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Analytic torsion for arithmetic locally symmetric manifolds and approximation of $L^2$-torsion

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In this paper we define a regularized version of the analytic torsion for quotients of a symmetric space of non-positive curvature by arithmetic lattices. The definition is based on the study of the renormalized trace of the corresponding heat operators, which is defined as the geometric side of the Arthur trace formula applied to the heat kernel. Then we study the limiting behavior of the analytic torsion as the lattices run through a sequence of congruence subgroups of a fixed arithmetic subgroup. Our main result states that for sequences of principal congruence subgroups, which converge to 1 at a fixed finite set of places and strongly acyclic flat bundles, the logarithm of the analytic torsion, divided by the index of the subgroup, converges to the $L^2$-analytic torsion.

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1. Introduction

The main purpose of this paper is to define a regularized analytic torsion for locally symmetric manifolds of finite volume which are quotients of a symmetric space of non-positive curvature by an arithmetic group and to study the limiting behavior of the renormalized analytic torsion for sequences of finite coverings of a given locally symmetric space. This is motivated by the applications of the Ray-Singer analytic torsion [58] of a compact manifold to the study of the growth of torsion in the cohomology of cocompact arithmetic groups by Bergeron and Venkatesh [10]. Other applications of a similar nature are given in [31], [48]. Since many important arithmetic groups are not cocompact, it is very desirable to extend these results to non-cocompact lattices. There exist some results for hyperbolic manifolds of finite volume [53], [55], [56], [49], [50].

1.1. Analytic torsion

To explain the approach in more detail, we briefly recall the definition of the Ray-Singer analytic torsion for a compact Riemannian manifold $X$ of dimension $d$. Let $\rho: \pi_1(X) \to \text{GL}(V)$ be a finite dimensional complex representation of the fundamental group of $X$ and let $E_\rho \to X$ be the flat vector bundle associated to $\rho$. Choose a Hermitian fiber metric in $E_\rho$. Let $\Delta_\rho(\rho)$ be the Laplace operator on $E_\rho$-valued $p$-forms with respect to the metrics on $X$ and in $E_\rho$. It is an elliptic differential operator, which is formally self-adjoint and non-negative. Let $h_\rho(\rho) := \dim \ker \Delta_\rho(\rho)$. Using the trace of the heat operator $e^{-t\Delta_\rho(\rho)}$, the zeta function $\zeta_\rho(s; \rho)$ of $\Delta_\rho(\rho)$ (see [61]) can be defined by

$$\zeta_\rho(s; \rho) := \frac{1}{\Gamma(s)} \int_0^\infty \left( \text{Tr} \left( e^{-t\Delta_\rho(\rho)} \right) - h_\rho(\rho) \right) t^{s-1} dt. \quad (1.1)$$

The integral converges for $\text{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane, which is holomorphic at $s = 0$. Then the analytic torsion $T_X(\rho) \in \mathbb{R}^+$, introduced by Ray and Singer [58], is defined by
\[
\log T_X(\rho) = \frac{1}{2} \sum_{p=1}^{d} (-1)^p p \frac{d}{ds} \zeta_p(s; \rho) \bigg|_{s=0}.
\] (1.2)

Bergeron and Venkatesh [10] have applied the analytic torsion in the case of a locally symmetric space \( X := \Gamma \backslash G/K \), where \( G \) is a semisimple Lie group of non-compact type, \( K \) a maximal compact subgroup and \( \Gamma \subset G \) a torsion free, cocompact discrete subgroup. The representations \( \rho \) of \( \Gamma \) considered in this case, are of the form \( \rho := \tau|_\Gamma \), where \( \tau \) is an irreducible finite-dimensional complex representation of \( G \). Denote by \( T_X(\tau) \) (resp. \( T_X^{(2)}(\tau) \)) the analytic torsion (resp. the \( L^2 \)-torsion [30], [32]) taken with respect to \( \tau|_\Gamma \). Let \( \tilde{X} := G/K \) be the universal covering of \( X \). Since the heat kernels of the corresponding Laplacians on \( \tilde{X} \) are \( G \)-invariant, one has

\[
\log T_X^{(2)}(\tau) = \text{vol}(X)t_X^{(2)}(\tau),
\] (1.3)

where \( t_X^{(2)}(\tau) \) is a constant that depends only on \( \tilde{X} \) and \( \tau \). One of the main results of [10] is the approximation of the \( L^2 \)-torsion by the renormalized analytic torsion of finite coverings of \( X \) under a certain non-degeneracy condition on \( \tau \). Representations which satisfy this condition are called strongly acyclic. Let \( X_i \to X \), \( i \in \mathbb{N} \), be a sequence of finite coverings of \( X \). Let \( \tau \) be strongly acyclic. Let \( \text{inj}(X_i) \) denote the injectivity radius of \( X_i \) and assume that \( \text{inj}(X_i) \to \infty \) as \( i \to \infty \). Then by [10, Theorem 4.5] one has

\[
\lim_{i \to \infty} \frac{\log T_{X_i}(\tau)}{\text{vol}(X_i)} = t_{\tilde{X}}^{(2)}(\tau).
\] (1.4)

Let \( \delta(\tilde{X}) := \text{rank}_C(G) - \text{rank}_C(K) = 1 \). The constant \( t_{\tilde{X}}^{(2)}(\tau) \) has been computed in [10, Proposition 5.2]. As a result it follows that \( t_{\tilde{X}}^{(2)}(\tau) \neq 0 \) if and only if \( \delta(\tilde{X}) = 1 \). Combined with the equality of analytic torsion and Reidemeister torsion [42], Bergeron and Venkatesh [10] used this result in the case \( \delta(\tilde{X}) = 1 \) to show that torsion in the cohomology of cocompact arithmetic groups grows exponentially.

1.2. The regularized trace of the heat operator

The definition of the analytic torsion is based on the compactness of the underlying manifold. Without this assumption, the heat operator \( e^{-t\Delta_p(\rho)} \) is in general not a trace class operator. So in order to generalize the above method to non-compact manifolds, the first problem is to define an appropriate regularized trace of the heat operators. For hyperbolic manifolds of finite volume a regularized trace has been defined by means of the renormalized trace of the heat kernel. This method goes back to Melrose [37] and has been used, for example, in [52], [12], [53], [45], [47], [48]. The method is based on the truncation of the heat kernel with the help of an appropriate height function. The upshot is that the renormalized trace of the heat kernel equals the geometric side of the Selberg trace formula applied to the heat kernel.
In the higher rank case we proceed in the same way as in the case of hyperbolic manifolds. To truncate the heat kernels we use Arthur’s truncation operator [2]. In [35] we have dealt with the case $G = \text{GL}(n)$. As in the case for hyperbolic manifolds, this renormalized trace of the heat kernel equals the geometric side of the Arthur trace formula applied to the heat kernel. We use this as the definition of the regularized trace of the heat operator for a general reductive group $G$. Let us explain this approach in more detail. To this end we pass to the adelic framework. To simplify notation we assume throughout the introduction that $G$ is a connected semisimple algebraic group defined over $\mathbb{Q}$. Assume that $G(\mathbb{R})$ is not compact. Let $K_\infty$ be a maximal compact subgroup of $G(\mathbb{R})$. Put $\tilde{X} = G(\mathbb{R})/K_\infty$. Let $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $\mathbb{A}_f$ the ring of finite adeles. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. We consider the adelic quotient

$$X(K_f) = G(\mathbb{Q}) \backslash (\tilde{X} \times G(\mathbb{A}_f)/K_f). \quad (1.5)$$

This is the adelic version of a locally symmetric space. In fact, $X(K_f)$ is the disjoint union of finitely many locally symmetric spaces, i.e., there exist lattices $\Gamma_i \subset G(\mathbb{R})$, $i = 1, \ldots, l$, such that

$$X(K_f) = \Gamma_1 \backslash \tilde{X} \sqcup \cdots \sqcup \Gamma_l \backslash \tilde{X} \quad (1.6)$$

(see Section 11). If $G$ is simply connected, then by strong approximation we have

$$X(K_f) = \Gamma \backslash \tilde{X},$$

where $\Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q})$. We will assume that $K_f$ is neat so that $X(K_f)$ is a manifold. Let $\nu : K_\infty \rightarrow \text{GL}(V_\nu)$ be a finite dimensional unitary representation. It induces a homogeneous Hermitian vector bundle $\tilde{E}_\nu$ over $\tilde{X}$, which is equipped with the canonical connection $\nabla^\nu$. Being homogeneous, it can be pushed down to every connected component $\Gamma_i \backslash \tilde{X}$ of $X(K_f)$ resulting in a locally homogeneous vector bundle over $X(K_f)$. Let $\tilde{\Delta}_\nu$ (resp. $\Delta_\nu$) be the associated Bochner-Laplace operator acting in the space of smooth sections of $\tilde{E}_\nu$ (resp. $E_\nu$). Using the kernel of the heat semigroup $e^{-t\tilde{\Delta}_\nu}$, $t > 0$, at the Archimedean place and the normalized characteristic function at the finite places, we get a smooth function $\phi_i^\nu$ on $G(\mathbb{A})$ which belongs to $\mathcal{C}(G(\mathbb{A}); K_f)$, the adelic version of the Schwartz space (see Section 2 for its definition). Let $J_{\text{geom}}$ be the geometric side of the Arthur trace formula introduced in [2]. Following [35, Definition 11.1], we define the regularized trace of $e^{-t\Delta_\nu}$ by

$$\text{Tr}_{\text{reg}} \left( e^{-t\Delta_\nu} \right) := J_{\text{geom}}(\phi_i^\nu) \quad (1.7)$$

In order to define the zeta function by the analogous formula (1.1) we need to determine the asymptotic behavior of $\text{Tr}_{\text{reg}} \left( e^{-t\Delta_\nu} \right)$ as $t \to 0$ and $t \to \infty$, respectively. To this end we use the Arthur trace formula.
Our first main result is concerned with the small time behavior of the regularized trace which is described in the following theorem.

**Theorem 1.1.** Let $G$ be a reductive algebraic group over $Q$, $K_f \subset G(\mathbb{A}_f)$ an open compact subgroup, and $\nu : K_\infty \to GL(V_\nu)$ a finite dimensional unitary representation of $K_\infty$. Let $\Delta_\nu$ be the associated Bochner-Laplace operator on the adelic quotient $X(K_f)$. Suppose that $K_f$ is neat, and let $r$ be the split semisimple rank of $G$. There exist constants $a_j, b_{ij} \in C$, $j \geq 0$, $1 \leq i \leq r$, depending on $\nu$ and $K_f$ such that

$$\text{Tr}_{reg}(e^{-t\Delta_\nu}) \sim t^{-d/2} \sum_{j=0}^\infty a_j t^{j} + t^{-(d-1)/2} \sum_{j=0}^\infty \sum_{i=1}^r b_{ij} t^{j/2}(\log t)^i$$

(1.8)

as $t \searrow 0$.

For $G = GL(n)$ or $SL(n)$ this result was proved in [35], and for hyperbolic manifolds in [44]. In these cases the arguments are simplified considerably.

To study the large time behavior we restrict attention to twisted Laplace operators, which are relevant for our purpose. Let $G$ be a reductive algebraic group over $Q$ as in the theorem. As before, let $\tau : G(\mathbb{R})^1 \to GL(V_\tau)$ be a finite dimensional complex representation (see (2.22) and the following discussion for the definition of $G(\mathbb{R})^1$). Let $\Gamma_i \backslash \tilde{X}$, $i = 1, \ldots, l$, be the components of $X(K_f)$. The restriction of $\tau$ to $\Gamma_i$ induces a flat vector bundle $E_{\tau,i}$ over $\Gamma_i \backslash \tilde{X}$. The disjoint union is a flat vector bundle $E_\tau$ over $X(K_f)$.

By [33] it is isomorphic to the locally homogeneous vector bundle associated to $\tau_{| K_\infty}$. We choose a $K_\infty$-invariant inner product in $V_\tau$. It gives rise to a fiber metric in the locally homogeneous vector bundle. On $X(K_f)$ we choose the metric which is induced from the restriction to $p$ of an invariant bi-linear form $B$ on $g$, which is positive definite on $p$ and negative definite on $t$. Let $\Delta_p(\tau)$ be the corresponding twisted Laplace operator on $p$-forms with values in $E_\tau$. Let $\text{Ad}_p : K_\infty \to GL(p)$ be the adjoint representation of $K_\infty$ on $p$, where $p = t^{1/2}$, and $\nu_p(\tau) = \lambda^p \text{Ad}_p \tau_{| K_\infty}$. Up to a vector bundle endomorphism, $\Delta_p(\tau)$ equals the Bochner-Laplace operator $\Delta_{\nu_p(\tau)}$. So $\text{Tr}_{reg}(e^{-t\Delta_p(\tau)})$ is well defined. The large time behavior of the regularized trace is described by the following proposition.

**Theorem 1.2.** Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup. Assume that $K_f$ is neat. Let $\tau \in \text{Rep}(G(\mathbb{R})^1)$ be irreducible. Assume that $\tau \not\equiv \tau_0$. Then we have

$$\text{Tr}_{reg}(e^{-t\Delta_p(\tau)}) = O(e^{-ct})$$

(1.9)

as $t \to \infty$ for all $p = 0, \ldots, d$.

1.3. Regularized analytic torsion

By Theorems 1.1 and 1.2 we can define the zeta function $\zeta_p(s; \tau)$ of $\Delta_p(\tau)$ as in (11.39), using the regularized trace of $e^{-t\Delta_p(\tau)}$ in place of the usual trace in (1.1). The
corresponding Mellin transform converges absolutely and uniformly on compact subsets of the half-plane $\text{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane by Theorems 1.1 and 1.2. Because of the presence of the log-terms in the expansion (1.8), the zeta function may have a pole at $s = 0$ so that we need to use the finite part of $\zeta_p(s; \tau)$ in the definition of the analytic torsion of $X(K_f)$. More precisely, for a meromorphic function $f(s)$ on $\mathbb{C}$ and $s_0 \in \mathbb{C}$, let $f(s) = \sum_{k \geq k_0} a_k(s - s_0)^k$ be the Laurent expansion of $f$ at $s_0$, and put $\text{FP}_{s=s_0} f := a_0$. Now we define the analytic torsion $T_{X(K_f)}(\tau) \in \mathbb{C} \setminus \{0\}$ by

$$
\log T_{X(K_f)}(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \left( \text{FP}_{s=0} \frac{\zeta_p(s; \tau)}{s} \right). \tag{1.10}
$$

**Remark 1.3.** Let $\delta(\widetilde{X}) = \text{rank}_\mathbb{C} G(\mathbb{R})^1 - \text{rank}_\mathbb{C} K_\infty$. We recall that in the co-compact case, the analytic torsion is trivial, unless $\delta(\widetilde{X}) = 1$. As the example of a hyperbolic manifold of even dimension shows [45], this does not need to be so in the non co-compact case.

### 1.4. Approximation of $L^2$-torsion

Let $\{K_j\}_{j \in \mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$. We say that $K_j \to 1$ as $j \to \infty$, if for every open compact subgroup $U$ of $G(\mathbb{A}_f)$ there exists $N \in \mathbb{N}$ such that $K_j \subset U$ for all $j \geq N$. We wish to study the limiting behavior of $\log T_{X(K_j)}(\tau) / \text{vol}(X(K_j))$ as $j \to \infty$. The main goal is to generalize the results of [36] to other reductive groups. For an adelic quotient $X := X(K_f)$ let $T_{X}^{(2)}(\tau)$ be the $L^2$-torsion [30], [32]. Since the heat kernels on $\widetilde{X}$ are $G(\mathbb{R})$-invariant, one has

$$
\log T_{X}^{(2)}(\tau) = \text{vol}(X)t_{X}^{(2)}(\tau), \tag{1.11}
$$

where $t_{X}^{(2)}(\tau)$ depends only on $\widetilde{X}$ and $\tau$. Based on the known results in the compact case [10], we make the following conjecture.

**Conjecture 1.4.** Let $\tau \in \text{Rep}(G(\mathbb{R})^1)$ be irreducible and assume that $\tau \not\approx \tau_0$. Let $\{K_j\}_{j \in \mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$ with $K_j \to 1$ as $j \to \infty$. Then

$$
\lim_{j \to \infty} \frac{\log T_{X(K_j)}(\tau)}{\text{vol}(X(K_j))} = t_{X}^{(2)}(\tau). \tag{1.12}
$$

In [36] we established this conjecture for principal congruence subgroups of $\text{GL}(n)$ and $\text{SL}(n)$. The constant $t_{X}^{(2)}(\tau)$ has been computed in [10, Proposition 5.2]. It is shown that $t_{X}^{(2)}(\tau) \neq 0$ if and only if $\delta(\widetilde{X}) = 1$.

Due to problems related to the geometric side of the trace formula, we are unable to prove Conjecture 1.4 in full generality. The problem is concerned with the global
coefficients in Arthur’s fine expansion of the geometric side. For GL(n) estimations of
the global coefficients are known [34]. For groups other than GL(n), currently very little
is known about these coefficients. That is why at moment we can only deal with a
restricted class of sequences of congruence groups which were considered in [16, Sect. 2].
Let $S$ be a finite set of primes. Let $\{K_j\}_{j \in \mathbb{N}}$ be a sequence of open compact subgroups
of $G(\mathbb{A}_f)$. We say that $\{K_j\}$ converges to 1 at $S$, denoted by $K_j \to 1$, if there exists
an open compact subgroup $K^S = \prod_{p \notin S} K_p$ of $G(\mathbb{A}_f^S)$ such that

$$K_j = K_{S,j} \times K^S, \quad \text{with} \quad K_{S,j} = \prod_{p \in S} K_{p,j}, \quad \text{and} \quad K_{S,j} \to 1, \quad (1.13)$$

where the latter condition means that for every open compact subgroup $U$ of $G_S := \prod_{p \in S} G(\mathbb{Q}_p)$ there exists $N_0 \in \mathbb{N}$ such that $K_{S,j} \subset U$ for $j \geq N_0$. An example is the
principal congruence subgroups $\Gamma_j := \Gamma(p^j)$ of $SL(n, \mathbb{Z})$ of level $p^j$, where $p$ is a fixed
prime. If $S = S_{\text{fin}}$, the set of all finite places, we will simply write $K_j \to 1$.

We need additional assumptions which are related to the spectral side of the trace
formula. Recall that in [23, Section 5], for a reductive group $G$ we introduced and studied
two natural properties, called (TWN) and (BD). The abbreviations stand for tempered winding numbers and bounded degree, respectively. These properties concern the behavior
of the intertwining operators associated to proper parabolic subgroups of $G$. They are
trivially satisfied if $G$ is anisotropic modulo the center. By [23, Prop. 5.5, Theorem 5.15]
they hold if $G$ equals GL(n) or SL(n). In many other cases they have been established in
[19], [21]. The main result is the following theorem.

**Theorem 1.5.** Let $G$ be a reductive algebraic group over $\mathbb{Q}$ which satisfies (TWN) and
(BD). Let $\tau \in \text{Rep}(G(\mathbb{R})^1)$ be irreducible and assume that $\tau \not\sim \tau_0$. Let $S$ be a finite set
of primes. Let $\{K_j\}_{j \in \mathbb{N}}$ be a sequence of open compact subgroups of $G(\mathbb{A}_f)$ with $K_j \to 1$
as $j \to \infty$. Let $X_j := X(K_j)$. Then

$$\lim_{j \to \infty} \frac{\log T_{X,j}(\tau)}{\text{vol}(X_j)} = l_{X}^{(2)}(\tau). \quad (1.14)$$

As mentioned above, we have $l_{X}^{(2)}(\tau) \neq 0$ if and only if $\delta(\tilde{X}) = 1$. Examples of groups
with $\delta(\tilde{X}) = 1$ are $SL(3)$, $SL(4)$ and $SO(p, q)$ with $pq$ odd. Examples of reductive groups
that satisfy properties (TWN) are given in [19] and property (BD) is discussed in [21].
Using [19, Theorem 3.11] and [21, Corollary 1], we get

**Corollary 1.6.**

(1) Let $G$ be an inner form of GL(n) or SL(n) defined over $\mathbb{Q}$. Let $\tau \in \text{Rep}(G(\mathbb{R})^1)$
be irreducible satisfying $\tau \not\sim \tau_0$. Let $S$ be a finite set of primes with $2 \notin S$. Then
(1.14) holds for every sequence \( \{K_j\}_{j \in \mathbb{N}} \) of open subgroups of a given open compact subgroup \( K_0 \) of \( G(\mathbb{A}_f) \) satisfying \( K_j \to 1 \) as \( j \to \infty \).

(2) Let \( G \) be a quasi-split classical group defined over \( \mathbb{Q} \). Let \( \tau \in \text{Rep}(G(\mathbb{R})^1) \) be irreducible satisfying \( \tau \not\cong \tau_\theta \). Let \( S \) be a finite set of primes with \( 2 \notin S \). Then (1.14) holds for every sequence of open subgroups \( \{K_j\}_{j \in \mathbb{N}} \) of a given open compact subgroup \( K_0 \) of \( G(\mathbb{A}_f) \) satisfying \( K_j \to 1 \) as \( j \to \infty \).

We note that among the classical groups, the special orthogonal groups \( \text{SO}(p, q) \) with \( pq \) odd satisfy \( \delta(\tilde{X}) = 1 \), which implies that \( t_X^{(2)}(\tau) \neq 0 \). Furthermore, \( \text{SO}(p, q) \) is quasi-split if \( |p - q| \leq 2 \).

Suppose that \( G \) satisfies strong approximation. Then Theorem 1.5 can be restated in terms of arithmetic subgroups of \( G(\mathbb{R}) \). Let \( K \subset G(\mathbb{A}_f) \) be an open compact subgroup of \( G(\mathbb{A}_f) \) and let \( \Gamma_K := G(\mathbb{Q}) \cap K \subset G(\mathbb{R}) \). Then \( \Gamma_K \) is a discrete subgroup of \( G(\mathbb{R}) \). By the strong approximation theorem [54, Theorem 7.12] it follows that the \( G(\mathbb{R}) \)-spaces \( G(\mathbb{Q}) \backslash G(\mathbb{A}) / K \) and \( \Gamma_K \backslash G(\mathbb{R}) \) are canonically isomorphic. A subgroup \( \Gamma \subset G(\mathbb{Q}) \) is called a congruence subgroup if there exists an open compact subgroup \( K_\Gamma \) of \( G(\mathbb{A}_f) \) such that \( \Gamma = G(\mathbb{Q}) \cap K_\Gamma \).

**Corollary 1.7.** Suppose that \( G \) is a simply connected \( \mathbb{Q} \)-simple group satisfying (TWN) and (BD) and that \( G(\mathbb{R})^1 \) is not compact. Let \( K_0 \subset G(\mathbb{A}_f) \) be a compact open subgroup and \( \Gamma_0 := G(F) \cap K_0 \). Let \( \{\Gamma_j\}_{j \in \mathbb{N}} \) be a sequence of congruence subgroups of \( \Gamma_0 \). Let \( S \) be a finite set of primes and assume that \( K_j \to 1 \) as \( j \to \infty \). Let \( \tau \in \text{Rep}(G(\mathbb{R})^1) \) be irreducible and assume that \( \tau \not\cong \tau_\theta \). Then

\[
\lim_{j \to \infty} \frac{\log T_{\Gamma_j \backslash \tilde{X}}(\tau)}{\text{vol}(\Gamma_j \backslash \tilde{X})} = t_X^{(2)}(\tau). \tag{1.15}
\]

The groups \( \text{SL}(n) \) are simple and simply connected and satisfy properties (TWN) and (BD) [23, Prop. 5.5, Theorem 5.15]. Moreover, for \( n \geq 3 \) every finite index subgroup of \( \text{SL}(n, \mathbb{Z}) \) is a congruence subgroup [9], [38]. Thus for every finite index subgroup \( \Gamma \) of \( \text{SL}(n, \mathbb{Z}) \), \( n \geq 3 \), there exists an open compact subgroup \( K_\Gamma \) of \( G(\mathbb{A}_f) \) such that \( \Gamma = G(\mathbb{Q}) \cap K_\Gamma \). By Corollary 1.7 we get

**Corollary 1.8.** Let \( \tilde{X}_n := \text{SL}(n, \mathbb{R}) / \text{SO}(n) \). Let \( \tau \in \text{Rep}(\text{SL}(n, \mathbb{R})) \) be irreducible and assume that \( \tau \not\cong \tau_\theta \). Let \( S \) be a finite set of primes. Let \( \{\Gamma_j\}_{j \in \mathbb{N}} \) be a sequence of finite index subgroups of \( \text{SL}(n, \mathbb{Z}) \) satisfying \( K_{\Gamma_j} \to 1 \) as \( j \to \infty \). Then

\[
\lim_{j \to \infty} \frac{\log T_{\Gamma_j \backslash \tilde{X}_n}(\tau)}{\text{vol}(\Gamma_j \backslash \tilde{X}_n)} = t_{\tilde{X}_n}^{(2)}(\tau). \tag{1.16}
\]

**Remark 1.9.** It is very desirable in the results above to remove the assumption that the sequence of open compact subgroups \( \{K_j\} \) converges to 1 at \( S \). This assumption is due to
the insufficient knowledge of the coefficients of the fine expansion of the geometric side of Arthur’s trace formula. For $\text{GL}(n)$ there exist appropriate estimations of the coefficients [34]. This was used in [36] to established (1.16) for the sequence of principal congruence subgroups $\{\Gamma(N)\}_{N \in \mathbb{N}}$ of $\text{SL}(n, \mathbb{Z})$ without any restriction, which was deduced from the corresponding result for $\text{GL}(n)$. Hoffmann [27] has described a conjectural approach to express the geometric side of the Arthur trace formula in terms of zeta integrals attached to prehomogeneous vector spaces. A successful completion of this program would lead to expressions of the coefficients in terms zeta functions attached to prehomogeneous vector spaces, which, in turn, would imply that the restriction on the sequences of congruence subgroups can be lifted in general. For certain unipotent classes in $\text{GL}(n)$ results have been proven in [13].

The approach described by Hoffmann [27] has been successfully carried out for classical groups of absolute rank up to two, especially for $\text{SL}(3)$, which is one of the cases where $t_{\tilde{X}}^{(2)}(\tau) \neq 0$. For $\text{SL}(3)$, the global coefficients are determined in [28] which can be used to get appropriate estimates for the global coefficients in the case of $\text{SL}(3)$. Thus for $n = 3$ we get

**Corollary 1.10.** Let $\tilde{X} := \text{SL}(3, \mathbb{R})/ \text{SO}(3)$. Let $\tau \in \text{Rep}(\text{SL}(3, \mathbb{R}))$ be irreducible and assume that $\tau \not\cong \tau_0$. Then for every sequence $\{\Gamma_j\}_{j \in \mathbb{N}}$ of finite index subgroups of $\text{SL}(3, \mathbb{Z})$ with $\text{vol}(\Gamma_j \setminus \tilde{X}) \to \infty$ we have

$$\lim_{j \to \infty} \frac{\log T_{\Gamma_j \setminus \tilde{X}}(\tau)}{\text{vol}(\Gamma_j \setminus \tilde{X})} = t_{\tilde{X}}^{(2)}(\tau) > 0. \quad (1.17)$$

**1.5. Method of proof**

Now we briefly explain our method to prove Theorems 1.1 and 1.2. To determine the asymptotic behavior of the regularized trace as $t \to +0$, we start with its definition (1.7) as the geometric side of the trace formula. Let $J_{\text{unip}}(f)$ be the unipotent contribution to the trace formula [5]. Then we show that $\phi_t^\nu$ can be replaced by a compactly supported function $\tilde{\phi}_t^\nu \in C^\infty(G(\mathbb{A})^1)$ with support in a sufficiently small neighborhood of 1 such that

$$\text{Tr}_{\text{reg}}(e^{-tA^\nu}) := J_{\text{unip}}(\tilde{\phi}_t^\nu) + O\left(e^{-c/t}\right) \quad (1.18)$$

as $t \to +0$. To analyze $J_{\text{unip}}(\tilde{\phi}_t^\nu)$, we use the fine geometric expansion [5] which expresses $J_{\text{unip}}(\tilde{\phi}_t^\nu)$ in terms of weighted orbital integrals. This reduces the proof of Theorem 1.1 to the study of weighted orbital integrals. In [35] we dealt with this problem for the group $\text{GL}(n)$. In this case all unipotent orbits are Richardson, which simplifies the analysis considerably. To deal with the weighted orbital integrals for an arbitrary reductive group $G$ we rely on [6]. Using the proof of [6, Corollary 6.2], we obtain an appropriate integral
expression for the weighted orbital integrals. Again the main issue is the analysis of the weight function and the proof that it has a certain logarithmic scaling behavior. Then, as in the case of GL(n) we insert a standard parametrix for the heat kernel into the weighted integral (5.10) and determine its asymptotic behavior as \( t \to 0 \). This leads to the proof of Theorem 1.1.

To prove Theorem 1.2, we use the spectral side of the trace formula. Let \( \phi_1^{\tau,p} \) be the function in \( \mathcal{C}(G(A)^1; K_f) \), which is defined in the same way as \( \phi_1^{\tau} \) in terms of the kernel of the heat operator \( e^{-t\Delta_p(\tau)} \) on the universal covering. Then by the trace formula we have

\[
\text{Tr}_{\text{reg}} \left( e^{-t\Delta_p(\tau)} \right) = J_{\text{spec}}(\phi_1^{\tau,p}).
\]

The key tool to deal with the spectral side is the refinement of the spectral expansion of the Arthur trace formula established in [22] (see Theorem 9.1). It states that

\[
J_{\text{spec}}(f) = \sum_{[M]} J_{\text{spec},M}(f), \quad f \in \mathcal{C}(G(A)^1).
\]

Here \([M]\) runs over the conjugacy classes of Levi subgroups of \( G \) and \( J_{\text{spec},M} \) is a distribution associated to \( M \). The term associated to \( G \) is \( \text{Tr} R_{\text{dis}}(f) \), where \( R_{\text{dis}} \) denotes the restriction of the regular representation of \( G(A)^1 \) in \( L^2(G(\mathbb{Q}) \backslash G(A)^1) \) to the discrete subspace. Using our assumption that \( \tau \not\equiv \tau_0 \), we obtain \( \dim \ker \Delta_p(\tau) = 0 \). Then it follows as in the compact case that there exists \( c > 0 \) such that

\[
\text{Tr} R_{\text{dis}}(\phi_1^{\tau,p}) = O(e^{-ct}), \quad \text{as} \quad t \to \infty.
\]

For a proper Levi subgroup \( M \) of \( G \), \( J_{\text{spec},M}(f) \) is an integral whose main ingredient are logarithmic derivatives of intertwining operators. The determination of the asymptotic behavior of \( J_{\text{spec},M}(\phi_1^{\tau,p}) \) as \( t \to \infty \) relies on two properties, one global and one local, of the intertwining operators. The global property is a uniform bound on the winding number of the normalizing factors of the intertwining operators in the co-rank one case. The bound that we need is (11.35), which was established in [42, Theorem 5.3]. The local property is concerned with the estimation of integrals of logarithmic derivatives of normalized local intertwining operators \( R_{\pi,v}(\tau,s) \), which are uniform in \( \tau \).

To prove Theorem 1.5 we follow the approach used in [36] in the case of SL(n). The new ingredients are the results of [19], [20] and [21] concerning the spectral side of the trace formula. These are estimations of the logarithmic derivatives of the global normalizing factors of the intertwining operators and bounds on the degrees of coefficients of local intertwining operators. At the moment, we can treat the geometric side only for \( G = \text{SL}(3) \) and \( G = \text{GL}(n) \) in all generality. For other reductive groups estimations of the global coefficients are not available. This is why we need to make restrictions on the sequences of congruence subgroups.
The paper is organized as follows. In Section 2 we fix notations and recall some basic facts. In Section 3 we begin with the study of the asymptotic expansion of the regularized trace of the heat operator. We show that for the derivation of the asymptotic expansion one can replace the geometric side of the trace formula by the unipotent contribution. Sections 4, 5 and 6 contain some preparatory material related to weighted unipotent orbital integrals. In Section 7 we show that the weighted unipotent orbital integrals with respect to test functions derived from the heat kernel admit an asymptotic expansion as \( t \to 0 \). In Section 8 we use this result combined with Arthur’s fine geometric expansion to prove Theorem 1.1. In Section 9 we recall the refined spectral expansion of Arthur’s trace formula. Section 10 is concerned with the study of logarithmic derivatives of local intertwining operators. In Section 11 we use the spectral side of the Arthur trace formula to prove Theorem 1.2 which concerns the large time asymptotic behavior of the regularized trace of the heat operator. Together with Theorem 1.1 this enables us to define the regularized analytic torsion. In the final section 12 we study the limiting behavior of the renormalized logarithm of the analytic torsion and prove Theorem 1.5.

2. Preliminaries

Let \( G \) be a reductive algebraic group defined over \( \mathbb{Q} \). We fix a minimal parabolic subgroup \( P_0 \) of \( G \) defined over \( \mathbb{Q} \) and a Levi decomposition \( P_0 = M_0 U_0 \), both defined over \( \mathbb{Q} \). Let \( \mathcal{F} \) be the set of parabolic subgroups of \( G \) which contain \( M_0 \) and are defined over \( \mathbb{Q} \). Let \( \mathcal{L} \) be the set of subgroups of \( G \) which contain \( M_0 \) and are Levi components of groups in \( \mathcal{F} \). For any \( P \in \mathcal{F} \) we write

\[
P = M_P N_P,
\]

where \( N_P \) is the unipotent radical of \( P \) and \( M_P \) belongs to \( \mathcal{L} \).

Let \( M \in \mathcal{L} \). Denote by \( A_M \) the \( \mathbb{Q} \)-split component of the center of \( M \). Put \( A_P = A_{M_P} \).

Let \( L \in \mathcal{L} \) and assume that \( L \) contains \( M \). Then \( L \) is a reductive group defined over \( \mathbb{Q} \) and \( M \) is a Levi subgroup of \( L \). We shall denote the set of Levi subgroups of \( L \) which contain \( M \) by \( \mathcal{L}^L(M) \). We also write \( \mathcal{F}^L(M) \) for the set of parabolic subgroups of \( L \), defined over \( \mathbb{Q} \), which contain \( M \), and \( \mathcal{P}^L(M) \) for the set of groups in \( \mathcal{F}^L(M) \) for which \( M \) is a Levi component. Each of these three sets is finite. If \( L = G \), we shall usually denote these sets by \( \mathcal{L}(M), \mathcal{F}(M) \) and \( \mathcal{P}(M) \).

Let \( X(M)_\mathbb{Q} \) be the group of characters of \( M \) which are defined over \( \mathbb{Q} \). Put

\[
a_M := \text{Hom}(X(M)_\mathbb{Q}, \mathbb{R}).
\]  

This is a real vector space whose dimension equals that of \( A_M \). Its dual space is

\[
a_M^* = X(M)_\mathbb{Q} \otimes \mathbb{R}.
\]
We shall write,
\[ a_P = a_{M_P}, \quad A_0 = A_{M_0} \quad \text{and} \quad a_0 = a_{M_0}. \] (2.20)

For \( M \in \mathcal{L} \) let \( A_M(\mathbb{R})^0 \) be the connected component of the identity of the group \( A_M(\mathbb{R}) \). Let \( W_0 = N_{G(Q)}(A_0)/M_0 \) be the Weyl group of \((G, A_0)\), where \( N_{G(Q)}(H) \) is the normalizer of \( H \) in \( G(Q) \). For any \( s \in W_0 \) we choose a representative \( w_s \in G(Q) \). Note that \( W_0 \) acts on \( \mathcal{L} \) by \( sM = w_sMw_s^{-1} \). For \( M \in \mathcal{L} \) let \( W(M) = N_{G(Q)}(M)/M \), which can be identified with a subgroup of \( W_0 \).

For any \( L \in \mathcal{L}(M) \) we identify \( a^*_L \) with a subspace of \( a^*_M \). We denote by \( a^L_M \) the annihilator of \( a^*_L \) in \( a_M \). Then \( r = \dim a^G_0 \) is the semisimple rank of \( G \). We set
\[ \mathcal{L}_1(M) = \{ L \in \mathcal{L}(M) : \dim a^L_M = 1 \} \]

and
\[ \mathcal{F}_1(M) = \bigcup_{L \in \mathcal{L}_1(M)} \mathcal{P}(L). \] (2.21)

We shall denote the simple roots of \((P, A_P)\) by \( \Delta_P \). They are elements of \( X(A_P)_Q \) and are canonically embedded in \( a^*_P \). Let \( \Sigma_P \subset a^*_P \) be the set of reduced roots of \( A_P \) on the Lie algebra of \( G \). For any \( \alpha \in \Sigma_M \) we denote by \( \alpha^\vee \in a_M \) the corresponding co-root. Let \( P_1 \) and \( P_2 \) be parabolic subgroups with \( P_1 \subset P_2 \). Then \( a^*_{P_2} \) is embedded into \( a^*_{P_1} \), while \( a_{P_2} \) is a natural quotient vector space of \( a_{P_1} \). The group \( M_{P_2} \cap P_1 \) is a parabolic subgroup of \( M_{P_2} \). Let \( \Delta_{P_1}^{P_2} \) denote the set of simple roots of \( (M_{P_2} \cap P_1, A_{P_1}) \). It is a subset of \( \Delta_{P_1} \). For a parabolic subgroup \( P \) with \( P_0 \subset P \) we write \( \Delta_P^{P_0} := \Delta_{P_0}^{P_1} \).

Let \( A \) (resp. \( A_f \)) be the ring of adeles (resp. finite adeles) of \( \mathbb{Q} \). We fix a maximal compact subgroup \( K = \prod_v K_v = K_\infty \cdot K_f \) of \( G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f) \). We assume that the maximal compact subgroup \( K \subset G(\mathbb{A}) \) is admissible with respect to \( M_0 \) [6, §1]. Let \( H_M : M(\mathbb{A}) \to a_M \) be the homomorphism given by
\[ e^{\langle \chi, H_M(m) \rangle} = |\chi(m)|_A = \prod_v |\chi(m_v)|_v \] (2.22)

for any \( \chi \in X(M) \) and denote by \( M(\mathbb{A})^1 \subset M(\mathbb{A}) \) the kernel of \( H_M \).

Let \( g \) and \( \mathfrak{t} \) denote the Lie algebras of \( G(\mathbb{R}) \) and \( K_\infty \), respectively. Let \( \theta \) be the Cartan involution of \( G(\mathbb{R}) \) with respect to \( K_\infty \). It induces a Cartan decomposition \( g = p \oplus \mathfrak{t} \). We fix an invariant bi-linear form \( B \) on \( g \) which is positive definite on \( p \) and negative definite on \( \mathfrak{t} \). This choice defines a Casimir operator \( \Omega \) on \( G(\mathbb{R}) \), and we denote the Casimir eigenvalue of any \( \pi \in \Pi(G(\mathbb{R})) \) by \( \lambda_\pi \). Similarly, we obtain a Casimir operator \( \Omega_{K_\infty} \) on \( K_\infty \) and write \( \lambda_\tau \) for the Casimir eigenvalue of a representation \( \tau \in \Pi(K_\infty) \) (cf. [11, §2.3]). The form \( B \) induces a Euclidean scalar product \( (X, Y) = -B(X, \theta(Y)) \) on \( g \) and all its subspaces. For \( \tau \in \Pi(K_\infty) \) we define \( \|\tau\| \) as in [14, §2.2]. Note that the
restriction of the scalar product $(\cdot, \cdot)$ on $\mathfrak{g}$ to $a_0$ gives $a_0$ the structure of a Euclidean space. In particular, this fixes Haar measures on the spaces $a_M^L$ and their duals $(a_M^L)^\ast$. We follow Arthur in the corresponding normalization of Haar measures on the groups $M(A)$ ([2, §1]).

Let $L^2_{\text{disc}}(A_M(R)^0M(Q)\setminus M(A))$ be the discrete part of $L^2(A_M(R)^0M(Q)\setminus M(A))$, i.e., the closure of the sum of all irreducible subrepresentations of the regular representation of $M(A)$. We denote by $\Pi_{\text{disc}}(M(A))$ the countable set of equivalence classes of irreducible unitary representations of $M(A)$ which occur in the decomposition of the discrete subspace $L^2_{\text{disc}}(A_M(R)^0M(Q)\setminus M(A))$ into irreducible representations.

Let $H$ be a topological group. We will denote by $\Pi(H)$ the set of equivalence classes of irreducible unitary representations of $H$.

Next we introduce the space $C(G(A)^1)$ of Schwartz functions. For any compact open subgroup $K_f$ of $G(A_f)$ the space $G(A)^1/K_f$ is the countable disjoint union of copies of $G(R)^1 = G(R) \cap G(A)^1$ and therefore, it is a differentiable manifold. Any element $X \in \mathcal{U}(g^{1\omega}_\xi)$ of the universal enveloping algebra of the Lie algebra $g^{1\omega}_\xi$ of $G(R)^1$ defines a left invariant differential operator $f \mapsto f \ast X$ on $G(A)^1/K_f$. Let $C(G(A)^1; K_f)$ be the space of smooth right $K_f$-invariant functions on $G(A)^1$ which belong, together with all their derivatives, to $L^1(G(A)^1)$. The space $C(G(A)^1; K_f)$ becomes a Fréchet space under the seminorms

$$
\|f \ast X\|_{L^1(G(A)^1)}, \ X \in \mathcal{U}(g^{1\omega}_\xi).
$$

Denote by $C(G(A)^1)$ the union of the spaces $C(G(A)^1; K_f)$ as $K_f$ varies over the compact open subgroups of $G(A_f)$ and endow $C(G(A)^1)$ with the inductive limit topology.

3. Asymptotic expansion of the regularized trace

Let $G$ be a reductive algebraic group over $Q$. We assume that its center $Z_G$ is $Q$-split and let $A_G$ be the identity component of $Z_G(R)$. Then $G(A) = G(A)^1 \times A_G$ and $G(R) = G(R)^1 \times A_G$ with $G(R)^1 = G(A)^1 \cap G(R)$, a semisimple real Lie group. Let $\theta : G(R) \rightarrow G(R)$ be a Cartan involution and $K_\infty = G(R)^0$ its fixed points. For each prime $p$ fix some maximal compact subgroup $K_p$ of $G(Q_p)$ and let $K_f = \prod_p K_p$. Let $K_f$ be a finite index subgroup of $K_f$ and $X(K_f) = G(Q) \setminus G(A)^1/K_\infty^0 \cdot K_f$, where $K_\infty^0$ is the connected identity component of $K_\infty$. Let $r$ denote the split semisimple rank of $G$ so that $r = \dim a_G^\circ$.

We recall the definition of the regularized trace. For that we adopt the notation from [35, §11-12]. Let $\nu : K_\infty \rightarrow GL(V_\nu)$ be a finite dimensional unitary representation. Let $\tilde{\Delta}_\nu$ be the Bochner-Laplace operator attached to $\nu$ on the universal covering $\tilde{X} = G(R)^1/K_\infty^0$ of $X(K_f)$. Let $H^\nu_F : G(R)^1 \rightarrow GL(V_\nu)$ be the convolution kernel associated with $\tilde{\Delta}_\nu$, and let $h^\nu_F = \text{Tr} H^\nu_F$. We extend $h^\nu_F$ to $G(R)$ by $h^\nu_F(ag) = h^\nu_F(g)$ for all $a \in A_G, g \in G(R)^1$. Let $1_{K_f} : G(A_f) \rightarrow \mathbb{C}$ be the characteristic function of $K_f$. Put
\[ \chi_{K_f} := \frac{1_{K_f}}{\text{vol}(K_f)} \]  

(3.1) 

and 

\[ \phi_t^\nu(g) = h_t^\nu(g_{\infty})\chi_{K_f}(g_f) \]

for \( g = g_{\infty} \cdot g_f \in G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f) \). Let \( J_{\text{geom}} \) denote the geometric side of Arthur’s trace formula. The regularized trace of \( e^{-t\Delta_v} \) is defined by 

\[ \text{Tr}_{\text{reg}}(e^{-t\Delta_v}) = J_{\text{geom}}(\phi_t^\nu). \]  

(3.2) 

This is well-defined because \( \phi_t^\nu \) and all its derivatives are in \( L^1(G(\mathbb{A})^1) \) so that \( J_{\text{geom}}(\phi_t^\nu) \) is well-defined by [22].

### 3.1. Reduction to unipotent distributions

The proof of Theorem 1.1 rests on an asymptotic expansion of certain unipotent distributions \( J_{M}(\mathcal{O}, \cdot) \), which will be introduced in §5 and which are defined only for compactly supported test functions. To state this result we first need to construct compactly supported test functions from \( \phi_t^\nu \).

Let \( d(\cdot, \cdot) : \hat{X} \times \hat{X} \to [0, \infty) \) be the geodesic distance on \( \hat{X} \), and put \( r(g_{\infty}) = d(g_{\infty}x_0, x_0) \) where \( x_0 = K_{\infty} \in \hat{X} \) is the base points. Let \( 0 < a < b \) be sufficiently small real numbers and let \( \beta : \mathbb{R} \to [0, \infty) \) be a smooth function supported in \([-b, b]\) such that \( \beta(y) = 1 \) for \( 0 \leq |y| \leq a \), and \( 0 \leq \beta(y) \leq 1 \) for \( |y| > a \). Define 

\[ \psi_t^\nu(g_{\infty}) = \beta(r(g_{\infty}))h_t^\nu(g_{\infty}) \]  

(3.3) 

and 

\[ \tilde{\phi}_t^\nu(g) = \psi_t^\nu(g_{\infty})\chi_{K_f}(g_f) \]  

(3.4) 

for \( g = g_{\infty} \cdot g_f \in G(\mathbb{A}) = G(\mathbb{R}) \cdot G(\mathbb{A}_f) \). Then \( \tilde{\phi}_t^\nu \in C_c^\infty(G(\mathbb{A})^1) \) and \( \psi_t^\nu \in C_c^\infty(G(\mathbb{R})^1) \). By [35, Proposition 12.1] there is some \( c > 0 \) such that for every \( 0 < t \leq 1 \) we have 

\[ |J_{\text{geom}}(\tilde{\phi}_t^\nu) - J_{\text{geom}}(\phi_t^\nu)| \ll e^{-c/t}. \]  

(3.5) 

We note that in [35, Sect. 12] we made the assumption that \( G = \text{GL}_n \) or \( G = \text{SL}_n \). However, the proof of the proposition holds for every reductive algebraic group \( G \). The next result reduces the considerations to the unipotent contribution to the geometric side. Before we state it, we recall the coarse geometric expansion of Arthur’s trace formula [8, §10]: Two elements \( \gamma_1, \gamma_2 \in G(\mathbb{Q}) \) are called coarsely equivalent if their semisimple parts (in the Jordan decomposition) are conjugate in \( G(\mathbb{Q}) \). Then for any \( f \in C_c^\infty(G(\mathbb{A})^1) \) we have
\[ J_{\text{geom}}(f) = \sum_{\sigma} J_\sigma(f), \]

where \( \sigma \) runs over the coarse equivalence classes in \( G(\mathbb{Q}) \), and the distribution \( J_\sigma \) is supported in the set of all \( g \in G(\mathbb{A})^1 \) whose semisimple part is conjugate in \( G(\mathbb{A}) \) to some semisimple element in \( \sigma \). If \( \sigma \neq \sigma' \), the supports of \( J_\sigma \) and \( J_{\sigma'} \) are disjoint. Note that the set of unipotent elements in \( G(\mathbb{Q}) \) constitute a single equivalence class \( \sigma_{\text{unip}} \) and we write \( J_{\text{unip}} = J_{\sigma_{\text{unip}}} \).

**Proposition 3.1.** If \( K_f \) is neat and the support of \( \beta \) is sufficiently small, then

\[ J_{\text{geom}}(\hat{\phi}_t^\nu) = J_{\text{unip}}(\hat{\phi}_t^\nu). \] (3.6)

**Proof.** Let \( \rho : G \rightarrow \text{GL}_N \) be a faithful representation of \( G \). For each prime \( p \) we can find \( \nu_p \geq 0 \) such that \( K_p \subseteq \rho^{-1}(\text{GL}_N(p^{-\nu_p}\mathbb{Z}_p)) \), and \( \nu_p = 0 \) for all but finitely many \( p \). Hence \( K_f \subseteq \rho^{-1}(\text{GL}_N(M^{-1}\mathbb{Z})) \) with \( M = \prod p^{\nu_p} \).

Let \( \chi : \text{GL}_N(\mathbb{A}) \rightarrow \mathbb{A}^N \) be defined by mapping elements of \( \text{GL}_N(\mathbb{A}) \) onto the sequence of coefficients of their characteristic polynomials (omitting the coefficient 1 of the highest degree monomial). Let \( f_\infty \in C_c^\infty(\text{GL}(\mathbb{R})^1) \). Suppose that \( g \in G(\mathbb{A})^1 \) is in the support of \( f_\infty \cdot 1_{K_f} \) and that the semisimple part of \( g \) is conjugate in \( G(\mathbb{A}) \) to some \( \sigma \in G(\mathbb{Q}) \). Then \( \chi(\rho(g)) = \chi(\rho(\sigma)) \in \mathbb{Q}^N \), and further \( \chi(\rho(g)) \in p^{-N\nu_p}\mathbb{Z}_p^N \) for every prime \( p \). Hence \( \chi(\rho(g)) \in M^{-N}\mathbb{Z}^N \). If we choose \( f_\infty \) to be supported in a sufficiently small, bi-\( K_{\infty} \)-invariant neighborhood of the identity of \( G(\mathbb{R})^1 \) (the support of \( f_\infty \) will possibly depend on \( M \)), we can arrange that \( \chi(\rho(g)) \in \mathbb{Z}^N \) so that the eigenvalues of \( \rho(g) \) are all algebraic integers. Shrinking the support of \( f_\infty \) even further if necessary, we can conclude the eigenvalues of \( \rho(g) \) must all be roots of unity. Otherwise, the group generated by the matrix \( \rho(g) \) would not be contained in a compact set. By assumption, \( K_f \) is neat so that these eigenvalues in fact need to be equal to 1. The semisimple part of \( \rho(g) \) therefore equals \( I_N \) (the identity matrix in \( \text{GL}_N(\mathbb{Q}) \)), that is, the semisimple part of \( g \) equals the identity in \( G(\mathbb{A})^1 \) so that \( g \) is unipotent.

By the discussion above on the coarse geometric expansion, we can now find a bi-\( K_{\infty} \)-invariant neighborhood \( \Omega \) of the identity in \( G(\mathbb{R})^1 \) such that whenever \( f_\infty \in C_c^\infty(\text{GL}(\mathbb{R})^1) \) is supported in \( \Omega \) we have \( J_{\text{geom}}(f) = J_{\text{unip}}(f) \) where \( f = f_\infty \cdot 1_{K_f} \in C_c^\infty(\text{GL}(\mathbb{A})^1) \). Hence if we choose the support of \( \beta \) in the definition of \( \hat{\phi}_t^\nu \) sufficiently small, we obtain the proposition. \( \square \)

In light of (3.5) and (3.6) we therefore only need to study the asymptotic expansion of \( J_{\text{unip}}(\hat{\phi}_t^\nu) \) as \( t \searrow 0 \) to prove Theorem 1.1. This will be done in the following sections by using the fine geometric expansion of \( J_{\text{unip}} \) involving weighted orbital integrals over the unipotent conjugacy classes in \( G(\mathbb{R}) \).
4. Preliminaries on unipotent conjugacy classes and integrals

Until §8 we will be concerned only with the real Lie group $G(\mathbb{R})^1$ so that we write $G_\infty$ for $G(\mathbb{R})^1$.

4.1. Notation

In abuse of our previous notation, we write $P_0 = M_0U_0 \subseteq G_\infty$ for the minimal parabolic $P_0(\mathbb{R}) \cap G_\infty = (M_0(\mathbb{R}) \cap G_\infty)(U_0(\mathbb{R}) \cap G_\infty)$ in $G_\infty$ until §8.

A parabolic subgroup $P$ of $G_\infty$ is called standard if it contains $P_0$, and semistandard if it contains $M_0$. A Levi subgroup $M$ in $G_\infty$ is called semistandard if it equals the Levi component containing $M_0$ of some semistandard parabolic subgroup. We write $\mathcal{L}$ for the set of semistandard Levi subgroups of $G_\infty$. If $M \in \mathcal{L}$, we write $\mathcal{L}(M)$ for the set of all $L \in \mathcal{L}$ with $M \subseteq L$ so that $\mathcal{L} = \mathcal{L}(M_0)$.

If $L \in \mathcal{L}$, then $P_0^L := P_0 \cap L = M_0(U_0 \cap L)$ is a minimal parabolic subgroup in $L$, and $P \mapsto P \cap L$ defines a surjective map from standard parabolic subgroups in $G_\infty$ to standard parabolic subgroups in $L$ (with respect to $P_0^L$). Similarly, we get surjective maps from semistandard parabolic and semistandard Levi subgroups in $G_\infty$ to such subgroups in $L$ (with respect to $M_0$). If $M, L \in \mathcal{L}$, $M \subseteq L$, we write $\mathcal{L}^L(M)$ for the semistandard Levi subgroups in $L$ containing $M$. Further, we write $\mathcal{F}^L(M)$ for the set of all semistandard parabolic subgroups in $L$ containing $M$. If $L = G_\infty$, we write $\mathcal{F}(M) = \mathcal{F}^{G_\infty}(M)$. Though this clashes with our global notation previously used, we hope that it will not lead to any confusion.

4.2. Unipotent conjugacy classes

We recall some basic facts on unipotent conjugacy classes and nilpotent orbits, which can for example be found in [57] and [6, §§5-6]. Let $M \in \mathcal{L}$ and let $\mathcal{O} \subseteq M$ be a unipotent $M$-conjugacy class in $M$, generated by $v_0 \in \mathcal{O}$. If $L \in \mathcal{L}(M)$, we write $\mathcal{O}^L$ for the union of unipotent conjugacy class in $L$ whose intersection with $v_0V(\mathbb{R})$ is open in $v_0V(\mathbb{R})$ for every parabolic subgroup $Q = MV \subseteq L$. This union is in fact finite, and the conjugacy classes are all contained in the same geometric unipotent conjugacy class (namely the induced conjugacy class) of $L$. Thus the dimension of $\mathcal{O}^L$ is well-defined.

Let $L \in \mathcal{L}(M)$, and let $\mathcal{O}_0^L \subseteq \mathcal{O}^L$ be one of the conjugacy classes. Let $P^L \subseteq L$ be a Jacobson–Morozov parabolic associated with $\mathcal{O}_0^L$ in $L$. We can choose $P^L$ to be standard, and we write $P^L = MLU^L$ for its Levi decomposition with $M^L \supseteq M_0$. Let $\mathfrak{l}$ denote the Lie algebra of $L$, and let

$$\mathfrak{l} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{l}_i$$

be the grading attached to the standard triple corresponding to our choice of Jacobson–Morozov parabolic $P^L$. Put $\mathfrak{u}^L_i := \bigoplus_{i > j} \mathfrak{l}_j$. Let $X_0 \in \mathfrak{l}_2$ such that $u := e^{X_0} \in \mathcal{O}_0^L$. As
explained in [6, §10] the Jacobson–Morozov parabolic can be chosen to be the same for each conjugacy class contained in $O^L$.

4.3. Measures on unipotent classes

We keep the notation from §4.2. To define distributions on the unipotent conjugacy classes, we need to fix measures. We fix once and for all some normalization of measures on $G_\infty$, on $K_\infty$, on the semistandard Levi subgroups in $G_\infty$, and on the unipotent radicals of the semistandard parabolic subgroups. We choose those measures such that they are compatible with respect to Iwasawa decomposition. We also fix a normalization of the measures on the vector spaces $I_i$.

Let $L_{u_0}$ be the centralizer of $u_0$ in $L$. Then $L_{u_0} \setminus L$ is diffeomorphic to $O^L_0$ and $L_{u_0}$ is unimodular being a unipotent group. The quotient measure on $L_{u_0} \setminus L$ (denoted by $d^u g$) defines an $L$-invariant measure on $O^L_0$ which in fact is a Radon measure on $O^L_0$ and has an explicit description:

Proposition 4.1 ([57]). There exists $c > 0$ and a polynomial $\varphi : I_2 \to \mathbb{C}$ of degree $\dim I_1$ such that if $f \in C^\infty_c(L)$, then

$$\int_{L_{u_0} \setminus L} f(g^{-1}u_0g) d^u g = c \int_{V_0 u^L_0} \int_{V_0} f_{K^L_\infty}(e^{X+Z})|\varphi(X)|^{1/2} dZ dX \tag{4.7}$$

where $V_0 \subseteq I_2$ is the orbit of $X_0$ under $M^L$ (a dense subset of $I_2$), and $f_{K^L_\infty}$ is defined by

$$f_{K^L_\infty}(g) := \int_{K^L_\infty} f(k^{-1}gk) dk, \ g \in L.$$

Note that we can replace $V_0$ by $I_2$ in the integral. Hence $|\varphi(X)|^{1/2} dZ dX$ defines invariant measures on every conjugacy class contained in $O^L$.

If we want to emphasize the dependence on $L$, we write $V^L_0, I^L, \varphi^L$ etc. If $Q = L V$ is a semistandard parabolic subgroup, we might also write $V^Q_0 := V^L_0, I^Q := I^L, \varphi^Q := \varphi^L$ etc.

4.4. Behavior under induction

Let $O \subseteq M, L \in \mathcal{F}(M)$ be as before. We can induce in stages, that is, $O^{G_\infty}_0 \subseteq O^{G_\infty}_0$. If $O^{G_\infty}_0 \subseteq O^{G_\infty}$ is one of the $G_\infty$-conjugacy classes, the invariant measure on $O^{G_\infty}_0$ is then given by a constant multiple of

$$\int_{N_Q I_2} \int_{L_{u^L_0}} \int_{I_2} f_{K^\infty}(e^{X+Z}n)|\varphi^L(X)|^{1/2} dZ dX dn \tag{4.8}$$

for any $f \in C^\infty_c(G_\infty)$ where $Q \in \mathcal{F}(L)$ is such that $L = L_Q$ is the Levi component of $Q$, $N_Q$ the unipotent radical of $Q$, and $f_{K^\infty}(g) = \int_{K^\infty} (k^{-1}gk) dk$. 

Remark 4.2. The dimension of a unipotent orbit can be computed in terms of the dimensions of the grading coming from the attached standard triple. More precisely, \( \dim \mathcal{O}^L = \dim \mathcal{O}_0^L = 2 \dim u_1^L + \dim I_1 \), see [17, Lemma 4.1.3] (this Lemma is proven in [17] only over \( \mathbb{C} \) but its proof stays valid over \( \mathbb{R} \)). Taking into account that \( \dim V_0^L = \dim I_2 \), and \( \deg \varphi^L = \dim I_1 \) we get

\[
\dim \mathcal{O}^L = 2 \left( \dim u_2^L + \dim V_0^L \right) + \deg \varphi^L,
\]

in particular, the dimension is even. Suppose that \( Q \in \mathcal{F}(M) \) is any semistandard parabolic subgroup with Levi component \( L \) and unipotent radical \( N \), \( Q = LN \). Then \( \dim \mathcal{O}^{G_\infty} = 2 \dim N + \dim \mathcal{O}^L \) so that

\[
\dim \mathcal{O}^{G_\infty} = 2 \left( \dim N + \dim u_2^L + \dim V_0^L \right) + \deg \varphi^L.
\]

5. Weighted unipotent integrals

5.1. Introduction

Arthur’s fine geometric expansion and his splitting formula (see §§8.1-8.2) describe \( J_{\text{unip}} \) as a linear combination of certain products of real and \( p \)-adic weighted unipotent integrals. For our purposes we need to be concerned only with the archimedean case for which we follow [6]: Let \( f \in C_\infty^\infty(G_\infty) \) and let \( \mathcal{O} \) be a unipotent conjugacy class in \( M \). The archimedean weighted orbital integrals \( J_M^{G_\infty}(f, \mathcal{O}) \) can be defined as sum of integrals over \( \mathcal{O}^{G_\infty} \) against certain non–invariant measure. Those non–invariant measures can be described as follows: Using the proof of [6, Corollary 6.2] we have

\[
J_M^{G_\infty}(f, \mathcal{O}) = \sum_{Q \in \mathcal{F}(M)} c(Q, \mathcal{O}) \int_{\mathcal{N}_Q} \int_{V_0^Q} \int f_{K_\infty}(e^{X+Z} n) w_{M,\mathcal{O}}^Q(e^{X+Z}) |\varphi^Q(X)|^{1/2} dZ dX dn dk,
\]

(5.9)

where the notation is as follows:

- \( f_{K_\infty}(g) = \int_{K_\infty} f(k^{-1} g k) dk \)
- \( w_{M,\mathcal{O}}^Q \) is a certain weight function discussed in [6]. We will study this weight function in more detail below,
- \( c(Q, \mathcal{O}) > 0 \) are suitable constants coming from the normalization of measures in (4.7) and (4.8),
- for \( Q \in \mathcal{F}(M) \), \( Q = L_QN_Q \) denotes its Levi decomposition with \( L_Q \) its Levi component, \( L_Q \supseteq M \), and \( N_Q \) its unipotent radical.

Set \( u_{Q,1} = u_1^{L_Q} \oplus n_Q \) with \( n_Q \) the Lie algebra of \( N_Q \). We extend \( \varphi^{L_Q} \) and \( w_{M,\mathcal{O}}^Q(\exp(\cdot)) \) to all of \( u_{Q,1} \) by projecting to \( u_1^{L_Q} \) along \( n_Q \). Then we can write the integral above also as
$$J_M^{G_\infty}(f, \mathcal{O}) = \sum_{Q \in \mathcal{F}(M)} c(Q, \mathcal{O}) \int_{u_{Q,1}} f_{\mathcal{K}_\infty}(e^Y) w_{M,\mathcal{O}}^Q(e^Y) |\varphi^Q(Y)|^{1/2} dY. \quad (5.10)$$

Note that for each $Q$, the integral is over the same union of unipotent orbit $\mathcal{O}^{G_\infty}$ but with different weight functions.

5.2. An asymptotic expansion of the weights

We now study the functions $w_{M,\mathcal{O}}^Q(\cdot)$ from (5.9) in more detail. For convenience of notation, we only consider the case $Q = G_\infty$. We write $w_{M,\mathcal{O}} = w_{M,\mathcal{O}}^{G_\infty}$.

We fix an embedding $\iota : G_\infty \hookrightarrow \text{GL}_n(\mathbb{R})^1$ that satisfies certain properties. Write $H = \iota(G_\infty)$, $S = \iota(T(\mathbb{R}))$, $N_0 = \iota(U(\mathbb{R}))$, where $T(\mathbb{R})$ is a maximal split torus in $G_\infty$, and $N_0(\mathbb{R})$ the unipotent radical of our fixed minimal parabolic subgroup of $G_\infty$. Then we assume $\iota$ to satisfy the following:

- $H$ is self-adjoint;
- $S$ is contained in the group of diagonal matrices $T_0 \subseteq \text{GL}_n(\mathbb{R})^1$;
- $N_0$ is contained in the group of unipotent upper triangular matrices $U_0 \subseteq \text{GL}_n(\mathbb{R})^1$;
- The restriction of the positive roots of $(T_0, U_0)$ to $S$ are positive roots of $(S, N_0)$; every root of $(S, N_0)$ is obtained this way.

The existence of such an embedding follows from [54, Proposition 3.13]. In the following we will write $\mathfrak{h}$ for the Lie algebra of $H$, $G_n = \text{GL}_n(\mathbb{R})^1$ and $\mathfrak{g}_n$ for the Lie algebra of $G_n$.

Let $P = MN$ be a semi-standard parabolic subgroup in $H$ and let $\mathcal{O} \subseteq M$ be a unipotent conjugacy class. Let $P_1 = MN_1$ be another semi-standard parabolic subgroup with the same Levi component $M$. Let $\varpi$ be a weight on $\mathfrak{a}_M/\mathfrak{a}_H$ that is an extremal weight for an irreducible representation $\Lambda_{\varpi}$ of $H$ on a finite dimensional vector space $V_{\varpi}$, defined over $\mathbb{R}$, which is also $P$-dominant. Let $H_P : H \rightarrow \mathfrak{a}_M$ be the Iwasawa projection.

Then for any $h \in H$, Arthur defines a weight function $v_P(\varpi, h)$ by $v_P(\varpi, h) = e^{-\varpi(H_P(h))}$ and as shown in [6, (3.3)], it satisfies

$$v_P(\varpi, h) = e^{-\varpi(H_P(h))} = \|\Lambda_{\varpi}(h^{-1})\phi_\varpi\|, \quad h \in H,$$

for $\phi_\varpi$ a unit vector in the representation space $V_{\varpi}$ of $\Lambda_{\varpi}$ with respect to a fixed norm $\| \cdot \|$ on $V_{\varpi}$.

Let $\pi = uv \in P_1$ with $u \in \mathcal{O}$ and $v \in N_1$. Let $a \in A_M$ be regular. Then there is a unique $n \in N_1$ such that

$$a\pi = n^{-1}aun. \quad (5.11)$$
Therefore, \( a \mapsto \Lambda_\varpi(n^{-1})\phi_\varpi \) is a rational function on a dense subset of, and hence on all of \( A_M/A_H \).

By Arthur’s construction \([6, \text{p. 238–239}]\) there exist unique integers \( k_\beta \geq 0 \) such that

\[
\lim_{a \to 1} \prod_{\beta \in \Sigma_P \cap \Sigma_{\varpi}} (a^\beta - a^{-\beta})^{k_\beta} v_P(\varpi, n)
\]

exists and is non-zero on a dense subset of \( \mathcal{O} N_1 \). Here \( \Sigma_P \) denotes the set of reduced roots of \( A_M \) on \( N \), and similarly, \( \Sigma_{\varpi} \) is the set of reduced roots on the opposite parabolic. The limit is in fact of the form \( \|W_\varpi(1, \pi)\| \) with \( W_\varpi(1, \pi) \in V_\varpi \) a polynomial on \( \mathcal{O} N_1 \).

The weight functions \( w_M(\pi) \) appearing in the weighted unipotent integrals are then of the form

\[
w_{M, \mathcal{O}}(\pi) = \sum_{\Omega} c_\Omega \prod_{\varpi \in \Omega} \log \|W_\varpi(1, \pi)\|
\]

where \( \Omega \) runs over all finite subsets of extremal weights of \( a_M \), and \( c_\Omega \in \mathbb{C} \) are coefficients which vanish for all but finitely many of the \( \Omega \)s. Note that \( w_{M, \mathcal{O}}(\pi) \) attains a finite value on a dense subset of \( \pi \)s.

### 5.3. Extending polynomials

We recall the notion of a Jacobson–Morozov parabolic subalgebra. Recall that \( \mathcal{O} \subseteq M \) denotes a unipotent conjugacy class, and let \( \mathcal{N} \subseteq \mathfrak{m} \) be the corresponding nilpotent orbit. By the Jacobson–Morozov Theorem \([57, \text{§2}]\) we can find an \( \mathfrak{sl}_2 \)-triple \((h_0, x_0, y_0)\) for \( \mathcal{N} \) in \( \mathfrak{m} \) with \( h_0 \) semisimple and \( x_0, y_0 \) nilpotent. We decompose \( \mathfrak{m} \) into eigenspaces under \( h_0 \), that is \( \mathfrak{m} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{m}_k \) with \( \mathfrak{m}_k = \{X \in \mathfrak{m} \mid [h_0, X] = kX\} \). Let \( \tilde{M} \) denote the centralizer of \( A_M \) in \( G_n \) and let \( \tilde{\mathfrak{m}} \subseteq \mathfrak{g} \) be its Lie algebra. Then \( \tilde{M} \) is a semi–standard Levi subgroup of \( G_n \) with \( M \subseteq \tilde{M} \), \( \tilde{\mathfrak{m}} \subseteq \tilde{\mathfrak{m}} \). Then \( h_0 \) also defines a grading on \( \tilde{\mathfrak{m}} \), \( \tilde{\mathfrak{m}} = \bigoplus_{k \in \mathbb{Z}} \tilde{\mathfrak{m}}_k \).

Let \( \mathfrak{q}^M = \bigoplus_{k \geq 0} \mathfrak{m}_k \subseteq \mathfrak{m} \) be the corresponding Jacobson–Morozov parabolic subalgebra. Let \( \mathfrak{v}^M = \bigoplus_{k > 0} \mathfrak{m}_k \) and \( \mathfrak{v}^M_1 = \bigoplus_{k > 1} \mathfrak{m}_k \). We also define \( \tilde{\mathfrak{v}}^M = \bigoplus_{k > 0} \tilde{\mathfrak{m}}_k \) and \( \tilde{\mathfrak{q}}^M = \bigoplus_{k \geq 0} \tilde{\mathfrak{m}}_k \) so that \( \mathfrak{v}^M \) is a vector subspace of \( \tilde{\mathfrak{v}}^M \).

Then each \( u \in \mathcal{O} \) can be written as \( k^{-1} e^{X} k \) for \( k \in K \cap M \) and a unique \( X \in \mathfrak{v}^M_1 \). Note that \( W_\varpi(1, \pi) \) is invariant under the conjugation by elements of \( K \cap M \) by \([6, (3.7)]\) so that we can assume that \( \pi = e^{X+Y} \) with \( X \in \mathfrak{v}^M_1 \) and \( Y \in \mathfrak{n}_1 \) where \( \mathfrak{n}_1 \) is the Lie algebra of \( N_1 \). By \([6, \text{p. 253}]\) \( W_\varpi(1, \pi) \) is a polynomial in \( X + Y \). Similarly, \( v_P(\varpi, e^{Y}) \) is a polynomial in \( Y \in \mathfrak{n}_1 \).

There are semistandard Levi subgroups \( \tilde{\mathcal{P}} = \tilde{M}\tilde{\mathcal{N}} \) and \( \tilde{\mathcal{P}}_1 = \tilde{M}\tilde{\mathcal{N}}_1 \) of \( G_n \) such that \( \mathcal{N} \subseteq \tilde{\mathcal{N}} \) and \( \mathcal{N}_1 \subseteq \tilde{\mathcal{N}}_1 \). Let \( \bar{\mathfrak{n}}_1 \) be the Lie algebra of \( \mathcal{N}_1 \) so that \( \mathfrak{n}_1 \) is a vector subspace of \( \bar{\mathfrak{n}}_1 \). We have a canonical vector space isomorphism \( \bar{\mathfrak{n}}_1 \cong \mathbb{R}^{\dim \bar{\mathfrak{n}}_1} \) via the coordinates given by the matrix entries. This also gives a canonical inner product on \( \bar{\mathfrak{n}}_1 \) so that we can find the orthogonal complement of \( \mathfrak{n}_1 \) in \( \bar{\mathfrak{n}}_1 \). We can therefore extend any polynomial on \( \mathfrak{n}_1 \) trivially to a polynomial on \( \bar{\mathfrak{n}}_1 \) along that complement. A similar construction holds.
for polynomials on $v_1^M$ so that they can be extended trivially to polynomials on $\tilde{v}_1^M$ as well.

In particular, we can extend $X + Y \mapsto W_\varpi(1, e^{X+Y})$ to a polynomial on all of $\tilde{v}_1^M + \tilde{n}_1$, and $Y \mapsto v_P(\varpi, e^Y)$ to a polynomial on all of $\tilde{n}_1$. Since $X + Y$ is nilpotent, $\log(id + X + Y)$ and $e^{X+Y}$ are finite series. Hence we can also consider $W_\varpi(1, id + X + Y) = W_\varpi(1, e^{\log(id + X + Y)})$ and $v_P(\varpi, id + Y) = v_P(\varpi, e^{\log(id + Y)})$ with $X \in \tilde{v}_1^M$ and $Y \in \tilde{n}_1$ which are again both polynomials.

5.4. In which the group is $GL(n)$

In this section we first prove a slightly more general version of [35, Lemma 7.2]. We change the notation for this section slightly: We assume that $H = G_n$ so that in particular, $P = MN$ and $P_1 = MN_1$ be semi–standard parabolic of $G_n$ with the same Levi component $M$. Let $a \in A_M$ be regular. Suppose that $\pi \in P_1$ is a unipotent element and write $\pi = u\nu$ with $u \in M$ and $\nu \in N_1$ unipotent. Then there is a unique $n \in N_1$ such that

$$a\pi = n^{-1}aun,$$

that is, we have a well-defined polynomial map $U_MN_1 \ni \pi \mapsto n \in N_1$ depending on $a$ where $U_M$ denotes the set of all unipotent elements in $M$.

Let $\Phi$ denote the set of all roots of $T_0$ on $\mathfrak{g}_n$. Let $\Phi_M \subseteq \Phi$ denote the subset of roots which are not trivial when restricted to $A_M$, and let $\Phi_1 \subseteq \Phi_M$ be the subset of roots that are positive with respect to $N_1$. Let $\Phi^+$ denote a choice of positive roots in $\Phi$ such that $\Phi_1 \subseteq \Phi^+$. Let $\mathfrak{n}' \subseteq \mathfrak{m}$ denote the nilpotent subalgebra corresponding to $\Phi^+ \setminus \Phi_1$. Then $\mathfrak{n}' \oplus \mathfrak{n}_1$ is the nilpotent radical of the minimal parabolic subalgebra of $\mathfrak{g}_n$ corresponding to $\Phi^+$.

If $Z \in \mathfrak{g}_n$ is any matrix and $\beta \in \Phi$ a root, we write $Z_\beta$ for the matrix entry of $Z$ corresponding to $\beta$. Note that the $\beta$-eigenspace in $\mathfrak{g}_n$ is of dimension 1, hence $Z_\beta$ is indeed a well-defined real number. For example, if $n = 3$ and $\Phi^+$ is the usual choice of positive roots for the diagonal torus $T_0$ corresponding to the upper triangular matrices, then if $\beta \in \Phi^+$ is the highest root, $Z_\beta$ is the matrix entry of $Z$ in the right upper corner.

Write $u = id + X_0$, $\nu = id + X$, and $n = id + Y$ with $X_0, X, Y$ suitable nilpotent matrices. Up to conjugation with $K \cap M$ we can assume that $X_0 \in \mathfrak{n}'$ which we will do from now on.

We show the following:

**Proposition 5.1.** Let $\beta \in \Phi_1$. Then for each subset $\alpha \subseteq \Phi_1$ there is a polynomial $Q_{\beta, \alpha}(Z; a^\alpha, \alpha \in \alpha)$, $Z := \pi = id = X_0 + X + X_0X$, such that:

- $$Y_\beta = \sum_{\alpha \subseteq \Phi_1} Q_{\beta, \alpha}(aZa^{-1}; a^\alpha, \alpha \in \alpha) \prod_{\alpha \in \alpha}(a^\alpha - 1),$$
where $\alpha$ runs over all subsets of $\Phi_1$;

- as a function in the matrix entries of $Z$, $Q_{\beta,\alpha}$ is a homogeneous polynomial of degree $\#\alpha$;
- if $Q_{\beta,\alpha}(aZa^{-1}; a^\alpha, \alpha \in \Omega)$ does not vanish identically, then for $X$ in general position, its limit as $a \to 1$ is non-zero.

**Proof.** We introduce a grading on the set $\Phi_1$: We say that $\beta \in \Phi_1$ has degree $k$, $k \geq 1$, if the $\beta$-coordinate of $A^k$ is non-zero, but that of $A^{k+1}$ is zero for a general matrix $A \in \mathfrak{n}_1$. We write $\Phi_1^{(k)}$ for the set of $\beta \in \Phi_1$ of degree $k$. Note that $\Phi_1^{(k)} = \emptyset$ when $k \geq n$.

We rearrange the relation (5.14) as follows:

$$\text{id} + X_0 + a(Z - X_0)a^{-1} = n^{-1}ana^{-1} + n^{-1}X_0ana^{-1}$$

$$= \text{id} + \sum_{k=1}^{n-1} (-1)^{k-1} Y^{k-1} [aYa^{-1} - Y] + X_0 + \sum_{k=1}^{n-1} (-1)^{k-1} Y^{k-1} [X_0aYa^{-1} - YX_0]$$

For $k \geq 1$, the non-zero entries in the matrices $Y^{k-1} [aYa^{-1} - Y]$ all correspond to $\beta$ in $\Phi_1^{(l)}$, $l \geq k$. Moreover, the matrix entry in

$$\sum_{k=1}^{n-1} (-1)^{k-1} Y^{k-1} [aYa^{-1} - Y]$$

(5.15)

corresponding to a root $\beta$ of degree $k$ is of the form

$$(a^\beta - 1)Y_\beta + \sum_{\alpha} (a^{\alpha_1} - 1)C_{\alpha} \prod_{\alpha \in \Omega} Y_\alpha$$

where the sum runs over all tuples $\alpha = (\alpha_1, \ldots)$ of pairwise different elements in $\bigcup_{l < k} \Phi_1^{(l)}$ such that $2 \leq \sum \deg \alpha_i \leq k$, and $C_{\alpha} \in \mathbb{R}$ are suitable coefficients. Note that this in particular means that the sum over the $\alpha$ in (5.15) contains only monomials of degree $\geq 2$, and is in fact empty if $k = 1$.

The sum $\sum_{k=1}^{n-1} (-1)^{k-1} Y^{k-1} [X_0aYa^{-1} - YX_0]$ has a similar structure as (5.15), except that each monomial has exactly one linear factor consisting of a matrix entry of $X_0$. In particular, as a polynomial in $Y$ and $X_0$, there is no linear factor, and the matrix entry corresponding to a root of degree $k$ has only factors consisting of $Y_\beta$ with $\deg \beta < k$ and $X_0$. Moreover, no matrix entry of

$$\sum_{k=1}^{n-1} (-1)^{k-1} Y^{k-1} [X_0aYa^{-1} - YX_0]$$

is divisible by any of the factors $a^\alpha - 1$ (unless it is identically 0), since in $[X_0aYa^{-1} - YX_0]$ the terms $a^\beta Y_\beta$ and $Y_\beta$ cannot occur with non–vanishing coefficient in the same matrix entry.
We can now argue inductively on the degree of $\beta$. If $\beta \in \Phi_1^{(1)}$, then

$$(a^\beta - 1)Y_\beta = a^\beta Z_\beta.$$ 

The assertion then follows by induction from the above description of the matrix entries. $\square$

5.5. Back to $H$

We return to the notation of §5.2 and (5.11). Write $u = \text{id} + X_0$, $\pi = \text{id} + Z$ and $n = \text{id} + Y$ with $X_0$, $Z$, and $Y$ nilpotent matrices.

**Proposition 5.2.** There exists $\nu \in \mathbb{R}$ such that

$$\|W_\varpi(1, \text{id} + sZ)\| = s^\nu \|W_\varpi(1, \pi)\|$$

for all $s > 0$.

**Remark 5.3.** Note that $\text{id} + sZ$ is not necessarily contained in $\mathcal{O}$ (it does not even have to be contained in $H$), but as we discussed above, we can extend $W_\varpi(1, \pi)$ to a polynomial on all of $\text{id} + \tilde{\pi}_1^M + \tilde{n}_1$.

**Proof.** We want to use Proposition 5.1. The element $a$ in (5.11) defines a semistandard Levi subgroup $\tilde{M}$ in $G_n$ by taking its centralizer in $G_n$. Then $a$ is a regular element of the center of $\tilde{M}$ and $\tilde{M} \subseteq \tilde{M}$. Moreover, as before, there are semistandard parabolic subgroups $\tilde{P} = \tilde{M} \tilde{N}$ and $\tilde{P}_1 = \tilde{M} \tilde{N}_1$ of $G_n$ such that $N \subseteq \tilde{N}$ and $N_1 \subseteq \tilde{N}_1$. As explained above, we can extend $v_P(\varpi, n)$ to a polynomial in $Y \in \tilde{n}_1$, $n = \text{id} + Y$. Each coordinate $Y_\beta$ of $Y$ can be described according to Proposition 5.1 so that under the change $Z \mapsto sZ$ the coordinate becomes

$$Y_\beta = \sum_{\alpha \subseteq \Phi_1} s^{\# \alpha} Q_{\beta, \alpha}(aZa^{-1}; a^\alpha, \alpha \in \alpha) \prod_{\alpha \in \alpha}(a^\alpha - 1).$$

Hence by definition of $\|W_\varpi(1, \pi)\|$ in (5.12) we can find $\nu \geq 0$ such that

$$\|W_\varpi(1, \text{id} + sZ)\| = s^\nu \|W_\varpi(1, \text{id} + Z)\|. \quad \square$$

Together with (5.13) this immediately implies the following:

**Corollary 5.4.** There exist

- constants $r, q \geq 0$,
- polynomials $p_1, \ldots, p_q : \tilde{v}_1^M \oplus \tilde{n}_1 \to \mathbb{R}$ which do not vanish on an open dense subset of $Z$ with $\text{id} + Z \in \mathcal{O}N_1$, and
• complex polynomials $Q_j$ in $q$-many variables, $j = 0, \ldots, r$,

such that for all $Z \in \mathfrak{v}^M \oplus \mathfrak{n}_1$ and $s > 0$ we have

$$w_{M,0} (\text{id} + sZ) = \sum_{i=0}^{r} (\log s)^i Q_i (\log |p_i(Z)|, \ldots, \log |p_q(Z)|).$$

6. Test functions

6.1. Linearizing the metric

Let

$$r(g) = d(g\mathbf{K}_\infty, \mathbf{K}_\infty), \quad (6.17)$$

where $d(\cdot, \cdot)$ denotes the geodesic distance function on $\tilde{X} = G_\infty/K_\infty$. We continue to assume that $G_\infty$ is a real semisimple Lie group, and we fix an embedding of $G_\infty$ into $\text{GL}_n(\mathbb{R})^1$ for some $n \geq 1$ as at the beginning of §5.2. From now on, we will identify $G_\infty$ and all its subgroups with their image in $\text{GL}_n(\mathbb{R})^1$ instead of writing $H$ etc., and the Lie algebra $\mathfrak{g}$ will be identified with its corresponding image in $\mathfrak{sl}_n(\mathbb{R})$. In addition to the properties satisfied by $G_\infty$ in §5.2, we can choose the embedding furthermore such that

• $\mathbf{K}_\infty \subseteq \mathbf{O}(n)$.
• $\mathfrak{a}_{\mathbb{C}}^G$ is contained in the subalgebra of diagonal matrices in $\mathfrak{sl}_n(\mathbb{R})$.
• If $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ is the Cartan decomposition of $\mathfrak{g}$, then $\mathfrak{s}$ is contained in the symmetric matrices in $\mathfrak{sl}_n(\mathbb{R})$, and $\mathfrak{k}$ in the skew-symmetric matrices.
• For $X = (X_{ij})_{i,j=1,\ldots,n} \in \mathfrak{g}$, put $\|X\|^2 = \sum_{i,j} |X_{ij}|^2$. Then $\|\cdot\|$ coincides with the norm on $\mathfrak{s}$ obtained from the Killing form on $\mathfrak{g}$. (More generally, we define $\|\cdot\|$ similarly on all of $\mathfrak{sl}_n(\mathbb{R})$.)
• The unipotent radical $U_0$ of the minimal parabolic subgroup $P_0$ is contained in the group of unipotent upper triangular matrices of $\text{GL}_n(\mathbb{R})^1$.

Let $d_n(\cdot, \cdot)$ denote the geodesic distance function on $\text{GL}_n(\mathbb{R})^1/\mathbf{O}(n)$, and let $r_n(g) = d_n(g\mathbf{O}(n), \mathbf{O}(n))$. By our choice of embedding, the Cartan decomposition on $G_\infty$ with respect to $K_\infty$ and on $\text{GL}_n(\mathbb{R})^1$ with respect to $\mathbf{O}(n)$ are compatible, that is, if $g = k_1ak_2 \in G_\infty = K_\infty A_0^G K_\infty$, then $k_1ak_2$ is also the Cartan decomposition of $g$ in $\text{GL}_n(\mathbb{R})^1$. Hence $r(g) = r(a)$ and $r_n(g) = r_n(a)$. By [24, (4.6.25)] we have $r(a) = \|\log a\|$ and $r_n(a) = \|\log a\|$ so that $r$ and $r_n$ coincide on $G_\infty \subseteq \text{GL}_n(\mathbb{R})^1$. This allows us to use results on the geodesic distance $r_n$ from [35] also for $r$. Recall from [35, Lemma 12.2] that if $g = I_n + X \in \text{GL}_n(\mathbb{R})^1$ with $X$ a nilpotent upper triangular matrix, then

$$r_n(I_n + X) = \frac{1}{4} \|X\|^2 + O(\|X\|^3) \quad (6.18)$$
as $X \to 0$. Here $I_n \in \text{GL}_n(\mathbb{R})$ denotes the identity matrix. Hence if $X$ varies over matrices such that $I_n + X \in G_\infty$, then the same is true for $r_n(I_n + X)$ by our above remark.

We also need to understand how $r(g)$ behaves as $g$ varies over unipotent matrices in $G_\infty$ that become unbounded:

**Lemma 6.1.** We have

$$r(g) \geq \frac{1}{2} \log \left( 1 + \frac{1}{n} \|g - I_n\|^2 \right)$$

for all unipotent $g \in G_\infty$.

**Proof.** Let $g \in G_\infty$ be unipotent and write $g = e^{Y_0}$ with $Y_0 \in \mathfrak{g}$ nilpotent. There exists $k \in K_\infty$ such that $Y := \text{Ad}(k)Y_0$ is an upper triangular nilpotent matrix. We can therefore write $e^Y = I_n + N$ for some nilpotent upper triangular $N \in \mathfrak{sl}_n(\mathbb{R})$. Let $X \in \mathfrak{a}_0^G$ and $k_1, k_2 \in K_\infty$ such that $g = k_1 e^X k_2$. Then $\|X\| = r(g) = r(k e^{Y_0} k^{-1}) = r(e^Y)$, where the first equality follows from [24, (4.6.25)]. Then

$$\text{tr} e^{2X} = \text{tr}(g'g) = \text{tr} e^{Y_0^t e^{Y_0}} = \text{tr} e^{Y^t} e^Y = \text{tr}(I_n + N)^t(I_n + N) = n + \text{tr} N^t N = n + \|N\|^2.$$

Let $X_1, \ldots, X_n$ be the diagonal entries of $X$. Then

$$\text{tr} e^{2X} = \sum_{j=1}^n e^{2X_j} \leq \sum_{j=1}^n e^{2\|X_j\|} \leq ne^{2\|X\|}.$$

Hence

$$e^{\|X\|} \geq (1 + \frac{1}{n} \|N\|)^{1/2}$$

so that

$$r(g) = \|X\| \geq \frac{1}{2} \log(1 + \frac{1}{n} \|N\|^2)$$

as asserted. $\square$

**Lemma 6.2.** Let $\tilde{u}_0$ denote the vector space of all nilpotent upper triangular matrices in $\mathfrak{gl}_n(\mathbb{R})$. Then there exists $x_0 > 0$ such that for all $X \in \tilde{u}_0$ with $\|X\| \geq x_0$ we have

$$\|e^X - I_n\| \geq \frac{1}{\sqrt{2}} \|X\|.$$

**Proof.** We consider a more abstract situation: Let $P : \mathbb{R}^N \to \mathbb{R}$ be a non-negative polynomial satisfying the following two properties:
• $P(x) \to \infty$ as $\|x\| \to \infty$, where $\|x\|$ denotes the usual Euclidean norm on $\mathbb{R}^N$, and

• $P(x)$ can be written as $P(x) = \|x\|^2 + \sum \alpha \alpha x^\alpha$ with $\alpha$ running over all multiindices $\alpha = (\alpha_1, \ldots, \alpha_N)$ of degree $3 \leq |\alpha| = \sum \alpha_i \leq \deg P$.

The function $u_0 \ni X \mapsto \|e^X - I_n\|^2$ is a polynomial exactly of this form. It will therefore suffice to show that for such polynomials we have $P(x) \geq \|x\|^2/2$ for $\|x\| \geq x_0$ for some $x_0 > 0$.

Fix $x \in \mathbb{R}^N$ with $\|x\| = 1$ and set $Q_x(t) = P(tx) - t^2$ for $t \in \mathbb{R}$. If $Q_x$ vanishes identically in $t$, we are done. If $Q_x$ does not vanish identically, then $Q_x$ is a non-trivial polynomial of degree $\geq 3$ in $t$. Since $P(tx) \to +\infty$ as $t \to \infty$, we must also have $Q_x(t) \to +\infty$ for $t \to \infty$. Hence there exists $t_x > 0$ such that $Q_x(t) \geq 0$ for all $t \geq t_x$, that is, $P(tx) \geq t^2$ for $t \geq t_x$. Since the set of all $x \in \mathbb{R}^N$ with $\|x\| = 1$ is compact, we can find $t_0 > 0$ such that $P(x) \geq \|x\|^2/2$ for all $x \in \mathbb{R}^N$ with $\|x\| \geq t_0$. This finishes the proof. \[\square\]

6.2. Asymptotic expansion of the heat kernel

We recall the following asymptotic expansion of $h_\nu^\nu$ which, for example, can be found in [35, Corollary 10.4]: If $\psi \in C^\infty(\mathbb{R})$ is non-negative, equals 1 around 0 and has sufficiently small support, then for any sufficiently large $N$ and any $0 < t \leq 1$ we have

$$h_\nu^\nu(g) = (4\pi t)^{-d/2} \psi(r(g)) \exp \left(-r(g)^2/(4t)\right) \sum_{n=0}^N a^\nu_n(g)t^n + O\left(t^{N+1-d/2}\right) \quad (6.19)$$

where $a^\nu_n \in C^\infty(G_\infty)$ are suitable functions and the implied constant depends on the function $\psi$ but not on $N$. We further note a uniform upper bound for $h_\nu^\nu$ (see, for example, [35, Corollary 10.2]): There exists $C > 0$ such that

$$|h_\nu^\nu(g)| \leq Ct^{-d/2}e^{-r(g)^2/(4t)} \quad (6.20)$$

for all $g \in G_\infty$ and all $0 < t \leq 1$.

7. Proof of Proposition 7.1

The unipotent distribution $J_{\text{unip}}(\tilde{\phi}_\nu^\nu)$ can be written as a linear combination of certain weighted unipotent integrals $J_M^G(O, \phi^\nu_\nu)$, see §8.1. It will turn out (see §8.2) that for us only the integrals $J_M^G(O, \psi^\nu_\nu)$ will be relevant. For these we have the following:

**Proposition 7.1.** Let $P = MU \in \mathcal{F}$, and let $O \subseteq M(\mathbb{R})$ be a unipotent conjugacy class in $M(\mathbb{R})$. Let $O^G \subseteq G(\mathbb{R})$ be the union of induced unipotent conjugacy class in $G(\mathbb{R})$ from
Moreover, the coefficients $b_{ij}$ are uniformly bounded.

We prove a slightly more general version of Proposition 7.1 which will make for a cleaner proof of Theorem 1.1 in the next section. Let $M \in \mathcal{L}$ and let $P_1 = M_1U_1 \in \mathcal{F}(M)$. If $f \in C^\infty(G_{\infty})$ define $f_{P_1} \in C^\infty(M_1)$ by

$$f_{P_1}(m) = \delta_{P_1}(m)^{1/2} \int_{K_\infty U_1} f(k^{-1}m) du dk$$

provided the right hand side is finite for any $m \in M_1$. If $f$ has compact support, $f_{P_1}$ is compactly supported as well.

If $\mathcal{O}$ is a unipotent conjugacy class in $M$, we can define $J_{M_1}^M(\mathcal{O}, f_{P_1})$ as before with $G_{\infty}$ replaced by $M_1$.

**Proposition 7.2.** Let $M \in \mathcal{L}$, $\mathcal{O} \subseteq M$ a unipotent conjugacy class in $M$, and $P_1 = M_1U_1$ a semistandard parabolic subgroup of $G$ with $M \subseteq M_1$. Let $d_{G_{\infty}}^2 = \dim \mathcal{O}G_{\infty}$, and let $r_{M_1}^M = \dim \alpha_{M_1}^M$. Then there exist constants $b_{ij} = b_{ij}(M, \mathcal{O}) \in \mathbb{C}$, $j \geq 0$, $0 \leq i \leq r_{M_1}^M$ such that for every $0 < t \leq 1$

$$J_{M_1}^M(\mathcal{O}, (\psi_t^\nu)_{P_1}) \sim t^{-d/2+2d_{G_{\infty}}^2/4} \sum_{j=0}^{r_{M_1}^M} \sum_{i=0}^{r_{M_1}^M} b_{ij} t^{i/2} (\log t)^i.$$  \hspace{1cm} (7.21)

Moreover, the coefficients $b_{ij}$ are uniformly bounded.

For the proof of this proposition we follow [35, §12] taking into account our results above. Suppose $\hat{Q} \subseteq M_1$ is a semistandard parabolic which contains $M$. Let $\hat{Q} = L_{\hat{Q}}N_{\hat{Q}}$ with $M \subseteq L_{\hat{Q}}$, and let $Q = \hat{Q}U_1$. Then $Q$ is a semistandard parabolic subgroup of $G$ with Iwasawa decomposition $Q = L_{\hat{Q}}N_{\hat{Q}}$, $N_{\hat{Q}} = N_{\hat{Q}}U_1$, that is, $Q$ and $\hat{Q}$ have the same Levi component $L_{\hat{Q}} = L_{\hat{Q}}$ containing $M$. Hence $\varphi^Q = \varphi^{\hat{Q}}$ and $w_{O,M}^Q = w_{O,M}^{\hat{Q}}$. Applying (5.10) to $J_{M_1}^M(\mathcal{O}, (\psi_t^\nu)_{P_1})$ and unfolding the definition of $(\psi_t^\nu)_{P_1}$, we obtained

$$J_{M_1}^M(\mathcal{O}, (\psi_t^\nu)_{P_1}) = \sum_{\mathcal{Q}} \mathcal{c}(Q, \mathcal{O}) \int_{u_{Q,1}} \psi_t^\nu(e^Y) w_{\hat{Q},\mathcal{O}}^Q(e^Y) |\varphi^{\hat{Q}}(Y)|^{1/2} dY$$  \hspace{1cm} (7.22)
where the sum runs over all $Q \in \mathcal{F}(M)$ in the image of the map $\tilde{Q} \mapsto Q$, and the constants depend on the normalization of measures. We therefore need to analyze integrals that are of the form as those on the right hand side of (7.22).

Fix $Q \in \mathcal{F}(M)$. We equip $u_{Q,1}$ with a Euclidean norm by fixing some isomorphism $u_{Q,1} \simeq \mathbb{R}^{\dim u_{Q,1}}$ which respects the direct sum decomposition $u_{Q,1} = u_1^{L_Q} \oplus u_Q$. Let $\epsilon > 0$ and let $B(\epsilon) \subseteq u_{Q,1}$ be the ball around 0 of radius $\epsilon$. Put $U(\epsilon) = u_{Q,1} \setminus B(\epsilon)$. We split the integral on the right hand side of (7.22) into integrals over $B(\epsilon)$ and $U(\epsilon)$.

Recall the function $\beta$ from the definition (3.3) of $\psi_t^\epsilon$. We assume that $\beta$ has sufficiently small support. Let $\epsilon > 0$ be such that $\beta(r(\exp(\cdot)))$ restricted to $B(\epsilon)$ is identically equal to 1. Fix $\psi \in C^\infty(\mathbb{R})$ as in §6.2 (for the expansion (6.19)) such that the support of the restriction of $\psi(r(\exp(\cdot)))$ to $u_{Q,1}$ is contained in $B(\epsilon)$.

We first show that the integral over $U(\epsilon)$ decays exponentially in $t^{-1}$ as $t \searrow 0$ and therefore does not contribute to the asymptotic expansion. Indeed, it follows from §5.5 and (6.20) that

$$
\left| \int_{U(\epsilon)} h_t^\psi(e^Y) |\varphi^Q(Y)|^{1/2} w_{M,\mathcal{O}}^Q(e^Y) \, dY \right| (7.23)
$$

can be bounded by a constant multiple of a finite sum of integrals of the form

$$
t^{-d/2} \int_{U(\epsilon)} e^{-r(e^Y)^2/(4t)} |\varphi^Q(Y)|^{1/2} \prod_{j=1,\ldots,J} \log \|p_j(e^Y - I)\| \, dY (7.24)
$$

where $p_j$, $j = 1, \ldots, J$ are suitable polynomials on the Lie algebra as the ones appearing in Corollary 5.4. In particular, $p_j(e^Y - I)$ does not vanish identically in $Y \in U(\epsilon)$.

The polynomial $\varphi^Q$ has degree $\dim \mathfrak{f}_1^Q$ so that $|\varphi^Q(Y)|^{1/2} \ll_{\epsilon} \|Y\|^{\dim \mathfrak{f}_1^Q/2}$. It follows from Lemma 6.1, that there are constants $c_\epsilon, c_1 > 0$ such that (7.24) is bounded by a constant multiple of

$$
e^{-c_\epsilon/t} \int_{u_{Q,1}} \|Y\|^{\dim \mathfrak{f}_1^Q} e^{-c_1 \log^2(1+\|Y\|)/t} \prod_{j=1,\ldots,J} \log \|p_j(e^Y - I)\| \, dY
$$

provided that $\epsilon$ is sufficiently small. Note that $p_j(e^Y - I)$ is again a polynomial in $Y$. By [35, Lemma 7.7] this integral converges. (In fact, [35, Lemma 7.7] only treats the case without the power of $\|Y\|$ in the integral, but it is immediate from the proof that a polynomial in $\|Y\|$ does not change the validity of the assertion.) Therefore, (7.24) is bounded by a constant multiple of $e^{-c_\epsilon/t}$, that is, it decays exponentially in $t^{-1}$ as $t \searrow 0$.

For the integral over $B(\epsilon)$ we use (6.19) and the expansion of $w_{M,\mathcal{O}}^Q$ from Corollary 5.4. Using (6.19) we get
\[ \int_{B(\varepsilon)} h_t^\nu(e^Y)|\varphi^Q(Y)|^{1/2}w_{M,Q}^Q(e^Y)\,dY 
\]

\[ = (4\pi t)^{-d/2} \sum_{n=0}^N t^n \int_{B(\varepsilon)} \exp\left(-r(e^Y)^2/(4t)\right) \psi(r(e^Y))a_n^\nu(e^Y)|\varphi^Q(Y)|^{1/2}w_{M,Q}^Q(e^Y)\,dY 
\]

\[ + O\left(t^{N+1-d/2}\right). \tag{7.25} \]

We now first proceed as in [35, §12.2] and use Taylor expansion of the functions \( \exp\left(-r(e^Y)^2/(4t)\right) \) and \( f(Y) := \psi(r(e^Y))a_n^\nu(e^Y)|\varphi^Q(Y)|^{1/2} \) in \( N = e^Y - I \) around 0. Note that except for \( \varphi^Q(Y) \) all involved functions depend only on \( r(e^Y) \), and can therefore be continued to smooth function on all of \( gl_n(\mathbb{R}) \). Since \( \varphi^Q(Y) \) is a polynomial, we can extend it to a polynomial on all of \( gl_n(\mathbb{R}) \) as well. Then, using (6.18), we can write for any \( K \geq 1 \) and any \( 0 < t \leq 1 \),

\[ \exp\left(-r(I + t^{1/2}N)^2/(4t)\right) = \exp\left(-\|N\|^2/4\right)\left(\sum_{k=0}^K t^{k/2}q_k(N) + R_K(t, N)\right) \]

where \( q_k \) are suitable polynomials of degree \( \leq 3nk \), \( q_0(N) = 1 \), and \( R_K(t, N) \) is a remainder term satisfying

\[ |R_K(t, N)| \leq c_1 t^{(K-1)/2}(1 + \|N\|)^{3nK} \]

for every \( 0 < t \leq 1 \) with \( c_1 > 0 \) some suitable constant.

Similarly,

\[ f(\log(I + t^{1/2}N)) = \sum_{l \leq L} b_l(N)t^{l/2} + Q_L(t, N) \]

where \( b_l \) is a polynomial in \( N \) of degree \( \leq l \), and \( Q_L(t, N) \) is a remainder term satisfying

\[ |Q_L(t, N)| \leq c_2 t^{(L+1)/2}(1 + \|N\|)^{L+1} \]

for all \( 0 < t \leq 1 \) and \( N \) with \( Y = \log(I + N) \in B(\varepsilon) \) with \( c_2 > 0 \) some absolute constant. Note that \( b_l(N) = 0 \) whenever \( l < \dim t_1^Q/2 \) since \( |\varphi^Q|^{1/2} \) is homogeneous in \( Y \) of degree \( \dim t_1^Q/2 \).

Hence the left hand side of (7.25) equals after a change of variables

\[ \sum_{k \leq K} \sum_{l \leq L} t^{(k+l)/2} t^{\dim u_{Q,1}/2} \int_{t^{-1/2}B(\varepsilon)} \exp(-\|N\|^2/4)q_k(N)b_l(N)w_M^Q(I + t^{1/2}N)\,dN + \Phi_{K,L}(t) \]

where \( B(\varepsilon) \) is the image of \( B(\varepsilon) \) in \( gl_n(\mathbb{R}) \) under \( Y \mapsto N = e^Y - I \), and with the remainder \( \Phi_{K,L}(t) \) satisfying
\[ |\Phi_{K,L}(t)| \leq c_3 t^{(L+K+1)/2}. \]

Note that the Jacobian of the change from \( Y \) to \( N \) is a polynomial in \( N \) with non-vanishing constant term that we absorb into the asymptotic expansion. Using the asymptotic expansion for the weight \( w^Q_M(I + t^{1/2}N) \), we can find coefficients \( c_{m,j,\Xi} \) such that the left hand side of (7.25) equals

\[
t^{-d/2}t^{\dim u_{Q,1}/2} \sum_{\dim l_1^Q \leq m \leq N} \sum_{j=0}^r t^{m/2} (\log t)^j \sum_{\Xi} \int_{t^{-1/2}B(\epsilon)} c_{m,j,\Xi}(Y) \prod_{k \in \Xi} \log \|p_k(Y)\| dY + \tilde{\Phi}_N(t)
\]

with the polynomials \( p_k \), \( k = 1, \ldots, q \), as in Corollary 5.4, \( \Xi \) running over (multi-)subsets of \( \{1, \ldots, q\} \) whose size is bounded by the polynomials \( Q_j \) appearing in Corollary 5.4, and \( \tilde{\Phi}_N(t) \) satisfying

\[
|\tilde{\Phi}_N(t)| \leq c_4 t^{(N-d+\dim l_1^Q + \dim u_{Q,1}+1)/2}.
\]

Now for \( 0 < t \leq 1 \) we have

\[
\sum_{\Xi} \int_{t^{-1/2}B(\epsilon)} c_{m,j,\Xi}(Y) \prod_{k \in \Xi} \log \|p_k(Y)\| dY = C_{m,j} + O \left( e^{-c_4/t} \right)
\]

which follows as in [35, §12]. Hence (7.25) equals

\[
t^{(-d+\dim l_1^Q + \dim u_{Q,1})/2} \sum_{m \leq N} \sum_{j=0}^r \tilde{C}_{m,j} t^{m/2} (\log t)^j + O \left( t^{(N-d+\dim l_1^Q + 1)/2} \right)
\]

for \( 0 < t \leq 1 \) where \( \tilde{C}_{m,j} = C_{m+\dim l_1^Q,j} \). Since \( d_0^{\infty} = 2(\dim u_{Q,1} + \dim l_1^Q) \), the assertion of the proposition follows.

8. Proof of Theorem 1.1

The proof of Theorem 1.1 is global so that we return to our normal notation. In particular, \( G \) denotes again a reductive algebraic group defined over \( \mathbb{Q} \). We fix a minimal parabolic subgroup \( P_0 \) of \( G \) (as an algebraic group over \( \mathbb{Q} \)), and write \( P_0 = M_0 U_0 \) with \( U_0 \) the unipotent radical of \( P_0 \) and \( M_0 \) the Levi subgroup of \( P_0 \). We call a parabolic subgroup of \( G \) standard if it contains \( P_0 \), and semistandard if it contains \( M_0 \). Let \( \mathcal{L} \) denote the set of all semistandard Levi subgroups of \( G \), that is, all \( M \subseteq G \) which are Levi components of semistandard parabolic subgroups.

8.1. Arthur’s fine geometric expansion

Let \( S \) be a finite set of places of \( \mathbb{Q} \), which includes the archimedean place, such that \( K_v = K_e \) for \( v \notin S \). Let \( G(\mathbb{Q}_S)^1 = G(\mathbb{Q}_S) \cap G(A)^1 \).
Let $M \in \mathcal{L}$. Following Arthur, we introduce an equivalence relation on the set of unipotent elements in $M(\mathbb{Q})$ that depends on the set $S$: Two unipotent elements $u, v \in M(\mathbb{Q})$ are equivalent if and only if $u$ and $v$ are conjugate in $M(\mathbb{Q}_S)$. We denote the equivalence class of $u$ by $[u]_S \subseteq M(\mathbb{Q})$ and let $\mathcal{U}_S^M$ denote the set of all such equivalence classes.

Note that two equivalent unipotent elements define the same unipotent conjugacy class in $M(\mathbb{Q}_S)$, so we can view $\mathcal{U}_S^M$ also as the set of unipotent conjugacy classes in $M(\mathbb{Q}_S)$ which have at least one $\mathbb{Q}$-rational representative, and we denote the corresponding conjugacy class by $[u]_S$ as well. This differs from our notation for unipotent conjugacy classes in $G(\mathbb{R})^1$ from the previous sections, but we now need to keep track of the dependence on $S$.

**Remark 8.1.**

(i) If $T \subseteq S$, then we get a well-defined map $\mathcal{U}_S^M \ni [u]_S \mapsto [u]_T \in \mathcal{U}_T^M$.

(ii) If $G = \text{GL}(n)$, the equivalence relation is independent of $S$ and is the same as conjugation in $M(\mathbb{Q})$.

For $[u]_S \in \mathcal{U}_S^M$ and $f_S \in C^\infty_c(G(\mathbb{Q}_S)^1)$, Arthur associates a weighted orbital integral $J^G_M([u]_S, f_S)$ [6] which is a distribution supported on the union of $G(\mathbb{Q}_S)$-conjugacy class induced from $[u]_S \subseteq M(\mathbb{Q}_S)$. If $S = \{\infty\}$, these distributions were discussed in §5. Let $1_{K^S} \in C^\infty_c(G(\mathbb{A}^S))$ be the characteristic function of $K^S$, if $f_S \in C^\infty_c(G(\mathbb{Q}_S)^1)$, then we write $f = f_S 1_{K^S} \in C^\infty_c(G(\mathbb{A})^1)$.

**Proposition 8.2** ([5], Corollary 8.3). There exist unique constants $a^M([u]_S, S) \in \mathbb{C}$, $[u]_S \in \mathcal{U}_S^M$, such that for all $f_S \in C^\infty_c(G(\mathbb{Q}_S)^1)$ we have

$$J_{\text{unip}}(f) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in \mathcal{U}_S^M} a^M([u]_S, S) J^G_M([u]_S, f_S).$$

(8.1)

8.2. The splitting formula

To understand the behavior of the distributions $J^G_M([u]_S, f_S)$ for our test functions, we want to apply our asymptotic expansion for the archimedean weighted integral. To that end we need to separate $\infty$ from the other places in $S$ which we will do by using Arthur’s splitting formula [8, (18.7)]: Suppose that $S = S_1 \cup S_2$ with $S_1, S_2$ non-empty and disjoint, and that $f_S$ is the restriction of a product $f_{S_1} f_{S_2}$ to $G(\mathbb{Q}_S)^1$ with $f_{S_j} \in C^\infty(G(\mathbb{Q}_{S_j}))$, $j = 1, 2$. Then

$$J^G_M([u]_S, f_S) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2) J^L_M([u]_{S_1}, f_{S_1, Q_1}) J^L_M([u]_{S_2}, f_{S_2, Q_2}),$$

(8.2)
where the notation is as follows: The $d^G_M(L_1, L_2) \in \mathbb{R}$ are certain constants which depend only on $M, L_1, L_2, G$ but not on $S$. In fact, $d^G_M(L_1, L_2)$ is non-zero only if the natural map $a^L_M \oplus a^L_M \rightarrow a^G_M$ is an isomorphism. The $Q_j$ are arbitrary elements in $\mathcal{P}(L_j)$, and

$$f_{S_j, Q_j}(m) = \delta_{Q_j}(m)^{1/2} \int_{K_{S_j}N_j(Q_{S_j})} f_{S_j}(k^{-1}mnk)\,dn\,dk$$  (8.3)

where $N_j$ is the unipotent radical of $Q_j$. Finally, $J^L_M([u]_{S_j}, \cdot)$ denotes the $S_j$-adic distribution supported on the union of $L_j(\mathbb{Q}_{S_j})$-conjugacy class induced from $[u]_{S_j} \subseteq M(\mathbb{Q}_{S_j})$ and defined as in [6].

### 8.3. Completion of the proof of Theorem 1.1

Let $S$ be as in §8.1 and write $S = \{\infty\} \cup S_0$. Then $K_f = K_{S_0}K^S$. Recall the definition of $\tilde{\phi}_t^\nu$ and $\psi_t^\nu$ from (3.4) and (3.3), respectively, so that

$$\tilde{\phi}_t^\nu = \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}} \cdot \mathbf{1}_{K^S}.$$

By Proposition 8.2 we have

$$J_{\text{unip}}(\tilde{\phi}_t^\nu) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in \mathcal{U}_M^S} a^M([u]_S, S)J^G_M([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}).$$

This is a finite sum and the number of summands is independent of $t$ because the support of $\psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}$ is independent of $t$. To prove Theorem 1.1 it therefore suffices to establish an asymptotic expansion of $J^G_M([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}})$ as $t \searrow 0$. We first apply the splitting formula (8.2) to this integral with $S_1 = \{\infty\}$ and $S_2 = S_0$. We get

$$J^G_M([u]_S, \psi_t^\nu \cdot \mathbf{1}_{K_{S_0}}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d^G_M(L_1, L_2)J^G_{L_1}([u]_\infty, \psi_t^\nu \cdot 1_{K_{S_0}, Q_1})J^G_{L_2}([u]_{S_0}, \psi_t^\nu \cdot 1_{K_{S_0}, Q_2}).$$

Again, this is a finite sum with number of summands independent of $t$, and $d^G_M(L_1, L_2)$ and $J^G_{L_1}([u]_{S_0}, \psi_t^\nu \cdot 1_{K_{S_0}, Q_2})$ constant in $t$. In combination with the asymptotic expansion of $J^G_{L_1}([u]_\infty, \psi_t^\nu \cdot 1_{K_{S_0}, Q_1})$ as $t \searrow 0$ from Proposition 7.1 we obtain Theorem 1.1

**Remark 8.3.** In the rank one case, the short time asymptotic expansion of the regularized trace of the heat operators can also be obtained by using methods of microlocal analysis [1, Theorem A.1]. In fact, this method works for the more general class of manifolds with cusps. It is a challenging problem to see if in the higher rank case the asymptotic expansion (1.8) can also be derived by methods of microlocal analysis.
9. The spectral side of the non-invariant trace formula

For the convenience of the reader we summarize in this section some basic facts about Arthur’s non-invariant trace formula. The trace formula is the equality

\[
J_{\text{geom}}(f) = J_{\text{spec}}(f), \quad f \in C_c^{\infty}(G(\mathbb{A})^1),
\]  

(9.1)
of the geometric side \(J_{\text{geom}}(f)\) and the spectral side \(J_{\text{spec}}(f)\) of the trace formula. The geometric side has been studied in the previous sections. Especially, in Proposition 3.1 we have shown that for a test function with sufficiently small support, the geometric side is reduced to the contribution of the unipotent equivalence class. Then in Proposition 8.2 we have recalled the fine geometric expansion of the unipotent contribution in terms of weighted orbital integrals.

In this section we recall the definition of the spectral side, and in particular the refinement of the spectral expansion obtained in [22].

The main ingredient of the spectral side is logarithmic derivatives of intertwining operators. We briefly recall the structure of the intertwining operators.

Let \(P = MU_P \in \mathcal{P}(M)\). Recall that we denote by \(\Sigma_P \subset a_P^*\), the set of reduced roots of \(A_M\) of the Lie algebra \(u_P\) of \(U_P\). Let \(\Delta_P\) be the subset of simple roots of \(P\), which is a basis for \((a_P^0)^*\). Write \(a_{P,+}^+\) for the closure of the Weyl chamber of \(P\), i.e.

\[
a_{P,+} = \{ \lambda \in a_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Sigma_P \} = \{ \lambda \in a_M^* : \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in \Delta_P \}.
\]

Denote by \(\delta_P\) the modulus function of \(P(\mathbb{A})\). Let \(\tilde{A}_2(P)\) be the Hilbert space completion of

\[
\{ \phi \in C^{\infty}(M(\mathbb{Q})U_P(\mathbb{A}) \backslash G(\mathbb{A})) : \delta_P^{-\frac{1}{2}} \phi(\cdot x) \in L^2_{\text{disc}}(A_M(\mathbb{R})^0M(\mathbb{Q}) \backslash M(\mathbb{A})), \forall x \in G(\mathbb{A}) \}
\]

with respect to the inner product

\[
(\phi_1, \phi_2) = \int_{A_M(\mathbb{R})^0M(\mathbb{Q})U_P(\mathbb{A}) \backslash G(\mathbb{A})} \phi_1(g)\overline{\phi_2}(g) \, dg.
\]

Let \(\alpha \in \Sigma_M\). We say that two parabolic subgroups \(P, Q \in \mathcal{P}(M)\) are adjacent along \(\alpha\), and write \(P|^{\alpha}Q\), if \(\Sigma_P \cap -\Sigma_Q = \{ \alpha \}\). Alternatively, \(P\) and \(Q\) are adjacent if the group \((P, Q)\) generated by \(P\) and \(Q\) belongs to \(\mathcal{F}_1(M)\) (see (2.21) for its definition). Any \(R \in \mathcal{F}_1(M)\) is of the form \((P, Q)\), where \(P, Q\) are the elements of \(\mathcal{P}(M)\) contained in \(R\). We have \(P|^{\alpha}Q\) with \(\alpha^\vee \in \Sigma_P^\vee \cap a_M^R\). Interchanging \(P\) and \(Q\) changes \(\alpha\) to \(-\alpha\).

For any \(P \in \mathcal{P}(M)\) let \(H_P : G(\mathbb{A}) \to a_P\) be the extension of \(H_M\) to a left \(U_P(\mathbb{A})\)- and right \(K\)-invariant map. Denote by \(\mathcal{A}^2(P)\) the dense subspace of \(\tilde{A}^2(P)\) consisting of its \(K\)- and \(\mathfrak{Z}\)-finite vectors, where \(\mathfrak{Z}\) is the center of the universal enveloping algebra of \(\mathfrak{g} \otimes \mathbb{C}\). That is, \(\mathcal{A}^2(P)\) is the space of automorphic forms \(\phi\) on \(U_P(\mathbb{A})M(\mathbb{Q}) \backslash G(\mathbb{A})\) such
that $\delta_P^{-\frac{1}{2}}\phi(\cdot;k)$ is a square-integrable automorphic form on $A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A})$ for all $k \in \mathbb{K}$. Let $\rho(P, \lambda), \lambda \in \mathfrak{a}_M^*, \epsilon$ be the induced representation of $G(\mathbb{A})$ on $\mathcal{A}^2(P)$ given by

$$(\rho(P, \lambda, y)\phi)(x) = \phi(xy)e^{(\lambda, H_P(xy) - H_P(x))}.$$  

It is isomorphic to the induced representation

$$\text{Ind}_{P}^{G(\mathbb{A})}(L_{\text{disc}}(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A})) \otimes e^{(\lambda, H_M(\cdot))}).$$

For $P, Q \in \mathcal{P}(M)$ let

$$M_{Q|P}(\lambda) : \mathcal{A}^2(P) \to \mathcal{A}^2(Q), \quad \lambda \in \mathfrak{a}_M^*,$$

be the standard intertwining operator [4, §1], which is the meromorphic continuation in $\lambda$ of the integral

$$[M_{Q|P}(\lambda)\phi](x) = \int_{U_Q(\mathbb{A}) \cap U_P(\mathbb{A}) \backslash U_Q(\mathbb{A})} \phi(nx)e^{(\lambda, H_P(nx) - H_Q(x))} \, dn, \quad \phi \in \mathcal{A}^2(P), \ x \in G(\mathbb{A}).$$

Given $\pi \in \Pi_{\text{disc}}(M(\mathbb{A}))$, let $\mathcal{A}_{\pi}^2(P)$ be the space of all $\phi \in \mathcal{A}^2(P)$ for which the function $M(\mathbb{A}) \ni x \mapsto \delta_P^{-\frac{1}{2}}\phi(xg), g \in G(\mathbb{A})$, belongs to the $\pi$-isotypic subspace of the space $L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))$. For any $P \in \mathcal{P}(M)$ we have a canonical isomorphism of $G(\mathbb{A}) \times (\mathfrak{g}_C, K_{\infty})$-modules

$$j_P : \text{Hom}(\pi, L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A}))) \otimes \text{Ind}_{P}^{G(\mathbb{A})}(\pi) \to \mathcal{A}_{\pi}^2(P).$$

If we fix a unitary structure on $\pi$ and endow $\text{Hom}(\pi, L^2(A_M(\mathbb{R})^0 M(\mathbb{Q}) \backslash M(\mathbb{A})))$ with the inner product $(A, B) = B^*A$ (which is a scalar operator on the space of $\pi$), the isomorphism $j_P$ becomes an isometry.

Suppose that $P^c|Q$. The operator $M_{Q|P}(\pi, s) : = M_{Q|P}(s\varpi)|_{\mathcal{A}_{\pi}^2(P)}$, where $\varpi \in \mathfrak{a}_M^*$ is such that $\langle \varpi, \alpha^\vee \rangle = 1$, admits a normalization by a global factor $n_\alpha(\pi, s)$ which is a meromorphic function in $s$. We may write

$$M_{Q|P}(\pi, s) \circ j_P = n_\alpha(\pi, s) \cdot j_Q \circ (\text{Id} \otimes R_{Q|P}(\pi, s))$$

where $R_{Q|P}(\pi, s) = \otimes_v R_{Q|P}(\pi_{v}, s)$ is the product of the locally defined normalized intertwining operators and $\pi = \otimes_v \pi_v$ [4, §6], (cf. [42, (2.17)]). In many cases, the normalizing factors can be expressed in terms automorphic $L$-functions [59], [60].

We now turn to the spectral side. Let $\mathcal{L} \supset M$ be Levi subgroups in $L, P \in \mathcal{P}(M)$, and let $m = \dim \mathfrak{a}_L^G$ be the co-rank of $L$ in $G$. Denote by $\mathfrak{b}_{P,L}$ the set of $m$-tuples $\beta = (\beta_1^\gamma, \ldots, \beta_m^\gamma)$ of elements of $\Sigma^\gamma$ whose projections to $\mathfrak{a}_L$ form a basis for $\mathfrak{a}_L^G$. For any $\underline{\beta} = (\beta_1^\gamma, \ldots, \beta_m^\gamma) \in \mathfrak{b}_{P,L}$ let $\text{vol}(\underline{\beta})$ be the co-volume in $\mathfrak{a}_L^G$ of the lattice spanned by $\underline{\beta}$ and let
\[ \Xi_L(\beta) = \{(Q_1, \ldots, Q_m) \in F_1(M)^m : \beta_i' \in a_M^{\ast}, \ i = 1, \ldots, m\} \]
\[ = \{(P_1, P'_1), \ldots, (P_m, P'_m) : \ P_i|^{\beta_i} P'_i, \ i = 1, \ldots, m\}. \]

For any smooth function \( f \) on \( a_M^{\ast} \) and \( \mu \in a_M^{\ast} \) denote by \( D_\mu f \) the directional derivative of \( f \) along \( \mu \in a_M^{\ast} \). For a pair \( P_1|^{\alpha} P_2 \) of adjacent parabolic subgroups in \( P(M) \) write
\[
\delta_{P_1|P_2}(\lambda) = M_{P_2|P_1}(\lambda) D_\omega M_{P_1|P_2}(\lambda) : A^2(P_2) \to A^2(P_2),
\]
where \( \omega \in a_M^{\ast} \) is such that \( \langle \omega, \alpha' \rangle = 1 \). Equivalently, writing \( M_{P_1|P_2}(\lambda) = \Phi(\langle \lambda, \alpha' \rangle) \)
for a meromorphic function \( \Phi \) of a single complex variable, we have
\[
\delta_{P_1|P_2}(\lambda) = \Phi((\lambda, \alpha'))^{-1}\Phi'(\langle \lambda, \alpha' \rangle).
\]

For any \( m \)-tuple \( X = (Q_1, \ldots, Q_m) \in \Xi_L(\beta) \) with \( Q_i = (P_i, P'_i), \ P_i|^{\beta_i} P'_i \), denote by \( \Delta_X(P, \lambda) \) the expression
\[
\frac{\text{vol}(\beta)}{m!} M_{P_1|P}(\lambda)^{-1}\delta_{P_1|P}(\lambda) M_{P'_1|P_2}(\lambda) \cdots \delta_{P_{m-1}|P_{m-1}}(\lambda) M_{P'_m|P_m}(\lambda) \delta_{P_m|P'_m}(\lambda) M_{P'_m|P}(\lambda).
\]

Recall the (purely combinatorial) map \( X_L : \mathfrak{B}_{P,L} \to F_1(M)^m \) with the property that \( X_L(\beta) \in \Xi_L(\beta) \) for all \( \beta \in \mathfrak{B}_{P,L} \) as defined in [22, pp. 179–180].

For any \( s \in W(M) \) let \( L_s \) be the smallest Levi subgroup in \( L(M) \) containing \( w_s \). We recall that \( a_{L_s} = \{ H \in a_M \mid sH = H \} \). Set
\[
\iota_s = |\text{det}(s-1)_a|^{\ast}. \]

For \( P \in F(M_0) \) and \( s \in W(M_P) \) let \( M(P, s) : A^2(P) \to A^2(P) \) be as in [3, p. 1300]. \( M(P, s) \) is a unitary operator which commutes with the operators \( \rho(P, \lambda, h) \) for \( \lambda \in i a_{L_s}^{\ast} \).

Finally, we can state the refined spectral expansion.

**Theorem 9.1 ([22]).** For any \( h \in C^\infty_c(G(\mathbb{A}) \)
the spectral side of Arthur’s trace formula is given by
\[
J_{\text{spec}}(h) = \sum_{[M]} J_{\text{spec}, M}(h), \quad (9.6)
\]

\( M \) ranging over the conjugacy classes of Levi subgroups of \( G \) (represented by members of \( L \)), where

---

1. Note that this definition differs slightly from the definition of \( \delta_{P_1|P_2} \) in [22].
2. The map \( X_L \) depends in fact on the additional choice of a vector \( \mu \in (a_M^{\ast})^m \) which does not lie in an explicit finite set of hyperplanes. For our purposes, the precise definition of \( X_L \) is immaterial.
\[ J_{\text{spec}, M}(h) = \frac{1}{|W(M)|} \sum_{s \in W(M)} t_s \sum_{\beta \in \mathfrak{C}^*} \int_{\mathfrak{l}_s^*} \text{Tr}(\Delta_{\mathcal{X}_L, \beta}(P, \lambda) M(P, s) \rho(P, \lambda, h)) \, d\lambda \]

with \( P \in \mathcal{P}(M) \) arbitrary. The operators are of trace class and the integrals are absolutely convergent with respect to the trace norm and define distributions on \( \mathcal{C}(G(A)^1) \).

10. Logarithmic derivatives of local intertwining operators

In this section we prove some auxiliary results about local intertwining operators. To begin with we recall some facts concerning local intertwining operators and normalizing factors. Let \( M \in \mathcal{L} \) and \( P, Q \in \mathcal{P}(M) \). Let \( v \) be a finite place of \( Q \) and \( K_v \) an open compact subgroup of \( G(Q_v) \). Let \( \pi_v \in \Pi(M(Q_v)) \). Given \( \lambda \in \mathfrak{a}_M^* \), let \( (I^0_Q(\pi_v, \lambda), \mathcal{H}_P(\pi_v)) \) denote the induced representation. Let \( \mathcal{H}^0_P(\pi_v) \subset \mathcal{H}_P(\pi_v) \) be the subspace of \( K_v \)-finite functions. Let

\[ J_Q|_P(\pi_v, \lambda) : \mathcal{H}^0_P(\pi_v) \to \mathcal{H}^0_Q(\pi_v) \]

be the local intertwining operator between the induced representations \( I^0_P(\pi_v, \lambda) \) and \( I^0_Q(\pi_v, \lambda) \) [59]. It is proved in [7], [15, Lecture 15] that there exist scalar valued meromorphic functions \( r_{Q|P}(\pi_v, \lambda) \) of \( \lambda \in \mathfrak{a}_P^* \) such that the normalized intertwining operators

\[ R_{Q|P}(\pi_v, \lambda) = r_{Q|P}(\pi_v, \lambda)^{-1} J_{Q|P}(\pi_v, \lambda) \]

satisfy the conditions \( (R_1) - (R_8) \) of Theorem 2.1 of [7]. We recall some facts about the local normalizing factors. First assume that \( v \) is a finite valuation of \( Q \) with corresponding prime number \( q_v \in \mathbb{N} \). Furthermore assume that \( \dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1 \) and \( \pi_v \) is square integrable. Let \( P \in \mathcal{P}(M) \) and let \( \alpha \) be the unique simple root of \( (P, A_M) \). Then Langlands [15, Lecture 15] has shown that there exists a rational function \( V_P(\pi_v, z) \) of one variable such that

\[ r_{P|F}(\pi_v, \lambda) = V_P(\pi_v, q_v^{-\lambda(\tilde{\alpha})}) \]

where \( \tilde{\alpha} \in \mathfrak{a}_M \) is uniquely determined by \( \alpha \). For the construction of \( V_P \) see also [42, Sect. 3]. We need the following lemma.

**Lemma 10.1.** Let \( M \in \mathcal{L} \) be such that \( \dim(\mathfrak{a}_M/\mathfrak{a}_G) = 1 \). There exists \( C > 0 \) such that for all \( P \in \mathcal{P}(M) \) and all \( \pi \in \Pi(M(Q_v)) \) the number of zeros of the rational function \( V_P(\pi, z) \) is less than or equal to \( C \).

**Proof.** First assume that \( \pi \) is square integrable. Then the corresponding statement for the number of poles was proved in [42, Lemma 3.1]. However, by [42, (3.6)] the number of zeros of \( V_P(\pi, z) \) agrees with the number of poles \( V_P(\pi, z) \). Now let \( \pi \) be tempered. It is
known that $\pi$ is an irreducible constituent of an induced representation $I^M_\mathcal{R}(\sigma)$ where $M_\mathcal{R}$ is an admissible Levi subgroup of $M$ and $\sigma \in \Pi(M_\mathcal{R}(\mathbb{Q}_v))$ is square integrable modulo $A_\mathcal{R}$. Then by [7, (2.2)] we are reduced to the square integrable case. In general $\pi$ is a Langlands quotient of an induced representation $I^M_R(\sigma, \mu)$, where $M_R$ is an admissible Levi subgroup of $M$, $\sigma \in \Pi_{\text{temp}}(M_R(\mathbb{Q}_v))$, and $\mu$ is a point in the chamber of $\mathfrak{a}_R^e/\mathfrak{a}_M^e$ attached to $R$. Now we use [7, (2.3)] to reduce the proof to the tempered case. \hfill $\Box$

The main goal of this section is to estimate the logarithmic derivatives of the normalized intertwining operators $R_{Q|P}(\pi, \lambda)$. For $G = \text{GL}(n)$ such estimates were derived in [51, Proposition 0.2]. The proof depends on a weak version of the Ramanujan conjecture, which is not available in general. Therefore we will establish only an integrated version of it, which however, is sufficient for our purpose. For $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ denote by $\mathcal{H}_P(\pi)$ the Hilbert space of the induced representation $I^P_\mathcal{R}(\pi, \lambda)$. Furthermore, for an open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and $\nu \in \Pi(K_\infty)$, denote by $\mathcal{H}_P(\pi)^{K_f}$ the subspace of vectors, which are invariant under $K_f$ and let $\mathcal{H}_P(\pi)^{K_f, \nu}$ denote the $\nu$-isotypical subspace of $\mathcal{H}_P(\pi)^{K_f}$. Let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Then $R_{Q|P}(\pi, \lambda)$ depends on a single variable $s \in \mathbb{C}$ and we will write

$$R'_{Q|P}(\pi, s_0) := \frac{d}{ds} R_{Q|P}(\pi, s)|_{s = s_0}$$

for any regular $s_0 \in \mathbb{C}$.

**Proposition 10.2.** Let $M \in \mathcal{L}$, and let $P, Q \in \mathcal{P}(M)$ be adjacent parabolic subgroups. Let $K_f \subset G(\mathbb{A}_f)$ be an open compact subgroup and let $\nu \in \Pi(K_\infty)$. Then there exists $C > 0$ such that

$$\int_{\mathbb{R}} \left\| R_{Q|P}(\pi, it)^{-1} R'_{Q|P}(\pi, it) \right\|_{\mathcal{H}_P(\pi)^{K_f, \nu}} (1 + |t|)^{-1} dt \leq C \quad (10.3)$$

for all $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$ with $\mathcal{H}_P(\pi)^{K_f, \nu} \neq 0$.

**Proof.** We may assume that $K_f$ is factorisable, i.e., $K_f = \prod_v K_v$. Let $S$ be the finite set of finite places such that $K_v$ is not hyperspecial. Since $P$ and $Q$ are adjacent, by standard properties of normalized intertwining operators [7, Theorem 2.1] we may assume that $P$ is a maximal parabolic subgroup and $Q = \overline{P}$, the opposite parabolic subgroup to $P$. By [7, Theorem 2.1, (R8)], $R_{Q|P}(\pi_v, s)^{K_v}$ is independent of $s$ if $v$ is finite and $v \notin S$. Thus we have

$$R_{P|P}(\pi, s)^{-1} R'_{P|P}(\pi, s)|_{\mathcal{H}_P(\pi)^{K_f, \nu}} = R_{P|P}(\pi, s)^{-1} R'_{P|P}(\pi, s)|_{\mathcal{H}_P(\pi)^{\nu}} + \sum_{v \in S} R_{P|P}(\pi_v, s)^{-1} R'_{P|P}(\pi_v, s)|_{\mathcal{H}_P(\pi_v)^{K_v}} \quad (10.4)$$
This reduces our problem to the operators at the local places. We distinguish between the archimedean and the non-archimedean case.

Case 1: \( v < \infty \). Define \( A_v : \mathbb{C} \to \text{End}(\mathcal{H}_P(\pi_v)^{K_v}) \) by

\[
A_v(q_v^{-s}) := R_{\mathbb{P}^1 P}(\pi_v, s)|_{\mathcal{H}_P(\pi_v)^{K_v}}.
\]

This is a meromorphic function with values in the space of endomorphisms of a finite dimensional vector space. It has the following properties. By the unitarity of \( R_{\mathbb{P}^1 P}(\pi_v, it) \), \( t \in \mathbb{R} \), it follows that \( A_v(z) \) is holomorphic for \( z \in S^1 \) and satisfies \( \|A_v(z)\| \leq 1, |z| = 1 \). By [7, Theorem 2.1], the matrix coefficients of \( A_v(z) \) are rational functions. As in [23, (14)] we get

\[
\int_{\mathbb{R}} \left\| R_{\mathbb{P}^1 P}(\pi_v, it)^{-1} R_{\mathbb{P}^1 P}(\pi_v, it) \right\|_{\mathcal{H}_P(\pi_v)^{K_v}} \left( 1 + |t| \right)^{-1} dt \leq \left( 2 + \frac{1}{2} \log q_v \right) \int_{S^1} \|A_v'(z)\| |dz|. \tag{10.5}
\]

As explained above, \( A_v \) satisfies the assumptions of [23, Corollary 5.18]. Denote by \( z_1, \ldots, z_m \in \mathbb{C} \setminus S^1 \) the poles of \( A_v(z) \). Then \( (z - z_1) \cdots (z - z_m)A_v(z) \) is a polynomial of degree \( n \) with coefficients in \( \text{End}(\mathcal{H}_P^{K_v}) \) and by [23, Corollary 5.18] we get

\[
\|A_v'(z)\| \leq \max \left( \max(n - m, 0) + \sum_{j : |z_j| > 1} \frac{|z_j|^2 - 1}{|z_j - z|^2}, \sum_{j : |z_j| < 1} 1 - \frac{|z_j|^2}{|z_j - z|^2} \right), \quad z \in S^1.
\]

To estimate the right hand side, we need the following lemma.

**Lemma 10.3.** There exists \( C > 0 \) such that for all \( z_0 \in \mathbb{C} \setminus S^1 \)

\[
\int_{S^1} \left| \frac{z_0^2 - 1}{|z - z_0|^2} \right| dz \leq C. \tag{10.6}
\]

**Proof.** First consider the case \( |z_0| < 1 \). If we change variables by \( z = e^{\imath \theta} \), then up to a constant, the integral equals the integral of the Poisson kernel for the unit disc over the unit circle. From the theory of the Poisson kernel it is well known that, as a function of \( z_0 \), this integral is a continuous function on the closed unit disc which is equal to 1 on the unit circle. Thus the lemma holds for \( |z_0| < 1 \). For \( |z_0| > 1 \) we use that

\[
\int_0^{2\pi} \frac{|z_0^2 - 1|}{|e^{\imath \theta} - z_0|^2} d\theta = \int_0^{2\pi} \frac{|z_0^{-1}|^2 - 1}{|e^{\imath \theta} - z_0^{-1}|^2} d\theta,
\]

which reduces the problem to the previous case. \( \Box \)

Next we estimate \( m \) and \( n \). First consider \( m \). Let \( J_{\mathbb{P}^1 P}(\pi_v, s) \) be the usual intertwining operator so that
where \( r_{\mathcal{P}|P}(\pi_v, s) \) is the normalizing factor [7]. By [59, Theorem 2.2.2] there exists a
polynomial \( p(z) \) with \( p(0) = 1 \) whose degree is bounded independently of \( \pi_v \), such that
\( p(q_v^{-s})J_{\mathcal{P}|P}(\pi_v, s) \) is holomorphic on \( \mathbb{C} \). To deal with the normalizing factor we use (10.2)
together with Lemma 10.1 to count the number of poles of \( r_{\mathcal{P}|P}(\pi_v, s)^{-1} \). This leads
to a bound for \( m \) which depends only on \( G \). To estimate \( n \) we fix an open compact
subgroup \( K_v \) of \( G(\mathbb{Q}_v) \). Our goal is now to estimate the order at \( \infty \) of any matrix
coefficient of \( R_{\mathcal{P}|P}(\pi_v, s) \) regarded as a function of \( z = q_v^{-s} \). Write \( \pi_v \) as Langlands
quotient \( \pi_v = J_R^M(\delta_v, \mu) \) where \( R \) is a parabolic subgroup of \( M \), \( \delta_v \) a square integrable
representation of \( M_R(\mathbb{Q}_v) \) and \( \mu \in (a_R^*/a_G^*)_\mathbb{C} \) with \( \text{Re}(\mu) \) in the chamber attached to
\( R \). Then by [7, p. 30] we have

\[
R_{\mathcal{P}|P}(\pi_v, s) = R_{\mathcal{P}|P(R)|P(R)}(\delta_v, s + \mu)
\]

with respect to the identifications described in [7, p. 30]. Here \( s \) is identified with a
point in \( (a_R^*/a_G^*)_\mathbb{C} \) with respect to the canonical embedding \( a_M^* \subset a_G^* \). Using again the
factorization of normalized intertwining operators we reduce the problem to the case of a
square-integrable representation \( \delta_v \). Moreover \( \delta_v \) has to satisfy \( [I_P^G(\delta_v, s)]_{K_v : 1} \geq 1 \).
By [62, Lemma 1] we have

\[
[I_P^G(\delta_v, s)]_{K_v : 1} \geq 1 \iff [\delta_v]_{K_v \cap M(\mathbb{Q}_v) : 1} \geq 1 \tag{10.7}
\]

Let \( \Pi_2(M(\mathbb{Q}_v)) \) be the space of square-integrable representations of \( M(\mathbb{Q}_v) \). This space
has a manifold structure [26], [62]. By [26, Theorem 10] the set of square-integrable
representations \( \Pi_2(M(\mathbb{Q}_v), K_v) \) of \( M(\mathbb{Q}_v) \) with \( [\delta_v]_{K_v \cap M(\mathbb{Q}_v) : 1} \geq 1 \) is a compact subset
of \( \Pi_2(M(\mathbb{Q}_v)) \). Under the canonical action of \( i a_M \), the set \( \Pi_2(M(\mathbb{Q}_v), K_v) \) decomposes
into a finite number of orbits. For \( \mu \in ia_M \) and \( \delta_v \in \Pi_2(M(\mathbb{Q}_v), K_v) \), let \( (\delta_v)_\mu \in
\Pi_2(M(\mathbb{Q}_v), K_v) \) be the result of the canonical action. Then it follows that

\[
R_{\mathcal{P}|P}((\delta_v)_\mu, \lambda) = R_{\mathcal{P}|P}(\delta_v, \lambda + \mu).
\]

In this way our problem is finally reduced to the consideration of the matrix coefficients of
\( R_{\mathcal{P}|P}(\pi_v, s)|_{K_v} \) for a finite number of representations \( \pi_v \). This implies that \( n \) is bounded
by a constant which is independent of \( \pi_v \). Together with Lemma 10.3 it follows that
for each finite place \( v \) of \( \mathbb{Q} \) and each open compact subgroup \( K_v \) of \( G(\mathbb{Q}_v) \) there exists
\( C_v > 0 \) such that

\[
\int_{\mathbb{R}} \left\| R_{\mathcal{P}|P}(\pi_v, it)^{-1}R_{\mathcal{P}|P}(\pi_v, it) \right\|_{\mathcal{H}_p(\pi_v)\mathbb{K}_v} (1 + |t|)^{-1} dt \leq C_v \tag{10.8}
\]

for all \( \pi_v \in \Pi(M(\mathbb{Q}_v)) \) with \( I_P^G(\pi_v)|_{\mathcal{H}_p(\pi_v)\mathbb{K}_v} \neq 0 \).
Case 2: \( v = \infty \). As above let \( M \in \mathcal{L} \) with \( \dim(a_M/a_G) = 1 \) and \( P \in \mathcal{P}(M) \). Let \( \pi_\infty \in \Pi(M(\mathbb{R})) \) and \( \nu \in \Pi(K_\infty) \). As explained in [51, Appendix], there exist \( w_1, ..., w_r \in \mathbb{C} \) and \( m \in \mathbb{N} \) such that the poles of \( R_{\mathcal{P}P}(\pi_\infty, s) \big|_{\mathcal{H}_p(\pi_\infty)_{\nu}} \) are contained in \( \cup_{j=1}^r \{ w_j - k : k = 1, ..., m \} \). Moreover, by [51, Proposition A.2] there exists \( c > 0 \) which depends only on \( G \), such that
\[
    r \leq c, \quad m \leq c(1 + \|\nu\|).
\]
Let \( A: \mathbb{C} \to \mathcal{H}_{\pi_\infty}(\nu) \) be defined by
\[
    A(z) := R_{\mathcal{P}P}(\pi_\infty, z) \big|_{\mathcal{H}_p(\pi_\infty)_{\nu}}
\]
and let \( b(z) = \prod_{j=1}^r \prod_{k=1}^m (z - w_j + k) \). Then it follows from \((R_0)\) of [7, Theorem 2.1] that \( b(z)A(z) \) is a polynomial function. Moreover, by unitarity of \( R_{\mathcal{P}P}(\pi_\infty, it) \), \( t \in \mathbb{R} \), we have \( \|A(it)\| \leq 1 \). Thus \( A(z) \) satisfies the assumptions of [23, Lemma 5.19]. Thus by [23, Lemma 5.19] and (10.9) we get
\[
    \int_{\mathbb{R}} \left\| R_{\mathcal{P}P}(\pi_\infty, it)^{-1} R'_{\mathcal{P}P}(\pi_\infty, it) \big|_{\mathcal{H}_p(\pi_\infty)_{\nu}} \right\|^2 (1 + |t|^2)^{-1} dt = \int_{\mathbb{R}} \|A'(it)\|^2 (1 + |t|^2)^{-1} dt
\]
\[
    \ll \sum_{j=1}^r \sum_{k=1}^m \frac{|\text{Re}(w_j) - k| + 1}{(|\text{Re}(w_j) - k| + 1)^2 + (\text{Im}(w_j))^2}
\]
\[
    \ll 1 + \|\nu\|.
\]
Combining (10.4), (10.8) and (10.10), the proposition follows. \( \Box \)

11. The analytic torsion

As before we consider a reductive algebraic group \( G \) over \( \mathbb{Q} \), an open compact subgroup \( K_f \) of \( G(\mathbb{A}_f) \) and the adelic quotient
\[
    X := X(K_f),
\]
defined by (1.5). Let \( K_\infty \subset G(\mathbb{R}) \) be a maximal compact subgroup and let \( \tilde{X} = G(\mathbb{R})/K_\infty \). Recall that there are finitely many arithmetic subgroups \( \Gamma_j \subset G(\mathbb{R}), \) \( j = 1, ..., l \), such that \( X(K_f) \) is the disjoint union of \( \Gamma_j \setminus \tilde{X} \), \( j = 1, ..., l \). To simplify the notation we prefer to work with a single component. This is, for example, satisfied if \( G \) is semisimple and simply connected. Then by strong approximation we have
\[
    X = \Gamma \setminus \tilde{X},
\]
(11.1)
where \( \Gamma = (G(\mathbb{R}) \times K_f) \cap G(\mathbb{Q}) \). However, all arguments go through without change for the general case.
11.1. The Hodge-Laplace operator and heat kernels

Let \( \tau \) be an irreducible finite-dimensional complex representation of \( G(\mathbb{R}) \) on \( V_\tau \). Let \( E_\tau \) be the flat vector bundle over \( X \) associated to the restriction of \( \tau \) to \( \Gamma \). Let \( \tilde{E}_\tau \) be the homogeneous vector bundle associated to \( \tau|_{K_\infty} \) and let \( E_\tau := \Gamma \backslash \tilde{E}_\tau \). There is a canonical isomorphism

\[
E_\tau \cong E_\tau \tag{11.2}
\]

[33, Proposition 3.1]. By [33, Lemma 3.1], there exists an inner product \( \langle \cdot, \cdot \rangle \) on \( V_\tau \), which is unique up to scaling, which satisfies

1. \( \langle \tau(Y)u, v \rangle = -\langle u, \tau(Y)v \rangle \) for all \( Y \in \mathfrak{f}, u, v \in V_\tau \)
2. \( \langle \tau(Y)u, v \rangle = \langle u, \tau(Y)v \rangle \) for all \( Y \in \mathfrak{p}, u, v \in V_\tau \).

Such an inner product is called admissible. We fix an admissible inner product. Since \( \tau|_{K_\infty} \) is unitary with respect to this inner product, it induces a metric on \( E_\tau \), and by (11.2) on \( E_\tau \), which we also call admissible. Let \( \Lambda^p(E_\tau) = \Lambda^p T^*(X) \otimes E_\tau \). Let

\[
\nu_p(\tau) := \Lambda^p \text{Ad}^\tau : K_\infty \to \text{GL}(\Lambda^p \mathfrak{p}^* \otimes V_\tau). \tag{11.3}
\]

Then by (11.2) there is a canonical isomorphism

\[
\Lambda^p(E_\tau) \cong \Gamma \backslash (G(\mathbb{R}) \times_{\nu_p(\tau)} (\Lambda^p \mathfrak{p}^* \otimes V_\tau)) \tag{11.4}
\]

of locally homogeneous vector bundles. Let \( \Lambda^p(X, E_\tau) \) be the space the smooth \( E_\tau \)-valued \( p \)-forms on \( X \). Let

\[
C^\infty(G(\mathbb{R}), \nu_p(\tau)) := \{ f : G(\mathbb{R}) \to \Lambda^p \mathfrak{p}^* \otimes V_\tau : f \in C^\infty, f(gk) = \nu_p(\tau)(k^{-1})f(g), \forall g \in G(\mathbb{R}), \forall k \in K_\infty \}, \tag{11.5}
\]

and

\[
C^\infty(\Gamma \backslash G(\mathbb{R}), \nu_p(\tau)) := \{ f \in C^\infty(G(\mathbb{R}), \nu_p(\tau)) : f(\gamma g) = f(g), \forall g \in G(\mathbb{R}), \forall \gamma \in \Gamma \}. \tag{11.6}
\]

The isomorphism (11.4) induces an isomorphism

\[
\Lambda^p(X, E_\tau) \cong C^\infty(\Gamma \backslash G(\mathbb{R}), \nu_p(\tau)). \tag{11.7}
\]

A corresponding isomorphism also holds for the spaces of \( L^2 \)-sections. Let \( \Delta_p(\tau) \) be the Hodge-Laplacian on \( \Lambda^p(X, E_\tau) \) with respect to the admissible metric in \( E_\tau \). Let \( R_\Gamma \)
denote the right regular representation of \( G(\mathbb{R}) \) in \( L^2(\Gamma \backslash G(\mathbb{R})) \). Let \( \Omega \in \mathcal{Z}(\mathfrak{g}_C) \) be the Casimir element. By [33, (6.9)] it follows that with respect to the isomorphism (11.7) one has

\[
\Delta_p(\tau) = -R_{\Gamma}(\Omega) + \tau(\Omega) \text{Id}.
\] (11.8)

Let \( \tilde{E}_\tau \rightarrow \tilde{X} \) be the lift of \( E_\tau \) to \( \tilde{X} \). There is a canonical isomorphism

\[
\Lambda^p(\tilde{X}, \tilde{E}_\tau) \cong C^\infty(G(\mathbb{R}), \nu_p(\tau)).
\] (11.9)

Let \( \tilde{\Delta}_p(\tau) \) be the lift of \( \Delta_p(\tau) \) to \( \tilde{X} \). Then again it follows from [33, (6.9)] that with respect to the isomorphism (11.9) we have

\[
\tilde{\Delta}_p(\tau) = -R(\Omega) + \tau(\Omega) \text{Id}.
\] (11.10)

Let \( e^{-t\tilde{\Delta}_p(\tau)} \) be the corresponding heat semigroup. Regarded as an operator in the Hilbert space \( L^2(G(\mathbb{R}), \nu_p(\tau)) \), it is a convolution operator with kernel

\[
H^{\tau,p}_t: G(\mathbb{R}) \rightarrow \text{End}(\Lambda^p p^* \otimes V_\tau)
\] (11.11)

which belongs to \( C^\infty \cap L^2 \) and satisfies the covariance property

\[
H^{\tau,p}_t(k^{-1}gk') = \nu_p(\tau)(k)^{-1}H^{\tau,p}_t(g)\nu_p(\tau)(k')
\] (11.12)

with respect to the representation (11.3). Moreover, for all \( q > 0 \) we have

\[
H^{\tau,p}_t \in (C^q(G(\mathbb{R})) \otimes \text{End}(\Lambda^p p^* \otimes V_\tau))^K \times K, 
\] (11.13)

where \( C^q(G(\mathbb{R})) \) denotes Harish-Chandra’s \( L^q \)-Schwartz space (see [46, Sect. 4]). Let \( h^{\tau,p}_t \in C^\infty(G(\mathbb{R})) \) be defined by

\[
h^{\tau,p}_t(g) = \text{Tr} H^{\tau,p}_t(g), \quad g \in G(\mathbb{R}).
\] (11.14)

Let \( \chi_{K_f} \) be defined by (3.1). We define \( \phi^{\tau,p}_t \in C^\infty(G(\mathbb{A})) \) by

\[
\phi^{\tau,p}_t(g_{\infty}g_f) := h^{\tau,p}_t(g_{\infty})\chi_{K_f}(g_f)
\] (11.15)

for \( g_{\infty} \in G(\mathbb{R}) \) and \( g_f \in G(\mathbb{A}_f) \). Following the definition given in (1.7), we define the regularized trace of \( e^{-t\tilde{\Delta}_p(\tau)} \) by

\[
\text{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) := J_{\text{geom}}(\phi^{\tau,p}_t).
\] (11.16)
11.2. The asymptotic behavior of the regularized trace

Our next goal is to determine the asymptotic behavior of $\text{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right)$ as $t \to 0$ and $t \to \infty$. For $G = \text{GL}(n)$ or $G = \text{SL}(n)$ this has been carried out in [35]. Concerning the asymptotic behavior as $t \to 0$, we have

**Lemma 11.1.** There exist $a_j, b_{ij} \in \mathbb{C}$, $j \in \mathbb{N}_0$, $i = 0, \ldots, r_j$, such that as $t \to 0$, there is an asymptotic expansion

$$\text{Tr}_{\text{reg}}\left(e^{-t\Delta_p(\tau)}\right) \sim t^{-d/2} \sum_{j=0}^{\infty} a_j t^j + t^{-(d-1)/2} \sum_{j=0}^{\infty} \sum_{i=0}^{r_j} b_{ij} t^{j/2}(\log t)^i.$$ 

**Proof.** Let $\Omega_{K_\infty} \in \mathcal{Z}(t\mathfrak{c})$ be the Casimir element of $K_\infty$. For $p = 0, \ldots, n$ put

$$E_p(\tau) := \nu_p(\tau)(\Omega_{K_\infty}),$$

which we regard as an endomorphism of $\Lambda^p p^* \otimes V_\tau$. It defines an endomorphism of $\Lambda^p T^*(X) \otimes E_\tau$. By [39, Proposition 1.1] and (11.10) we have

$$\Delta_p(\tau) = \Delta_{\nu_p(\tau)} + \tau(\Omega)\text{Id} - E_p(\tau).$$

Let $\nu_p(\tau) = \oplus_{\nu \in \Pi(K_\infty)} m(\nu)\nu$ be the decomposition of $\nu_p(\tau)$ into irreducible representations. This induces a corresponding decomposition of the homogeneous vector bundle

$$\tilde{E}_{\nu_p(\tau)} = \bigoplus_{\nu \in \Pi(K_\infty)} m(\nu)\tilde{E}_{\nu}.$$  

(11.17)

With respect to this decomposition we have

$$E_p(\tau) = \bigoplus_{\nu \in \Pi(K_\infty)} m(\nu)\sigma(\Omega_{K_\infty})\text{Id}_{V_{\nu}},$$

(11.18)

where $\nu(\Omega_{K_\infty})$ is the Casimir eigenvalue of $\nu$ and $V_{\nu}$ the corresponding representation space. Let $\Delta_{\nu}$ be the Bochner-Laplace operator associated to $\nu$. By (11.17) we get a corresponding decomposition of $C^\infty(\tilde{X}, \tilde{E}_{\nu_p(\tau)})$ and with respect to this decomposition we have

$$\tilde{\Delta}_{\nu_p(\tau)} = \bigoplus_{\nu \in \Pi(K_\infty)} m(\nu)\tilde{\Delta}_{\nu}. $$

(11.19)

This shows that $\tilde{\Delta}_{\nu_p(\tau)}$ commutes with $E_p(\tau)$, and therefore we have

$$H^\tau_{t, p} = e^{-t(\tau(\Omega) - E_p(\tau))} \circ H^\nu_p(t).$$

(11.20)
Let $H_t^\nu$ be the kernel of $e^{-t\Delta^\nu}$ and $h_t^\nu := \text{Tr} H_t^\nu$. Then it follows from (11.18) and (11.19) that
\begin{equation}
    h_t^{\nu,p} = \sum_{\nu \in \Pi(K_\infty)} m(\nu) e^{-t(\tau(\Omega) - \nu(\Omega_{K_\infty}))} h_t^\nu.
\end{equation}

Using the definition of $\phi_t^{\nu,p}$ and $\phi_t^\nu$, respectively, we get
\begin{align*}
    \text{Tr}_{\text{reg}}(e^{-t\Delta^\nu}(\tau)) &= J_{\text{geom}}(\phi_t^{\nu,p}) = \sum_{\nu \in \Pi(K_\infty)} m(\nu) e^{-t(\tau(\Omega) - \nu(\Omega_{K_\infty}))} J_{\text{geom}}(\phi_t^\nu) \\
    &= \sum_{\nu \in \Pi(K_\infty)} m(\nu) e^{-t(\tau(\Omega) - \nu(\Omega_{K_\infty}))} \text{Tr}_{\text{reg}}(e^{-t\Delta^\nu}).
\end{align*}

Applying Theorem 1.1 concludes the proof. \(\square\)

To study the asymptotic behavior as $t \to \infty$ we use the Arthur trace formula (9.1). By [22, Corollary 1], $J_{\text{spec}}$ is a distribution on $C(G(\mathbb{A}); K_f)$ (see section 2 for its definition) and by [18, Theorem 7.1], $J_{\text{geom}}$ is continuous on $C(G(\mathbb{A}); K_f)$. This implies that (9.1) holds for $\phi_t^{\nu,p}$ and we have
\begin{equation}
    \text{Tr}_{\text{reg}}(e^{-t\Delta^\nu}(\tau)) = J_{\text{spec}}(\phi_t^{\nu,p}). \quad (11.22)
\end{equation}

Now we apply Theorem 9.1 to study the asymptotic behavior as $t \to \infty$ of the right hand side. First we have
\begin{equation}
    J_{\text{spec}}(\phi_t^{\nu,p}) = \sum_{[M]} J_{\text{spec},M}(\phi_t^{\nu,p}), \quad (11.23)
\end{equation}

where the sum ranges over the conjugacy classes of Levi subgroups of $G$ and $J_{\text{spec},M}(\phi_t^{\nu,p})$ is given by (9.7).

To analyze these terms, we proceed as in [35, Section 13]. Let $M \in \mathcal{L}$ and $P \in \mathcal{P}(M)$. We use the notation introduced in section 9. Recall that the discrete subspace $L^2_{\text{dis}}(A_M(\mathbb{R})^0 M(Q)\backslash M(\mathbb{A}))$ splits as the completed direct sum of its $\pi$-isotypic components for $\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))$. We have a corresponding decomposition of $\hat{A}^2(P)$ as a direct sum of Hilbert spaces $\hat{\oplus}_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \hat{A}^2_{\pi}(P)$. Similarly, we have the algebraic direct sum decomposition
\begin{equation}
    \mathcal{A}^2(P) = \bigoplus_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \mathcal{A}_\pi^2(P),
\end{equation}

where $\mathcal{A}_\pi^2(P)$ is the $K$-finite part of $\hat{A}_\pi^2(P)$. Let $A^2_{\pi}(P)^{K_f}$ be the subspace of $K_f$-invariant functions in $A^2_{\pi}(P)$, and for any $\sigma \in \Pi(K_\infty)$ let $A^2_{\pi}(P)^{K_f,\sigma}$ be the $\sigma$-isotypic subspace of $A^2_{\pi}(P)^{K_f}$. Recall that $A^2_{\pi}(P)^{K_f,\sigma}$ is finite dimensional [43, Prop. 3.5].
For $P, Q \in \mathcal{P}(M)$ let $M_{Q|P}(\lambda)$ be the intertwining operator (9.2). Denote by $M_{Q|P}(\pi, \lambda)$ the restriction of $M_{Q|P}(\lambda)$ to $A^2_{\pi}(P)$. Recall that the operator $\Delta_X(P, \lambda)$, which appears in the formula (9.7), is defined by (9.5). Its definition involves the intertwining operators $M_{Q|P}(\lambda)$. If we replace $M_{Q|P}(\lambda)$ by its restriction $M_{Q|P}(\pi, \lambda)$ to $A^2_{\pi}(P)$, we obtain the restriction $\Delta_X(P, \pi, \lambda)$ of $\Delta_X(P, \lambda)$ to $A^2_{\pi}(P)$. Similarly, let $\rho_{\pi}(P, \lambda)$ be the induced representation in $A^2_{\pi}(P)$. Fix $\beta \in \mathcal{B}_{P,L_s}$ and $s \in W(M)$. Then for the integral on the right of (9.7) with $h = \phi_{t,P}^{s,\pi}$ we get

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\xi),i(\mathfrak{a}_{L_s}^{\xi})),} \int \text{Tr} \left( \Delta_X(\beta)(P, \pi, \lambda)M(P, \pi, s)\rho_{\pi}(P, \lambda, \phi_{t,P}^{s,\pi}) \right) d\lambda. \quad (11.24)$$

In order to deal with the integrand, we need the following result. Let $\pi$ be a unitary admissible representation of $G(\mathbb{R})$. Let $A : \mathcal{H}_\pi \to \mathcal{H}_\pi$ be a bounded operator which is an intertwining operator for $\pi|_{K_\infty}$. Then $A \circ \pi(h_\pi^\nu)$ is a finite rank operator. Define an operator $\tilde{A}$ on $\mathcal{H}_\pi \otimes V_\nu$ by $\tilde{A} := A \otimes \text{Id}$. Then by [35, (9.13)] we have

$$\text{Tr}(A \circ \pi(h_\pi^\nu)) = e^{i(\pi(\Omega) - \nu(\Omega_{K_\infty}))} \text{Tr}(\tilde{A}|_{(\mathcal{H}_\pi \otimes V_\nu)^{K_\infty}}). \quad (11.25)$$

We will apply this to the induced representation $\rho_{\pi}(P, \lambda)$. Let $P, Q \in \mathcal{P}(M)$ and $\nu \in \Pi(K_\infty)$. Assume that $(A^2_{\pi}(P)^{K_f} \otimes V_\nu)^{K_\infty} \neq 0$. Denote by $\bar{M}_{Q|P}(\pi, \nu, \lambda)$ the restriction of

$$M_{Q|P}(\pi, \lambda) \otimes \text{Id} : A^2_{\pi}(P) \otimes V_\nu \to A^2_{\pi}(P) \otimes V_\nu$$

to $(A^2_{\pi}(P)^{K_f} \otimes V_\nu)^{K_\infty}$. Denote by $\Delta_X(\beta)(P, \pi, \nu, \lambda)$ and $\bar{M}(P, \pi, \nu, s)$ the corresponding restrictions. Let $m(\pi)$ denote the multiplicity with which $\pi$ occurs in the regular representation of $M(\xi)$ in $L^2_{\text{dis}}(A_M(\mathbb{R})^0 M(Q)\backslash M(A))$. Then

$$\rho_{\pi}(P, \lambda) \cong \bigoplus_{i=1}^{m(\pi)} \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\pi \otimes e^{i\lambda}). \quad (11.26)$$

Fix positive restricted roots of $\mathfrak{a}_P$ and let $\rho_{\alpha_P}$ denote the corresponding half-sum of these roots. For $\xi \in \Pi(M(\mathbb{R}))$ and $\lambda \in \mathfrak{a}_P^*$ let

$$\pi_{\xi,\lambda} := \text{Ind}_{P(\mathbb{R})}^{G(\mathbb{R})}(\xi \otimes e^{i\lambda})$$

be the unitary induced representation. Let $\xi(\Omega_M)$ be the Casimir eigenvalue of $\xi$. Define a constant $c(\xi)$ by

$$c(\xi) := -\langle \rho_{\alpha_P}, \rho_{\alpha_P} \rangle + \xi(\Omega_M). \quad (11.27)$$

Then for $\lambda \in \mathfrak{a}_P^*$ one has

$$\pi_{\xi,\lambda}(\Omega) = -\|\lambda\|^2 + c(\xi) \quad (11.28)$$
(see [29, Theorem 8.22]). Let

$$T_p := \{ \nu \in \Pi(K_\infty) : [\nu_p(\tau) : \nu] \neq 0 \}. \quad (11.29)$$

Using (11.21), (11.26) and (11.25), it follows that (11.24) is equal to

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \sum_{\nu \in T_p} m(\pi) e^{-t(\tau(\Omega)-c(\pi_\infty))}$$

$$\cdot \int_{i(a_{L_s}^2)^*} e^{-t\|\lambda\|^2} \text{Tr} \left( \Delta_{X_{L_s}(\beta)}(P, \pi, \nu, \lambda) \tilde{M}(P, \pi, \nu, s) \right) d\lambda. \quad (11.30)$$

Since $M(P, \pi, s)$ is unitary, (11.30) can be estimated by

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \sum_{\nu \in T_p} m(\pi) \dim(A^2_\pi(P)^{K_f, \nu})$$

$$\cdot e^{-t(\tau(\Omega)-c(\pi_\infty))} \int_{i(a_{L_s}^2)^*} e^{-t\|\lambda\|^2} \|\Delta_{X_{L_s}(\beta)}(P, \pi, \nu, \lambda)\| d\lambda. \quad (11.31)$$

For $\pi \in \Pi(M(\mathbb{A}))$ denote by $\lambda_{\pi_\infty}$ the Casimir eigenvalue of the restriction of $\pi_\infty$ to $M(\mathbb{R})$. Given $\lambda > 0$, let

$$\Pi_{\text{dis}}(M(\mathbb{A}); \lambda) := \{ \pi \in \Pi(M(\mathbb{A})) : |\lambda_{\pi_\infty}| \leq \lambda \}.$$ 

Let $d = \dim(M(\mathbb{R})^1/K_{N_\infty}^M$. If we use [41, Theorem 0.1] and argue in the same way as in the proof of [43, Proposition 3.5] it follows that for every $\nu \in \Pi(K_\infty)$ there exists $C > 0$ such that

$$\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}); \lambda)} m(\pi) \dim(A^2_\pi(P)^{K_f, \nu}) \leq C(1 + \lambda^{2d}) \quad (11.32)$$

for all $\lambda \geq 0$. Next we estimate the integral in (11.31). We use the notation of section 9. Let $\beta = (\beta_1^\nu, \ldots, \beta_m^\nu)$ and $X_{L_s}(\beta) = (Q_1, \ldots, Q_m) \in \Xi_{L_s}(\beta)$ with $Q_i = (P_i, P_i^\dagger), P_i|\beta| P_i^\dagger, i = 1, \ldots, m$. Using the definition (9.5) of $\Delta_{X_{L_s}(\beta)}(P, \pi, \nu, \lambda)$, it follows that we can bound the integral by a constant multiple of

$$\dim(\nu) \int_{i(a_{L_s}^2)^*} e^{-t\|\lambda\|^2} \prod_{i=1}^m \left\| \delta_{P_i|P_i^\dagger}(\lambda) \left| A^2_\pi(P_i)^{K_f, \nu} \right. \right\| d\lambda, \quad (11.33)$$

where $\delta_{P_i|P_i^\dagger}(\lambda)$ is defined by (9.4). We introduce new coordinates $s_i := \langle \lambda, \beta_i^\nu \rangle, i = 1, \ldots, m$, on $(a_{L_s,C}^2)^*$. Using (9.3) and (9.4), we can write
\[
\delta_{P_i|P_i'}(\lambda) = \frac{n'_{\beta_i}(\pi, s_i)}{n_{\beta_i}(\pi, s_i)} + j_{P_i'} \circ (\text{Id} \otimes R_{P_i|P_i'}(\pi, s_i)^{-1} R_{P_i|P_i'}(\pi, s_i)) \circ j_{P_i'}^{-1}.
\]  \tag{11.34}

Put
\[
\mathcal{A}^2_\pi(P)^{K_f, \mathcal{T}_p} = \bigoplus_{\nu \in \mathcal{T}_p} \mathcal{A}^2_\pi(P)^{K_f, \nu},
\]
where \(\mathcal{T}_p\) is defined by (11.29). It follows from [42, Theorem 5.3] that there exist \(N, k \in \mathbb{N}\) and \(C > 0\) such that
\[
\int \left| \frac{n'_{\beta_i}(\pi, s)}{n_{\beta_i}(\pi, s)} \right| (1 + |s|^2)^{-k} ds \leq C(1 + \lambda^2_{\pi, \infty})^N, \quad i = 1, \ldots, m,
\]  \tag{11.35}
for all \(\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))\) with \(\mathcal{A}^2_\pi(P)^{K_f, \mathcal{T}_p} \neq 0\). Combining (11.34), (11.35) and Proposition 10.2, it follows that we have
\[
\int e^{-t|\lambda|^2} \prod_{i=1}^{m} \left\| \delta_{P_i|P_i'}(\lambda) \right\|_{\mathcal{A}^2_\pi(P)^{K_f, \nu}} d\lambda \ll (1 + \lambda^2_{\pi, \infty})^{mN}
\]
for all \(t \geq 1\), and \(\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))\) with \(\mathcal{A}^2_\pi(P)^{K_f, \mathcal{T}_p} \neq 0\). Thus (11.31) can be estimated by a constant multiple of
\[
\sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \sum_{\nu \in \mathcal{T}_p} m(\pi) \dim \left( \mathcal{A}^2_\pi(P)^{K_f, \nu} \right) (1 + \lambda^2_{\pi, \infty})^N e^{-t(\tau(\Omega) - c(\pi, \infty))}.
\]  \tag{11.36}

Now we can proceed as in [35]. For the convenience of the reader we recall the arguments. By (11.27) we have
\[
\tau(\Omega) - c(\pi, \infty) = \tau(\Omega) + \|\rho_a\|^2 - \lambda_{\pi, \infty}.
\]  \tag{11.37}
Together with [35, Lemma 13.2], it follows that there exists \(\lambda_0 > 0\) such that
\[
\tau(\Omega) - c(\pi, \infty) \geq |\lambda_{\pi, \infty}|/2
\]
for all \(\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))\) with \(\mathcal{A}^2_\pi(P)^{K_f, \mathcal{T}_p} \neq 0\) and \(|\lambda_{\pi, \infty}| \geq \lambda_0\). We decompose the sum over \(\pi\) in (11.36) into two summands \(\Sigma_1(t)\) and \(\Sigma_2(t)\), where in \(\Sigma_1(t)\) the summation runs over all \(\pi\) with \(|\lambda_{\pi, \infty}| \leq \lambda_0\). Using (11.32), it follows that for \(t \geq 1\)
\[
\Sigma_2(t) \ll e^{-t|\lambda_0|/2}.
\]
Since \(\Sigma_1(t)\) is a finite sum, both in \(\pi\) and \(\nu\), it follows from [35, Lemma 13.1] that there exists \(c > 0\) such that
for $t \geq 1$. Recall that [35, Lemma 13.1] requires that $P = MAN$ is a proper parabolic subgroup of $G$. Putting everything together we obtain the following result.

**Lemma 11.2.** Let $\tau \in \text{Rep}(G(\mathbb{R}))$. Assume that $\tau \not\cong \tau_0$. Let $M$ be a proper Levi subgroup of $G$. There exists $c > 0$ such that

$$J_{\text{spec}, M}(\phi_{t}^{\tau,p}) = O(e^{-ct})$$

for $t \geq 1$.

It remains to deal with the case $M = G$. This has been done already in [35] for an arbitrary reductive group $G$. Using [35, (13.34)] and the considerations following this equality, we get

$$J_{\text{spec}, G}(\phi_{t}^{\tau,p}) = O(e^{-ct}) \quad (11.38)$$

for some $c > 0$. Combined with Lemma 11.2 we obtain

**Proposition 11.3.** There exists $c > 0$ such that

$$J_{\text{spec}}(\phi_{t}^{\tau,p}) = O(e^{-ct})$$

for all $t \geq 1$ and $p = 0, \ldots, n$.

Applying the trace formula (9.1), we get

$$\text{Tr}_{\text{reg}} \left( e^{-t\Delta_{p}(\tau)} \right) = O(e^{-ct}), \quad \text{as } t \to \infty,$$

which is the proof of Theorem 1.2. The asymptotic expansion as $t \to +0$ is provided by Lemma 11.1.

Thus the corresponding zeta function $\zeta_{p}(s; \tau)$, defined by the Mellin transform

$$\zeta_{p}(s; \tau) := \frac{1}{\Gamma(s)} \int_{0}^{\infty} \text{Tr}_{\text{reg}} \left( e^{-t\Delta_{p}(\tau)} \right) t^{s-1} \, dt \quad (11.39)$$

is holomorphic in the half-plane $\text{Re}(s) > d/2$ and admits a meromorphic extension to the whole complex plane. It may have a pole at $s = 0$. Let $f(s)$ be a meromorphic function on $\mathbb{C}$. For $s_{0} \in \mathbb{C}$ let

$$f(s) = \sum_{k \geq k_{0}} a_{k} (s - s_{0})^{k}$$
be the Laurent expansion of \( f \) at \( s_0 \). Put \( \text{FP}_{s=s_0} := a_0 \). Now we define the analytic torsion \( T_X(\tau) \in \mathbb{R}^+ \) by

\[
\log T_X(\tau) = \frac{1}{2} \sum_{p=0}^{d} (-1)^p p \left( \text{FP}_{s=0} \frac{\zeta_p(s; \tau)}{s} \right). \tag{11.40}
\]

**Remark 11.4.** In the case of \( G = \text{GL}(3) \) we have determined the coefficients of the log-terms in [35]. The calculation shows that the zeta functions definitely have a pole at \( s = 0 \). However, the combination \( \sum_{p=1}^{5} (-1)^p p \zeta_p(s; \tau) \) turns out to be holomorphic at \( s = 0 \) and we can instead define the logarithm of the analytic torsion by

\[
\log T_{X(K_f)}(\tau) = \frac{d}{ds} \left( \frac{1}{2} \sum_{p=1}^{5} (-1)^p p \zeta_p(s; \tau) \right) \bigg|_{s=0}.
\]

Put

\[
K(t, \tau) := \sum_{p=1}^{d} (-1)^p p \text{Tr}_{\text{reg}} \left( e^{-t \Delta_p(\tau)} \right). \tag{11.41}
\]

Then \( K(t, \tau) = O(e^{-ct}) \) as \( t \to \infty \) and the Mellin transform

\[
\int_{0}^{\infty} K(t, \tau) t^{s-1} dt
\]

converges absolutely and uniformly on compact subsets of \( \text{Re}(s) > d/2 \) and admits a meromorphic extension to \( \mathbb{C} \). Moreover, by (11.40) we have

\[
\log T_X(\tau) = \text{FP}_{s=0} \left( \frac{1}{\Gamma(s)} \int_{0}^{\infty} K(t, \tau) t^{s-1} dt \right). \tag{11.42}
\]

Let \( \delta(\tilde{X}) := \text{rank}_\mathbb{C}(G(\mathbb{R})^1) - \text{rank}_\mathbb{C}(K_\infty) \). Let \( \Gamma \subset G(\mathbb{R})^1 \) be a torsion free, co-compact lattice. Let \( X = \Gamma \backslash \tilde{X} \). If \( \dim X \) is even or \( \delta(\tilde{X}) \geq 2 \), then \( T_X(\tau) = 1 \) for every \( \tau \in \text{Rep}(G(\mathbb{R})^1) \) [46, Prop. 4.2]. We note that the proof of [46, Prop. 4.2] for the case \( \delta(\tilde{X}) \geq 2 \) contains a mistake, which is the claim that the Grothendieck group of admissible representations is generated by induced representations. This is only true for \( \text{GL}(n, \mathbb{R}) \). However, the proof can be fixed by proceeding as in the proof of [40, Corollary 2.2]. As the example in [45] shows, in the non-compact case the analytic torsion need not be trivial if \( \dim X \) is even. Hence one may guess that this is also true if \( \delta(\tilde{X}) \geq 2 \). We consider here the contribution of the discrete spectrum to the analytic torsion and study when it vanishes. Let
\[ \phi^*_t := \sum_{p=1}^{d} (-1)^p p \phi^T_p \quad \text{and} \quad k^*_t := \sum_{p=1}^{d} (-1)^p p k^T_p. \]

Then by (11.16) we have
\[ K(t, \tau) = J_{\text{spec}}(\phi^*_t). \quad \text{(11.43)} \]

For \( \pi \in \Pi(G(\mathbb{R})) \) let \( \Theta_\pi \) be the global character. Then the contribution of the discrete spectrum is given by
\[ J_{\text{spec},G}(\phi^*_t) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}))} m(\pi) \dim \left( \mathcal{H}^{K_f}_{\pi}(\phi^*_t) \right) \Theta_{\pi_{\infty}}(k^*_t). \quad \text{(11.44)} \]

If \( \dim X \) is even, we can follow the proof of [46, Prop. 4.2] to show that \( J_{\text{spec},G}(\phi^*_t) = 0 \). We conjecture that \( J_{\text{spec},G}(\phi^*_t) = 0 \), if \( \delta(X) \geq 2 \). In [35, Lemma 13.5] it was shown that this holds for \( G = \text{GL}(n) \), i.e., \( J_{\text{spec,GL}(n)}(\phi^*_t) = 0 \), if \( n \geq 5 \).

12. Approximation of \( L^2 \)-analytic torsion

In this section we prove the Theorem 1.5. We begin with some notation.

Let \( G \) be a reductive algebraic group over \( \mathbb{Q} \). Throughout this section we assume that \( G \) satisfies two conditions, called (TWN) and (BD), which were introduced in [23, Definition 5.2, Definition 5.9]. These conditions imply appropriate estimations for the logarithmic derivatives of the intertwining operators, which occur on the spectral side of the trace formula. Property (TWN) is a global condition concerning the scalar-valued normalizing factors of the intertwining operators, while (BD) is a condition for the local intertwining operators. For (BD) we need additional assumptions introduced in [21, Definition 2]. Let \( S \) be a set of primes. We say that \( G \) satisfies property (BD) for \( S \), if the local groups \( G(\mathbb{Q}_p) \), \( p \in S \), satisfy (BD) with a uniform value of \( C \) (see [21] for details).

For families of open compact subgroups of \( G(\mathbb{A}_f) \) we will need a certain non-degeneracy condition introduced in [20, Definition 1.3] which we recall next. For any reductive group \( H \) let \( H(\mathbb{A})^+ \) be the image of the map \( H^{sc} \to H(\mathbb{A}) \), where \( H^{sc} \) is the simply connected cover of the derived group of \( H \). Define \( H(\mathbb{A}_f)^+ \) analogously. A family \( K \) of compact open subgroups of \( G(\mathbb{A}_f) \) is called non-degenerate, if for any \( \mathbb{Q} \)-simple normal subgroup \( H \) of \( G \) we have \( \text{vol}_{H(\mathbb{A}_f)^+}(K \cap H(\mathbb{A}_f)^+) \to 0 \), \( K \in K \).

For an open compact subgroup \( K_M \subset M(\mathbb{A}_f) \) let \( \mu_{K_M}^M \) be the measure on \( \Pi(M(\mathbb{R})^1) \) defined by
\[ \mu_{K_M}^M = \frac{\text{vol}(K_M)}{\text{vol}(M(\mathbb{Q}) \setminus M(\mathbb{A})^1)} \cdot \sum_{\pi \in \Pi(M(\mathbb{A})^1)} \dim \text{Hom}_{M(\mathbb{A})^1}(\pi, L^2(M(\mathbb{Q}) \setminus M(\mathbb{A})^1)) \dim \pi_{K_M}^{K_M} \delta_{\pi_{\infty}}. \quad \text{(12.1)} \]
In the notation of [23] this is the measure \( \mu_{K,\mathcal{M}}^{\mathcal{M},\mathcal{S},\infty} \), where \( \mathcal{S},\infty = \{ \infty \} \). We will consider a collection of measures on \( \Pi(M(\mathcal{R})^1) \) which are polynomially bounded. For the definition we refer to [20, Sect. 4.1].

Let \( K \subset G(\mathbb{A}_f) \) be a compact open subgroup and let \( X(K) \) be the adelic quotient \((1.5)\). Let \( \tau \in \text{Rep}(G(\mathbb{R})^1) \) be irreducible and assume that \( \tau \) satisfies \( \tau \not\equiv \tau \circ \theta \), where \( \theta \) is the Cartan involution. Let \( E_{\tau,K} \rightarrow X(K) \) be the flat vector bundle over \( X(K) \) associated to \( \tau \). Let \( \Delta_{p,K}(\tau) \) be the Laplace operator acting in the space of \( E_{\tau,K} \)-valued \( p \)-forms on \( X(K) \). Let \( T_{X(K)}(\tau) \) be the regularized analytic torsion.

Recall the definition of \( T_f \) from (11.29). Let \( \mathcal{C}(G(\mathbb{A})^1)_{T_f} \) be the subspace of \( \mathcal{C}(G(\mathbb{A})^1) \) consisting of all \( f \in \mathcal{C}(G(\mathbb{A})^1) \) whose \( \mathbf{K}_\infty \times \mathbf{K}_\infty \)-types are contained in \( T_f \times T_f \).

To establish Theorem 1.5, we follow the approach of [36]. Recall the expression of \( T_{X(K)}(\tau) \) by (11.42) and (11.41). This reduces the problem to the consideration of the Mellin transform of the regularized trace of the heat operators. We write

\[
\int_0^\infty \text{Tr}_\text{reg} \left( e^{-t\Delta_{p,K}(\tau)} \right) t^{s-1} dt = \int_0^T \text{Tr}_\text{reg} \left( e^{-t\Delta_{p,K}(\tau)} \right) t^{s-1} dt + \int_T^\infty \text{Tr}_\text{reg} \left( e^{-t\Delta_{p,K}(\tau)} \right) t^{s-1} dt. \tag{12.2}
\]

To begin we estimate the second integral on the right hand side. The first auxiliary result is the following proposition.

**Proposition 12.1.** Let \( \tau \in \text{Rep}(G(\mathbb{R})^1) \) be irreducible. Assume that \( \tau \not\equiv \tau_\theta \). Let \( K_0 \subset G(\mathbb{A}_f) \) be a compact open subgroup. Then there exist \( C, c > 0 \) such that

\[
\frac{1}{\text{vol}(X(K))} | \text{Tr}_\text{reg} \left( e^{-t\Delta_{p,K}(\tau)} \right) | \leq Ce^{-ct} \tag{12.3}
\]

for all \( t \geq 1, p = 0, \ldots, d \), and open subgroups \( K \subset K_0 \).

**Proof.** Let \( h_t^{\tau,p} \) be defined by (11.14). Then by (11.22), the definition of \( \phi_t^{\tau,p} \) by (11.15) and Theorem 9.1 we have

\[
\text{Tr}_\text{reg} \left( e^{-t\Delta_{p,K}(\tau)} \right) = \sum_{[M]} \text{J}_{\text{spec},M}(h_t^{\tau,p} \otimes \chi_K), \tag{12.4}
\]

where \([M]\) runs over the conjugacy classes of Levi subgroups of \( G \). Let \( M \) be a proper standard Levi subgroup, \( P \in \mathcal{P}(M) \), and let \( \text{level}(K, G_M^+) \) be defined by [20, (3.1)]. Let \( l \in \mathbb{N} \) and \( \varepsilon > 0 \). By [36, Lemma 6.3] there exist \( C, c > 0 \) such that

\[
| \text{J}_{\text{spec},M}(h_t^{\tau,p} \otimes \chi_K) | \leq Ce^{-ct} \sum_{\pi \in \Pi_{\text{dis}}(M(\mathbb{A}))} \sum_{\nu \in T_f} \dim(\mathcal{A}_\pi^2(P,M^+)(1+|\lambda_{\pi,\infty}|^{-l})\text{level}(K, G_M^+)^{\varepsilon} \tag{12.5}
\]

for all $t \geq 1$, $p = 0, \ldots, d$, and open compact subgroups $K \subset G(A_f)$. To estimate the double series we proceed as in the proof of Proposition 4.4 in [20]. First we need to introduce some additional notation. Let $\mathcal{T}_{p, M} \subset \Pi(K_{M, \infty})$ be the finite set of all irreducible components of the restriction of elements of $\mathcal{T}_p$ to $K_{M, \infty}$. By Frobenius reciprocity only those $\pi \in \Pi_{\text{dis}}(M(A))$ such that $\pi_{\infty}$ contains a $K_{M, \infty}$-type in $\mathcal{T}_{p, M}$ can contribute to the right-hand side of (12.5). Denote by $\Pi_{\text{dis}}(M(A))^{\mathcal{T}_{p, M}}$ the corresponding subset of $\Pi_{\text{dis}}(M(A))$. Now consider the dimension of the spaces of automorphic forms appearing on the right hand side of (12.5). We have

$$\dim \mathcal{A}^2_\pi(P)^{K, \nu} = m_\pi \dim \text{Ind}_{P(A)}^{G(A)}(\pi)^{K, \nu} = m_\pi \dim \left(\text{Ind}_{P(F)}^{G(F)}(\pi_\infty)\right)^{\nu} \dim \left(\text{Ind}_{P(A_f)}^{G(A_f)}(\pi_f)\right)^K,$$

where

$$m_\pi = \dim \text{Hom}(\pi, L^2_{\text{dis}}(A_M M(Q) \setminus M(A))).$$

Now observe that $\dim \left(\text{Ind}_{P(F)}^{G(F)}(\pi_\infty)\right)^{\nu} \leq (\dim \nu)^2$. Denote by $\proj_M : P(A_f) \to M(A_f)$ the canonical projection. For $\gamma \in G(A_f)$ and an open compact subgroup $K \subset G(A_f)$ let

$$K^\gamma_M := \proj_M(P(A_f) \cap \gamma K \gamma^{-1}).$$

Then we have

$$\dim \left(\text{Ind}_{P(A_f)}^{G(A_f)}(\pi_f)\right)^K = \sum_{\gamma \in P(A_f) \setminus G(A_f)/K} \dim \pi^{K^\gamma_M}_f.$$

Summarizing we get

$$J_{\text{spec}, \mathcal{M}}(h_{\mathcal{T}_p}^T \otimes \chi K) \ll_{p, \epsilon} e^{-c t} \text{level}(K, G_M^+) \epsilon \cdot \sum_{\gamma \in P(A_f) \setminus G(A_f)/K} \sum_{\pi \in \Pi_{\text{dis}}(M(A))^{\mathcal{T}_{p, M}}} (1 + |\lambda_{\pi, \infty}|)^{-t} m_\pi \dim \pi^{K^\gamma_M}_f.$$  

(12.6)

According to the proof of [20, Corollary 4.9] the set of measures $\{\mu^{K^\gamma_M}_{M, K}\}$, where $K$ ranges over the open subgroups of $K_0$ and $\gamma \in G(A_f)$, is polynomially bounded. Using [20, (4.1)], it follows there exists $N \in \mathbb{N}$, depending only on $\mathcal{T}_{p, M}$, such that

$$\text{vol}_M(K^\gamma_M) \sum_{\pi \in \Pi_{\text{dis}}(M(A))^{\mathcal{T}_{p, M}}} (1 + |\lambda_{\pi, \infty}|)^{-N} m_\pi \dim \pi^{K^\gamma_M}_f \ll_{p, \epsilon} 1$$  

(12.7)

for all $K \subset K_0$ and $\gamma \in G(A_f)$. For any $f \in C_c(G(A_f))$ define $\mathcal{O} I_p(f)$ by
Arguing for where \( \nu \in \Pi(K_\infty) \) we have \([\pi_\infty|_{K_\infty} : \nu] \leq \dim(\nu) \) [29, Theorem 8.1]. Thus there exists \( C > 0 \) such that

\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(h_t^{\tau,p} \otimes \chi_K)| \leq C \text{vol}(K) \sum_{\pi \in \Pi_{\text{dis}}(G(A)^1)^\tau} m_\pi \dim(\pi^K) e^{(\pi_\infty(\Omega) - \tau(\Omega))}.
\]

for all \( t > 0 \) and open subgroups \( K \subset K_0 \). Note that in our notation \( \lambda_{\pi_\infty} = \pi_\infty(\Omega) \). Arguing as in the proof of Lemma 6.3 in [36], it follows that there exists \( c > 0 \) and for all \( l \in \mathbb{N} \) there exists \( C_l > 0 \) such that

\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(h_t^{\tau,p} \otimes \chi_K)| \leq \frac{C}{\text{vol}(K)} \sum_{\pi \in \Pi_{\text{dis}}(G(A)^1)^\tau} m_\pi \dim(\pi^K) e^{(\pi_\infty(\Omega) - \tau(\Omega))}.
\]
\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(h_t^{\tau,p} \otimes \chi_K)| \leq C_l \text{vol}(K) e^{-ct} \\
\times \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A}))} m_\pi (1 + |\lambda_{\pi,\infty}|)^{-l} \dim(\pi_f^K) 
\]
for all \( t \geq 1 \) and open subgroups \( K \subset K_0 \). Now observe that (12.7) holds also for \( M = G \), which implies that there exist \( C, c > 0 \) such that

\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(h_t^{\tau,p} \otimes \chi_K)| \leq C e^{-ct} \quad (12.13)
\]
for all \( t \geq 1, p = 0, \ldots, d \), and open subgroups \( K \subset K_0 \). Combining (12.4), (12.11) and (12.13), the proof is completed. \( \square \)

Using Proposition 12.1 it follows that there exist \( C, c > 0 \) such that

\[
\frac{1}{\text{vol}(X(K))} \left| \int_{T}^{\infty} \text{Tr}_{\text{reg}} \left( e^{-t\Delta_p,K(\tau)} \right) t^{-1} dt \right| \leq C e^{-cT} \quad (12.14)
\]
for all \( T \geq 1, p = 0, \ldots, d \), and open subgroups \( K \subset K_0 \).

Now we turn to the estimation of the first integral on the right hand side of (12.2). In order to study the short time behavior of the regularized trace of the heat operator with the help of the trace formula, we first need to show that we can replace \( h_t^{\tau,p} \) by an appropriate compactly supported test function without changing the asymptotic behavior as \( t \to 0 \). Let \( r(g) \) be the function on \( G_\infty \), which is defined by (6.17). For \( R > 0 \) let

\[
B_R := \{ g \in G_\infty : r(g) < R \}.
\]
We need the following auxiliary lemma.

**Lemma 12.2.** There exist \( C, c > 0 \) such that

\[
\int_{G_\infty} e^{-r^2(g)/t} dg \leq C e^{ct}
\]
for all \( t > 0 \).

**Proof.** For \( G = \text{GL}_n \) this was proved in [36, Lemma 7.1]. Let \( d_n(\cdot, \cdot) \) denote the geodesic distance function on \( GL_n(\mathbb{R})^1 / \text{O}(n) \), and let \( r_n(g) = d_n(g \text{O}(n), \text{O}(n)) \). At the beginning of section 6 we have shown that with respect to our choice of the embedding \( G_\infty \subset GL_n(\mathbb{R})^1 \), \( r \) and \( r_n \) coincide on \( G_\infty \). Thus the lemma follows from the case of \( \text{GL}_n \). \( \square \)
Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$ such that $f(u) = 1$, if $u \leq 1/2$, and $f(u) = 0$, if $u \geq 1$. Let $\varphi_R \in \mathcal{C}_c^\infty(G_\infty)$ be defined by

$$\varphi_R(g) := f \left( \frac{r(g)}{R} \right). \quad (12.15)$$

Then we have $\text{supp} \varphi_R \subset B_R$. Extend $\varphi_R$ to $G(\mathbb{R})$ by

$$\varphi_R(g \infty z) = \varphi_R(g \infty), \quad g \infty \in G_\infty, \quad z \in A_G(\mathbb{R})^0.$$

Define $\tilde{h}_{t,R}^{\tau,p} \in \mathcal{C}_c^\infty(G(\mathbb{R}))$ by

$$\tilde{h}_{t,R}^{\tau,p}(g \infty) := \varphi_R(g \infty) h_{t}^{\tau,p}(g \infty), \quad g \infty \in G(\mathbb{R}). \quad (12.16)$$

Then the restriction of $\tilde{h}_{t,R}^{\tau,p} \otimes \chi_K$ to $G(A)^1$ belongs to $\mathcal{C}_c^\infty(G(A)^1)$.

**Proposition 12.3.** There exist constants $C_1, C_2, C_3 > 0$ such that

$$\frac{1}{\text{vol}(X(K))} \left| J_{\text{spec}}(h_t^{\tau,p} \otimes \chi_K) - J_{\text{spec}}(\tilde{h}_{t,R}^{\tau,p} \otimes \chi_K) \right| \leq C_1 e^{-C_2 R^2/t + C_3 t}$$

for all open subgroups $K \subset K_0$, $p = 0, \ldots, d$, $t > 0$ and $R \geq 1$.

Proposition 12.3 allows us to replace $h_t^{\tau,p}$ by a compactly supported function.

**Proof.** Let $\psi_R := 1 - \varphi_R$. Then

$$J_{\text{spec}}(h_t^{\tau,p} \otimes \chi_K) - J_{\text{spec}}(\tilde{h}_{t,R}^{\tau,p} \otimes \chi_K) = J_{\text{spec}}(\psi_R h_t^{\tau,p} \otimes \chi_K).$$

Now we use the refined spectral expansion (9.7). Let $M \in \mathcal{L}$, and let $J_{\text{spec,M}}$ be the distribution on the right hand side of (9.7), which corresponds to $M$. Assume that $M \neq G$. We note that $\psi_R h_t^{\tau,p} \otimes \chi_K$ belongs to $\mathcal{C}(G(A)^1)^\tau_\rho$ and [20, Prop. 4.4] extends to $\mathcal{C}(G(A)^1)^\tau_\rho$. We recall that by [20, Proposition] the set of measures $\{\mu_{K,M}^{\gamma}\}$, where $K$ ranges over open subgroups of $K_0$ and $\gamma \in G(A_f)$, is polynomial bounded. Using [20, Proposition 4.4] and [20, Lemma 4.6] it follows that there exist $k \in \mathbb{N}$ and $\delta > 0$ such that

$$J_{\text{spec,M}}(\psi_R h_t^{\tau,p} \otimes \chi_K) \ll_{\tau_\rho} \text{vol}(K) \|\psi_R h_t^{\tau,p}\|_k \text{level}(K; G_+^{\infty})^{-\delta} \quad (12.17)$$

for all $t > 0$, $p = 0, \ldots, d$, and open subgroups $K \subset K_0$. The norm $\| \cdot \|_k$ is defined in [20, Sect. 3.1]. Let $\nabla^l : \mathcal{C}_c^\infty([T^*(G(\mathbb{R})^1)]^{\otimes (l-1)}) \to \mathcal{C}_c^\infty([T^*(G(\mathbb{R})^1)]^{\otimes l})$, $l \in \mathbb{N}$, be the extension of the exterior derivative $d : \mathcal{C}_c^\infty(G(\mathbb{R})^1) \to \mathcal{C}_c^\infty(T^*(G(\mathbb{R})^1))$, using the connection on $T^*(G(\mathbb{R})^1)$. There exists $C > 0$ such that the norm
\[
\|h\|_k \leq C \sum_{l=0}^{k} \|\nabla^l h\|_{L^1(G(\mathbb{R})^1)}, \quad h \in C(G(\mathbb{A})^1).
\]

Using Lemma 12.2 we can proceed exactly as in the proof of Proposition 7.2 in [36] and estimate \(\|\psi_Rh_t^{r,p}\|_k\). Combined with (12.17) it follows that there exist \(C_1, C_2, C_3 > 0\) such that
\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},M}(\psi_Rh_t^{r,p} \otimes \chi_K)| \leq C_1 e^{-C_2 R^2/t + C_3 t} \text{level}(K; G^+_M)^{-\delta} \quad (12.18)
\]
for all open subgroups \(K \subset K_0, \ p = 0, \ldots, d, \ t > 0\) and \(R \geq 1\).

It remains to deal with the case \(M = G\). In this case we have
\[
J_{\text{spec},G}(\psi_Rh_t^{r,p} \otimes \chi_K) = \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A})^1)} m_\pi \Tr \pi(\psi_Rh_t^{r,p} \otimes \chi_K)
\]
\[
= \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A})^1)} m_\pi \dim(\pi^K) \Tr \pi_\infty(\psi_Rh_t^{r,p}).
\]

If we argue as in the proof of Proposition 7.2 in [36, pp. 335], we get
\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(\psi_Rh_t^{r,p} \otimes \chi_K)| \leq C_1 e^{-C_2 R^2/t + C_3 t} \text{vol}(K) \sum_{\pi \in \Pi_{\text{dis}}(G(\mathbb{A})^1)} m_\pi \dim(\pi^K) (1 + |\lambda_{\pi^K}|)^{-k}.
\]

By [20, Proposition 4.7], the collection of measures \(\{\mu^K_N\}_{N \in \mathbb{N}}\), where \(K\) ranges over open subgroups of \(K_0\), is polynomially bounded hence we have (12.7), which also holds for \(M = G\). Hence there exist \(C_1, C_2, C_3 > 0\) such that
\[
\frac{1}{\text{vol}(X(K))} |J_{\text{spec},G}(\psi_Rh_t^{r,p} \otimes \chi_K)| \leq C_1 e^{-C_2 R^2/t + C_3 t}
\]
for all open subgroups \(K \subset K_0, \ t > 0, \ p = 0, \ldots, d,\) and \(R \geq 1\). This completes the proof of the proposition. \(\square\)

Now recall that
\[
\Tr_{\text{reg}} \left( e^{-t \Delta_{p,K}(\tau)} \right) = J_{\text{spec}}(h_t^{r,p} \otimes \chi_K).
\]
For \(R > 0\) let \(\varphi_R \in C_c^\infty(G(\mathbb{R})^1)\) be the function defined by (12.15). By Proposition 12.3 we have
\[
\Tr_{\text{reg}} \left( e^{-t \Delta_{p,K}(\tau)} \right) = J_{\text{spec}}(\varphi_Rh_t^{r,p} \otimes \chi_K) + r_R(t), \quad (12.19)
\]
where \(r_R(t)\) is a function of \(t \in [0, T]\) which satisfies
\[
\mathcal{F}_P(s) = \left( \frac{1}{s} \Gamma(s) \right) \int_0^T r_R(t) t^{s-1} dt = \int_0^T r_R(t) t^{-1} dt.
\]

Moreover
\[
\frac{1}{\text{vol}(\mathcal{X}(K))} \left| \int_0^T r_R(t) t^{-1} dt \right| \leq C_1 \int_0^T e^{C_2 R^2/t + C_3 t} t^{-1} dt \leq C_1 e^{C_4 R^2/T + C_3 T} \int_0^{T/R^2} e^{-C_4/t} t^{-1} dt. \tag{12.21}
\]

Now put \( R = T^2 \) and let
\[
h_{\tau, p, t, T} := \varphi_T h_{\tau, p, t}.
\]

Then it follows from (12.19) and (12.21) that there exist \( C, c > 0 \) such that
\[
\frac{1}{\text{vol}(\mathcal{X}(K))} \left| \mathcal{F}_P(s) \right| \leq C_1 \int_0^T e^{C_2 R^2/t + C_3 t} t^{-1} dt \leq C_1 e^{C_4 R^2/T + C_3 T} \int_0^{T/R^2} e^{-C_4/t} t^{-1} dt. \tag{12.23}
\]

for \( T \geq 1, p = 0, \ldots, d, \) and open subgroups \( K \subset K_0 \). Using the trace formula, we are reduced to deal with
\[
\mathcal{F}_P(s) \left( \int_0^T J_{\text{geom}}(h_{\tau, p, t, T} \otimes \chi_K) t^{s-1} dt \right).
\]

Now we fix \( T > 1 \). Let \( \varphi \in C_0^\infty(G(\mathbb{R})^1) \) be such that \( \varphi(g) = 1 \) in a neighborhood of \( 1 \in G(\mathbb{R})^1 \) and \( \text{supp} \varphi \subset B_T \). Put
\[
\tilde{h}_{\tau, p, t} = \varphi h_{\tau, p, t}.
\]

We consider test functions with \( \tilde{h}_{\tau, p, t} \) at the infinite place and \( \chi_K \) at the finite places. By Proposition (3.1) we have
\[ J_{\text{geom}}(\tilde{h}_t^{\tau,p} \otimes \chi_{K_j}) = J_{\text{unip}}(\tilde{h}_t^{\tau,p} \otimes \chi_{K_j}) \] (12.24)

for all \( K_j \) provided we choose the support of \( \tilde{h}_t^{\tau,p} \) sufficiently small (this choice depends on \( K_0 \) only).

In order to deal with the unipotent contribution we need to restrict the sequences of subgroups as described in the introduction. We fix a finite set \( S_1 \) of primes. Let \( \{K_j\}_{j \in \mathbb{N}} \) be sequence of open subgroups of \( K_0 \) satisfying \( K_j \rightarrow 1 \). Recall that this means that \( K_j = K_{S_1,j} \times K_{S_1}^{S_1} \) with \( K_{S_1}^{S_1} = \prod_{p \notin S_1} K_p \) a fixed open compact subgroup of \( G(A_{S_1}) \) and \( K_{S_1,j} = \prod_{p \in S_1} K_{p,j} \) with \( K_{S_1,j} \rightarrow 1 \) as \( j \rightarrow \infty \). Let \( S_0 \subset \mathfrak{s}_{\text{fin}} \setminus S_1 \) such that \( K_p = K_p \) for \( p \in \mathfrak{s}_{\text{fin}} \setminus (S_0 \cup S_1) \). Thus

\[ K_{S_1}^{S_1} = \prod_{p \in S_0} K_p \times \prod_{p \in \mathfrak{s}_{\text{fin}} \setminus (S_0 \cup S_1)} K_p. \]

We note that \( \{K_j\}_{j \in \mathbb{N}} \) is a non-degenerate sequence in the sense defined above. Put

\[ S := \{\infty\} \cup S_0 \cup S_1. \]

Note that \( K_j = \prod_v K_v \) with \( K_v = K_v \) for \( v \notin S \). Hence by the fine geometric expansion (8.1) we have

\[
J_{\text{unip}}(\tilde{h}_t^{\tau,p} \otimes \chi_{K_j}) = \sum_{M \in \mathcal{L}} \sum_{[u]_S \in U_S^M} a^M([u]_S, S)J^G_M([u]_S, \tilde{h}_t^{\tau,p} \otimes \chi_{K_j})
\]

\[ = \text{vol}(G(\mathbb{Q}) \setminus G(\mathbb{A})^1 / K_j)\tilde{h}_t^{\tau,p}(1)
+ \sum_{(M,[u]_S) \neq (G,\{1\})} a^M([u]_S, S)J^G_M([u]_S, \tilde{h}_t^{\tau,p} \otimes \chi_{K_j}). \] (12.25)

Concerning the volume factor in the first summand on the right hand side, we used that \( \chi_{K_j} = 1_{K_j} / \text{vol}(K_j) \). To deal with the first term on the right hand side, we note that \( \tilde{h}_t^{\tau,p}(1) = h_t^{\tau,p}(1) \). Furthermore, by [46, (5.11)] there is an asymptotic expansion

\[ h_t^{\tau,p}(1) \sim \sum_{j=0}^{\infty} a_j t^{-d/2+j} \] (12.26)

as \( t \rightarrow 0 \), and by [46, (5.16)] there exists \( c > 0 \) such that

\[ h_t^{\tau,p}(1) = O(e^{-ct}) \] (12.27)

as \( t \rightarrow \infty \). From (12.26) and (12.27) follows that the integral

\[ \int_0^\infty h_t^{\tau,p}(1)t^{s-1}dt \] (12.28)
converges in the half-plane $\text{Re}(s) > d/2$ and admits a meromorphic extension to $\mathbb{C}$ which is holomorphic at $s = 0$. The same is true for the integral over $[0, T]$ and we get

$$\text{FP}_{s=0} \left( \frac{1}{s \Gamma(s)} \int_0^T h_t^{(1)}(1)t^{s-1} \, dt \right) = \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty h_t^{(1)}(1)t^{s-1} \, dt \right) \bigg|_{s=0} + O(e^{-cT}). \quad (12.29)$$

Now recall the definition of the $L^2$-analytic torsion [30], [32]. For $t > 0$ let

$$K^{(2)}(t, \tau) := \sum_{p=1}^d (-1)^p p h_t^{(1)}(1).$$

Put

$$t^{(2)}_X(\tau) := \frac{d}{ds} \left( \frac{1}{\Gamma(s)} \int_0^\infty K^{(2)}(t, \tau) t^{s-1} \, dt \right) \bigg|_{s=0}.$$  

To summarize, we get

$$\frac{1}{2} \sum_{p=1}^d (-1)^p p \text{FP}_{s=0} \left( \frac{1}{s \Gamma(s)} \int_0^T h_t^{(1)}(1)t^{s-1} \, dt \right) = t^{(2)}_X(\tau) + O(e^{-cT}) \quad (12.30)$$

for $T \geq 1$. We observe that by [46, (5.20)], the $L^2$-analytic torsion $T_{X(K_j)}^{(2)}(\tau) \in \mathbb{R}^+$ is given by

$$\log T_{X(K_j)}^{(2)}(\tau) = \text{vol}(X(K_j)) \cdot t^{(2)}_X(\tau). \quad (12.31)$$

Next we consider the weighted orbital integrals on the right hand side of (12.25). Note that by definition of $\chi_{K_j}$ we have

$$J_G^M([u]_S, \tilde{h}_t^{(1)} \otimes \chi_{K_j}) = \frac{1}{\text{vol}(K_j)} J_G^M([u]_S, \tilde{h}_t^{(1)} \otimes 1_{K_j}).$$

To deal with the integral on the right hand side, we use the splitting formula (8.2). Put $S_f := S \setminus \{\infty\} = S_0 \cup S_1$. Then we get

$$J_G^M([u]_S, \tilde{h}_t^{(1)} \otimes \chi_{K_j}) = \sum_{L_1, L_2 \in \mathcal{L}(M)} d_{M}^{L_1}(L_1, L_2) J_G^M([u]_\infty, \tilde{h}_t^{(1)} \chi_{L_2 Q_1}) J_G^M([u]_{S_f}, (1_{K_j})_{S_f, Q_2}). \quad (12.32)$$

Using (11.21) and Proposition (7.2), it follows that as $t \to 0$, $J_G^M([u]_\infty, \tilde{h}_t^{(1)} \chi_{L_2 Q_1})$ has an asymptotic expansion of the form (7.21). This implies that the integral
converges absolutely and uniformly on compact subsets of the half plane $\text{Re}(s) > d/2$ and admits a meromorphic extension to $s \in \mathbb{C}$. Put

\[
A_{M}^{L}([u]_{\infty}, T) := \text{FP}_{s=0} \left( \frac{1}{s\Gamma(s)} \int_{0}^{T} J_{M}^{L}([u]_{\infty}, \tilde{\eta}_{t,Q_{1}}^{\tau,p}) t^{s-1} dt \right). \tag{12.34}
\]

By (12.32) it follows that the Mellin transform of $J_{M}^{G}([u]_{S}, \tilde{\eta}_{t}^{\tau,p} \otimes \chi_{K_{j}})$ as a function of $t$ is a meromorphic function on $\mathbb{C}$, and we get

\[
\text{FP}_{s=0} \left( \frac{1}{s\Gamma(s)} \int_{0}^{T} J_{M}^{G}([u]_{S}, \tilde{\eta}_{t}^{\tau,p} \otimes \chi_{K_{j}}) t^{s-1} dt \right) = \sum_{L_{1}, L_{2} \in \mathcal{L}(M)} d_{M}^{G}(L_{1}, L_{2}) A_{M}^{L}( [u]_{\infty}, T) J_{M}^{L_{2}}( [u]_{S}, 1_{K_{j}}) s_{f}, Q_{2} ), \tag{12.35}
\]

where $(1_{K_{j}}) s_{f}, Q_{2}$ is defined by (8.3). Next we deal with the orbital integral on the right hand side. Let $L \in \mathcal{L}$ and let $Q = LV$ be a semistandard parabolic subgroup. We first consider the case that the $K_{j}$ are principal congruence subgroups $K_{j}$ with $N_{j} \rightarrow \infty$ as $j \rightarrow \infty$. Using the definition (8.3) and the fact that $K_{j}$ is a normal subgroup of $K_{f}$, we have

\[
1_{K}(N_{j}) s_{f}, Q(m) = \delta_{Q}(m)^{1/2} \text{vol}(K_{f}) \int_{V(Q_{s_{f}})} 1_{K}(N_{j}) s_{f}(mv) dv
\]

for any $m \in L(A_{f})$. Hence $1_{K}(N_{j}) s_{f}, Q(m) = 0$ unless $m \in K^{L}(N_{j}) s_{f} = K(N_{j}) s_{f} \cap L(A_{f})$. Now if $m \in K^{L}(N_{j}) s_{f}$, we have $mv \in K(N_{j}) s_{f}$ if and only if $v \in K(N_{j}) s_{f}$. Hence

\[
1_{K}(N_{j}) s_{f}, Q(m) = \text{vol}(K_{f}) 1_{K^{L}(N_{j}) s_{f}}(m)
\]

which converges to 0 for any $m \neq 1$.

We now go back to the general case of a sequence of open compact subgroups $K_{j}$ of $K_{f}$ that converges to 1 at $s_{f}$. Let $N_{S}$ denote the product of all primes in $s_{f}$. By definition, for each $m \geq 0$, there exists $J_{m}$ such that $K_{j} \subseteq K(N_{S}^{m})$ for all $j \geq J_{m}$. Using the normality of $K(N_{S}^{m})$ again, we also have $\{ k^{-1} x k \mid k \in K, x \in K_{j} \} \subseteq K(N_{S}^{m})$ for all $j \geq J_{m}$. Using the computation for the principal congruence subgroups, we can therefore conclude that $1_{K_{j}, s_{f}, Q}$ also converges pointwise to 0 away from 1. Now we argue as in [16, Lemma 7] to conclude that for $[u]_{s_{f}} \neq 1$, 

\[
\int_{0}^{T} J_{M}^{L}([u]_{\infty}, \tilde{\eta}_{t,Q_{1}}^{\tau,p}) t^{s-1} dt \tag{12.33}
\]
\[
\lim_{j \to \infty} J_M^L([u]_{S_f}, 1_{K_j, S_f}) = 0. \tag{12.36}
\]

Denote by \( J_{\text{unip} - \{1\}}(\tilde{h}_t^{r,p} \otimes \chi_{K_j}) \) the sum on the right hand side of (12.25). Combining our results, we have proved

**Lemma 12.4.** For every \( T \geq 1 \) we have

\[
\frac{1}{\text{vol}(X(K_j))} \text{FP}_{s=0} \left( \frac{1}{s \Gamma(s)} \int_0^T J_{\text{unip} - \{1\}}(\tilde{h}_t^{r,p} \otimes \chi_{K_j}) t^{s-1} dt \right) \to 0 \text{ as } j \to \infty.
\]

Now we can complete the proof of Theorem 1.5. Let

\[
K_j(t, \tau) := \frac{1}{2} \sum_{k=1}^d (-1)^k k \text{Tr}_{\text{reg}} \left( e^{-t \Delta_{k,j}(\tau)} \right)
\]

where the \( \Delta_{k,j}(\tau) \) denotes the twisted Laplace operator on \( k \) forms with values in \( T_\tau \) over \( X(K_j) \). Let \( T > 0 \). By (1.10) and (11.39) we have

\[
\log T_{X(K_j)}(\tau) = \text{FP}_{s=0} \left( \frac{1}{s \Gamma(s)} \int_0^T K_j(t, \tau) t^{s-1} dt \right) + \int_T^\infty K_j(t, \tau) t^{-1} dt. \tag{12.37}
\]

By (12.14) there exist \( C, c > 0 \) such that

\[
\frac{1}{\text{vol}(X(K_j))} \left| \int_T^\infty K_j(t, \tau) t^{-1} dt \right| \leq C e^{-cT}. \tag{12.38}
\]

for all \( T \geq 1 \) and \( n \in \mathbb{N} \).

Now we turn to the first term on the right hand side. Let \( \tilde{h}_t^{r,p} \in C_c^\infty(G(\mathbb{R})^1) \) be defined by (12.22). Put

\[
K_j(t, \tau; T) := \frac{1}{2} \sum_{k=1}^d (-1)^k k J_{\text{geom}}(\tilde{h}_t^{r,k} \otimes \chi_{K_j}).
\]

Using (12.23) and the trace formula, it follows that, up to a term of order \( O(e^{-cT}) \), we can replace \( K_j(t, \tau) \) by \( K_j(t, \tau, T) \) in the first term on the right hand side of (12.37). Let

\[
K_{\text{unip} - \{1\}, j}(t, \tau; T) := \frac{1}{2} \sum_{p=1}^d (-1)^p p J_{\text{unip} - \{1\}}(\tilde{h}_t^{r,p} \otimes \chi_{K_j}). \tag{12.39}
\]
By [16, Lemma 5] and (12.25) it follows that for every \( T \geq 1 \) there exists \( J_0(T) \in \mathbb{N} \) such that

\[
K_j(t, \tau; T) = \frac{\text{vol}(X(K_j))}{2} \sum_{k=1}^{d} (-1)^k k h_{t, T} k + K_{\text{unip} - \{1\}, j}(t, \tau; T)
\]

for \( j \geq J_0(T) \). Using (12.30) it follows that for every \( T \geq 1 \) there exists \( J_0(T) \in \mathbb{N} \) such that

\[
\frac{1}{\text{vol}(X(K_j))} \int_0^T K_j(t, \tau)t^{s-1}dt = t_X^{(2)}(\tau) + \frac{1}{\text{vol}(X(K_j))} \int_0^T K_{\text{unip} - \{1\}, j}(t, \tau)t^{s-1}dt + O(e^{-cT}) (12.40)
\]

for \( j \geq J_0(T) \). For the last step we have to specialize to sequences of groups \( \{K_j\}_{j \in \mathbb{N}} \) which converge to 1 at \( S_1 \). Applying Lemma 12.4 we get that for every \( T \geq 1 \) and \( \varepsilon > 0 \) there exist constants \( C_1(T), C_2, c > 0 \) and \( J_0(T) \in \mathbb{N} \) such that

\[
\frac{1}{\text{vol}(X(K_j))} \int_0^T K_j(t, \tau)t^{s-1}dt - t_X^{(2)}(\tau) \leq C_1(T)\varepsilon + C_2e^{-cT} \quad \text{(12.41)}
\]

for \( j \geq J_0(T) \). Combined with (12.37), (12.38) and (12.14), Theorem 1.5 follows.

12.1. Proof of Corollary 1.10

Let \( G = \text{SL}(3) \). We proceed as in the proof of Theorem 1.5. The starting point is the decomposition (12.2) of the Mellin transform of the regularized trace of the heat operator. Concerning the second integral, the estimation (12.14) holds in general. We only have to deal with the first integral. As above, using (12.23) and (12.24), this can be reduced to the consideration of the unipotent contribution.

To this end we use the description of the unipotent contribution to the geometric side of the trace formula given in [28, Appendix B]. To begin with we need to introduce some notation.

For \( z \in Z(\mathbb{Q}) = \{ \pm I_3 \} \) let \( \mathfrak{o}_z \) be the \( O \)-equivalence class in \( G(\mathbb{Q}) \), and for \( x \in G(\mathbb{Q}) \) denote by \( \{x\}_G \) the \( G(\mathbb{Q}) \)-conjugacy class of \( x \). The set \( \mathfrak{o}_z \) is divided into three classes

\[
\mathfrak{o}_z = O_{\text{tri}, z} \cup O_{\text{min}, z} \cup O_{\text{reg}, z},
\]

where \( O_{\text{tri}, z} = \{ z \} \).
\[ O_{\text{min},z} = \{zu(0,1,0)\}_G, \quad \text{and} \quad O_{\text{reg},z} = \bigcup_{\alpha \in \mathbb{Q}^\times/(\mathbb{Q}^\times)^3} \{zu(1,0,\alpha)\}_G \] (12.43)

(cf. [25, §3]). For \( T_0 \in a_1^+ \) and \( f \in C_\infty^c(G(\mathbb{A})^1) \) let \( J_{T_0}^{O_{\text{tri},z}}(f) \), \( J_{T_0}^{O_{\text{min},z}}(f) \), and \( J_{T_0}^{O_{\text{reg},z}}(f) \) be the integrals defined in [28, Appendix A.3]. Then the unipotent contribution \( J_{0z}^{T_0}(f) \) is given by

\[
J_{0z}^{T_0}(f) = J_{O_{\text{tri},z}}^{T_0}(f) + J_{O_{\text{min},z}}^{T_0}(f) + J_{O_{\text{reg},z}}^{T_0}(f). \quad (12.44)
\]

Let \( S \) be a finite set of places of \( \mathbb{Q} \) containing the Archimedean place. For a character \( \chi = \prod_v \chi_v \) of \( \mathbb{Q}^\times \backslash \mathbb{A}^1 \), where \( \chi_v \) is a character of \( \mathbb{Q}^\times_v \), which is unramified outside of \( S \) let

\[
L^S(s, \chi) := \prod_{p \not\in S} L_p(s, \chi_p) \quad (12.45)
\]

be the partial Hecke \( L \)-function outside of \( S \). It is well known that \( L^S(s, \chi) \) is absolutely convergent and holomorphic for \( \text{Re}(s) > 1 \), and admits a meromorphic extension to the entire complex plane, which in fact is entire when \( \chi \) is not equal to the trivial character \( \chi_0 \) on \( \mathbb{Q}^\times \backslash \mathbb{A}^1 \). If \( S = \{ \infty \} \), we write \( L(s, \chi) \) for \( L^S(s, \chi) \). For \( \chi = \chi_0 \) we further write \( \zeta^S(s) := L^S(s, \chi_0) \) so that \( \zeta(s) = \zeta(\infty)(s) \) coincides with the Riemann zeta function.

Now we are ready to describe the global coefficients. Let

\[
L^S(s, \chi) = \alpha_{-1}^S(\chi)(s-1)^{-1} + \alpha_0^S(\chi) + \alpha_1^S(\chi)(s-1) + \ldots
\]

be the Laurent expansion of \( L^S(s, \chi) \) around \( s = 1 \). We write \( \alpha_k^S(\chi) = 0 \) if \( \chi \neq \chi_0 \). For a semistandard Levi subgroup \( H \) of \( \text{SL}(3) \) denote by \( \text{vol}_H \) the volume of \( H(\mathbb{Q}) \backslash H(\mathbb{A})^1 \).

Let

\[
u := u(0, 1, 0) \quad \text{and} \quad \mu_x := u(1, 0, x).
\]

For abbreviation put

\[ u(x_{12}, x_{13}, x_{23}) := \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix}, \quad x_{12}, x_{13}, x_{23} \in \mathbb{R}.\]

Let \( M_0 \) be the standard minimal Levi subgroup and \( M \) the \((2, 1)\) standard Levi subgroup of \( \text{SL}(3) \). The corresponding global coefficients then are according to [28, Appendix B] given by

\[ a_{\text{SL}(3)}(S, \nu) = \text{vol}_M \zeta^S(2)^{-1}(\zeta^S)'(2) \]

and
\[ a^{\text{SL}(3)}(S, \mu_x) = \frac{\text{vol}_M M_0}{3\alpha_1^2} \left( \alpha_1^S \alpha_{-1}^S + \sum_{\chi \neq \chi_0} \chi(x) \alpha_0^S(\chi) \alpha_0^S(\chi^{-1}) \right) \]

where the sum runs over all cubic characters \( \chi : \mathbb{Q}^* \backslash \mathbb{A}^1 \to \mathbb{C}^* \) that are unramified outside of \( S \). Next we describe the integrals on the right hand side of (12.44). For the first one we have \( J_{O_{\text{tri},z}}^T(f) = \text{vol}_G f(z) \). Let

\[ J_{O_{\text{min},z}}^T(f) = \text{vol}_M J_{M}^T(z, f) + a^{\text{SL}(3)}(S, \nu) J_G(z \nu, f). \]  

be the weighted orbital integrals defined in [28, Appendix A.3] which coincide with Arthur’s weighted orbital integrals. Let \( f \in C_c^\infty(G(Q_S)) \). By [28, Theorem B.1] the integral \( J_{O_{\text{min},z}}^T(f) \) converges absolutely and we have

\[ J_{O_{\text{min},z}}^T(f) = \text{vol}_M J_{M}^T(z, f) + \frac{\text{vol}_M}{2} \alpha_0^S J_{M}^T(z \mu_0, f) + \frac{\text{vol}_M}{3} \sum_{x \in \mathbb{Q}^*/(\mathbb{Q}^* \cap (\mathbb{Q}_S^*)^3)} a^{\text{SL}(3)}(S, \mu_x) J_G(z \mu_x, f). \]

The global coefficients are estimated by the following lemma.

**Lemma 12.5.** There exists a constant \( c > 0 \), and for each \( \epsilon > 0 \) a constant \( c_\epsilon > 0 \), both \( c \) and \( c_\epsilon \) independent of \( S \) and \( x \), such that

\[ \left| a^{\text{SL}(3)}(S, \nu) \right| \leq c \]

and

\[ a^{\text{SL}(3)}(S, \mu_x) \leq c_\epsilon N_S^\epsilon (\log N_S)^2 \]

where \( N_S = \prod_{p \in S_{\text{fin}}} p \).

**Proof.** The first assertion immediately follows from the fact that \( \zeta^S(2)^{-1}(\zeta^S)'(2) \) is bounded from above independently of \( S \).

For the second assertion using standard estimates for the derivatives of Hecke \( L \)-functions at 1 (see, e.g., [63, Theorem 8.18]) one can easily establish that for every Hecke character \( \chi \) of \( \mathbb{Q} \) that is unramified outside of \( S \), and every \( k \in \{-1, 0, 1\}, \)

\[ \alpha_k^S(\chi) \leq c(\log N_S)^{k+1} \]
where \( c > 0 \) is a suitable constant that does not depend on \( S \) or \( \chi \) and \( N_S := \prod_{p \in S} p \).

Finally, the number of cubic Hecke characters of \( \mathbb{Q} \) that are unramified outside of \( S \) is bounded from above by \( 9^{\text{fin}} \) since \( |Q_p^\times/(Q_p^\times)^3| \leq 9. \)

Now we have to deal with the weighted orbital integrals with test function \( f := \tilde{h}_t^\tau p \otimes \chi_K_j \). Except for \( J_G(z\mu_x, \tilde{h}_t^\tau p \otimes \chi_K_j) \), the orbital integrals are the same as the corresponding integrals for \( \text{GL}(3) \). These integrals can be treated as in [36]. Using the splitting formula (8.2), we are reduced to the study of weighted orbital integrals of the form \( J_M^L(u, 1_{K_p}) \), for Levi subgroups \( M \subseteq L \subseteq G \), \( u \in M(\mathbb{Q}) \) and \( p \) any prime. The remaining integral \( J_G(z\mu_x, \tilde{h}_t^\tau p \otimes \chi_K_j) \) does not contain any weight factor and can be dealt with directly. The integrals \( J_M^L(u, 1_{K_p}) \) are estimated by the following lemma.

**Lemma 12.6.** There exist \( c, R > 0 \) such that for all Levi subgroups \( M \subseteq L \subseteq G \) and all unipotent elements \( u \in M(\mathbb{Q}) \) with \( (u, M) \neq (1, L) \), we have
\[
J_M^L(u, 1_{K_p}) \leq cp^{-R}
\]
for every prime \( p \).

**Proof.** This can easily be seen from the explicit description of the weighted orbital integrals in [28, Appendix A and B]. In fact, most weighted orbital integrals are the same as in the \( \text{GL}(3) \) case so that the estimate for those follows from [36]. The only remaining integrals are those described just before [28, Theorem B.2] and do not contain any logarithmic weight functions. It is easily seen from their explicit description in [28] that the asserted bound also holds for them. \( \square \)

Combining Lemma 12.5, Lemma 12.6, and the fact that \( |Q_p^\times/(Q_p^\times)^3| \leq 9 \), Corollary 1.10 follows as [36, Theorem 1.1] in [36, §10].

**Data availability**

No data was used for the research described in the article.

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**References**
