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Generalized Bernstein Functions

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Abstract

A class of functions called generalized Bernstein functions is studied. The fundamental properties of this class are given and its relation to generalized Stieltjes functions via the Laplace transform is investigated. The subclass of generalized Thorin-Bernstein functions is characterized in different ways. Examples of generalized Bernstein functions include incomplete gamma functions, Lerch’s transcendent and some hypergeometric functions.

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1 Introduction

In this paper we investigate the so-called generalized Bernstein functions of order $\lambda$. These classes of functions contain the class of Bernstein functions. A Bernstein function is by definition a non-negative function $g$ on $(0, \infty)$ for which $g'$ is completely monotonic. It is known that $g$ is a Bernstein function if and only if

$$g(x) = \alpha x + \beta + \int_0^\infty (1 - e^{-xt}) d\nu(t),$$

where $\alpha$ and $\beta$ are non-negative numbers, and $\nu$, called the Lévy measure, is a positive measure on $(0, \infty)$ satisfying $\int_0^1 t d\nu(t) < \infty$ and $\int_1^\infty d\nu(t) < \infty$. We refer to [11, Chapter 1] for information on these classes of functions.

Definition 1.1 A non-negative $C^\infty$-function $g$ on $(0, \infty)$ called a generalized Bernstein function of order $\lambda > 0$ if $x^{1-\lambda} g'(x)$ is a completely monotonic function. The class of these functions is denoted by $\mathcal{B}_\lambda$. 
Any generalized Bernstein function admits an integral representation of the form in the proposition below.

**Proposition 1.2** Let \( g \) be a non-negative \( C^\infty \)-function on \((0, \infty)\). Then \( g \) belongs to \( B_\lambda \) if and only if
\[
 g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^{xt} e^{-u} u^{\lambda-1} du \frac{d\mu(t)}{t^\lambda}, \quad x > 0,
\]
for non-negative constants \( \alpha \) and \( \beta \) and a positive measure \( \mu \) on \((0, \infty)\) making the integral converge.

The proof of this proposition can be found in [10]. Let us mention that the triple \((\alpha, \beta, \mu)\) is uniquely determined by \( f \) and that the above integral converges for any \( x > 0 \) exactly when
\[
 \int_0^\infty \frac{d\mu(t)}{(1+t)^\lambda} < \infty.
\]

The last assertion follows using that (for fixed \( x > 0 \))
\[
 \int_0^{xt} e^{-u} u^{\lambda-1} du
\]
behave like \( t^\lambda \) as \( t \to 0 \) and that it remains bounded as \( t \to \infty \).

When \( \lambda = 1 \), we have
\[
 \int_0^{xt} e^{-u} u^{\lambda-1} du = 1 - e^{-xt},
\]
and (1) reduces to the well-known integral representation of a Bernstein function with the corresponding Lévy measure being \( d\mu(t)/t \).

The representation in Proposition 1.2 can be written as
\[
 g(x) = \alpha x^\lambda + \beta + \int_0^\infty \gamma(\lambda, xt) \frac{d\mu(t)}{t^\lambda},
\]
where \( \gamma(a, z) \) is the incomplete gamma function defined by
\[
 \gamma(a, z) = \int_0^z t^{a-1} e^{-t} dt.
\]

The motivation for our investigations in this work comes from applications to special functions, see Examples 3.4, 3.11 and 4.3 about Lerch’s transcendent, the Lomax distribution and the hypergeometric function \( _2F_1 \).
Generalized Bernstein functions of order $\lambda$ are closely related to generalized Stieltjes functions of order $\lambda$, see Theorem 3.1. For fixed $\lambda > 0$ a function $f : (0, \infty) \to \mathbb{R}$ is called a generalized Stieltjes function of order $\lambda$ if

$$f(x) = \int_0^\infty \frac{d\mu(t)}{(x+t)^\lambda} + c,$$

where $\mu$ is a positive measure on $[0, \infty)$ making the integral converge for $x > 0$ and $c \geq 0$. The set of generalized Stieltjes functions of order $\lambda$ is denoted $S_{\lambda}$.

In [12] Sokal obtained a characterization of the class $S_{\lambda}$ in terms for positivity of a sequence of operators. A characterization without mentioning this sequence of operators was found in [10]. It reads: $f \in S_{\lambda}$ if and only if the function $c_k^\lambda(f)$ is completely monotonic for all $k = 0, 1, \ldots$, where

$$c_k^\lambda(f)(x) = x^{1-\lambda}(x^{\lambda-1}f(x))^{(k)}.$$ (5)

For additional information on these classes see e.g. [9].

We remark that $f$ is a generalized Stieltjes function of order $\lambda$ of the form (4) if and only if

$$f(x) = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt} t^{\lambda-1} \kappa(t) \, dt + c, \quad x > 0,$$ (6)

where $\kappa$ is a completely monotonic function. In the affirmative case, $\kappa(t) = \int_0^\infty e^{-ts} \, d\mu(s)$. See [9, Lemma 2.1]. This characterization shows also that $S_{\lambda_1} \subset S_{\lambda_2}$ for $\lambda_1 < \lambda_2$.

The paper is organized as follows. In the next section we give some basic results about generalized Bernstein functions. Section 3 contains our main results, followed in Section 5 by some additional remarks. In Section 4 we introduce and investigate generalized Thorin-Bernstein functions.

## 2 Basic results

An immediate consequence of Proposition 1.2 is the following.

**Corollary 2.1** If $g \in B_{\lambda}$ then $g(x)/x^\lambda$ is completely monotonic.

**Proof.** Indeed, we have

$$\frac{g(x)}{x^\lambda} = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty \int_0^t e^{-xu} u^{\lambda-1} \, du \, \frac{d\mu(t)}{t^\lambda}, \quad x > 0,$$
and this completes the proof. □

In the next proposition we gather some simple properties of generalized Bernstein functions and their integral representation.

**Proposition 2.2** Let \( g \in \mathcal{B}_{\lambda} \) with the representation (1). Then

(i) \( \alpha = \lim_{x \to \infty} g(x)/x^{\lambda} \),

(ii) \( \beta = \lim_{x \to 0} g(x) \),

(iii) \( g \) is bounded if and only if \( \alpha = 0 \) and \( \int_0^\infty d\mu(t)/t^\lambda < \infty \). In the affirmative case we have

\[
\lim_{x \to \infty} g(x) = \beta + \Gamma(\lambda) \int_0^\infty \frac{d\mu(t)}{t^\lambda}.
\]

**Proof.** If \( g \) has the representation (1) then it easily follows that

\[
x^{1-\lambda}g'(x) = \lambda \alpha + \int_0^\infty e^{-xt} d\mu(t) \to \lambda \alpha
\]
as \( x \to \infty \). But then \( g(x)/x^{\lambda} \to \alpha \) because of l'Hospital’s rule. Thus (i) is proved. Since the integrand

\[
\int_0^{xt} e^{-u} u^{\lambda-1} du
\]
is increasing as a function of \( x \), (ii) follows at once by dominated convergence. If \( g \) is bounded then clearly \( \alpha = 0 \) and then one may conclude by the monotone convergence theorem that \( \int_0^\infty d\mu(t)/t^\lambda < \infty \) and that the asserted relation holds. The converse in (iii) is trivial. □

**Proposition 2.3** If \( \lambda_1 < \lambda_2 \) then \( \mathcal{B}_{\lambda_1} \subset \mathcal{B}_{\lambda_2} \).

**Proof.** If \( x^{1-\lambda_1}g'(x) \) is completely monotonic then

\[
x^{1-\lambda_2}g'(x) = x^{\lambda_1-\lambda_2} x^{1-\lambda_1} g'(x)
\]
is also completely monotonic. □

Concerning pointwise convergence of a sequence of functions from \( \mathcal{B}_{\lambda} \) we notice the following result.
Proposition 2.4 Suppose that \( \{ f_n \} \) is a sequence from \( B_\lambda \) and that \( f_n \) converges pointwise to \( f \). Then \( f \in B_\lambda \) and \( f_n^{(k)} \to f^{(k)} \) locally uniformly in \( (0, \infty) \) for any \( k \geq 0 \). Furthermore, if \( f_n \) corresponds to the triple \( (\alpha_n, \beta_n, \mu_n) \) and \( f \) corresponds to \( (\alpha, \beta, \mu) \) then \( \mu_n \to \mu \) vaguely in \( (0, \infty) \), and \( \lambda \alpha_n \epsilon_0 + \mu_n \to \lambda \alpha \epsilon_0 + \mu \) vaguely in \([0, \infty)\).

Remark 2.5 It should be remarked that we cannot assert that \( \alpha_n \to \alpha \) or \( \beta_n \to \beta \); see [11, Example 3.9].

Proof. Since \( f_n(x)/x^\lambda \) is a completely monotonic function by Corollary 2.1 and \( f_n(x)/x^\lambda \to f(x)/x^\lambda \) for all \( x > 0 \) we obtain that \( f(x)/x^\lambda \) is completely monotonic. We also know that

\[
\frac{f_n'(x)}{x^\lambda} - \lambda \frac{f_n(x)}{x^{\lambda+1}} = \left( \frac{f_n(x)}{x^\lambda} \right)' - n \left( \frac{f(x)}{x^\lambda} \right)' = \frac{f'(x)}{x^\lambda} - \lambda \frac{f(x)}{x^{\lambda+1}}.
\]

It follows from this that \( f_n'(x)/x^{\lambda-1} \to f'(x)/x^{\lambda-1} \), and since \( f_n'(x)/x^{\lambda-1} \) is completely monotonic, so is \( f'(x)/x^{\lambda-1} \). Therefore, \( f \in B_\lambda \).

Furthermore, \( \partial_x^k (f_n(x)/x^\lambda) \) converges locally uniformly to \( \partial_x^k (f(x)/x^\lambda) \) and from this it easily follows (using an inductive argument) that \( \partial_x^k f_n \to \partial_x^k f \) locally uniformly.

Finally, since \( f_n'(x)/x^{\lambda-1} \to f'(x)/x^{\lambda-1} \) it follows that

\[
\int_0^\infty e^{-xt} d(\lambda \alpha_n \epsilon_0 + \mu_n)(t) \to \int_0^\infty e^{-xt} d(\lambda \alpha \epsilon_0 + \mu)(t),
\]

and the assertions on vague convergence are consequences of this. \( \square \)

3 Main results

We can now state our main results.

Theorem 3.1 The following holds:

(a) If \( g \) belongs to the class \( B_\lambda \) then \( xL(g)(x) \) belongs to \( S_\lambda \).

(b) If \( f \) belongs to \( S_\lambda \) then there exists a function \( g \in B_\lambda \) such that \( f(x)/x = L(g)(x) \).

Proof. Suppose that \( g \) belongs to \( B_\lambda \) and let

\[
f(x) = \int_0^\infty e^{-xt} g(t) \, dt.
\]
By integration by parts it is obtained that
\[ f(x) = \frac{g(0)}{x} + \frac{1}{x} \int_0^\infty e^{-xt} g'(t) \, dt. \]

Therefore,
\[ xf(x) = g(0) + \int_0^\infty e^{-xt} t^{\lambda-1} \left( t^{1-\lambda} g'(t) \right) \, dt, \]
and this shows that \( xf(x) \) belongs to \( S_\lambda \). This verifies (a).

To verify (b), suppose that \( f : (0, \infty) \to [0, \infty) \) belongs to \( S_\lambda \). Then
\[ f(x) = \alpha + \int_0^\infty e^{-xt} t^{\lambda-1} h(t) \, dt, \]
for some \( \alpha \geq 0 \) and some completely monotonic function \( h \). (Notice that \( t^{\lambda-1} h(t) \) is integrable at the origin.) Then
\[
\frac{f(x)}{x} = \alpha \mathcal{L}(dt)(x) + \mathcal{L}(dt)(x)(t^{\lambda-1} h(t))(x)
\]
\[ = \mathcal{L}(\alpha dt + dt \ast (t^{\lambda-1} h(t)))(x). \]

Defining \( g(t) = \alpha + (ds \ast (s^{\lambda-1} h(s)))(t) \) we have
\[ g(t) = \alpha + \int_0^t s^{\lambda-1} h(s) \, ds \]
and it follows that \( g \in B_\lambda \). \( \square \)

We remark that the representing measure of the completely monotonic function \( h \) in the proof above is the measure representing \( f \) as a generalized Stieltjes function of order \( \lambda \).

**Corollary 3.2** We have
\[
\bigcap_{\lambda > 0} B_\lambda = \{ \text{the constant functions with values in } [0, \infty) \}. 
\]

**Proof.** Suppose that \( h \in \bigcap_{\lambda > 0} B_\lambda \). Then according to (a) in Theorem 3.1 \( x\mathcal{L}(h)(x) \in S_\lambda \) for all \( \lambda > 0 \). However, \( \bigcap_{\lambda > 0} S_\lambda \) is exactly the set of constant and non-negative functions on \( (0, \infty) \) (see [7]). This gives
\[ \mathcal{L}(h)(x) = \frac{c}{x} = \mathcal{L}(c)(x), \]
so that \( h(t) = c \) for \( t > 0 \) by the injectivity of the Laplace transform. \( \square \)
Theorem 3.3 Let $\mu$ be a positive measure on $(0, \infty)$ such that (2) holds and let $c \geq 0$. Then the function
\[ b(t) = \int_0^t s^{\lambda-1} L(\mu)(s) \, ds + c \] (7)
belongs to $B_{\lambda}$.

Conversely, if $b \in B_{\lambda}$ then there is a positive measure $\mu$ on $(0, \infty)$ for which (2) holds and $c \geq 0$ such that $b$ has the representation (7).

Proof. The first claim follows at once using Fubini’s theorem. A slightly different approach is to show that $b'(t) = t^{\lambda-1} L(\nu)(t)$. Conversely, if $b \in B_{\lambda}$ then according to Theorem 3.1, $xL(b)(x)$ belongs to $S_{\lambda}$ and can thus be written as
\[ xL(b)(x) = \int_0^\infty e^{-xt^{\lambda-1}\kappa(t)} \, dt + c, \]
for some completely monotonic function $\kappa$ and some $c \geq 0$. This gives us $L(b) = L(t^{\lambda-1}\kappa(t) * 1 + c)$, from which the representation of $b$ follows. \( \square \)

Examples 3.4 1) The incomplete gamma function $x \mapsto \gamma(\lambda, x)$ belongs to $B_{\lambda}$.

2) Lerch’s transcendent is defined by
\[ \Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \quad |z| < 1, \]
with natural restrictions on the parameters $s$ and $a$. Putting $a = \lambda$ and letting $s > 0$ it has analytic extension to the cut plane $\mathbb{C} \setminus [1, \infty)$ and
\[ \Phi(z, s, \lambda) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} \, dt, \quad |z| < 1. \]

(See e.g. [6, 25.14.5].) Elementary computations show that (for $|z| < 1$)
\[ \Phi(-z, s, \lambda) = \frac{1}{\Gamma(s)} \int_0^1 \frac{(\log(1/u))^{s-1} u^{\lambda-1}}{1 + zu} \, du \] (8)\[ = \frac{1}{\Gamma(s)} \int_1^\infty \frac{(\log(u))^{s-1} u^{-\lambda}}{u + z} \, du. \] (9)

Relation (9) shows that $\Phi(1-x, s, \lambda)$ can be extended to a Stieltjes function.

When $s = 1$, relation (8) yields
\[ \Phi(-x, s, \lambda) = x^{-\lambda} \int_0^x \frac{u^{\lambda-1}}{1 + u} \, du, \]
showing that $x^\lambda \Phi(-x, 1, \lambda)$ is a generalized Bernstein function of order $\lambda$. This may also be seen directly from the definition of $\Phi(-x, 1, \lambda)$ as an infinite sum (when $0 \leq x < 1$). This example can be generalized. We have, see [4, 3.194(1)],

$$
\int_0^x \frac{u^{\lambda-1}}{(1+u)^\nu} du = \frac{x^\lambda}{\lambda} {}_2F_1(\nu, \lambda; 1 + \lambda; -x),
$$

showing that $x^\lambda {}_2F_1(\nu, \lambda; 1+\lambda; -x)$ (when $\nu > 0$) can be extended to function in $B_\lambda$.

Next we consider the closure, in the vague topology, of the union of the classes $B_\lambda$. For the reader’s convenience we collect a few facts about this topology on the space of Radon measures, denoted by $M_+([0, \infty))$ on $[0, \infty)$. A net of Radon measures $\{\mu_\alpha\}_{\alpha \in A}$ converges vaguely to a Radon measure $\mu$ if

$$
\int_0^\infty f(x) \, d\mu_\alpha(x) \to \int_0^\infty f(x) \, d\mu(x)
$$

for all continuous functions $f$ of compact support in $[0, \infty)$. It is known that $M_+([0, \infty))$ is metrizable and that the set of so-called molecular measures (finite combinations of point masses) is dense in $M_+([0, \infty))$ in this topology. We also remark that a sequence of probability measures $\{\mu_n\}$ converges weakly to a probability measure $\mu$ if (10) holds for any continuous and bounded function $f$.

**Proposition 3.5** Let $\mu$ be a Radon measure on $[0, \infty)$. Then there is a sequence $\{b_n\}$ of functions from $\bigcup_{\lambda>0} B_\lambda$ such that

$$
b_n'(x) \, dx \to_n \mu
$$

vaguely.

In the next lemma we show that any pointmass is the weak limit of a concrete sequence of derivatives of generalized Bernstein functions. This lemma is probably known to specialists but for the reader’s convenience we have included the proof here. The probability distribution $g_n(x) \, dx$ in the lemma is the gamma distribution with shape parameter $n$ and scale parameter $1/n$.

**Lemma 3.6** Let $g_n(x) = n^n x^{n-1} e^{-nx} / \Gamma(n)$. Then the sequence of probabilities $\{g_n(x) \, dx\}$ converges weakly to the point mass at 1.
Proof. Let \( f \) be a continuous and bounded function on \([0, \infty)\), and let \( \epsilon > 0 \). Choose \( \delta > 0 \) such that \( |f(x) - f(1)| < \epsilon \) for \( |x - 1| < \delta \), and choose an upper bound \( C > 0 \) on \(|f|\) on \([0, \infty)\). This gives
\[
\left| \int_0^\infty f(x)g_n(x) \, dx - f(1) \right| \leq 2C \int_0^{1-\delta} g_n(x) \, dx + \epsilon + 2C \int_{1+\delta}^\infty g_n(x) \, dx,
\]
and it is thus enough to verify
\[
\int_0^a g_n(x) \, dx \to n \, 0, \quad \text{for } a < 1, \quad (11)
\]
\[
\int_a^\infty g_n(x) \, dx \to n \, 0, \quad \text{for } a > 1. \quad (12)
\]
These integrals are related to the incomplete gamma functions \( \gamma(a, z) \) (see (3)) and
\[
\Gamma(a, z) = \int_z^\infty t^{a-1} e^{-t} \, dt.
\]
Indeed for \( a < 1 \) we obtain
\[
\int_0^a g_n(x) \, dx = \frac{1}{\Gamma(n)} \gamma(n, an).
\]
The asymptotic behaviour of \( \gamma(n, an) \) as \( n \to \infty \) is given by
\[
\gamma(n, an) \sim -(an)^ne^{-an}\left\{ \frac{1}{(a-1)n} - \frac{an}{(a-1)^3n^3} + \cdots \right\},
\]
(see e.g. [6, Eq. 8.11.6]), and therefore
\[
\int_0^a g_n(x) \, dx \sim \frac{(an)^ne^{-an}}{\Gamma(n)(1-a)n} = \frac{n^n}{\Gamma(n)e^n\sqrt{n}(1-a)^n}\frac{a^ne^{-an}}{\sqrt{n(1-a)}}.
\]
Stirling’s formula, and the fact that \( ae^{-a} < 1 \), making \( a^n e^{-an} = (ae^{-a})^n \) tend to zero, yields (11). The relation (12) can be verified in a similar way, using the asymptotic behaviour of \( \Gamma(n, an) \).

Proof of Proposition 3.5. As explained above it is enough to consider any measure of the form
\[
\nu = \sum_{j=1}^k \alpha_j \delta_{x_j},
\]
where \( \alpha_j \geq 0 \) and \( x_j \geq 0 \). Lemma 3.6 shows that \( v \) is the vague limit of a sequence of measures of the form

\[
\sum_{j=1}^{k} \frac{\alpha_j}{x_j} g_n(x/x_j) \, dx,
\]

with \( g_n(x) = n^n x^{n-1} e^{-nx}/\Gamma(n) \). (If \( x_j = 0 \) we use \( \epsilon_{1/\ell} \to \epsilon_0 \) vaguely and approximate \( \epsilon_{1/\ell} \).) We thus see that any Radon measure is vaguely approximated by measures of the form

\[
\sum_{j=1}^{k} \beta_j g_n(x/x_j) \, dx,
\]

where \( \beta_j > 0 \) and \( x_j > 0 \). Finally we notice that

\[
\sum_{j=1}^{k} \beta_j g_n(x/x_j) = \sum_{j=1}^{k} \frac{n^n}{\Gamma(n)} x_j^{1-n} x^{n-1} e^{-nx/x_j}
\]

and that

\[
b_n(x) = \sum_{j=1}^{k} \frac{n^n}{\Gamma(n)} x_j^{1-n} \int_{0}^{x} t^{n-1} e^{-nt/x_j} \, dt \in \bigcup_{a>0} B_a.
\]

This completes the proof. \( \square \)

It is easy to show that if \( \mu_n \to_n \mu \) vaguely for \( \mu_n, \mu \in M_+([0, \infty)) \) then \( \nu_n = \mu_n([0, x]) \, dx \) converges vaguely to \( \nu = \mu([0, x]) \, dx \). Hence we have obtained that the distribution function of any Radon measure on \([0, \infty)\) is the vague limit of functions from \( \bigcup_{\lambda>0} B_\lambda \).

In general it is known that \( \{ \mu \in M_+([0, \infty)) \mid \mu([0, \infty)) \leq A \} \) is vaguely compact. Hence we also obtain the following proposition.

**Proposition 3.7** For any given \( A > 0 \) the closure of the set

\[
\{ g \in \bigcup_{\lambda>0} B_\lambda \mid \lim_{x \to \infty} g(x) \leq A \}
\]

is vaguely compact.

Let us end the discussion about the closure of the union of generalized Bernstein functions by considering pointwise convergence. It is clear that the limit of a pointwise convergent sequence from \( \bigcup B_\lambda \) is increasing. Conversely, we notice the following proposition.
Proposition 3.8 For any increasing function $\phi : [0, \infty) \to [0, \infty)$ there is a sequence $\{b_m\}$ from $\cup B_\lambda$ such that $b_m \to \phi$ pointwise except on a countable set.

Proof. We choose an increasing sequence of non-negative simple functions $\{s_n\}$ (functions with only finitely many different values) such that $s_n(x) \to \phi(x)$ for all $x \geq 0$. For any simple function $s$ we may (arguing as in the proof of Proposition 3.5) choose $b_k \in \cup B_\lambda$ such that $|b_k(x) - s(x)| \to 0$ for all except finitely many $x \geq 0$ (the discontinuities of $s$). Thus there is a sequence $\{b_m\}$ from $\cup B_\lambda$ such that $b_m \to \phi$ pointwise except on a countable set. □

Next some subclasses of generalized Bernstein functions are studied. We begin with a characterization of the bounded generalized Bernstein functions, based on Proposition 2.2.

Proposition 3.9 The following are equivalent for $f : (0, \infty) \to [0, \infty)$:

(i) $f \in B_\lambda$ is bounded;

(ii) $f$ has the representation

\[ f(x) = c - x^\lambda \int_0^\infty p(v)v^{\lambda-1}e^{-xv}dv, \]

where $p$ is a non-negative, increasing and bounded function on $(0, \infty)$, and $c \geq \Gamma(\lambda)p(\infty)$.

Proof. If (i) holds and $f$ has the representation (1) then according to Proposition 2.2, $\alpha = 0$ and the function

\[ p(v) = \int_0^v \frac{d\mu(t)}{t^\lambda} \]

is non-negative, increasing and bounded. Standard manipulations yield

\[ f(x) = \beta + \Gamma(\lambda)p(\infty) - x^\lambda \int_0^\infty p(v)v^{\lambda-1}e^{-xv}dv. \]

If (ii) holds then a change of variable gives us

\[ f(x) = c - \int_0^\infty p(u/x)u^{\lambda-1}e^{-u}du \geq c - p(\infty)\Gamma(\lambda) \geq 0, \]

so $f$ is non-negative and bounded. Furthermore, writing $p(v) = \int_0^v d\sigma(s)$ we have, by Fubini’s theorem,

\[ f(x) = c - \int_0^\infty \int_{sx}^\infty u^{\lambda-1}e^{-u}du d\sigma(s). \]
so that

\[ f'(x) = \int_{0}^{\infty} s(sx)^{\lambda-1}e^{-sx}d\sigma(s) = x^{\lambda-1} \int_{0}^{\infty} s^\lambda e^{-sx}d\sigma(s). \]

Thus \( f \in \mathcal{B}_\lambda \). This completes the proof. \( \square \)

**Remark 3.10** An ordinary Bernstein function \( f \) is bounded if and only if \( f = c - g \), where \( c \geq 0 \) and \( g \) is completely monotonic, bounded, and with \( \lim_{x \to \infty} g(x) = 0 \), see [11, Proposition 3.10]. This characterization follows from Proposition 3.9, noticing that when \( \lambda = 1 \) the integral

\[ \int_{0}^{\infty} p(v)v^{\lambda-1}e^{-xv}dv \]

in (ii) is a so-called completely monotonic function of order 1. This is equivalent to its product with \( x \) being completely monotonic. See [8].

**Example 3.11** The function \( F_\lambda \) defined by

\[ F_\lambda(x) = 1 - \lambda x^\lambda e^x \Gamma(-\lambda, x) \]

is bounded and belongs to \( \mathcal{B}_\lambda \). Indeed, from [6, 8.6.4] it follows that

\[ \Gamma(-\lambda, x) = \frac{x^{-\lambda}e^{-x}}{\Gamma(\lambda + 1)} \int_{0}^{\infty} \frac{t^\lambda}{x + t} e^{-t} dt \]

\[ = \frac{e^{-x}}{\Gamma(\lambda + 1)} \int_{0}^{\infty} \frac{t^\lambda}{1 + t} e^{-xt} dt \]

so that

\[ F_\lambda(x) = 1 - \frac{1}{\Gamma(\lambda)} \int_{0}^{\infty} \frac{t}{1 + t} t^{\lambda-1} e^{-xt} dt \]

and the assertion follows from Proposition 3.9. We remark that \( F_\lambda \) is the distribution function of a randomized Lomax distribution in which the scale parameter is Gamma distributed with form parameter \( \lambda \). Lomax distributions are heavy-tailed probability distributions used in business, economics, and actuarial science.

A function \( f \) is called a generalized complete Bernstein function of order \( \lambda > 0 \) if

\[ f(x) = ax^\lambda + b + \int_{0}^{\infty} \int_{0}^{xt} u^{\lambda-1}e^{-u}du\varphi(t) dt, \quad (13) \]
where $\varphi(t)$ is completely monotonic. This class of functions is denoted by $\mathcal{CB}_\lambda$.

The integral (13) converges exactly when $\varphi(t)t^\lambda/(1+t)^\lambda$ is integrable on $(0, \infty)$. For $\lambda = 1$ this is the class of complete Bernstein functions in [11, Definition 6.1]. The complete Bernstein functions $f$ are characterized by $f(x)/x \in \mathcal{S}_1$, see [11, Theorem 6.2]. We shall show that this characterization does not hold for other values of $\lambda$.

**Proposition 3.12** If $f \in \mathcal{CB}_\lambda$ then $f(x)/x^\lambda \in \mathcal{S}_\lambda$. Conversely, if for a function $f : (0, \infty) \to (0, \infty)$ we have $f(x)/x^\lambda \in \mathcal{S}_\lambda$ of the special form

$$f(x)/x^\lambda = \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-xt}t^{\lambda-1}\Phi(t) \, dt + \alpha,$$

where $\alpha \geq 0$, and $\Phi(t) = \int_0^\infty \varphi(s) \, ds + \beta$, $\varphi$ being completely monotonic and $\beta \geq 0$, makes the integral converge then $f \in \mathcal{CB}_\lambda$.

**Proof.** Applying the substitution $v = u/x$ in the inner integral and using Fubini’s theorem yields immediately

$$f(x)/x^\lambda = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty \int_0^x e^{-uv}v^{\lambda-1} \varphi(t) \, dv \, dt$$

or

$$= \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty e^{-uv}v^{\lambda-1} \int_v^\infty \varphi(t) \, dt \, dv.$$ 

The inner integral is a completely monotonic function of $v$ and this completes the proof of the first assertion. The second assertion is proved by retracing the steps above. \qed

### 4 Generalized Thorin-Bernstein functions

According to [11] a Bernstein function is called a Thorin-Bernstein function if its Lévy measure has a density $m$ for which $tm(t)$ is completely monotonic (or $m$ is completely monotonic of order 1). We say that a function $g \in \mathcal{B}_\lambda$ is a generalized Thorin-Bernstein function of order $\lambda$ if

$$g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^{xt} u^{\lambda-1} e^{-u} \varphi(t) \, dt \, du,$$

where $t\varphi(t)$ is completely monotonic. This class of functions is denoted by $\mathcal{TB}_\lambda$. The following theorem characterizes these functions in different ways, and is a generalization of parts of [11, Theorem 8.2].
In [3] a class of Thorin-Bernstein functions \( f \) for which \( f' \) belongs to \( \mathcal{S}_\lambda \) is considered, see also [2]. We stress that this class is not the same as the class of generalized Thorin-Bernstein functions introduced in this section.

**Theorem 4.1** For a function \( g : (0, \infty) \to [0, \infty) \) the following are equivalent.

(i) \( g \) is a generalized Thorin-Bernstein function of order \( \lambda \);

(ii) \( g \) admits the integral representation

\[
g(x) = \alpha x^\lambda + \beta + \int_0^\infty \left( \frac{x}{x+s} \right)^\lambda \frac{h(s)}{s} ds,
\]

where \( \alpha \geq 0, \beta \geq 0 \) and \( h \) is non-negative and increasing;

(iii) \( x^{1-\lambda}g'(x) \) belongs to \( \mathcal{S}_\lambda \) and the limit \( \lim_{x \to 0^+} g(x) \) exists.

Proof. (i) \( \Rightarrow \) (ii): (i) means that

\[
g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^x u^{\lambda-1} e^{-u} du \varphi(t) dt,
\]

where

\[
\varphi(t) = \int_0^\infty e^{-ts} h(s) ds
\]

and \( h \) is increasing. The representation of \( g \) is now rewritten:

\[
g(x) = \alpha x^\lambda + \beta + \int_0^\infty \int_0^x u^{\lambda-1} e^{-u} du \int_0^\infty e^{-ts} h(s) ds dt
\]

\[
= \alpha x^\lambda + \beta + \int_0^\infty \int_0^x u^{\lambda-1} e^{-u} du e^{-ts} dt h(s) ds
\]

\[
= \alpha x^\lambda + \beta + x^\lambda \int_0^\infty \int_0^\infty \int_0^t v^{\lambda-1} e^{-xv} dv e^{-ts} dt h(s) ds
\]

\[
= \alpha x^\lambda + \beta + x^\lambda \int_0^\infty \int_0^\infty \int_0^v e^{-ts} dt v^{\lambda-1} e^{-xv} dv h(s) ds
\]

\[
= \alpha x^\lambda + \beta + x^\lambda \int_0^\infty \int_0^\infty \frac{1}{s} e^{-vs} dt v^{\lambda-1} e^{-xv} dv h(s) ds
\]

\[
= \alpha x^\lambda + \beta + x^\lambda \int_0^\infty \int_0^\infty e^{-v(s+x)} v^{\lambda-1} dv \frac{h(s)}{s} ds
\]

\[
= \alpha x^\lambda + \beta + \Gamma(\lambda) \int_0^\infty \left( \frac{x}{s+x} \right)^\lambda \frac{h(s)}{s} ds.
\]
(ii) ⇒ (iii): If $g$ has the representation in (ii) we write $h$ as

$$h(s) = \int_0^s d\sigma(u),$$

for some positive measure $\sigma$ on $[0, \infty)$. Differentiation and Fubini’s theorem yield

$$g'(x) = \lambda \alpha + \lambda \int_0^\infty \frac{h(s)}{(x + s)^{\lambda + 1}} ds$$

$$= \lambda \alpha + \int_0^\infty \frac{d\sigma(u)}{(x + u)^\lambda}.$$  

Since $g'$ is non-negative, $g$ is increasing and since it is also non-negative the limit must exist.

(iii) ⇒ (i): We have, with a completely monotonic function $\varphi$,

$$g'(x) = x^{\lambda - 1} \int_0^\infty e^{-xt} t^{\lambda - 1} \varphi(t) dt + cx^{\lambda - 1}.$$  

Therefore, and because $\lim_{x \to 0^+} g(x)$ exists,

$$g(x) - g(0) = \int_0^x s^{\lambda - 1} \int_0^\infty e^{-st} t^{\lambda - 1} \varphi(t) dt ds + \frac{c}{\lambda} x^{\lambda}$$

$$= \int_0^\infty \int_0^x s^{\lambda - 1} e^{-st} ds t^{\lambda - 1} \varphi(t) dt + \frac{c}{\lambda} x^{\lambda}.$$  

This gives, using the substitution $u = ts/x$ in the inner integral,

$$g(x) = g(0) + \frac{c}{\lambda} x^{\lambda} + \int_0^\infty \int_0^x v^{\lambda - 1} e^{-v} dv \frac{\varphi(t)}{t} dt.$$  

Here, the function $t \cdot \varphi(t)/t$ is completely monotonic, and this completes the proof. \(\square\)

**Corollary 4.2** If $g'(x)x^{1-\lambda}$ belongs to $S_\lambda$ and the limit $\lim_{x \to 0} g(x)$ exists then $g(x)x^{-\lambda}$ belongs to $S_\lambda$.

**Proof.** From the proof of Theorem 4.1 ((iii) ⇒ (i)) it follows that

$$\frac{g(x)}{x^{\lambda}} = \frac{g(0)}{x^{\lambda}} + \frac{c}{\lambda} + \int_0^t \int_0^u v^{\lambda - 1} e^{-vx} dv \frac{\varphi(t)}{t} dt$$

$$= \frac{g(0)}{x^{\lambda}} + \frac{c}{\lambda} + \int_0^\infty \int_u^\infty \frac{\varphi(t)}{t} dt u^{\lambda - 1} e^{-ux} du.$$  

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Since
\[ \Phi(u) = \int_{u}^{\infty} \frac{\varphi(t)}{t} \, dt \]
is completely monotonic it follows that \( g(x)x^{-\lambda} \) belongs to \( S_\lambda \). 
\[ \square \]

We remark that Corollary 4.2 follows also from Proposition 3.12.

**Example 4.3** Suppose that \( \nu > \mu > \lambda > 0 \). The formula \([4, 3.197(9)]\) can be written
\[ B(\mu - \lambda, \lambda)x^{\lambda/2}F_1(\nu, \mu - \lambda; \mu; 1 - x) = \int_{0}^{\infty} \left( \frac{x}{x + s} \right)^{\lambda} \frac{h(s)}{s} \, ds, \]
where \( h(s) = s^\lambda (1 + s)^{\nu - \lambda} \), and \( B \) denotes the Beta function. Since the function \( h \) is increasing we see from Theorem 4.1 that \( x^{\lambda/2}F_1(\nu, \mu - \lambda; \mu; 1 - x) \) can be extended to a generalized Thorin-Bernstein function of order \( \lambda \).

**Remark 4.4** The representation in (ii) of Theorem 4.1 can using \( h(s) = \int_{0}^{s} d\sigma(u) \) be written in the form
\[ g(x) = \alpha x^\lambda + \beta + \int_{0}^{\infty} \int_{0}^{\infty} \left( \frac{x}{x + s} \right)^{\lambda} \frac{1}{s} \, ds \, d\sigma(u). \]
The measure \( \sigma \) is thus a generalization of the so-called Thorin measure for Thorin-Bernstein functions. Notice furthermore that if \( g \) is written in this form then an easy computation shows that \( \sigma \) is the representing measure for the function \( x^{1-\lambda}g'(x) \) in \( S_\lambda \):
\[ g'(x) = \alpha x^{\lambda - 1} + x^{\lambda - 1} \int_{0}^{\infty} \frac{d\sigma(u)}{(x + u)^\lambda}. \]
The following corollary is an easy consequence of Theorem 4.1.

**Corollary 4.5** The following statements are equivalent for a function \( g : (0, \infty) \to (0, \infty) \):

(i) \( g(x) = e^{-f(x)} \), where \( f \) is a generalized Thorin-Bernstein function of order \( \lambda \);

(ii) \( -x^{1-\lambda}(\log g)'(x) \) belongs to \( S_\lambda \) and \( \lim_{x \to 0} g(x) \leq 1 \).

**Proposition 4.6** Let \( \{g_n\} \) be a sequence of generalized Thorin-Bernstein functions of order \( \lambda \) and suppose that \( g_n \to g \) pointwise on \( (0, \infty) \). Then \( g \) is also a generalized Thorin-Bernstein function of order \( \lambda \).
Proof. We know that $x^{1-\lambda} g_n'(x)$ and $x^{-\lambda} g_n(x)$ are in $\mathcal{S}_\lambda$. In particular, $x^{-\lambda} g_n(x)$ and hence also $x^{-\lambda} g(x)$ are completely monotonic and

$$x^{-\lambda} g'_n(x) - \lambda x^{-\lambda-1} g_n(x) = (x^{-\lambda} g_n(x))' \to (x^{-\lambda} g(x))' = x^{-\lambda} g'(x) - \lambda x^{-\lambda-1} g(x).$$

We get from this, $x^{1-\lambda} g_n'(x) \to x^{1-\lambda} g'(x)$, and since $\mathcal{S}_\lambda$ is closed under pointwise convergence, $x^{1-\lambda} g'(x) \in \mathcal{S}_\lambda$. Finally, the limit $\lim_{x \to 0} g(x)$ exists since $g$ is non-negative and non-decreasing. $\square$

Remark 4.7 It is clear that $\mathcal{TB}_\lambda \subset \mathcal{CB}_\lambda \subset x^\lambda \mathcal{S}_\lambda$ and that

$$\frac{g(x)}{x^\lambda} = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty \frac{1}{(x+s)^\lambda} \frac{h(s)}{s} ds, \quad \text{for } g \in \mathcal{TB}_\lambda,$$

$$\frac{g(x)}{x^\lambda} = \alpha + \frac{\beta}{x^\lambda} + \int_0^\infty \frac{1}{(x+s)^\lambda} \frac{d\sigma(s)}{s}, \quad \text{for } g \in \mathcal{CB}_\lambda.$$

The difference is simply that the representing measure $\sigma$ for $g(x)x^{-\lambda}$ has an increasing density wrt. Lebesgue measure when $g \in \mathcal{TB}_\lambda$.

5 Additional results and comments

5.1 Generalized Bernstein functions of order at most one

The class of Bernstein functions is characterized in [11, Theorem 3.6] as those functions $f$ for which $g \circ f$ is completely monotonic whenever $g$ is completely monotonic, or for which $e^{-tf}$ is completely monotonic for all $t > 0$. Proposition 5.2 extends this characterization to generalized Bernstein functions of order less than or equal to 1.

Before stating this characterization notice that if $f$ is a positive and increasing $C^1$-function on $(0, \infty)$ and $0 < \lambda < \nu$ then

$$\int_r^x f'(t) t^{\nu-\lambda} \, dt = f(x)x^{\nu-\lambda} - f(r)r^{\nu-\lambda} - (\nu - \lambda) \int_r^x f(t)t^{\nu-\lambda-1} \, dt.$$

As $r \to 0$ we have $f(r)r^{\nu-\lambda} \to 0$ and $\int_r^x f(t)t^{\nu-\lambda-1} \, dt \to \int_0^x f(t)t^{\nu-\lambda-1} \, dt$ since $f$ is positive and increasing. This shows that

$$F(x) = \int_0^x f'(t) t^{\nu-\lambda} \, dt$$

is a well defined function. The following lemma is now easy to obtain.
Lemma 5.1 Suppose $0 < \lambda < \nu$. The following are equivalent for a non-negative and increasing $C^1$-function $f$ on $(0, \infty)$.

(i) $f \in B_\lambda$,

(ii) the corresponding function $F$ in (14) belongs to $B_\nu$.

Proposition 5.2 Suppose $\lambda \in (0, 1]$. The following are equivalent for a non-negative and increasing $C^1$-function $f$ on $(0, \infty)$.

(i) $f \in B_\lambda$,

(ii) $x \mapsto e^{-tF(x)}$ is completely monotonic for all $t > 0$, where

$$F(x) = \int_0^x f'(s)s^{1-\lambda}ds$$

(iii) $g \circ F$ (with $F$ as in (ii)) is completely monotonic whenever $g$ is completely monotonic.

Proof. This is immediate from Lemma 5.1 (with $\nu = 1$) and [11, Theorem 3.6].

In the next proposition the functions having a primitive in $B_\lambda$ are characterized, and this result holds for all $\lambda > 0$.

Proposition 5.3 For a function $g : (0, \infty) \to [0, \infty)$ and $\lambda > 0$ the following are equivalent.

(i) There exists a function $f \in B_\lambda$ such that $f'(x) = g(x)$.

(ii) The function $g$ is of the form $g(x) = x^{\lambda-1}L(\mu)(x)$ for a positive measure $\mu$ on $[0, \infty)$ such that $\int_0^\infty d\mu(t)/(t + 1)^\lambda < \infty$.

Proof. If (i) holds and $f$ has the representation in Proposition 1.2 then

$$\frac{g(x)}{x^{\lambda-1}} = \frac{f'(x)}{x^{\lambda-1}} = \lambda \alpha + \int_0^\infty e^{-xt}d\mu(t),$$

with $\int_0^\infty d\mu(t)/(t + 1)^\lambda < \infty$. Hence (ii) holds. Conversely, if (ii) holds we simply define $f$ as

$$f(x) = \int_0^\infty \int_0^{xt} u^{\lambda-1}e^{-u}du \frac{d\mu(t)}{t^\lambda} + \frac{\mu(\{0\})}{\lambda}x^\lambda$$

and notice that $f \in B_\lambda$ with

$$f'(x) = x^{\lambda-1}\left(\mu(\{0\}) + \int_0^\infty e^{-xt}d\mu(t)\right) = g(x).$$

This completes the proof.

□
5.2 Composition and convolution of Bernstein functions

We have seen in Corollary 2.1 that $g(x)/x^\lambda$ is completely monotonic for any $g \in B_\lambda$. This makes it possible to prove the following proposition, generalizing [11, Corollary 3.7(vi)].

**Proposition 5.4** Let $f, g \in B_\lambda$ and suppose that $a, b \geq 0$ such that $a + b \leq 1$. Then $x \mapsto f(x^a)g(x^b)$ belongs to $B_\lambda$.

**Proof.** Let $b(x) = f(x^a)g(x^b)$. A computation shows that

$$
1 - \lambda b'(x) = ax^{(a+b-1)\lambda} \frac{f'(x^a)g(x^b)}{(x^a)^{\lambda-1}(x^b)^{\lambda}} + bx^{(a+b-1)\lambda} \frac{g'(x^b)}{(x^b)^{\lambda-1}}
$$

The factor $f'(x^a)/(x^a)^{\lambda-1}$ is completely monotonic since it is the composition of the Bernstein function $x \mapsto x^a$ and a completely monotonic function. (See [11, Theorem 3.6].) The same thing holds for the factor $g(x^b)/(x^b)^{\lambda}$. Since also $x \mapsto x^{(a+b-1)\lambda}$ is completely monotonic the first summand is completely monotonic, and so is the second term. □

**Proposition 5.5** Suppose that $\lambda \leq 1$, $f \in B_1$ and $g \in B_\lambda$. Then $f \circ g \in B_\lambda$.

**Proof.** This follows from the relation

$$
\frac{f'(g(x))g'(x)}{x^{\lambda-1}} = \frac{f'(g(x))g'(x)}{x^{\lambda-1}},
$$

noting that $g \in B_1$ so that the first term is completely monotonic. The second term is completely monotonic by definition. □

**Remark 5.6** (1) Proposition 5.5 does not hold for $\lambda > 1$. [Consider $f(x) = 1 - e^{-x} \in B_1$ and $g(x) = x^2 \in B_2$.]

(2) Suppose that $\lambda \leq 1$. If $f \in B_\lambda$ and $g \in B_1$ then $f \circ g$ need not be in $B_\lambda$. [Consider $f(x) = \sqrt{x} \in B_{1/2}$ and $g(x) = 1 + x - e^{-x} \in B_2$.]

**Proposition 5.7** Suppose that $\lambda \geq 1$, $f \in B_\lambda$ and $g \in B_1$. Then

(i) If $\lambda$ is an integer then $f \circ g \in B_\lambda$.

(ii) If $g(x)/x$ is logarithmically completely monotonic then $f \circ g \in B_\lambda$.

(A function $f : (0, \infty) \to (0, \infty)$ is called logarithmically completely monotonic if $-(\log f)'$ is completely monotonic, see [1].)
Proof of Proposition 5.7. This follows from the relation
\[
\frac{f'(g(x))g'(x)}{x^{\lambda-1}} = \frac{f'(g(x))}{g(x)^{\lambda-1}} \left( \frac{g(x)}{x} \right)^{\lambda-1} g'(x),
\]
noting that \( g \in B_1 \) so that the first and third term are completely monotonic. The second term is completely monotonic if \( \lambda \) is an integer since \( g(x)/x \) is completely monotonic. If \( g(x)/x \) is logarithmically completely monotonic then any non negative power is completely monotonic.

\[ \square \]

Remark 5.9 The composition of two functions from \( B_\lambda \) need not even belong to \( B_{\lambda^2} \). This can be seen by considering \( h \circ h \), where \( h(x) = 1 - (x + 1)e^{-x} \in B_2 \).

Remark 5.8 (1) If \( \lambda \geq 1 \) and \( x \mapsto f(x^\lambda) \in B_1 \) then \( f \in B_{1/\lambda} \). [Proof: \( f'(x^\lambda)x^{\lambda-1} \) is completely monotonic and \( x \mapsto x^{1/\lambda} \) is Bernstein.] The converse implication need not hold. [Example: \( f(x) = \int_0^x t^{-1/2}e^{-t} dt \in B_{1/2} \) but \( f(x^2) \notin B_1 \)].

(2) If \( \lambda \leq 1 \) and \( f \in B_{1/\lambda} \) then \( x \mapsto f(x^\lambda) \in B_1 \). [Proof: \( f'(x)x^{1-1/\lambda} \) is completely monotonic and \( x \mapsto x^{1/\lambda} \) is Bernstein.] The converse implication need not hold. [Example: \( f(x) = 1 - e^{-x^2} \notin B_2 \) but \( f(x^{1/2}) \in B_1 \)].

Proposition 5.10 Suppose that \( f_1 \in B_{\lambda_1} \) and \( f_2 \in B_{\lambda_2} \). Then \( (f_1 * f_2)' \in B_{\lambda_1 + \lambda_2} \).

Proof. By Theorem 3.1 we have \( xL(f_1)(x) \in S_{\lambda_1} \) and \( xL(f_2)(x) \in S_{\lambda_2} \).

Hence
\[
x^2L(f_1 * f_2)(x) = x^2L(f_1)(x)L(f_2)(x) \in S_{\lambda_1 + \lambda_2},
\]
so that \( xL(f_1 * f_2)(x) = L(\phi)(x) \) for some \( \phi \in B_{\lambda_1 + \lambda_2} \). This means that \( f_1 * f_2 = \phi * 1 \) so we obtain \( (f_1 * f_2)' = \phi \). \[ \square \]

5.3 A finite case

In [10] certain classes, denoted by \( C^N_\lambda \), were investigated. In order to be more consistent with the notation in the present paper we relabel these classes as \( C^N_\lambda \). They are defined by
\[
C^N_\lambda = \{ f \in C^\infty((0, \infty)) \mid c^\lambda_0(f) \text{ is completely monotonic for } k = 0, \ldots, N \},
\]
where \( c^\lambda_k(f) \) is defined in (5). Examples of functions in \( C^N_\lambda \setminus C^{N+1}_\lambda \) can be found via the so-called \( N \)-monotonic functions, see [5]. A function \( p : (0, \infty) \to \mathbb{R} \) is called \( N \)-monotonic if \( p \in C^N((0, \infty)) \) and satisfies
\[
(-1)^k p^{(k)}(x) \geq 0, \text{ for } k = 0, \ldots, N.
\]
The result is stated as follows. The proof follows directly from [10, Theorem 1.6].

**Proposition 5.11** Assume that 

\[ p \in C^{N+1}((0, \infty)) \]  

is \( N \)-monotonic but not \( N + 1 \) monotonic, and that  

\[ \int_{0}^{\infty} e^{-xt} t^{\lambda-1} |p^{(k)}(t)| \, dt < \infty \]

for \( 0 \leq k \leq N + 1 \). Then the function \( f \) given by  

\[ f(x) = \int_{0}^{\infty} e^{-xt} t^{\lambda-1} p(t) \, dt, \quad x > 0, \]

belongs to \( C_{\lambda}^{N} \setminus C_{\lambda}^{N+1} \).

We denote by \( B_{\lambda}^{N} \) the class  

\[ B_{\lambda}^{N} = \{ \nu \in M_{+}((0, \infty)) \mid xL(\nu)(x) \in C_{\lambda}^{N} \} \]

and characterize it in the following proposition. (The notation \( \partial \mu \) means the derivative of a measure \( \mu \) on \((0, \infty)\) in the sense of distributions. See e.g. [10].)

**Proposition 5.12** Let \( N \geq 1 \). The following properties are equivalent for a measure \( \nu \) in \( M_{+}((0, \infty)) \).

(a) \( \nu \in B_{\lambda}^{N} \).

(b) \( \nu \) is absolutely continuous wrt. Lebesgue measure, with density  

\[ \sigma(t) = c + \int_{0}^{t} s^{\lambda-1} d\mu(s), \]

where \( \mu \in M_{+}((0, \infty)) \) and \( c \geq 0 \), and furthermore \( \mu_k \equiv (-1)^k \partial^k \mu \) is in \( M_{+}((0, \infty)) \) and  

\[ \int_{0}^{\infty} e^{-xs}s^{\lambda-1} d\mu_k(s) < \infty \]

for \( k = 0, \ldots, N \) and all \( x > 0 \).

**Proof.** Suppose that \( \nu \in B_{\lambda}^{N} \). Then \( xL(\nu)(x) \in C_{\lambda}^{N} \) so that by [10, Theorem 1.6]  

\[ xL(\nu)(x) = c + \int_{0}^{\infty} e^{-xs}s^{\lambda-1} d\mu(s), \]
where \( \mu \in M_+((0, \infty)) \) and \( c \geq 0 \), and furthermore \( \mu_k \equiv (-1)^k \partial^k \mu \) is in \( M_+((0, \infty)) \). Therefore,

\[
\mathcal{L}(\nu) = \mathcal{L} \left( c + \int_0^t s^{\lambda-1} d\mu(s) \right),
\]
so that \( \nu \) has the representation in (b).

If (b) holds then

\[
x \mathcal{L}(\nu)(x) = c + x \int_0^\infty e^{-xt} \int_0^t s^{\lambda-1} d\mu(s) \, dt = c + x \int_0^\infty \int_s^\infty e^{-xt} s^{\lambda-1} d\mu(s) \, dt = c + \int_0^\infty e^{-xs} s^{\lambda-1} d\mu(s).
\]

Again, according to [10, Theorem 1.6] this shows that \( x \mathcal{L}(\nu)(x) \in C^N_\lambda \). \( \square \)

**Remark 5.13** We obtain from the above that \( \nu \in \cap_{N=1}^\infty B^N_\lambda \) if and only if \( x \mathcal{L}(\nu)(x) \in \cap_{N=1}^\infty C^N_\lambda \), and this is the case if and only if \( x \mathcal{L}(\nu)(x) \) is in \( S_\lambda \) (see [10, Corollary 1.4]), or by Theorem 3.1, \( \nu \in B_\lambda \). We also see that Theorem 3.3 is a limit case of Proposition 5.12.

In the next proposition we determine for which functions \( f \in S_{\lambda+1} \) the product \( xf(x) \) belongs to \( S_\lambda \).

**Proposition 5.14** The following are equivalent for a function \( f : (0, \infty) \to (0, \infty) \):

(a) \( x \mapsto xf(x) \in S_\lambda \).

(b) \( f \in S_{\lambda+1} \) and \( \mathcal{L}(\mu) \in C^1_\lambda \), where \( \mu \) is the representing measure for \( f \) in \( S_{\lambda+1} \).

**Proof.** Suppose that (a) holds. Then

\[
f(x) = xf(x) \frac{1}{x} \in S_\lambda \cdot S_1 \subseteq S_{\lambda+1},
\]
and also \( \lim_{x \to \infty} f(x) = 0 \). We write

\[
f(x) = \int_0^\infty \frac{d\mu(t)}{(t+x)^{\lambda+1}} = \mathcal{L} \left( \frac{1}{\Gamma(\lambda+1)} t^\lambda \varphi(t) \right)(x),
\]
\[
xf(x) = \int_0^\infty \frac{d\nu(t)}{(t+x)^{\lambda}} + c = \mathcal{L} \left( \frac{1}{\Gamma(\lambda)} t^{\lambda-1} \psi(t) \right)(x) + c,
\]

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where $\varphi = L(\mu)$ and $\psi = L(\nu)$. Thus we have

$$L\left(\frac{1}{\Gamma(\lambda + 1)} t^\lambda \varphi(t)\right) = L\left(\left(\frac{1}{\Gamma(\lambda)} t^{\lambda - 1} \psi(t) + c\epsilon_0\right) * 1\right),$$

so that

$$\frac{1}{\Gamma(\lambda + 1)} t^\lambda \varphi(t) = \left(\frac{1}{\Gamma(\lambda)} t^{\lambda - 1} \psi(t) + c\epsilon_0\right) * 1,$$

showing that not only $\varphi$ but also $t^{1-\lambda}(t^\lambda \varphi(t))'$ is completely monotonic, i.e. $\varphi \in C^1_\lambda$.

Conversely, if $f \in S_{\lambda+1}$ with $\varphi = L(\mu) \in C^1_\lambda$ then $\varphi$ and $t^{1-\lambda}(t^\lambda \varphi(t))'$ are in particular decreasing so that partial integration yields (for $r > 0$)

$$\int_r^\infty e^{-xt} t^\lambda \varphi(t) \, dt = \left[-\frac{1}{x} e^{-xt} t^\lambda \varphi(t)\right]_r^\infty + \frac{1}{x} \int_r^\infty e^{-xt} (t^\lambda \varphi(t))' \, dt.$$

Letting $r \to 0$ in this relation and using monotone convergence gives us that $t^\lambda \varphi(t)$ has a finite (and non-negative) limit $c$ as $t \to 0$ and that the integral on the right-hand side converges. Hence

$$\Gamma(\lambda + 1) f(x) = \frac{c}{x} + \frac{1}{x} \int_0^\infty e^{-xt} t^{\lambda - 1} \left(t^{1-\lambda}(t^\lambda \varphi(t))'\right) \, dt,$$

showing that $xf(x)$ belongs to $S_\lambda$. \hfill $\Box$

### References


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