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Reserve-dependent Management Actions in Life Insurance

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Abstract

In a setup of with-profit life insurance including bonus, we study the calculation of the market reserve, where Management Actions such as investment strategies and bonus allocation strategies depend on the reserve itself. Since future bonus payments depend on the retrospective savings account, the introduction of Management Actions that depend on the (prospective) market reserve results in an entanglement of retrospective and prospective reserves. We study the complications that arise due to the interdependence between retrospective and prospective reserves, and characterize the market reserve by a partial differential equation (PDE). We reduce the dimension of the PDE in the case of linearity, and furthermore, we suggest an approximation of the market reserve based on the forward rate. The quality of the approximation is studied in a numerical example.

1 Introduction

In this paper, we study the calculation of the market reserve of a with-profit life insurance contract in a setup, where the so-called Management Actions have a complex structure. The market reserve is the expected present value of future guaranteed and non-guaranteed payments from the insurer to the insured, and the Management Actions influence the payments of a life insurance contract, for instance through the investment strategy and the bonus allocation strategy. Especially, the non-guaranteed payments are influenced by future Management Actions. The life insurance company takes many considerations into account, when deciding on its Management Actions, and the decisions depend on the financial situation of the company, which is measured by the balance sheet. A fair
redistribution of bonus is of great importance in with-profit life insurance, such that the policyholders who contributed to the surplus receive a reasonable amount of bonus. In order to fairly model the future bonus allocation strategy, we need a sophisticated model that takes the entire balance sheet into account. In our model, we allow the future Management Actions to depend on all balance sheet items, and the dependence on the market reserve complicates the setup.

The modelling of bonus in with-profit life insurance is studied in Norberg (1999), Stefensen (2006) and Asmussen and Stefensen (2020). We extend the model from Asmussen and Steffensen (2020) to allow for a broader range of investment and bonus allocation strategies, and characterize the prospective market reserve within this model. The core of the model is the surplus that arise due to the prudent assumptions about the interest rate and insurance risks on which payments are specified at initialization of the life insurance contract. By legislation, the surplus is to be paid back to the policyholders as bonus. We use the bonus scheme spoken of as additional benefits, where bonus is used to buy more insurance, and therefore the savings account of the insurance contract is influenced by bonus in terms of dividend payments. This results in a link between the savings account and the guaranteed payments, which is different from the setup in Steffensen (2006), where dividends only depend on the surplus, and guaranteed payments are not influenced by dividends. With the introduction of Management Actions that depend on the market reserve, the stochastic differential equation of the retrospective savings account and the retrospective surplus depend on the prospective market reserve. This paper studies the complications that arise due to the interdependence between retrospective and prospective reserves caused by the structure of the Management Actions. The result is a characterization of the market reserve by a partial differential equation (PDE) for a general model of the financial market with methods inspired by . We reduce the dimension of the PDE under the assumption of linearity of the dividend strategy with calculations similar to those in Steffensen (2006), and suggest an approximation of the market reserve based on the forward interest rate. The quality of the approximation is studied in a numerical example.

Christiansen et al. (2014) study reserve-dependence in benefits and costs in a life insurance setup without bonus, and characterize the prospective market reserve by a Thiele differential equation. The inclusion of bonus in our setup prevents us from applying the results from Christiansen et al. (2014). The results in this paper combine the modelling of bonus in life insurance from Asmussen and Steffensen (2020) with reserve-dependence from Christiansen et al. (2014). From a simulation point-of-view, the entanglement of retrospective and prospective reserves is notoriously difficult to handle, and Nyegaard et al. (2021) propose a simulation method to disentangle the problem based on intrinsic
values. Bruhn and Lollike (2021) and Ahmad et al. (2021) focus on a projection model of the retrospective savings account and surplus in a setting similar to ours, but where the dividend strategy is restricted to depend on the state of the insured, the savings account and the surplus. The inclusion of the prospective market reserve in the specification of the dividend strategy makes projection of the savings account and the surplus with the methods developed in Bruhn and Lollike (2021) and Ahmad et al. (2021) impossible.

The structure of the paper is the following. In Section 2, we present the setup of with-profit life insurance including bonus, introduce a model of the financial market, define the assets and liabilities of the insurance company, and link Management Actions in terms of investments and dividends to the market reserve. Calculation of the market reserve is studied in Section 3, we derive the PDE, and study the case of linearity. A numerical example in Section 4 emphasizes the practical applications of our result.

2 Reserves in Life Insurance

We introduce the setup of with-profit life insurance including bonus from Asmussen and Steffensen (2020) in a general financial market. Two decompositions of the liabilities of the insurer is presented, and we link Management Actions in terms of dividends and the investment strategy to the liabilities.

2.1 Setup

We consider the classical model of a life insurance contract, as presented in for instance Norberg (1991), where a Markov process \( Z = (Z(t))_{t \geq 0} \) on a finite state space \( J \) describes the state of the policyholder of a life insurance contract. Payments in the contract link with sojourns in states and transitions between states.

The transition probabilities of \( Z \) are given by

\[
p_{ij}(s, t) = \mathbb{P}(Z(t) = j \mid Z(s) = i),
\]

for \( i, j \in J \) and \( s \leq t \). We assume that the transition intensities

\[
\mu_{ij}(t) = \lim_{h \downarrow 0} \frac{1}{h} p_{ij}(t, t + h),
\]

exist for \( i, j \in J, i \neq j \) and are suitably regular. The process \( N^k(t) \) counts the number of jumps of \( Z \) into state \( k \in J \) up to and including time \( t \).

\[
N^k(t) = \# \{ s \in (0, t] \mid Z(s-) \neq k, Z(s) = k \},
\]
where \( Z(s-) = \lim_{h \downarrow 0} Z(s - h) \). Let \( \mathcal{F}_Z^t = (\mathcal{F}_t^Z)_{t \geq 0} \) be the natural filtration generated by the state process \( Z \).

We consider a general financial market, where the insurance company invests in a money market account governed by the interest rate \( r \) and \( K \) traded assets. The financial market is assumed to be free of arbitrage resulting in the existence of a (not necessarily unique) martingale measure \( Q \). All quantities in the model of the financial market are modelled directly under the martingale measure.

The interest rate is modelled as a diffusion process with dynamics

\[
dr(t) = \alpha_r(t, r(t))dt + \sigma_r(t, r(t))dW_r(t),
\]

where \( W_r \) is a Brownian motion under the risk neutral measure \( Q \), and \( \alpha_r : [0, \infty) \times \mathbb{R} \to \mathbb{R} \) and \( \sigma_r : [0, \infty) \times \mathbb{R} \to (0, \infty) \) are deterministic and sufficiently regular functions.

The general market consists of a money market account with dynamics

\[
dS_0(t) = r(t)S_0(t)dt,
\]

and \( K \) traded assets \( S(t) = (S_1(t), \ldots, S_K(t))^T \) with dynamics

\[
dS(t) = r(t)S(t)dt + \tilde{\sigma}(t, S(t), r(t))dW(t),
\]

where \( W(t) = (W_1(t), \ldots, W_M(t))^T \) is a \( M \)-dimensional Brownian motion under \( Q \) independent of \( W_r(t) \), and where

\[
\tilde{\sigma}(t, s, r) = \begin{pmatrix}
\sigma_{11}(t, s, r) \cdot s_1 & \sigma_{12}(t, s, r) \cdot s_1 & \ldots & \sigma_{1M}(t, s, r) \cdot s_1 \\
\sigma_{21}(t, s, r) \cdot s_2 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\sigma_{K1}(t, s, r) \cdot s_K & \sigma_{K2}(t, s, r) \cdot s_K & \ldots & \sigma_{KM}(t, s, r) \cdot s_K
\end{pmatrix},
\]

for \( s \in \mathbb{R}^K \) and sufficiently regular and deterministic functions \( \sigma_{ij} : [0, \infty) \times \mathbb{R}^K \times \mathbb{R} \to (0, \infty) \). The natural filtration generated by the financial market is \( \mathcal{F}_S^t = (\mathcal{F}_t^S)_{t \geq 0} \), and the combined information about the state process \( Z \) and the financial market at time \( t \) is given by \( \mathcal{F}_t = \mathcal{F}_t^S \cup \mathcal{F}_t^Z \). We assume independence between the state process \( Z \) and the financial market. With this specification of the financial market, the interest rate \( r(t) \) and the traded assets \( S(t) \) are Markov processes, and the ideas presented in this paper rely on the Markov property of the financial market. Our results generalize directly to any financial market, that is Markov and independent of the state process \( Z \).
Furthermore, we assume the existence of a suitable regular forward interest rates \( u \mapsto f(t,u) \) for \( t \geq 0 \), which satisfies
\[
\mathbb{E}^Q \left[ e^{-\int_t^s r(u)\,du} \middle| \mathcal{F}_t^S \right] = e^{-\int_t^s f(t,u)\,du},
\]
and \( f(t,t) = r(t) \) for all \( 0 \leq t \leq s \). The forward interest rate \( u \mapsto f(t,u) \) is measurable with respect to \( \mathcal{F}_t^S \).

The insurance company invests in an account \( G \) that consists of investments in the money market account and in the traded assets. We assume that the proportion of \( G \) invested in risky asset \( k \) is given by \( q_k(t) \). The account \( G \) has dynamics
\[
dG(t) = \left( 1 - \sum_{k=1}^K q_k(t) \right) G(t) \frac{dS_0(t)}{S_0(t)} + \sum_{k=1}^K q_k(t) G(t) \frac{dS_k(t)}{S_k(t)}
= r(t) G(t) dt + G(t) q(t)^T \sigma(t, S(t), r(t)) dW(t),
\]
where \( q(t) = (q_1(t), ..., q_K(t))^T \), and
\[
\sigma(t, s, r) = \begin{pmatrix}
\sigma_{11}(t, s, r) & \sigma_{12}(t, s, r) & \cdots & \sigma_{1M}(t, s, r) \\
\sigma_{21}(t, s, r) & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\sigma_{K1}(t, s, r) & \sigma_{K2}(t, s, r) & \cdots & \sigma_{KM}(t, s, r)
\end{pmatrix}.
\]

2.2 With-profit Life Insurance

In with-profit life insurance, payments specified in the insurance contract are based on prudent assumptions about insurance risks and the return in the financial market. We denote these assumptions the first order (technical) basis. The first order basis consists of the technical interest rate \( r^* \) and the technical transition intensities \( \mu^*_{ij}, i, j \in \mathcal{J}, i \neq j \). Assumptions about interest rate and transition intensities in the first order basis are prudent compared to the expectation of the actual development of the market interest rate and transition intensities. The actual future development of the market interest rate and the market transition intensities \( \mu_{ij} \) is unknown and needs to be modelled. Throughout, we assume that the market transition intensities are modelled in advance, and consider \( \mu_{ij} \) as externally given, which is also practise in for instance Danish life insurance industry. The model of the market interest rate is specified in Equation (1).

Due to the prudent first order basis, a surplus arises which by product design is to be paid back to the policyholders in terms of bonus. The redistribution of bonus is governed by legislation (in Denmark denoted Kontributionsbekendtgørelsen), and life insurance companies have certain degrees of freedom in the redistribution of bonus denoted as part of
the Management Actions of the company. We use the bonus scheme spoken of as additional benefits where bonus is used to buy more insurance. Inspired by Asmussen and Steffensen (2020) Chapter 6, the payments of the insurance contract consist of two types of payments. The payment stream \( B_1 \) represents payments not regulated by bonus, and \( B_2 \) represents the profile of payments regulated by bonus. The payment streams contain benefits less premiums of the insurance contract

\[
dB_i(t) = dB_i^{Z(t)}(t) + \sum_{k: k \neq Z(t-)} b_{i}^{Z(t-)}(t) dN^k(t),
\]

and

\[
dB_i^j(t) = b_{i}^j(t)dt + \Delta B_i^j(t)d\epsilon_n(t),
\]

for \( j \in \mathcal{J} \) and where \( \epsilon_n(t) = 1_{\{t \geq n\}} \) is the Dirac measure, \( b_{i}^j \) denotes continuous payments during sojourn in state \( j \), and \( b_{i}^{jk} \) denotes the single payment upon transition from state \( j \) to state \( k \). There is a lump sum payment of size \( \Delta B_i^j(n-) \) just before the contract terminates at time \( n \). Other lump sum payments at fixed time points during sojourns in states are disregarded in this setup. We assume that the payment functions \( b_{i}^j, b_{i}^{jk} \) and \( \Delta B_i^j \) are deterministic and sufficiently regular.

The technical reserve for the payment stream \( B_i \) for \( i = 1, 2 \) in this setup is the present value of future payments discounted with the technical interest rate

\[
V_i^{*Z(t)}(t) = \mathbb{E}^* \left[ \int_t^n e^{-\int_t^s r^*(u)du} dB_i(s) \bigg| Z(t) \right],
\]

where \( \mathbb{E}^* \) implies that we use the first order transition intensities in the distribution of \( Z \). See Asmussen and Steffensen (2020) Chapter 6 Section 4 for the dynamics of \( V_i^{*Z(t)}(t) \).

Bonus is distributed from the insurance company to the insured through a dividend payment stream \( D \). With the bonus scheme additional benefits, bonus is used to buy more insurance, and we denote by \( Q(t) \) the number of payment processes \( B_2 \) bought up to time \( t \). Additional benefits are bought under the technical basis, and as we then use dividends to buy \( B_2(t) \) at the price of \( V_2^*(t) \), we must have that

\[
dD(t) = V_2^{*Z(t)}(t)dQ(t).
\]

The policyholder experiences the payment stream with dynamics

\[
dB(t) = dB_1(t) + Q(t)dB_2(t),
\]

which is the payment process guaranteed at time \( t \).
2.3 Assets and Liabilities

The assets, \( U(t) \), of the insurance company are given by past premiums less benefits accumulated with the capital gains from investing in \( G \), which consists of investments in the money market account, \( S_0 \), and the risky assets, \( S \),

\[
U(t) = -\int_0^t \frac{G(t)}{G(s)} (dB_1(s) + Q(s)dB_2(s)).
\]  

We assume that \( U(0) = 0 \).

We consider two decompositions of the liabilities of the insurance company. One decomposition is in the savings account of the policyholder and the surplus. The savings account \( X \) of an insurance contract is the technical value of future payments guaranteed at time \( t \), i.e.

\[
X(t) = V_1^{*z(t)}(t) + Q(t)V_2^{*z(t)}(t).
\]

The savings account \( X(t) \) depends on the process \( Q(t) \), which denotes the number of payment processes \( B_2 \) bought up to time \( t \). We can express \( Q(t) \) in terms of the savings account and link the payment stream experienced by the policyholder to the savings account

\[
\frac{dB(t)}{dt} = \frac{dB(t, X(t))}{dt} = b^Z(t, X(t))dt + \Delta B^Z(t, X(t))d\epsilon_n(t) + \sum_{k: k \neq Z(t)} b^Z(t, X(t))dN^k(t),
\]

where

\[
b^j_t(t, x) = b^j_1(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b^j_2(t),
\]

\[
\Delta B^j(t, x) = \Delta B^j_1(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} \Delta B^j_2(t),
\]

\[
b^{jk}(t, x) = b^{jk}_1(t) + \frac{x - V_1^{*j}(t)}{V_2^{*j}(t)} b^{jk}_2(t).
\]

Note that since \( V_i^{*j}(n-) = \Delta B^j_i(n-) \), the lump sum payment at termination of the contract is equal to the savings account, \( \Delta B^j(n-, x) = x \).

The surplus \( Y \) is the difference between the assets and the savings account

\[
Y(t) = U(t) - X(t).
\]
We assume that the proportion of the account $G$ invested in the risky asset $S_k$ can be written in the form
\[ q_k(t) = \frac{\tilde{\pi}_k(t)Y(t)}{U(t)}, \]
where $\tilde{\pi}_k$ is a sufficiently regular process. The investment strategy of the insurance company is $\tilde{\pi}(t) = (\tilde{\pi}_1(t), ..., \tilde{\pi}_K(t))^T$. Hence, the proportion of $G$ invested in the risky asset $k$ is proportional to the surplus divided by the assets, leading to a larger investment if the surplus is large compared to the savings account.

**Proposition 1.** The savings account, $X$, and the surplus, $Y$, have dynamics
\[
dX(t) = r^*(t)X(t)dt - dB(t, X(t)) + dD(t) + \sum_{k: k \neq Z(t-)} R^{*Z(t)-k}(t, X(t-))(dN^k(t) - \mu^*_Z(t-k)(t)dt),
\]
\[
dY(t) = r(t)Y(t)dt + Y(t)\tilde{\pi}(t)^T\sigma(t, S(t), r(t))dW(t) - dD(t) + c^Z(t)(t, X(t))dt,
\]
where
\[
R^{*jk}(t, x) = b^{jk}(t, x) + \chi^{jk}(t, x) - x,
\]
\[
\chi^{jk}(t, x) = V^{sk}_1(t) + \frac{x - V^{sj}_1(t)}{V^{sj}_2(t)}V^{sk}_2(t),
\]
\[
c^j(t, x) = (r(t) - r^*(t))x + \sum_{k: k \neq j} (\mu^*_j(t) - \mu_{jk}(t))R^{*jk}(t, x).
\]

**Proof.** See Asmussen and Steffensen (2020) Chapter 6 Section 7, for the dynamics of the savings account. For the surplus, insert the dynamics of the account $G$ from Equation (3) and of the savings account
\[
dY(t) = -dG(t)\int_0^t \frac{1}{G(s)}(dB_1(s) + Q(s)dB_2(s)) - (dB_1(t) + Q(t)dB_2(t)) - dX(t)
\]
\[
= r(t)\left( -\int_0^t \frac{G(t)}{G(s)}(dB_1(s) + Q(s)dB_2(s)) - X(t) \right)dt + r(t)X(t)dt
\]
\[
+ \left( -\int_0^t \frac{G(t)}{G(s)}(dB_1(s) + Q(s)dB_2(s)) \right)\tilde{\pi}(t)^TY(t)\frac{U(t)}{U(r)}\sigma(t, S(t), r(t))dW(t)
\]
\[
- r^*(t)X(t)dt - dD(t) - \sum_{k: k \neq Z(t-)} R^{*Z(t)-k}(t, X(t-))(dN^k(t) - \mu^*_Z(t-k)(t)dt)
\]
\[
= r(t)Y(t)dt + Y(t)\tilde{\pi}(t)^T\sigma(t, S(t), r(t))dW(t) + c^Z(t)(t, X(t))dt - dD(t)
\]
\[
- \sum_{k: k \neq Z(t-)} R^{*Z(t)-k}(t, X(t-))(dN^k(t) - \mu_Z(t-k)(t)dt),
\]
which completes the proof.

Similar to Asmussen and Steffensen (2020), we disregard the last martingale term from the dynamics of the surplus, since the policyholder is only meant to participate in the systematic risks and the systematic contributions to the surplus given by \( c^j(t, x) \). Since the insured participate in the systematic risks, it is reasonable that parts of the surplus belongs to the policyholders. The insurance company is exposed to both systematic and unsystematic risks, and therefore it is fair that a part of the surplus belongs to the insurance company.

Based on the principle of equivalence on the technical basis, a natural constraint is that the savings account and the surplus are equal to zero at initialization of the contract i.e. \( X(0^-) = 0 \) and \( Y(0^-) = 0 \). This assumption implies that the savings account and the surplus are retrospective reserves. Hence, the decomposition of the liabilities into the savings account and the surplus is a decomposition based on retrospective reserves. Another decomposition of the liabilities is based on prospective reserves, and the natural constraint on the prospective reserves is that they are equal to zero at termination of the insurance contract. The prospective reserves are the market value of the guaranteed payments, the market value of future bonus payments, also denoted Future Discretionary Benefits (FDB), and the future profits.

Uncertainties in the future payments arise from two different types of risk. There is the risk associated with the state of the insured described by the state process \( Z \), and the risk from investments in the risky asset. Inspired by Asmussen and Steffensen (2020) Chapter 6 Section 3, we valuate the risk associated with \( Z \) under the physical measure \( P \) due to diversification, and evaluate financial risks under the risk neutral measure \( Q \) determined by the financial market. Therefore, valuation of future payments is performed under the product measure \( P \otimes Q \).

The market value of the guaranteed payments, \( V^{g,Z(t)}(t) \), is the expected present value of the future payments that are guaranteed the insured at time \( t \)

\[
V^{g,Z(t)}(t) = \mathbb{E}^{P \otimes Q} \left[ \int_t^\infty e^{-\int_t^u r(u)\,du} \left( dB_1(s) + Q(t)dB_2(s) \right) \bigg| \mathcal{F}_t \right].
\]

**Remark 1.** We can express the market value of the guaranteed payments in terms of the
savings account

\[ V_{g,Z}(t) = \mathbb{E}^{\mathcal{F}_t} \left[ \int_t^n e^{-f_s^* r(u) du} \left( dB_1(s) + Q(t) dB_2(s) \right) \bigg| \mathcal{F}_t \right] \]

\[ = \mathbb{E}^{\mathcal{F}_t} \left[ \int_t^n e^{-f_s^* r(u) du} dB_1(s) \bigg| Z(t), \mathcal{F}_t^S \right] + Q(t) \cdot \mathbb{E}^{\mathcal{F}_t} \left[ \int_t^n e^{-f_s^* r(u) du} dB_2(s) \bigg| Z(t), \mathcal{F}_t^S \right] \]

\[ = \mathbb{E}^P \left[ \int_t^n e^{-f_s^* f(t,u) du} dB_1(s) \bigg| Z(t), r(t) \right] + Q(t) \cdot \mathbb{E}^P \left[ \int_t^n e^{-f_s^* f(t,u) du} dB_2(s) \bigg| Z(t), r(t) \right], \]

since \( Q(t) \) is measurable with respect to \( \mathcal{F}_t \), \( Z \) is Markov, and \( Q(t) \) is a function of \( X(t) \). The forward rate is inserted in the discount factor due to the fact that the state process \( Z \) and the financial market are independent, such that the market value of the payment streams \( dB_1 \) and \( dB_2 \) consists of the valuation of risks associated with \( Z \) only and can be performed under \( P \) independent of the financial market.

The market value of the future bonus payments (FDB) is

\[ V^{b,Z}(t) = \mathbb{E}^{\mathcal{F}_t} \left[ \int_t^n e^{-f_s^* r(u) du} \left( Q(s) - Q(t) \right) dB_2(s) \bigg| \mathcal{F}_t \right]. \]

The market reserve is the expected present value under the market basis of future guaranteed and non-guaranteed payments, and therefore it is the sum of the market value of the guaranteed payments and the market value of the future bonus payments

\[ V^Z(t) = V^{g,Z}(t) + V^{b,Z}(t) \]

\[ = \mathbb{E}^{\mathcal{F}_t} \left[ \int_t^n e^{-f_s^* r(u) du} \left( dB_1(s) + Q(s) dB_2(s) \right) \bigg| \mathcal{F}_t \right]. \]

Future profit is the difference between the assets and the market reserve, \( V^{p,Z}(t) = U(t) - V^Z(t) \). The market reserve is the expected present value of future payments from the insurance company to the insured, while future profit is the expected present value of payments allotted the insurance company for taking on risks.

Note that in the first decomposition of the liabilities, the sum of the retrospective savings account and surplus is equal to the assets, and in the second decomposition, the sum of the prospective market reserve and future profit is equal to the assets. Hence, \( U(t) = X(t) + Y(t) = V^{p,Z}(t) + V^Z(t) \).
2.4 Reserve-dependent Dividends and Investments

Calculation of the balance sheet items requires a specification of the investment strategy and the dividend payment stream. These are part of the Management Actions of the insurance company, and the determination of the investment strategy and the dividend payment stream holds certain degrees of freedom.

We assume that dividends are allocated continuously such that
\[ dD(t) = \delta(t)dt, \]
where \( \delta \) is the dividend strategy of the insurance company.

When deciding the investment strategy and the dividend allocation strategy, the insurance company considers its financial situation in terms of relations between balance sheet items. Therefore, an attractable model of the Management Actions includes the possibility that dividends and investments depend on all balance sheet items.

We consider a setup where the investment strategy of the insurance company depends on the savings account, the surplus, and the market reserve
\[ \pi_k(t) = \pi_k(t, X(t), Y(t), V_Z(t)), \quad (6) \]
for deterministic and sufficiently regular functions \( \pi_k, \ k = 1, ..., K \). In the same way, we allow dividends to depend on the savings account, the surplus, and the market reserve
\[ \delta(t) = \delta_Z(t, X(t), Y(t), V_Z(t)), \quad (7) \]
for a deterministic and sufficiently regular function \( \delta \). Due to the relations between the balance sheet items, the specifications of the investment strategy and the dividend strategy above also allow investments and dividends to depend on the assets, \( U(t) \), the market value of guaranteed payments, \( V_g.Z(t) \), the market value of future bonus payments, \( V_b.Z(t) \), and future profits, \( V_p.Z(t) \). It is reasonable to assume that the dividend process depends on FDB, since it is likely that the amount of bonus depends on the reserve of future bonus.

With this specification of the investment strategy and the dividend strategy there is a forward-backward entanglement of the prospective market reserve in the retrospective savings account and surplus, since the investment strategy and the dividend strategy appear in the dynamics from Proposition 1.

3 Calculation of the Market Reserve

The setup with investments and dividends linked to all balance sheet items is attractable, since the Management Actions of the insurance company may depend on the entire balance
Calculation of the market reserve within this setup is complicated due to the interdependence between retrospective and prospective reserves. We characterize the market reserve by a partial differential equation (PDE), and consider the case of linearity that leads to a reduction in the dimension of the PDE.

3.1 Partial Differential Equation of the Market Reserve

Informally, \( (X(t), Y(t), Z(t)) \) is seen to be Markov with the specification of the investment strategy and the dividend strategy in Equations (6) and (7), since the dynamics of the savings account and the surplus depend on \( (X(t), Y(t), Z(t)) \) and the financial market, which is Markov. Hence, with a slight misuse of notation where \( V \) is now a function and not a stochastic process, we write the market reserve as

\[
V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) = E^{P\otimes Q}\left[ \int_t^n e^{-\int_s^t r(u)du} (dB_1(s) + Q(s)dB_2(s)) \bigg| X(t), Y(t), r(t), S(t), Z(t) \right]. (8)
\]

Since the surplus depends on the stochastic interest rate and the traded assets, the market reserve also depends on \( r(t) \) and \( S(t) \).

**Proposition 2.** Assume that \( V^j(t, x, y, r, s) \) is sufficiently differentiable. Then the market reserve satisfies the following partial differential equation

\[
\frac{\partial}{\partial t} V^j(t, x, y, r, s) = r V^j(t, x, y, r, s) - b^j(t, x) - \sum_{k \neq j} R^{jk}(t, x, y, r, s) \mu_{jk}(t) - D_x V^j(t, x, y, r, s)
- D_y V^j(t, x, y, r, s) - D_r V^j(t, x, y, r, s) - D_s V^j(t, x, y, r, s),
\]

\[V^j(n-, x, y, r, s) = x,\] (9)
where
\[
\mathcal{D}_x V^j(t, x, y, r, s) = \frac{\partial}{\partial x} V^j(t, x, y, r, s) (r^*(t) x - b(t, x) + \delta(t, x, y, V^j(t, x, y, r, s)) - \sum_{k: k \neq j} R^{jk}(t, x) \mu_{jk}(t)),
\]
\[
\mathcal{D}_y V^j(t, x, y, r, s) = \frac{\partial}{\partial y} V^j(t, x, y, r, s) (ry - \delta(t, x, y, V^j(t, x, y, r, s)) + c^j(t, x)) + \frac{1}{2} \frac{\partial^2}{\partial y^2} V^j(t, x, y, r, s) y^2 \sigma_y^2,
\]
\[
\mathcal{D}_r V^j(t, x, y, r, s) = \frac{\partial}{\partial r} V^j(t, x, y, r, s) \alpha_r(t, r) + \frac{1}{2} \frac{\partial^2}{\partial r^2} V^j(t, x, y, r, s) \sigma_r^2(t, r),
\]
\[
\mathcal{D}_s V^j(t, x, y, r, s) = \sum_{k=1}^K \frac{\partial}{\partial s_k} V^j(t, x, y, r, s) r s_k + \frac{1}{2} \sum_{k=1}^K \sum_{l=1}^K \frac{\partial^2}{\partial s_k \partial s_l} V^j(t, x, y, r, s) s_k s_l \sum_{m=1}^M \sigma_{km}(t, r, s) \sigma_{lm}(t, s, r),
\]
\[
R^{jk}(t, x, y, r, s) = b^{jk}(t, x) + V^k(t, \chi^{jk}(t, x), y, r, s) - V^j(t, x, y, r, s),
\]
and
\[
\sigma_y^2 = \pi(t, x, y, V^j(t, x, y, r, s))^T \sigma(t, s, r) \sigma(t, s, r)^T \pi(t, x, y, V^j(t, x, y, r, s)).
\]

Conversely, if a function \( V^j(t, x, y, r, s) \) satisfies the partial differential equation above, it is indeed the market reserve defined in Equation (8).

**Proof.** See Appendix A.

The boundary condition is due to the lump sum payment at time \( n - \).

**Remark 2.** In the Black-Scholes model of the financial market, where the interest rate is constant and deterministic, \( r(t) = r \in \mathbb{R} \), and the volatility is constant, \( \sigma(t, s, r) = \sigma > 0 \), the state-wise market reserve is a function of the savings account and the surplus, independent of the traded asset, \( S \). The function \( V^j(t, x, y) \) satisfies a partial differential equation equal to the partial differential equation in Proposition 2, but where \( \mathcal{D}_r V^j(t, x, y, r, s) = \mathcal{D}_s V^j(t, x, y, r, s) = 0 \) and \( \sigma_y = \sigma \pi(t, x, y, V^j(t, x, y)) \). This result also applies in the Black-Scholes model with a deterministic and time dependent interest rate \( r(t) \).

In order to calculate the market reserve, we must solve the PDE from Proposition 2 for all values of \( j, x, y, r \) and \( s \), which is computationally demanding if even possible. One way to reduce the dimension of the PDE is to assume a more specific model for the financial market, which is the case in Remark 2. Another approach is to study the special case of linearity in the dividend strategy.
3.2 Linearity

The payment stream, \( dB(t, x) \), and the sum-at-risk, \( R^{jk}(t, x) \), are by construction linear in the savings account. Therefore, the dynamics of the savings account and the surplus from Proposition 1 are linear in the savings account, the surplus and the market reserve if and only if the investment strategy from Equation (6) and the dividend strategy from Equation (7) are linear.

**Proposition 3.** Assume that the dividend strategy from Equation (7) is in the form

\[
\delta^j(t, x, y, r, v) = \delta^j_0(t, r) + \delta^j_1(t, r) \cdot y + \delta^j_2(t, r) \cdot v,
\]

for deterministic functions \( \delta^j_0, \delta^j_1, \delta^j_2 \). Then the market reserve is given by

\[
V^j(t, x, y, r) = h^j_0(t, r) + h^j_1(t, r) \cdot x + h^j_2(t, r) \cdot y,
\]

where the functions \( h_0, h_1, \) and \( h_2 \) satisfy the system of partial differential equations stated in Appendix B.

**Proof.** Since the function \( V^j(t, x, y, r) = h^j_0(t, r) + h^j_1(t, r) \cdot x + h^j_2(t, r) \cdot y \) satisfies the partial differential equation in Proposition 2, when \( h^j_0(t, r), h^j_1(t, r) \) and \( h^j_2(t, r) \) satisfy the system of partial differential equations in Appendix B for all \( j \in J \), Proposition 2 gives the result.

It is worth noticing that linearity of the dividend strategy is enough to make sure that the market reserve does not depend on the risky assets, \( S \), and therefore the result applies for any choice of investment strategy. The existence of a solution is not certain, but if the system of partial differential equations has a solution, Proposition 2 gives that Equation (10) is in fact the market reserve. The linear structure of the market reserve in Equation (10) is similar to the results in Steffensen (2006), where linearity of the surplus in the dividend strategy is inherited in the prospective reserve.

The result in Proposition 3 reduces the dimension of the PDE of the market reserve compared to the case without linearity. This simplifies the calculation of the market reserve, since it is less computational heavy to solve the system of PDE’s for the \( h \) functions for all values of \( r \), compared to finding the solution for all values of \( x, y, r \) and \( s \).

**Remark 3.** In the Black-Scholes model of the financial market, still under the assumption of linearity of the dividend strategy, the market reserve has representation

\[
V^j(t, x, y) = h^j_0(t) + h^j_1(t) \cdot x + h^j_2(t) \cdot y,
\]
where the functions \( h_0, h_1 \) and \( h_2 \) satisfy a system of ordinary differential equations. Hence, despite the forward-backward entanglement of the market reserve in the savings account and the surplus, the market reserve can be calculated as the solution to a system of backward ordinary differential equations in this case. The result also apply in the case, where the interest rate is time-dependent and deterministic.

From a computational point of view, it is demanding to solve PDE’s, and therefore it is a desirable result that a combination of linearity of the dividend strategy and the Black-Scholes model of the financial market, reduces the dimension of PDE from Proposition 2 in such a way that we are able to calculate the market reserve as the solution to a system of ordinary differential equations. The ordinary differential equations in Remark 3 fit into the class of Riccati equations. It is not certain that Riccati equations have solutions, but if a solution exists it is relatively easy to solve the system of ordinary differential equations numerically. The existence of solutions highly depends on the choice of the dividend strategy. With the choice in Example 1 below, we actually have an analytical solution.

**Example 1.** When dividends are equal to the surplus contribution from Proposition 1

\[ \delta^j(t, x, y, v) = c^j(t, x), \]

the dividends are linear in the savings account and the market reserve is given by

\[ V^j(t, x, y, r) = x, \]

since the functions \( h_0^j(t, r) = h_2^j(t, r) = 0 \) and \( h_1^j(t, r) = 1 \) solve the partial differential equations from Appendix B for all \( j \in \mathcal{J} \). Hence, the market reserve is equal to the savings account. In this case, the technical basis become redundant since the surplus that arise due to the prudent technical basis is immediately distributed as dividends to the savings account. In this case, the surplus, \( Y(t) \), is equal to zero, and the same holds for future profits.

For the majority of dividend strategies, an analytical expression for the market reserve is difficult to obtain, and the market reserve must be calculated numerically.

### 3.3 Approximation of the Market Reserve

In general, it is computationally more demanding to solve PDE’s compared to solving ODE’s by numerical methods, and there exist more precise methods for solving ODE’s. Under the assumption of linearity in the dividend strategy, we are able to calculate the market reserve as the solution to a system of backwards PDE’s by Proposition 3. In a
Black-Scholes model of the financial market, we actually obtain a system of ODE’s by Remark 3.

It may be desirable to lose some accuracy in order to decrease computation time by making approximations that result in ODE’s instead of PDE’s. Therefore, we aim to approximate the model with a stochastic interest rate in Equation (1) by a Black-Scholes model. To do this, we replace the stochastic interest rate with the deterministic forward interest rate. Due to linearity in the dividend strategy, the market reserve does not depend on the risky assets, $S$, by Proposition 3, and therefore we only approximate the stochastic interest rate. When calculating the market reserve, this corresponds to approximating the solution of the PDE’s for the $h$ functions from Proposition 3 by the solution to a system of ODE’s based on the forward interest rate. We consider the approximation

$$r(t) \approx f(0, t),$$

$$h_i^j(t, r) \approx \tilde{h}_i^j(t),$$

for $i = 0, 1, 2$ and $j \in \mathcal{J}$. The functions $\tilde{h}_i^j$ satisfy the system of ODE’s given by the equations in Appendix B, where $h_i^j(t, r)$ is replaced by $\tilde{h}_i^j(t)$, $r$ is replaced by $f(0, t)$, and it is noted that $\frac{\partial}{\partial r} \tilde{h}_i^j(t) = 0$.

In a setup without bonus, the market reserve is the expected present value of future payments discounted by the forward interest rate. Therefore, we consider the forward interest rate an appropriate approximation for the stochastic interest rate. We investigate the quality of this approximation in a numerical example.

### 4 Numerical Study

In this section, we emphasize the practical applications of our results in a numerical example. Within a survival model with a stochastic interest rate and linearity in the dividend strategy, we solve the PDE’s in Proposition 3 and compare the resulting market reserve with the solution of the ODE’s obtained by approximating with the forward interest rate as described in Section 3.3.

![Figure 1: Survival model in the numerical example](https://ssrn.com/abstract=3988059)

The survival model is illustrated in Figure 1, where state 0 corresponds to alive and state 1 corresponds to dead. We consider an insured male at age $a_0$ at initialization of the insurance contract, and the insurance contract consists of a life annuity regulated by
bonus, which is paid by a single premium of \( V_{r^*(0)}(0) \) at time 0. Then the savings account at time 0 is equal to the single premium, \( X(0) = V_{r^*(0)}(0) \), and \( dB_1 = 0 \), since all payments are regulated by bonus. The payment process is

\[
dB(t, X(t)) = 1_{\{Z(t)=0\}} \frac{X(t)}{V_{r^*(0)}(t)} \frac{b_0(t)}{V_{r^*(0)}(t)} dt.
\]

We assume the interest rate from Equation (1) follows a Vasicek model with dynamics

\[
 dr(t) = (\phi + \psi r(t)) dt + \sqrt{\theta} dW(t),
\]

where \((W(t))_{t \geq 0}\) is a Brownian motion under the risk-neutral measure \( \mathbb{Q} \). Let \( u \mapsto f(t, u) \) be the forward interest rate for \( t \geq 0 \).

The components in this example are stated in Table 1. The parameters in the interest rate model are inspired by Falden and Nyegaard (2021), and the technical mortality rate is the same as in Bruhn and Løkke (2021). The market mortality rate in this example is chosen to be \( 1.1 \cdot \mu^*(t) \), such that the technical basis is prudent compared to the market basis. The premium is determined according to the principle of equivalence.

<table>
<thead>
<tr>
<th>Component</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Age of policyholder, ( a_0 )</td>
<td>65</td>
</tr>
<tr>
<td>Termination, ( n )</td>
<td>45</td>
</tr>
<tr>
<td>Premium</td>
<td>15.22021</td>
</tr>
<tr>
<td>Annuity, ( b_0^*(t) )</td>
<td>1</td>
</tr>
<tr>
<td>( Z(0) )</td>
<td>0</td>
</tr>
<tr>
<td>( \mu_{01}(t) )</td>
<td>( 0.0005 + 10^{5.6+0.04(t+a_0)-10} )</td>
</tr>
<tr>
<td>( \mu_{01}(t) )</td>
<td>( 1.1 \cdot \mu^*_0(t) )</td>
</tr>
<tr>
<td>( r^*(t) )</td>
<td>0.01</td>
</tr>
<tr>
<td>( r(0) )</td>
<td>0.05</td>
</tr>
<tr>
<td>( \phi )</td>
<td>0.008127</td>
</tr>
<tr>
<td>( \psi )</td>
<td>-0.162953</td>
</tr>
<tr>
<td>( \theta )</td>
<td>0.000237</td>
</tr>
</tbody>
</table>

There are only dividends in state 0, since upon death all payments cancel. We assume
the dividend process is in the form
\[
\delta^0(t) = \lambda_1(t) c^0(t, X(t)) + \lambda_2(t) V^{h,0}(t) \\
= \lambda_1(t) X(t) \left( r(t) - \rho^*(t) + \mu_0(t) - \rho_{01}(t) \right) \\
+ \lambda_2(t) \left( V^0(t, X(t), Y(t)) - \frac{X(t)}{V_2^{00}(t)} V_2^{g,0}(t, r(t)) \right),
\]
where \( V_2^{g,0}(t, r) = \mathbb{E}^P \left[ \int_t^n e^{-\int_t^u f(t, u)du} dB_B(s) \mid Z(t), r(t) \right] \) and \( c^0 \) is the surplus contribution from Proposition 1.

The case where \( \lambda_1(t) = 1 \) and \( \lambda_2(t) = 0 \) corresponds to Example 1, and in this case the market reserve is equal to the savings account. It is reasonable to assume that \( \lambda_1(t) \in (0, 1) \), since a part of the surplus contribution is then immediately distributed as dividends.

We let \( \lambda_1(t) = 0.5 \) and \( \lambda_2(t) = 0.05 \), hence half of the surplus contribution and 5\% of FDB are distributed as bonus.

In this example, the partial differential equations from Proposition 3 result in \( h^0_0(t, r) = 0 \), \( h^0_2(t, r) = 0 \) and
\[
\frac{\partial}{\partial t} h^0_1(t, r) = - h^0_1(t, r)^2 \lambda_2(t) - \frac{b_0^0(t)}{V_{2^0}^0(t)} \\
\quad + h^0_1(t, r) \left( (1 - \lambda_1(t)) \left( r - \rho^*(t) + \mu_0(t) - \rho_{01}(t) \right) + \frac{b_0^0(t)}{V_{2^0}^0(t)} + \frac{\lambda_2(t) V_{2}^{g,0}(t, r)}{V_{2^0}^0(t)} \right) \\
- (\phi + \psi r) \frac{\partial}{\partial r} h^0_1(t, r) - \frac{\theta}{2} \frac{\partial^2}{\partial r^2} h^0_1(t, r),
\]
for the model with stochastic interest rate, which reduces to an ordinary differential equation when inserting the forward interest rate
\[
\frac{d}{dt} \tilde{h}^0_1(t) = - \tilde{h}^0_1(t)^2 \lambda_2(t) - \frac{b_0^0(t)}{V_{2^0}^0(t)} \\
\quad + \tilde{h}^0_1(t) \left( (1 - \lambda_1(t)) \left( f(0, t) - \rho^*(t) + \mu_0(t) - \rho_{01}(t) \right) + \frac{b_0^0(t)}{V_{2^0}^0(t)} + \frac{\lambda_2(t) V_{2}^{g,0}(t)}{V_{2^0}^0(t)} \right),
\]
\( \tilde{h}^0_1(n) = 1 \).

The partial differential equation for the function \( h_1 \) is solved numerically using the Explicit finite difference method, and the ordinary differential equation for the function \( \tilde{h}_1 \) is solved numerically using the Runge Kutta forth-order method.

We calculate the market reserve at time zero by computing the function \( h_1 \) as the solution to the PDE and by solving the ODE for \( \tilde{h}_1 \) based on the deterministic forward interest rate. The results are presented in Table 2.
Figure 2: The retrospective and the prospective decomposition of the liabilities at time 0 in the numerical example. The market reserve is calculated using the PDE method.

<table>
<thead>
<tr>
<th>PDE solution</th>
<th>ODE solution</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>13.16423</td>
<td>13.24555</td>
<td>0.00618</td>
</tr>
</tbody>
</table>

In this example, there is a small difference in the value of the market reserve at time zero. When we approximate with the forward interest rate, the market reserve is larger than in the model with the stochastic interest rate. Hence, the approximation method is conservative from an accounting point-of-view.

The decomposition of the liabilities based on retrospective and prospective reserves, respectively, at time 0 for this example is illustrated in Figure 2. The surplus is equal to zero at initialization of the contract, and therefore the retrospective decomposition only consists of the savings account. The market value of the guaranteed payments constitutes around two thirds of the prospective decomposition, and FDB is almost equal to future profits.

In order to get a better understanding of the difference between the two methods to calculate the market reserve, we compare the function \( t \mapsto \tilde{h}_1(t) \), which is the solution of the ODE, to the mean, the 2.5\%-quantile, and the 97.5\%-quantile of the stochastic process \( t \mapsto h_1(t, r(t)) \), since the market reserve is \( V^0(t, x, r) = h_0^0(t, r) \cdot x \), and the approximated market reserve is \( \tilde{V}^0(t, x) = \tilde{h}_1^0(t) \cdot x \). We compute \( \mathbb{E}[h_1(t, r(t))] \) by simulating 1000 interest rate paths, simulated with an Euler scheme based on the dynamics of the interest rate in Equation (11), interpolate the solution to the PDE of \( h_1 \) over \( r \), consider the function for each simulated interest rate path and calculate the empirical mean.

In this example, the market reserve is decreasing since benefits are paid out immediately.
Figure 3: Illustration of the functions $\tilde{h}_1$ and $h_1$. The red is $\tilde{h}_1$ based on the forward interest rate, the blue is $h_1$ based on the simulated interest rate paths, where the mean is the solid line, and the 2.5%-quantile and the 97.5%-quantile is the dashed line.

after the premium payment at time 0. The market reserve at time $t$ is $V(t,X(t)) = h_0^0(t,r) \cdot X(t)$, and therefore the development of the $h^0_1$ functions in Figure 3 does not have a one-to-one correspondence with the development of the market reserve. Based on the values of $E[h_1(t,r(t))]$ and $\tilde{h}_1^0(t)$ in Figure 3, the development of the market reserve is similar to the development of the savings account, which is also a decreasing process in this example. When the contract terminates, the market reserve equals zero since there are no future payments. The payment of $X(n)$ at termination of the insurance contract results in the boundary conditions $h_1^1(n,r) = \tilde{h}_1^0(n) = 1$, and is consistence with $V(n,X(n)) = 0$, since

$$X(n) = Q(n)V_2^*(n) = 0.$$

The approximation $\tilde{h}_1(t)$ is in general larger than, but close to $E[h_1(t,r(t))]$. Therefore based on this example, we consider the approximation with the forward interest rate reasonable, since $\tilde{h}_1^0$ is close to the estimated mean and within the 95% confidence interval of $h_1^0(t,r(t))$. The computation time for solving the ODE is significantly lower than for solving the PDE, and therefore the approximation is useful if one can accept the relative difference.
A Proof of Proposition 2

Construct a martingale $m$ as

$$m(t) = \mathbb{E}^{P\otimes Q}\left[ \int_0^t e^{-\int_0^s r(u)du} (dB_1(s) + Q(s)dB_2(s)) \bigg| \mathcal{F}_t \right]$$

$$= \int_0^t e^{-\int_0^s r(u)du} (dB_1(s) + Q(s)dB_2(s)) + e^{-\int_0^s r(u)du} V^{Z(t)}(t, X(t), Y(t), r(t), S(t)).$$

The dynamics of $m$ are

$$dm(t) = e^{-\int_0^s r(u)du} \left( dB_1(t) + Q(t)dB_2(t) + r(t)V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt 
+ dV^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \right).$$

By the multidimensional Itô formula, we have the dynamics of the market reserve

$$dV^{Z(t)}(t, X(t), Y(t), r(t), S(t))$$

$$= \frac{\partial}{\partial t} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt$$

$$+ \mathcal{D}_x V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt$$

$$+ \mathcal{D}_y V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt + \frac{\partial}{\partial y} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))$$

$$\times Y(t)\pi(t, X(t), Y(t), V^{Z(t)}(t, X(t), Y(t), r(t), S(t)))^T \sigma(t, S(t), r(t))dW(t)$$

$$+ \mathcal{D}_x V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt$$

$$+ \frac{\partial}{\partial r} V^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))dW_r(t)$$

$$+ \mathcal{D}_y V^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt$$

$$+ \sum_{k=1}^K \frac{\partial}{\partial s_k} V^{Z(t)}(t, X(t), Y(t), r(t), S(t)) S_k(t) \sum_{m=1}^M \sigma_{km}(t, S(t), r(t))dW_m(t)$$

$$+ \sum_{k:k\neq Z(t-)} \left( V^k(t, X^{Z(t-)}(t), Y(t), r(t), S(t)) - V^{Z(t-)}(t, X(t-), Y(t), r(t), S(t)) \right) dN^k(t).$$

(12)
Combining this, the dynamics of \( m(t) \) are
\[
dm(t) = e^{-\int_0^t r(u)du} \left( b^Z(t)(t, X(t)) + r(t)V^Z(t)(t, X(t), Y(t), r(t), S(t)) \right) \\
+ \frac{\partial}{\partial t} V^Z(t)(t, X(t), Y(t), r(t), S(t)) + \mathbb{D}_x V^Z(t)(t, X(t), Y(t), r(t), S(t)) \\
+ \mathbb{D}_y V^Z(t)(t, X(t), Y(t), r(t), S(t)) \\
+ \mathbb{D}_r V^Z(t)(t, X(t), Y(t), r(t), S(t)) \\
+ \mathbb{D}_s V^Z(t)(t, X(t), Y(t), r(t), S(t)) \\
+ \sum_{k=k\neq Z(t)} R^{Z(t)k}(t, X(t), Y(t), r(t), S(t)) \mu_{Z(t)k}(t) \right) dt \\
+ e^{-\int_0^t r(u)du} \Delta B^Z(t)(t, X(t)) d\epsilon_n(t) + e^{-\int_0^t r(u)du} dM(t),
\]
where \( M \) is a martingale with dynamics
\[
dM(t) = \frac{\partial}{\partial t} V^Z(t)(t, X(t), Y(t), r(t), S(t)) \sigma_r(t, r(t))dW_r(t) + \frac{\partial}{\partial y} V^Z(t)(t, X(t), Y(t), r(t), S(t)) \\
\times Y(t) \pi(t, X(t), Y(t), V^Z(t)(t, X(t), Y(t), r(t), S(t)))^T \sigma_r(t, r(t))dW(t) \\
+ \sum_{k=1}^K \frac{\partial}{\partial s_k} V^Z(t)(t, X(t), Y(t), r(t), S(t)) S_k(t) \sum_{m=1}^M \sigma_{km}(t, S(t), r(t)) dW_m(t) \\
+ \sum_{k=k\neq Z(t)} R^{Z(t)k}(t, X(t), Y(t), r(t), S(t))(dN^k(t) - \mu_{Z(t)k}(t) dt).
\]

Since \( e^{-\int_0^t r(u)du} dM(t) \) also are the dynamics of a martingale and since \( m(t) \) is a martingale, the term in front of \( dt \) in the dynamics of \( m(t) \) must be equal to zero for all \( t, X(t), Y(t), r(t), S(t) \) which results in the partial differential equation for the market reserve. Due to the lump sum payment at time \( n- \), \( \Delta B(n-, X(n-)) = X(n-) \), the boundary condition of the partial differential equation is \( V^j(n, x, y, r, s) = x \).

Now, assume that a function \( \tilde{V}^j(t, x, y, r, s) \) satisfies the partial differential equation in Equation (9). We show that this function is in fact the market reserve in Equation (8). Consider an investment strategy and dividend strategy given by
\[
\tilde{\pi}_k(t) = \pi_k(t, X(t), Y(t), \tilde{V}^Z(t)(t, X(t), Y(t), r(t), S(t))),
\]
\[
dD^Z(t)(t) = \delta(t, X(t), Y(t), \tilde{V}^Z(t)(t, X(t), Y(t), r(t), S(t))) dt,
\]
for \( k = 1, \ldots, K \).

The multidimensional Itô formula, the dynamics from Equation (12) with \( \tilde{V} \) inserted instead of \( V \), and the fact that \( \tilde{V} \) satisfies the partial differential equation in Equation
(9) yield that
\[
\begin{align*}
&\text{d}\left( e^{-\int_0^t r(u)du} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \right) \\
= &- r(t)\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))dt + e^{-\int_0^t r(u)du} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)) \\
= &e^{-\int_0^t r(u)du} \left( \sum_{k,k \neq Z(t-)} \bar{R}^{Z(t-)-k}(t, X(t-), Y(t), r(t), S(t)) \left( d\bar{N}^{k}(t) - \mu_{Z(t-)-k}(t)dt \right) \\
&- b^{Z(t)}(t, X(t))dt - \sum_{k,k \neq Z(t-)} b^{Z(t)-k}(t, X(t-))d\bar{N}^{k}(t) \\
&+ \frac{\partial}{\partial y} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))Y(t) \\
&\times \pi(t, X(t), Y(t), \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t)))^T \sigma(t, S(t), r(t))dW(t) \\
&+ \frac{\partial}{\partial r} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))\sigma_r(t, r(t))dW_r(t) \\
&+ \sum_{k=1}^K \frac{\partial}{\partial s_k} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))S_k(t) \sum_{m=1}^M \sigma_{km}(t, S(t), r(t))dW_m(t) \right).
\end{align*}
\]

Integrating over the interval [t, n) and taking the $\mathbb{P} \otimes \mathbb{Q}$ expectation conditioning on $\mathcal{F}_t$ give that
\[
\begin{align*}
e^{-\int_0^n r(u)du} \bar{V}^{Z(n-)}(n, X(n-), Y(n-), r(n-), S(n-)) - e^{-\int_0^t r(u)du} \bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))
&= - \mathbb{E}^{\mathbb{P} \otimes \mathbb{Q}} \left[ \int_t^n e^{-\int_0^s r(u)du} \left( b^{Z(s)}(s, X(s))ds + \sum_{k,k \neq Z(s-)} b^{Z(s)-k}(s, X(s-))d\bar{N}^{k}(s) \right) \bigg\vert \mathcal{F}_t \right],
\end{align*}
\]

since the remaining terms in the dynamics of $\bar{V}^{Z(t)}(t, X(t), Y(t), r(t), S(t))$ are martingales with respect to the filtration $\mathcal{F}$. Multiplying by $-\exp(-\int_0^t r(u)du)$ and including the boundary condition at time $n-$ in the payment stream gives that $\bar{V}^{Z(t, x, y, r, s)}$ is the market reserve.
 Partial differential equations for $h$-functions

\[
\begin{align*}
\frac{\partial}{\partial t} h_0^j(t, r) &= r h_0^j(t, r) - b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t) - \sum_{k: k \neq j} \mu_{jk}(t) \\
&\quad \times \left( b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t) + h_0^j(t, r) + h_1^j(t, r) \left( V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t) \right) - h_0^j(t, r) \right) \\
&\quad - h_1^j(t, r) \left( - b_1^j(t) + \frac{V_1^{*j}(t)}{V_2^{*j}(t)} b_2^j(t) + \delta_1^j(t) + \delta_3^j(t) h_0^j(t, r) \right) \\
&\quad - \sum_{k: k \neq j} \mu_{jk}(t) \left( b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t) \right) \\
&\quad - h_1^j(t, r) \left( - \delta_1^j(t) - \delta_3^j(t) h_0^j(t, r) \right) \\
&\quad + \sum_{k: k \neq j} \left( \mu_{jk}(t) - \mu_{jk}(t) \right) \left( b_1^{jk}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} b_2^{jk}(t) + V_1^{*k}(t) - \frac{V_1^{*j}(t)}{V_2^{*j}(t)} V_2^{*k}(t) \right) \\
&\quad - \frac{\partial}{\partial r} h_0^j(t, r) \alpha_r(t, r) - \frac{1}{2} \frac{\partial^2}{\partial r^2} h_0^j(t, r) \sigma_r^2(t, r), \\
&h_0^j(n, r) = 0,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} h_1^j(t, r) &= r h_1^j(t, r) - \frac{1}{V_2^{*j}(t)} b_2^j(t) - \sum_{k: k \neq j} \mu_{jk}(t) \left( \frac{1}{V_2^{*j}(t)} b_2^{jk}(t) + h_0^j(t, r) \frac{1}{V_2^{*j}(t)} V_2^{*k}(t) - h_1^j(t, r) \right) \\
&\quad - h_1^j(t, r) \left( - V_1^{*j}(t) b_2^j(t) + \delta_1^j(t) + \delta_3^j(t) h_0^j(t, r) \right) \\
&\quad - \sum_{k: k \neq j} \mu_{jk}(t) \left( \frac{1}{V_2^{*j}(t)} b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)} V_2^{*k}(t) - 1 \right) \\
&\quad - h_1^j(t, r) \left( - \delta_1^j(t) - \delta_3^j(t) h_0^j(t, r) + r - V_1^{*j}(t) \right) \\
&\quad + \sum_{k: k \neq j} \left( \mu_{jk}(t) - \mu_{jk}(t) \right) \left( \frac{1}{V_2^{*j}(t)} b_2^{jk}(t) + \frac{1}{V_2^{*j}(t)} V_2^{*k}(t) - 1 \right) \\
&\quad - \frac{\partial}{\partial r} h_1^j(t, r) \alpha_r(t, r) - \frac{1}{2} \frac{\partial^2}{\partial r^2} h_1^j(t, r) \sigma_r^2(t, r), \\
&h_1^j(n, r) = 1,
\end{align*}
\]

\[
\begin{align*}
\frac{\partial}{\partial t} h_2^j(t, r) &= - \sum_{k: k \neq j} \mu_{jk}(t) \left( h_0^j(t, r) - h_1^j(t, r) \right) - h_1^j(t, r) \left( \delta_2^j(t) + \delta_3^j(t) h_0^j(t, r) \right) \\
&\quad + h_2^j(t, r) \left( \delta_2^j(t) + \delta_3^j(t) h_0^j(t, r) \right) - \frac{\partial}{\partial t} h_2^j(t, r) \alpha_r(t, r) - \frac{1}{2} \frac{\partial^2}{\partial r^2} h_2^j(t, r) \sigma_r^2(t, r), \\
&h_2^j(n, r) = 0.
\end{align*}
\]
References


