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Many nonlinear methods have been proposed for constrained locally adaptive smoothing, such as local monotonicity [21], regularization [9], and wavelet denoising. These filters may be followed by multiscale-gradient estimation for edge detection, and represent an interesting higher complexity alternative to the linear smoothing filters studied here.

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REFERENCES

Information Measures in Scale-Spaces

Jon Sporring and Joachim Weickert

Abstract—This correspondence investigates Rényi’s generalized entropies under linear and nonlinear scale-space evolutions of images. Scale-spaces are useful computer vision concepts for both scale analysis and image restoration. We regard images as densities and prove monotony and smoothness properties for the generalized entropies. The scale-space extended generalized entropies are applied to global scale selection and size estimations. Finally, we introduce an entropy-based fingerprint description for textures.

Index Terms—Scale-space, Shannon–Wiener entropy, Rényi’s generalized entropies, scale selection, size estimation, texture.

I. INTRODUCTION

In recent years multiscale techniques have gained a lot of attention in the image processing community. Typical examples are pyramid and wavelet decompositions. They represent images at a small scale-resolution.

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number of scales and have proven their use for many image-processing tasks. Another important class of multiscale techniques consists of so-called scale-space representations [1]–[4]. They embed an original image into a continuous family of subsequently simpler versions. Many scale-spaces can be formulated as the evolution of the initial image under a suitable linear or nonlinear diffusion process. Such an image evolution is useful for tasks such as feature extraction, scale selection, and segmentation, see [5]–[7] and references therein.

Information-theoretical concepts such as the Shannon–Wiener entropy [8], [9], Rényi’s generalized entropies [10]–[12], and the Kullback–Leibler distance [13] have made contributions to image analysis; for instance, Brink and Pendock [14], Brink [15], and Sahoo et al. [17] have used them for local image thresholding, and Vehel et al. [17] and Chaudhuri and Sarkar [18] study images in a multifractal setting. It is not difficult to see that the generalized entropies, the multifractal spectrum, the gray-value moments and the Kullback–Leibler distance and increasing the scale should simplify the image without creating spurious structures. Since a scale-space creates a hierarchy of the image features, it constitutes an important step from a pixel-related image description to a semantical image description.

It has been shown that partial differential equations are the suitable framework for scale-spaces [32]. The oldest and best studied scale-space obtains a simplified version \( p(\mathbf{x}, t) \) of \( p(\mathbf{x}) \) as the solution of the linear diffusion process with \( p(\mathbf{x}) \) as initial value
\[
0 = \partial_t p + \Delta p \quad \text{in} \quad \mathbb{R}^2
\]
\[
p(\mathbf{x}, 0) = p(\mathbf{x})
\]
where \( \mathbf{x} = (x_1, x_2)^T \). It is well known from the mathematical literature that the solution \( p(\mathbf{x}, t) \) can be calculated by convolving \( p(\mathbf{x}) \) with a Gaussian of standard deviation \( \sigma = \frac{\sqrt{2t}}{10} \)
\[
p(\mathbf{x}, t) = \left( G_t \ast p \right)(\mathbf{x})
\]
\[
G_t(\mathbf{x}) := \frac{1}{4\pi t} e^{-|\mathbf{x}|^2/(4t)}.
\]
This process is called Gaussian scale-space or linear scale-space. It was first discovered by Lijima [1], [2] and became popular two decades later through the work of Witkin [3] and Koenderink [4]. A detailed treatment of the various aspects of Gaussian scale-space theory can be found in [5], [33], [7], and the references therein.

Unfortunately, Gaussian smoothing also blurs and dislocates semantically important features such as edges. This has triggered the study of nonlinear scale-spaces. Perona and Malik [34] proposed to replace the linear diffusion equation (1) by the nonlinear diffusion process
\[
0 = \nabla \cdot \left( g(|\nabla p|) \nabla p \right)
\]
where \( \nabla = (\partial_x, \partial_y)^T \), and the diffusivity \( g(|\nabla p|) \) is a decreasing function of \( |\nabla p| \). The idea is to regard \( |\nabla p| \) as an edge detector and to encourage smoothing within a region over smoothing across boundaries. Thus locations where the gradient is large have a large likelihood of being an edge, and the diffusivity is reduced.

In our experiments we consider a nonlinear diffusion process where the diffusivity is given by
\[
g(|\nabla p|) := \frac{1}{\sqrt{1 + |\nabla p|^2/\lambda^2}} \quad (\lambda > 0).
\]
Such a choice guarantees that the nonlinear diffusion filter is well-posed. This is one of the simplest representative of nonlinear scale-spaces. Overviews of other nonlinear scale-spaces can be found in [6] and [23].

### III. Generalized Entropies

Let us now consider a discrete image \( \mathbf{p} = (p_1, \ldots, p_N)^T \), where \( p_i > 0 \) for all \( i \). Note that a single index is used for the two-dimensional enumeration of pixels. Its family of generalized entropies

Then in Section III we will investigate a scale-space extension of the generalized entropies. Finally, in Section IV we will describe some applications in image processing. A conclusion is given in Section V.
is defined as

$$S_\alpha(p) := \frac{1}{1-\alpha} \log \sum_{i=1}^N p_i^\alpha$$

(7)

for $\alpha \neq 1$. The limit from left and right at $\alpha = 1$ is the Shannon–Wiener entropy

$$S_1(p) = -\sum_{i=1}^N p_i \log p_i$$

(8)

and we might as well consider it as part of the continuum. The parameter $\alpha$ is called information order.

Let the vector-valued function $p(t) = (p_1(t), \ldots, p_N(t))^T$ be the linear or nonlinear scale-space extension, where the continuous parameter $t$ denotes scale. These scale-spaces can be obtained by a spatial discretization of (1) or (5) with reflecting boundary conditions.

We will now discuss some details of the mathematical structure of the generalized entropies.

**Proposition III.1:** The generalized entropies are decreasing in $\alpha$.

**Proof:** Follows immediately from [10], [36].

**Proposition III.2:** The generalized entropies $S_\alpha(p(t))$ are increasing in $t$ for $\alpha > 0$, constant for $\alpha = 0$, and decreasing for $\alpha < 0$. For $t \to \infty$, they converge to $S_0$.

**Proof:** The proof is based on a result from [23, Theorem 5]. For a discrete image $p(t)$, which is obtained from a spatially discrete diffusion scale-space, the following holds. The expression

$$\Phi(p(t)) := \sum_{i=1}^N r(p_i(t))$$

(9)

is decreasing in $t$ for every smooth convex function $r$. Moreover, $\lim_{t \to \infty} p_i(t) = 1/N$ for all $i$.

Using this we first prove the monotony of $S_\alpha$, with respect to $t$.

Let $\alpha > 1$ and $s > 0$. Since $r(s) = s^\alpha$ satisfies

$$r''(s) = \alpha(\alpha - 1)s^{\alpha-2} > 0$$

(10)

it follows that $r$ is convex. Thus

$$\Phi(p(t)) = \sum_{i=1}^N r(p_i(t)) = \sum_{i=1}^N p_i^\alpha(t)$$

(11)

is decreasing in $t$ and

$$S_\alpha(p(t)) = \frac{1}{1-\alpha} \log \Phi(p(t))$$

(12)

is increasing in $t$.

Similar reasoning can be applied to establish monotony for the cases $0 < \alpha < 1$ and $\alpha < 0$.

For $\alpha = 1$ we obtain the Shannon–Wiener entropy for which monotony has already been shown in [23].

Let $\alpha = 0$. Then

$$S_0(p(t)) = \log \sum_{i=1}^N p_i^0(t) = \log N = \text{const.} \quad \forall t.$$  

(13)

To verify the asymptotic behavior of the generalized entropies we utilize $\lim_{t \to \infty} p_i(t) = 1/N$. For $\alpha \neq 1$ this gives

$$\lim_{t \to \infty} S_\alpha(p(t)) = \frac{1}{1-\alpha} \log \sum_{i=1}^N \frac{1}{N^\alpha} = \log N = S_0$$

(14)

and $\alpha = 1$ yields

$$\lim_{t \to \infty} S_1(p(t)) = -\sum_{i=1}^N \frac{1}{N} \log \frac{1}{N} = \log N = S_0.$$  

(15)

This completes the proof.

The following smoothness results constitute the basis for studying derivatives of generalized entropies as will be done in Section IV.

**Proposition III.3:** The generalized entropies are $C_\infty$ for $\alpha \neq 1$ and at least $C^1$ in $\alpha = 1$. For linear scale-space they are $C_\infty$ in $t$, and for the nonlinear scale-space they are $C^1$ in $t$.

**Proof:** In order to prove smoothness with respect to $\alpha$, we first consider the case $\alpha \neq 1$. Then $S_\alpha$ is the product of the two $C_\infty$ functions $1/(1-\alpha)$ and $\log \sum_{i=1}^N p_i^\alpha$, and thus also $C_\infty$ in $\alpha$.

The smoothness in $\alpha = 1$ is verified by applying l'Hôpital’s rule. Straightforward calculations show that

$$\lim_{\alpha \to 1} \frac{\partial S_\alpha}{\partial \alpha} = \frac{\sum_{i=1}^N p_i \log p_i}{2}.$$  

(16)

Thus $\partial S_\alpha/\partial \alpha$ exists and $S_\alpha$ is in $C^1$.

For linear scale-space, $C_\infty$ in $t$ follows directly from the fact that $G_t(x)$ is in $C_\infty$ with respect to $t$. In the nonlinear case, $C^1$ in $t$ is a consequence of the fact that the solution $p(t)$ is in $C^1$ with respect to $t$. This is proven in [23, Theorem 4].

Fig. 1 illustrates the monotony of the generalized entropies both in scale and order for both scale-spaces. The figures have been created by finite difference algorithms which preserve the monotony properties established in this section [37].

**IV. EXPERIMENTS**

We will in this section demonstrate some applications for the generalized entropies in image processing. We will consider the change of entropies by logarithmic scale

$$c_\alpha(p(t)) := \frac{\partial S_\alpha(p(t))}{\partial (\log t)}$$

(17)

since this appears to be the natural parameter (at least for linear scale-space) [4], [38], [5, Sec. 8.7.1], [30]. We emphasize that the generalized entropies are global measures and are thus best suited for images with homogeneous textures.

**A. Shannon–Wiener Entropy and Zooming**

This section analyzes the zooming behavior of the Shannon–Wiener entropy in linear scale-space.

Fig. 2 (top left and right) shows images from a laboratory experiment. The camera is placed fronto-parallel to a plane with a simple texture: pieces of paper with discs arranged in a regular manner. A sequence is produced as a series of increasing zoom values. In Fig. 2 (bottom) we plot the scale $\sigma = \sqrt{2t}$ of the point of maximum entropy change against the mean size of the discs. As can be seen, the relation is close to linear. It appears that in linear scale-space the point of maximal entropy change by logarithmic scale corresponds to the size of the dominating image structures.

**B. Spatial Extent of Structures**

In this section we show that the scaling behavior in linear scale-space carries over to the generalized entropies, and that they can be used to simultaneously measure the size of light and dark structures. We shall also see that the latter cannot be done with the Shannon–Wiener entropy.

The idea is as follows: The definition of the generalized entropies implies that entropies for large positive $\alpha$ focus on high gray-values (white areas), while for large negative value they analyze low gray-values (dark areas).

We expect that $c_\alpha(p(t))$ is especially high for structures of diameter $d$, when the variance $\sigma^2 = t/2$ of the Gaussian is close to the variance
of the structures. Let us, for simplicity, consider a random variable with uniform probability density function whose support is a disc of diameter $d$. Its variance is

$$
\sigma^2 = \int_0^{2\pi} \int_0^{d/2} \frac{r^2}{\pi(d/2)^2} r \, dr \, d\phi = \frac{d^2}{8}. \quad (18)
$$

Hence we expect a light (or dark) structure of diameter $d$ to have a significant entropy change by logarithmic scale at time $\sigma^2/2 = d^2/16$. This size estimate remains qualitatively correct for nondisc structures. In this case, it gives the size of the largest minimal diameter.

Fig. 3 shows the result of a performance analysis. The size estimate (18) has been applied to a number of simple sinusoidal images with structures (half-wavelengths) between 1 and 257 pixels. It can be seen in the bottom graph, for sufficiently large structures, that the estimated sizes are close to the true size. Although by definition, the generalized entropies are not symmetric in order, both have a similar scaling behavior which is close to linear.

In Fig. 4 we show an experiment on a texture with a more complicated periodicity. This real image has been created by the Belousov–Zhabotinsky reaction [39]. From orders $\pm 20$ we find dominating low-intensity values corresponding to a diameter 7.2, while the dominating high-intensity values suggest structures of diameter 3.5. From this we conclude that the distance between the light spiral arms in the mean is approximately 7.2 pixels, and the width of the spiral arms is approximately 3.5 pixels. In spite of the fact that the disc model (18) is not very appropriate for the line-like structure, the size estimates are in the correct order of magnitude.

The Shannon–Wiener entropy cannot be used for size estimation since it is a mixture of information from both light and dark areas. Thus is does not allow for a distinction between fore- and background.

C. Fingerprints for Entropies in Scale-Space

Sections IV-A and IV-B have shown that the scales of extremal entropy change carry significant information for selected information orders. Thus it would be interesting to introduce a compact description of the extremal changes for the continuum of information orders. In analogy with edge analysis in linear scale-space [40] we call such a description a fingerprint image. In Fig. 5 fingerprint images for two textures are given, both in the linear and nonlinear scale-space. The fingerprint lines are the extrema of $c_\alpha(p(t))$ in $t$. Our monotony results immediately imply the following consequences: If there is only one fingerprint line for a given positive order, then it corresponds to a maximum (likewise, to a minimum for negative orders); see also Fig. 3. For almost all orders there will be an odd number of fingerprint lines, that correspond to alternating maxima and minima.

It appears that the location of the fingerprint lines is more stable over information orders for the nonlinear scale-space than for the
linear one. Due to the reduced diffusivity of the nonlinear scale-space, the fingerprint lines are shifted toward higher scales.

V. CONCLUSIONS

In this correspondence we have investigated entropies as a means for extracting information from scale-spaces. This has lead to the following contributions.

- Monotony and smoothness properties for the Shannon–Wiener entropy and Rényi’s generalized entropies have been proven for the linear and nonlinear diffusion scale-spaces. The proofs hold also for all other nonlinear diffusion scale-spaces treated in [23].
- We have illustrated that the generalized entropies can be used to perform size measurements for periodic textures. This is not possible with the Shannon–Wiener entropy. We have proceeded to define a fingerprint image for entropies in scale-space and analyzed some of its basic properties. The localization of the fingerprint lines can be improved using nonlinear instead of linear scale-space.

The following topics appear promising for future work.

- In the context of texture analysis, it would be interesting to perform an in-depth study on the relation between the fingerprint topology and the structure of the texture.
- This correspondence has focused on the maximal entropy change by scale, however, indicates especially stable scales with respect to evolution time. We expect these scales to be good candidates for stopping times in nonlinear diffusion scale-spaces.
- The entropies in this correspondence are global measures. For topics such as focus-of-attention it would be interesting to study local variants of them.

It should be emphasized that the analysis carried out in this correspondence is directly transferable to the analysis of multifractals, gray-value moments, and gray-value histograms.

APPENDIX

RELATIONS TO GRAY-VALUE MOMENTS, HISTOGRAMS, AND MULTIFRACTAL SPECTRA

The gray-value moments of an image are defined as

$$m_\alpha(p) = \sum_{i=1}^{N} p_i^\alpha.$$  \hspace{1cm} (19)

From the definition of $S_\alpha$ in (7) it is clear that there is a one-to-one relation to $m_\alpha$.

Let the image $(p_1, \ldots, p_N)^T$ consist of $M$ distinct gray-values $v_1, \ldots, v_M$ occurring $f_1, \ldots, f_M$ times. We may use this gray-value histogram $f$ to rewrite the moments as

$$m_\alpha(p) := \sum_{j=1}^{M} f_j v_j^\alpha.$$  \hspace{1cm} (20)
Fig. 3. Scaling behavior and size estimation with generalized entropies. Top Left: Test image generated by \(257^{-2}(1 + 0.6 \cos(\omega_1) \cos(\omega_2))\) with \(\omega = 9\pi/257\). Top Right: The corresponding \(E_{\alpha}(p(t))\) curves for \(\alpha = \pm 100\). Top curve is for positive order and bottom curve for negative order. Bottom: A double logarithmic plot of the true size versus the estimated size for various \(\omega\). The straight line depicts the truth, the circles the estimation from order 100, and the crosses for order \(-100\).

Fig. 4. Left: Spiral generated by a chemical reaction. Right: Entropy changes for orders 20 (top curve) and \(-20\) (bottom curve).
Considering the moments $m_0, \cdots, m_{M-1}$ gives the relation
\[
\begin{bmatrix}
m_0 \\
m_1 \\
m_2 \\
\vdots \\
m_{M-1}
\end{bmatrix} = \begin{bmatrix}1 & 1 & \cdots & 1 \\
v_1 & v_2 & \cdots & v_M \\
v_1^2 & v_2^2 & \cdots & v_M^2 \\
\vdots & \vdots & \ddots & \vdots \\
v_1^{M-1} & v_2^{M-1} & \cdots & v_M^{M-1}
\end{bmatrix} \begin{bmatrix}f_1 \\
f_2 \\
f_3 \\
\vdots \\
f_M
\end{bmatrix}.
\]

The system matrix is the so-called Vandermonde matrix. By induction over $M$ the determinant can be shown to be $\prod_{1 \leq n < m \leq M} (v_m - v_n)$.

Since $v_j, j = 1, \cdots, M$ are distinct, the matrix is invertible (but ill-conditioned). Thus there is a one-to-one relation between the moments $m_0, \cdots, m_{M-1}$ and the histogram $f_1, \cdots, f_M$.

The equivalence of the multifractal spectrum and the generalized entropies is discussed in [42].

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