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ON NON-SURJECTIVE WORD MAPS ON $\text{PSL}_2(\mathbb{F}_q)$

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Abstract. Jambor–Liebeck–O'Brien showed that there exist non-proper-power word maps which are not surjective on $\text{PSL}_2(\mathbb{F}_q)$ for infinitely many $q$. This provided the first counterexamples to a conjecture of Shalev which stated that if a two-variable word is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$. Motivated by their work, we construct new examples of these types of non-surjective word maps. As an application, we obtain non-surjective word maps on the absolute Galois group of $\mathbb{Q}$.

1. Introduction

A word in $k$ variables is an expression of the form

$$w(x_1, \ldots, x_k) = \prod_{j=1}^t x_{i_j}^{\varepsilon_j},$$

where $i_j \in [1, k]$, for each $j \in [1, t]$ and $\varepsilon_j = \pm 1$. Given a word $w$ in $k$ variables, and a group $G$, one has the verbal mapping

$$w : G \times \cdots \times G \rightarrow G,$$

defined by,

$$w(g_1, \ldots, g_k) = \prod_{j=1}^t g_{i_j}^{\varepsilon_j}.$$

See Segal [Seg09, Chapter 1]. It was first shown by Liebeck–Shalev [LS01, Theorem 1.6], that for a given non-trivial word $w$, each element of each large enough finite simple group $G$ can be expressed as a product of $c(w)$ values of $w$ in $G$ unless $w$ is trivial on $G$, with $c(w)$ only depending on $w$. In recent years, a lot of research has been devoted to studying the $c(w)$’s. For instance, it has since been established that $c(w) = 2$. This follows from the works of Larsen–Shalev [LS09], Shalev [Sha09], Larsen–Shalev–Tiep [LST11]. Some words actually have $c(w) = 1$ (i.e., they are surjective) and in fact, it was a long-standing conjecture to show that the commutator word has $c(w) = 1$ (the Ore’s conjecture). This was resolved by Liebeck–O’Brien–Shalev–Tiep [LOST10]. There are other words which are surjective. On the other hand, it can be easily seen that, if we take $w = x_1^n$ and $G$ is a finite simple group with $\gcd(|G|, n) > 1$, then $c(w) > 1$. Shalev conjectured that if $w(x_1, x_2)$ is not a proper power of a non-trivial word, then the corresponding word map is surjective on $\text{PSL}_2(\mathbb{F}_q)$ for all sufficiently large $q$, see [BGG12, Conjecture 8.3]. This conjecture was recently disproved by Jambor–Liebeck–O’Brien [JLO13, Theorem 1]. They gave examples of an infinite family of non-proper-power words which are non-surjective on an infinite family of finite simple groups. The words that they constructed have word lengths $3q - 1$ where $q \geq 5$ is a prime.
In this note, our main motivations are to extend this class of non-proper-power words and also to construct examples in other word lengths.

1.1. Statement of results. In [JLO13, Theorem 1], for primes \( q \geq 5 \), the words of the form

\[ x_1^2[x_1^{-2}, x_2^{-1}]^{2q-1} \]

(having word length \( 3q - 1 \)) have been considered and the non-surjectivity of the induced word map on \( \text{PSL}_2(\mathbb{F}_{p^n}) \) has been established for primes \( p \) and integers \( n \) satisfying certain suitable conditions. Our principal result (Theorem 3.1) considers words that are not in the purview of [JLO13, Theorem 1]. It focuses on the words of the form

\[ x_1^2(x_1^2 x_2 x_1^{-2} x_2^{-1})^{r-1} \]

(having word length \( 3r - 1 \)) where \( r \geq 5 \) is any odd integer not divisible by 3, and the words of the form

\[ x_1^{-2}(x_1^2 x_2 x_1^{-2} x_2^{-1})^{r-1} \]

(having word length \( 3r - 5 \)) where \( r \geq 7 \) is any odd integer such that \( r + 1 \) is not divisible by 3. In Theorem 3.1, we establish the non-surjectivity of the induced word map on \( \text{PSL}_2(\mathbb{F}_p) \) for primes \( p \) and integers \( n \) satisfying appropriate conditions. Next, we show that there exist primes \( p \) and integers \( n \) such that these conditions hold (see Proposition 3.2). In §3.2, we combine Theorem 3.1 and Proposition 3.2 with the results on the inverse Galois problem for \( \text{PSL}_2(\mathbb{F}_p) \) (obtained by Shih [Shi74] and Zywina [Zyw15]) to show that the above-mentioned words are non-surjective on the absolute Galois group of the number fields having degree coprime to 6, see Proposition 3.4.

2. Identities involving trace polynomial of word maps

For any word \( w \) in the free group \( F_2 \), it was observed by Vogt [Vog89], and Fricke and Klein [FK65] that the trace of \( w(x, y) \) can be expressed as a polynomial in terms of the traces of \( x, y, xy \) for any \( 2 \times 2 \) matrices \( x, y \) with determinant 1. More precisely, there exists a polynomial \( \tau(w) \in \mathbb{Z}[s, t, u] \) such that for any field \( K \) and any \( x, y \in \text{SL}_2(K) \), the trace of \( w(x, y) \) is equal to the polynomial \( \tau(w) \) evaluated at \( s = \text{tr}(x), t = \text{tr}(y), u = \text{tr}(xy) \). For a proof of this result, we refer to the works of Horowitz [Hor72, Theorem 3.1], Plesken–Fabiańska [PF09, Theorem 2.2] and Jambor [Jam15, Theorem 2.2].

Let \( y_1 \) denote one of the words

\[ x_1^2 x_2 x_1^{-2} x_2^{-1}, \quad x_1^2 x_2 x_1^{-2} x_2^{-1}. \]

For \( k \in \mathbb{Z} \) with \( k \neq 1 \), let \( y_k \) denote the \( k \)-th power of the word \( y_1 \).

**Lemma 2.1.** If

\[ \tau(x_1^2 y_{k-1}) = \tau(x_1^{-2} y_k) \]  \hspace{1cm} (2.1)

holds for \( k = a, a + 1 \) with \( a \in \mathbb{Z} \), then it holds for \( k = a + 2 \). If Equation (2.1) holds for \( k = b, b - 1 \) with \( b \in \mathbb{Z} \), then it holds for \( k = b - 2 \). For \( k = 0, 1 \), Equation (2.1) holds. Consequently, Equation (2.1) holds for any \( k \in \mathbb{Z} \).
Proof. If Equation (2.1) holds for $k = a, a + 1$, then it holds for $k = a + 2$ since
\[
\tau(x_1^{-2}y_{a+2}) = \tau(y_{a+2}x_1^{-2}) \\
= \tau(y_1^2y_{a}x_1^{-2}) \\
= \tau(y_1)\tau(y_{a+1}x_1^{-2}) - \tau(y_ax_1^{-2}) \\
= \tau(y_1)\tau(x_1^{-2}y_{a+1}) - \tau(x_1^{-2}y_a) \\
= \tau(y_1)\tau(x_1^{2}y_a) - \tau(x_1^{2}y_{a-1}) \\
= \tau(y_1)\tau(y_{a-1}x_1^{2}) - \tau(y_{a-1}x_1^{2}) \\
= \tau(y_1^2y_{a-1}x_1^{2}) \\
= \tau(y_{a+1}x_1^{2}) \\
= \tau(x_1^{2}y_{a+1})
\]
hold. If Equation (2.1) holds for $k = b, b - 1$, then it holds for $k = b - 2$ since
\[
\tau(x_1^{-2}y_{b-2}) = \tau(y_{b-2}x_1^{-2}) \\
= \tau(y_1)\tau(y_{b-1}x_1^{-2}) - \tau(y_1x_1^{-2}) \\
= \tau(y_1)\tau(y_{b-2}x_1^{2}) - \tau(y_{b-2}x_1^{2}) \\
= \tau(y_1)\tau(x_1^{-2}y_{b-2}) - \tau(x_1^{-2}y_b) \\
= \tau(y_1)\tau(x_1^{2}y_{b-2}) - \tau(x_1^{2}y_{b-1}) \\
= \tau(y_1)\tau(y_{b-3}x_1^{2}) - \tau(y_{b-3}x_1^{2}) \\
= \tau(y_{b-2}x_1^{2}) - \tau(y_{b-2}x_1^{2}) \\
= \tau(y_{b-3}x_1^{2}) \\
= \tau(x_1^{2}y_{b-3})
\]
hold.

Note that Equation (2.1) holds for $k = 1$ since
\[
\tau(x_1^{2}) = \tau(x_2x_1^{2}x_2^{-1}) \\
= \tau(x_1^{-2}y_1).
\]

It follows that
\[
\tau(x_1^{2}y_{-1}) = \tau(y_{-1}x_1^{-2}) \\
= \tau(y_1x_1^{-2}) \\
= \tau(x_1^{-2}y_1) \\
= \tau(x_1^{2}) \\
= \tau(x_1^{-2}).
\]

So, Equation (2.1) holds for $k = 0$. Consequently, Equation (2.1) holds for any $k \in \mathbb{Z}$. □
Lemma 2.2. For any integer \( k \geq 1 \), the equalities
\[
\tau(x_1^2 y_{\pm k}) = \tau(x_1^2 y_k)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k_{\pm}} (-1)^{k_{\pm} - i} \tau(y_i) + (-1)^{k_{\pm}} \right)
\]
\[
= (\tau(x_1)^2 - 2) \prod_{i=1}^{k_{\pm}} (\tau(y_1) + \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})
\]
hold where \( k_{\pm} = k - \frac{1}{2} \pm \frac{1}{2} \).

Proof. The first equality follows since
\[
\tau(x_1^2 y_{-k}) = \tau(y_k x_1^{-2})
\]
\[
= \tau(x_1^{-2} y_k)
\]
hold for any integer \( k \). Note that for any integer \( k \),
\[
\tau(x_1^2 y_k) = \tau(x_1)\tau(x_1 y_k) - \tau(y_k)
\]
\[
= \tau(x_1)(\tau(x_1^{-1})\tau(x_1^{-1} y_k) - \tau(x_1^{-2} x_1 y_k)) - \tau(y_k)
\]
\[
= \tau(x_1)\tau(x_1^{-1})\tau(y_k) - \tau(x_1)\tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= \tau(x_1)^2 \tau(y_k) - \tau(x_1^{-1})\tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= (\tau(x_1^2) + 2)\tau(y_k) - \tau(x_1^{-1})\tau(x_1^{-1} y_k) - \tau(y_k)
\]
\[
= \tau(x_1^2)\tau(y_k) - (\tau(x_1^{-1})\tau(x_1^{-1} y_k) - \tau(y_k))
\]
\[
= \tau(x_1^2)\tau(y_k) - \tau(x_1^2 y_k)
\]
\[
= \tau(x_1^2)\tau(y_k) - \tau(x_1^2 y_k - 1)
\]
hold where the second last equality follows from Lemma 2.1. It follows that for any integer \( k \geq 1 \),
\[
\tau(x_1^2 y_k) = \tau(x_1^2 y_k) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k((-1)^k \tau(x_1^2 y_k) - \tau(x_1^2)) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k}((-1)^i \tau(x_1^2 y_i) - (-1)^{i-1} \tau(x_1^2 y_{i-1})) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k}((-1)^i(\tau(x_1^2 y_i) + \tau(x_1^2 y_{i-1}))) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k}((-1)^i \tau(x_1^2)\tau(y_i)) + (-1)^k \tau(x_1^2)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k}(-1)^{k-i} \tau(y_i) + (-1)^k \right)
\]
and
\[
\tau(x_1^2 y_{-k}) = \tau(x_1^2 y_{-k}) - (-1)^k \tau(x_1^2) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k((-1)^k \tau(x_1^2 y_{-k}) - \tau(x_1^2)) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i \tau(x_1^2 y_{-i}) - (-1)^{i-1} \tau(x_1^2 y_{-(i-1)})) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i(\tau(x_1^2 y_{-i}) + \tau(x_1^2 y_{-(i-1)}))) + (-1)^k \tau(x_1^2)
\]
\[
= (-1)^k \sum_{i=1}^{k} ((-1)^i(\tau(x_1^2) \tau(y_{-i+1}))) + (-1)^k \tau(x_1^2)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k} (-1)^{k-i} \tau(y_{-i+1}) + (-1)^k \right)
\]
\[
= \tau(x_1^2) \left( \sum_{i=2}^{k} (-1)^{k-i} \tau(y_{-i+1}) + 2(-1)^{k-1} + (-1)^k \right)
\]
\[
= \tau(x_1^2) \left( \sum_{i=2}^{k-1} (-1)^{(k-1)-i} \tau(y_i) + (-1)^{k-1} \right)
\]
\[
= \tau(x_1^2) \left( \sum_{i=1}^{k-1} (-1)^{(k-1)-i} \tau(y_i) + (-1)^{k-1} \right)
\]

hold. This proves that
\[
\tau(x_1^2 y_{\pm k}) = \tau(x_1^2) \left( \sum_{i=1}^{k} (-1)^{k-i} \tau(y_i) + (-1)^k \right).
\]

The final equality follows from [JLO13, Lemma 2.1].

\[\square\]

3. Non-surjectivity of word maps

In this section, we study non-surjective word maps on \( \text{PSL}_2(\mathbb{F}) \) for certain finite fields \( \mathbb{F} \), and as an application, we obtain non-surjective word maps on the absolute Galois group of certain number fields.

3.1. Non-surjective maps on \( \text{PSL}_2(\mathbb{F}) \). For any positive integer \( m \), let \( \zeta_m \) denote the root of unity \( e^{2\pi i/m} \).

**Theorem 3.1.** Let \( k \) be an integer with \( k_\pm \geq 1 \). Let \( p \) be a prime and \( n \) be a positive integer such that the following conditions hold.

(1) The integer 2 is not a square modulo \( p \).
(2) The integer \( n \) is odd.
(3) The inertia degree \( f_i \) of \( p \) in the extension \( \mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}) \) does not divide \( n \) for any \( 1 \leq i \leq k_{\pm} \).

Then the word map \((x, y) \mapsto w(x, y)\) is not surjective on \( \text{PSL}_2(\mathbb{F}_{p^n}) \) where \( w \) denotes one among the words
\[
x_1^{\pm 2}y_k, x_1^2y_{\pm k}.
\]

Proof. By Lemma 2.2, the trace polynomial of the word \( w \) factors as
\[
(s^2 - 2) \prod_{i=1}^{k_{\pm}} (\tau(y_1) + \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})
\]
over \( \mathbb{Z} [\zeta_{2k_{\pm}+1} + \zeta_{2k_{\pm}+1}^{-1}] \). If some element of \( \text{SL}_2(\mathbb{F}_{p^n}) \) lies in the image of the induced word map on \( \text{SL}_2(\mathbb{F}_{p^n}) \), then the word polynomial \( \tau(w) \) will vanish at some point of \( \mathbb{F}_{p^n}^3 \). The polynomial \( X^2 - 2 \) has no root in \( \mathbb{F}_{p^n} \). So one of the factors of the product
\[
\prod_{i=1}^{k_{\pm}} (\tau(y_1) + \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i})
\]
with coefficients in \( \mathbb{F}_{p^n}^3_{\langle f_1, \ldots, f_{k_{\pm}} \rangle} \), vanishes at a point of \( \mathbb{F}_{p^n}^3 \). Thus \( \zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i} \) is contained in \( \mathbb{F}_{p^n} \) for some \( 1 \leq i \leq k_{\pm} \). This is impossible since \( f_i \) does not divide \( n \) for all \( i \). This proves the result.

Note that if there exists an integer \( n \) such that the condition (3) in Theorem 3.1 holds, then the inertia degree of \( p \) in the extension \( \mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}) \) is at least two for any \( 1 \leq i \leq k_{\pm} \). If \( k_{\pm} \equiv 1 \) (mod 3), then the inertia degree of \( p \) in the extension \( \mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}) \) is one for \( i = \frac{2k_{\pm}+1}{3} \leq k_{\pm} \). So, \( k_{\pm} \not\equiv 1 \) (mod 3) is a necessary condition for having an integer \( n \) satisfying the condition (3) in Theorem 3.1. We prove that this congruence condition is also sufficient to guarantee the existence of primes \( p \) and integers \( n \geq 1 \) satisfying the conditions in Theorem 3.1.

**Proposition 3.2.** Let \( k \) be an integer such that \( k_{\pm} \geq 1 \) and \( k_{\pm} \not\equiv 1 \) (mod 3). Then there are infinitely many primes \( p \) such that the integer 2 is not a square modulo \( p \) and the inertia degree of \( p \) in the extension \( \mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}) \) is at least two for any \( 1 \leq i \leq k_{\pm} \).

Proof. Note that if \( p \) is prime such that \( p^2 \not\equiv 1 \) modulo each divisor of \( 2k_{\pm} + 1 \) larger than 1, then the inertia degree of any such prime in the extension \( \mathbb{Q}(\zeta_{2k_{\pm}+1}^i + \zeta_{2k_{\pm}+1}^{-i}) \) is at least 2 for any \( 1 \leq i \leq k_{\pm} \). Thus, it suffices to show that the primes \( p \) such that 2 is not a square modulo \( p \) is coprime to \( 2k_{\pm} + 1 \) and \( p^2 \not\equiv 1 \) modulo each divisor of \( 2k_{\pm} + 1 \) larger than 1, have density
\[
\frac{1}{2} \prod_{p \mid 2k_{\pm} + 1} \left( 1 - \frac{3}{p} \right).
\]
This follows since such primes are precisely the primes \( p \) satisfying the following conditions:
\[
p \equiv 3, 5 \pmod{8}
\]
and \( p^2 \not\equiv 1 \) modulo each prime divisor of \( 2k_{\pm} + 1 \). \( \square \)

**Corollary 3.3.** If \( k \) is an integer such that \( k_{\pm} \geq 1 \) and \( k_{\pm} \not\equiv 1 \) (mod 3), then there are infinitely many primes \( p \) and integers \( n \geq 1 \) satisfying the conditions in Theorem 3.1.
3.2. Non-surjective maps on absolute Galois group of number fields. It would be interesting to look at non-surjective word maps on the absolute Galois group of number fields, i.e., on the group Gal($\overline{\mathbb{Q}}/K$), where $\overline{\mathbb{Q}}$ denotes an algebraic closure of the field of rational numbers $\mathbb{Q}$, and $K/\mathbb{Q}$ is a finite extension. The inverse Galois problem has been solved by Shih for the group PSL$_2(\mathbb{F}_p)$ for any odd prime $p$ such that 2, 3 or 7 is a quadratic non-residue modulo $p$ [Shi74]. Recently, it has been solved by Zywina for the group PSL$_2(\mathbb{F}_p)$ for any prime $p \geq 5$ [Zyw15]. Thus, for any prime $p \geq 5$, there exists a finite Galois extension $L/\mathbb{Q}$ whose Galois group is isomorphic to PSL$_2(\mathbb{F}_p)$. We use this result to obtain the following.

**Proposition 3.4.** If $K$ is a number field such that its degree over $\mathbb{Q}$ is coprime to 6, then the word map $(x, y) \mapsto w(x, y)$ is not surjective on the absolute Galois group of $K$ where $w$ denotes one among the words

$$x_1^{\pm 2}y_k, x_1^2y_{\pm k},$$

where $k$ is an integer with $k_\pm \geq 1$ and $k_\pm \not\equiv 1 \pmod{3}$.

**Proof.** Let $n$ denote the degree of $K$ over $\mathbb{Q}$. Let $p \geq 5$ be a prime such that $p \equiv 2 \pmod{n}$. Note that $p(p^2-1) \equiv 6 \pmod{n}$. Since $n$ is coprime to 6, it follows that $p(p^2-1)$ is coprime to $n$. Thus, the total number of elements of PSL$_2(\mathbb{F}_p)$ is coprime to $n$. Let $L$ be a Galois extension of $\mathbb{Q}$ such that the Galois group of this extension is isomorphic to PSL$_2(\mathbb{F}_p)$. Note that the fields $K, L$ are linearly disjoint. Hence, the Galois group of the extension $KL/K$ is isomorphic to PSL$_2(\mathbb{F}_p)$. From Theorem 3.1 and Proposition 3.2, it follows that the word map $(x, y) \mapsto w(x, y)$ is not surjective on Gal($\overline{\mathbb{Q}}/K$) where $w$ denotes one among the above-mentioned words. $\square$

4. Further Questions

The connection between the length of the word map and its surjectivity or its non-surjectivity is not yet well understood.

**Question 4.1.** Does there exist a non-proper-power odd length word which is non-surjective on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$?

In fact, one can ask a more refined question about the possible lengths of the words inducing non-surjective maps on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$.

**Question 4.2.** Consider the set $A$ consisting of the lengths of the word $w$ in $F_2$ such that $w$ is a non-proper-power word and is non-surjective on PSL$_2(\mathbb{F}_q)$ for infinitely many $q$. What can we say about $|\mathbb{N} \setminus A|$ or about $\mathbb{N} \setminus A$?

Note that $A$ contains $3q-1$ for any prime $q \geq 5$ by [JLO13, Theorem 1]. From Theorem 3.1 and Proposition 3.2, it follows that the set $A$ contains $3r-1$ for any odd integer $r \geq 5$ not divisible by 3 and $A$ contains $3r-5$ for any odd integer $r \geq 7$ such that $r+1$ is not divisible by 3. Thus, $A$ contains almost all positive integers which are congruent to $\pm 2, \pm 4 \pmod{18}$.

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