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ON NON-SURJECTIVE WORD MAPS ON $\text{PSL}_2(\mathbb{F}_q)$

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Abstract. Jambor–Liebeck–O’Brien showed that there exist non-proper-power word maps which are not surjective on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$. This provided the first counterexamples to a conjecture of Shalev which stated that if a two-variable word is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$. Motivated by their work, we construct new examples of these types of non-surjective word maps. As an application, we obtain non-surjective word maps on the absolute Galois group of $\mathbb{Q}$.

1. Introduction

A word in $k$ variables is an expression of the form

$$w(x_1, \ldots, x_k) = \prod_{j=1}^{t} x_{i_j}^{\varepsilon_j},$$

where $i_j \in [1, k]$, for each $j \in [1, t]$ and $\varepsilon_j = \pm 1$. Given a word $w$ in $k$ variables, and a group $G$, one has the verbal mapping

$$w : G \times \cdots \times G \to G,$$

defined by,

$$w(g_1, \ldots, g_k) = \prod_{j=1}^{t} g_{i_j}^{\varepsilon_j}.$$

See Segal [Seg09, Chapter 1]. It was first shown by Liebeck–Shalev [LS01, Theorem 1.6], that for a given non-trivial word $w$, each element of each large enough finite simple group $G$ can be expressed as a product of $c(w)$ values of $w$ in $G$ unless $w$ is trivial on $G$, with $c(w)$ only depending on $w$. In recent years, a lot of research has been devoted to studying the $c(w)$'s. For instance, it has since been established that $c(w) = 2$. This follows from the works of Larsen–Shalev [LS09], Shalev [Sha09], Larsen–Shalev–Tiep [LST11]. Some words actually have $c(w) = 1$ (i.e., they are surjective) and in fact, it was a long-standing conjecture to show that the commutator word has $c(w) = 1$ (the Ore’s conjecture). This was resolved by Liebeck–O’Brien–Shalev–Tiep [LOST10]. There are other words which are surjective. On the other hand, it can be easily seen that, if we take $w = x_1^n$ and $G$ is a finite simple group with $\gcd(|G|, n) > 1$, then $c(w) > 1$. Shalev conjectured that if $w(x_1, x_2)$ is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$, see [BGG12, Conjecture 8.3]. This conjecture was recently disproved by Jambor–Liebeck–O’Brien [JLO13, Theorem 1]. They gave examples of an infinite family of non-proper-power words which are non-surjective on an infinite family of finite simple groups. The words that they constructed have word lengths $3q - 1$ where $q \geq 5$ is a prime.

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In this note, our main motivations are to extend this class of non-proper-power words and also to construct examples in other word lengths.

1.1. Statement of results. In [JLO13, Theorem 1], for primes \( q \geq 5 \), the words of the form 
\[
x_1^2[x_1^{-2}, x_2^{-1}]^{\frac{q-1}{2}}
\]
(having word length \( 3q - 1 \)) have been considered and the non-surjectivity of the induced word map on \( \text{PSL}_2(\mathbb{F}_p^{n}) \) has been established for primes \( p \) and integers \( n \) satisfying certain suitable conditions. Our principal result (Theorem 3.1) considers words that are not in the purview of [JLO13, Theorem 1]. It focuses on the words of the form 
\[
x_1^2(x_1^2 x_2^1 x_1^2 x_2^{-1})^{\frac{r-1}{2}}
\]
(having word length \( 3r - 1 \)) where \( r \geq 5 \) is any odd integer not divisible by 3, and the words of the form 
\[
x_1^{-2}(x_1^2 x_2^1 x_1^2 x_2^{-1})^{\frac{r-1}{2}}
\]
(having word length \( 3r - 5 \)) where \( r \geq 7 \) is any odd integer such that \( r + 1 \) is not divisible by 3. In Theorem 3.1 we establish the non-surjectivity of the induced word map on \( \text{PSL}_2(\mathbb{F}_p^{n}) \) for primes \( p \) and integers \( n \) satisfying appropriate conditions. Next, we show that there exist primes \( p \) and integers \( n \) such that these conditions hold (see Proposition 3.2). In §3.2 we combine Theorem 3.1 and Proposition 3.2 with the results on the inverse Galois problem for \( \text{PSL}_2(\mathbb{F}_p) \) (obtained by Shih [Shi74] and Zywina [Zyw15]) to show that the above-mentioned words are non-surjective on the absolute Galois group of the number fields having degree coprime to 6, see Proposition 3.4.

2. Identities involving trace polynomial of word maps

For any word \( w \) in the free group \( F_2 \), it was observed by Vogt [Vog89], and Fricke and Klein [FK65] that the trace of \( w(x, y) \) can be expressed as a polynomial in terms of the traces of \( x, y, xy \) for any \( 2 \times 2 \) matrices \( x, y \) with determinant 1. More precisely, there exists a polynomial \( \tau(w) \in \mathbb{Z}[s, t, u] \) such that for any field \( K \) and any \( x, y \in \text{SL}_2(K) \), the trace of \( w(x, y) \) is equal to the polynomial \( \tau(w) \) evaluated at \( s = \text{tr}(x), t = \text{tr}(y), u = \text{tr}(xy) \). For a proof of this result, we refer to the works of Horowitz [Hor72, Theorem 3.1], Plesken–Fabiańska [PF09, Theorem 2.2] and Jambor [Jam15, Theorem 2.2].

Let \( y_1 \) denote one of the words 
\[
x_1^2 x_2^1 x_1^{-2}, \quad x_1 x_2^{-1}.
\]
For \( k \in \mathbb{Z} \) with \( k \neq 1 \), let \( y_k \) denote the \( k \)-th power of the word \( y_1 \).

Lemma 2.1. If
\[
\tau(x_1^2 y_{k-1}) = \tau(x_1^{-2} y_k)
\]
holds for \( k = a, a + 1 \) with \( a \in \mathbb{Z} \), then it holds for \( k = a + 2 \). If Equation (2.1) holds for \( k = b, b - 1 \) with \( b \in \mathbb{Z} \), then it holds for \( k = b - 2 \). For \( k = 0, 1 \), Equation (2.1) holds. Consequently, Equation (2.1) holds for any \( k \in \mathbb{Z} \).
Proof. If Equation (2.1) holds for $k = a, a + 1$, then it holds for $k = a + 2$ since
\[
\tau(x_1^{-2}y_{a+2}) = \tau(y_{a+2}x_1^{-2}) = \tau(y_1^2y_a^{-2}) = \tau(y_1)\tau(y_{a+1}x_1^{-2}) - \tau(y_a^{-2}x_1) = \tau(y_1)\tau(x_1^{-2}y_{a+1}) - \tau(x_1^{-2}y_a) = \tau(y_1)\tau(x_1^2y_a) - \tau(x_1^2y_{a-1}) = \tau(y_1)\tau(y_a^2x_1^2) - \tau(y_{a-1}x_1^2) = \tau(y_1^2y_{a-1}x_1^2) = \tau(y_{a+1}x_1^2)
\]
hold. If Equation (2.1) holds for $k = b, b - 1$, then it holds for $k = b - 2$ since
\[
\tau(x_1^{-2}y_{b-2}) = \tau(y_{b-2}x_1^{-2}) = \tau(y_1)\tau(y_1y_{b-2}x_1^{-2}) - \tau(y_1^2y_{b-2}x_1^{-2}) = \tau(y_1)\tau(y_1x_1^2) - \tau(y_1x_1^2) = \tau(y_1)\tau(x_1^{-2}y_{b-1}) - \tau(x_1^{-2}y_b) = \tau(y_1)\tau(x_1^2y_{b-2}) - \tau(x_1^2y_{b-1}) = \tau(y_1)\tau(y_b^{-2}x_1^2) - \tau(y_b^{-1}x_1^2) = \tau(y_1)\tau(y_{b-2}x_1^2) - (\tau(y_1)\tau(y_1y_{b-1}x_1^2) - \tau(y_2y_{b-1}x_1^2)) = \tau(y_1)\tau(y_{b-2}x_1^2) - (\tau(y_1)\tau(y_{b-2}x_1^2) - \tau(y_{b-3}x_1^2)) = \tau(y_{b-3}x_1^2) = \tau(x_1^2y_{b-3})
\]
hold.

Note that Equation (2.1) holds for $k = 1$ since
\[
\tau(x_1^2) = \tau(x_2x_1^2x_2^{-1}) = \tau(x_1^2y_1).
\]
It follows that
\[
\tau(x_1^2y_{-1}) = \tau(y_1^{-1}x_1^{-2}) = \tau(y_1x_1^2) = \tau(x_1^2y_1) = \tau(x_1^2) = \tau(x_1^{-2}).
\]
So, Equation (2.1) holds for $k = 0$. Consequently, Equation (2.1) holds for any $k \in \mathbb{Z}$. □
Lemma 2.2. For any integer $k \geq 1$, the equalities
\[
\tau(x_1^2 y_\pm) = \tau(x_1^\pm y_k)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k_\pm} (-1)^{k_\pm - i} \tau(y_i) + (-1)^{k_\pm} \right)
\]
\[
= (\tau(x_1)^2 - 2) \prod_{i=1}^{k_\pm} (\tau(y_i) + \zeta_{2k_\pm + 1}^i + \zeta_{2k_\pm + 1}^{-i})
\]
hold where $k_\pm = k - \frac{1}{2} \pm \frac{1}{2}$.

Proof. The first equality follows since
\[
\tau(x_1^2 y_{-k}) = \tau(y_k x_1^{-2})
\]
\[
= \tau(x_1^{-2} y_k)
\]
hold for any integer $k$. Note that for any integer $k$,
\[
\tau(x_1^2 y_k) = \tau(x_1) \tau(x_1 y_k) - \tau(y_k)
\]
\[
= \tau(x_1) (\tau(x_1^{-1}) \tau(x_1^{-1} x_1 y_k) - \tau(x_1^{-2} x_1 y_k)) - \tau(y_k)
\]
\[
= \tau(x_1) \tau(x_1^{-1}) \tau(y_k) - \tau(x_1) \tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= \tau(x_1)^2 \tau(y_k) - \tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= (\tau(x_1^2) + 2) \tau(y_k) - \tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= \tau(x_1^2) \tau(y_k) - (\tau(x_1^{-1}) \tau(x_1^{-1} y_k) - \tau(y_k))
\]
\[
= \tau(x_1^2) \tau(y_k) - \tau(x_1^{-2} y_k)
\]
\[
= \tau(x_1^2) \tau(y_k) - \tau(x_1^{-2} y_{k-1})
\]
hold where the second last equality follows from Lemma 2.1. It follows that for any integer $k \geq 1$,
\[
\tau(x_1^2 y_k) = \tau(x_1^2 y_k) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k ((-1)^k \tau(x_1^2 y_k) - \tau(x_1^2)) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i \tau(x_1^2 y_i) - (-1)^{i-1} \tau(x_1^2 y_{i-1})) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i \tau(x_1^2 y_i) + \tau(x_1^2 y_{i-1})) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i \tau(x_1^2) \tau(y_i)) + (-1)^k \tau(x_1^2)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k} (-1)^{k-i} \tau(y_i) + (-1)^k \right)
\]
and

\[ \tau(x_1^2 y_{-k}) = \tau(x_1^2 y_{-k}) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2) \]
\[ = (-1)^k((-1)^k \tau(x_1^2 y_{-k}) - \tau(x_1^2)) + (-1)^k \tau(x_1^2) \]
\[ = (-1)^k \sum_{i=1}^{k}((-1)^i \tau(x_1^2 y_{-i}) - (-1)^{i-1} \tau(x_1^2 y_{-(i-1)})) + (-1)^k \tau(x_1^2) \]
\[ = (-1)^k \sum_{i=1}^{k}((-1)^i(\tau(x_1^2 y_{-i}) + \tau(x_1^2 y_{-(i-1)}))) + (-1)^k \tau(x_1^2) \]
\[ = (-1)^k \sum_{i=1}^{k}((-1)^i(\tau(x_1^2 \tau(y_{-i+1}))) + (-1)^k \tau(x_1^2) \]
\[ = \tau(x_1^2) \left( \sum_{i=1}^{k}(-1)^{k-i} \tau(y_{-i+1}) + (-1)^k \right) \]
\[ = \tau(x_1^2) \left( \sum_{i=1}^{k}(-1)^{k-i} \tau(y_{-i+1}) + 2(-1)^{k-1} + (-1)^k \right) \]
\[ = \tau(x_1^2) \left( \sum_{i=1}^{k-1}(-1)^{(k-1)-i} \tau(y_i) + (-1)^{k-1} \right) \]
\[ = \tau(x_1^2) \left( \sum_{i=1}^{k-1}(-1)^{(k-1)-i} \tau(y_i) + (-1)^{k-1} \right) \]

hold. This proves that

\[ \tau(x_1^2 y_{\pm k}) = \tau(x_1^2) \left( \sum_{i=1}^{k_{\pm}}(-1)^{k_{\pm}-i} \tau(y_i) + (-1)^{k_{\pm}} \right). \]

The final equality follows from \cite{JLO13} Lemma 2.1.

\[ \square \]

3. Non-surjectivity of word maps

In this section, we study non-surjective word maps on \( \text{PSL}_2(\mathbb{F}) \) for certain finite fields \( \mathbb{F} \), and as an application, we obtain non-surjective word maps on the absolute Galois group of certain number fields.

3.1. Non-surjective maps on \( \text{PSL}_2(\mathbb{F}) \). For any positive integer \( m \), let \( \zeta_m \) denote the root of unity \( e^{2\pi i/m} \).

**Theorem 3.1.** Let \( k \) be an integer with \( k_{\pm} \geq 1 \). Let \( p \) be a prime and \( n \) be a positive integer such that the following conditions hold.

(1) The integer 2 is not a square modulo \( p \).
(2) The integer $n$ is odd.

(3) The inertia degree $f_i$ of $p$ in the extension $\mathbb{Q}(\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-i})$ does not divide $n$ for any $1 \leq i \leq k_\pm$.

Then the word map $(x, y) \mapsto w(x, y)$ is not surjective on $\text{PSL}_2(\mathbb{F}_{p^n})$ where $w$ denotes one among the words

$$x_1^{\pm 2}y_k, x_{2}^{\pm 2}y_k.$$ 

Proof. By Lemma 2.2, the trace polynomial of the word $w$ factors as

$$(s^2 - 2) \prod_{i=1}^{k_\pm} (\tau(y_1) + \zeta_{2k_\pm+1}^i + \zeta_{2k_\pm+1}^{-i})$$

over $\mathbb{Z}[\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-1}]$. If some element of $\text{SL}_2(\mathbb{F}_{p^n})$ lies in the image of the induced word map on $\text{SL}_2(\mathbb{F}_{p^n})$, then the word polynomial $\tau(w)$ will vanish at some point of $\mathbb{F}_{p^n}^3$. The polynomial $X^2 - 2$ has no root in $\mathbb{F}_{p^n}$. So one of the factors of the product

$$\prod_{i=1}^{k_\pm} (\tau(y_1) + \zeta_{2k_\pm+1}^i + \zeta_{2k_\pm+1}^{-i})$$

with coefficients in $\mathbb{F}_{p^n_{\prod_{i=1}^{k_\pm} \zeta_{2k_\pm+1}}}$, vanishes at a point of $\mathbb{F}_{p^n}^3$. Thus $\zeta_{2k_\pm+1}^i + \zeta_{2k_\pm+1}^{-i}$ is contained in $\mathbb{F}_{p^n}$ for some $1 \leq i \leq k_\pm$. This is impossible since $f_i$ does not divide $n$ for all $i$. This proves the result. \[\Box\]

Note that if there exists an integer $n$ such that the condition (3) in Theorem 3.1 holds, then the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-i})$ is at least two for any $1 \leq i \leq k_\pm$. If $k_\pm \equiv 1 \pmod{3}$, then the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-i})$ is one for $i = \frac{2k_\pm+1}{3} \leq k_\pm$. So, $k_\pm \not \equiv 1 \pmod{3}$ is a necessary condition for having an integer $n$ satisfying the condition (3) in Theorem 3.1. We prove that this congruence condition is also sufficient to guarantee the existence of primes $p$ and integers $n \geq 1$ satisfying the conditions in Theorem 3.1.

Proposition 3.2. Let $k$ be an integer such that $k_\pm \geq 1$ and $k_\pm \not \equiv 1 \pmod{3}$. Then there are infinitely many primes $p$ such that the integer 2 is not a square modulo $p$ and the inertia degree of $p$ in the extension $\mathbb{Q}(\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-i})$ is at least two for any $1 \leq i \leq k_\pm$.

Proof. Note that if $p$ is prime such that $p^2 \equiv 1$ modulo each divisor of $2k_\pm + 1$ larger than 1, then the inertia degree of any such prime in the extension $\mathbb{Q}(\zeta_{2k_\pm+1} + \zeta_{2k_\pm+1}^{-i})$ is at least 2 for any $1 \leq i \leq k_\pm$. Thus, it suffices to show that the primes $p$ such that 2 is not a square modulo $p$, $p$ is coprime to $2k_\pm + 1$ and $p^2 \not \equiv 1$ modulo each divisor of $2k_\pm + 1$ larger than 1, have density

$$\frac{1}{2} \times \prod_{p | 2k_\pm+1} \left(1 - \frac{3}{p}\right).$$

This follows since such primes are precisely the primes $p$ satisfying the following conditions:

$p \equiv 3, 5 \pmod{8}$

and $p^2 \not \equiv 1$ modulo each prime divisor of $2k_\pm + 1$. \[\Box\]

Corollary 3.3. If $k$ is an integer such that $k_\pm \geq 1$ and $k_\pm \not \equiv 1 \pmod{3}$, then there are infinitely many primes $p$ and integers $n \geq 1$ satisfying the conditions in Theorem 3.1.
3.2. Non-surjective maps on absolute Galois group of number fields. It would be interesting to look at non-surjective word maps on the absolute Galois group of number fields, i.e., on the group Gal($\overline{\mathbb{Q}}/K$), where $\overline{\mathbb{Q}}$ denotes an algebraic closure of the field of rational numbers $\mathbb{Q}$, and $K/\mathbb{Q}$ is a finite extension. The inverse Galois problem has been solved by Shih for the group PSL$_2(\mathbb{F}_p)$ for any odd prime $p$ such that $2, 3$ or $7$ is a quadratic non-residue modulo $p$ [Shi74]. Recently, it has been solved by Zywina for the group PSL$_2(\mathbb{F}_p)$ for any prime $p \geq 5$ [Zyw15]. Thus, for any prime $p \geq 5$, there exists a finite Galois extension $L/\mathbb{Q}$ whose Galois group is isomorphic to PSL$_2(\mathbb{F}_p)$. We use this result to obtain the following.

**Proposition 3.4.** If $K$ is a number field such that its degree over $\mathbb{Q}$ is coprime to $6$, then the word map $(x, y) \mapsto w(x, y)$ is not surjective on the absolute Galois group of $K$ where $w$ denotes one among the words $x_1^{\pm 2}y_k, x_1^2y_{\pm k}$, where $k$ is an integer with $k_\pm \geq 1$ and $k_\pm \not\equiv 1 \pmod{3}$.

**Proof.** Let $n$ denote the degree of $K$ over $\mathbb{Q}$. Let $p \geq 5$ be a prime such that $p \equiv 2 \pmod{n}$. Note that $p(p^2-1) \equiv 6 \pmod{n}$. Since $n$ is coprime to $6$, it follows that $p(p^2-1)$ is coprime to $n$. Thus, the total number of elements of PSL$_2(\mathbb{F}_p)$ is coprime to $n$. Let $L$ be a Galois extension of $\mathbb{Q}$ such that the Galois group of this extension is isomorphic to PSL$_2(\mathbb{F}_p)$. Note that the fields $K, L$ are linearly disjoint. Hence, the Galois group of the extension $KL/K$ is isomorphic to PSL$_2(\mathbb{F}_p)$. From Theorem 3.1 and Proposition 3.2, it follows that the word map $(x, y) \mapsto w(x, y)$ is not surjective on Gal($\overline{\mathbb{Q}}/K$) where $w$ denotes one among the above-mentioned words. □

4. Further Questions

The connection between the length of the word map and its surjectivity or its non-surjectivity is not yet well understood.

**Question 4.1.** Does there exist a non-proper-power odd length word which is non-surjective on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$?

In fact, one can ask a more refined question about the possible lengths of the words inducing non-surjective maps on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$.

**Question 4.2.** Consider the set $A$ consisting of the lengths of the word $w$ in $F_2$ such that $w$ is a non-proper-power word and is non-surjective on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$. What can we say about $|N \setminus A|$ or about $\mathbb{N} \setminus A$?

Note that $A$ contains $3q - 1$ for any prime $q \geq 5$ by [JLO13, Theorem 1]. From Theorem 3.1 and Proposition 3.2, it follows that the set $A$ contains $3r - 1$ for any odd integer $r \geq 5$ not divisible by $3$ and $A$ contains $3r - 5$ for any odd integer $r \geq 7$ such that $r + 1$ is not divisible by $3$. Thus, $A$ contains almost all positive integers which are congruent to $\pm 2, \pm 4 \pmod{18}$.

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