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The linear response of stellar systems does not diverge at marginal stability

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ABSTRACT

The linear response of a stellar system’s gravitational potential to a perturbing mass comprises two distinct contributions. Most famously, the system will respond by forming a polarization ‘wake’ around the perturber. At the same time, the perturber may also excite one or more ‘Landau modes’, i.e. coherent oscillations of the entire stellar system which are either stable or unstable depending on the system parameters. The amplitude of the first (wake) contribution is known to diverge as a system approaches marginal stability. In this paper, we consider the linear response of a homogeneous stellar system to a point mass moving on a straight line orbit. We prove analytically that the divergence of the wake response is in fact cancelled by a corresponding divergence in the Landau mode response, rendering the total response finite. We demonstrate this cancellation explicitly for a box of stars with Maxwellian velocity distribution. Our results imply that polarization wakes may be much less efficient drivers of secular evolution than previously thought. More generally, any prior calculation that accounted for wakes but ignored modes – such as those based on the Balescu-Lenard equation – may need to be revised.

Key words: gravitation – plasmas – galaxies: kinematics and dynamics – galaxies: spiral.

1 INTRODUCTION

Many of the crowning achievements of galactic dynamics – such as our understanding of spiral structure (Lin & Shu 1964; Julian & Toomre 1966; Lynden-Bell & Kalnajs 1972; Sellwood & Carlberg 2014), of dynamical friction (Chandrasekhar 1943), of galaxy stability (Palmer 1994), and of bar-halo interactions (Tremaine & Weinberg 1984; Weinberg 1985) – are underpinned, explicitly or otherwise, by the theory of linear response. At the most basic level, linear response theory answers the question: how does a stellar system respond to a weak gravitational perturbation? This perturbation can take various forms; depending on the context, bars, spiral modes, molecular clouds, infalling satellites, dark matter substructure, and internal Poisson noise all constitute legitimate perturbers of a galaxy (Weinberg 2001; Pichon & Aubert 2006). Though most of the conclusions we draw in this paper will be valid for systems undergoing arbitrary perturbations, for concreteness and concision we will focus here on systems perturbed by a single point mass.

The foundational study of a stellar system perturbed by a heavy point mass is Chandrasekhar’s (1943) paper on dynamical friction. Chandrasekhar showed that, to linear order in the perturber mass, the stellar system responds by forming a ‘wake’ behind the perturber. This wake is said to ‘polarize’ or ‘dress’ the gravitational potential of the perturber, typically increasing its effective mass, in a manner analogous to Debye shielding in electrostatic plasma (Ichimaru 1973). The second-order effect is that the gravitational field of the wake backreacts on the perturber, producing an effective frictional drag. Chandrasekhar’s picture of wake-drive friction now underlies huge swathes of astrophysical theory concerning galaxy mergers, supermassive black hole coalescence, the nature of dark matter, and so on (e.g. Begelman, Blandford & Rees 1980; Boylan-Kolchin, Ma & Quataert 2008; Lancaster et al. 2020).

Moreover, several authors have recognized that the linear, wake-like response described above is particularly prominent if the stellar system in question is only weakly stable. By ‘stable’ here we mean that the Landau modes – i.e. the set of self-consistent, coherent oscillations that the system can support in the absence of a perturber – are all exponentially damped; the word ‘weakly’ means that the associated damping time-scale is long (in a sense we define more precisely below). The key point is that in weakly stable systems, the wake response can be strongly amplified by its own self-gravity.

This amplification is important for the dynamical friction experienced by a sinking satellite (Weinberg 1989). It also has a profound effect upon the response of galactic discs to co-orbiting perturbers like molecular clouds (D’Onghia, Vogelsberger & Hernquist 2013; Sellwood 2021). For instance, in a classic analysis, Julian & Toomre (1966) inserted a point mass into a local model of a stable, shearing stellar disc. Working in the time-asymptotic limit, they found that the wake induced by the point mass was much more massive in discs with smaller Toomre $Q$ (see their table 1; also Binney 2020). In other words, the less stable they made the disc, the more strongly amplified was the response. These self-amplified shearing wakes are still one of the primary explanations offered for the existence of spiral structure in galaxies. Naively, however, one might expect that as the system is driven towards marginal stability ($Q \rightarrow 1$) the linear wake response will diverge.

\begin{itemize}
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\end{itemize}
Similarly, and importantly for this paper, Weinberg (1993) and Magorrian (2021) have studied the response of a homogeneous, periodic stellar system to a point mass perturber moving on a straight-line orbit. Both authors derived an expression for the linear wake-like perturbation to the system’s distribution function in the time-asymptotic limit (i.e. after waiting a time long compared to the damping time of all Landau modes) – see equation (22) of Weinberg (1993), and equation (40) of Magorrian (2021). However, these expressions manifestly diverge if one of the system’s Landau mode frequencies tends towards the (real) frequency of the perturbation, which is possible as the system approaches marginal stability. More broadly, many ‘quasi-linear’ kinetic theories that purport to describe the secular evolution of stellar systems predict a rate of evolution that is proportional to the product of two linear wake-like terms (e.g. Binney & Lacey 1988; Fouvry & Bar-Or 2018, and equation (79) of Magorrian 2021). It has been claimed that this rate will diverge as the system approaches marginal stability, and this behaviour has been likened to the phenomenon of critical opalescence (e.g. Chavanis 2023).

These divergent wake results are worrying, since they predict that near marginal stability, the linear response of stellar systems can become arbitrarily (and hence non-linearly) large. Taken at face value, this suggests that linear and quasi-linear theories are incapable of describing accurately the dynamics of weakly stable stellar systems. However, the divergences can all be traced back to an erroneous time-asymptotic assumption, in a regime where the required time-scale separation does not actually exist. Precisely, the above authors were (explicitly or otherwise) assuming that all the Landau modes of the stellar system had decayed. But as a system approaches marginal stability, the time-scale for this decay becomes infinitely long. Said differently, near marginal stability, a wake calculation on its own is an incomplete description of the finite-time linear response: one should also account directly for the contribution of the system’s Landau modes. As is well-known in plasma theory, this Landau mode contribution also diverges as a system is driven towards marginal stability (e.g. Regisser & Oberman 1968; Hatori 1969; Oberman 1970). The key point is that the divergence in the Landau mode contribution cancels the divergence in the wake contribution, such that the total linear response – the only physically relevant quantity – is rendered finite. Not only is the divergence cured, but the amplitude of the total linear response is much smaller than one would naively predict based on the wake calculation alone, meaning linear theory need not necessarily be abandoned.

The purpose of this paper is to demonstrate this basic fact in a simple stellar-dynamical context. To this end, we consider an initially unperturbed, homogeneous box of stars, and calculate the linear response of its gravitational potential to an externally imposed point-mass perturber moving on a straight-line orbit. We include the Landau mode contribution to the response, and thereby prove that the total response is always finite regardless of the system’s stability properties. We also demonstrate the cancellation of the wake and mode divergences explicitly for a box of stars with initially Maxwellian velocity distribution.

We present our calculations in §2, and summarize in §3.

2 LINEAR RESPONSE THEORY

We consider a system of equal-mass stars interacting via Newtonian gravity and subject to an external potential force. Let \( f(r, v, t) \) be the distribution function (DF), which we normalize in such a way that the average mass contained inside the infinitesimal phase-space volume element \( dr dv \) around the point \((r, v)\) at time \( t \) is equal to \( f(r, v, t) dr dv \). We will assume that the system is collisionless, in which case the DF obeys the Vlasov equation

\[
\frac{\partial f}{\partial t} + v \cdot \frac{\partial f}{\partial r} - \nabla \Phi^{\text{tot}} \cdot \frac{\partial f}{\partial v} = 0.
\]

(1)

The total potential \( \Phi^{\text{tot}} \) consists of two parts,

\[
\Phi^{\text{tot}}(r, t) = \Phi(r, t) + \Phi^{\text{ext}}(r, t).
\]

(2)

The internal potential \( \Phi \) satisfies the Poisson equation

\[
\nabla^2 \Phi = 4 \pi G \int dv \ f.
\]

(3)

The external potential \( \Phi^{\text{ext}} \) will be left unspecified for now.

Let us assume further that our system is confined to a periodic box of volume \( V \). Then we can develop any function on phase space as a Fourier series: for instance,

\[
f(r, v, t) = \sum_k \exp(ik \cdot r) f_k(v, t),
\]

(4)

where the wavenumbers \( k \) have components \( k_i = 2 \pi n_i / L_i \) where \( n_i \) is an integer and \( L_i \) is the length of the \( i \)th side of the box\(^2\), and

\[
f_k(v, t) = \frac{1}{V} \int dr \exp(-ik \cdot r) f(r, v, t),
\]

(5)

are the spatial Fourier coefficients. We may also define the Laplace transform of these coefficients as

\[
\hat{f}_k(v, z) = \int_0^\infty dt \exp(izt) f_k(v, t).
\]

(6)

This is defined for all complex frequencies \( z \) with \( \text{Im} \ z \geq \eta \), where \( \eta \) is larger than the growth rate of the most unstable Landau mode of the system (see below) or, if the system is stable, larger than zero.

We assume a spatially uniform background. Further, we assume that the fluctuations in the DF are small compared to, and evolve much more rapidly than, the spatial average \( f_0 \), i.e. \( f_k \ll f_0 \) for \( k \neq 0 \). Then we may linearize the Vlasov equation (1) and Fourier-Laplace transform the result to arrive at

\[
i(k \cdot v - z) \hat{f}_k(v, z) - i k \frac{\partial f_0}{\partial v} \hat{\Phi}_k^0(z) = \hat{f}_k(v, 0).
\]

(7)

To simplify matters, we will assume that at \( t = 0 \), there are no fluctuations in the DF, i.e. \( f_k(v, 0) = 0 \). The initial internal potential fluctuation is then just \( \Phi_k^0(0) = 0 \), so that \( \Phi_k^{\text{tot}}(0) = \Phi_k^{\text{ext}}(0) \). Solving equation (7) for \( \hat{f}_k(v, z) \) and inserting the result into the linearized Poisson equation

\[
- k^2 \hat{\Phi}_k(z) = 4\pi G \int dv \ \hat{f}_k(v, z)
\]

(8)

then yields a little bit of algebra the equation

\[
\epsilon_k(z) \Phi_k^{\text{tot}}(z) = \Phi_k^{\text{ext}}(z).
\]

(9)

\(^2\)Our results also hold in the special case of infinite homogeneous systems \( V \to \infty \), by replacing \( V^{-1} \sum_k \to (2\pi)^{-3} \int dk \).
where \( \epsilon_k(z) = 1 + \frac{4\pi G}{k^2} \int \frac{dv}{k \cdot v - z} k \cdot \frac{\partial f_0}{\partial v} \) (Im \( z > 0 \))

(10)
is the dispersion function, which for Im \( z \leq 0 \) is defined through analytic continuation (see below). The zeroes of \( \epsilon_k(z) \) correspond to the frequencies of the Landau modes that the system supports.

The total gravitational potential in the time domain is given by the inverse Laplace transform:

\[
\Phi_k^{\text{tot}}(t) = \int_{i\gamma_\text{tot} \to -\infty} dz \frac{\exp(-izt)}{2\pi} \Phi_k^{\text{ext}}(z).
\]

(11)

As we remarked above, \( \eta \) is larger than the growth rate of the most unstable Landau mode or, if the system is stable, larger than zero. This means that by construction, \( \epsilon_k(z) \neq 0 \) along the integration contour in equation (11). From equation (9) it thus follows that

\[
\Phi_k^{\text{tot}}(t) = \int_{i\gamma_\text{tot} \to -\infty} dz \frac{\exp(-izt)}{2\pi} \frac{\Phi_k^{\text{ext}}(z)}{\epsilon_k(z)}.
\]

(12)

2.1 Response of an unperturbed system to a point mass moving at constant velocity

Suppose \( \Phi^{\text{ext}} \) is generated by an externally imposed point mass \( m \) moving on a straight-line trajectory \( r = r_p + v_p t \) for constant \( r_p \) and \( v_p \). Then the external potential \( \Phi^{\text{ext}} \) satisfies

\[
\nabla^2 \Phi^{\text{ext}} = 4\pi G m \delta(r - r_p - v_p t).
\]

(13)

From this it follows that the Fourier coefficients of the external potential are

\[
\Phi_k^{\text{ext}}(t) = \exp(-ik \cdot v_p t) \Phi_k^{\text{ext}}(0),
\]

(14)

where

\[
\Phi_k^{\text{ext}}(0) = -\frac{4\pi G m}{k^2} \exp(-ik \cdot r_p).
\]

(15)

The Laplace transform of equation (14) is

\[
\Phi_k^{\text{ext}}(z) = \frac{\Phi_k^{\text{ext}}(0)}{ik \cdot v_p - z}.
\]

(16)

Inserting this into equation (12) yields the total gravitational potential in the form of

\[
\Phi_k^{\text{tot}}(t) = \int_{i\gamma_\text{tot} \to -\infty} dz \frac{\exp(-izt)}{2\pi i (k \cdot v_p - z) \epsilon_k(z)} \Phi_k^{\text{ext}}(0).
\]

(17)

2.2 The response at late times

The integral in equation (17) can be carried out by closing the integration contour with a very large semicircle in the lower half plane on which the integrand vanishes (Thorne & Blandford 2017). Provided that the integrand's only singularities are poles, the inverse Laplace transform is then equal to \( 2\pi i \) times the sum of residues of the integrand at those poles. In order to follow this procedure, we must analytically continue the dispersion function \( \epsilon_k(z) \), defined in equation (10) for \( \text{Im} \, z > 0 \), into the lower half of the complex plane. This is achieved by taking the integral over the parallel component of the stellar velocity \( k \cdot v/k \) along the so-called Landau contour, which is deformed in such a way that it always passes below the point \( k \cdot v = z \) (e.g. Binney & Tremaine 2008).

This by

\[
\epsilon_k(z) = 1 + \frac{4\pi G}{k^2} \int \frac{dv}{(k \cdot v - z)^\gamma} k \cdot \frac{\partial f_0}{\partial v} \quad (\text{Im} \, z > 0)
\]

(18)

which unlike equation (10) is defined for all \( z \).

The dispersion function as given in equation (18) is analytic in the whole complex plane, so its zeroes (i.e. the Landau mode frequencies) are all isolated.\(^3\) Let us assume first that the system is stable, which means that all Landau mode frequencies are located in the lower half of the complex plane. The exponential factor in equation (17) then ensures that the contributions from the residue at each Landau mode frequency eventually decays. Asymptotically, the only non-zero contribution comes from the residue at \( z = k \cdot v_p \), which is given by

\[
\Phi_k^{\text{tot}}(t) = \frac{1}{\epsilon_k(k \cdot v_p)} \Phi_k^{\text{ext}}(t) \quad \text{for} \quad t \gg -1/\gamma_k,
\]

(19)

where \( \gamma_k < 0 \) is the growth rate of the most weakly damped mode (i.e. all other modes have imaginary parts which are more negative than \( \gamma_k \)). Typically one interprets (19) by saying that the external potential \( \Phi_k^{\text{ext}}(t) \) has been ‘dressed’ by collective effects, which are encapsulated in the factor \( 1/\epsilon_k(k \cdot v_p) \). The part of (19) which comes from the perturbed distribution of stars – i.e. the right-hand side of (19) minus \( \Phi_k^{\text{ext}}(t) \) – is typically referred to as the ‘wake’.

As a shorthand, in the following subsection we will refer to the entire right-hand side of (19) as the ‘wake’ contribution to the potential.

The dressed potential as given in equation (19) clearly diverges if \( \epsilon_k(z) \) has a zero at the real frequency \( z = k \cdot v_p \). A necessary condition for this is that the system is marginally stable, i.e. \( \gamma_k = 0 \). Put differently, if one considers a marginally stable system, one can always choose a perturbing velocity \( v_p \) such that the right-hand side of (19) is infinite. The key point of our paper is that the divergence of (19) as the system approaches marginal stability is spurious, and stems from the neglect of the Landau mode potential itself—or, equivalently, from the erroneous assumption that one can always wait a time \( t \) long compared to \(-1/\gamma_k \) such that (19) becomes valid. No such \( t \) exists for marginally stable systems, and even for very weakly stable systems, \(-1/\gamma_k \) may be comparable to other timescales of interest such as the relaxation time, rendering (19) invalid for practical purposes.

To cure the divergence, let us assume that the frequency of the system’s most weakly damped mode is a single, simple zero of \( \epsilon_k(z) \).\(^4\) This means that, without loss of generality, we can

\(^3\)Technically, \( \epsilon_k(z) \) can also exhibit other unusual features, e.g. branch cuts associated with singularities of \( f_0(v) \) for complex \( v \). These give rise to behaviour different from either the wakes or Landau modes considered here (e.g. Lee & Shadwick 2023). We ignore these complications.

\(^4\)The symmetry \( \epsilon_k(z) = \epsilon_k(-z) \) implies that if \( \omega_k + i\gamma_k \) is a Landau mode frequency, then so is \( -\omega_k + i\gamma_k \). Thus, in general, there is not only a single most weakly damped mode at each \( k \), but rather a pair of most weakly damped modes. However, an exception to this occurs if \( \omega_k = 0 \), in which case there is only a single most weakly damped mode. This is true for the kappa distribution

\[
f(v) = \frac{\rho}{(2\pi)^{3/2} \Gamma(k - 1/2)} \frac{\Gamma(\kappa + 1)}{\Gamma(3/2)} [1 + v^2/(2\kappa \sigma^2)]^{-\kappa/2}.
\]

(20)

which includes both the Maxwellian (\( \kappa \to \infty \)) and the Lorentzian (\( \kappa = 1 \)) as limiting cases. The zeroes of \( \epsilon_k(z) \) for these DFs are also all simple. Since these are the cases of most interest to us, we will assume a single, simple mode hereafter; one can straightforwardly generalize our treatment further if desirable (Oberman 1970).
\[ \varepsilon_k(z) = (z - z_k) \alpha_k(z) \quad \text{with} \quad \alpha_k(z_k) \neq 0, \quad (21) \]

cf. Bălescu (1963). Note that \( \varepsilon'_k(z_k) = \alpha_k(z_k) \), where \( \varepsilon'_k(z) \) denotes the derivative of \( \varepsilon_k(z) \) with respect to \( z \). We insert equation (21) into equation (17), and again close the contour with a large semicircle in the lower half plane, but this time we include the contribution both from the residue at \( z = k \cdot v_p \) and from the residue at \( z = z_k \). The result is

\[
\Phi_k(t) = \frac{1}{\varepsilon_k(k \cdot v_p)} - \frac{\exp(i(k \cdot v_p - z_k)t)}{(k \cdot v_p - z_k)\varepsilon_k(z_k)} \frac{\Phi_k(t)}{k \cdot v_p}(t). \quad (22)
\]

This equation is valid for times \( t \gg -1/\gamma_k \), where \( \gamma_k \) is the growth rate of the second most weakly damped mode. The first term on the right-hand side is the same as in (19). It represents the dressed potential of the perturber and, as we know, it diverges as \( z_k \to k \cdot v_p \). The second term is the potential of the Landau mode that the perturber has excited. It also diverges as \( z_k \to k \cdot v_p \), but it does so in a way that precisely cancels the divergence of the first term. Indeed, letting \( z_k \to k \cdot v_p \) yields

\[
\lim_{z_k \to k \cdot v_p} \Phi_k(t) = -\left[ \frac{\alpha_k'(k \cdot v_p)}{\alpha_k'(k \cdot v_p)} + \frac{i t}{\alpha_k'(k \cdot v_p)} \right] \Phi_k(t). \quad (23)
\]

We stress that equation (22) is valid for stable and unstable systems, and provides a smooth transition between them. It also provides sensible results in limiting cases. For strongly stable systems, the damping time-scale \(-1/\gamma_k \) is short, so the Landau mode term decays rapidly and we soon recover the dressed ‘wake’ potential (19). On the other hand, for unstable systems the Landau mode eventually dominates. Of course, in the unstable case, (22) eventually breaks down once the Landau mode has grown so large that non-linear effects become important.

Before moving on, let us simplify equation (22) by supposing that (i) the perturbing mass point is either at rest or the wave number \( k \) is perpendicular to its velocity, so \( k \cdot v_p = 0 \), and (ii) the most weakly damped eigenmode has a purely imaginary frequency \( z_k = i \gamma_k \). Then

\[
\Phi_k(t) = \frac{1}{\varepsilon_k(0)} + \frac{\exp(\gamma_k t)}{\varepsilon_k(z_k)} \Phi_k(t). \quad (24)
\]

We will use this expression in the next subsection.

### 2.3 Maxwellian stellar system

Let us suppose that the background DF is Maxwellian, i.e.

\[
f_0(\Psi) = \frac{\rho_0}{(2\pi)^{3/2} \sigma^3} \exp \left( -\frac{\Psi^2}{2\sigma^2} \right), \quad (25)
\]

where \( \sigma > 0 \) is the velocity dispersion. The dispersion relation (18) is given by

\[
\varepsilon_k(z) = 1 - \frac{k^2}{k^2} W\left( \frac{z}{k \sigma} \right), \quad (26)
\]

\[5\text{Note that the time dependence of the second term on the right-hand side of (23) is } i t \exp(-i z_k t). \text{ When carrying out the inverse Laplace transform using the calculus of residues, such a time-dependence is the signature of a second-order pole (see Appendix A in Bălescu 1963). This is sensible because the two simple poles at } z = k \cdot v_p \text{ and } z = z_k \text{ merge to become a second-order pole as } z_k \to k \cdot v_p. \]

\[6\text{Note that for } k \cdot v_p = 0, \text{ equation (16) implies that } \Phi_k(0) = 0. \]

Figure 1. Top: growth rate of the most weakly damped (or most rapidly growing) mode of a Maxwellian stellar system as a function of dimensionless wavelength \( k_\beta/k \). Bottom: we suppose the Maxwellian system is perturbed by a point mass, and plot the ratio of the total potential to the perturber potential, equation (28), for different times \( t \). The black dashed line shows the naive wake result that ignores the Landau mode contribution (equation 19).

\[
\gamma_k = \sqrt{\frac{8}{\pi}} (k_\beta - k_\sigma) \sigma. \quad (29)
\]

Obviously the system is stable (unstable) for \( k > k_1 \) (\( k < k_1 \)). Substituting (29) into equation (28) and taking the limit of marginal stability, \( k \to k_1 \), yields

\[
\lim_{k \to k_1} \frac{\Phi_k(t)}{\Phi_k(0)} = \frac{2}{\pi} + \sqrt{\frac{2}{k \sigma}}. \quad (30)
\]

For wavenumbers that are significantly different from \( k_1 \) we can compute the growth rate \( \gamma_k \) numerically using a root-finding algorithm, and hence calculate \( \Phi_k(t) \) using equation (28). We show the result of this calculation in Fig. 1. The top panel shows the growth rate \( \gamma_k \) as a function of scale \( k_\beta/k \); as expected, the \( k \)-space divides into stable \((k_\beta/k < 1)\) and unstable \((k_\beta/k > 1)\) regions. In the bottom panel

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we plot with different coloured lines the ratio of the total potential to external potential, \( \Phi_k^{\text{tot}}(t)/\Phi_k^{\text{ext}}(t) \), computed using equation (28) at different times \( t \). With a black dashed line we show the (time-independent) part of (28) that arises by considering only the first term (the wake contribution) and ignoring the second term (the Landau mode). In other words, the black dashed line is the naive result that one would find by simply applying (19) in the weakly stable regime (as is often done in Balescu-Lenard calculations, e.g. Fouvry, Bar-Or & Chavanis 2019). We see clearly that this contribution is divergent for marginally stable systems (\( k/k \to 1 \)), and severely overestimates the linear response for weakly stable systems (say \( k/k \approx 0.95 \)) at early times. This emphasizes the fact that even for stable systems, the form (19) of the external potential is only valid for times \( t \gg 1/\gamma_k \), which becomes infinite as marginal stability is approached.

Note that we do not recover the exact initial condition \( \Phi_k^{\text{ini}}(0) = \Phi_k^{\text{ext}}(0) \) for all \( k \) in Fig. 1 (see the blue line in the bottom panel). Instead, \( \Phi_k^{\text{ini}}(0)/\Phi_k^{\text{ext}}(0) \) equals unity only for \( k/k = 0 \), and then decreases slowly with increasing \( k/k \). The reason for this is that when we wrote down equation (22), we accounted only for the system’s most weakly damped Landau mode and ignored the contributions from any other modes. Had we included all the modes, \( k \), we would find \( \Phi_k^{\text{ini}}(0)/\Phi_k^{\text{ext}}(0) = 1 \) for all \( k \). We have checked this claim explicitly for a Lorentzian stellar system in equation (20) with \( \kappa = 1 \), for which all modes can be calculated analytically. Apart from this minor technical detail, the linear response behaviour of the Lorentzian stellar system does not differ quantitatively that of the Maxwellian, so we do not report the details here.

Finally, in Fig. 2 we fix \( k_i \sigma = 5 \), and show explicitly the contributions from the two individual terms in equation (28), namely

\[
\Phi_k^{\text{wake}}(t) = \frac{\Phi_k^{\text{ext}}(t)}{1 - k^2/k^2}, \quad \Phi_k^{\text{mode}}(t) = -\frac{\exp(-iz\delta_k t) \Phi_k^{\text{ext}}(t)}{1 - k^2/k^2 - z^2_k/(k\sigma)^2}, \quad (31)
\]

as well as their sum. We see that in the strongly stable regime \( k/k \ll 1 \), the mode contribution is negligible, and the wake contribution is just equal to the perturber potential \( (\Phi_k^{\text{ini}}(t)/\Phi_k^{\text{ext}}(0) \to 1) \). This makes sense since at small scales, self-gravity is unimportant, so \( \epsilon_k(z) \approx 1 \) for all \( z \). On the other hand, in the strongly unstable regime \( k/k \gg 1 \), the exponentially growing mode contribution dominates the total potential. Both the ‘wake’ and ‘mode’ contributions individually diverge as \( k/k \) approaches unity from either side, but their sum is perfectly well-defined for all \( k \). None of our calculations exhibit any sharp feature at marginal stability.

\section{Summary}

The self-consistent response of a gravitating system to a weak perturbation is a fundamental problem of stellar dynamics. The most basic mathematical tool with which this problem may be attacked is linear response theory (Palmer 1994; Nelson & Tremaine 1999; Binney & Tremaine 2008). However, several previous calculations based on the dressed ‘wakes’ that perturbations induce in stellar systems have suggested, implicitly or explicitly, that the linear response potential actually diverges if the system in question happens to approach marginally stable (see the Introduction for examples). Such an obviously unphysical result might lead one to conclude that, for very weakly stable stellar systems at least, linear theory ought to be abandoned. More optimistically, one might hope that the divergence can be cured (e.g. by non-linear effects), but that wakes are nevertheless very efficient means to amplify gravitational perturbations, and hence drive rapid secular evolution in galaxies (Fouvry et al. 2015; Chavanis 2023).

In this paper, we demonstrated that for finite times, the linear response theory is in fact not divergent at all, regardless of a stellar system’s stability properties. To do this, we calculated the linear gravitational potential response of a homogeneous stellar system to a point mass perturber. The result naturally decomposes into a ‘wake’-like part associated with polarization (‘dressing’) of the perturber potential, and an exponentially growing or decaying response which is associated with the excitation of the system’s Landau modes. We showed analytically that as the system approaches marginal stability (i.e. as the imaginary part of its most weakly damped Landau mode frequency approaches zero), the contributions from the dressed wake and the mode both individually diverge, but their sum, which is the only physically relevant quantity, does not. This fact was already understood many years ago in plasma physics (e.g. Register & Oberman 1968; Hatori 1969; Oberman 1970), but does not seem to have been appreciated in stellar dynamics until now. As an example, we considered the linear response of a Maxwellian stellar system to an orbiting point mass. We illustrated explicitly the cancellation of wake and mode divergences in the vicinity of marginal stability at any finite time, and showed that the total potential response behaves smoothly at all scales \( k \). The key point is that the divergent result (19) employed in previous studies is only valid on time-scales long compared to the decay time of all Landau modes, and for marginally stable systems no such time-scale exists. While we have focused exclusively on homogeneous stellar systems perturbed by a single point mass, one may straightforwardly extend our results to inhomogeneous stellar systems, to multiple perturbers, and to perturbations that are internally generated rather than externally imposed. Such calculations will form part of our future work; here we simply state the generic result that the finite-time linear response of stellar systems never diverges as a result of the system approaching marginal stability. This implies that the \textit{total}, self-gravitating response of galaxies may be much weaker than would be naively calculated based on dressed wakes alone. This is a good thing from the point of view of the validity of linear response theory, which would of course be rendered invalid if the the fluctuations in weakly stable systems really were extremely (i.e. non-linearly) large. On the other hand, it also suggests that wake-driven secular evolution of galaxies may be much less efficient than previously thought, and that the rate of this evolution may be severely overestimated by kinetic schemes that explicitly ignore the direct contribution from Landau modes, like Balescu-Lenard theory (Heyvaerts 2010; Chavanis 2012; Hamilton 2021). In upcoming work, we will present a unified kinetic theory that accounts for both types of response.

In conclusion, any previous calculation of the evolution of a stellar system that accounted for strongly amplified wakes, but ignored the explicit contribution from Landau modes, may need to be revised.

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DATA AVAILABILITY

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