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Derezinski, Jan; Gass, Christian; Ruba, Blazej Teofil

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Generalized integrals and point interactions

Jan Dereziński¹, Christian Gaß¹ and Błażej Ruba²

¹ Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland
² Department of Mathematics, University of Copenhagen, Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark
E-mail: jan.derezinski@fuw.edu.pl

Abstract. First we recall a method of computing scalar products of eigenfunctions of a Sturm-Liouville operator. This method is then applied to Macdonald and Gegenbauer functions, which are eigenfunctions of the Bessel, resp. Gegenbauer operators. The computed scalar products are well defined only for a limited range of parameters. To extend the obtained formulas to a much larger range of parameters, we introduce the concept of a generalized integral. The (standard as well as generalized) integrals of Macdonald and Gegenbauer functions have important applications to operator theory. Macdonald functions can be used to express the integral kernels of the resolvent (Green functions) of the Laplacian on the Euclidean space in any dimension. Similarly, Gegenbauer functions appear in Green functions of the Laplacian on the sphere and the hyperbolic space. In dimensions 1,2,3 one can perturb these Laplacians with a point potential, obtaining a well defined self-adjoint operator. Standard integrals of Macdonald and Gegenbauer functions appear in the formulas for the corresponding Green functions. In higher dimensions the Laplacian perturbed by point potentials does not exist. However, the corresponding Green function can be generalized to any dimension by using generalized integrals.

The following notes are a short version of our papers [7, 8].

1. Bilinear integrals of eigenfunctions of Sturm-Liouville operators
Consider a Sturm-Liouville operator
\[ C := -\rho(r)^{-1}\left(\partial_r p(r)\partial_r + q(r)\right) \]
acting on functions on an interval \([a, b]\). C is formally symmetric for the bilinear scalar product with the density \(\rho\):
\[ \langle f | g \rangle := \int_a^b f(r)g(r)\rho(r)dr. \]
Let us describe a method to compute the scalar product of two eigenfunctions of \(C\).

First, consider eigenfunctions \(f_i\) corresponding to two distinct eigenvalues \(E_i\), \(i = 1, 2\). That is, suppose that \(Cf_i = E_i f_i\). Then the following is true:
\[ \int_a^b f_1(r)f_2(r)\rho(r)dr = \frac{W(b) - W(a)}{E_1 - E_2} \]
where \(W(r) := f_1(r)p(r)f_2'(r) - f_1'(r)p(r)f_2(r)\) is the Wronskian.
(1) is sometimes called Green’s identity, or the integrated Lagrange identity. Note that if \( a, b \) are singular points of the corresponding differential equation, then the right hand side of (1) can often be easily evaluated.

Using an appropriate limiting procedure we can also often evaluate (1) for \( f = f_1 = f_2 \):

\[
\langle f | f \rangle = \int_a^b f(r)^2 \rho(r)dr.
\] (2)

2. Bilinear integrals of Macdonald and Gegenbauer functions

The following two families of Sturm-Liouville operators are especially important for applications [17, 14, 13]:

the Bessel operator:
\[
B_\alpha := -\frac{1}{r} \partial_r r \partial_r + \frac{\alpha^2}{r^2},
\]

and the Gegenbauer operator:
\[
G_\alpha := -(1-w^2)^{-\alpha} \partial_w (1-w^2)^{\alpha+1} \partial_w.
\]

The modified Bessel equation is the eigenequation of \( B_\alpha \) with eigenvalue \(-1\). The (standard) Bessel equation is its eigenequation for eigenvalue 1. The separation of variables in the Laplacian on the Euclidean space \( \mathbb{R}^d \) leads to the Bessel operator on \([0, \infty[\) with density \( \rho = 2r \) and \( \alpha = \frac{d-2}{2} \).

The Gegenbauer equation is the eigenequation of \( G_\alpha \) with eigenvalue \( \lambda^2 - (\alpha + \frac{1}{2})^2 \). The Gegenbauer operator on \([-1, 1]\) with density \( \rho(w) = (1-w^2)^{\frac{d-2}{2}} \) arises when we separate variables in the Laplacian on the sphere \( S^d \). We obtain the Gegenbauer operator on \([1, \infty[\) with density \( \rho(w) = (w^2 - 1)^{\frac{d-2}{2}} \) when we separate variables in the Laplacian on the hyperbolic space \( \mathbb{H}^d \).

Scaled Macdonald functions \( K_{\alpha}(br) \) are exponentially decaying eigenfunctions of the Bessel operator with eigenvalue \(-b^2\). Applying the above method for \( a > 0, b > 0 \) we find

\[
\int_0^\infty K_{\alpha}(ar)K_{\alpha}(br)2rdr = \frac{\pi((a/b)^\alpha - (b/a)^\alpha)}{\sin(\pi \alpha)(a^2 - b^2)}, \quad |\text{Re}(\alpha)| < 1, \quad \alpha \neq 0; \quad (3)
\]

\[
\int_0^\infty K_0(ar)K_0(br)2rdr = \frac{2 \ln \frac{a}{b}}{a^2 - b^2}, \quad (4)
\]

\[
\int_0^\infty K_{\alpha}(br)^22rdr = \frac{\pi \alpha}{b^2 \sin(\pi \alpha)}, \quad |\text{Re}(\alpha)| < 1, \quad \alpha \neq 0; \quad (5)
\]

\[
\int_0^\infty K_0(br)^22rdr = \frac{1}{b^2}. \quad (6)
\]

The first formula follows directly by Green’s identity. The next three identities are obtained by applying the \textit{de l’Hôpital rule} to \( \alpha = 0 \) and \( \alpha = b \).

The identities (3)–(6) can be found in standard collections of integrals, such as [9].

The Gegenbauer equation is the special case of the hypergeometric equation with the symmetry \( w \rightarrow -w \) and the singular points at \(-1, 1, \infty\):

\[
(1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \left(\alpha + \frac{1}{2}\right)^2)f(w) = 0.
\]

In the present context, the Gegenbauer equation is arguably more convenient than the equivalent, but more frequently encountered associated Legendre equation.
We will use two kinds of Gegenbauer functions: one is characterized by its asymptotics \( \sim \frac{1}{\Gamma(\alpha + 1)} \) at 1:

\[
S_{\alpha, \pm \lambda}(w) := \sum_{j=0}^{\infty} \left( \frac{1}{2} + \alpha + \lambda \right) \frac{1}{j!} \left( \frac{1-w}{2} \right)^j,
\]

where \( (z)_n \) is the Pochhammer symbol. The other has the asymptotics \( \sim \frac{1}{w^{2+\alpha+\lambda} \Gamma(\lambda+1)} \) at \( \infty \):

\[
Z_{\alpha, \lambda}(w) := \frac{1}{(w \pm 1)^{\frac{1}{2} + \alpha + \lambda}} \sum_{j=0}^{\infty} \left( \frac{1}{2} + \lambda \right) \frac{1}{j!} \left( \frac{2}{1 \pm w} \right)^j.
\]

We note the identities

\[
S_{\alpha, \lambda}(w) = S_{\alpha, -\lambda}(w), \quad Z_{\alpha, \lambda}(w) = \frac{Z_{-\alpha, \lambda}(w)}{(w^2 - 1)_{\frac{1}{2}}^\alpha},
\]

as well as the slightly more subtle Whipple identity:

\[
Z_{\alpha, \lambda}(w) := (w^2 - 1)^{-\frac{1}{2} + \frac{\lambda}{2} - \frac{\alpha}{2}} S_{\lambda, \alpha} \left( \frac{w}{(w^2 - 1)_{\frac{1}{2}}} \right),
\]

\[
S_{\alpha, \lambda}(w) := (w^2 - 1)^{-\frac{1}{2} + \frac{\lambda}{2} - \frac{\alpha}{2}} Z_{\lambda, \alpha} \left( \frac{w}{(w^2 - 1)_{\frac{1}{2}}} \right), \quad \text{Re}(w) > 0,
\]

where \( (w^2 - 1)_{\frac{1}{2}}^{\alpha} := (w - 1)^\alpha (w + 1)^\alpha \) and we use the principal branch of the power function.

Here are the basic bilinear integrals of Gegenbauer functions. We assume \(|\text{Re}(\alpha)| < 1, \alpha \neq 0\) and \(\text{Re}(\lambda) > 0\):

\[
\int_{-1}^{1} S_{\alpha, i\beta_1}(w) S_{\alpha, i\beta_2}(w) (1 - w^2)^\alpha dw = \frac{2^{2\alpha+2}}{(\beta_1^2 - \beta_2^2) \sin \pi \alpha} \left( \frac{\cosh(\pi \beta_1)}{\Gamma(\frac{1}{2} + \alpha + i \beta_1) \Gamma(\frac{1}{2} + \alpha + i \beta_2)} - (\beta_1 \leftrightarrow \beta_2) \right)
\]

\[
\int_{1}^{\infty} Z_{\alpha, \lambda_1}(w) Z_{\alpha, \lambda_2}(w) (w^2 - 1)^\alpha dw = \frac{2^{\lambda_1 + \lambda_2 + 1}}{(\lambda_1^2 - \lambda_2^2) \sin \pi \alpha} \left( \frac{1}{\Gamma(\frac{1}{2} - \alpha + \lambda_1) \Gamma(\frac{1}{2} + \alpha + \lambda_2)} - (\lambda_1 \leftrightarrow \lambda_2) \right).
\]

Applying the de l'Hôpital rule we extend these identities to \( \alpha = 0, \lambda_1 = \lambda_2, \) and \( \beta_1 = \beta_2, \) see [7]. In contrast to the well-known integrals over Macdonald functions, the integrals (7) and (8) seem to be new.
3. Generalized integral
The integrals (3)–(6) and (7)–(8) are divergent for $|\text{Re}(\alpha)| \geq 1$. Using the generalized integral, (3)–(6) and (8) can be extended to all $\alpha \in \mathbb{C}$, and (7) can be extended to $\text{Re}(\alpha) > -1$. The concept of the generalized integral can be traced back to Hadamard [10, 11] and Riesz [16]; see [12, 15] for modern expositions.

Note that several variations of the generalized integral are possible. In our note we restrict ourselves to functions which are non-integrable near a finite point and have a finite number of homogeneous singularities at this point. In some of the above references one can also find other variations of the generalized integral applicable to functions which are non-integrable near infinity and/or exhibit almost homogeneous singularities, i.e., singularities that are homogeneous up to powers of logarithms.

Without loss of generality, we can put the singular point to 0. We then say that a function $f$ on $]0, \infty[$ is integrable in the generalized sense if it is integrable on $]1, \infty[$ and if there exists a finite set $\Omega \subset \mathbb{C}$ and $f_k \in \mathbb{C}$, $k \in \Omega$, such that $f - \sum_{k \in \Omega} f_k r^k$ is integrable on $]0, 1[$. We define the generalized integral as

$$\text{gen} \int_0^\infty f(r) \, dr := \sum_{k \in \Omega \setminus \{-1\}} \frac{f_k}{k+1} + \int_0^1 \left( f(r) - \sum_{k \in \Omega} f_k r^k \right) \, dr + \int_1^\infty f(r) \, dr.$$ 

For $f \in L^1[0, \infty]$ the generalized and standard integrals coincide:

$$\text{gen} \int_0^\infty f(r) \, dr = \int_0^\infty f(r) \, dr.$$

If $f_{-1} \neq 0$, the generalized integral has a scaling anomaly but it is always invariant with respect to a power transformation:

$$\text{gen} \int_0^\infty f(r) \, dr = \text{gen} \int_0^\infty f(\alpha u) \, \alpha du + f_{-1} \ln(\alpha),$$

$$\text{gen} \int_0^\infty f(r) \, dr = \text{gen} \int_0^\infty f(u^\alpha) \, \alpha u^{\alpha - 1} du.$$

Under more general coordinate transformations $g(u)$, which are smooth and preserve the leading scaling behavior of the integrand (i.e., $g(0) = 0$ and $g'(0) \neq 0$), the generalized integral transforms as

$$\text{gen} \int_0^\infty f(g(u)) g'(u) du - \text{gen} \int_0^\infty f(r) \, dr$$

$$= - f_{-1} \ln g'(0) + \sum_{l \in (\mathbb{N}+1) \cap \Omega} \frac{f_{-l}}{(l-1)(l-1)!} \left. \frac{d^{l-1}}{du^{l-1}} \left( \frac{u}{g(u)} \right)^{l-1} \right|_{u=0}.$$

That is to say, the generalized integral transforms non-trivially under such transformations if there is an $n \in \mathbb{N}$ such that $f_{-n} \neq 0$. For this reason, we call the generalized integral anomalous if $f_{-n} \neq 0$ for some $n \in \mathbb{N}$.

Non-anomalous generalized integrals have much better properties than anomalous ones. They are often easy to compute: one just applies analytic continuation.

A systematic analysis of properties of the generalized integral and a proof of (9) can be found in [7]. To our knowledge, the formula (9) is new.
4. Bilinear generalized integrals of Macdonald and Gegenbauer functions

Let \( a, b > 0 \). For \( \alpha \notin \mathbb{Z} \) generalized integrals of Macdonald functions are analytic continuations of standard integrals:

\[
gen \int_0^\infty K_\alpha(ar)K_\alpha(br)2rdr = \frac{\pi}{\sin(\pi\alpha)} \left( \frac{a}{b} \right)^\alpha - \left( \frac{b}{a} \right)^\alpha,
\]

\[
\gen \int_0^\infty K_\alpha(br)^22rdr = \frac{\pi\alpha}{b^2\sin(\pi\alpha)}.
\]

They have poles at \( \alpha \in \mathbb{Z} \).

For \( \alpha \in \mathbb{Z} \) the generalized integrals are anomalous and more complicated to compute. In particular, they do not coincide with the finite parts of the above expressions:

\[
\gen \int_0^\infty K_\alpha(ar)K_\alpha(br)2rdr = (-1)^\alpha 2^{\frac{\alpha}{2}} \frac{(a^2 - b^2)^{\alpha/2} \ln \left( \frac{a}{b} \right)}{a^2 - b^2} \sum_{k=0}^{[\alpha]-1} \left( \frac{a}{b} \right)^{2k-\alpha+1} \left( \psi(1+k) + \psi(\lvert \alpha - k \rvert) \right);
\]

\[
\gen \int_0^\infty K_\alpha(br)^22rdr = \frac{(-1)^\alpha}{b^2} \left( [\alpha] \ln \left( \frac{b^2}{4} \right) + 1 + 2[\alpha] (1 - \psi(1+|\alpha|)) \right).
\]

Similarly we can compute generalized bilinear integrals of \( S_{\alpha,\lambda}, Z_{\alpha,\lambda} \). All these formulas together with their derivations can be found in [7].

5. Convergence of Gegenbauer functions and their integrals

As \( \beta, \lambda \to \infty \), Gegenbauer functions converge to Macdonald functions in the following sense:

\[
\frac{\pi e^{-\pi \beta} (\sin \theta)^{\alpha + \frac{1}{2}}}{2^\alpha \beta^{\alpha + \frac{1}{2}}} S_{\alpha,\pm \beta}(-\cos \theta) = (\theta \beta)^{-\alpha} K_\alpha(\beta \theta)(1 + O(\beta^{-1})�);
\]

\[
\frac{\sqrt{\pi} \Gamma \left( \frac{1}{2} - \alpha + \lambda \right) (\sinh \theta)^{\alpha + \frac{1}{2}}}{2^{\lambda + \frac{1}{2}} \beta^{\alpha + \frac{1}{2}}} Z_{\alpha,\lambda}(\cosh \theta) = (\lambda \theta)^{-\alpha} K_\alpha(\lambda \theta)(1 + O(\lambda^{-1})�).
\]

The generalized integrals of Gegenbauer functions converge to the corresponding generalized integrals of Macdonald functions:

\[
\frac{\pi^2 e^{-2\pi \beta} \beta^{2\alpha}}{2^{2\alpha}} \gen \int_{-1}^1 S_{\alpha,\beta}(w)^2(1 - w^2)^\alpha d2w = \left( 1 + O\left( \frac{1}{\beta} \right) \right) \gen \int_0^\infty K_\alpha(\beta r)^22rdr;
\]

\[
\frac{\pi \Gamma \left( \frac{1}{2} + \alpha + \lambda \right)^2}{2^{2\lambda + 1} \lambda^{2\alpha}} \gen \int_1^\infty Z_{\alpha,\lambda}(w)^2(w^2 - 1)^\alpha d2w = \left( 1 + O\left( \frac{1}{\lambda} \right) \right) \gen \int_0^\infty K_\alpha(\lambda r)^22rdr.
\]

The convergence of these generalized integrals is straightforward in the non-anomalous case. In the anomalous case one has to choose the variables carefully, which we did:

\[
2rdr = dr^2, \quad 2(\cosh r - 1) \simeq r^2, \quad 2(1 - \cos r) \simeq r^2.
\]

Note that the generalized integral is invariant with respect to the change of variables \( r \to r^2 \), but not with respect to scaling. Proofs of these convergence statements can be found in [7].
6. Laplacian on the Euclidean space, the hyperbolic space and the sphere

In the remaining part of our manuscript we describe an application of generalized integrals to operator theory. These applications will involve point interactions of the Laplacian. We start by recalling some basic information about Green functions of Laplacians. The following formulas are well-known, confer for example [5, 8].

Consider the **Laplacian** $\Delta_d$ on the **Euclidean space** $\mathbb{R}^d$. Let $G_d(z; x, x')$ be the **Euclidean Green function**, that is the integral kernel of the resolvent $(-z - \Delta_d)^{-1}$. For $\Re \beta > 0$, we have

$$G_d(-\beta^2; x, x') = \frac{1}{(2\pi)^{\frac{d}{2}}} \left( \frac{\beta}{|x - x'|} \right)^{-\frac{d}{2}} K_{\frac{d}{2} - 1}(\beta|x - x'|).$$

The **hyperbolic space** is

$$\mathbb{H}^d := \{ x \in \mathbb{R}^{1,d} \mid |x| = 1 \}$$

where $|x| = x^0 y^0 - x^1 y^1 - \cdots - x^d y^d$ is the **Minkowskian pseudoscalar product**. The hyperbolic distance between $x, x' \in \mathbb{H}^d$ is given by $\cosh(r) = |x|x'|$.

Let $\Delta^h_d$ denote the **Laplace-Beltrami operator** on $\mathbb{H}^d$ and let $G^h_d(z; x, x')$ be the **hyperbolic Green function**, that is, the integral kernel of $(-z - \Delta^h_d - (\frac{d-1}{2})^2)^{-1}$. Then

$$G^h_d(-\beta^2; x, x') = \frac{\sqrt{\pi} \Gamma\left(\frac{d-1}{2} + \beta\right)}{\sqrt{2(2\pi)^{\frac{3}{2}}} 2^{\frac{d}{2}}} S_{\frac{d}{2} - 1, \beta}(|x|x'|).$$

The **unit sphere** is

$$S^d := \{ x \in \mathbb{R}^{1,d+1} \mid (x|x) = 1 \},$$

where $(x|y) = x^0 y^0 + x^1 y^1 + \cdots + x^d y^d$ is the **Euclidean scalar product**. The spherical distance between $x, x' \in S^d$ is given by $\cos(r) = (x|x')$. Let $\Delta^s_d$ denote the **Laplace-Beltrami operator** on $S^d$. Let $G^s_d(z; x, x')$ be the **spherical Green function**, that is, the integral kernel of $(-z - \Delta^s_d + (\frac{d-1}{2})^2)^{-1}$. Then

$$G^s_d(-\beta^2; x, x') = \frac{\Gamma\left(\frac{d}{2} - 1 + i\beta\right) \Gamma\left(\frac{d}{2} - 1 - i\beta\right)}{2^{d-1} \pi^{\frac{d}{2}}} S_{\frac{d}{2} - 1, i\beta}(-(x|x')).$$

We remark that the integral kernels of spectral projections of the discussed operators may be expressed explicitly in terms of special functions of the same type as for resolvents, see e.g. [8].

Let us also mention some basic properties of operators. Let $H$ be a self-adjoint operator and let $G(-\rho) := (\rho + H)^{-1}$ be its resolvent. Then $G(-\rho)$ satisfies

$$\begin{align*}
(H + \rho)G(-\rho) &= \mathbb{1}, \\
G(-\rho)^* &= G(-\rho^*), \\
\frac{d}{d\rho}G(-\rho) &= -G(-\rho)^2.
\end{align*}$$

(12) is called the **resolvent formula in the differential form**.

7. Laplacian with point interactions

Let us now try to define the Laplacian with a perturbation localized in a single point. For simplicity, we display details only for the case of the Euclidean space. We look for an operator $-\Delta^\gamma_d$, which is a self-adjoint extension of $-\Delta_d$ restricted to $C_c^\infty(\mathbb{R}^d \setminus \{0\})$. The parameter $\gamma \in \mathbb{R} \cup \{\infty\}$ will parametrize these self-adjoint extensions.
Actually, instead of $-\Delta_d^\gamma$ it is more convenient to look for its resolvent
\[ G_d^\gamma(-\rho) = (-\Delta_d^\gamma + \rho)^{-1}. \]
By the conditions (10), (11) and (12) its integral kernel $G_d^\gamma(-\rho, x, x')$ should satisfy
\[ (-\Delta + \rho)G_d^\gamma(-\rho, x, x') = \delta(x - x'), \quad x \neq 0, \]
\[ G_d^\gamma(-\rho, x, x') = G_d^\gamma(-\rho, x', x), \]
\[ \partial_\rho G_d^\gamma(-\rho, x, x') = -\int G_d^\gamma(-\rho, y)G_d^\gamma(-\rho, y, x')dy. \]
These conditions are solved by a Krein-type resolvent
\[ G_d^\gamma(-\rho, x, x') = G_d(-\rho, x, x') + \frac{G_d(-\rho, x, 0)G_d(-\rho, 0, x')}{\gamma + \Sigma_d(\rho),} \]
where
\[ \partial_\rho \Sigma_d(\rho) = \int_{\mathbb{R}^d} G_d(-\rho, 0, y)^2dy, \]
and $\gamma$ is an arbitrary constant. In dimensions $d = 1, 2, 3$ the above integral is finite and we obtain
\[ \Sigma_d(\beta^2) = \begin{cases} -\frac{1}{2\pi}, & d = 1; \\ \frac{\ln(\beta^2)}{4\pi}, & d = 2; \\ \frac{\beta}{2\pi}, & d = 3. \end{cases} \]
Thus we obtain formulas for the Green functions with a point potential in dimensions $d = 1, 2, 3$ parametrized by a real parameter $\gamma \in \mathbb{R}$:
\[ G_d^\gamma(-\beta^2; x, x') = \begin{cases} e^{-\beta|x-x'|} + \frac{e^{-\beta|x|}e^{-\beta|x'|}}{(2\beta)^2(\gamma - \frac{d}{2})}, & d = 1; \\ \frac{K_0(\beta|x-x'|)}{2\pi} + \frac{K_0(\beta|x|)K_0(\beta|x'|)}{(2\pi)^2(\gamma + \frac{\ln(\beta^2)}{4\pi})}, & d = 2; \\ \frac{e^{-\beta|x-x'|}}{4\pi|x-x'|} + \frac{e^{-\beta|x|}e^{-\beta|x'|}}{(4\pi)^2|x||x'|^{(\gamma + \frac{d}{2})}}, & d = 3. \end{cases} \]
For dimensions $d = 1, 3$ we used the fact that for half-integer parameters the Macdonald function reduces to elementary functions. Of course, the above construction is well known from the literature [1, 2, 3] and often used in the physics literature.

The operators $-\Delta_d^\gamma$, strictly speaking, have no analogs for $d \geq 4$. However, the functions $G_d^\gamma(-\rho; x, x')$ can be generalized to $d \geq 4$ using generalized integrals:
\[ \partial_\rho \Sigma_d(\rho) = \frac{(\beta^2)^{\frac{d}{2}-1}\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma(\frac{d}{2})}\gen \int_0^\infty K_{\frac{d}{2}-1}(\sqrt{\rho}r)^d2rdr. \]
We obtain
\[ \Sigma_d(\beta^2) = \begin{cases} \frac{(-1)^{\frac{d+1}{2}}\beta^{d-2}}{(4\pi)^{\frac{d}{2}}(2\frac{1}{2})!}, & d \text{ odd}; \\ \frac{(-1)^{\frac{d+1}{2}}\beta^{d-2}}{(4\pi)^{\frac{d}{2}}(2\frac{1}{2})!} + \frac{\beta^2}{2\psi\left(\frac{d}{2}\right) + \ln\frac{\pi^2}{4}}, & d \text{ even}. \end{cases} \]
Thus for each dimension \(d\) we obtain a family of Green functions

\[
G^d_\gamma(-\beta^2; x, x') = \frac{1}{(2\pi)^d} \left( \frac{\beta}{|x-x'|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\beta|x-x'|) + \frac{1}{(2\pi)^d} \left( \frac{\beta^2}{|x-x'|} \right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\beta|x'|) K_{\frac{d}{2}-1}(\beta|x|) \frac{\gamma + \Sigma_d(\beta^2)}{\gamma + \Sigma_d(\beta^2)},
\]

(13)
describing point interaction of strength controlled by the parameter \(\gamma\).

A similar analysis can be performed for the hyperbolic and spherical Green functions in all dimensions \([8]\).

One can ask what is the meaning of \(G^d_\gamma(-\beta^2; x, x')\) in dimensions \(d \geq 4\), for which it is not the kernel of a bounded operator (and in particular, not a kernel of a resolvent of a self-adjoint operator). Our expectation is as follows. Suppose that \(V\) is a potential on \(\mathbb{R}^d, \mathbb{H}^d\) or \(\mathbb{S}^d\), possibly strong but with a small support. Consider the Schrödinger operator \(-\Delta_d + V\). Let

\[
G^V(-\beta^2) = (\beta^2 - \Delta_d + V)^{-1}
\]

be its resolvent with the integral kernel \(G^V(-\beta^2; x, x')\). Then far from the support of \(V\) we can approximate \(G^V(-\beta^2; x, x')\) by \(G^d(-\beta^2; x, x')\) as in (13), possibly adding to \(\Sigma_d(\beta^2)\) a polynomial in the energy of degree \(< \frac{d}{2}-1\) if \(d\) is odd and \(< \frac{d}{2} - 2\) if \(d\) is even. Thus possible infrared behaviors of \(G^V(-\beta^2; x, x')\) are controlled by a finite number of parameters (coefficients of the above mentioned polynomials).

The choice given by the zero polynomial that is defined by the above generalized integrals can be viewed as a “standard reference point”. The generalized integral computed in different coordinates, which are related to \(r^2\) by a well-behaved coordinate transformation in the sense of (9), will yield a different choice of polynomial.

The observation that a well-behaved change of variables will, by (9), only change this polynomial is particularly important in the curved cases, where we chose \(2(\cosh r - 1)\), resp. \(2(1 - \cos r)\) as integration variables instead of \(r^2\).

This is analogous to the renormalization of quantum quantum field theory using dimensional regularization, see e.g. [4].

The above situation resembles the idea often expressed in the context of quantum field theory and of the theory of critical phenomena, attributed to Keneth Wilson: for large distances correlation functions have a universal behavior independent of the details of the interaction, described by few parameters.

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