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ON NON-SURJECTIVE WORD MAPS ON $\text{PSL}_2(\mathbb{F}_q)$

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Abstract. Jambor–Liebeck–O’Brien showed that there exist non-proper-power word maps which are not surjective on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$. This provided the first counterexamples to a conjecture of Shalev which stated that if a two-variable word is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$. Motivated by their work, we construct new examples of these types of non-surjective word maps. As an application, we obtain non-surjective word maps on the absolute Galois group of $\mathbb{Q}$.

1. Introduction

A word in $k$ variables is an expression of the form

$$w(x_1, \ldots, x_k) = \prod_{j=1}^{t} x_{i_j}^{\varepsilon_j},$$

where $i_j \in [1, k]$, for each $j \in [1, t]$ and $\varepsilon_j = \pm 1$. Given a word $w$ in $k$ variables, and a group $G$, one has the verbal mapping

$$w : G \times \cdots \times G \to G,$$

defined by,

$$w(g_1, \ldots, g_k) = \prod_{j=1}^{t} g_{i_j}^{\varepsilon_j}.$$  

See Segal [Seg09 Chapter 1]. It was first shown by Liebeck–Shalev [LS01 Theorem 1.6], that for a given non-trivial word $w$, each element of each large enough finite simple group $G$ can be expressed as a product of $c(w)$ values of $w$ in $G$ unless $w$ is trivial on $G$, with $c(w)$ only depending on $w$. In recent years, a lot of research has been devoted to studying the $c(w)$’s. For instance, it has since been established that $c(w) = 2$. This follows from the works of Larsen–Shalev [LS09], Shalev [Sha09], Larsen–Shalev–Tiep [LST11]. Some words actually have $c(w) = 1$ (i.e., they are surjective) and in fact, it was a long-standing conjecture to show that the commutator word has $c(w) = 1$ (the Ore’s conjecture). This was resolved by Liebeck–O’Brien–Shalev–Tiep [LOST10]. There are other words which are surjective. On the other hand, it can be easily seen that, if we take $w = x_1^n$ and $G$ is a finite simple group with $\gcd(|G|, n) > 1$, then $c(w) > 1$. Shalev conjectured that if $w(x_1, x_2)$ is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$, see [BGG12 Conjecture 8.3]. This conjecture was recently disproved by Jambor–Liebeck–O’Brien [JLO13 Theorem 1]. They gave examples of an infinite family of non-proper-power words which are non-surjective on an infinite family of finite simple groups. The words that they constructed have word lengths $3q - 1$ where $q \geq 5$ is a prime.

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In this note, our main motivations are to extend this class of non-proper-power words and also to construct examples in other word lengths.

1.1. Statement of results. In \cite[Theorem 1]{JLO13}, for primes $q \geq 5$, the words of the form
\[
x_1^2[x_1^{-2}, x_2^{-1}]^{q-1}_2
\]
(having word length $3q - 1$) have been considered and the non-surjectivity of the induced word map on $\operatorname{PSL}_2(\mathbb{F}_{p^n})$ has been established for primes $p$ and integers $n$ satisfying certain suitable conditions. Our principal result (Theorem 3.1) considers words that are not in the purview of \cite[Theorem 1]{JLO13}. It focuses on the words of the form
\[
x_1^2(x_1^2 x_2 x_1^{-2} x_2^{-1})^{r-1}_2
\]
(having word length $3r - 1$) where $r \geq 5$ is any odd integer not divisible by 3, and the words of the form
\[
x_1^{-2}(x_1^2 x_2 x_1^{-2} x_2^{-1})^{r-1}_2
\]
(having word length $3r - 5$) where $r \geq 7$ is any odd integer such that $r + 1$ is not divisible by 3. In Theorem 3.1 we establish the non-surjectivity of the induced word map on $\operatorname{PSL}_2(\mathbb{F}_p)$ for primes $p$ and integers $n$ satisfying appropriate conditions. Next, we show that there exist primes $p$ and integers $n$ such that these conditions hold (see Proposition 3.2). In §3.2, we combine Theorem 3.1 and Proposition 3.2 with the results on the inverse Galois problem for $\operatorname{PSL}_2(\mathbb{F}_p)$ (obtained by Shih \cite{Shi74} and Zywina \cite{Zyw15}) to show that the above-mentioned words are non-surjective on the absolute Galois group of the number fields having degree coprime to 6, see Proposition 3.4.

2. Identities involving trace polynomial of word maps

For any word $w$ in the free group $F_2$, it was observed by Vogt \cite{Vog89}, and Fricke and Klein \cite{FK65} that the trace of $w(x, y)$ can be expressed as a polynomial in terms of the traces of $x, y, xy$ for any $2 \times 2$ matrices $x, y$ with determinant 1. More precisely, there exists a polynomial $\tau(w) \in \mathbb{Z}[s, t, u]$ such that for any field $K$ and any $x, y \in \operatorname{SL}_2(K)$, the trace of $w(x, y)$ is equal to the polynomial $\tau(w)$ evaluated at $s = \text{tr}(x), t = \text{tr}(y), u = \text{tr}(xy)$. For a proof of this result, we refer to the works of Horowitz \cite[Theorem 3.1]{Hor72}, Plesken–Fabiańska \cite[Theorem 2.2]{PF09} and Jambor \cite[Theorem 2.2]{Jam15}.

Let $y_1$ denote one of the words
\[
x_1^2 x_2 x_1^{-2} x_2^{-1}, \quad x_1^2 x_2 x_1^{-2} x_2^{-1}.
\]
For $k \in \mathbb{Z}$ with $k \neq 1$, let $y_k$ denote the $k$-th power of the word $y_1$.

Lemma 2.1. If
\[
\tau(x_1^2 y_{k-1}) = \tau(x_1^{-2} y_k)
\]
holds for $k = a, a + 1$ with $a \in \mathbb{Z}$, then it holds for $k = a + 2$. If Equation (2.1) holds for $k = b, b - 1$ with $b \in \mathbb{Z}$, then it holds for $k = b - 2$. For $k = 0, 1$, Equation (2.1) holds. Consequently, Equation (2.1) holds for any $k \in \mathbb{Z}$. 
Proof. If Equation (2.1) holds for \(k = a, a + 1\), then it holds for \(k = a + 2\) since

\[
\begin{align*}
\tau(x_1^{-2}y_{a+2}) &= \tau(y_{a+2}x_1^{-2}) \\
&= \tau(y_1^2y_{a+1}^{-2}) \\
&= \tau(y_1^2)(y_{a+1}x_1^{-2}) - \tau(y_{a+1}^{-2}) \\
&= \tau(y_1^2)(x_1^{-2}y_{a+1}) - \tau(y_{a+1}^{-2}) \\
&= \tau(y_1^2)(y_{a+1}^2y_{a+1}) - \tau(y_{a+1}^{-2}) \\
&= \tau(y_1^2)(y_{a+1}^2x_1^{-2}) - \tau(y_{a+1}^{-2}) \\
&= \tau(y_1^2y_{a+1}^{-2}) \\
&= \tau(y_{a+1}x_1^{-2}) \\
&= \tau(x_1^{-2}y_{a+1})
\end{align*}
\]

hold. If Equation (2.1) holds for \(k = b, b - 1\), then it holds for \(k = b - 2\) since

\[
\begin{align*}
\tau(x_1^{-2}y_{b-2}) &= \tau(y_{b-2}x_1^{-2}) \\
&= \tau(y_1)(y_{b-2}x_1^{-2}) - \tau(y_{b-2}x_1^{-2}) \\
&= \tau(y_1)(y_{b-1}x_1^{-2}) - \tau(y_{b-2}x_1^{-2}) \\
&= \tau(y_1)(x_1^{-2}y_{b-1}) - \tau(x_1^{-2}y_{b-1}) \\
&= \tau(y_1)(y_{b-1}^2y_{b-2}) - \tau(y_{b-1}x_1^{-2}) \\
&= \tau(y_1)(y_{b-1}^2y_{b-2}) - (\tau(y_1)(y_{b-1}x_1^{-2}) - \tau(y_{b-1}x_1^{-2})) \\
&= \tau(y_1)(y_{b-2}x_1^{-2}) - \tau(y_1)(y_{b-2}x_1^{-2}) - \tau(y_{b-3}x_1^{-2}) \\
&= \tau(y_{b-3}x_1^{-2}) \\
&= \tau(x_1^{-2}y_{b-3})
\end{align*}
\]

hold.

Note that Equation (2.1) holds for \(k = 1\) since

\[
\begin{align*}
\tau(x_1^2) &= \tau(x_1^{2x_1^{-2}x_2^{-1}}) \\
&= \tau(x_1^{-2}y_1).
\end{align*}
\]

It follows that

\[
\begin{align*}
\tau(x_1^2y_{-1}) &= \tau(y_{-1}^{-1}x_1^{-2}) \\
&= \tau(y_{-1}^{-1}x_1^{-2}) \\
&= \tau(y_1x_1^{-2}) \\
&= \tau(x_1^{-2}y_1) \\
&= \tau(x_1^{-2}y_1) \\
&= \tau(x_1^{-2}).
\end{align*}
\]

So, Equation (2.1) holds for \(k = 0\). Consequently, Equation (2.1) holds for any \(k \in \mathbb{Z}\). \(\square\)
Lemma 2.2. For any integer $k \geq 1$, the equalities
\[\tau(x_1^2 y_{\pm k}) = \tau(x_1^2 y_k)\]
\[= \tau(x_1^2) \left( \sum_{i=1}^{k_{\pm}} (-1)^{k_{\pm} - i} \tau(y_i) + (-1)^{k_{\pm}} \right)\]
\[= (\tau(x_1)^2 - 2) \prod_{i=1}^{k_{\pm}} (\tau(y_i) + \zeta_{2k_{\pm}+1} + \zeta_{2k_{\pm}+1}^{-1})\]
hold where $k_{\pm} = k - \frac{1}{2} \pm \frac{1}{2}$.

Proof. The first equality follows since
\[\tau(x_1^2 y_{-k}) = \tau(y_k x_1^{-2})\]
\[= \tau(x_1^{-2} y_k)\]
hold for any integer $k$. Note that for any integer $k$,
\[\tau(x_1^2 y_k) = \tau(x_1) \tau(x_1 y_k) - \tau(y_k)\]
\[= \tau(x_1)(\tau(x_1^{-1}) \tau(x_1^{-1} x_1 y_k) + \tau(x_1^{-2} x_1 y_k)) - \tau(y_k)\]
\[= \tau(x_1) \tau(x_1^{-1}) \tau(y_k) - \tau(x_1) \tau(x_1^{-1} y_k) - \tau(y_k)\]
\[= \tau(x_1)^2 \tau(y_k) - \tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k)\]
\[= (\tau(x_1^2) + 2) \tau(y_k) - \tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k)\]
\[= \tau(x_1^2) \tau(y_k) - (\tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k))\]
\[= \tau(x_1^2) \tau(y_k) - \tau(x_1^{-2} y_k)\]
\[= \tau(x_1^2) \tau(y_k) - \tau(x_1^{-2} y_{-k})\]
hold where the second last equality follows from Lemma 2.1. It follows that for any integer $k \geq 1$,
\[\tau(x_1^2 y_k) = \tau(x_1^2 y_k) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2)\]
\[= (-1)^k((-1)^k \tau(x_1^2 y_k) - \tau(x_1^2)) + (-1)^k \tau(x_1^2)\]
\[= (-1)^k \sum_{i=1}^{k}((-1)^i \tau(x_1^2 y_i) - (-1)^{i-1} \tau(x_1^2 y_{i-1})) + (-1)^k \tau(x_1^2)\]
\[= (-1)^k \sum_{i=1}^{k}((-1)^i (\tau(x_1^2 y_i) + \tau(x_1^2 y_{i-1}))) + (-1)^k \tau(x_1^2)\]
\[= (-1)^k \sum_{i=1}^{k}((-1)^i \tau(x_1^2) \tau(y_i)) + (-1)^k \tau(x_1^2)\]
\[= \tau(x_1^2) \left( \sum_{i=1}^{k}(-1)^{k-i} \tau(y_i) + (-1)^k \right)\]
and
\[
\tau(x_1^2 y_{-k}) = \tau(x_1^2 y_{-k}) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2) \\
= (-1)^k((-1)^k (x_1^2 y_{-k}) - \tau(x_1^2)) + (-1)^k \tau(x_1^2) \\
= (-1)^k \sum_{i=1}^k ((-1)^i \tau(x_1^2 y_{-i}) - (-1)^{i-1} \tau(x_1^2 y_{-(i-1)})) + (-1)^k \tau(x_1^2) \\
= (-1)^k \sum_{i=1}^k ((-1)^i (\tau(x_1^2 y_{-i}) + \tau(x_1^2 y_{-(i-1)}))) + (-1)^k \tau(x_1^2) \\
= (-1)^k \sum_{i=1}^k ((-1)^i (\tau(x_1^2) \tau(y_{-i+1}))) + (-1)^k \tau(x_1^2) \\
= \tau(x_1^2) \left( \sum_{i=1}^k (-1)^{k-i} \tau(y_{-i+1}) + (-1)^k \right) \\
= \tau(x_1^2) \left( \sum_{i=2}^k (-1)^{k-i} \tau(y_{-i+1}) + 2(-1)^{k-1} + (-1)^k \right) \\
= \tau(x_1^2) \left( \sum_{i=1}^{k-1} (-1)^{(k-1)-i} \tau(y_{-i}) + (-1)^{k-1} \right) \\
= \tau(x_1^2) \left( \sum_{i=1}^{k-1} (-1)^{(k-1)-i} \tau(y_i) + (-1)^{k-1} \right)
\]

hold. This proves that
\[
\tau(x_1^2 y_{\pm k}) = \tau(x_1^2) \left( \sum_{i=1}^{k_{\pm}} (-1)^{k_{\pm}-i} \tau(y_i) + (-1)^{k_{\pm}} \right).
\]
The final equality follows from [JLO13, Lemma 2.1].

\[\square\]

3. Non-surjectivity of word maps

In this section, we study non-surjective word maps on PSL$_2(\mathbb{F})$ for certain finite fields $\mathbb{F}$, and as an application, we obtain non-surjective word maps on the absolute Galois group of certain number fields.

3.1. Non-surjective maps on PSL$_2(\mathbb{F})$. For any positive integer $m$, let $\zeta_m$ denote the root of unity $e^{2\pi i/m}$.

**Theorem 3.1.** Let $k$ be an integer with $k_{\pm} \geq 1$. Let $p$ be a prime and $n$ be a positive integer such that the following conditions hold.

1. The integer 2 is not a square modulo $p$. 


(2) The integer $n$ is odd.

(3) The inertia degree $f_i$ of $p$ in the extension $\mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$ does not divide $n$ for any $1 \leq i \leq k_{\pm}$.

Then the word map $(x, y) \mapsto w(x, y)$ is not surjective on $\text{PSL}_2(\mathbb{F}_{p^n})$ where $w$ denotes one among the words

$$x_1^{\pm 2} y_k, x_1^2 y_{\pm k}.$$ 

Proof. By Lemma 2.2, the trace polynomial of the word $w$ factors as

$$(s^2 - 2) \prod_{i=1}^{k_{\pm}} (\tau(y_i) + \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$$

over $\mathbb{Z}[\zeta_{2k_{\pm}+1} + \zeta_{2k_{\pm}+1}^{-1}]$. If some element of $\text{SL}_2(\mathbb{F}_{p^n})$ lies in the image of the induced word map on $\text{SL}_2(\mathbb{F}_{p^n})$, then the word polynomial $\tau(w)$ will vanish at some point of $\mathbb{F}_{p^n}$. The polynomial $X^2 - 2$ has no root in $\mathbb{F}_{p^n}$. So one of the factors of the product

$$\prod_{i=1}^{k_{\pm}} (\tau(y_i) + \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$$

with coefficients in $\mathbb{F}_{p^n}$, vanishes at a point of $\mathbb{F}_{p^n}^3$. Thus $\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}$ is contained in $\mathbb{F}_{p^n}$ for some $1 \leq i \leq k_{\pm}$. This is impossible since $f_i$ does not divide $n$ for all $i$. This proves the result. 

Note that if there exists an integer $n$ such that the condition (3) in Theorem 3.1 holds, then the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$ is at least two for any $1 \leq i \leq k_{\pm}$. If $k_{\pm} \equiv 1 \pmod{3}$, then the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$ is one for $i = \frac{2k_{\pm}+1}{3} \leq k_{\pm}$. So, $k_{\pm} \not\equiv 1 \pmod{3}$ is a necessary condition for having an integer $n$ satisfying the condition (3) in Theorem 3.1. We prove that this congruence condition is also sufficient to guarantee the existence of primes $p$ and integers $n \geq 1$ satisfying the conditions in Theorem 3.1.

**Proposition 3.2.** Let $k$ be an integer such that $k_{\pm} \geq 1$ and $k_{\pm} \not\equiv 1 \pmod{3}$. Then there are infinitely many primes $p$ such that the integer $2$ is not a square modulo $p$ and the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$ is at least two for any $1 \leq i \leq k_{\pm}$.

*Proof.* Note that if $p$ is prime such that $p^2 \not\equiv 1 \pmod{2k_{\pm}+1}$ larger than 1, then the inertia degree of any such prime in the extension $\mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})$ is at least 2 for any $1 \leq i \leq k_{\pm}$. Thus, it suffices to show that the primes $p$ such that 2 is not a square modulo $p$, $p$ is coprime to $2k_{\pm}+1$ and $p^2 \not\equiv 1 \pmod{2k_{\pm}+1}$, have density

$$\frac{1}{2} \times \prod_{p|2k_{\pm}+1} \left(1 - \frac{3}{p}\right).$$

This follows since such primes are precisely the primes $p$ satisfying the following conditions:

$$p \equiv 3, 5 \pmod{8}$$

and $p^2 \not\equiv 1 \pmod{2k_{\pm}+1}$. 

**Corollary 3.3.** If $k$ is an integer such that $k_{\pm} \geq 1$ and $k_{\pm} \not\equiv 1 \pmod{3}$, then there are infinitely many primes $p$ and integers $n \geq 1$ satisfying the conditions in Theorem 3.1.
3.2. Non-surjective maps on absolute Galois group of number fields. It would be interesting to look at non-surjective word maps on the absolute Galois group of number fields, i.e., on the group $\text{Gal}(\mathbb{Q}/K)$, where $\mathbb{Q}$ denotes an algebraic closure of the field of rational numbers $\mathbb{Q}$, and $K/\mathbb{Q}$ is a finite extension. The inverse Galois problem has been solved by Shih for the group $\text{PSL}_2(\mathbb{F}_p)$ for any odd prime $p$ such that $2, 3$ or $7$ is a quadratic non-residue modulo $p$ [Shi74]. Recently, it has been solved by Zywina for the group $\text{PSL}_2(\mathbb{F}_p)$ for any prime $p \geq 5$ [Zyw15]. Thus, for any prime $p \geq 5$, there exists a finite Galois extension $L/\mathbb{Q}$ whose Galois group is isomorphic to $\text{PSL}_2(\mathbb{F}_p)$. We use this result to obtain the following.

**Proposition 3.4.** If $K$ is a number field such that its degree over $\mathbb{Q}$ is coprime to $6$, then the word map $(x, y) \mapsto w(x, y)$ is not surjective on the absolute Galois group of $K$ where $w$ denotes one among the words $x_1^{\pm 2}y_k, x_1^2y_{\pm k}$, where $k$ is an integer with $k_\pm \geq 1$ and $k_\pm \not\equiv 1 \pmod{3}$.

**Proof.** Let $n$ denote the degree of $K$ over $\mathbb{Q}$. Let $p \geq 5$ be a prime such that $p \equiv 2 \pmod{n}$. Note that $p(p^2-1) \equiv 6 \pmod{n}$. Since $n$ is coprime to $6$, it follows that $p(p^2-1)$ is coprime to $n$. Thus, the total number of elements of $\text{PSL}_2(\mathbb{F}_p)$ is coprime to $n$. Let $L$ be a Galois extension of $\mathbb{Q}$ such that the Galois group of this extension is isomorphic to $\text{PSL}_2(\mathbb{F}_p)$. Note that the fields $K, L$ are linearly disjoint. Hence, the Galois group of the extension $KL/K$ is isomorphic to $\text{PSL}_2(\mathbb{F}_p)$. From Theorem 3.1 and Proposition 3.2, it follows that the word map $(x, y) \mapsto w(x, y)$ is not surjective on $\text{Gal}(\mathbb{Q}/K)$ where $w$ denotes one among the above-mentioned words. \hfill $\square$

4. Further Questions

The connection between the length of the word map and its surjectivity or its non-surjectivity is not yet well understood.

**Question 4.1.** Does there exist a non-proper-power odd length word which is non-surjective on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$?

In fact, one can ask a more refined question about the possible lengths of the words inducing non-surjective maps on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$.

**Question 4.2.** Consider the set $A$ consisting of the lengths of the word $w$ in $F_2$ such that $w$ is a non-proper-power word and is non-surjective on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$. What can we say about $|\mathbb{N} \setminus A|$ or about $\mathbb{N} \setminus A$?

Note that $A$ contains $3q - 1$ for any prime $q \geq 5$ by [JLO13, Theorem 1]. From Theorem 3.1 and Proposition 3.2, it follows that the set $A$ contains $3r - 1$ for any odd integer $r \geq 5$ not divisible by 3 and $A$ contains $3r - 5$ for any odd integer $r \geq 7$ such that $r + 1$ is not divisible by 3. Thus, $A$ contains almost all positive integers which are congruent to $\pm 2, \pm 4 \pmod{18}$.

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