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THE LIE COALGEBRA OF MULTIPLE POLYLOGARITHMS

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ABSTRACT. We use Goncharov’s coproduct of multiple polylogarithms to define a Lie coalgebra over an arbitrary field. It is generated by symbols subject to inductively defined relations, which we think of as functional relations for multiple polylogarithms. In particular, we have inversion relations and shuffle relations. We relate our definition to Goncharov’s Bloch groups, and to the concrete model for $\mathcal{L}(F)_{\leq 4}$ by Goncharov and Rudenko.

1. INTRODUCTION

1.1. The motivic Lie coalgebra. For a field $F$, one expects the existence of a graded Lie coalgebra $\mathcal{L}(F)$ such that the weight $n$ part of its Chevalley-Eilenberg complex $\wedge^n(\mathcal{L}(F))$ computes the motivic cohomology groups $H^i_M(F, \mathbb{Z}(n))$ (see e.g. Gon94). The existence of this so-called motivic Lie coalgebra is known for number fields Gon05. It is desirable to have a concrete description. Goncharov Gon94 conjectures that $\wedge^n(\mathcal{L}(F))$ is rationally quasi-isomorphic to the Bloch complex $\Gamma(F, n)$ defined in Gon95. This complex has the form

\begin{equation}
B_n(F) \xrightarrow{\delta} \cdots \xrightarrow{\delta} B_{n-k}(F) \otimes \wedge^k(F^*) \xrightarrow{\delta} \cdots \xrightarrow{\delta} B_2(F) \otimes \wedge^{n-2}(F^*) \xrightarrow{\delta} \wedge^n(F^*),
\end{equation}

with each group $B_k(F)$ being generated by symbols $\{x\}_k$ with $x \in F \cup \{\infty\}$ subject to relations that may be thought of as polylogarithm relations. The rightmost $\delta$ takes $\{x\}_2 \otimes a$ to $x \wedge (1-x) \wedge a$, and the others take $\{x\}_k \otimes a$ to $\{x\}_{k-1} \otimes x \wedge a$. Goncharov conjectures that $\mathcal{L}(F)_n = B_n(F)$ for $n \leq 3$, but for $n = 4$, $\mathcal{L}(F)_4$ is larger. An explicit model for $\wedge^n(\mathcal{L}(F))_{n \leq 4}$ is given in Goncharov-Rudenko GR18 (see Section 6 for a brief summary).

1.2. Multiple polylogarithms and Goncharov’s coproduct. The classical polylogarithms $\text{Li}_n(x)$ have multivariable generalizations $\text{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$ called multiple polylogarithms. Their properties are extensively studied by Goncharov Gon01 Gon05. He shows that one may view $\text{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$ and $\log(x)$ as elements $\text{Li}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)$ and $\log^\mathcal{E}(x)$ of a certain Hopf algebra (the Hopf algebra of framed rational Hodge structures Gon05 appendix). In fact, they generate a Hopf subalgebra of this Hopf algebra, and Goncharov gives an explicit formula for the coproduct. It is best described using the generating series

\begin{equation}
\text{Li}^\mathcal{E}(x_1,\ldots,x_d|t_1,\ldots,t_d) = \sum_{n_i \geq 1} \text{Li}^\mathcal{E}_{n_1,\ldots,n_d}(x_1,\ldots,x_d)t_1^{n_1-1}\cdots t_d^{n_d-1},
\end{equation}

\begin{equation}
x^t = \sum_{n \geq 0} \frac{1}{n!} \log^\mathcal{E}(x)^n t^n.
\end{equation}

The coproduct of a generating series is defined termwise (fixing the $t_i$).

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**Theorem 1.1 ([Gon05 Prop. 6.1]).** The coproduct $\Delta$ of $\text{Li}^G(x_1, \ldots, x_d|t_1, \ldots, t_d)$ is given by

$$
\sum_{i_1 \to i_2, \ldots, i_k \to i_{k+1}} \prod_{a=0}^{k} (-1)^{j_a-i_a} x_{i_a}^{j_a} \left( \text{Li}^G(x_{j_a-1}, x_{j_a-2}, \ldots, x_{i_a-1}|t_{j_a} - t_{j_a-1}, \ldots, t_{j_a} - t_{i_a}) \right) 
$$

The sum is over all $k = 0, \ldots, d$, and all sequences $\{i_a\}_{a=0}^{k+1}$ and $\{j_a\}_{a=0}^{k}$ with

$$i_a \leq j_a < i_{a+1}, \quad j_0 = i_0 = 0, \quad i_{k+1} = d + 1,$$

and by definition we have $x_{i \to j} = \prod_{r=1}^{j-i} x_r$ (note that this convention differs from Goncharov’s, but works better in this paper) and $\text{Li}^G(\emptyset|\emptyset) = 1$.

**Example 1.2.** When $d = 2$, the $\{i_a\}$ and $\{j_a\}$ satisfying (4) are:

$$
i_0, j_0 | 1, 2 | 3, \quad i_0, j_0 | 1, 1 | 3, \quad i_0, j_0 | 1, 2 | 3, \quad i_0, j_0 | 0, 2, 2 | 3, \quad i_0, j_0 | 0, 0, 3 | 3,
$$

and it follows that

$$
\Delta \text{Li}^G(x_1, x_2|t_1, t_2) = \text{Li}^G(x_1, x_2|t_1, t_2) \otimes x_1^{t_1} x_2^{t_2} + \text{Li}^G(x_1 x_2|t_1) \otimes (x_1 x_2)^{t_1} \text{Li}^G(x_2|t_2 - t_1) - \text{Li}^G(x_1 x_2|t_2) \otimes (x_1 x_2)^{t_2} \text{Li}^G(x_1^{-1}|t_2 - t_1) + \text{Li}^G(x_2|t_2) \otimes \text{Li}^G(x_1|t_1)x_2^{t_2} + 1 \otimes \text{Li}^G(x_1, x_2|t_1, t_2).
$$

One can then compute $\Delta \text{Li}^G_{n_1, n_2}(x_1, x_2)$ as the coefficient of $t_1^{n_1-1} t_2^{n_2-1}$ of the right-hand side of (6).

Given a graded Hopf algebra $H$, we have a Lie coalgebra $L = \frac{H^*}{[H^*, H^*]}$ with cobracket induced by the coproduct. We are thus interested in the quotient by products. The following elementary corollary of Theorem 1.1 is our main motivation.

**Corollary 1.3.** Modulo products and constants, $\Delta \text{Li}^G(x_1, \ldots, x_d|t_1, \ldots, t_d)$ can be written as

$$
\text{Li}^G(x_1, \ldots, d|t_1, \ldots, d) \otimes \sum_{p=1}^{d} t_p \log^G(x_p) +
$$

$$
\sum_{p=2}^{d} \text{Li}^G(x_p, \ldots, d|t_p, \ldots, d) \otimes \text{Li}^G(x_1, \ldots, p-1|t_1, \ldots, p-1) +
$$

$$
\sum_{1 \leq p < q \leq d} \text{Li}^G(x_1, \ldots, p-q, \ldots, d|t_1, \ldots, p, q+1, \ldots, d) \otimes \text{Li}^G(x_{p+1}, \ldots, q|t_{p+1}, \ldots, q - t_p) +
$$

$$
\sum_{1 \leq p < q \leq d} (-1)^{q-p} \text{Li}^G(x_1, \ldots, p-q, \ldots, d|t_1, \ldots, p-1, q, \ldots, d) \otimes \text{Li}^G(x_{q-1}, x_{q-2}, \ldots, x_{p-1}|t_q - t_{q-1}, \ldots, p)
$$

Where $x_1, \ldots, p-q, \ldots, d$ is shorthand for $(x_1, \ldots, x_{p-1}, \prod_{r=p}^{q} x_r, x_{q+1}, \ldots, x_d)$ and dots indicate that indices increase (or decrease) by 1. We stress that the product is from $p$ to $q$, not to $q - 1$. 

Proof. One easily checks that the only sequences \( \{i_\alpha\} \) and \( \{j_\alpha\} \) for which the corresponding term in (3) does not involve products are
\[
\begin{align*}
(0, 0, 0, 0, 1, 1, 2, 2, \ldots, d, d, d + 1) \\
0, 0, 1, 1, \ldots, p, p, p + 1, p + 1, 1, 2, 2, \ldots, d, d, d + 1) \\
i_0 j_0 i_1 j_1 i_2 j_2 i_3 j_3 \ldots i_k j_k i_{k+1} (0, 0, 0, 1, 1, \ldots, 1, 1, \ldots, d, d, d + 1)
\end{align*}
\]
and that the corresponding terms are exactly those given in (7). \( \Box \)

1.3. Real single valued polylogarithms. One has single valued variants \( L_n(x) \) of the classical polylogarithms (see e.g. [DB12, Zag91])
\[
L_n(z) = \begin{cases} 
\text{Re} & \text{if } n \text{ is odd} \\
\text{Im} & \text{if } n \text{ is even}
\end{cases} \bigg( \sum_{r=0}^{n-1} \frac{2^r B_r}{r!} \ln^{n-r}(z)(\log |z|)^r \bigg).
\]
The following result justifies thinking about the relations in \( B_n(F) \) as polylogarithm relations:

**Theorem 1.4** ([Zag91, Prop. 3]). For any element \( \alpha = \sum n_i f_i(t) \in B_n(\mathbb{C}(t)) \) with \( \delta(\alpha) = 0 \) in \( B_{n-1}(\mathbb{C}(t)) \otimes \mathbb{C}(t)^2 \) we have
\[
\sum n_i L_n(f_i(t)) = \text{constant}.
\]
There are also single valued analogues \( L_{n_1,\ldots,n_d} \) of the multiple polylogarithms \( \text{Li}_{n_1,\ldots,n_d} \). There is no closed formula, but they can be computed from a variation matrix [Zha02]. We shall here only need that \( L_{n_1,\ldots,n_d} \) is defined on
\[
S_d(\mathbb{C}) = \left\{ (x_1, \ldots, x_d) \in (\mathbb{C}^*)^d \mid \prod_{r=i}^j x_r \neq 1 \text{ for all } i \leq j \in \{1, \ldots, d\} \right\}
\]
and that
\[
\lim_{x_i \to 0} L_{n_1,\ldots,n_d}(x_1, \ldots, x_d) = 0 \quad \text{for all } i.
\]
The functions \( L_n(x) \) also satisfy that \( \lim_{x \to \infty} L_n(x) = 0 \) for \( n > 1 \), but in higher depth the limit as \( x \) tends to \( \infty \) is no longer 0.

1.4. Structure of the paper. In Section 2 we define a purely symbolic coalgebra \( \mathcal{L}^\text{symb}(F) \) with no relations. We have \( \mathcal{L}^\text{symb}_{n_1,\ldots,n_d}(F) = F^* \), and \( \mathcal{L}^\text{symb}_{n_1,\ldots,n_d} \) is generated by symbols \( [x_1, \ldots, x_d]_{n_1,\ldots,n_d} \) with \( (x_1, \ldots, x_d) \in S_d(F) \), where \( d \) and \( n_1, \ldots, n_d \) are positive integers. Thinking of a symbol as a polylogarithm we define the cobracket as in Corollary 1.3 and show directly that \( \delta^2 = 0 \). In Section 3 we inductively define a group of relations \( R_n(F) \) and define
\[
\mathcal{L}_n(F) = \mathcal{L}^\text{symb}_{n_1,\ldots,n_d}(F)/R_n(F).
\]
The definition mimics Goncharov’s definition of relations in \( B_n(F) \), and the proof that the cobracket takes relations to 0 in \( \wedge^2(\mathcal{L}(F)) \) follows Goncharov as well. Section 3.1 gives some basic examples of relations, and Section 3.2 discusses the problem of defining symbols when \( (x_1, \ldots, x_d) \notin S_d(F) \). For example, \( \{1, 1\}_{1,1} \) is not well defined. In Section 3.3 we speculate that \( \mathcal{L}(F) \) is the motivic Lie coalgebra \( \mathcal{L}(F) \) and conjecture a generalization of Theorem 1.4 which justifies thinking of \( R_n(F) \) as polylogarithm relations. Section 4 discusses the inversion relations, which are inspired by Goncharov’s inversion relations for multiple polylogarithms.
particular, we show that in \( L(F) \) one can express each symbol \([x_d^{-1}, \ldots, x_1^{-1}]_{n_d, \ldots, n_1}\) in terms of symbols involving only non-inverted \( x_i \). Section 4.1 shows that one can use this to define an alternative cobracket on \( L_{\text{symb}}(F) \) without inverted \( x_i \). The relations are the same. The alternative one may in fact be more natural (see e.g. Remark 5.3). Section 5 briefly discusses the shuffle product relations showing that at least in low depth a shuffle product is 0 in \( L(F) \). Finally, Section 6 relates our work to that of Goncharov and Rudenko, who gave a concrete model for \( L_{n \leq 4}(F) \).

**Remark 1.5.** Although our work is heavily inspired by the work of Goncharov, it does not require any of Goncharov’s results, except for motivation.

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## 2. A purely symbolic Lie coalgebra

For a field \( F \), and a positive integer \( d \), let

\[
S_d(F) = \left\{ (x_1, \ldots, x_d) \in (F^*)^d \mid \prod_{r=i}^{j} x_r \neq 1 \text{ for all } i \leq j \in \{1, \ldots, d\} \right\}.
\]

We wish to define a graded Lie coalgebra

\[
L_{\text{symb}}(F) = \bigoplus_{n=1}^{\infty} L_{n_{\text{symb}}}(F).
\]

We first define \( L_{1_{\text{symb}}}(F) = F^* \), which we shall identify with the abelian group generated by symbols \([x]_1\) with \( x \in F \setminus \{0, 1\} \) and \([x]_0\) for \( x \in F \setminus \{0, 1\} \) subject to the relations

\[
[x]_1 = -[1-x]_0, \quad [xy]_0 = [x]_0 + [y]_0.
\]

For \( n > 1 \), define \( L_{n_{\text{symb}}}(F) \) to be generated by symbols \([x_1, \ldots, x_d]_{n_1, \ldots, n_d}\) with

\[
(x_1, \ldots, x_d) \in S_d(F), \quad d, n_1, \ldots, n_d \in \mathbb{Z}_+, \quad n_1 + \cdots + n_d = n.
\]

We refer to \( n_1 + \cdots + n_d \) as the *weight* and \( d \) as the *depth* of a symbol, but stress that \([x]_0\) is in weight 1, not 0. We think of \([x_1, \ldots, x_d]_{n_1, \ldots, n_d}\) as representing \( L_{n_1, \ldots, n_d}(x_1, \ldots, x_d) \) and \([x]_0\) as representing \( \log(x) \).

### 2.1. The cobracket on \( L_{\text{symb}}(F) \)

As in [2] we define

\[
[x_1, \ldots, x_d|t_1, \ldots, t_d] = \sum_{n_i \geq 1} [x_1, \ldots, x_d]_{n_1, \ldots, n_d} t_1^{n_1-1} \cdots t_d^{n_d-1},
\]
and define $\delta : \mathbb{L}^{\text{symb}}(F) \to \wedge^2(\mathbb{L}^{\text{symb}}(F))$ to be zero in weight 1, and otherwise given by \cite{7}. It is convenient to write $\delta = \delta_1 + \delta_2 + \delta_3 + \delta_4$, where for $X = [x_1, \ldots, d][t_1, \ldots, d]$

$$\delta_1(X) = X \wedge \left( \sum_{p=1}^{d} t_p[x_p]_0 \right)$$

(19)

$$\delta_2(X) = \sum_{p=2}^{d} [x_p, \ldots, d][t_p, \ldots, d] \wedge [x_1, \ldots, p-1][t_1, \ldots, p-1]$$

$$\delta_3(X) = \sum_{1 \leq p < q \leq d} [x_1, \ldots, p-q, \ldots, d][t_1, \ldots, p, q+1, \ldots, d] \wedge [x_{p+1}, \ldots, q][t_{p+1}, \ldots, q - t_p]$$

$$\delta_4(X) = \sum_{1 \leq p < q \leq d} (-1)^{q-p}[x_1, \ldots, p-q, \ldots, d][t_1, \ldots, p-1, q, \ldots, d] \wedge [x_{q-1}^{-1}, \ldots, x_p^{-1}, t_q - t_{q-1}, \ldots, p].$$

**Example 2.1.** In depth 1, we have $\delta[x|t] = [x|t] \wedge [x|0]$, from which it follows that $\delta[x]_n = [x]_{n-1} \wedge [x]_0$ for $n > 1$. Note the case $\delta[x]_2 = [x]_1 \wedge [x]_0 = x \wedge (1-x) \in \wedge^2(F^*)$. We thus recover the boundary map in the Bloch complex.

**Example 2.2.** In depth 2, $\delta[x_1, x_2|t_1, t_2]$ equals (compare with \cite{8})

$$\delta[x_1, x_2|t_1, t_2] = [x_1, x_2|t_1, t_2] + [t_1[x_1]_0 + t_2[x_2]_0] + [x_2[t_2]_1 + [x_1|t_1] + [x_1x_2|t_1] \wedge [x_2|t_2 - t_1] - [x_1x_2|t_2] \wedge [x_1^{-1}|t_2 - t_1].$$

In particular, $\delta[x_1, x_2]_{r,s}$ is given by

$$[x_1, x_2]_{r-1,s} \wedge [x_1]_0 + [x_1, x_2]_{r,s-1} \wedge [x_2]_0 + [x_2]_s \wedge [x_1]_r$$

$$+ \sum_{i=1}^{r} (-1)^{r-i} \left( \frac{r + s - 1 - i}{s - 1} \right) [x_1 x_2]_i \wedge [x_2]_{r+s-i}$$

$$+ \sum_{i=1}^{s} (-1)^{s-i} \left( \frac{r + s - 1 - i}{r - 1} \right) [x_1 x_2]_i \wedge [x_1^{-1}]_{r+s-i}.$$

In the case where $r$ or $s$ is 1, the symbols $[x_1, x_2]_{0,s}$ and $[x_1, x_2]_{r,0}$ are interpreted as 0.

**Remark 2.3.** In our definition of $\wedge^2(\mathbb{L}^{\text{symb}}(F))$, $x \wedge x$ is identically 0, not 2-torsion.

**Theorem 2.4.** The cobracket above makes $\mathbb{L}^{\text{symb}}(F)$ into a Lie coalgebra.

**Proof.** We must show that $\delta^2 = 0$. To do this, it is enough to show that

$$\delta_1^2 = 0, \quad \delta_2^2 = 0, \quad \delta_1 \delta_3 + \delta_3 \delta_1 = 0,$$

$$\delta_3^2 + \delta_2 \delta_3 + \delta_3 \delta_2 = 0, \quad \delta_2 \delta_4 + \delta_4 \delta_2 + \delta_4^2 + \delta_3 \delta_4 + \delta_4 \delta_3 = 0.$$

The proof that $\delta_1^2 = 0$ is elementary. The remaining equalities are all straightforward, so for brevity, we prove only that $\delta_2^2 = 0$, and that $\delta_1 \delta_4 + \delta_4 \delta_1 = 0$. Firstly, $\delta_2^2([x_1, \ldots, d][t_1, \ldots, d])$ is given by

$$\sum_{2 \leq p \leq d} \delta_2[x_p, \ldots, d][t_p, \ldots, d] \wedge [x_1, \ldots, p-1][t_1, \ldots, p-1] - \sum_{2 \leq p \leq d} [x_p, \ldots, d][t_p, \ldots, d] \wedge \delta_2[x_1, \ldots, p-1][t_1, \ldots, p-1]$$

$$- \sum_{2 \leq p \leq d} \sum_{p+1 \leq r \leq d} [x_r, \ldots, d][t_r, \ldots, d] \wedge [x_1, \ldots, r-1][t_1, \ldots, r-1] \wedge [x_1, \ldots, p-1][t_1, \ldots, p-1]$$

$$- \sum_{2 \leq p \leq d} \sum_{2 \leq r \leq p-1} [x_p, \ldots, d][t_p, \ldots, d] \wedge [x_r, \ldots, p-1][t_r, \ldots, p-1] \wedge [x_1, \ldots, r-1][t_1, \ldots, r-1] = 0.$$
Similarly, \( \delta_1 \delta_4([x_1, \ldots, d|t_1, \ldots, d]) \) equals
\[
\sum_{1 \leq p < q \leq d} (-1)^{q-p} \delta_1[x_{q-p-1, \ldots, q}|t_q - t_{q-1, \ldots, p}] \\
- \delta_1[x_{q-p-1, \ldots, q}|t_q - t_{q-1, \ldots, p}]
\]
which equals
\[
\sum_{1 \leq p < q \leq d} (-1)^{q-p} [x_{1, \ldots, p-1, q, \ldots, d}|t_{1, \ldots, p-1, q}|t_q - t_{q-1, \ldots, p}] \\
\left( \sum_{1 \leq r \leq p-1} t_r[x_r]_{0} + t_q[x_p \cdots x_q]_{0} + \sum_{q+1 \leq r \leq d} t_r[x_r]_{0} \right) \land \left( \sum_{p \leq r \leq q-1} (t_q - t_r)[x_r]_{1} \right)
\]
\[
- \sum_{1 \leq p < q \leq d} (-1)^{q-p} [x_{1, \ldots, p-1, q, \ldots, d}|t_{1, \ldots, p-1, q}|t_q - t_{q-1, \ldots, p}] \\
\left( \sum_{1 \leq r \leq d} t_r[x_r]_{0} \right).
\]
Finally,
\[
\delta_4 \delta_1([x_1, \ldots, d|t_1, \ldots, d]) = \delta_4[x_{1, \ldots, d}|t_1, \ldots, d] \land \left( \sum_{p=1}^{d} t_p[x_p]_{0} \right)
\]
\[
= \sum_{1 \leq p < q \leq d} (-1)^{q-p} [x_{1, \ldots, p-1, q, \ldots, d}|t_{1, \ldots, p-1, q}|t_q - t_{q-1, \ldots, p}] \\
\left( \sum_{p=1}^{d} t_p[x_p]_{0} \right),
\]
and it follows that \( \delta_1 \delta_4 + \delta_4 \delta_1 = 0. \)

**Remark 2.5.** Theorem 2.4 also holds without requiring that the tuples are in \( S_d(F) \) (same proof). The problem with arbitrary tuples arises when defining the relations; see Section 3.2.

### 2.2. Terms involving zero or infinity

We shall also define elements \([x_1, \ldots, d]_{n_1, \ldots, n_d}\) when some of the \( x_i \) are 0 or \( \infty \), but we still require that all consecutive products \( \prod_{r=1}^{n} x_r \) are well defined and not 1 (\( \infty x = \infty \neq 1; 0 \infty \) is undefined). When some \( x_i \) are 0, \([x_1, \ldots, d]_{n_1, \ldots, n_d}\) is defined to be zero (this is motivated by \( \| \)). When some \( x_i \) are \( \infty \), the definition is more subtle and we refer to Section 4. All we need for now is that \( [\infty]_n = 0 \) for \( n > 1 \) (and undefined for \( n=1 \)).

### 3. The relations

We now define groups \( R_n(F) \) of relations in \( L_n^{\text{symb}}(F) \). We can then define
\[
\mathbb{L}_n(F) = \mathbb{L}_n^{\text{symb}}(F)/R_n(F).
\]
The definition is inductive starting with the definition of \( R_1(F) \) to be the trivial group, so that \( \mathbb{L}_1(F) = F^* \). Suppose \( n > 1 \) and that \( \mathbb{L}_k(K) \) has been defined for all fields \( K \) and all \( k < n \). Then
\[
\wedge^2 ([L(K)])_n = \bigoplus_{k+l=n} \mathbb{L}_k(K) \land \mathbb{L}_l(K)
\]
is also well defined for all \( K \). Let
\[
A_n(K) = \text{Ker} \left( \delta : \mathbb{L}^{\text{symb}}(K) \rightarrow \wedge^2(\mathbb{L}(K))_n \right)
\]

Mimicking Goncharov’s definition of relations in the Bloch complexes [Gon95, p. 221] we wish to define \( R_n(F) \subset \mathbb{L}^{\text{symb}}(F) \) to be generated by elements \( \alpha(p) - \alpha(q) \), where \( p \) and \( q \) are points on a connected (geometrically irreducible) smooth curve \( X \) over \( F \) with function field \( F(X) \) and \( \alpha \) is an element in \( A_n(F(X)) \). The only problem with this is our requirement that all tuples be well defined (allowing for 0 and \( \infty \); see Section 2.2).

Each element \( \alpha \in \mathbb{L}^{\text{symb}}(F(X)) \) is a linear combination of terms \( [f_1, \ldots, f_d]_{n_1, \ldots, n_d} \), which we refer to as terms of \( \alpha \).

**Definition 3.1.** We say that \( \alpha \in \mathbb{L}^{\text{symb}}(F(X)) \) is well defined at \( x \in X \) if all products \( \prod_{r=1}^j f_r(x) \) are defined and distinct from 1, for each term \( [f_1, \ldots, f_d]_{n_1, \ldots, n_d} \) of \( \alpha \).

By induction, we see that if \( \alpha \in A_n(F(X)) \), we can rearrange the terms \( \delta(\alpha) \) as a linear combination of terms \( (\delta(q) - \delta(q')) \otimes y, \) where \( Y \) is a smooth connected curve over \( F(X) \), \( q \) and \( q' \) are points in \( Y \), \( \beta \in A_k(F(X)(Y)) \) and \( y \in \mathbb{L}^{\text{symb}}(F(X)) \).

**Definition 3.2.** All \( \alpha \in A_2 \) are proper. For \( n > 2 \), \( \alpha \in A_n(F(X)) \) is proper if whenever \( \alpha \) is well defined at \( x \in X \), there is an arrangement of \( \delta(\alpha) \) as above with \( \delta(q), \delta(q') \), and \( y \) all well defined at \( x \) (no cancelation of undefined terms) with \( \beta \) proper.

**Definition 3.3.** The group \( R_n(F) \) is generated by elements of the form \( \alpha(p) - \alpha(q) \), where \( \alpha \in A_n(F(X)) \) is proper and well defined at \( p, q \in X \).

We now prove that \( \mathbb{L}(F) \) is also a Lie coalgebra. To do this we must prove that the cobracket respects the relations. The proof is similar to [Gon95, Lemma 1.16].

**Theorem 3.4.** The map \( \delta : \mathbb{L}^{\text{symb}}(F) \rightarrow \wedge^2(\mathbb{L}(F))_n \) takes \( R_n(F) \) to 0.

**Proof.** It is enough to show that for any proper \( \alpha \in A_n(F(X)) \), the element \( \delta(\alpha(p)) \) is zero in \( \wedge^2(\mathbb{L}(F))_n \) for all \( p \in X \) where \( \alpha(p) \) is defined. Fix such \( X, \alpha, \) and \( p \) and write \( \delta(\alpha) \) in \( \wedge^2(\mathbb{L}^{\text{symb}}(F(X))) \) as a linear combination of elements \( (\beta(q) - \beta(q')) \wedge y \) as above. It follows that \( \delta(\alpha(p)) \) is a linear combination of elements of the form \( (\beta(q)(p) - \beta(q')(p)) \otimes y(p), \) and the result follows by showing that \( \beta(q)(p) - \beta(q')(p) \) is in \( R_k(F) \). Let \( \beta_p \) be the element in \( \mathbb{L}^{\text{symb}}(F(Y(p))) \) obtained from \( \beta \) by restriction to the fiber \( Y(p) \) over \( p \). Since \( \beta \) is in \( A_k(F(X)(Y(p))) \), an induction argument shows that \( \beta_p \) is in \( A_k(F(Y(p))) \) and is proper. Note that \( q \) and \( q' \) can be regarded as maps \( X \rightarrow Y, \) so restriction to \( p \in X \) determines points \( q_p \) and \( q'_p \) in \( Y(p) \). We then have
\[
\beta(q)(p) - \beta(q')(p) = \beta_p(q_p) - \beta_p(q'_p) \in R_k(F).
\]

This concludes the proof. \( \Box \)

3.1. **Basic examples of relations.** In the following examples we repeatedly use the fact that for \( \alpha \in \mathbb{L}^{\text{symb}}(F(x)) \), \( \delta(\alpha) = 0 \in \wedge^2(\mathbb{L}(F(x))) \) implies that \( \alpha(x) \) is constant in \( \mathbb{L}(F) \).

**Example 3.5.** Let \( \alpha_n = 2([x]_n + (-1)^n[x^{-1}]_n) \in \mathbb{L}^{\text{symb}}(F) \). By induction, \( \alpha_n \in A_n(F(x)) \) when \( n \geq 2 \), so \( \alpha_n(x) \) is constant in \( \mathbb{L}_n(F) \). Hence, \( \alpha_n(x) = \alpha_n(0) = 2[0]_n + 2(-1)^n[\infty]_n = 0 \). Note, however, that \( \alpha_1(x) = -2[x]_0 \).

**Example 3.6.** One easily shows that \( \alpha = [x, y]_{1,1} + [x]_2 - \frac{[x(y)]}{1-xy} \) is in \( A_2(F(x, y)) \). Considering the specialization at \( (x, y) = (0, y) \) it follows that \( [x, y]_{1,1} + [x]_2 - \frac{[x(y)]}{1-xy} = 0 \in \mathbb{L}_2(F) \) whenever all terms are defined. This means that every element in \( \mathbb{L}_2(F) \) can be expressed using only terms in depth 1.
Example 3.7 (Five term relation). Similarly, we obtain that

\[(31) \quad [x]_2 + [y]_2 - [xy]_2 - \frac{y(1-x)}{1-xy} [x]_2 - \frac{x(1-y)}{1-xy} [y]_2 = 0 \in \mathbb{L}_2(F).\]

Proposition 3.8. Modulo 2-torsion every element in \(\mathbb{L}_3(F)\) can be written in terms of depth 1 symbols.

Proof. A simple computation shows that the elements

\[(32) \quad [x, y]_{2,1} + [x]_3 + \frac{1 - y}{1 - xy} [y]_3 + [xy]_3 + \frac{-xy}{1-xy}[1 - y]_3 - \frac{x(1-y)}{1-xy} [y]_3 \]

and

\[(33) \quad [x, y, z]_{1,1,1} - \frac{y}{1 - y} [x]_3 + \frac{1 - x}{1 - xz} [y]_3 + \frac{xy(1-z)}{1 - xz} [z]_3 - [1 - x]_3 + \frac{-y(1-x)}{1 - y} [x]_3 + \frac{-y(1-z)}{1 - y} [y]_3 - \frac{x(1-y)(1-z)}{(1-xyz)(1-y)} [z]_3\]

have vanishing cobracket modulo 2-torsion. Specializing at \((0, y)\) and \((x, 0, z)\), respectively, proves the result for \([x, y]_{2,1}\) and \([x, y, z]_{1,1,1}\). Finally, \([x, y]_{1,2} + [y, x]_{2,1} + [xy]_3\) also has vanishing cobracket, concluding the proof. \(\square\)

Remark 3.9. With more work one can show that (up to torsion) every element in \(\mathbb{L}_4(F)\) can be expressed in terms of depth 1 symbols. By Example 3.6 we should have for any \(a\)

\[(34) \quad \frac{x-a}{x-a} [x]_1 + [x]_2 - [a]_2 = 0.\]

Setting \(x = 1\), we would thus get that \([1, 1]_{1,1} + [1]_2 = [a]_2\) for any \(a\), which would imply that \(\mathbb{L}_2(F) = 0\).

Remark 3.10. We believe that the reason for this is that \(L_{1,1}(x, y)\) does not have a limit as \(x\) and \(y\) tend to 1. One has similar issues with symbols such as \([x_1, 0, \infty]_{1,1,1}\).

Remark 3.11. One can give meaning to some additional symbols, but these should always be expressible in terms of symbols in \(S_d(F)\). For example, one may define \([x, x^{-1}]_{1,1} = -[x]_2\). We shall not pursue this here.

3.2. The issue of arbitrary symbols. The following argument shows that not all symbols can be meaningfully defined. By Example 3.6 we should have for any \(a\)

\[(35) \quad \frac{x-a}{x-a} [x]_1 + [x]_2 - [a]_2 = 0.\]

Setting \(x = 1\), we would thus get that \([1, 1]_{1,1} + [1]_2 = [a]_2\) for any \(a\), which would imply that \(\mathbb{L}_2(F) = 0\).

Remark 3.10. We believe that the reason for this is that \(L_{1,1}(x, y)\) does not have a limit as \(x\) and \(y\) tend to 1. One has similar issues with symbols such as \([x_1, 0, \infty]_{1,1,1}\).

Remark 3.11. One can give meaning to some additional symbols, but these should always be expressible in terms of symbols in \(S_d(F)\). For example, one may define \([x, x^{-1}]_{1,1} = -[x]_2\). We shall not pursue this here.

3.3. Conjectures and speculation. Conjecture 3.12 below is a natural generalization of Theorem 1.4. Consider the map

\[(36) \quad r: \mathbb{L}^{\text{symb}}(\mathbb{C}) \to \mathbb{R}, \quad (x_1, \ldots, x_n)_{n_1, \ldots, n_d} \to \mathcal{L}_{n_1, \ldots, n_d}(x_1, \ldots, x_d), \quad [x]_0 \mapsto \log(|x|).\]

Conjecture 3.12. If \(\alpha \in \mathbb{L}(\mathbb{C}(t))\) is such that \(\delta(\alpha) = 0\), then \(r(\alpha(t)) = 0\) is constant in \(t\).

Remark 3.13. We have verified Conjecture 3.12 numerically for many examples in weight 4 and lower.
We also speculate more boldly that $\mathbb{L}(F)$ is rationally isomorphic to the motivic Lie coalgebra, so that the weight $n$ part of the complex

\begin{equation}
\mathbb{L}(F) \xrightarrow{\delta} \wedge^2(\mathbb{L}(F)) \xrightarrow{\delta \text{id} - \text{id} \wedge \delta} \wedge^3(\mathbb{L}(F)) \xrightarrow{\delta} \ldots
\end{equation}

should compute motivic cohomology, rationally. For a finite field $F_q$ with $q$ elements the only motivic cohomology groups are

\begin{equation}
H^1_{\mathcal{M}}(F_q, \mathbb{Z}(n)) = K_{2n-1}(F_q) = \mathbb{Z}/(q^n - 1)\mathbb{Z}.
\end{equation}

Assuming that $R_2(F_q)$ is generated by five term relations and that $R_3(F_q)$ is generated by the 31 term relations in [Zic19], one can compute $H^1(\wedge^*(\mathbb{L}(F_q)))$ for $n = 2$ and 3. When $q > 3$, experimental evidence suggests that \textit{integrally}, one has

\begin{equation}
|H^1(\wedge^*(\mathbb{L}(F_q)))|_{n=2,3} = \frac{|H^1_{\mathcal{M}}(F_q, \mathbb{Z}(n))|}{|F_q|}
\end{equation}

where $=_{2,3}$ means equality up to small powers of 2 and 3. The equality appears to hold on the nose when $n = 2$.

4. Inversion Relations

In the following we ignore 2-torsion. It is well known (see e.g. [Lew91]) that the classical polylogarithms satisfy the inversion relation

\begin{equation}
\text{Li}_n(x) + (-1)^n \text{Li}_n(x^{-1}) = -\frac{(2\pi i)^n}{n!} B_n \left( \frac{\log x}{2\pi i} \right),
\end{equation}

where $B_n(x) = \sum_{k=0}^n \binom{n}{k} B_{n-k} x^k$ are the Bernoulli polynomials. Thinking of a symbol $[x]_n$ as $\text{Li}_n(x)$ modulo products and powers of $\pi i$, we would thus expect $[x]_n + (-1)^n [x^{-1}]_n$ to be zero in $\mathbb{L}(F)$. This was proved in Example 3.5.

Goncharov [Gon01, Sec. 2.6] extended the classical inversion formula to multiple polylogarithms. For example, one has

\begin{equation}
\begin{aligned}
\text{Li}_{n_1,n_2}(x_1, x_2) + \ldots + (-1)^{n_1+n_2} \text{Li}_{n_2,n_1}(x_2^{-1}, x_1^{-1}) + \ldots + (-1)^{n_1} \text{Li}_{n_1}(x_1^{-1}) \text{Li}_{n_2}(x_2) \\
+ \sum_{p+q=n_1} \frac{(2\pi i)^p}{p!} (-1)^q \binom{q+n_2-1}{n_2-1} B_p \left( \frac{\log(x_1 x_2)}{2\pi i} \right) \text{Li}_{q+n_2}(x_2) \\
+ \sum_{p+q=n_2} \frac{(2\pi i)^p}{p!} (-1)^{n_2} \binom{n_1+q-1}{n_1-1} B_p \left( \frac{\log(x_1 x_2)}{2\pi i} \right) \text{Li}_{n_1+q}(x_1^{-1}) = 0,
\end{aligned}
\end{equation}

which suggests that we should have

\begin{equation}
\begin{aligned}
[x_1, x_2]_{n_1,n_2} + \ldots + (-1)^{n_1+n_2} [x_2^{-1}, x_1^{-1}]_{n_2,n_1} \\
+ (-1)^{n_1} \binom{n_1+n_2-1}{n_2-1} [x_2]_{n_1+n_2} + \ldots + (-1)^{n_1} \binom{n_1+n_2-1}{n_1-1} [x_1^{-1}]_{n_1+n_2} = 0 \in \mathbb{L}_{n_1+n_2}(F).
\end{aligned}
\end{equation}

More generally, Goncharov’s work suggests that $[x_d^{-1}, \ldots, x_1^{-1}]_{n_d, \ldots, n_1}$ should be expressible using terms where no $x_i$ is inverted. Inspired by Goncharov’s work we make the following definition.
Definition 4.1. The inversion map is defined inductively on power series by the formula

\begin{align}
\text{inv}([x_1, \ldots, d|t_1, \ldots, d]) &= -(-1)^d [x_1, \ldots, d|t_1, \ldots, d] + \frac{(-1)^d}{t_1} [x_2, \ldots, d|t_2, \ldots, d] \\
&\quad - \frac{(-1)^d}{t_1} [x_2, \ldots, d|t_2, \ldots, d - t_1] - \frac{1}{t_d} \text{inv}([x_1, \ldots, d-1|t_1, \ldots, d-1]) + \frac{1}{t_d} \text{inv}([x_1, \ldots, d-1|t_1, \ldots, d-1 - t_d]).
\end{align}

The induction starts with the formula \(\text{inv}[x|t] = [x|t] + [x]_0\).

We wish to prove the following:

Theorem 4.2. We have \([x^{-1}_{d, \ldots, 1} - t_{d, \ldots, 1}] = \text{inv}[x_1, \ldots, d|t_1, \ldots, d] \in \mathbb{L}(F)\). In particular,

\begin{align}
(44) \quad (-1)^d(-1)^{n_1 + \cdots + nd}[x^{-1}_1, \ldots, x^{-1}_d]_{n_1, \ldots, n_d} = \text{inv}[x_1, \ldots, x_d]_{n_1, \ldots, n_d}.
\end{align}

Example 4.3. For \(d = 1\), we have \((-1)^{n+1}[x^{-1}]_n = [x]_n\) for \(n > 1\) and \([x^{-1}]_1 = [x]_1 + [x]_0\). These hold by Example 3.5.

Example 4.4. For \(d = 2\), \(\text{inv}[x_1, x_2|t_1, t_2]\) equals

\begin{align}
(45) \quad -[x_1, x_2|t_1, t_2] + \frac{1}{t_1} [x_2|t_2] - \frac{1}{t_1} [x_2|t_2 - t_1] - \frac{1}{t_2} \text{inv}[x_1|t_1] + \frac{1}{t_2} \text{inv}[x_1|t_1 - t_2].
\end{align}

The coefficient \(t_1^{n_1-1}t_2^{n_2-1}\) of the equation in Theorem 4.2 gives (42).

Recall that the proof that \([x]_n + (-1)^n[x^{-1}]_n = 0\) used that the symbol \([\infty]_n\) is well defined and equal to 0. Unfortunately, \(\mathcal{L}_{n_1, \ldots, n_d}(x_1, \ldots, x_d)\) does not tend to 0 when some of the \(x_i\) tend to \(\infty\), so we do not expect to be able to simply declare \([x_1, \ldots, x_d]_{n_1, \ldots, n_d}\) to be 0 when some of the \(x_i\) are \(\infty\). Instead we take the equality in Theorem 4.2 as a definition.

Definition 4.5. If some of the \(x_i\) are \(\infty\), we define

\begin{align}
(46) \quad [x_1, \ldots, x_d]_{n_1, \ldots, n_d} = (-1)^d(-1)^{n_1 + \cdots + n_d} \text{inv}([x^{-1}_d, \ldots, x^{-1}_1]_{n_1, \ldots, n_d})
\end{align}

when \(n_1 + \cdots + n_d > 1\).

For example we have

\begin{align}
(47) \quad [x_1, \infty]_{n_1, n_2} = (-1)^{n_2} \left(\frac{n_1 + n_2}{n_1 - 1}\right) [x_1]_{n_1 + n_2}, \quad [\infty, x_2]_{n_1, n_2} = -(1)^{n_1} \left(\frac{n_1 + n_2}{n_2 - 1}\right) [x_2]_{n_1 + n_2}
\end{align}

Remark 4.6. In weight 1, symbols \([\infty]_1\) and \([\infty]_0\) are not defined.

Proof of Theorem 4.2. The structure of the proof is similar to that of the depth 1 inversion relation in Example 3.5. Suppose by induction that it holds in depth less than \(d\) and in weight less than \(n\). The case of depth 1 is the relation \([x]_n + (-1)^n[x^{-1}]_n = 0\). For \(X = [x_1, \ldots, d|t_1, \ldots, d]\) let

\begin{align}
(48) \quad X^{-1} = [x^{-1}_{d, \ldots, 1} - t_{d, \ldots, 1}].
\end{align}

We must prove that \(X^{-1} = \text{inv}(X)\). We first prove by induction that \(\delta(X^{-1})_n = \delta(\text{inv} X)_n\), which implies that \(X^{-1} - \text{inv}(X)\) is constant. Setting \(x_1, \ldots, x_d\) equal to zero then gives the result by Definition 4.5. In order to prove that \(\delta(X^{-1})_n = \delta(\text{inv}(X))_n\), we consider all possible terms involved. It will be convenient to think of the \(x_i\) appearing in each term as formal variables, and not elements in \(F\). It thus makes sense to talk about inverted terms (those that involve inverses \(x_i^{-1}\)) and regular terms (those that don’t involve inverses). Any identity in the free abelian group on terms (or their wedge products) gives rise to an identity in \(L^{\text{sym}}(F)\) (or \(\wedge^2(L^{\text{sym}}(F))\) by
assigning values to the $x_i$. We shall consider three distinct operations on terms which we all think of as inversions:

$$\text{inv}, \quad X \mapsto X^{-1}, \quad \iota,$$

where inv is defined by (43), $X \mapsto X^{-1}$ by (48), and $\iota$ by fixing regular terms and replacing an inverted term $X^{-1}$ by $\text{inv}(X)$. Letting

$$A = [x_{2,\ldots,d}t_{2,\ldots,d}], \quad B = [x_{2,\ldots,d}t_{2,\ldots,d} - t_1],$$

$$C = [x_{1,\ldots,d-1}t_{1,\ldots,d-1}], \quad D = [x_{1,\ldots,d-1}t_{1,\ldots,d-1} - t_d],$$

we claim that the following holds in $\mathbb{L}^{\text{symb}}(F)$:

$$\iota(\delta(X^{-1}) + (-1)^d\delta(X)) = \iota \delta\left(\frac{(-1)^d}{t_1}A - \frac{(-1)^d}{t_1}B - \frac{1}{t_d}C^{-1} + \frac{1}{t_d}D^{-1}\right).$$

Since the cobracket of $X$ consists entirely of terms of lower depth and weight, where $Y^{-1} = \text{inv}(Y)$ by induction, it follows that one has

$$\delta(X^{-1})_n = \delta\left(- (-1)^dX + \frac{(-1)^d}{t_1}A - \frac{(-1)^d}{t_1}B - \frac{1}{t_d}C^{-1} + \frac{1}{t_d}D^{-1}\right)_n = \delta \text{inv}(X)_n$$

in $\mathbb{L}_n(F)$. It remains to prove (51). Using the shorthands

$$Y_{r,s} = [x_{r,s}|t_{r,s} - \iota u], \quad Y_{r,s;u} = [x_{r,s}|t_{r,s} - \iota u]$$

$$Y_{r,s}^{p,q} = [x_{r,p,q}|t_{r,p,q} + \iota u], \quad Y_{r,s;u}^{p,q} = [x_{r,p,q}|t_{r,p,q} + \iota u]$$

$$Z_{r,s}^{p,q} = [x_{r,p,q}|t_{r,p,q}], \quad Z_{r,s;u}^{p,q} = [x_{r,p,q}|t_{r,p,q} - \iota u]$$

we have

$$\delta(X^{-1}) + (-1)^d\delta(X) = \left(X + (-1)^dX^{-1}\right) \land \sum_{p=1}^d t_p[x_p]0 +$$

$$\sum_{2 \leq p \leq d} (-1)^p \left(Y_{p,d} + (-1)^{d-p+1}Y_{p,d}^{-1}\right) \land Y_{1,p-1}^{-1} +$$

$$\sum_{2 \leq p \leq d} Y_{p,d} \land \left(Y_{1,p-1} + (-1)^{p-1}Y_{1,p-1}^{-1}\right) +$$

$$\sum_{1 \leq p < q \leq d} \left(Y_{p,q}^{p,q} + (-1)^{d-q+p}(Y_{p,q}^{p,q})^{-1}\right) \land Y_{p+1,q,p}^{-1} +$$

$$\sum_{1 \leq p < q \leq d} (-1)^{q-p} \left(Z_{p,q}^{p,q} + (-1)^{d-q+p}(Z_{p,q}^{p,q})^{-1}\right) \land Y_{p+1,q}^{-1}.$$

For any regular term $W$ we can define $A_W, B_W, C_W, D_W$ as in (50). If $W$ has depth $d_W$ the definitions of $\iota$ and inv imply that

$$\iota(W^{-1} + (-1)^{d_W}W) = \iota\left(\frac{(-1)^{d_W}}{t_1}A_W - \frac{(-1)^{d_W}}{t_1}B_W - \frac{1}{t_{d_W}}C_W^{-1} + \frac{1}{t_{d_W}}D_W\right).$$

Plugging this into (54), it is now straightforward to match up the terms with those on the righthand side of (51). This concludes the proof. \qed

**Remark 4.7.** Even if we don’t allow any symbols with $x_i = \infty$ we still expect Theorem 4.2 to hold, but only up to torsion. For example, one can show that $6([x]_2 + [x^{-1}]_2) = 0 \in \mathbb{L}_2(F)$. 

4.1. A coalgebra without inverted terms. As an aside, we now show that if we change \( \delta \) by replacing each term \( x_{q-1}^{-1}, \ldots, x_p^{-1}|t_q - t_{p, \ldots, q-1} \) by \( \text{inv}[x_{p, \ldots, q-1}|t_{p, \ldots, q-1} - t_q] \), we obtain another cobracket \( \delta' \) on \( \mathbb{L}^\text{symb}(F) \). Moreover, the groups \( R_n(F) \) defined using \( \delta \) or \( \delta' \) are the same. All that remains is to show that \( \delta'^2 = 0 \). As in the proof of Theorem 4.2 the argument is purely symbolic, and makes use of the operation \( \iota \).

**Lemma 4.8.** For a regular term \( X \) we have

\[
\iota \delta(X) = \delta'(X), \quad \iota \delta(X^{-1}) = \delta'(\text{inv}(X)).
\]

**Proof.** The first equality follows immediately from the definition of \( \delta' \). The second holds in depth 1, and the general case follows by induction from \( |X| \).

**Theorem 4.9.** \( \delta' \) is a cobracket on \( \mathbb{L}^\text{symb}(F) \), i.e. \( \delta'^2 = 0 \).

**Proof.** Let’s write \( \delta(X) \) in the form \( \sum X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge X_4^{(j)^{-1}} \). By Lemma 4.8 we have

\[
0 = \iota \delta^2(X) = \iota \delta \left( \sum_i X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge X_4^{(j)^{-1}} \right)
\]

\[
= \sum_i \iota \delta X_1^{(i)} \wedge X_2^{(i)} - X_1^{(i)} \wedge \iota \delta X_2^{(i)} + \sum_j \iota \delta X_3^{(j)} \wedge \text{inv} X_4^{(j)} - X_3^{(j)} \wedge \iota \delta X_4^{(j)^{-1}}
\]

\[
= \sum_i \delta' X_1^{(i)} \wedge X_2^{(i)} - X_1^{(i)} \wedge \delta' X_2^{(i)} + \sum_j \delta' X_3^{(j)} \wedge \text{inv} X_4^{(j)} - X_3^{(j)} \wedge \delta' \text{inv} X_4^{(j)}
\]

\[
= \delta' \left( \sum_i X_1^{(i)} \wedge X_2^{(i)} + \sum_j X_3^{(j)} \wedge \text{inv} X_4^{(j)} \right) = \delta^2(X).
\]

This proves the result. \( \square \)

5. Shuffle relations

Since the multiple polylogarithms are iterated integrals they satisfy shuffle relations (see e.g. \cite[Sec. 2.5]{Gon01}). For example, one has

\[
\text{Li}_{n_1}(x_1) \text{Li}_{n_2}(x_2) = \text{Li}_{n_1+n_2}(x_1, x_2) + \text{Li}_{n_2,n_1}(x_2, x_1) + \text{Li}_{n_1+n_2}(x_1 x_2).
\]

This motivates us to define **shuffle products**

\[
[x_1]_{n_1} [x_2]_{n_2} = [x_1, x_2]_{n_1+n_2} + [x_2, x_1]_{n_2+n_1} + [x_1 x_2]_{n_1+n_2} \in \mathbb{L}^{n_1+n_2}(F).
\]

In terms of power series this is equivalent to

\[
[x_1|t_1][x_2|t_2] = [x_1, x_2|t_1, t_2] + [x_2, x_1|t_2, t_1] + \frac{1}{t_1 - t_2}([x_1 x_2|t_1] - [x_1 x_2|t_2])
\]

Similarly, we define a shuffle product

\[
[x_1, x_2|t_1, t_2][x_3|t_3] = [x_1, x_2, x_3|t_1, t_2, t_3] + [x_1, x_3, x_2|t_1, t_3, t_2] + [x_3, x_1, x_2|t_3, t_1, t_2]
\]

\[
+ \frac{1}{t_1 - t_3}([x_1 x_3, x_2|t_1, t_2] - [x_1 x_3, x_2|t_3, t_2]) + \frac{1}{t_2 - t_3}([x_1, x_2 x_3|t_1, t_2] - [x_1, x_2 x_3|t_1, t_3])
\]

corresponding to the shuffle relations for \( \text{Li}_{n_1+n_2}(x_1, x_2) \text{Li}_{n_3}(x_3) \).

**Theorem 5.1.** The shuffle products \( [x_1|t_1][x_2|t_2] \) and \( [x_1, x_2|t_1, t_2][x_3, t_3] \) are zero in \( \mathbb{L}(F) \).
Proof. A simple computation shows that
\begin{equation}
\delta([x_1|x_2]|t_1|t_2|t_2)) = ([x_1|x_1]|t_1|t_2) \wedge (t_1|x_1|t_2) + t_2|x_1|t_2|t_2) \in \wedge^2(\mathbb{L}(F)).
\end{equation}
Assuming by induction that \((\delta)_{n−1} = 0\) it follows that \((\delta)_{n} = 0\). Setting \(x_1 = x_2 = 0\) shows that the constant is 0. Another straightforward computation shows that
\begin{equation}
\delta([x_1,x_2|t_1|t_2]|x_3|t_3|t_3) = ([x_1,x_2|t_1|t_2]|x_3|t_3|t_3) \wedge \sum_{i=1}^{3} t_i|x_1|t_1|t_2|t_3 + [x_2|t_2] \wedge ([x_1|t_1]|x_3|t_3) + ([x_1|x_2|t_1]|x_3|t_3|t_3) \wedge [x_2|t_2|t_1] − ([x_1|x_2|t_2]|x_3|t_3|t_3) \wedge ([x_1|t_1|t_2] + [x_1|t_1] − [x_1|x_2|t_2]|x_3|t_3|t_2),
\end{equation}
so the same argument as above shows that \([x_1,x_2|t_1|t_2]|x_3|t_3|t_3 = 0\) as well.

Remark 5.2. We can similarly define shuffle products in arbitrary depth. We conjecture that they are all zero in \(\mathbb{L}(F)\). Proving this would require showing that the cobracket of a shuffle relation is given in terms of shuffle relations in lower weight and depth. This appears to be the case.

Remark 5.3. It seems worth noting that if one replaces \(\delta\) by \(\delta'\), \((\delta)_{n} = 0\) hold not just in \(\wedge^2(\mathbb{L}(F))\), but also \(\wedge^2(\mathbb{L}_{\text{symb}}(F))\) (and not just modulo 2-torsion).

6. The Goncharov–Rudenko Lie coalgebra

Goncharov and Rudenko [GR18] also considered the problem of explicitly constructing the motivic Lie-coalgebra. They give an explicit construction in weight 4 and less. We denote their co-algebra \(\mathbb{L}_{\text{GR}}(F)\). They define \(\mathbb{L}_{\text{GR}}(F) = F^\ast\), \(\mathbb{L}_{\text{GR}}(F) = \mathcal{B}_2(F)\), and define (for \(n = 3, 4\)) \(\mathbb{L}_{\text{GR}}(F)\) to be the group generated by symbols \(\{x\}_n\) and \(\{x,y\}_{n−1,1}\) subject to certain relations, most notably a relation \(Q_n\). The relationship with their work and ours is given in Theorem 6.1 below. Its proof is straightforward, but long, so we omit it. The main point is to show that \(Q_n\) maps to an element in \(\mathcal{R}_n(F)\). We stress that their symbol \(\{x,y\}_{n−1,1}\) does not correspond to our symbol \(\{x,y\}_{n−1,1}\). We find our symbol more natural since it directly relates to the multiple polylogarithm (their symbol is related to motivic correlators).

Theorem 6.1. There is a surjective map \(\mathbb{L}_{\text{GR}}(F)_{\mathbb{Q}} \to \mathbb{L}_{\leq 4}(F)_{\mathbb{Q}}\) taking \(\{x\}_n\) to \(\{x\}_n\), \(\{x,y\}_{2,1}\) to \(−[x/y, y]_{2,1} − [x]_3 − [y]_3\) and \(\{x,y\}_{3,1}\) to \(−[x/y, y]_{3,1} − [x]_4 + [y]_4\). The map preserves the cobracket.

Remark 6.2. We suspect that the map is an isomorphism, but we do not know if all of our relations can be expressed in terms of the \(Q_n\) relations.

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