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Interactions of quantum systems with pulses of quantized radiation: From a cascaded master equation to a traveling mode perspective

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The interaction of a propagating pulse of quantum radiation with a localized quantum system can be described by a cascaded master equation with a distinct initially populated input and a finally populated output field mode [Kiilerich and Mølmer, Phys. Rev. Lett. 123, 123604 (2019)]. By transformation to an appropriate interaction picture, we break the cascaded nature of the master equation and recover an effective time-dependent interaction with a lossless single mode and a supplementary lossy mode. The former closely represents the traveling pulse, while the latter constitutes a non-Markovian component in the exchange of quanta between the scatterer and the quantized field. The transformed master equation offers important insights into the system dynamics, and it permits numerically efficient solutions.

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I. INTRODUCTION

A full quantum analysis of the interaction of a discrete quantum system with a cavity field may often assume (near) resonance with only a single standing-wave field mode whose mode function is unaffected by the interaction. The interaction then implies only the joint evolution of a single field oscillator and the discrete quantum system, as described, e.g., in the Jaynes-Cummings model.

Numerous proposals exist to employ interactions with simple quantum systems with the purpose to manipulate and prepare quantum states, such as number states, coherent states, squeezed states, and Schrödinger-cat states of a single mode of radiation. The motivation is often that such states can be propagated in space and thus be used for probing, communication, and transfer of states and operations in quantum networks [1]. But how does a localized quantum system, such as a single two-level atom, interact with a traveling pulse that initially occupies only a single mode of radiation? If the temporal profile of this pulse at the location of the atom is \( u(t) \), do we obtain a time-dependent interaction of the same form as for an atom flying through a cavity and exploring the position-dependent field strength of the resonant eigenmode? That is, is the system correctly or to a good approximation described by the Hamiltonian \( \hat{H}_{\text{JC}}(t) = i\sqrt{\gamma}(u^*(\gamma)\sigma^+ - u(t)\sigma^-) \)?

\[
\hat{H}_{\text{JC}}(t) = i\sqrt{\gamma}(u^*(\gamma)\sigma^+ - u(t)\sigma^-), \tag{1}
\]

where \( \sqrt{\gamma}u(t) \) specifies the time-dependent complex coupling strength and the raising and lowering operators of the field \( \hat{a}^\dagger, \hat{a} \) and the two-level system \( \sigma^+, \sigma^- \) account for the coherent exchange of excitation between the atom and the traveling pulse?

The answer to this question is complicated by the fact that in the absence of a cavity, the propagating field explores a continuum of frequency modes and, by dispersive and absorptive effects, the interaction with a scatterer may change the temporal shape of the field mode function in a manner that is entangled with its quantum state of excitation [see Fig. 1(a)]. Multiple analyses have addressed different aspects of this multiphoton, multimode problem [2–12].

As a simpler problem, we may enquire what the quantum state occupying a definite pulse mode after the interaction is. Applying the theory of a cascaded quantum system [13,14], that problem was treated in Refs. [15,16] by incorporating the incident pulse of quantized radiation as the output from an upstream virtual single-mode cavity that gradually releases its quantum state content in the form of a pulse. The quantum state of any specific output pulse mode may, in a similar manner, be associated with the asymptotic final content of a virtual downstream filter cavity. This experimentally inspired construction of input and output wave-packet modes leads to a simple density-matrix formalism with time-dependent couplings of the scatterer to two discrete cavity modes [see Fig. 1(b)]. The problem thus takes a quite different form than suggested by Eq. (1).

The cascaded master equation describes dynamics where the upstream cavity (representing the incident pulse) leaks all its quanta, while the downstream cavity gradually acquires the entire fraction of the output field that populates the specified output pulse mode function.

In this article we transform this master equation to the interaction picture with respect to the linear transfer of quanta between the upstream and downstream cavities, as they would occur in the absence of the scatterer. This transformation

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breaks the cascaded nature of the master equation such that the new, initially populated and initially empty field modes both experience Jaynes-Cummings-like coupling to the scatterer [see Fig. 1(c)]. As a consequence of the limited exchange of quanta between the transformed modes and the scatterer during the finite pulse, the numerical solution of the master equation is much less demanding than in the original Schrödinger picture.

This article is structured as follows: In Sec. II the virtual cavities and cascaded system formalism are presented. In Sec. III we introduce the interaction picture and the corresponding master equation for the case of identical input and output modes and derive the simple, Jaynes-Cummings-like form of the interaction Hamiltonian. In Sec. IV we extend the interaction picture to describe different input and output pulses due, for example, to the effect of dispersion caused by an empty cavity. As a numerical example we analyze the creation of squeezed states and Schrödinger-cat states by the scattering of pulses on a cavity with a Kerr nonlinearity.

II. QUANTUM INTERACTIONS WITH A LIGHT PULSE: A VIRTUAL CAVITY APPROACH

Consider a local quantum system described by the Hamiltonian $\hat{H}_0(t)$ and Lindblad dissipation terms $\hat{L}_i$, with $i = 1, \ldots, n$. Its interaction with an incident quantized radiation field is governed by $V = i\sqrt{\gamma}\hat{c}^{\dagger}\hat{b}_{in}(t) - \hat{c}\hat{b}_{in}(t)$, where $\hat{c}^{\dagger}$ is the lowering (raising) operator of the system accompanying absorption (emission) of a quantum of radiation. This interaction involves the input field continuum operators $\hat{b}_{in}(t)$, which can be viewed as the Fourier transform of the frequency eigenmode annihilation operators. If the incident radiation is restricted to a single pulse, described by a square-integrable mode function $u(t)$, the interaction can be accounted for by an effective cascaded-system master equation [15]. Here the quantum pulse is described as if it leaks from an upstream virtual cavity with a coherent out-coupling strength $g_a(t)$, where

$$\begin{equation}
g_a(t) = \frac{u^*(t)}{\sqrt{1 - \int_0^t dt' |u(t')|^2}}.
\end{equation}$$

Likewise, the component of outgoing radiation that eventually occupies an arbitrary wave-packet mode $v(t)$ can be picked up by a virtual downstream filter cavity with a coherent in-coupling strength $g_s(t)$,

$$\begin{equation}
g_s(t) = -\frac{v^*(t)}{\sqrt{\int_0^t dt' |v(t')|^2}}.
\end{equation}$$

All other output modes are reflected by the $v$ cavity, and they are, in our formalism, described as Markovian loss. According to input-output theory [13], the output field after reflection by the $v$ cavity is thus governed by the annihilation operator

$$\begin{equation}
\hat{b}_{out}(t) = \hat{b}_{in}(t) + g_a^*(t)\hat{a}_u + \sqrt{\gamma}\hat{c} + g_s^*(t)\hat{a}_v,
\end{equation}$$

which represents the interference between the amplitudes of the incident vacuum field operator and fields emitted by the scatterer and the two cavities. Detection of an outgoing photon is thus accompanied by the action of a single quantum jump operator, equivalent to the appearance of a Lindblad damping term,

$$\begin{equation}
\hat{L}_0(t) = \sqrt{\gamma}\hat{c} + g_a^*(t)\hat{a}_u + g_s^*(t)\hat{a}_v,
\end{equation}$$

in the master equation for the joint state of the quantum scatterer and the two virtual cavities,

$$\begin{equation}
d\rho/dt = \frac{i}{\hbar}[\hat{H}(t), \rho] + \sum_{i=0}^n D[\hat{L}_i] \rho.
\end{equation}$$

Here the Hamiltonian is formed both by the system part $\hat{H}_0(t)$ and by the coherent interactions between the different components,

$$\begin{equation}
\hat{H}(t) = \hat{H}_0(t) + \hat{L}_1(t) + \frac{i}{2}\sqrt{\gamma}g_a(t)\hat{a}_u \hat{c} + g_s(t)\hat{a}_v \hat{c}^{\dagger} - \text{H.c.},
\end{equation}$$

and the master-equation terms $D[\hat{L}_i] \rho = -\frac{1}{2}[\hat{L}_i, \hat{L}_i^\dagger], \hat{L}_i^\dagger \hat{L}_i$ apply both for the outgoing field loss, represented by $\hat{L}_0$ in (5), and for the damping terms $\{\hat{L}_{ai} = a_{1i} \ldots a_{ni}\}$ acting on the quantum scatterer. Note that both the Hamiltonian and the $\hat{L}_0^\dagger \hat{L}_0$ product terms in the master equation give rise to cross terms between the field and scatterer operators $\hat{a}_{ui}$ and $\hat{c}^{(i)}$, and $\hat{c}^{(i)}$, and the Hamiltonian and the dissipative terms conspire to cancel all contributions that cause excitation transfer towards the upstream cavity (see Ref. [16] for further elaboration). This is a key property of the cascaded system master equation, built into its formal derivation [13,14].

Reflection by a one-sided cavity is treated in a similar manner, and it is also possible to treat multioutput situations where a scatterer causes both transmission and reflection [16].
If retardation effects can be neglected, more complex networks can also be treated [17], while non-Markovian effects of retardation and time delays engage the multimode character of the field [18].

### III. TIME-DEPENDENT MODES: AN INTERACTION PICTURE

The introduction of separate upstream and downstream virtual cavity modes is the most crucial element that distinguishes our analysis of the interaction with a traveling pulse from the interaction with a (single) real cavity mode of radiation. In particular, even if we assume the same outgoing wave packet as we inject into the system, it enforces a two-mode rather than a single-mode treatment of the problem, and the bare propagation of a light pulse without a scatterer amounts to the transfer of the quantum state between these modes.

In the interaction picture, the bare system Hamiltonian \( \hat{H}_i(t) \) and damping terms \( \hat{L}_i \) and \( \hat{L}_o \), with \( i > 0 \), are not affected by the transformation to the interaction picture, which concerns only the pulse mode operators. We thus obtain

\[
\hat{H}_i = \hat{H}_S + \frac{i \sqrt{\gamma}}{2} [g_a(t)M_{11}(t) - g_c(t)M_{21}(t)]\hat{a}_{e,0}(0)^\dagger \hat{c} + [g_c^*(t)M_{22}(t) - g_a^*(t)M_{12}(t)]\hat{c}^\dagger \hat{a}_{e,0}(0) - \text{H.c.} \quad (11)
\]

and

\[
\hat{L}_{o,i} = \sqrt{\gamma}\hat{c} + g_a^*(t)[M_{11}(t)\hat{a}_{e,0}(0) + M_{12}(t)\hat{a}_{c,0}(0)] + g_c^*(t)[M_{21}(t)\hat{a}_{e,0}(0) + M_{22}(t)\hat{a}_{c,0}(0)]. \quad (12)
\]

In the remaining text, we shall omit the index \( I \) and assume that field annihilation operators with a time argument \( t \) refer to the interaction picture.

#### A. Identical incoming and outgoing modes

We examine first the case where the outgoing field mode is equal to the incoming one, \( \nu(t) = \upsilon(t) \), which suggests introducing

\[
\sin^2 \theta(t) = \int_0^t \left| \upsilon(t') \right|^2 dt' \quad (13)
\]

with \( \left| \upsilon(t) \right|^2 = d \sin^2 \theta/dt = 2 \sin \theta \cos \theta \theta/dt \), since this allows us to rewrite

\[
g_a^*(t)g_c(t) = -\left| \upsilon(t) \right|^2 \sin \theta(t) \cos \theta(t) = -2 \frac{d\theta}{dt}. \quad (14)
\]

We thus obtain the simple coefficient matrix

\[
M(t) = \begin{pmatrix}
\cos \theta(t) & -\sin \theta(t) \\
\sin \theta(t) & \cos \theta(t)
\end{pmatrix},
\]

and we can rewrite the interaction-picture Hamiltonian (11) as

\[
\hat{H}_i = \hat{H}_S + i \sqrt{\gamma} \left[ \upsilon(t) \hat{a}_{c,0}(0)^\dagger \hat{c} - \upsilon(t)^\dagger \hat{a}_{c,0}(0) \right] + i \frac{\sqrt{\gamma}}{2} \left[ \cos \theta(t) + \tan \theta(t) \right] \left[ \upsilon(t)^\dagger \hat{a}_{c,0}(0)^\dagger \hat{c} - \upsilon(t) \hat{a}_{c,0}(0) \right]. \quad (15)
\]

and the Lindblad operator (12) of field loss as

\[
\hat{L}_{o,0} = \sqrt{\gamma} \left[ \cos \theta(t) + \tan \theta(t) \right] \upsilon(t) \hat{a}_{c,0}(0). \quad (16)
\]

Remarkably, the first interaction term in Eq. (15) is exactly the Jaynes-Cummings Hamiltonian (1) that one might have anticipated by simple arguments, but the interaction with the traveling pulse is supplemented by the coupling to a second mode of the field, and the dissipation of the system (16) also includes this ancillary mode.

A visual representation of the transformation to the interaction picture is presented in Fig. 1, where Fig. 1(a) shows the pulse incident upon the scatterer and the possible emission of a multimode field. Figure 1(b) illustrates the elements of our cascaded master equation in the Schrödinger picture, which employs the release of the incident and recapture of an outgoing pulse mode by virtual cavities with time-dependent coupling strengths \( \sqrt{\gamma} g_a(t) \) and \( \sqrt{\gamma} g_c(t) \) to the scatterer. Figure 1(c) shows the coupling to the time-evolved modes in the interaction picture, where the leftmost cavity represents...
the propagating pulse mode and the rightmost cavity represents an orthogonal superposition of the two modes shown in Fig. 1(b).

In the absence of the coupling to the central system (setting $\gamma = 0$) the Hamiltonian vanishes in the interaction picture, and nothing happens to the quantum content of the time-dependent left and right modes in the interaction picture in Fig. 1(c). When $\gamma \neq 0$, during the finite interaction time, there may be only a limited exchange of quanta between the scatterer and the time-dependent modes, which is in stark contrast to the complete emptying and partial filling of the $u$- and $v$-cavity modes in the Schrödinger picture represented by Fig. 1(b). Thus, only a narrow range of Fock states may be populated at any time, which allows a truncation of the Hilbert space, as discussed in Sec. III C.

B. Direction of radiation propagation in the interaction picture

In our cascaded-open-system approach [15,16], it is assumed that the light travels—and the central system emits—only in the direction from the initially populated $u$ cavity towards the output mode $v$ cavity.

In the Schrödinger picture, terms in the Hamiltonian and Lindblad damping terms thus exactly cancel upstream propagation of excitations from the scatterer and the $v$ cavity. The same formal cancellation, however, is less obvious in the interaction picture, where an initial excitation of the $v$ cavity, indeed, decays instantly, and to derive this result, we consider the mean amplitude $\beta$ in the $v$ cavity mode, which decays at the rate

$$\frac{d\beta}{dt} = -\frac{|g_v|^2}{2}\beta.$$  \hspace{1cm}(17)

Solving this equation by quadrature, we obtain

$$\int_{\beta_1}^{\beta_2} \frac{d\beta}{\sqrt{-\frac{|g_v|^2}{2}\beta}} = -\frac{1}{2} \int_0^t dt' |g_v(t')|^2.$$  \hspace{1cm}(18)

Now, we note that $|g_v(t')|^2$ equals $|\nu(t')|^2$ divided by $F(t) = \int_0^t dt' |\nu(t')|^2$, and we define $f(t) = |\nu(t)|^2 = dF/dt$ to obtain

$$\int_{\beta_1}^{\beta_2} \frac{d\beta}{\sqrt{-\frac{1}{2} F(t)}} = \ln \frac{\beta_2}{\beta_1} = \frac{1}{2} \int_0^t dt' \frac{f(t')}{F(t')}.$$  \hspace{1cm}(19)

The last term is readily evaluated and yields

$$\ln \frac{\beta(t)}{\beta(0)} = -\frac{1}{2} \ln \left( \frac{F(t)}{F(0)} \right) \Rightarrow \beta(t) = \beta(0) \sqrt{\frac{F(0)}{F(t)}}.$$  \hspace{1cm}(20)

Since $F(t)$ vanishes for $t = 0$ and is finite for any $t > 0$, any initial excitation in the $v$ cavity, indeed, decays instantly, and there is no upstream propagation.

C. Rabi oscillations with a quantum pulse

To show the advantages of the interaction-picture transformation, we consider a simple example. Atoms excited by coherent monochromatic light show sinusoidal Rabi oscillations of their excited-state population, $P_{22}(t) = \sin^2(\Omega t/2)$. In cavity QED, similar oscillations occur for an atom interacting with a quantized field [19], and we can now also investigate how an atom behaves when exposed to a resonant pulse $u(t)$ with quantum light.

We want to address the excitation of both the atom and the traveling pulse, and we consider the output field mode $v(t) = u(t)$, described by the Hamiltonian (15) and the Lindblad operator (16). We use a normalized Gaussian pulse

$$u(t) = \frac{1}{\sqrt{2\pi t_p^2}} \exp \left( -\frac{(t - t_p)^2}{2t_p^2} \right).$$  \hspace{1cm}(21)

with $t_p = 4\gamma^{-1}$ and $\tau = \gamma^{-1}$. With this choice Eq. (13) yields

$$\theta(t) = \sin^{-1} \left\{ \frac{1}{2} \left( \text{erf} \left( \frac{t - t_p}{\tau} \right) + \text{erf} \left( \frac{t_p}{\tau} \right) \right) \right\},$$  \hspace{1cm}(22)

where erf denotes the error function.

Figure 2 shows the time evolution, starting with a ground-state atom and an incident pulse prepared in a Fock state with $n = 20$ photons. It is clear from the behavior of $\langle \hat{c}^\dagger \hat{c} \rangle$ in the bottom panel that the two-level system undergoes Rabi oscillations before the interaction is over and the system relaxes to the ground state. The purpose of this example is to show the drastic numerical advantage of the interaction picture. In the Schrödinger-picture calculation, the $u$-cavity mode is completely emptied of its initial excitation, which is partially retrieved by the $v$-cavity mode, and the quantum state of the field hence explores the tensor product of two 21-dimensional Hilbert spaces. In the interaction picture, the time-evolving oscillator modes explore only a few different Fock states. The top panel in Fig. 2 thus shows how the time-dependent $u$ mode
loses just 1–2 photons while the $v$ mode acquires less than unit excitation. Only the 14–20-photon Fock states are reasonably populated in the $u$ mode, while the same is true for the 0–4-photon Fock states in the $v$ mode. Truncation of the Hilbert space to these relevant number-state components thus easily yields a 20-fold reduction in dimension and a corresponding 400-fold reduction in the number of density-matrix elements, a factor that would be even larger for larger incident photon numbers and results in much shorter computing times. In this example, the computation time was reduced from 14.5 min using the Schrödinger-picture calculation to merely 10 seconds using the interaction-picture approach and using the same hardware.

IV. SCATTERING ON A CAVITY, THREE-MODE INTERACTION PICTURE

In the previous section, we transformed the master equation to an interaction picture with respect to the coupling term between virtual $u$ and $v$ input and output cavity modes. If the scatterer is itself an optical cavity or a system contained inside a cavity, it is possible to apply an interaction picture with respect to the coupling terms among all these cavities. For a linear coupling of three cavities modes described by the Hamiltonian $\hat{H}_0$,

$$\hat{H}_0(t) = \frac{i}{2}[\sqrt{\gamma}g_u(t)\hat{a}_u^{\dagger}\hat{c} + \sqrt{\gamma}g_v^*(t)\hat{c}^{\dagger}\hat{a}_v + g_u(t)g_v^*(t)\hat{c}^{\dagger}\hat{a}_u - \text{H.c.}],$$

(23)

we expand the field operators in the interaction picture as a vector,

$$\begin{pmatrix} \hat{a}_u(t) \\ \hat{c}(t) \\ \hat{a}_v(t) \end{pmatrix} = \begin{pmatrix} \hat{a}_u(0) \\ \hat{c}(0) \\ \hat{a}_v(0) \end{pmatrix} e^{\text{M}(t)},$$

(24)

where the operators at $t = 0$ refer to the Schrödinger picture and the $3 \times 3$ coefficient matrix $\text{M}(t)$ equals the identity matrix at $t = 0$ and solves the equation

$$\frac{d}{dt} \text{M}(t) = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{\gamma}g_u(t) & g_u(t)g_v^*(t) \\ -\sqrt{\gamma}g_u^*(t) & 0 & \sqrt{\gamma}g_v^*(t) \\ -g_u^*(t)g_v(t) & -\sqrt{\gamma}g_v(t) & 0 \end{pmatrix} \text{M}(t).$$

(25)

This linear system of equations is readily solved and yields the transformation of the cascaded master equation to the interaction picture.

A. Three-mode interaction picture with an empty cavity

We assume three cavities (two virtual, one real) with the ladder operators $\hat{a}_u$, $\hat{c}$, and $\hat{a}_v$ and a Gaussian $u$ pulse as specified in Sec. III C. In the absence of any further interactions, the transformation to the interaction picture handles all the dynamics, and no quanta ever leave the interaction picture $\hat{a}_u$ mode, while the interaction picture $\hat{c}$ and $\hat{a}_v$ are “dark” modes that never become populated during the interaction, again allowing efficient numerical calculations of otherwise large Hilbert spaces.

Note that the output mode after scattering on a single mode cavity with linewidth $\gamma$ and no internal losses is given in frequency space by $v(\omega) = r(\omega)u(\omega)$ [15], where

$$r(\omega) = \frac{i(\omega - \omega_c) + \frac{\gamma}{2}}{i(\omega - \omega_c) - \frac{\gamma}{2}}.$$  

(26)

At late times, the interaction-picture operator $\hat{a}_u(t)$ accounts for all the incident photons which now populate the pulse mode transformed according to (26).

B. Three-mode interaction picture with a Kerr nonlinear cavity: Squeezing of a light pulse

As a nontrivial example of the application of the three-mode interaction picture, we introduce a nonlinear Kerr effect in the $c$ cavity,

$$H_c(t) = K:\text{c}^\dagger(t)\text{c}(t)^2,$$

(27)

where $K$ is a constant. The nonlinear Kerr interaction $\text{c}^\dagger(t)\text{c}(t)^2$ [20] acts on a coherent state by phase shifting each Fock state by an amount proportional to $n^2$. This effectively stretches the complex phase-space distribution of the state and transforms the coherent state into a squeezed state. This picture is readily applied to an intracavity field, but a long pulse incident on a cavity may at no time have all its photons inside the cavity and thus not be subject to the full nonlinear interaction, while a short pulse is spectrally broad and may reflect without even entering the cavity.

Our theory takes the spatial propagation properly into account, and our interaction picture restricts the evolution of the quantum states of the field to the number states initially occupied. We have found that using $K = 0.02\gamma$, this effect transforms a coherent state with $\alpha = 4$ and the pulse parameters defined in Sec. III C into a squeezed state at the end of the interaction. States with larger coherent-state amplitudes become more squeezed, while larger values of the Kerr interaction strength $K$ may distort the mode shape and cause deterioration of the single-mode character and hence loss of squeezing.

We quantify the degree of squeezing by minimizing the variance of field quadratures rotated by the phase angle $\phi$

$$\text{Var}(\phi) = \left<[\Delta(\cos\phi\hat{x} - \sin\phi\hat{p})]^2\right>.$$  

(28)

For the case of an interaction with $K = 0.02\gamma$ and $\alpha = 4$, the minimum uncertainty arises for the angle $\phi = 0.52$ rad, where the variance is $\left<[\Delta(\cos\phi\hat{x} - \sin\phi\hat{p})]^2\right> \approx 0.245$, which is squeezed compared to the coherent state value of 0.5. Figure 3 shows the Wigner function indeed resembles that of a squeezed state along an axis rotated clockwise by $\phi = 0.52$ rad $\approx 30^\circ$. In Fig. 4, the variance is shown as a function of $\phi$ for different values of the Kerr interaction strength, and we observe that the stronger Kerr effect leads to a smaller degree of (single-mode) squeezing.

C. Three-mode interaction picture with a Kerr nonlinear cavity: Turning a coherent pulse into a Schrödinger-cat state

The value of $n^2$ can be written as $4m$ and $4m + 1$ for even and odd $n$, respectively, where $m$ is an integer. For a sufficiently strong Kerr interaction, the accumulated different phase factors attain the values $i$ on odd Fock states and 1
on even Fock states and hence transfer a coherent state in a single-mode cavity into a Schrödinger-cat state [21].

In our setup, only the component of the pulse which is inside the nonlinear e-mode cavity experiences the Kerr interaction, which must be strong enough to yield the discrete phase differences to form the cat state yet weak enough that it does not ruin the single-mode character of the pulse. This poses a useful application of the interaction-picture calculation, which focuses on the field content of the traveling pulse and takes the linear dispersion of the pulse shape by translation, which allows one to consider the field content of the traveling pulse. It does not ruin the single-mode character of the pulse. This phase difference between even and odd n should be \( \pi/2 \), and hence, \( K \) must satisfy

\[
K = \frac{\pi}{2} \left( \int_0^T \langle M_{21}(t')^4 \rangle^{-1} \right),
\]

where \( T \) is the duration of the pulse. Assuming an incident Gaussian wave packet with the same parameters as in Sec. III C, the numerical evaluation of the integral over \( M_{21}(t) \) yields a required value of \( K \simeq 1.3 \) to form the Schrödinger-cat state. When we solve the three-mode problem numerically in the interaction picture with a sufficiently strong interaction, however, we find a significant loss of population of the \( \hat{a}_u \) pulse modes (see Fig. 5). Figure 6 shows the Wigner function of the quantum state of the traveling pulse mode at different times during the interaction. It is evident that the loss of amplitude also prevents the formation of the cat state.

It seems that the large value needed for \( K \) is not compatible with the preservation of the single-mode character of the quantum field. Instead of looking at only a single interaction, we therefore propose to let the light pulse pass the...
nonlinear cavity several times (or pass through a sequence of several such cavities) and thus accumulate the nonlinear phase shift from repeated weak Kerr interactions with a small value of $K$. To ensure that the input pulse has the same initial Gaussian shape at each interaction, one may need to reshape the pulse, e.g., by sum-frequency generation [22]. The weak Kerr interaction at each passage transforms the quantum state of the field only slightly, while maintaining its single-mode character, and after sufficiently many passages, the light pulse attains the Schrödinger-cat quantum state.

To create the cat by $N$ transmission steps, Eq. (31) can now be relaxed to

$$N \cdot K \int_0^T dt' |M_{21}(t')|^4 = \frac{\pi}{2}. \quad (32)$$

Using an interaction strength $K = 0.01\gamma$, a Gaussian wave packet with the same parameters as in Sec. III C, and the numerical evaluation of the integral $\int_0^T dt' |M_{21}(t')|^4 \approx 1.180\gamma^{-1}$, we estimate the number of interactions required to create a cat state from an input coherent state as

$$N = \frac{\pi}{2 \times 0.01\gamma \times 1.180\gamma^{-1}} \approx 133. \quad (33)$$

This is in good agreement with the full numerical calculation, presented in Fig. 7, which shows the development of different nonclassical states in the process to reach the final Yurke-Stoler cat state.

V. CONCLUSION AND OUTLOOK

An effective cascaded open quantum system approach can be used to describe the initial preparation, the interaction, and the final analysis of how a pulse of quantum radiation interacts with, e.g., an atom. The propagation of quantum pulses in free space and among cavities can be solved exactly in the Heisenberg picture by a linear transformation of the mode operators. This constitutes a good starting point for an interaction-picture treatment of the cascaded master-equation theory. The interaction between the propagating light and localized scatterers thus takes the conventional Jaynes-Cummings form of a time-dependent single-mode interaction, while it also provides the interaction with an auxiliary oscillator. These terms together ensure the exact analysis assuming only the standard Born-Markov treatment of couplings to the continuum of free-space (or guided) modes.

Our theory is simple to implement directly with standard master-equation solvers, and since the main part of the dynamics is already taken care of by the transformation to the interaction picture, it considerably simplifies the numerical calculations. We note that our analysis may equally apply to other bosonic fields (light, microwaves, matter waves, sound and spin waves). For a wide range of wave phenomena, (linear) dispersion plays an important role and may be treated by a suitable interaction picture along the same lines as our treatment of the transmission through a single linear cavity. Our method thus permits dealing separately with the effects of propagation and of quantum interactions.

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