_1) = r\) can be reduced to a restriction. We use this result to show that for \(\dim(U_1) = r-1\), tensors of the form (1) are essentially all honest partial degenerations that can occur. Furthermore, we construct honest partial degenerations of \(\langle r \rangle\) for any \(r\) to non-concise tensors. These no-go results show that restriction, partial degenerations and degenerations are in fact three different notions.

1.3. Aided restriction and aided rank. The starting point of the second part of this work is the fact that any degeneration can be turned into a restriction using interpolation. It is known that if \(T \succeq S\) then \(T \boxtimes (d+1) \succeq S\) where \(\boxtimes\) is the Kronecker product of tensors. In Section 4, we study the case where the supporting tensor is a matrix instead of a unit tensor. More precisely, for a tensor \(T \in U_1 \otimes U_2 \otimes U_3\), define

\[
T^{p} = T \boxtimes (1,1,p) \in U_1 \otimes (U_2 \otimes \mathbb{C}^p) \otimes (U_3 \otimes \mathbb{C}^p);
\]

Lemma 4.2 shows that if \(T \succeq S\) then there exists a \(p\) such that \(T^{p} \succeq S\). We call \(1,1,p\) the aiding matrix and \(p\) the rank of the aiding matrix. We study upper bounds on the rank of an aiding matrix in terms of approximation and error degree of a degenerations and a partial degenerations.
In Section 4.1, we show that for partial degenerations of approximation degree \( d \) and error degree \( e \), the rank of the aiding matrix can be equal to \( \min\{d+1, e+1\} \). More precisely, if \( T \in_d S \) then \( T^{\#d+1} \geq S \) and \( T^{\#e+1} \geq S \). Even more strikingly we show that if \( T \in U_1 \otimes U_2 \otimes U_3 \), where the space \( U_1 \otimes U_2 \otimes U_3 \) is prehomogenous under the action of \( \text{GL}(U_2) \times \text{GL}(U_3) \), and if the orbit of \( T \) is dense, then \( T^{\#2} \geq S \) for any other tensor \( S \in U_1 \otimes U_2 \otimes U_3 \). Using methods from algebraic geometry, we generalize this observation to the setup where the orbit closure is a lower dimensional variety.

It turns out that these findings are in strong contrast to the ease of degenerations that are not partial degenerations. To see this, in Section 4.2, we develop a method to lower bound the minimal possible rank of an aiding matrix. This relies on a variant of the substitution method \([\text{AFT}11]\). In Section 4.2, we define the notion of aided rank as

\[
R^p(T) = \min\{r : \langle r \rangle^p \geq T \}.
\]

When \( p = 1 \), this reduces to the notion of tensor rank \([\text{Lan}17, \text{Prop. 5.1.2.1}]\). We show that one can generalize the substitution method to give lower bounds on aided rank and on the minimal possible rank of an aiding matrix for several examples of degeneration. For example, for the degeneration

\[ (2^k) \ni W^\otimes k \]

we show that \( R^{2^k-1}(W^\otimes k) \geq 2^k + 1 \). In other words, the minimal rank \( p \) of an aiding matrix turning the degeneration in equation (2) into a restriction is \( 2^k \). Note that for this example, the no-go result for partial degenerations \( \text{Theorem 3.14} \) gives that the degeneration cannot be realized as a partial degeneration. Also note that the minimal possible rank of an aiding matrix differs from our naive upper bound from \( \text{Theorem 4.2} \) only by a factor of \( 1/2 \).

1.4. Conclusion and outlook. In this work, we introduce and study for the first time partial degenerations which is the natural intermediate notion of restriction and degeneration of tensors. We believe that studying partial degenerations can yield deeper insight in the theory of restriction and degeneration of tensors and thereby also deeper insight into combinatorial and complexity theoretic questions as well as physical processes. We identify some open problems arising from this work:

1. Consider the tensor

\[
\lambda = e_1 \wedge e_2 \wedge e_3 \otimes e_3 \otimes e_3 \in \mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3
\]

where \( \wedge \) is the antisymmetric product. In \([\text{CLVW}20]\), the fact that \( (2, 2, 2) \ni \lambda \) was used to construct an efficient representation in the ‘projected entangled pair’ (PEPS) formalism of the ‘resonating valence bond state’ (RVB state) \([\text{And}73]\). It is known that \( (2, 2, 2) \notin \lambda \) and we conjecture that a partial degeneration is not possible either.

2. Consider a degeneration

\[
(A_1(\epsilon) \otimes A_2(\epsilon) \otimes A_3(\epsilon)) T = \epsilon^d S + \sum_{i=1}^e \epsilon^{d+i} S_i.
\]

To the best of our knowledge there are no known examples where the approximation degree \( d \) is provably bounded from below (e.g. \( d \geq 2 \)). We propose that it might be more tangible to show bounds like this in the context of partial degenerations.

3. The fact that degenerations can be turned in to restrictions by interpolating with a unit tensor (see \( \text{Theorem 4.1} \)) is the key ingredient that allows one to use border rank in the study of matrix multiplication complexity, see e.g. \([\text{BCLR}79, \text{CW}87]\). It has been used more recently to construct more efficient tensor network representations of quantum states \([\text{CLVW}20]\) and for many other theoretical applications like showing that tensor rank and tensor border rank are not multiplicative under
the tensor product [CJZ18, CGJ19]. It is interesting to ask for similar applications when interpolating with a matrix instead of a unit tensor as in Lemma 4.2.

(4) In this work, we focused on tensors of order three over complex numbers. Many of our results generalize to the more general setup of k-fold tensor products of vector spaces over any field and it would be interesting to further explore these situations. For higher order tensor products, one potentially has a hierarchy of partial degenerations, depending on the number of linear maps depending on ε.

(5) In the recent work [JLV22], the authors prove uniqueness results and polynomial time algorithms for r-aided rank decompositions. They generalize their results to so-called (X, W)-decompositions. Many of our results on r-aided rank also generalize to this set-up and it would be interesting to further explore this direction.

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2. Preliminaries

In this section we revisit the theory of restrictions and degenerations of tensors. We introduce a few special tensors that will be used in the paper and mention their relation to algebraic complexity theory and quantum information theory. For a thorough introduction to geometric aspects of degeneration of tensors we refer to [Lan12]. More details about the relation to algebraic complexity theory can be found in [BCS97, Bla13]. An introduction to tensor networks can be found in [CPGSV21].

Let $U_1, U_2, U_3$ and $V_1, V_2, V_3$ be complex vector spaces of dimensions $u_1, u_2, u_3$ and $v_1, v_2, v_3$, respectively, and consider tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$. We say that $T$ restricts to $S$ and write $T \supseteq S$ if there are linear maps $A_i : U_i \rightarrow V_i$ such that $S = (A_1 \otimes A_2 \otimes A_3)T$. We say that $T$ degenerates to $S$ and write $T \preceq S$ if there are linear maps $A_i(\epsilon) : U_i \rightarrow V_i$ depending polynomially in $\epsilon$ such that

$$(A_1(\epsilon) \otimes A_2(\epsilon) \otimes A_3(\epsilon))T = \epsilon^dS + \epsilon^{d+1}S_1 + \ldots + \epsilon^{d+e}S_e$$

for some natural numbers $d, e$ and some tensors $S_1, \ldots, S_e$. The quantities $d$ and $e$ are called approximation degree and error degree, respectively. Sometimes, we want to keep track of $d$ and $e$ and write $T \preceq^d S$.

Recall the Kronecker product of tensors. Fixing bases and letting $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ be specified in these bases by coefficients $T_{i_1,i_2,i_3}$ and $S_{j_1,j_2,j_3}$, that is,

$$T = \sum_{i_1,i_2,i_3=1}^{u_1,u_2,u_3} T_{i_1,i_2,i_3} e_{i_1} \otimes e_{i_2} \otimes e_{i_3}, \quad S = \sum_{j_1,j_2,j_3=1}^{v_1,v_2,v_3} S_{j_1,j_2,j_3} e_{j_1} \otimes e_{j_2} \otimes e_{j_3}.$$

We define their Kronecker product $T \otimes S \in (U_1 \otimes V_1) \otimes (U_2 \otimes V_2) \otimes (U_3 \otimes V_3)$ as

$$T \otimes S = \sum_{i_1,i_2,i_3,j_1,j_2,j_3} T_{i_1,i_2,i_3} S_{j_1,j_2,j_3} e_{i_1} \otimes e_{j_1} \otimes (e_{i_2} \otimes e_{j_2}) \otimes (e_{i_3} \otimes e_{j_3}).$$

We will use the short notation $T \otimes^k$ for $T \otimes \cdots \otimes T$ where the Kronecker product was taken $k$ times.

Another way of combining two tensors $T$ and $S$ in equation (3) into one is the direct sum. For that, pick a basis $e_1, \ldots, e_{u_1+v_1}$ of $U_1 \oplus V_1$ for $i = 1, 2, 3$ and define the tensor
For example, the bilinear map induced by $T$ we will in particular use matrix multiplication tensors of the form $T^\ast = T \otimes (1, 1, p)$. The tensors mentioned in this section have natural applications in algebraic complexity theory. A tensor $T \in U_1 \otimes U_2 \otimes U_3$ naturally defines a bilinear map $T : U_1^* \times U_2^* \to U_3$. For example, the bilinear map induced by $\langle r \rangle$ multiplies two $r$-dimensional vectors entry-wise. The induced bilinear map for the matrix multiplication tensor $\langle m, n, p \rangle$ is the matrix multiplication, mapping two matrices, of size $m \times n$ and $n \times p$ respectively, to (the transpose of) their product, which is a matrix of size $m \times p$. In this context, the rank of a tensor
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encodes the number of scalar multiplications needed to evaluate the associated bilinear map. For example, a major open problem in algebraic complexity theory is to determine the exponent of matrix multiplication $\omega$. This is the minimal $\omega$ such that for any $\epsilon > 0$ one can multiply $n \times n$ matrices using $O(n^{\omega+\epsilon})$ multiplications; it can be defined in terms of rank and border rank as the minimum $\omega$ such that $R((n,n,n))$ or equivalently $\overline{R}((n,n,n))$ is in $O(n^\omega)$ [BCS97, Blä13].

The tensors introduced so far have a natural interpretation in quantum information theory as well. In quantum physics, tensors correspond to multiparticle quantum states and the notion of restriction is known as conversion under stochastic local operations and classical communication (SLOCC). In this context, the unit tensor is known as the unnormalized Greenberger-Horn-Zeilinger (GHZ) state. The $W$ tensor also plays an important role: The fact that (2) $\not\sim W$ and $W \not\sim (2)$ was used in [DVC00] to observe that three qubits can be genuinely three-party entangled in two inequivalent ways. The tensor $(1,1,p)$ has a natural interpretation as well: It describes a quantum state on three parties where the second and third parties share an Einstein-Podolsky-Rosen pair (EPR pair) on $p$ levels. For any tensor $T$, the tensor $T^{op}$ is the overall (unnormalized) quantum state when all three parties share a quantum state associated to $T$ and the second and third party in addition are given an EPR pair on $p$ levels.

3. Partial degeneration

In this section we introduce and study the notion of partial degeneration which is a natural intermediate notion between restriction and degeneration. In Section 3.1 we will define partial degeneration. After that we review in Section 3.2 a known example of a partial degeneration. In Section 3.3 we will recall a property of tensors called compressibility and demonstrate with an example how this can be used to rule out restriction. We will see more examples in Section 3.4 using the theory of prehomogenous tensor spaces. Finally, in Section 3.5 we will study situations where no honest partial degeneration exist.

3.1. Definition and motivation. The main concept of this section is the following special version of degeneration, intermediate between restriction and fully general degeneration.

**Definition 3.1.** Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ be tensors. We say that $T$ degenerates partially to $S$ and write $T \bowtie S$ if there is a linear map $A_1 : U_1 \to V_1$ and linear maps $A_2(\epsilon), A_3(\epsilon)$ with entries in the polynomial ring $\mathbb{C}[\epsilon]$ such that

$$(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))T = \epsilon^d S + \epsilon^{d+1} S_1 + \cdots + \epsilon^{d+r} S_r.$$

We sometimes write $T \bowtie^d S$ to keep track of $d$ and $\epsilon$. We call a partial degeneration $T \bowtie S$ an honest partial degeneration if $T$ does not restrict to $S$.

It is clear that every restriction is a partial degeneration and every partial degeneration is a degeneration. This raises the following two questions:

1. Can every partial degeneration $T \bowtie S$ be realized as a restriction $T \geq S$?
2. Can every degeneration $T \bowtie S$ be realized as a partial degeneration $T \bowtie S$?

We point out that only allowing one of the three linear maps to depend on $\epsilon$ provides the same notion as restriction:

**Remark 3.2.** Let $S \in U_1 \otimes U_2 \otimes U_3$, $S \in V_1 \otimes V_2 \otimes V_3$ be tensors and suppose there are linear maps $A_1, A_2, A_3(\epsilon)$ with $A_1 : U_1 \to V_1$, and $A_3(\epsilon)$ depending polynomially in $\epsilon$ such that $S = \lim_{\epsilon \to 0} \frac{1}{\epsilon d} (A_1 \otimes A_2 \otimes A_3(\epsilon)(T))$. Then $S$ is a restriction of $T$. To see this, write $A_3(\epsilon) = \epsilon^d A_{3,d} + \cdots + \epsilon^{d+r} A_{3,r}$ with $A_{3,d} : U_3 \to V_3$. It is immediate that $S = A_1 \otimes A_2 \otimes A_{3,d}(T)$; this expresses $S$ as a restriction of $T$. 

In Sections 3.2, 3.3 and 3.4 we provide families of examples demonstrating that the answer to question in (1) is negative. In Section 3.5, we show that the answer to question (2) is negative as well.

3.2. Strassen’s tensor. In [Str87], a first example of a partial degeneration was found: Let $U_1 \simeq \mathbb{C}^{r-1}$ and $U_2 \simeq U_3 \simeq \mathbb{C}^q$ and consider the tensor

$$\text{Str}_q = \sum_{i=1}^{q-1} e_i \otimes e_i \otimes e_q + e_i \otimes e_q \otimes e_i \in U_1 \otimes U_2 \otimes U_3.$$ 

It is not hard to see that $R(\text{Str}_q) = 2q - 2$. On the other hand, it is a partial degeneration of the unit tensor $\langle q \rangle$ via

$$c\text{Str}_q = \sum_{i=1}^{q-1} e_i \otimes (e_q + e e_i) \otimes (e_q + e e_i) - \left( \sum_{i=1}^{q-1} e_i \right) \otimes e_q \otimes e_q + \mathcal{O}(e^2).$$

In Section 3.5, we will show that these are essentially all partial degenerations of $\langle r \rangle$ that can be found in $U_1 \otimes U_2 \otimes U_3$ with $\dim(U_1) = r - 1$.

3.3. Compressibility of tensors and matrix multiplication. In this section we will find a family of examples of partial degeneration of the $2 \times 2$-matrix multiplication tensor. One challenge of finding an honest partial degeneration is to show that this partial degeneration is actually not a restriction. To achieve that, we recall the notion of compressibility [LM18].

**Definition 3.3.** A tensor $T \in U_1 \otimes U_2 \otimes U_3$ is $(a_1, a_2, a_3)$-compressible if there are linear maps $A_i : U_i \to U_i$ of rank $a_i$ such that $(A_1 \otimes A_2 \otimes A_3)T = 0$.

Geometrically, $T$ is $(a_1, a_2, a_3)$-compressible if there are linear subspaces $U'_j \subseteq U^*_j$ with dim $U'_j = a_i$, such that, as a trilinear map $T|_{U'_1 \otimes U'_2 \otimes U'_3} \equiv 0$. In coordinates, this is equivalent to the existence of bases of the spaces $U_1, U_2$ and $U_3$ such that, in these bases, $T_{i_1,i_2,i_3} = 0$ if $i_j \geq \dim U_i - a_j$. We visualize the concept of an $(a_1, a_2, a_3)$-compressible tensor in Figure 2.

The following technical result will become handy later.

**Lemma 3.4.** Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$. Let $T \geq S$ and let $S$ be concise. Moreover, assume that $S$ is $(a_1, a_2, a_3)$-compressible. Then, $T$ is also $(a_1, a_2, a_3)$-compressible.

**Proof.** By assumption, there are maps $A_i$ with rank $a_i$ such that $(A_1 \otimes A_2 \otimes A_3)T = 0$. As $S$ is concise, the restriction maps $M_i$ must be surjective where $S = (M_1 \otimes M_2 \otimes M_3)T$. Therefore, the maps $(A_1M_1, A_2M_2)$ and $A_3M_3$ also have rank $a_1, a_2$ and $a_3$, respectively. Since $(A_1M_1 \otimes A_2M_2 \otimes A_3M_3)T = (A_1 \otimes A_2 \otimes A_3)S = 0$ the claim follows. \qed
Lemma 3.4 can be used to exclude restrictions $T \geq S$ if $T$ is less compressible than $S$. An example of a tensor that is not “very compressible” is the matrix multiplication tensor.

**Lemma 3.5.** The two by two matrix multiplication tensor $(2,2,2)$ is not $(2,3,3)$-compressible.

**Proof.** Note that any $4 \times 4$ matrix $M = (M_{(u,v),(x,y)})^2_{u,v,x,y=1}$ (labelled by double indices) induces a linear endomorphism of the space of $2 \times 2$ matrices via

$$M : x \mapsto M.x = (M.x)_{i,j=1,2}, \quad (M.x)_{i,j} = \sum_{k,l=1,2} M_{(i,j),(k,l)}x_{k,l}.$$ 

Recall that $(2,2,2)$ corresponds to calculating the four bilinear forms $z_{j,i} = x_{i1}y_{1j} + x_{i2}y_{2j}$ for $i,j = 1,2$, that is, the entries of $(x \cdot y)^T$ where the entries of $x$ and $y$ are regarded as variables.

Let $S = (A_1 \otimes A_2 \otimes A_3)(2,2,2)$ be a restriction of the two-by-two matrix multiplication tensor. Interpreting $A_1, A_2$ and $A_3$ as $4 \times 4$ matrices, an easy calculation shows that the four bilinear forms corresponding to the tensor $S$ are the four entries of the transpose of

$$A_3, ((A_1.x) \cdot (A_2.y)).$$

Now, let the rank of $A_1$ and $A_2$ be at least $3$ and the rank of $A_3$ be at least $2$. It is clear that the space of all $A_1.x$ for $x \in M_{2 \times 2}$ is at least 3 dimensional (the same holds for $A_2$). It is well-known that every subspace of $M_{2 \times 2}$ of dimension at least 3 must contain an invertible matrix. Choosing $x_0 \in M_{2 \times 2}$ such that $A_1x_0$ is invertible, we see that the space of matrices of the form $(A_1.x_0) \cdot (A_2.y)$ for $y \in M_{2 \times 2}$ contains three linearly independent matrices. Hence, since we assumed that $A_3$ has rank $\geq 2$, we see that equation (4) cannot be identically $0$. This finishes the proof. □

The following technical result is a generalization of a fact well-known in the tensor network literature and will help us to construct partial degenerations of the matrix multiplication tensor.

**Lemma 3.6.** Let $V_1, V_2, V_3$ be vector spaces with dimensions $v_1, v_2, v_3$ and let $S \in V_1 \otimes V_2 \otimes V_3$. Then, we have $(m,n,p) \geq S$ if and only if there are a natural number $d$ and matrices

$$\alpha_1(\epsilon), \ldots , \alpha_{v_1}(\epsilon) \in \mathbb{C}[\epsilon]^{m \times n}, \beta_1(\epsilon), \ldots , \beta_{v_2}(\epsilon) \in \mathbb{C}[\epsilon]^{n \times p}, \gamma_1(\epsilon), \ldots , \gamma_{v_3}(\epsilon) \in \mathbb{C}[\epsilon]^{p \times m}$$

such that $\epsilon^d S_{i,j,k} = \text{tr}(\alpha_i(\epsilon)\beta_j(\epsilon)\gamma_k(\epsilon)) + \mathcal{O}(\epsilon^{d+1})$.

Moreover, if the matrices $\alpha_1(\epsilon) \ldots , \alpha_{v_1}(\epsilon)$ can be chosen constant in $\epsilon$ we have $T \in S$. If all matrices in (5) can be chosen constant in $\epsilon$, we have $(m,n,p) \geq S$.

With this, we are now ready to find honest partial degenerations of $(2,2,2)$.

**Proposition 3.7.** Every concise tensor $S \in \mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ that is $(3,3,3)$-compressible is an honest partial degeneration of $(2,2,2)$.

**Proof.** Fixing bases, we can write our tensor $S$ as

$$S = \sum_{i,j,k=1}^{3,4,4} S_{i,j,k} e_i \otimes e_j \otimes e_k$$

such that $S_{i,j,k} = 0$ whenever both $j$ and $k$ are $\geq 2$.

From Lemma 3.6, it suffices to find $2 \times 2$ matrices

$$\alpha_1, \ldots , \alpha_3 \in \mathbb{C}^{m \times n}, \beta_1(\epsilon), \ldots , \beta_4(\epsilon) \in \mathbb{C}[\epsilon]^{n \times p}, \gamma_1(\epsilon), \ldots , \gamma_4(\epsilon) \in \mathbb{C}[\epsilon]^{p \times m}$$

such that

$$\epsilon^d S_{i,j,k} = \text{tr}(\alpha_i(\epsilon)\beta_j(\epsilon)\gamma_k(\epsilon)) + \mathcal{O}(\epsilon^{d+2}).$$

Choosing matrices
one easily verifies that (6) is fulfilled.

Since $S$ is concise and is $(3,3,3)$-compressible we conclude with Lemma 3.4 and 3.5 that $S$ is an honest partial degeneration of $(2,2,2)$.

**Remark 3.8.** Lemma 3.5 implies that no concise tensor which is $(2,3,3)$-compressible is a restriction of $(2,2,2)$. One might ask, if Proposition 3.7 still holds if we relax the condition on $S$ to being $(2,3,3)$-compressible. This turns out to be not true: In fact, it has been shown that the closure of the set of all degenerations of $(2,2,2)$ in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ has dimension 31 [BLG21]. A simple calculation – the code for which can be found in Appendix A – shows that the orbit closure of a generic element of $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ which is $(2,3,3)$-compressible has dimension at least 37.

3.4. **Prehomogeneous spaces.** In this section, we will see more examples of partial degenerations by making a connection to the well-studied theory of prehomogeneous tensor spaces.

**Definition 3.9.** Let $G$ be a group acting on a vector space $V$. We say that $V$ is prehomogenous under the action of $G$ if there is an element $T \in V$ such that $G.T$ is dense in $V$ with respect to the Zariski topology, i.e. $\overline{G.T} = V$.

Consider the space $U_1 \otimes U_2 \otimes U_3$ where the $U_i$ are vector spaces of dimension $u_i$. Clearly, if $T, S \in U_1 \otimes U_2 \otimes U_3$ such that $S$ is in the orbit closure of $T$ under the action of $\text{GL}(U_2) \times \text{GL}(U_3)$, then $T \triangleright S$. Hence, if $U_1 \otimes U_2 \otimes U_3$ is prehomogenous under the action of $\text{GL}(U_2) \times \text{GL}(U_3)$, and $T$ is an element of the dense orbit then all tensors $S \in U_1 \otimes U_2 \otimes U_3$ are partial degenerations of $T$.

Prehomogeneity of $U_1 \otimes U_2 \otimes U_3$ under the action of $\text{GL}(U_2) \times \text{GL}(U_3)$ only depends on the dimensions of the vector spaces involved and is easy to check [SK77, Ven19].

**Theorem 3.10.** Assume $u_2 \leq u_3$. Define $\lambda(u_1) = \frac{u_1 + \sqrt{u_1^2 - 4}}{2}$. Then, the space $U_1 \otimes U_2 \otimes U_3$ is prehomogenous under the action of $\text{GL}(U_2) \times \text{GL}(U_3)$ if and only if $u_3 > \lambda(u_1)u_2$.

Hence, for any choices of $u_1, u_2, u_3$ satisfying the conditions in Theorem 3.10, there is an element $T \in U_1 \otimes U_2 \otimes U_3$ such that for all $S \in U_1 \otimes U_2 \otimes U_3$ it holds $T \triangleright S$.

To show that there exists $S$ which is not a restriction of $T$, we use the following well-known statement. For completeness, we will include a proof.

**Lemma 3.11.** Let $T, S \in U_1 \otimes U_2 \otimes U_3$ be tensors. Assume $S$ is concise. Then $T \geq S$ if and only if $T$ and $S$ lie in the same $\text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$-orbit.

**Proof.** By definition, $T \geq S$ holds if and only if there are linear maps $A_1, A_2$ and $A_3$ such that $(A_1 \otimes A_2 \otimes A_3)T = S$. It is easy to see that such a tensor $S$ cannot be concise if any of the three maps is not invertible: In fact, assume without loss $A_1$ not invertible, that is, there exists a non-zero vector $u_1 \in U_1$ not contained in the image of $A_1$. But then the dual
$u^1 \in U_1^*$ of this vector is in the kernel of $S$ when interpreted as linear map $S : U_1^* \to U_2 \otimes U_3$. In other words, $S$ is not concis. This proves the claim.

**Theorem 3.12.** Let $U_1, U_2, U_3$ have dimensions $u_1, u_2, u_3$ such that $\lambda(u_1)u_2 < u_3 < u_1u_2$, where $\lambda(u_1) = \frac{u_1+\sqrt{u_1^2-4}}{2}$. Then there exist tensors $T,S \in U_1 \otimes U_2 \otimes U_3$ such that $T \not\leq S$ but $T \not\geq S$.

**Proof.** We know from Theorem 3.10 that the space $U_1 \otimes U_2 \otimes U_3$ is prehomogeneous under $\text{GL}(U_2) \times \text{GL}(U_3)$. Let $T$ be a tensor in the dense $\text{GL}(U_3) \times \text{GL}(U_3)$-orbit, so that $T \not\leq S$ for every $S \in U_1 \otimes U_2 \otimes U_3$.

Let $p = u_1u_2 - u_3$. Note that $u_1 - 1 \leq \lambda(u_1) < u_1$, so $\lambda(u_1)u_2 < u_3 < u_1u_2$ implies that $0 < p < u_2$. Define the tensor $S \in U_1 \otimes U_2 \otimes U_3$ as

$$S = \sum_{i=1}^{u_1-1} e_i \otimes \left( \sum_{j=1}^{u_2} e_j \otimes e_{(i-1)u_2+j} \right) + e_{u_1-1} \otimes \left( \sum_{j=1}^{u_2-p} e_j \otimes e_{(u_1-1)u_2+j} \right)$$

It is not hard to see that the tensor $S$ is concis.

To show that $T$ and $S$ lie in different $\text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$-orbits, we compute the dimensions of these orbits. Denote $G = \text{GL}(U_1) \times \text{GL}(U_2) \times \text{GL}(U_3)$.

For $T$ we have $U_1 \otimes U_2 \otimes U_3 \not\leq G \cdot T \supset [\text{GL}(U_2) \times \text{GL}(U_3)] \cdot T = U_1 \otimes U_2 \otimes U_3$, hence $G \cdot T = U_1 \otimes U_2 \otimes U_3$ and $\dim G \cdot T = u_1u_2u_3$.

For $S$, the dimension of the orbit $G \cdot S$ can be found as $\dim G \cdot S = \dim G - \dim \text{Stab}_G(S)$. The stabilizer $\text{Stab}_G(S)$ is isomorphic to $P(1, u_1) \times P(u_2 - p, u_2)$ where $P(a,b) \subset \text{GL}b$ is the parabolic group preserving a subspace of dimension $a$. Indeed, let $S_{i} \in U_2 \otimes U_3$ be the slices of $S$ corresponding to the standard basis, that is, $S = \sum_{i=1}^{u_1} e_i \otimes S_i$. Note that $\text{rk}(S_i) = u_2$ for $i < u_1$ and $\text{rk}(S_{u_1}) = u_2 - p$. Moreover, a nonzero linear combination $\sum_{i=1}^{u_1} \alpha_i S_i$ has rank $u_1 - 1$ if and only if $\alpha_i = 0$ for $i \leq u_1 - 1$. It follows that $(A \otimes B \otimes C)S = S$, then $A$ preserves the 1-dimensional subspace $(e_{(u_1-1)})$. Therefore, we have $a_{u_1-1} (B \otimes C)S_{u_1} = S_{u_1}$ and it follows that $B$ preserves the $(u_1 - p)$-dimensional subspace $(e_1, \ldots, e_{u_2-p+1})$, which is the image of $S_{u_1}$ considered as a linear map $U_3^* \to U_2$. Now, given $A$ and $B$ which preserve the required subspaces, the map $C$ such that $(A \otimes B \otimes C)S = S$ always exists and is unique. To prove this, note that $S$ considered as a linear map $U_3^* \to U_1 \otimes U_2$ is an isomorphism between $U_3^*$ and the subspace $\{(e_1 \ldots, e_{u_1-1}) \otimes U_2 \otimes (e_{u_1}) \otimes (e_1, \ldots, e_{u_2-p})\} \subset U_1 \otimes U_2$. Thus, $C$ can be found as the contragredient map to $A \otimes B$ restricted to this subspace.

From the description of $\text{Stab}_G(S)$ it follows that

$$\dim \text{Stab}_G(S) = (u_1^2 - u_1 + 1) + (u_2^2 - p(u_2 - p))$$

and

$$\dim G \cdot S = u_2^3 + (u_1 - 1) + p(u_2 - p) = u_3(u_1u_2 - p) + (u_1 - 1) + p(u_2 - p) = u_1u_2u_3 - p(u_3 - u_2) + u_1 - 1 - p^2 < u_1u_2u_3 - u_3 + u_2 + u_1 - 2 < u_1u_2u_3.$$ 

The last inequality holds because $u_2$ cannot be equal to 1 under the assumptions of the theorem, and thus $u_3 \geq (u_1 - 1)u_2 > u_1 - 2 + u_2$.

It follows that the orbits of $T$ and $S$ are distinct and thus $T \not\leq S$ by Lemma 3.11. \hfill $\Box$

The proof of Theorem 3.12 can be used to find concrete examples for partial degenerations: In fact, the proof of Theorem 3.10 gives a recursive way of constructing elements with dense orbit. A closed formula for elements of $U_1 \otimes U_2 \otimes U_3$ that have dense orbit on the other hand is not known. To see more concrete examples of partial degenerations, we now focus on tensors $T \in \mathbb{C}^2 \otimes \mathbb{C}^m \otimes \mathbb{C}^n$. Clearly, Theorem 3.10 tells us that this space is $\text{GL}(\mathbb{C}^m) \times \text{GL}(\mathbb{C}^n)$-prehomogeneous whenever $m \neq n$. Fixing as basis $e_0, e_1$ of $\mathbb{C}^2$, we can write our tensor as $T = e_0 \otimes T_0 + e_1 \otimes T_1$ where $T_0, T_1 \in \mathbb{C}^m \otimes \mathbb{C}^n$ can be thought of as...
$m \times n$ matrices. In that way, our tensor is uniquely specified by a tuple of matrices $[T_0, T_1]$ which one often calls the \textit{matrix pencil} associated with $T$.

The fact that matrix pencil spaces are prehomogeneous has been known for a long time. In particular, we know an explicit element whose orbit is dense from \cite{Pok86}.

**Theorem 3.13.** For $m < n$ the action of $GL(C^m) \times GL(C^n)$ on $C^2 \otimes C^m \otimes C^n$ is prehomogeneous, that is, there is a tensor $T \in C^2 \otimes C^m \otimes C^n$ such that its orbit closure is the whole space:

$$(GL(C^m) \times GL(C^n) ) \cdot T = C^2 \otimes C^m \otimes C^n$$

In particular, for any tensor $T$ with dense orbit, every other tensor is a partial degeneration of $T$. A particular choice of a tensor $T$ with dense orbit is the tensor associated with the matrix pencil $[I_0, I_1]$ where

$$I_0 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix}, \quad I_1 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

For example, let $n = m + 1$. According to Theorem 3.12, the pencil

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad S_1 = \begin{pmatrix} 0 & 0 & \ldots & 0 & 1 \\ 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots \\ 0 & 0 & \ldots & 0 & 0 \end{pmatrix}$$

is an honest partial degeneration of the dense orbit element in equation (7).

3.5. **A no-go result for the unit tensor.** In this section we will see that under certain circumstances, partial degenerations do not exist even when degenerations do. Recall that we defined the unit tensor as $\langle r \rangle = e_1 \otimes e_1 \otimes e_1 + \cdots + e_r \otimes e_r \otimes e_r$. We first show that there are no proper partial degenerations of the unit tensor if the constant map has full rank. Note that the rank condition on the constant map cannot be dropped: Already in Section 3.2, we saw an example of an honest partial degeneration where the constant map has rank $r - 1$. We will use our no-go result to prove a classification result for this set-up.

**Proposition 3.14.** Let $S \in V_1 \otimes V_2 \otimes V_3$ be any tensor. If $\langle r \rangle \geq S$ via degeneration maps $A_1, A_2(\epsilon)$ and $A_3(\epsilon)$ where the constant map $A_1$ is of full rank $r$ then $\langle r \rangle \geq S$.

**Proof.** It is clear that we can assume $\dim(V_1) = r$ and that $A_1$ is invertible.

Assume

$$S = \lim_{\epsilon \to 0} (\text{id} \otimes A_2(\epsilon) \otimes A_3(\epsilon))(\langle r \rangle)$$

is a degeneration where the first map is the identity. That is, we have

$$S = e_1 \otimes M_1 + \ldots + e_r \otimes M_r$$

where $M_i = \lim_{\epsilon \to 0} A_2(\epsilon) e_i \otimes A_3(\epsilon) e_i$. Hence, it is clear that for all $i$, $M_i$ must be a rank one matrix as limit of rank one matrices. But clearly, a tensor of the form in equation (8) where the $M_i$ are rank one is a restriction of $\langle r \rangle$.

Now, let $S = \lim_{\epsilon \to 0} (A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))(\langle r \rangle)$ be any partial degeneration of $\langle r \rangle$. From before, we know that $\hat{S} = (A_1^{-1} \otimes \text{id} \otimes \text{id})S = \lim_{\epsilon \to 0} (\text{id} \otimes A_2(\epsilon) \otimes A_3(\epsilon))(\langle d \rangle)$ is a restriction of $\langle d \rangle$. Hence, the same holds for $\hat{S} = (A_1 \otimes \text{id} \otimes \text{id})\hat{S}$. This finishes the proof. \hfill $\square$

**Remark 3.15.** We note that the result in Proposition 3.14 does not apply to degenerations. Proposition 3.14 in particular says that if $V_1$ is has dimension $r$ and $S \in V_1 \otimes V_2 \otimes V_3$ concise we cannot have an honest partial degeneration $\langle r \rangle \geq S$ (else the constant map
would be invertible by conciseness of $S$). But, for example, it is well-known that the unit tensor \((2)\) does not restrict but degenerate to $W = e_1 \otimes e_1 + e_2 \otimes e_1 + e_2 \otimes e_1 + e_3 \otimes e_1 \otimes e_1$ which is concise in the same space as \((2)\). Hence, $W$ is an honest degeneration of \((2)\) but not a partial degeneration. We note that the same holds for the degenerations \((2^k) \otimes W^{\otimes k}\) for all $k$.

It is clear that one cannot drop the condition that $A_1$ has full rank: in Section 3.2, we saw that Strassen’s tensor $\text{Str}_r$ is an example of a partial degeneration of \((r)\) where $A_1$ has rank $r - 1$. In fact, we can use Proposition 3.14 to prove the following characterization of all partial degenerations of \((r)\) where the constant map has rank $r - 1$.

**Proposition 3.16.** Let $T \in U_1 \otimes U_2 \otimes U_3$ with $\dim(U_1) = r - 1$ be a concise tensor such that \((r) \bowtie T\) and \((r) \not\bowtie T\). Then for some $q$ such that $3 \leq q \leq r$ the tensor $T$ decomposes as

$$T = S_q + X_{r-q}$$

where $\text{Str}_q \geq S_q$ and \((r-q) \geq X_{r-q}\).

**Proof.** Suppose \((r) \bowtie T\) via a partial degeneration

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^d} (A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r \rangle = T \in U_1 \otimes U_2 \otimes U_3.$$

Since $T$ is concise, the map $A_1$ has rank equal to $\dim(U_1) = r - 1$. Note that $A_1$ can be factored as $A_1 = AM_q DP$ where $A: C^{r-1} \to U_1$ is invertible, $M_q: C^r \to C^{r-1}$ is defined as

$$M_q: \begin{cases} e_i \mapsto e_i \\ e_r \mapsto e_1 + \cdots + e_{q-1} \end{cases}$$

for $1 \leq i \leq r - 1$

with $1 \leq q \leq r$, $D: C^r \to C^r$ is diagonal, and $P: C^r \to C^r$ is a permutation matrix. Indeed, suppose $\pi \in S_r$ is a permutation such that $A_1 e_{\pi(1)}, \ldots, A_1 e_{\pi(r-1)}$ are linearly independent and $A_1 e_{\pi(r)} = \lambda_1 A_1 e_1 + \cdots + \lambda_{q-1} A_1 e_{q-1}$ with nonzero $\lambda_1, \ldots, \lambda_{q-1}$. Defining $A$ to be the map $A: e_i \mapsto \lambda_i A_1 e_{\pi(i)}$, $D = \text{diag}(\lambda_1^{-1}, \ldots, \lambda_{q-1}^{-1}, \ldots, 1)$, and $P$ the permutation matrix corresponding to $\pi^{-1}$, we get the required factorization.

Note that \((DP \otimes \text{id} \otimes \text{id})\langle r \rangle = (\text{id} \otimes DP^{-1} \otimes P^{-1})\langle r \rangle\). Now we can rearrange the partial degeneration $(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r \rangle$ as

$$(A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon))\langle r \rangle = (A \otimes \text{id} \otimes \text{id})(\text{id} \otimes A_2(\epsilon)DP^{-1} \otimes A_3(\epsilon)P^{-1})(M_q \otimes \text{id} \otimes \text{id})\langle r \rangle$$

This means that if \((r) \bowtie T\), then up to a change of basis $T$ is a partial degeneration of

$$(M_q \otimes \text{id} \otimes \text{id})\langle r \rangle = \sum_{i=1}^{q-1} e_i \otimes (e_i \otimes e_i + e_r \otimes e_r) + \sum_{i=q}^{r-1} e_i \otimes e_i \otimes e_i$$

with identity map on the first factor.

Define $H_q = \sum_{i=1}^{q-1} e_i \otimes (e_i \otimes e_i + e_q \otimes e_q) \in C^{r-1} \otimes C^q \otimes C^q$. We have $(M_q \otimes \text{id} \otimes \text{id})\langle r \rangle \simeq H_q \otimes (r-q)$. Using Proposition 3.14, we see that $T = S_q + X_{r-q}$ where $S_q$ is a partial degeneration of $H_q$ and $X_{r-q}$ is a restriction of $(r-q)$. It remains to analyse partial degenerations of $H_q$.

So, consider a partial degeneration

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon^d} (\text{id} \otimes B(\epsilon) \otimes C(\epsilon))H_q.$$

Define $b_1(\epsilon) = B(\epsilon)e_1$ and $c_1(\epsilon) = C(\epsilon)e_1$. Suppose $b_q(\epsilon) = b_{q,\mu} e_{\mu} + b_{q,\mu+1} e_{\mu+1} + \ldots$ After a basis change we may assume $b_{q,\mu} = c_q$. Define

$$E(\epsilon): \begin{cases} e_i \mapsto e_i, & i < q \\ e_q \mapsto e^{-\mu} b_q(\epsilon) \end{cases}$$
We have $\lim_{\epsilon \to 0} E(\epsilon) = \text{id}$, so by changing $B(\epsilon)$ to $E(\epsilon)^{-1}B(\epsilon)$ we obtain a partial degeneration for the same tensor $S_q$ with $b_q(\epsilon) = e^\mu e_q$. Using the same argument, we can assume without loss of generality that $c_q(\epsilon) = -e^\epsilon e_q$. In this situation we have

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon} (\text{id} \otimes B(\epsilon) \otimes C(\epsilon))H_p = \sum_{i=1}^{q-1} e_i \otimes \left( \frac{1}{\epsilon} b_i(\epsilon) \otimes c_i(\epsilon) - e^{\lambda+\mu-d} e_q \otimes e_q \right).$$

In case $\lambda + \mu > d$, we clearly have

$$S_q = \lim_{\epsilon \to 0} \sum_{i=1}^{q} e_i \otimes \left( \frac{1}{\epsilon} b_i(\epsilon) \otimes c_i(\epsilon) \right).$$

In this case $S_q$ is a partial degeneration of $(q-1)$ and by Proposition 3.14, we can choose the $b_i(\epsilon)$ and $c_i(\epsilon)$ constant in $\epsilon$ and obtain $(q-1) \geq S_q$ which yields $T \leq (r-1) \leq (r)$. If $\lambda + \mu < d$, we must have

$$b_i(\epsilon) = e^\sigma e_0 + \tilde{b}_i(\epsilon)$$

$$c_i(\epsilon) = e^\tau e_0 + \tilde{c}_i(\epsilon)$$

with $\sigma + \tau = \lambda + \mu$ so that

$$S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{i=1}^{q-1} e_i \otimes \left( e^\sigma \tilde{b}_i(\epsilon) \otimes e_q + e^\tau e_q \otimes \tilde{c}_i(\epsilon) \right).$$

For each $i = 1, \ldots, q-1$, the limit

$$e^* S_q = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{i=1}^{q-1} e_i \otimes \left( e^\sigma \tilde{b}_i(\epsilon) \otimes e_q + e^\tau e_q \otimes \tilde{c}_i(\epsilon) \right)$$

must exist and is of the form $b_i \otimes e_q + e_q \otimes c_i$ for some $b_i \in U_2$ and $c_i \in U_3$. Consequently, $S_q = \sum_{i=1}^{q-1} e_i \otimes (b_i \otimes e_q + e_q \otimes c_i)$ is a restriction of $\text{Str}_q$.

Finally, consider the case $\lambda + \mu = d$. Here it holds that

$$S_q = \lim_{\epsilon \to 0} \sum_{i=1}^{q-1} e_i \otimes \left( \frac{1}{\epsilon} b_i(\epsilon) \otimes c_i(\epsilon) - e^{\lambda+\mu-d} e_q \otimes e_q \right) =$$

$$= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \sum_{i=1}^{q-1} e_i \otimes (b_i(\epsilon) \otimes c_i(\epsilon)) - \sum_{i=1}^{q-1} e_i \otimes e_q \otimes e_q$$

In this case $S_q$ is a partial degeneration of $(q)$, and applying Proposition 3.14, we see that $(q) \geq S_q$ and $(r) \geq T$. We obtain that the only case where $( r ) \notin T$ is when $T = S_q + X_{r-q}$ with $S_q \leq \text{Str}_q$ and $X_{r-q} \leq (r-q)$ for some $q$ such that $1 \leq q \leq r$. We can exclude cases $q = 1$ and $q = 2$ because in these cases $\text{Str}_q \leq (q)$. \hfill \Box

4. AIDED RESTRICTION AND AIDED RANK

A related notion to partial degenerations is the notion of aided rank which we will introduce in Section 4.1. There, we will also explain its relation to partial degenerations.

In Section 4.2, we will present a generalization of the method to lower bound rank in [AFT11] and use it in Section 4.3 to calculate the aided rank for tensor powers of the $W$-tensor.

4.1. AIDED RESTRICTION AND INTERPOLATION. In this section we will introduce the notion of aided rank and show its relations to partial degenerations. For any tensor $T \in U_1 \otimes U_2 \otimes U_3$ recall the notation

$$T^{\otimes p} = T \otimes (1,1,p).$$

Recall the following interpolation result which is based on ideas introduced in [BCLR79].

**Theorem 4.1.** Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ such that $T \preceq_d S$. Then, $T \otimes (e+1) \geq S$ and $T \otimes (2d+1) \geq S$. 


We start by observing that for one can use a unit matrix instead of a unit tensor to interpolate degenerations. We use notation from matrix multiplication in order to write this matrix as $(1,1,p)$ where $p$ is the rank of the unit matrix.

**Lemma 4.2.** Consider tensors $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ and assume $T \succeq S$. Then,

$$T^{w_{u_2}u_3} \succeq S.$$  

**Proof.** Fixing bases of the vector spaces involved, can write

$$T^{w_{u_2}u_3} = \sum_{i,j,k,l} T_{i,j,k} e_i \otimes (e_j \otimes e_l) \otimes (e_k \otimes e_l)$$

where $i = 1, \ldots, u_1$, $j = 1, \ldots, u_2$ and $k,l = 1, \ldots, u_3$.

Letting $\Pi_3 : U_3 \otimes U_3 \to \mathbb{C}$ be the linear functional that maps $e_j \otimes e_l$ to $1$ if $j = l$ and $0$ else, we see by applying $\Pi_3$ on the third tensor factor that

$$T^{w_{u_3}} = \sum_{i,j,k} t_{i,j,k} e_i \otimes (e_j \otimes e_k) = \tilde{T}.$$  

The tensor $\tilde{T}$ is just the tensor $T$ seen as a bipartite tensor in $U_1 \otimes (U_2 \otimes U_3)$. We can also interpret $S$ as bipartite tensor $S \in V_1 \otimes (V_2 \otimes V_3)$ and know $\tilde{T} \succeq \tilde{S}$. In fact, since degeneration and restriction are equivalent for tensors on two factors, we have $\tilde{T} \succeq \tilde{S}$.

Again fixing bases, we have

$$S^{w_{u_3}} = \sum_{i,j,k,l} S_{i,j,k} e_i \otimes (e_j \otimes e_k \otimes e_i) \otimes e_i.$$  

As before, we can now define $\Pi_{U_3} : V_3 \otimes V_3 \to \mathbb{C}$ which maps $e_k \otimes e_l$ to $1$ if $k = l$ and $0$ else. Applying $\Pi_{V_3}$ to the second tensor factor we see $S^{w_{u_3}} \succeq S$.

After all, we have seen

$$T^{w_{(u_3-v_3)}} = (T^{w_{u_3}})^{w_{v_3}} \succeq \tilde{T}^{w_{v_3}} \succeq \tilde{S}^{w_{v_3}} \succeq S.$$  

**Remark 4.3.** The proof technique for Lemma 4.2 is inspired from the physical interpretation of tensors: Considering the tensors as three party quantum states, we used two EPR-pairs to teleport the particle at the third party to the second party and back.

The main question we ask is for a degeneration $T \succeq S$, how big must $p$ be such that $T^{w_p} \succeq S$. We will see that the minimal $p$ necessary to turn the degeneration into a restriction $T^{w_p} \succeq S$ can be chosen drastically smaller if the degeneration is a partial degeneration. On the other hand, we will calculate $p$ precisely for the degeneration $\langle 2^k \rangle \succeq W^{2^k}$ where we know from Theorem 3.14 that no partial degeneration exists. As it will turn out, here the minimal $p$ differs from the naive bound in Lemma 4.2 only by a factor of $\frac{1}{2}$. To simplify further discussions, let us introduce the following notation.

**Definition 4.4** (Aided rank). Let $S \in V_1 \otimes V_2 \otimes V_3$ and fix $p \geq 1$. We define the aided rank of $S$ as

$$R^p(T) = \min \{ r : \langle r \rangle^{w_p} \succeq S \}.$$  

Clearly, we have $R^1(T) = R(T)$. Lemma 4.2 shows that $R(S) = r$ implies that there is some $q$ such that $R^q(S) = r$. To find better bounds on the minimal $p$, we now show a variation of Theorem 4.1.

**Proposition 4.5.** Let $T \in U_1 \otimes U_2 \otimes U_3$ and $S \in V_1 \otimes V_2 \otimes V_3$ and assume $T \succeq S$. Then,

$$T^{w_{d+1}} \succeq S \text{ and } T^{w_{e+1}} \succeq S.$$
Proof. Say, the partial degeneration is given by
\[ (A_1 \otimes A_2(\epsilon) \otimes A_3(\epsilon)) T = \epsilon^d S + \sum_{i=1}^{e} \epsilon^{d+i} S_i. \]
Note that we can discard powers of \( \epsilon \) higher than \( d \) and write
\[ A_2(\epsilon) = \sum_{i=0}^{d} \epsilon^i A_{2,i}, \quad A_3(\epsilon) = \sum_{i=0}^{d} \epsilon^i A_{3,i}. \]
We then observe
\[ S = A_1 \otimes \left( \sum_{i=0}^{d} A_{2,i} \otimes A_{3,d-i} \right) T \]
and therefore
\[ A_1 \otimes \left( \sum_{i=0}^{d} A_{2,i} \otimes e^i_j \right) \otimes \left( \sum_{i=0}^{d} A_{3,d-i} \otimes e^i_j \right) T^{*d+1} = S \]
which shows \( T^{*d+1} \geq S \).
In order to see \( T^{*e+1} \geq S \), not that for \( \epsilon > 0 \), we can rewrite equation (9) as
\[ (A_1 \otimes (A_2(\epsilon)/\epsilon^d) \otimes A_3(\epsilon)) T = S + \epsilon S_1 + \cdots + \epsilon^e S_e = q(\epsilon). \]
Using Lagrange interpolation we can pick \( \alpha_0, \ldots, \alpha_e \neq 0 \) such that
\[ q(\epsilon) = \sum_{j=0}^{e} \frac{\epsilon - \alpha_m}{\alpha_j - \alpha_m}. \]
By writing \( \mu_j = \prod_{m \neq j} \frac{\alpha_m - \alpha_j}{\alpha_j - \alpha_m} \), we therefore get \( S = q(0) = q(\alpha_0)\mu_0 + \cdots + q(\alpha_e)\mu_e \). Note that the \( q(\alpha_j) \) are all restrictions of \( T \) where the first restriction map can be chosen to be \( A_1 \). With that, 
\[ S = q(0) = \left( A_1 \otimes \left( \sum_{j=0}^{e} \frac{\alpha_j}{\alpha_d} A_2(\alpha_j) \otimes e^j_j \right) \otimes \left( \sum_{j=0}^{e} A_3(\alpha_j) \otimes e^j_j \right) \right) T^{*e+1} \]
which finishes the proof. \( \square \)

In particular, we can exclude partial degeneration with a certain approximation degree if we can lower bound the aided rank of a tensor. Note that in the case of prehomogenous spaces, we can find an even better bound.

**Proposition 4.6.** Assume that \( U_1 \otimes U_2 \otimes U_3 \) is prehomogenous under the action of \( GL(U_2) \times GL(U_3) \) and let \( T \) be an element with dense orbit. Then, for all \( S \in U_1 \otimes U_2 \otimes U_3 \) it holds that
\[ T^{*2} \geq S. \]

**Proof.** Consider the affine degree 1 curve \( L \) parametrized by \( L(\epsilon) = T + \epsilon(S - T) \). It is clear that both \( T \) and \( S \) lie on \( L \). Clearly, the linear span of any two distinct points on \( L \) contains all points on \( L \). The orbit of \( T \) is open in \( U_1 \otimes U_2 \otimes U_3 \). Hence, the intersection of \( L \) and the complement of the orbit of \( T \) is a closed subset of \( L \), that is, a finite collection of points. Hence, there exists a second point \( \tilde{T} \) in the orbit of \( T \) on \( L \). Writing \( \tilde{T} = (id \otimes M_2 \otimes M_3)T \), and \( S = \lambda \tilde{T} + \mu \tilde{T}, \) we observe
\[ [id \otimes (Aid \otimes e^*_1 + \mu M_2 \otimes e^*_2) \otimes (id \otimes e^*_1 + M_3 \otimes e^*_2)] T^{*2} = S \]
which proves the claim. \( \square \)

Note that Proposition 4.6 supports the intuition that in the case of partial degenerations, the minimal aiding rank \( q \) turning it into a restriction is small. In Section 3.4 we saw that whenever \( T \in U_1 \otimes U_2 \otimes U_3 \) has a dense orbit under the action of \( GL(U_2) \times GL(U_3) \) it holds for all \( S \in U_1 \otimes U_2 \otimes U_3 \) that \( T^{*2} \geq S \).
We prove two variants of Proposition 4.6 in the case where the $\GL(U_2) \times \GL(U_3)$ is not dense. For that, we use an argument introduced in [CGJ19], where one exploits the genericity of certain linear spaces intersecting the orbit to reconstruct elements on the linear space. In the proofs of Proposition 4.7 and Proposition 4.8 we use some basic notation and results from algebraic geometry, we refer to [Har92] for details. In particular, we refer to [CGJ19, Remark 4.4] for a characterization of the degree of an algebraic variety in terms of intersection multiplicity for a generic line.

**Proposition 4.7.** Let $T \in U_1 \otimes U_2 \otimes U_3$; let $\Omega_T = (\GL(U_2) \times \GL(U_3)) \cdot [T] \subseteq \PP(U_1 \otimes U_2 \otimes U_3)$ be the orbit of the point $[T]$ and let $X_T = \overline{\Omega_T}$ be its Zariski-closure. Suppose $X_T$ is a hypersurface in $\PP(U_1 \otimes U_2 \otimes U_3)$. Let $S \in U_1 \otimes U_2 \otimes U_3$ be an element such that $\mult_{X_T}([S]) \leq \deg(X_T) - 2$. Then

$$T^{\bullet 2} \geq S.$$ 

**Proof.** Let $m = \mult_{X_T}([S])$. In particular $m = 0$ if $[S] \notin X_T$. Let $L$ be a generic line through $[S]$. The genericity assumption guarantees that $X_T \cap L$ is a $0$-dimensional scheme of degree $\deg(X_T)$; by Bertini’s Theorem [GH94, Sec. 1.1, p.137] this scheme has a component of degree $m$ supported at $[S]$ and $\deg(X_T) - m$ distinct points. Moreover, by genericity $L \cap (X_T \setminus \Omega_T) = \emptyset$, therefore all intersection points in $L \cap X_T$, except possibly $S$, belong to $\Omega_T$. By assumption $\deg(X_T) - m \geq 2$, so there exist two distinct points $[T_1], [T_2] \notin S$ in $L \cap \Omega_T$. Since $[T_1], [T_2]$ are distinct, they span the line $L$; therefore there exist scalars $\lambda_1, \lambda_2 \in \mathbb{C}$ such that $S = \lambda_1 T_1 + \lambda_2 T_2$. Since $[T_1], [T_2] \notin \Omega_T$, there exist $g_2^{(1)} \otimes g_3^{(1)}, g_2^{(2)} \otimes g_3^{(2)} \in \GL(U_2) \times \GL(U_3)$ such that $T_i = g_2^{(i)} \otimes g_3^{(i)} T$. Consider the restriction of $T^{\bullet 2} = T \otimes (1, 1, 2)$ to $U_1 \otimes U_2 \otimes U_3$ defined by the maps

$$M_1 = \id_{U_1} : U_1 \to U_1$$
$$M_2 = \lambda_1 g_2^{(1)} \otimes e_1^* + \lambda_2 g_2^{(2)} \otimes e_2^* : U_2 \otimes \mathbb{C}^2 \to U_2$$
$$M_3 = g_3^{(1)} \otimes e_1^* + g_3^{(2)} \otimes e_2^* : U_3 \otimes \mathbb{C}^2 \to U_3.$$

It is immediate that $M_1 \otimes M_2 \otimes M_3(T^{\bullet 2}) = S$. \[\square\]

**Proposition 4.8.** Let $T \in U_1 \otimes U_2 \otimes U_3$ be concise in the first factor; let $\Omega_T = (\GL(U_2) \times \GL(U_3)) \cdot [T] \subseteq \PP(U_1 \otimes U_2 \otimes U_3)$ be the orbit of the point $[T]$ and let $X_T = \overline{\Omega_T}$ be its Zariski-closure. Let $c = \codim X_T$ be the codimension of $X_T$ in $\PP(U_1 \otimes U_2 \otimes U_3)$. Let $S \notin X_T$. Then

$$T^{\bullet (c+1)} \geq S.$$ 

**Proof.** Since $T$ is concise in the first factor, the variety $X_T$ is not linearly degenerate. In particular $\deg(X_T) \geq c + 1$ [Har92, Corollary 18.12]. Let $L$ be a generic linear space through $S$ of dimension $c$. Bertini’s Theorem, together with the genericity assumption, guarantees that $L \cap X_T$ is a set of $\deg(X_T)$ distinct points and by genericity they all belong to $\Omega_T$. Moreover, it is easy to see that $L \cap X_T$ is not linearly degenerate in $L$. Therefore, there exist $[T_0], \ldots, [T_c] \in L \cap X_T$ which span $L$, and in particular there exist $\lambda_0, \ldots, \lambda_c$ such that $S = \lambda_0 T_0 + \cdots + \lambda_c T_c$. Since $T_j \in \Omega_T$ for every $j$, there exists $g_2^{(j)} \otimes g_3^{(j)} \in \GL(U_2) \times \GL(U_3)$ such that $g_2^{(j)} \otimes g_3^{(j)} T = T_j$. 

Consider the restriction of \( T^{(c+1)} = T \otimes 1, 1, c+1 \) to \( U_1 \otimes U_2 \otimes U_3 \) defined by the maps
\[
M_1 = \text{id}_{U_1} : U_1 \to U_1
\]
\[
M_2 = \sum_{j=0}^{c} \left( \lambda_{j_2}^{(j)} \otimes e_j^* \right) : U_2 \otimes \mathbb{C}^{c+1} \to U_2
\]
\[
M_3 = \sum_{j=0}^{c} \left( g_{j_3}^{(j)} \otimes e_j^* \right) : U_3 \otimes \mathbb{C}^{c+1} \to U_3.
\]
It is immediate that \( M_1 \otimes M_2 \otimes M_3(T^{(c+1)}) = S. \)

4.2. A substitution method for aided rank. In this section we will give a method to lower bound aided rank. Our method builds on a known method from [AFT11]. We will use it to calculate aided ranks of powers of the \( W \)-tensor. We start by mentioning the following technical fact without proof.

Lemma 4.9. Let \( V \) be a vector space and \( U \) be a finite dimensional subspace of dimension \( d \) of \( V \). If \( U \) is contained in the span of vectors \( u_1, \ldots, u_d \), then all \( u_i \) must be elements of \( U \).

The second lemma gives a useful characterization of restrictions of \( \langle n \rangle^{wp} \) in terms of flattenings. It is a simple generalization of a well-known characterization of tensor rank, see for example [Lan12, Theorem 3.1.1.1]

Lemma 4.10. Let \( S \in V_1 \otimes V_2 \otimes V_3 \) be any tensor and fix some natural number \( p \). Then we have
\[
R^{wp}(S) = \min \{ r : S(V_1^*) \subseteq V_2 \otimes V_3 \text{ spanned by } r \text{ matrices of rank } \leq p \}.
\]
Proof. If \( \langle r \rangle^{wp} \geq S \) we can write \( S = a_1 \otimes M_1 + \cdots + a_r \otimes M_r \) for matrices \( M_i \) of rank at most \( p \), in other words, \( S(V_1^*) \) is spanned by \( r \) matrices \( M_1 \ldots M_r \) of rank at most \( p \).

On the other hand, assume \( S(V_1^*) = \langle N_1 \ldots N_r \rangle \) for matrices \( N_i \) of rank at most \( p \). Fixing a basis of \( V_1 \) the tensor \( S \) is given by \( \sum_{i=1}^{n_1} e_i \otimes P_i \) where \( P_i = S(e_i^*) \). Since \( S(V_1^*) \) is spanned by the \( N_j \), we can find coefficients \( \lambda_{ij} \) such that \( P_i = \sum_{j=1}^{r} \lambda_{ij} N_j \). Hence,
\[
S = \sum_{j=1}^{r} \left( \sum_{i=1}^{n_1} \lambda_{ij} e_i \right) \otimes N_j.
\]
Noticing that this is a restriction of \( \langle r \rangle^{wp} \) finishes the proof.

We can use Lemma 4.10 to generalize a method to lower bound rank of tensors introduced in [AFT11].

Theorem 4.11. Let \( S \in V_1 \otimes V_2 \otimes V_3 \) and say, \( \dim(V_1) = v_1 \). Fixing a basis \( e_1 \ldots e_{v_1} \) of \( V_1 \), we write
\[
S = \sum_{i=1}^{v_1} e_i \otimes M_i
\]
for matrices \( M_i \in V_2 \otimes V_3 \) and assume \( M_1 \neq 0 \). Moreover, for complex numbers \( \lambda_2, \ldots, \lambda_{v_1} \) define
\[
\hat{S}(\lambda_2, \ldots, \lambda_{v_1}) = \sum_{j=2}^{v_1} e_j \otimes (M_j - \lambda_j M_1).
\]
We have
(i) There exist \( \lambda_2, \ldots, \lambda_{v_1} \in \mathbb{C} \) such that
\[
R^{wp}(\hat{S}(\lambda_2, \ldots, \lambda_{v_1})) \leq R^{wp}(S) - 1.
\]
(ii) Assume that \( M_1 \) has rank at most \( p \). Then, for all \( \lambda_2, \ldots, \lambda_{v_1} \)
\[
R^{wp}(\hat{S}(\lambda_2, \ldots, \lambda_{v_1})) \geq R^{wp}(S) - 1.
\]
Hence, if $M_1$ has rank at most $p$, we always find $\lambda_2, \ldots, \lambda_{v_1}$ such that

$$R^p(\hat{S}(\lambda_2, \ldots, \lambda_{v_1})) = R^p(S) - 1.$$  

**Proof.** Let $r = R^p(T)$, that is, $S(V_1^*)$ is contained in the span of $r$ matrices of rank at most $p$. Denote these matrices by $X_1, \ldots, X_r$, and write

$$M_i = \sum_{j=1}^r \mu_{i,j}X_j \text{ for } i = 1 \ldots v_1.$$  

Without loss of generality assume that $\mu_{1,1} \neq 0$ and set $\lambda_j = \frac{\mu_{i,j}}{\mu_{1,1}}$. We easily see that

$$\hat{S}(\lambda_2, \ldots, \lambda_{v_1})(V_1^*) \subset (X_2, \ldots, X_r),$$  

and therefore

$$R^p(\hat{S}(\lambda_2, \ldots, \lambda_{v_1})) \leq R^p(S) - 1.$$  

That shows the first claim.

On the other hand, if $M_1$ has rank at most $p$ and $Y_1, \ldots, Y_s$ span $\hat{S}(\lambda_2, \ldots, \lambda_{v_1})$, then clearly $M_1, Y_1, \ldots, Y_s$ will spans $\hat{S}(V_1^*)$, which shows the second claim. $\square$

In the next section we will see how one can use Theorem 4.11 to calculate aided ranks.

### 4.3. Aided rank of Kronecker powers of the $W$-tensor.

Let $V_1, V_2$ and $V_3$ be two dimensional with fixed bases $e_1, e_2$. In this section, we will use the method developed in Section 4.2 to calculate the aided rank of powers of the $W$-tensor

$$W = e_1 \otimes e_1 \otimes e_2 + e_1 \otimes 2 \otimes e_1 + e_2 \otimes e_1 \otimes e_1 \in V_1 \otimes V_2 \otimes V_3.$$  

The main result of this section is the following.

**Proposition 4.12.** For the $k$’th Kronecker power of the $W$ tensor $W^{\otimes k} \in V_1^{\otimes k} \otimes V_2^{\otimes k} \otimes V_3^{\otimes k}$ it holds that

$$R^p(W^{\otimes k}) = \begin{cases} 2^k & \text{if } p \geq 2^k \\ 2^k & \text{if } p < 2^k. \end{cases}$$  

**Proof.** It is also clear that for any $r < 2^k$, $(r)^p \not\subset W^{\otimes k}$ independently of $p$. On the other hand, $W(V_1^*)$ is the span of $e_1 \otimes e_2 + e_2 \otimes e_1$ and $e_1 \otimes e_1$, in other words, $R^2(W) = 2$. Consequently, we know that $(2^k)^{\otimes 2^k} \supseteq W^{\otimes k}$ for all $k$, in other words, $R^{2^k}(W^{\otimes k}) \leq 2^k$.

We will now use Theorem 4.11 to show that $(2^k)^{\otimes 2^k-1} \not\subset W^{\otimes k}$ which will finish the proof. On can verify easily that – thinking of the elements of $V_2^{\otimes k} \otimes V_3^{\otimes k}$ as $2^k \times 2^k$ matrices – all matrices in $W^{\otimes k}((V_1^{\otimes k})^*)$ are of the form

$$(10) \begin{pmatrix} \ast & x_0 \\ x_0 & \ast & \cdots & 0 \end{pmatrix}.$$  

That is, all matrices in $W^{\otimes k}((V_1^{\otimes k})^*)$ have the same entry $x_0$ in all antidiagonal entries and zeros in all entries below the antidiagonal. Now, assume for some $p$ that $(2^k)^p \not\subset W^{\otimes k}$. By Lemma 4.10, there are matrices $N_i$ of rank at most $p$ such that

$$(11) \quad W^{\otimes k}((V_1^{\otimes k})^*) \subseteq \langle N_1, \ldots, N_{2^k} \rangle.$$  

As $W^{\otimes k}$ is concise, $W^{\otimes k}((V_1^{\otimes k})^*)$ has dimension $2^k$. Therefore, by Lemma 4.9, the $N_i$ are elements of $W^{\otimes k}((V_1^{\otimes k})^*)$. We observe that a matrix of the form (10) with $x_0 \neq 0$ has full rank $2^k$. That is, if the matrices $N_i$ have rank $p < 2^k$ and are elements of $W^{\otimes k}((V_1^{\otimes k})^*)$, their span only contains matrices with zeros on the antidiagonal. That is, equation (11) cannot be satisfied if all $N_i$ have rank at most $p < 2^k$, that is,

$$(2^k)^p \not\subset W^{\otimes k} \text{ if } p < 2^k.$$  

In other words, $R^p(W^{\otimes k}) > 2^k$ for $p < 2^k$. $\square$
In particular we see that the minimal rank of an aiding matrix turning the degeneration \((2^k) \not\subseteq W^{\oplus k}\) into a restriction differs from the bound in Proposition 4.2 only by a factor of \(\frac{1}{2}\).

REFERENCES


The following Macaulay2 [GS] code gives a lower bound of 37 on the dimension of the orbit of a generic tensor in $\mathbb{C}^3 \otimes \mathbb{C}^4 \otimes \mathbb{C}^4$ which is $(2,3,3)$-compressible. This code is an adjustment of code that can be found at https://fulges.github.io/code/BDG-DimensionTNS.html.

```plaintext
V_1 = QQ[v_(1,1)..v_(1,3)]
V_2 = QQ[v_(2,1)..v_(2,4)]
V_3 = QQ[v_(3,1)..v_(3,4)]
W_1 = QQ[w_(1,1)..w_(1,3)]
W_2 = QQ[w_(2,1)..w_(2,4)]
W_3 = QQ[w_(3,1)..w_(3,4)]

ALL = V_1**V_2**V_3**W_1**W_2**W_3

M_1 = sub(random(QQ^4,QQ^4),ALL)
M_2 = mutableMatrix(ALL,4,4)
M_3 = mutableMatrix(ALL,4,4)

for i from 0 to 3 do(
   M_2_(0,i)=random(QQ);
   M_2_(i,0)=random(QQ);
   M_3_(0,i)=random(QQ);
   M_3_(i,0)=random(QQ);
)
M_2 = matrix M_2
M_3 = matrix M_3

T = 0
for i from 1 to a do(
   for j from 1 to b do(
      for k from 1 to c do(
         T = T + M_i_(j-1,k-1)*w_(1,i)*w_(2,j)*w_(3,k);
      );
   );
)--T is now (2,3,3)-compressible with random entries

-- a random point in Hom(W1,V1) + Hom(W2,V2) + Hom(W3,V3)
-- the rank of the differential of the parametrization map at randHom
-- will provide a lower bound on dim of the orbit closure of our tensor
randHom = flatten flatten apply(3,j->
tolist apply(1..di_(j+1),i ->w_(j+1,i)=>sub(random(1,V_(j+1)),ALL)));

-- compute the image of the differential
-- LL will be a list of elements of multidegree (1,1,1),
-- which are to be interpreted as elements of V1 \otimes V2 \otimes V3
-- generating the image of the differential of the parametrization map
LL = flatten for i from 1 to 3 list (ww = sub(vars(W_i),ALL);
vv = sub(vars(V_i),ALL);
   flatten entries (sub( (vv ** diff(ww,Tused)),randHom)));
minGen = mingens (ideal LL);
orbitdim = numcols(minGen) --37
```
APPENDIX B. PARTIAL DEGENERATIONS OF THE UNIT TENSOR

We have seen in Proposition 3.14 that the unit tensor \( r \) does not admit partial degenerations where the constant map \( A_1 \) is full rank. However, we also saw that in the case that \( A_1 \) has rank \( r - 1 \) there are honest partial degenerations which we classify in Proposition 3.16. In this appendix, we see that also in the realm of matrix pencils examples exist. For that, we consider for simplicity only matrix pencils in \( C^{2} \otimes C^{m} \otimes C^{m+1} \). Recall that the matrix pencil \( T \) given in (7) has a dense orbit under the action of \( GL_m \times GL_{m+1} \). It is well known (see for example Theorem 3.11.1.1 in [Lan12]) that this pencil has rank \( m + 1 \). On the other hand, it is known that the maximal rank of a tensor in \( C^{2} \otimes C^{m} \otimes C^{m+1} \) is \( \lfloor \frac{3m}{2} \rfloor \). Hence, we can find tensors \( S \) in \( C^2 \otimes C^m \otimes C^{m+1} \) with \( (m + 1) > T \triangleright S, R(S) > m + 1 \) and consequently \( (m + 1) \triangleright S \) but \( (m + 1) \not\triangleright S \).

To see an explicit example, let us construct for every \( m \) a matrix pencil of rank greater or equal to \( m + 1 \) to which \( (m) \) degenerates partially. For this, we recall the following well-known result about the rank of matrix pencils [Gri78, Jä79].

**Proposition B.1.** Consider \( p_1 \times q_1 \) matrices \( T'_1, T'_2 \) and \( p_2 \times q_2 \) matrices \( T'_{1''}, T'_{2''} \). Let \( T' \) be the tensor corresponding to the matrix pencil \([T'_1, T'_2]\) and similar for \( T'' \) and write \( T \in C^{2} \otimes C^{p_1+p_2} \otimes C^{q_1+q_2} \) for the tensor corresponding to the matrix pencil

\[
\begin{bmatrix}
(T'_1 & T'_{1''}) \\
(T'_2 & T'_{2''})
\end{bmatrix}.
\]

Then, it holds that

\[R(T) = R(T') + R(T'').\]

We will now construct a partial degeneration of \( (m) \) and will show using Proposition B.1 that it has rank at least \( m + 1 \). Applying the linear map

\[A_1 : U \rightarrow C^2, \epsilon_k \mapsto \epsilon_1 + ke_2\]

we see that \( (m) \) restricts to the tensor corresponding to the matrix pencil \([id_m, \text{diag}(1 \ldots m)]\).

Since the matrix

\[
M = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
2 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
2 & 1 & \cdots & 1 \\
m & m - 1 & 1 & m
\end{pmatrix}
\]

has \( m \) different eigenvalues \( 1, \ldots, m \), we deduce that also the tensor associated to the matrix pencil \([id_m, M]\) is a restriction of \( (m) \).

For any \( 1 < k < m \) define \( S_{k,m} \) to be the tensor corresponding to the matrix pencil

\[
(12) \quad \begin{bmatrix}
(id_{k-1}, 0) & 0 & 0 \\
0 & (id_{m-k}) & 0 \\
0 & 0 & J_1 \\
0 & 0 & 0 & J_2
\end{bmatrix}.
\]

where

\[
J_1 = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
k - 1 & 1 & \cdots & 1 \\
m & m - 1 & 1 & m
\end{pmatrix},
\]

\[
J_2 = \begin{pmatrix}
1 & 1 & \cdots & 1 \\
\vdots & \ddots & \ddots & \ddots \\
k + 1 & 1 & \cdots & 1 \\
m & m - 1 & 1 & m
\end{pmatrix}
\]

One verifies that applying the degeneration maps

\[A_2(\epsilon) = \text{diag}(1, \ldots, 1, \epsilon, \ldots, \epsilon), \quad A_3(\epsilon) = \text{diag}(\epsilon, \ldots, \epsilon, 1, \ldots, 1)\]

for \( k, m-k \) and \( k+1, m-k-1 \) respectively.
the tensor corresponding to the matrix pencil $[\text{id}_k, \text{diag}(1 \ldots m)]$ results in $\epsilon S_{k,m} + O(\epsilon^2)$. In particular, $(m) \triangleright S_{k,m}$.

From Lemma B.1, we know that

\begin{equation}
R(S_{k,m}) = R(S_{k,m}^1) + R(S_{k,m}^2)
\end{equation}

where $S_{k,m}^1$ corresponds to $[(\text{id}_{k-1}, 0), J_1]$ and $S_{k,m}^2$ to $[(0 \quad \text{id}_{m-k})^T, J_2]$, respectively. Using flattenings, one can now verify that the two pencils in equation (13) have ranks $k$ and $m - k + 1$, respectively, which shows $R(S_{k,m}) \geq m + 1$. Hence, it is not a restriction of $(m)$.

**Appendix C. The aided rank of the Coppersmith Winograd tensor**

In this appendix, we demonstrate with further examples how to compute aided rank using Theorem 4.11, we are going to calculate the aided ranks of the Coppersmith Winograd (CW) tensors. The study of these tensors was a crucial tool in the breakthrough result [CW87] bounding the exponent of matrix multiplication $\omega$ from above by 2.376.

**Definition C.1.** Let $V_1 \cong V_2 \cong V_3 \cong \mathbb{C}^{m^2}$ and fix a basis $e_0 \ldots e_{q+1}$. The $q$’th CW tensor is the symmetric tensor

\begin{equation}
T_{CW,q} = \sum_{i=1}^{q} e_0 \otimes e_i \otimes e_i + e_{q+1} \otimes e_0 \otimes e_0 +
\end{equation}

\begin{equation}
\sum_{j=1}^{q} e_j \otimes e_0 \otimes e_j + e_0 \otimes e_{q+1} \otimes e_0 +
\end{equation}

\begin{equation}
\sum_{k=1}^{q} e_k \otimes e_k \otimes e_0 + e_0 \otimes e_0 \otimes e_{q+1} \in V_1 \otimes V_2 \otimes V_3.
\end{equation}

We want to calculate $R^{\star p}(T_{CW,q})$ for any $p$ and $q$.

**Proposition C.2.** For $p \geq 2$, the $p$-aided rank of the $q$’th Coppersmith Winograd tensor is given by

\begin{equation}
R^{\star p}(T_{CW,q}) = q + 1 + \left[ \frac{q + 2}{p} \right].
\end{equation}

**Proof.** Writing

\begin{equation}
M(x_0, \ldots, x_{q+1}) = \begin{pmatrix}
x_{q+1} & x_1 & \ldots & x_q & x_0 \\
x_1 & x_0 & & & \\
\vdots & & \ddots & & \\
x_q & & & x_0 \\
x_0 & & & & 0
\end{pmatrix},
\end{equation}

we have $T_{CW,q}(V_1^*) = \{M(x_0, \ldots, x_{q+1}) : x_0, \ldots, x_{q+1} \in \mathbb{C}\}$. Note that $T_{CW,q}$ is concise. Hence, we have $R^{\star p}(T_{CW,q}) \geq q + 2$ for any $p \in \mathbb{N}$. Moreover, it is clear that $R^{\star p}(T_{CW,q}) \leq q + 2$ whenever $p \geq q + 2$ which gives $R^{\star p}(T_{CW,q}) = q + 2$ for all $p \geq q + 2$.

For $p \leq q + 1$, we will use Theorem 4.11.

Say, $p \geq 2$. Interpreting $V_2 \otimes V_3$ as space of $q + 2 \times q + 2$ matrices, we have

\begin{equation}
T_{CW,q} = \sum_{i=0}^{q+1} e_i \otimes M(x_i = 1, x_j = 0 \text{ for } i \neq j),
\end{equation}

The matrix $M(0, \ldots, 0, 1)$ has rank 1, hence we can find $\lambda_0^{(1)}, \ldots, \lambda_q^{(1)}$ using Theorem 4.11 such that

\begin{equation}
T_{CW,q}^{(1)} = \sum_{i=0}^{q} e_i \otimes \left( M(x_i = 1, x_j = 0 \text{ for } i \neq j) - \lambda_i^{(1)} M(0, \ldots, 0, 1) \right)
\end{equation}

\begin{equation}
=: M^{(1)}(x_i = 1, x_j = 0 \text{ for } i \neq j)$$
satisfies
\[ R^{\bullet p}(T^{(1)}_{CW,q}) = R^{\bullet p}(T_{CW,q}) - 1. \]

Note that the matrices \( M^{(1)}(x_0, \ldots, x_q) \) have the form
\[
M^{(1)}(x_0, \ldots, x_q) = \begin{pmatrix}
* & x_1 & \cdots & x_q & x_0 \\
x_1 & x_0 & & \\
\vdots & & & \\
x_q & & & x_0 \\
x_0 & & & 0
\end{pmatrix}.
\]

Still, \( M^{(1)}(0, \ldots, 0, 1) \) has only non-zero entries in the first column or in the first row. That is, it has rank less than or equal to \( p \), hence we can apply Theorem 4.11 again and obtain \( \lambda_0^{(2)}, \ldots, \lambda_q^{(2)} \) such that
\[
T^{(2)}_{CW,q} = \sum_{i=0}^{q-1} |x_i| \otimes \left( M^{(1)}(x_i = 1, x_j = 0 \text{ for } i \neq j) - \lambda_i^{(2)} M^{(1)}(0, \ldots, 0, 1) \right)\]

satisfies
\[ R^{\bullet p}(T^{(2)}_{CW,q}) = R^{\bullet p}(T^{(1)}_{CW,q}) - 1. \]

Again, we see that the elements of \( T^{(2)}_{CW,q}(V_1^*) \) have the form
\[
M^{(2)}(x_0, \ldots, x_{q-1}) = \begin{pmatrix}
* & x_1 & \cdots & * & x_0 \\
x_1 & x_0 & & & \\
\vdots & & & & \\
* & & & x_0 & \\
x_0 & & & 0 &
\end{pmatrix}
\]

Repeating this procedure \( q + 2 \) times leads to
\[
T^{(q+1)}_{CW,q} = |0| \otimes \left( M^{(q)}(1, 0) - \lambda^{(q+1)} M^{(q)}(0, 1) \right)\]

with
\[
M^{(q+1)} = \begin{pmatrix}
* & * & \cdots & * & 1 \\
* & 1 & \cdots & & \\
\vdots & & & & \\
* & & & 1 & \\
1 & & & & 0
\end{pmatrix}
\]

By Theorem 4.11 we reduced the aided rank by exactly 1 in each step yielding
\[ R^{\bullet p}(T^{(q+1)}_{CW,q}) = R^{\bullet p}(T^{(q)}_{CW,q}) - 1 = \cdots = R^{\bullet p}(T_{CW,q}) - (q + 1). \]

As \( M^{(q+1)} \) has rank \( q + 2 \) it follows \( R^{\bullet p}(T^{(q+1)}_{CW,q}) = \left\lfloor \frac{q+2}{p} \right\rfloor \) and with that
\[
R^{\bullet p}(T_{CW,q}) = q + 1 + \left\lfloor \frac{q+2}{p} \right\rfloor.
\]

We can also find the following upper bound on the aided rank of \( T^{\# 2}_{CW,q} \).

**Proposition C.3.** It holds that
\[
R^{\bullet p^2}(T^{\# 2}_{CW,q}) \leq q^2 + 4q + 3 + \left\lfloor \frac{(q+2)^2}{p^2} \right\rfloor.
\]
In particular, there are choices of $m$, $p$ and $q$ such that $\langle m \rangle^p \not\geq T_{CW,q}$ but $(\langle m \rangle^p)^{\otimes 2} \geq (T_{CW,q})^{\otimes 2}$.

Proof. Let us write

$$N(x_0, \ldots, x_{q+1}, y_0, \ldots, y_{q+1}) = \begin{pmatrix} x_{q+1} \cdot M(y) & x_1 \cdot M(y) & \cdots & x_q \cdot M(y) & x_0 \cdot M(y) \\ x_1 \cdot M(y) & x_0 \cdot M(y) & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ x_q \cdot M(y) & \cdots & x_0 \cdot M(y) & 0 \\ x_0 \cdot M(y) & \cdots & 0 & 0 \end{pmatrix}$$

where the matrices $M(y)$ are as in the proof of Proposition C.2 given by

$$M(y) = M(y_0, \ldots, y_{q+1}) = \begin{pmatrix} y_{q+1} & y_1 & \cdots & y_q & y_0 \\ y_1 & y_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ y_q & \cdots & y_0 & 0 \\ y_0 & & & 0 \end{pmatrix}. $$

With this, we have

$$(T_{CW,q})^{\otimes 2} = \sum_{i,j=0}^{q+1} (e_i \otimes e_j) \otimes N(x_i = 1, y_j = 1).$$

and consequently,

$$(T_{CW,q})^{\otimes 2} ((V_1^{\otimes 2})^*) = \{ N(x, y) : x, y \in \mathbb{C}^{q+2} \}.$$ 

The rank of the matrix $N(x, y)$ depends on these vectors $x$ and $y$.

(i) If $x = y = e_0$, the matrix $N(x, y)$ has rank $(q + 2)^2$.

(ii) If $x = e_0$ and $y = e_{q+1}$ or if $x = e_{q+1}$ and $y = e_0$, the matrix $N(x, y)$ has rank $q + 2$.

(iii) If $x = e_0$ and $y \in \{ e_1, \ldots, e_q \}$ or if $x \in \{ e_1, \ldots, e_q \}$ and $y = e_0$ the matrix $N(x, y)$ has rank $2(q + 2)$.

(iv) If $x = e_{q+1}$ and $y \in \{ e_1, \ldots, e_q \}$ or if $x \in \{ e_1, \ldots, e_q \}$ and $y = e_{q+1}$ the matrix $N(x, y)$ has rank 2.

(v) If $x = y = e_{q+1}$, the matrix $N(x, y)$ has rank 1.

(vi) If $x \in \{ e_1, \ldots, e_q \}$ and $y \in \{ e_1, \ldots, e_q \}$ the matrix $N(x, y)$ has rank 4.

Hence, to generate $(T_{CW,q})^{\otimes 2} (A^*)$, we need to generate 1 matrix of rank 1, 2$q$ matrices of rank 2, 2$q$ matrices of rank 4, 2 matrices of rank $q + 2$, 2$q$ matrices of rank $2(q + 2)$ and 1 matrix of rank $(q + 2)^2$. Assuming $p^2 \geq 2(q + 2)$, we will need at most

$$q^2 + 4q + 3 + \left[ \frac{(q + 2)^2}{p^2} \right]$$

matrices of rank $p^2$ to generate $(T_{CW,q})^{\otimes 2} ((V_1^{\otimes 2})^*)$. In other words,

$$R^{p^2} (T_{CW,q}^{\otimes 2}) \leq q^2 + 4q + 3 + \left[ \frac{(q + 2)^2}{p^2} \right].$$

To find $m, p$ and $q$ such that $\langle m \rangle^p \not\geq T_{CW,q}$ but $(\langle m \rangle^p)^{\otimes 2} \geq (T_{CW,q})^{\otimes 2}$, we choose $p$ and $q$ such that $\left[ \frac{q^2 + 2}{p+1} \right] < \left[ \frac{q^2 + 2}{p} \right]$ and $m = q + 1 + \left[ \frac{q^2 + 2}{p} \right]$. By construction, we have $\langle m \rangle^p \not\geq T_{CW,q}$. We have found an example whenever

$$q^2 + 4q + 3 + \left[ \frac{(q + 2)^2}{p^2} \right] \leq \left( q + 1 + \left[ \frac{q + 2}{p + 1} \right] \right)^2.$$
To see an explicit example pick \( q = 11 \) and \( p = 6 \). We have
\[
R^{66}(T_{CW,11}) = 11 + 1 + \left[ \frac{11 + 2}{6} \right] = 15
\]
\[
R^{77}(T_{CW,11}) = 11 + 1 + \left[ \frac{11 + 2}{7} \right] = 14,
\]
that is, \( (14)^{77} \geq T_{CW,11} \) but \( (14)^{66} \nmid T_{CW,11} \). From equation (14), we get
\[
R^{66}(T_{CW,11}^{2^2}) \leq 11^2 + 4 \cdot 11 + 3 + \left[ \frac{13^2}{6^2} \right] = 173 \leq 14^2 = 196.
\]
That gives
\[
\langle 196 \rangle^{36} \geq \langle 173 \rangle^{36} \geq T_{CW,11}^{2^2}.
\]
\[\square\]