Exploiting Twistor Techniques for One-loop QCD Amplitudes

E. J. Bjerrum-Bohr, N.; dunbar, David; Ita, Harald

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Exploiting Twistor Techniques for One-loop QCD Amplitudes

N. E. J. Bjerrum-Bohr\textsuperscript{a}, David C. Dunbar\textsuperscript{ab} and Harald Ita\textsuperscript{a} SWAT-06/465
\textsuperscript{a}Department of Physics, University of Wales Swansea
\textsuperscript{b}Presented by David C. Dunbar at Loop and Legs 2006

1. Introduction

Recently, a “weak-weak” duality between massless gauge theory and a topological string theory propagating in twistor space has been proposed in ref. \cite{1}. This duality implies surprising structure within the S-matrix of gauge theories. In supersymmetric theories this has been exploited to facilitate considerable progress in computing scattering amplitudes. The application of these ideas to QCD has taken longer, however recently progress has been made in developing techniques which can be applied to compute one-loop gluon scattering amplitudes \cite{2,3,4}. In this talk we discuss and review the work of ref. \cite{5}. We aim to establish recursion relations in the number of scattering gluons in an one loop amplitude.

2. Twistor Inspired Techniques: Tree calculations

The link to twistor string theory is clearest if we express amplitudes in terms of spinor variables by replacing the massless momentum by $p_{a\dot{a}} = \lambda_a\bar{\lambda}_{\dot{a}}$ where $p_{a\dot{a}} = (\sigma^\mu)_{a\dot{a}}p_\mu$ and use the spinor helicity formalism \cite{6} for the polarisation vectors. The BCFW on-shell recursion relations \cite{7} for tree amplitudes are one of the remarkable formalisms which have arisen from the duality. The recursion relations rely on the analytic structure of the amplitude after it has been continued to a function in the complex plane $A(z)$ by shifting the (spinorial) momentum of two reference legs,

\begin{equation}
\lambda^1_a \rightarrow \lambda^1_a + z\lambda^2_a, \quad \bar{\lambda}^2_{\dot{a}} \rightarrow \bar{\lambda}^2_{\dot{a}} - z\bar{\lambda}^1_{\dot{a}}.
\end{equation}

These shifts are equivalent to a shift in the momenta 

\begin{equation}
p^1_{a\dot{a}} \rightarrow p^1_{a\dot{a}} + z\lambda^2_a\bar{\lambda}^2_{\dot{a}}, \quad p^2_{a\dot{a}} \rightarrow p^2_{a\dot{a}} - z\lambda^1_a\bar{\lambda}^1_{\dot{a}}.
\end{equation}

By integrating $A(z)/z$ over a contour at infinity and by assuming $A(z) \rightarrow 0$, the unshifted amplitude $A(0)$ can be determined from the residues of the function $A(z)/z$. The poles of this function are at $z = 0$ and at $z_i$ given by the factorisations of $A(z)$ when $P^2(z) = 0$ for some intermediate propagator $i/P^2$, with the residue given by the product of the two tree amplitudes. A recursion relation is thus obtained which gives the $n$-point amplitude as a sum over lower point functions \cite{7}

\begin{equation}
A(0) = \sum_i \hat{A}_k(z_i) \times \frac{i}{P_i^2} \times \hat{A}_{n-k+1}(z_i).
\end{equation}

The summation only includes factorisations where the two shifted legs 1 and 2 are on opposite sides of the pole. The tree amplitudes are evaluated at the value of $z$ such that the shifted pole term vanishes, i.e. $P_i(z_i)^2 = 0$.

The technique also extends and in fact the correctness of the MHV formalism \cite{8}, the other influential output from the twistor duality, can be derived from this approach \cite{9,10}. The BCFW recursion relations differ from the well established Berends-Giele recursion relations \cite{11} in that they are on-shell.

Although the duality relates string theory to $\mathcal{N} = 4$ Super-Yang-Mills theory, the techniques inspired by the duality have much wider applicability. For gluonic tree amplitudes, this is not so surprising since the tree amplitudes for gluonic scattering coincide in QCD and $\mathcal{N} = 4$ SYM. More surprisingly, the techniques may be applied to tree amplitudes with massive particles \cite{12} and...
3. One-Loop QCD Amplitudes

A one-loop amplitude for massless particles can be expanded in the form

\[ A = \sum_i c_i I_i^1 + \sum_i d_i I_i^3 + \sum_i e_i I_i^2 + R, \quad (3.1) \]

where \( I_i^1 \) are scalar integral \( n \)-point integrals and \( R \) denotes rational terms. Loop amplitudes contain logarithmic (and dilogarithmic) terms which would contain cuts in the complex plane when shifted. Thus the entire amplitude is not suitable for a recursion relation. However, recursion relations may be used on parts of the amplitude:

A) The rational terms

\[ R \equiv A - \left( \sum_i c_i I_i^1 + \sum_i d_i I_i^3 + \sum_i e_i I_i^2 \right). \quad (3.2) \]

B) The rational coefficients of the integral functions \( c_i, d_i \) and \( e_i \).

The two approaches are complementary rather than competing. In both cases, to apply a recursion relation the key is an understanding of the singularity structure in the shifted coefficients \( R(z), c_i(z), d_i(z) \) or \( e_i(z) \), which can be inferred from the factorisation properties [16] of the full amplitude as \( P^2 \to 0 \),

\[ A_n^{1\text{-loop}} \xrightarrow{P^2 \to 0} \sum_{h=\pm} \left[ A_n^{1\text{-loop}} \frac{i}{P^2} A_{n-m+1}^{\text{tree}} + A_n^{\text{tree}} \frac{i}{P^2} A_{n-m+1}^{\text{tree}} \mathcal{F}_n \right] \]

\[ + A_{m+1}^{\text{tree}} \frac{i}{P^2} A_{n-m+1}^{1\text{-loop}} + A_{m+1}^{\text{tree}} \frac{i}{P^2} A_{n-m+1}^{\text{tree}} \mathcal{F}_n \]. \quad (3.3) \]

For the case of the coefficient \( c_i(z) \) we obtain a recursion relation analogous to that for tree amplitudes,

\[ c_n(0) = \sum_{\alpha, h} A_{n-m+1}^{h}(z_\alpha) \frac{i}{P^\alpha} c_{m+1}(z_\alpha), \]

\[ \quad (3.4) \]

where \( A_{n-m+1}^{h}(z_\alpha) \) and \( c_{m+1}(z_\alpha) \) are shifted tree amplitudes and coefficients evaluated at the residue value \( z_\alpha \) and \( h \) denotes the helicity of the intermediate state.

In order to have a valid bootstrap the shifted coefficient has to vanishes as \(|z| \to \infty\); otherwise there would be a dropped boundary term. We can, however, impose criteria to prevent this from happening. Consider an integral and consider the unitarity cut which isolates the cluster on which the recursion will be performed, i.e. the one with the two shifted legs.

The dashed line in this figure indicates the cut. The recursion is to be performed with the two shifted legs from the right-most cluster. Then simple criteria for a valid recursion are:

1. The shifted tree amplitude on the side of the cluster undergoing recursion vanishes as \(|z| \to \infty\).

2. All loop momentum dependent kinematic poles are unmodified by the shift.

Note that these are sufficient and not necessary conditions.

4. Complication: Spurious Singularities

In addition to physical singularities, pieces of amplitudes also contain spurious singularities. A spurious singularity is a singularity that does not appear in the full amplitude but which is present only in some parts of the amplitude. Typical examples are co-planar singularities such as \( \frac{1}{(z^2 - \beta \gamma z)} \) which vanishes when \( P = \alpha k_2 + \beta k_3 \). Such singularities are common in the coefficients of integral functions. These are not singularities of the full amplitude since, on the singularity,
the integral functions are not independent but combine to cancel. For example, for six-point
kinematics, the product \( \langle 2 | P_{234} [5] \rangle \) vanishes when
\( t_{234} t_{612} - s_{34} s_{61} = 0 \). At this point the functions
\( \ln(s_{34}/t_{234}) \) and \( \ln(s_{61}/t_{612}) \) are no longer inde-
pendent and the combination
\[
\frac{a_1}{\langle 2 | P_{234} [5] \rangle} \ln(s_{34}/t_{234}) + \frac{a_2}{\langle 2 | P_{234} [5] \rangle} \ln(s_{61}/t_{612}),
\]
(4.1)
is non-singular provided that \( a_1 = a_2 \) evaluated at the singularity. Some spurious singu-
larities can be controlled by the choice of ba-
sis functions. For example expressions such as
\( \ln(r)/(1 - r)^3 \) will typically appear in amplitudes
where \( r \) is the ratio of two momentum invariants.
These expressions have unphysical singularities at
\( r = 1 \) which cancel when combined with similar
singularities in the rational terms. If we con-
sider for a basis integral function the combination
\( L_2(r) = \ln(r) - (r - 1)/(1 - r)^3 \) which is
finite as \( r \rightarrow 1 \) then both the cut-constructible
and rational terms will be individually free of this
spurious singularity.

5. Supersymmetric Decomposition of
QCD Amplitudes

In general we shall always examine color-
decomposed amplitudes. Let \( A_{n}^{[j]} \) denote the
leading in color partial amplitude for gluon scat-
tering due to an (adjoint) particle of spin \( J \) in
the loop. The three choices we are interested in
are gluons (\( J = 1 \)), adjoint fermions (\( J = 1/2 \))
and adjoint scalars (\( J = 0 \)). It is considerably
easier to calculate the contributions due to su-
persymmetric matter multiplets together with the
complex scalar. The three types of supersymmet-
ric multiplet are the \( N = 4 \) multiplet and the
\( N = 1 \) vector and matter multiplets. We can
obtain the amplitudes for QCD from the super-
symmetric contributions via
\[
A_{n}^{[1]} = A_{n}^{N=4} - 4 A_{n}^{N=1 \, \text{chiral}} + A_{n}^{[0]},
A_{n}^{[1/2]} = A_{n}^{N=1 \, \text{chiral}} - A_{n}^{[0]},
\]
(5.1)
The contribution from massless quark scattering
can be obtained from these trivially. When we
compute amplitudes in supersymmetric theories
we are calculating parts of the QCD amplitude -
although the process is incomplete unless we can
obtain the non-supersymmetric contribution \( A_{n}^{[0]} \)
For \( N = 4 \) SYM, cancellations lead to con-
siderable simplifications in the loop momentum
integrals. This is manifest in the “string-based
approach” of computing loop amplitudes [17]. As
a result, \( N = 4 \) one-loop amplitudes can be
expressed simply as a sum of scalar box-integral
functions [18]. The box-coefficients are “cut-
constructible” [18]. That is they may be deter-
mined by an analysis of the cuts where the tree
amplitudes are the normal four dimensional one.
This allows a variety of techniques to be used in
evaluating these. Originally an analysis of unit-
ary cuts was used to determine the coefficients
firstly for the MHV case [18] and secondly for
the remaining six-point amplitudes [19]. Twistor
inspired techniques, combined with the applica-
tion of cut-constructibility have been developed
rapidly over the past year [20,21,22]. For theo-
ries with less supersymmetry the amplitudes are
also cut-constructible and, although more compi-
cated, significant progress has also been made for
these theories [23,24,25]. Consequently when we
wish to obtain QCD amplitudes, in many cases,
the remaining component is that for a scalar cir-
culating in the loop. In a numerical or semi-
umerical computation the scalar component is
also the simplest component to obtain [26]. This
piece is, in principle, cut-constructible provided
one performs cuts in exactly in the dimensional
regulating parameter [27]; alternately one can
split into cut-constructible pieces [19,28] plus
rational terms and establish recursion relations
for the rational [2].

6. Example: Split Helicity Amplitudes

As an example, let us consider the “split heli-
city” amplitude where the negative helicity gluons
in the colour-ordered amplitude are all adjacent.
\( A_{n}^{1- \text{loop}}(1^-, 2^-, \ldots, r^-, r + 1^+, \ldots, n^+) \).
This helicity amplitude has several simplifying
features and the tree amplitude in known from
the BCFW techniques [29]. The \( N = 4 \) com-
ponent of this amplitude can most easily be ob-
amplitudes do contain such functions but they are entirely determined by the $\mathcal{N} = 4$ component.

Consider a generic triangle or bubble integral function. Such a function will contain at least one massive corner. The external legs on this corner will be of split helicity. (If the external legs on this corner had the same helicity, then the internal legs would both by necessity be of the opposite helicity. This tree amplitude vanishes in $D = 4$ for the scalar and fermionic states and does hence not contribute to the case we are considering.) Let these legs be $a^-, \cdots, r^-, (r + 1)^+, \cdots, b^+$. It can then be shown that the shift

$$\lambda_{r+1} \rightarrow \lambda_{r+1} + z\lambda_r, \quad \bar{\lambda}_r \rightarrow \bar{\lambda}_r - z\bar{\lambda}_{r+1}, \quad (6.1)$$

satisfies the sufficiency conditions for a recursion relation. Starting from the known five and six point functions we can then build the result A) for the $\mathcal{N} = 1$ chiral multiple and B) for the cut-constructible part of the scalar contribution for a split helicity partial amplitude. We present here the results for the case with precisely three negative helicity gluons - the NMHV amplitudes.

For the amplitudes with an $\mathcal{N} = 1$ chiral multiplet running in the loop the result is,

$$A_n^{\mathcal{N}=1 \text{ chiral}}(1^-, 2^-, 3^-, 4^+, 5^+, \cdots, n^+) =$$

$$\frac{A_{\text{tree}}}{2} \left[ K_0[s_{n1}] + K_0[s_{34}] \right] - \frac{i}{2} \sum_{r=1}^{n-1} \hat{d}_{n,r} \frac{L_0[t_{3,r}/t_{2,r}]}{t_{2,r}} + \sum_{r=4}^{n-2} \hat{g}_{n,r} \frac{L_0[t_{2,r}/t_{2,r+1}]}{t_{2,r+1}} + \sum_{r=4}^{n-2} \hat{h}_{n,r} \frac{L_0[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}} \quad (6.2)$$

where,

$$\hat{d}_{n,r} = \frac{\langle 3|K_3,r K_2,r|1 \rangle^2 \langle 3|K_3,r [k_2, K_2,r] K_2,r|1 \rangle}{\langle 2|K_2,r, r|2|K_2,r, r+1 \rangle P_{3,r} P_{r+1,1} t_{2,r} t_{3,r}}$$

$$\hat{g}_{n,r} = \sum_{j=4}^{r} \frac{\langle 3|K_3,j K_2,j P|1 \rangle \langle 3|K_3,j K_2,j \tilde{P}|1 \rangle}{\langle 2|K_2,j, j|2|K_2,j, j+1 \rangle \times \langle 3|K_3,j K_2,j [k_{r+1}, K_2,r]|1 \rangle}{P_{3,j} P_{r+1,1} t_{3,j} t_{2,j}}$$

$$\hat{h}_{n,r} = (-1)^n \hat{g}_{n,n-r+2} \rangle_{(123..n)} \langle (321..n)} \quad (6.4)$$

The contribution for a scalar in the loop is

$$A_n^{[3]}(1^-, 2^-, 3^-, 4^+, 5^+, \cdots, n^+) = \frac{1}{3} A_n^{\mathcal{N}=1 \text{ chiral}} - \frac{i}{3} \sum_{r=4}^{n-1} \frac{L_2[t_{3,r}/t_{2,r}]}{t_{2,r}^{r+1}} + \sum_{r=4}^{n-2} \frac{L_2[t_{3,r}/t_{3,r+1}]}{t_{3,r+1}} + \text{rational}, \quad (6.3)$$

where,

$$\tilde{P} = k_{r+1} K_{r+1,1} + P = k_{r+1} K_{r+1,1}.$$

The rational pieces of the scalar contribution remain to be calculated.

The formulas for the split helicity amplitudes with an arbitrary number of negative helicity gluons are given explicitly in ref. [5].

7. Conclusions

The past two years have seen significant progress in the computation of loop amplitudes in gauge theories. Although, these techniques have arisen in the context of highly supersymmetric theories the process of applying them to more general theories such as QCD is underway.
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