Information measures in scale-spaces

Sporring, Jon; Weickert, Joachim

Published in:
IEEE Transactions on Information Theory

Publication date:
1999

Document version
Early version, also known as pre-print

Citation for published version (APA):
Many nonlinear methods have been proposed for constrained locally adaptive smoothing, such as local monotonicity [21], regularization [9], and wavelet denoising. These filters may be followed by multiscale-gradient estimation for edge detection, and represent an interesting higher complexity alternative to the linear smoothing filters studied here.

ACKNOWLEDGMENT

The authors are grateful to R. Kozick for his assistance with the EM algorithm, and to F. Gin material related to the CRB’s.

REFERENCES


Information Measures in Scale-Spaces

Jon Sporring and Joachim Weickert

Abstract— This correspondence investigates Rényi’s generalized entropies under linear and nonlinear scale-space evolutions of images. Scale-spaces are useful computer vision concepts for both scale analysis and image restoration. We regard images as densities and prove monotony and smoothness properties for the generalized entropies. The scale-space extended generalized entropies are applied to global scale selection and size estimations. Finally, we introduce an entropy-based fingerprint description for textures.

Index Terms— Scale-space, Shannon–Wiener entropy, Rényi’s generalized entropies, scale selection, size estimation, texture.

I. INTRODUCTION

In recent years multiscale techniques have gained a lot of attention in the image processing community. Typical examples are pyramid and wavelet decompositions. They represent images at a small scale.

Manuscript received February 10, 1998; revised October 15, 1998. This work was funded in part by the Real World Computing Partnership, the Danish National Research Council, and the EU-TMR Project VIRGO.

J. Sporring was with the Image Group, Department of Computer Science, University of Copenhagen, DK-2100 Copenhagen, Denmark. He is now with the Foundation for Research and Technology–Hellas (FORTH), GR 711 10 Heraklion, Crete, Greece.

J. Weickert is with the Image Group, Department of Computer Science, University of Copenhagen, DK-2100 Copenhagen, Denmark. Publisher Item Identifier S 0018-9448(99)02269-5.
number of scales and have proven their use for many image- 
processing tasks. Another important class of multiscale techniques 
consists of so-called scale-space representations [1]–[4]. They embed 
an original image into a continuous family of subsequently simpler 
versions. Many scale-spaces can be formulated as the evolution of the 
initial image under a suitable linear or nonlinear diffusion process. 
Such an image evolution is useful for tasks such as feature extraction, 
scale selection, and segmentation, see [5]–[7] and references therein. 

Information-theoretical concepts such as the Shannon–Wiener en-
tropy [8], [9], Rényi’s generalized entropies [10]–[12], and the 
Kullback–Leibler distance [13] have made contributions to image 
analysis; for instance, Brink and Pendock [14], Brink [15], and 
Sahoo et al. [17] have used them for local image thresholding, and 
Vehel et al. [17] and Chaudhuri and Sarkar [18] study images in 
a multifractal setting. It is not difficult to see that the generalized 
entropies, the multifractal spectrum, the gray-value moments and the 
gray-value histogram itself are equivalent representations: they can 
be transformed into each other by one-to-one mappings. More details 
can be found in the Appendix. 

Since scale-spaces simplify images, it is only natural to investigate 
their simplification properties in terms of information measures. 
Already in 1949, Shannon mentioned that the Shannon–Wiener 
entropy decreases under averaging transformations [8, p. 52]. In 1993, 
Illner and Neunzert [19] studied a biased diffusion process, where 
the original image evolves toward a background image b along a 
path where its Kullback–Leibler distance with respect to b increases 
monotonically. Jägersand [20] used the Kullback–Leibler distance in 
linear scale-space for focus of attention. Oomes and Snoeren [21] 
used the entropy relative to a background measure to estimate the 
size of objects in images. Sporring [22] applied the Shannon–Wiener 
entropy in linear scale-space to perform scale selection in textures 
and showed the monotone behavior using concepts from thermodynamics. 
Weickert [23] proved monotony of the Shannon–Wiener entropy 
in linear and nonlinear diffusion scale-spaces by regarding it as a 
Lyapunov functional. Lyapunov functionals have been used for 
scale-space synchronization [24] and for a uniform sampling of the 
scale axis with respect to its information content [25]. The fractal 
dimension in scale-spaces has been investigated by Peleg et al. 
[26], Barth et al. [27], and Pei et al. [28]. Relations between Shan-
non–Wiener entropy and multiscale concepts in terms of wavelets 
have been established by Kim and Brooks [29], where inequality 
theory was applied to propose optimal measures for feature-directed 
segmentation. 

The present correspondence extends previous work in this field by 
studying both theoretical aspects and the practical potential of 
generalized entropies in a linear and nonlinear multiscale setting. 
Generalized entropies are complete in the sense that they allow for 
a reconstruction of the gray-value histogram (see the Appendix). A 
scale-space extension is used to complement the entropies with spatial 
information. We prove monotony and smoothness properties with 
respect to the information order and the scale parameter. We use the 
scale-space behavior of generalized entropies for scale selection and 
size estimation, and we introduce a fingerprint-like description for 
textures. The results indicate that our extensions broaden the potential 
use of entropy methods in image analysis. Some preliminary results 
in this correspondence have been presented at conferences [22], [30]. 

Throughout this correspondence we identify an image by its two-
dimensional distribution of light on a rectangular image domain. It 
should be noted that this representation is invariant under multiplic-
ation with, but not under addition of, a constant. It is important to note 
that this two-dimensional distribution is not the gray-value histogram. 
The outline of our correspondence is as follows. In Section II we 
will give a brief introduction to linear and nonlinear scale-spaces. 

Then in Section III we will investigate a scale-space extension of the 
generalized entropies. Finally, in Section IV we will describe some 
applications in image processing. A conclusion is given in Section V. 

II. A SHORT INTRODUCTION TO SCALE-SPACES 

The images considered in this work are all discrete, but for 
simplicity we will in this section introduce two scale-spaces in the 
continuous setting. Discrete scale-space aspects are discussed by 
Lindeberg [5] for the linear framework, and by Weickert [23] for the 
nonlinear setting. Scale-spaces can be considered as an alternative to 
traditional smoothing methods from statistics [31]. 

In scale-space theory one embeds an image \( p(x) : \mathbb{R}^2 \rightarrow \mathbb{R} \) into a continuous family \( \{ p(x,t) | t \geq 0 \} \) of gradually smoother versions of it. The original image corresponds to the scale \( t = 0 \), and increasing the scale should simplify the image without creating spurious structures. Since a scale-space creates a hierarchy of the image features, it constitutes an important step from a pixel-related image description to a semantical image description. 

It has been shown that partial differential equations are the suitable 
framework for scale-spaces [32]. The oldest and best studied scale-

space obtains a simplified version \( p(x,t) \) of \( p(x) \) as the solution of the linear diffusion process with \( p(x) \) as initial value 

\[
\frac{\partial p}{\partial t} = \nabla \cdot (\nabla p + \beta \nabla^2 p) 
\] 

where \( x = (x_1, x_2)^T \). It is well known from the mathematical 
literature that the solution \( p(x,t) \) can be calculated by convolving 
\( p(x) \) with a Gaussian of standard deviation \( \sigma = \sqrt{2t} \) 

\[
p(x,t) = \left( G_t * p \right)(x) 
\] 

\[
G_t(x) := \frac{1}{4\pi t} e^{-|x|^2/(4t)}. 
\] 

This process is called Gaussian scale-space or linear scale-space. 
It was first discovered by Lijima [1], [2] and became popular two 
decades later through the work of Witkin [3] and Koenderink [4]. 
A detailed treatment of the various aspects of Gaussian scale-space 
three can be found in [5], [33], [7], and the references therein. 

Unfortunately, Gaussian smoothing also blurs and dislocates seman-
tically important features such as edges. This has triggered the 
study of nonlinear scale-spaces. Perona and Malik [34] proposed to 
replace the linear diffusion equation (1) by the nonlinear diffusion 
process 

\[
\frac{\partial p}{\partial t} = \nabla \cdot (g(\nabla p) \nabla p) 
\] 

where \( \nabla = (\partial_x, \partial_y)^T \), and the diffusivity \( g(\nabla p) \) is a decreasing function of \( |\nabla p| \). The idea is to regard \( |\nabla p| \) as an edge detector 
and to encourage smoothing within a region over smoothing across 
boundaries. Thus locations where the gradient is large have a large 
likelihood of being an edge, and the diffusivity is reduced. 

In our experiments we consider a nonlinear diffusion process where 
the diffusivity is given by [35] 

\[
g(\nabla p) := \frac{1}{\sqrt{1 + |\nabla p|^2/\lambda^2}} \quad (\lambda > 0). 
\] 

Such a choice guarantees that the nonlinear diffusion filter is well-
posed. 

This is one of the simplest representative of nonlinear scale-spaces. 
Overviews of other nonlinear scale-spaces can be found in [6] and 
[23].
The generalized entropies are decreasing in $\alpha$. 

**Proposition III.2:** The generalized entropies $S_{\alpha}(p(t))$ are increasing in $t$ for $\alpha > 0$, constant for $\alpha = 0$, and decreasing for $\alpha < 0$. For $t \to \infty$, they converge to $S_0$.

**Proof:** The proof is based on a result from [23, Theorem 5]. For a discrete image $p(t)$, which is obtained from a spatially discrete diffusion scale-space, the following holds. The expression

$$\Phi(p(t)) := \sum_{i=1}^{N} r(p_i(t))$$

is decreasing in $t$ for every smooth convex function $r$. Moreover, $\lim_{t \to \infty} p_i(t) = 1/N$ for all $i$.

Using this we first prove the monotony of $S_{\alpha}$ with respect to $t$.

Let $\alpha > 1$ and $s > 0$. Since $r(s) = s^\alpha$ satisfies

$$r'(s) = \alpha s^{\alpha-1} > 0$$

it follows that $r$ is convex. Thus

$$S_\alpha(p(t)) = \frac{1}{1-\alpha} \log \Phi(p(t))$$

is increasing in $t$ and

$$S_\alpha(p(t)) = \lim_{t \to \infty} \frac{1}{1-\alpha} \log \Phi(p(t))$$

is increasing in $t$.

Similar reasoning can be applied to establish monotony for the cases $0 < \alpha < 1$ and $\alpha < 0$.

For $\alpha = 1$ we obtain the Shannon–Wiener entropy for which monotony has already been shown in [23].

Let $\alpha = 0$. Then

$$S_0(p(t)) = \sum_{i=1}^{N} p_i^0(t) = \log N = \text{const. \forall t}$$

To verify the asymptotic behavior of the generalized entropies we utilize $\lim_{t \to \infty} p_i(t) = 1/N$. For $\alpha \neq 1$ this gives

$$S_{\alpha}(p(t)) \approx \frac{1}{1-\alpha} \log \sum_{i=1}^{N} p_i^\alpha = \log N = S_0$$

and $\alpha = 1$ yields

$$S_1(p(t)) = \sum_{i=1}^{N} \log \frac{1}{N} = \log N = S_0$$

This completes the proof.

The following smoothness results constitute the basis for studying derivatives of generalized entropies as will be done in Section IV.

**Proposition III.3:** The generalized entropies are $C^\infty$ for $\alpha \neq 1$ and at least $C^1$ in $\alpha = 1$. For linear scale-space they are $C^\infty$ in $t$, and for the nonlinear scale-space they are $C^1$ in $t$.

**Proof:** In order to prove smoothness with respect to $\alpha$, we first consider the case $\alpha \neq 1$. Then $S_\alpha$ is the product of the two $C^\infty$ functions $1/(1-\alpha)$ and $\log \sum_{i=1}^{N} p_i^\alpha$, and thus also $C^\infty$ in $\alpha$.

The smoothness in $\alpha = 1$ is verified by applying l'Hôpital’s rule. Straightforward calculations show that

$$\lim_{\alpha \to 1} \frac{\partial S_\alpha}{\partial \alpha} = \frac{1}{2} \left( \sum_{i=1}^{N} p_i \log p_i \right)^2 - \left( \sum_{i=1}^{N} p_i \log p_i \right)^2$$

Thus $\partial S_{\alpha}/\partial \alpha$ exists and $S_{\alpha}$ is in $C^1$.

For linear scale-space, $C^\infty$ in $t$ follows directly from the fact that $G_t(x)$ is in $C^\infty$ with respect to $t$. In the nonlinear case, $C^1$ in $t$ is a consequence of the fact that the solution $p(t)$ is in $C^1$ with respect to $t$. This is proven in [23, Theorem 4].

Fig. 1 illustrates the monotony of the generalized entropies both in scale and order for both scale-spaces. The figures have been created by finite difference algorithms which preserve the monotony properties established in this section [37].

IV. EXPERIMENTS

We will in this section demonstrate some applications for the generalized entropies in image processing. We will consider the change of entropies by logarithmic scale

$$c_{\alpha}(p(t)) := \frac{\partial S_{\alpha}(p(t))}{\partial (\log t)}$$

since this appears to be the natural parameter (at least for linear scale-space) [4], [38], [5, Sec. 8.7.1], [30]. We emphasize that the generalized entropies are global measures and are thus best suited for images with homogeneous textures.

A. Shannon–Wiener Entropy and Zooming

This section analyzes the zooming behavior of the Shannon–Wiener entropy in linear scale-space.

Fig. 2 (top left and right) shows images from a laboratory experiment. The camera is placed fronto-parallel to a plane with a simple texture: pieces of paper with discs arranged in a regular manner. A sequence is produced as a series of increasing zoom values. In Fig. 2 (bottom) we plot the scale $\sigma = \sqrt{\nu}$ of the point of maximum entropy change against the mean size of the discs. As can be seen, the relation is close to linear. It appears that in linear scale-space the point of maximal entropy change by logarithmic scale corresponds to the size of the dominating image structures.

B. Spatial Extent of Structures

In this section we show that the scaling behavior in linear scale-space carries over to the generalized entropies, and that they can be used to simultaneously measure the size of light and dark structures. We shall also see that the latter cannot be done with the Shannon–Wiener entropy.

The idea is as follows: The definition of the generalized entropies implies that entropies for large positive $\alpha$ focus on high gray-values (white areas), while for large negative value they analyze low gray-values (dark areas).

We expect that $c_{\alpha}(p(t))$ is especially high for structures of diameter $d$, when the variance $\sigma^2 = d/2$ of the Gaussian is close to the variance.
of the structures. Let us, for simplicity, consider a random variable with uniform probability density function whose support is a disc of diameter $d$. Its variance is

$$\sigma^2 = \int_0^{2\pi} \int_0^{d/2} \frac{r^2}{\pi(d/2)^2} r \, dr \, d\phi = \frac{d^2}{8}. \quad (18)$$

Hence we expect a light (or dark) structure of diameter $d$ to have a significant entropy change by logarithmic scale at time $\sigma^2 / 2 = d^2 / 16$. This size estimate remains qualitatively correct for nondisc structures. In this case, it gives the size of the largest minimal diameter.

Fig. 3 shows the result of a performance analysis. The size estimate (18) has been applied to a number of simple sinusoidal images with structures (half-wavelengths) between 1 and 257 pixels. It can be seen in the bottom graph, for sufficiently large structures, that the estimated sizes are close to the true size. Although by definition, the generalized entropies are not symmetric in order, both have a similar scaling behavior which is close to linear.

In Fig. 4 we show an experiment on a texture with a more complicated periodicity. This real image has been created by the Belousov–Zhabotinsky reaction [39]. From orders $\pm 20$ we find dominating low-intensity values corresponding to a diameter 7.2, while the dominating high-intensity values suggest structures of diameter 3.5. From this we conclude that the distance between the light spiral arms in the mean is approximately 7.2 pixels, and the width of the spiral arms is approximately 3.5 pixels. In spite of the fact that the disc model (18) is not very appropriate for the line-like structure, the size estimates are in the correct order of magnitude.

The Shannon–Wiener entropy cannot be used for size estimation since it is a mixture of information from both light and dark areas. Thus it does not allow for a distinction between fore- and background.

C. Fingerprints for Entropies in Scale-Space

Sections IV-A and IV-B have shown that the scales of extremal entropy change carry significant information for selected information orders. Thus it would be interesting to introduce a compact description of the extremal changes for the continuum of information orders. In analogy with edge analysis in linear scale-space [40] we call such a description a fingerprint image. In Fig. 5 fingerprint images for two textures are given, both in the linear and nonlinear scale-space. The fingerprint lines are the extrema of $c_\alpha(p(t))$ in $t$. Our monotony results immediately imply the following consequences: If there is only one fingerprint line for a given positive order, then it corresponds to a maximum (likewise, to a minimum for negative orders); see also Fig. 3. For almost all orders there will be an odd number of fingerprint lines, that correspond to alternating maxima and minima. This can be seen, for instance, in the middle right graph in Fig. 5. For information order 60, the leftmost line is a maximum followed by alternating minima and maxima.

It appears that the location of the fingerprint lines is more stable over information orders for the nonlinear scale-space than for the
linear one. Due to the reduced diffusivity of the nonlinear scale-space, the fingerprint lines are shifted toward higher scales.

V. CONCLUSIONS

In this correspondence we have investigated entropies as a means for extracting information from scale-spaces. This has lead to the following contributions.

- Monotony and smoothness properties for the Shannon–Wiener entropy and Rényi’s generalized entropies have been proven for the linear and nonlinear diffusion scale-spaces. The proofs hold also for all other nonlinear diffusion scale-spaces treated in [23].
- We have illustrated that the generalized entropies can be used to perform size measurements for periodic textures. This is not possible with the Shannon–Wiener entropy. We have proceeded to define a fingerprint image for entropies in scale-space and analyzed some of its basic properties. The localization of the fingerprint lines can be improved using nonlinear instead of linear scale-space.

The following topics appear promising for future work.
- In the context of texture analysis, it would be interesting to perform an in-depth study on the relation between the fingerprint topology and the structure of the texture.
- This correspondence has focused on the maximal entropy change by scale, however, indicates especially stable scales with respect to evolution time. We expect these scales to be good candidates for stopping times in nonlinear diffusion scale-spaces.
- The entropies in this correspondence are global measures. For topics such as focus-of-attention it would be interesting to study local variants of them.

It should be emphasized that the analysis carried out in this correspondence is directly transferable to the analysis of multifractals, gray-value moments, and gray-value histograms.

APPENDIX

RELATIONS TO GRAY-VALUE MOMENTS, HISTOGRAMS, AND MULTIFRACTAL SPECTRA

The gray-value moments of an image are defined as

$$m_\alpha(p) = \sum_{i=1}^{N} p_i^{\alpha}$$

(19)

From the definition of $S_\alpha$ in (7) it is clear that there is a one-to-one relation to $m_\alpha$.

Let the image $\{p_1, \ldots, p_N\}$ consist of $M$ distinct gray-values $v_1, \ldots, v_M$ occurring $f_1, \ldots, f_M$ times. We may use this gray-value histogram $f$ to rewrite the moments as

$$m_\alpha(p) = \sum_{j=1}^{M} f_j v_j^{\alpha}.$$  

(20)
Fig. 3. Scaling behavior and size estimation with generalized entropies. Top Left: Test image generated by $257^{-2}(1 + 0.6 \cos(\omega x_1) \cos(\omega x_2))$ with $\omega = 9\pi/257$. Top Right: The corresponding $e_{\alpha}(s(t))$ curves for $\alpha = \pm 100$. Top curve is for positive order and bottom curve for negative order. Bottom: A double logarithmic plot of the true size versus the estimated size for various $\omega$. The straight line depicts the truth, the circles the estimation from order 100, and the crosses for order $-100$.

Fig. 4. Left: Spiral generated by a chemical reaction. Right: Entropy changes for orders $20$ (top curve) and $-20$ (bottom curve).
Considering the moments $m_0, \ldots, m_{M-1}$ gives the relation
\[
\begin{bmatrix}
  m_0 \\
  m_1 \\
  m_2 \\
  \vdots \\
  m_{M-1}
\end{bmatrix} = \begin{bmatrix}
  1 & 1 & \cdots & 1 \\
  v_1 & v_2 & \cdots & v_M \\
  v_1^2 & v_2^2 & \cdots & v_M^2 \\
  \vdots & \vdots & \ddots & \vdots \\
  v_1^{M-1} & v_2^{M-1} & \cdots & v_M^{M-1}
\end{bmatrix} \begin{bmatrix}
  f_1 \\
  f_2 \\
  f_3 \\
  \vdots \\
  f_M
\end{bmatrix}
\]
The system matrix is the so-called Vandermonde matrix. By induction over $M$ the determinant can be shown to be $\prod_{1 \leq n < m \leq M} (v_m - v_n)$. Since $v_j, j = 1, \ldots, M$ are distinct, the matrix is invertible (but ill-conditioned). Thus there is a one-to-one relation between the moments $m_0, \ldots, m_{M-1}$ and the histogram $f_1, \ldots, f_M$.

The equivalence of the multifractal spectrum and the generalized entropies is discussed in [42].

ACKNOWLEDGMENT

We thank P. Johansen, M. Nielsen, L. Florack, O. F. Olsen, and R. Maas for many discussions on this topic. P. G. Sørensen from the Department of Chemistry at the University of Copenhagen [39] has supplied the spiral image in Fig. 4. The images in Fig. 5 are from the ‘VisTex’ collection [41].