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Grubb, Gerd

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Resolvents for fractional-order operators with nonhomogeneous local boundary conditions

Gerd Grubb

Department of Mathematical Sciences, Copenhagen University, Universitetsparken 5, DK-2100 Copenhagen, Denmark

Abstract

For 2α-order strongly elliptic operators $P$ generalizing $(-\Delta)^{a}$, $0 < a < 1$, the homogeneous Dirichlet problem on a bounded open set $\Omega \subset \mathbb{R}^{n}$ has been widely studied. Pseudodifferential methods have been applied by the present author when $\Omega$ is smooth; this is extended in a recent joint work with Helmut Abels showing exact regularity theorems in the scale of $L^{q}$-Sobolev spaces $H^{a}_{q}$ for $1 < q < \infty$, when $\Omega$ is $C^{\tau+1}$ with a finite $\tau > 2a$. We now develop this into existence-and-uniqueness theorems (or Fredholm theorems), by a study of the $L^{p}$-Dirichlet realizations of $P$ and $P^{*}$, showing that there are finite-dimensional kernels and cokernels lying in $C^{\alpha}(\Omega)$ with suitable $\alpha > 0$, $d(x) = \text{dist}(x, \partial\Omega)$. Similar results are established for $P - \lambda I$, $\lambda \in \mathbb{C}$. The solution spaces equal $a$-transmission spaces $H^{a}_{q}(t \Omega)$. Moreover, the results are extended to nonhomogeneous Dirichlet problems prescribing the local Dirichlet trace $(u/d^{a-1})|_{\partial\Omega}$. They are solvable in the larger spaces $H^{a-1}_{q}(t \Omega)$. Furthermore, the nonhomogeneous problem with a spectral parameter $\lambda \in \mathbb{C}$,

$$Pu - \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^{n} \setminus \Omega, \quad (u/d^{a-1})|_{\partial\Omega} = \varphi \text{ on } \partial\Omega,$$

is for $q < (1 - a)^{-1}$ shown to be uniquely resp. Fredholm solvable when $\lambda$ is in the resolvent set resp. the spectrum of the $L^{2}$-Dirichlet realization.

E-mail address: grubb@math.ku.dk.

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The results open up for applications of functional analysis methods. Here we establish solvability results for evolution problems with a time-parameter \( t \), both in the case of the homogeneous Dirichlet condition, and the case where a nonhomogeneous Dirichlet trace \( (u(x,t)/dx^a(x))|_{x \in \partial \Omega} \) is prescribed.

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0. Introduction

Fractional-order operators \( P \) have been studied extensively in recent years, the most prominent example being the fractional Laplacian \((-\Delta)^a (0 < a < 1)\) of order \( 2a \). They are of interest in Probability and Finance, as well as in Differential Geometry and Mathematical Physics.

Let \( P \) be a classical pseudodifferential operator of order \( 2a \), strongly elliptic with even symbol. From its definition on \( \mathbb{R}^n \), one can define its action on open subsets \( \Omega \subset \mathbb{R}^n \) in several ways (this is not obvious since \( P \) is generally nonlocal); the most common choice is to let \( P \) act on functions \( u \) defined on \( \mathbb{R}^n \) but vanishing on \( \mathbb{R}^n \setminus \Omega \) (i.e., supported in \( \bar{\Omega} \)), and restrict \( Pu \) to \( \Omega \) afterwards. This leads to the homogeneous restricted fractional Dirichlet problem

\[
Pu = f \text{ on } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \Omega. \tag{0.1}
\]

The strategies to study this include methods from potential theory and singular integral operator theory, probabilistic methods, and pseudodifferential methods.

The present author has worked with pseudodifferential methods [19–24], leading to satisfactory results in cases of \( C^\infty \)-domains \( \Omega \) and operators depending smoothly on \( x \). Other methods have allowed far less smoothness of \( \Omega \) (and in some cases of \( P \)). To breake this gap, we have in [2] with Helmut Abels worked out a theory that systematically allows \( \Omega \) to be \( C^{1+\tau} \) and \( P \) to have a \( C^\tau \)-smooth dependence on \( x \), for finite positive \( \tau \), leading to results in Sobolev-type spaces \( H^s_\partial \) (Bessel-potential spaces) with parameter \( s \) limited by \( \tau \), and with corollaries in Hölder spaces. We work in this paper with such operators and domains, under the basic hypothesis that \( \tau > 2a \); this is replaced by \( \tau > 2a + 1 \) if a nonhomogeneous boundary condition enters.

An important point in the present investigations is to enhance the regularity results of [2] with genuine solvability results: Theorems about existence and uniqueness of solutions, or, when relevant, Fredholm solvability. The point of departure is here the properties of the \( L_2 \)-Dirichlet realization \( P_{D,2} \) of \( P \) defined variationally from the sesquilinear form \( \int_\Omega Pu \bar{v} \, dx \) on \( \dot{H}^a(\bar{\Omega}) \) (the functions in the Sobolev space \( H^a(\mathbb{R}^n) \) supported in \( \bar{\Omega} \)). By compact embeddings, \( P_{D,2} \) has a discrete spectrum \( \Sigma \subset \mathbb{C} \) consisting of eigenvalues \( \lambda \) with finite dimensional eigenspaces \( N_\lambda \). The following issues will be addressed:
1) The regularity of eigenfunctions $u_\lambda$, i.e., nontrivial solutions of

$$Pu_\lambda = \lambda u_\lambda \text{ on } \Omega, \quad u_\lambda = 0 \text{ on } \mathbb{R}^n \setminus \Omega,$$

(0.2)

$\lambda \in \mathbb{C}$. It is shown that the possible $\lambda$-values belong to $\Sigma$, and the eigenfunctions lie in $d^\alpha C^\alpha(\overline{\Omega})$ for suitable $\alpha > 0$; $d = \text{dist}(x, \partial \Omega)$. See Theorem 4.4 and Corollary 4.5 below. The regularity of eigenfunctions has been studied earlier for $(-\Delta)^a$ e.g. in Chen and Song [7], Servadei and Valdinoci [40] and Ros-Oton and Serra [37], for smooth $P$ and $\Omega$ in Grubb [20], and for $\Omega$ equal to a ball in Dyda, Kuznetsov and Kwasnicki [10]. We also describe the eigenfunctions for $P^*$, in Theorem 4.15.

2) The structure of the $L_q$-Dirichlet realization $P_{D,q}$ of $P$ for functions satisfying (0.1), $1 < q < \infty$. In particular an investigation of $P^*$ and a proof that $P_{D,q}$ in $L_q(\Omega)$ and $(P^*)_{D,q'}$ in $L_{q'}(\Omega)$ are adjoints ($\frac{1}{q} + \frac{1}{q'} = 1$), and are Fredholm operators with the same kernel and cokernel as in the case $q = 2$. See Theorem 4.16 1° below. We are not aware of other studies of $P_{D,q}$ for $q \neq 2$ in nonsmooth cases with the precision that $D(P_{D,q})$ equals the $a$-transmission space $H_q^{a(2a)}(\overline{\Omega})$.

3) A treatment of the resolvent problem for $P_{D,q}$:

$$(P - \lambda)u = f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

(0.3)

showing unique solvability when $\lambda \in \mathbb{C} \setminus \Sigma$ and Fredholm solvability when $\lambda \in \Sigma$, in appropriate Sobolev-type function spaces. See Theorem 4.16 2° and Theorem 4.17 below. Also results with a higher regularity parameter $s > 0$, including Hölder spaces, are obtained, see Corollary 4.9 and Theorem 4.18.

4) A treatment of the local nonhomogeneous Dirichlet problem for $P$:

$$Pu = f \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

(0.4)

$$(u/d^{a-1})|_{\partial \Omega} = \varphi \text{ on } \partial \Omega,$$

showing regularity results (Theorem 3.7 and Corollary 3.9), and unique or Fredholm solvability (Theorem 5.1 and Corollary 5.2), in $H_q^s$-related function spaces and Hölder spaces.

5) A treatment of local nonhomogeneous Dirichlet problems with a spectral parameter $\lambda \in \mathbb{C}$:

$$Pu - \lambda u = f \text{ in } \Omega,$$

$$u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,$$

(0.5)

$$(u/d^{a-1})|_{\partial \Omega} = \varphi \text{ on } \partial \Omega,$$

when $q < (1 - a)^{-1}$. There is unique solvability when $\lambda \in \mathbb{C} \setminus \Sigma$, Fredholm solvability when $\lambda \in \Sigma$. See Theorem 5.4 below. The study of such problems was initiated by

6) Solvability of evolution problems

$$Pu(x, t) + \partial_t u(x, t) = f(x, t) \text{ on } \Omega \times I, \quad I = [0, T[, \quad u(x, t) = 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \quad u(x, 0) = 0;$$

possibly with a nonhomogeneous boundary condition

$$(u(x, t)/d^{a-1}(x))|_{x \in \partial\Omega} = \psi(x, t) \text{ on } \partial\Omega \times I.$$  

See Theorems 6.2–6.5 below. The results without condition (0.7) are a straightforward extension of results for smooth cases shown in [22], [23]; there are earlier results for $x$-independent operators on nonsmooth domains with Hölder estimates by Ros-Oton with Fernandez-Real and Vivas [11], [38]. Evolution problems prescribing the nonhomogeneous boundary condition (0.7) have to our knowledge not been studied before.

A large part of the results is new even for $(-\Delta)^a$.

It is a pervading fact in all these results that the exact operator domains are found; they have the form of $a$-transmission spaces $H^a_q(\Omega)$ in cases with homogeneous Dirichlet condition, and $(a - 1)$-transmission spaces $H^a_q(\Omega)$ in cases with nonhomogeneous Dirichlet condition.

It should be noted that there exist several interpretations of what a nonhomogeneous Dirichlet condition could be. A frequently studied possibility is to prescribe an exterior value of $u$,

$$u = g \text{ on } \mathbb{R}^n \setminus \Omega;$$

then the problem can be reduced to the homogeneous Dirichlet problem by subtraction of a suitable extension of $g$ to $\Omega$ (as described e.g. in [18]). Problems with the condition (0.8) are global, involving all of $\mathbb{R}^n$. Our choice of nonhomogeneous Dirichlet condition is

$$(u/d^{a-1})|_{\partial\Omega} = \varphi \text{ on } \partial\Omega;$$

it is localized to $\partial\Omega$, even pointwise. In Section 1 below, we explain by comparison with $\Delta$ why this choice is natural for $(-\Delta)^a$.

Plan of the paper: Section 1 introduces the local nonhomogeneous Dirichlet condition for $2a$-order operators. Section 2 sets up the terminology, introducing function spaces, pseudodifferential operators, and the special $\mu$-transmission spaces and their role in the definition of weighted boundary values. Section 3 recalls the regularity result for the
homogeneous Dirichlet problem known from [2], and establishes regularity results for the nonhomogeneous Dirichlet problem. In Section 4, realizations $P_{D,q}$ of the homogeneous Dirichlet problem for $P$ in $L_q(\Omega)$ ($1 < q < \infty$) are studied. Regularity of eigenfunctions is shown, also for $P^*$, and the resolvent $(P_{D,q} - \lambda)^{-1}$ is set up for $\lambda \notin \Sigma$, where $\Sigma$ denotes the spectrum of $P_{D,2}$. Fredholm properties are established for $P_{D,q} - \lambda$ when $\lambda \in \Sigma$. Section 5 shows the unique or Fredholm solvability of the nonhomogeneous Dirichlet problem, and shows how $P$ can be replaced by $P - \lambda$ when $q < (a - 1)^{-1}$. Finally, Section 6 treats evolution problems in cases where there is a uniform norm estimate of the resolvent.

1. A simple introduction to the local nonhomogeneous Dirichlet problem for $(-\Delta)^a$

1.1. Standard elliptic boundary problems

First recall some facts about the Laplacian $A = -\Delta$ (they are also true for strongly elliptic second-order differential operators $A$ with smooth coefficients, satisfying $\text{Re}(Au,u) > 0$ when $u \in C^\infty_0(\Omega) \setminus \{0\}$).

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. For simplicity, consider solvability just in a $C^\infty$ setting. For any $\mu > -1$ we define the space $\mathcal{E}_\mu$ by:

$$\mathcal{E}_\mu(\overline{\Omega}) = e^+ d_0^\mu C^\infty(\overline{\Omega}),$$

where $d_0(x)$ is a function equal to $\text{dist}(x, \partial \Omega)$ on a neighborhood of $\Omega$, extended as a positive $C^\infty$-function to the rest of $\Omega$. ($d_0$ can be replaced by an equivalent function $d$, see (2.2)ff. below.) Here $e^+$ denotes extension by zero on $\mathbb{R}^n \setminus \Omega$; it is relevant in the consideration of nonlocal operators. We shall also use the notation $r^+$ that indicates restriction from $\mathbb{R}^n$ to $\Omega$.

The functions in $\mathcal{E}_0$ have Taylor expansions in the neighborhood of each boundary point, where $x$ equals $(x', x_n)$ in local coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+$:

$$u(x) = u_0(x') + u_1(x')x_n + \frac{1}{2} u_2(x')x_n^2 + \ldots \text{ for } x_n > 0;$$

(1.2)

here $u_0 = \gamma_0 u = \lim_{x_n \to 0^+} u(x', x_n)$, and $u_j = \gamma_j u = \gamma_0(\partial^j_{x_n} u)$, for $j = 1, 2, \ldots$, with the usual notation for boundary values.

For integer values $k \geq 1$, the functions in $\mathcal{E}_k(\overline{\Omega})$ have Taylor expansions at the boundary, like (1.2) but skipping the first $k$ terms. For example for $u \in \mathcal{E}_1(\overline{\Omega})$,

$$u(x) = u_1(x')x_n + \frac{1}{2} u_2(x')x_n^2 + \ldots \text{ for } x_n > 0.$$

(1.3)

It is well-known that the nonhomogeneous Dirichlet problem for $A$:

$$Au = f \text{ on } \Omega, \quad \gamma_0 u = \varphi \text{ on } \partial \Omega,$$

(1.4)
is uniquely solvable for any \( f \in C^\infty(\overline{\Omega}) \), \( \varphi \in C^\infty(\partial\Omega) \), with \( u \in \mathcal{E}_0(\overline{\Omega}) \simeq C^\infty(\overline{\Omega}) \).

In particular, the homogeneous Dirichlet problem for \( A \):

\[
Au = f \text{ on } \Omega, \quad \gamma_0 u = 0 \text{ on } \partial\Omega,
\]

is uniquely solvable for any \( f \in C^\infty(\overline{\Omega}) \); here \( u \in \mathcal{E}_1(\overline{\Omega}) \), satisfying (1.3).

1.2. The fractional Laplacian

Now consider \( P = (-\Delta)^a \) with \( 0 < a < 1 \). It is of order \( 2a \), and the symbol \( p(\xi) = |\xi|^{2a} \) is even in \( \xi \) (satisfies \( p(-\xi) = p(\xi) \)). It was proved in \([30,31]\) (and presented in detail in \([19]\)) by a fine analysis of what happens at the boundary, that \( r^+ P \) has a good meaning on \( \mathcal{E}_{a+k}(\overline{\Omega}) \) with \( k \) integer \( \geq -1 \), mapping

\[
r^+ P : \mathcal{E}_{a+k}(\overline{\Omega}) \to C^\infty(\overline{\Omega}).
\]

The homogeneous restricted Dirichlet problem for \( P \) is generally agreed to be the problem

\[
Pu = f \text{ on } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}.
\]

It is well-known (by a variational argument) that this problem is uniquely solvable when \( u \) is a priori sought in \( \dot{H}^a(\overline{\Omega}) = \{ u \in H^a(\mathbb{R}^n) \mid u = 0 \text{ on } \mathbb{R}^n \setminus \Omega \} \) and \( f \) is given in \( L^2(\Omega) \). The regularity question is about how a higher regularity of \( f \) implies a higher regularity of \( u \).

It is shown in \([19]\) that when \( f \in C^\infty(\overline{\Omega}) \), the solution of (1.7) is in fact in \( \mathcal{E}_a(\overline{\Omega}) \). Thus \( r^+ P \) defines a homeomorphism:

\[
r^+ P : \mathcal{E}_a(\overline{\Omega}) \xrightarrow{\sim} C^\infty(\overline{\Omega}).
\]

Note that, by multiplication by \( d^a \), one has from (1.2) in local coordinates:

when \( u \in \mathcal{E}_a \), \( u(x) = v_0(x')x_n^a + v_1(x')x_n^{a+1} + \frac{1}{2} v_2(x')x_n^{a+2} + \ldots \) for \( x_n > 0 \),

where \( v_0 = \gamma_0(u/x_n^a) \), \( v_1 = \gamma_1(u/x_n^a) \), etc. Then \( \mathcal{E}_a \) has a role parallel to that of \( \mathcal{E}_1 \) in the standard homogeneous Dirichlet problem (1.5), cf. (1.3).

Analogously to the nonhomogeneous standard Dirichlet problem (1.4) we now consider \( \mathcal{E}_{a-1} \), which will have a role parallel to that of \( \mathcal{E}_0 \) in the following nonhomogeneous local Dirichlet problem for \( P \):

\[
Pu = f \text{ on } \Omega, \quad u = 0 \text{ on } \mathbb{R}^n \setminus \overline{\Omega}, \quad \gamma_0(u/d^{a-1}) = \varphi,
\]

where \( u \) is sought in \( \mathcal{E}_{a-1} \). Indeed, the boundary behavior of functions in \( \mathcal{E}_{a-1} \) is
when \( u \in \mathcal{E}_a-1 \),
\[
u(x) = w_0(x')x_n^{-1} + w_1(x')x_n^{-1} + \frac{1}{2} w_2(x')x_n^{-1} + \ldots \text{ for } x_n > 0, \quad (1.11)
\]
where \( w_0 = \gamma_0(u/x_n^{-1}), \ w_1 = \gamma_1(u/x_n^{-1}) \), etc. Note that the expansion is similar to that in (1.9), the only difference being that the coefficient \( w_0 \) vanishes there, i.e.,
\[
\mathcal{E}_a \text{ is the subset of } \mathcal{E}_a-1 \text{ where } \gamma_0(u/x_n^{-1}) = 0. \quad (1.12)
\]

Using that a given \( \varphi \in C^\infty(\partial\Omega) \) can be lifted to a function \( z \in \mathcal{E}_a-1 \) such that \( \gamma_0(z/d_a^{-1}) = \varphi \) (namely, locally, \( z(x) = \varphi(x')x_n^{-1} \)), we get immediately the unique solvability of the nonhomogeneous problem (1.10) from the solvability of the homogeneous problem (1.7).

For generalizations \( P \) of the fractional Laplacian with smooth, even symbol, one finds the same results. This shows the interesting fact that \( \mathcal{E}_a \) is universal as the solution space for the homogeneous Dirichlet problem, and that \( \mathcal{E}_a-1 \) similarly plays a universal role for our nonhomogeneous Dirichlet problem.

Let us mention briefly that one can also define a local Neumann condition for \( P \) in analogy with the standard Neumann condition for \( A \), namely by prescribing \( \gamma_1(u/d_a^{-1}) \). Also here there are general solvability results; more details are found in e.g. [18], [21].

Note that the functions \( u \) in \( \mathcal{E}_a-1 \) blow up like \( d_a^{-1} \) at the boundary at the points where \( \gamma_0(u/d_a^{-1}) \) does not vanish; this is a natural fact in the theory.

### 1.3. Results in Sobolev spaces

There is now the question of how these problems are treated in more general function spaces. For example, in terms of \( L_2 \)-Sobolev spaces, the homogeneous Dirichlet problem for the Laplacian (1.5) is solved in \( H^2(\Omega) \cap H^0_0(\Omega) \) when \( f \in L_2(\Omega) \), and the nonhomogeneous Dirichlet problem for the Laplacian (1.4) is solved in \( H^2(\Omega) \) when \( f \in L_2(\Omega), \ \varphi \in H^2(\partial\Omega) \). The same spaces enter when \( \Delta \) is replaced by a strongly elliptic second-order differential operator \( A \) with smooth coefficients.

There are corresponding results for \( (-\Delta)^a \): The homogeneous Dirichlet problem (1.7) is solved in \( H^{a(2a)}(\Omega) \) when \( f \in L_2(\Omega) \), and the nonhomogeneous Dirichlet problem (1.10) is solved in \( H^{(a-1)(2a)}(\Omega) \) when \( f \in L_2(\Omega), \ \varphi \in H^{a+\frac{1}{2}}(\partial\Omega) \). Here the so-called transmission spaces \( H^{a(s)}(\Omega) \) and \( H^{(a-1)(s)}(\Omega) \) enter; they were defined in [19] (building on [30]), and are important since they give exact information. \( H^{a(2a)}(\Omega) \) takes the place of \( H^2(\Omega) \cap H^0_0(\Omega) \) and contains the space \( \mathcal{E}_a(\Omega) \), whereas \( H^{(a-1)(2a)}(\Omega) \) takes the place of \( H^2(\Omega) \) and contains \( \mathcal{E}_{a-1}(\Omega) \). (Their definition is recalled in the general preliminaries section below.) Also here the spaces are universal; the same spaces enter when \( (-\Delta)^a \) is replaced by a strongly elliptic pseudodifferential operator \( P \) of order \( 2a \) with even symbol.

**Remark 1.1.** The nonhomogeneous Dirichlet problem (1.10) for \( (-\Delta)^a \) was proposed simultaneously and independently in the works [19] and Abatangelo [1], in two very
different formulations. Ours was in the style explained above, whereas Abatangelo formulated the problem in its relation to a Green’s function and a representation of the solution by a sum of integrals over $\Omega$ and $\partial\Omega$. In [1], the boundary value is somewhat implicitly formulated as a term $Eu$ defined by integrals, and only appears in the form $c\gamma_0(u/d^{a-1})$ in the case where $\Omega$ is a ball. The word “large solution” is introduced to underline the blow-up (like $d^{a-1}$) that solutions with nonzero continuous boundary data will have at the boundary.

It has been known to many people as an accepted fact (or folklore) that prescribing $Eu$ is equivalent to prescribing $\gamma_0(u/d^{a-1})$. A proof that $Eu$ is proportional to $\gamma_0(u/d^{a-1})$ for the fractional Laplacian is included as Appendix A.1 below. A different proof is given in App. B of [6], with another proportionality factor (see Theorem. A.4ff. below).

2. Preliminaries

2.1. Function spaces

The space $C^k(\mathbb{R}^n) \equiv C^k_\mathbb{R}(\mathbb{R}^n)$ consists of $k$-times differentiable functions with uniform norms $\|u\|_{C^k} = \sup_{|\alpha| \leq k, x \in \mathbb{R}^n} |D^\alpha u(x)|$ ($k \in \mathbb{N}_0$), and the Hölder spaces $C^\tau(\mathbb{R}^n)$, $\tau = k + \sigma$ with $k \in \mathbb{N}_0$, $0 < \sigma < 1$, also denoted $C^{k,\sigma}(\mathbb{R}^n)$, consists of function with norms $\|u\|_{C^\tau} = \|u\|_{C^k} + \sup_{|\alpha| = k, x \neq y} |D^\alpha u(x) - D^\alpha u(y)|/|x - y|^\sigma$. The latter definition extends to Lipschitz spaces $C^{k,1}(\mathbb{R}^n)$. There are similar spaces over subsets of $\mathbb{R}^n$. We denote $C^\infty(\mathbb{R}^n) = \bigcap_{k \in \mathbb{N}} C^k_\mathbb{R}(\mathbb{R}^n)$.

The halfspaces $\mathbb{R}^n_\pm$ are defined by $\mathbb{R}^n_\pm = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}$, with points denoted $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1})$. For a given real function $\zeta \in C^{1+\tau}(\mathbb{R}^{n-1})$ (some $\tau > 0$), we define the curved halfspace $\mathbb{R}^n_\zeta$ by

$$\mathbb{R}^n_\zeta = \{ x \in \mathbb{R}^n \mid x_n > \zeta(x') \};$$

(2.1)

it is a $C^{1+\tau}$-domain. (The function $\zeta$ was denoted $\gamma$ in [2]; we change the name to avoid confusion with the notation for trace operators $\gamma_j$.)

By a bounded $C^{1+\tau}$-domain $\Omega$ we mean the following: $\Omega \subset \mathbb{R}^n$ is open and bounded, and every boundary point $x_0$ has an open neighborhood $U$ such that, after a translation of $x_0$ to 0 and a suitable rotation, $U \cap \Omega$ equals $U \cap \mathbb{R}^n_\zeta$ for a function $\zeta \in C^{1+\tau}(\mathbb{R}^{n-1})$ with $\gamma(0) = 0$.

Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from $\mathbb{R}^n$ to $\Omega$ resp. $\overline{\Omega} = \mathbb{R}^n \setminus \partial\Omega$) is denoted $r_\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from $\Omega$ resp. $\overline{\Omega}$ to $\mathbb{R}^n$) is denoted $e_\pm$. (The notation is also used for $\Omega = \mathbb{R}^n_\zeta$.) Restriction from $\overline{\mathbb{R}^n_\pm}$ to $\overline{\mathbb{R}^n_0}$ resp. $\partial\Omega$ is denoted $\gamma_0$.

When $\Omega$ is a $C^{1+\tau}$-domain, we denote by $d(x)$ (as in [19, Def. 2.1] for the $C^\infty$-case) a function that is $C^{1+\tau}$ on $\overline{\Omega}$, positive on $\Omega$ and vanishes only to the first order on $\partial\Omega$ (i.e., $d(x) = 0$ and $\nabla d(x) \neq 0$ for $x \in \partial\Omega$). On bounded sets it satisfies near $\partial\Omega$:

$$C^{-1}d_0(x) \leq d(x) \leq Cd_0(x)$$

(2.2)
with $C > 0$, where $d_0(x)$ equals $\text{dist}(x, \partial \Omega)$ on a neighborhood of $\partial \Omega$ and is extended as a correspondingly smooth positive function on $\Omega$. When $\tau \geq 1$, $d_0$ itself can be taken $C^{1+\tau}$ (as explained e.g. in [2]), then moreover, $d/d_0$ is a positive $C^{\tau}$-function on $\Omega$.

We take $d_0(x) = x_n$ in the case of $\mathbb{R}^n_+$. For $\mathbb{R}^n$, the function $d(x) = x_n - \zeta(x')$ satisfies (2.2) when the extension of $d_0(x)$ is suitably chosen for large $x_n$ (cf. e.g. [2]).

The Bessel-potential spaces $H^s_q(\mathbb{R}^n)$ are defined for $s \in \mathbb{R}$, $1 < q < \infty$, by

$$H^s_q(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_q(\mathbb{R}^n) \},$$

where $\mathcal{F}$ is the Fourier transform $\hat{u}(\xi) = \mathcal{F}u(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} u(x) \, dx$, and the function $\langle \xi \rangle$ equals $(|\xi|^2 + 1)^{\frac{1}{2}}$. For $q = 2$, this is the scale of Sobolev spaces, where the index 2 is usually omitted. $S'(\mathbb{R}^n)$ is the Schwartz space of temperate distributions, the dual space of $S(\mathbb{R}^n)$ (the space of rapidly decreasing $C^\infty$-functions).

For $s \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, the spaces $H^s_q(\mathbb{R}^n)$ are also denoted $W^s_q(\mathbb{R}^n)$ or $W^{s,q}(\mathbb{R}^n)$ in the literature. We moreover need to refer to the Besov spaces $B^s_{q,q}(\mathbb{R}^n)$, also denoted $B^s_q(\mathbb{R}^n)$, that coincide with the $W^s_q$-spaces when $s \in \mathbb{R} \setminus \mathbb{N}$. They necessarily enter in connection with boundary value problems in an $H^s_q$-context, because they are the correct range spaces for trace maps $\gamma_j u = (\partial_{n}^j u)|_{x_n=0}$:

$$\gamma_j : H^s_q(\mathbb{R}^n_+), \overline{B}^s_q(\mathbb{R}^n_+) \to B^{s-j-\frac{1}{q}}_q(\mathbb{R}^{n-1}) \text{, for } s - j - \frac{1}{q} > 0,$$

(cf. (2.5)), surjectively and with a continuous right inverse; see e.g. the overview in the introduction to [13]. For $q = 2$, the two scales $H^s_q$ and $B^s_q$ are identical, but for $q \neq 2$ they are related by strict inclusions: $H^s_q \subset B^s_q$ when $q > 2$, $H^s_q \supset B^s_q$ when $q < 2$.

Along with the spaces $H^s_q(\mathbb{R}^n)$ defined in (2.3), there are the two scales of spaces associated with $\Omega$ for $s \in \mathbb{R}$:

$$\overline{H}^s_q(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in H^s_q(\mathbb{R}^n) \}, \text{ the restricted space,}$$

$$\dot{H}^s_q(\Omega) = \{ u \in H^s_q(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega} \}, \text{ the supported space;}$$

here $\text{supp } u$ denotes the support of $u$ (the complement of the largest open set where $u = 0$). $\overline{H}^s_q(\Omega)$ is in other texts often denoted $H^s_q(\Omega)$ or $H^s_q(\mathbb{R}^n)$, and $\dot{H}^s_q(\Omega)$ may be indicated with a ring, zero or twiddle; the current notation stems from Hörmander [31, App. B.2]. There is an identification of $\overline{H}^s_q(\Omega)$ with the dual space of $\dot{H}^{s-q}(\Omega)$, $\frac{1}{q'} = 1 - \frac{1}{q}$, in terms of a duality extending the sesquilinear scalar product $\langle f, g \rangle = \int_{\Omega} f \overline{g} \, dx$.

Besides for the $H^s_q$ and $B^s_q$-spaces, there are in [18] for $C^\infty$-domains established the relevant results in many other scales of spaces, namely Besov spaces $B^s_{p,q}$ for $1 \leq p, q \leq \infty$ and Triebel-Lizorkin spaces $F^s_{p,q}$ (for the same $p, q$ but with $p < \infty$). Here we just want to mention the Hölder-Zygmund scale $B^{\infty, \infty}$, also denoted $C^s$. The space $C^s$ identifies with the Hölder space $C^s$ when $s \in \mathbb{R} \setminus \mathbb{N}$, and for positive integer $k$ satisfies $C^{k-\varepsilon} \supset C_k^s \supset C^{k-1,1} \supset C^k_b$ for small $\varepsilon > 0$; moreover, $C^0_\infty \supset L_\infty \supset C^0_\bullet$ (with strict inclusions everywhere). Similarly to (2.5), we denote the spaces of restricted, resp. supported elements
\[ \mathcal{C}^s_\ast(\Omega) = \{ u \in \mathcal{D}'(\Omega) \mid u = r^+ U \text{ for some } U \in C^s_\ast(\mathbb{R}^n) \}, \]

\[ \dot{\mathcal{C}}^s_\ast(\overline{\Omega}) = \{ u \in C^s_\ast(\mathbb{R}^n) \mid \text{supp } u \subset \overline{\Omega} \}. \] (2.6)

The star can be omitted when \( s \in \mathbb{R}_+ \setminus \mathbb{N} \) (then we shall often write \( \mathcal{C}^s(\Omega) \) in the more established notation \( C^s(\overline{\Omega}) \)). Hölder spaces over \( C^{1+\tau} \)-domains \( \Omega \) are used in [2].

### 2.2. Pseudodifferential operators

A pseudodifferential operator \((\psi\text{do})\) \( P \) on \( \mathbb{R}^n \) is defined from a function \( p(x, \xi) \) on \( \mathbb{R}^n \times \mathbb{R}^n \), called the symbol, by

\[ Pu = \text{Op}(p(x, \xi)) u = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} p(x, \xi) \hat{u}(\xi) \, d\xi = \mathcal{F}_{\xi \rightarrow x}^{-1}(p(x, \xi) \mathcal{F}u(\xi)), \] (2.7)

using the Fourier transform \( \mathcal{F} \). An introduction to \( \psi\text{do} \)'s is given e.g. in [15, Ch. 7–8]. A description with more references and an inclusion of results for operators with nonsmooth symbols can be found in [2]. We shall here just give a quick summary of definitions and consequences that we need in the present paper.

The space \( S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \) of symbols \( p \) of order \( m \in \mathbb{R} \) consists of the complex \( C^\infty \)-functions \( p(x, \xi) \) such that \( \partial_x^\alpha \partial_\xi^\beta p(x, \xi) \) is \( O(|\xi|^{m-|\alpha|}) \) for all \( \alpha, \beta \), for some \( m \in \mathbb{R} \), with global estimates in \( x \in \mathbb{R}^n \). \( P \) is then of order \( m \). It maps \( H^s_q(\mathbb{R}^n) \) continuously into \( H^{s-m}_q(\mathbb{R}^n) \) for all \( s \in \mathbb{R} \).

\( P \) with symbol \( p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \) is said to be classical when \( p \) has an asymptotic expansion \( p(x, \xi) \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi) \) with \( p_j \) homogeneous in \( \xi \) of degree \( m - j \) for all \( |\xi| \geq 1 \) and \( j \in \mathbb{N}_0 \), such that

\[ \partial_x^\alpha \partial_\xi^\beta (p(x, \xi) - \sum_{j < J} p_j(x, \xi)) = O(|\xi|^{m-\alpha-J}) \text{ for all } \alpha, \beta \in \mathbb{N}_0^n, J \in \mathbb{N}_0. \] (2.8)

The space of classical symbols is denoted \( S^m(\mathbb{R}^n \times \mathbb{R}^n) \). For a complete theory one adds to these operators the smoothing operators (mapping any \( H^s_q(\mathbb{R}^n) \) into \( \bigcap t H^t_q(\mathbb{R}^n) \)), regarded as operators of order \(-\infty\). (For example, \((-\Delta)^a\) fits into the calculus when it is written as \( \text{Op}(\{1 - \eta(\xi)\}|\xi|^{2a}) + \text{Op}(\eta(\xi)|\xi|^{2a}) \), where \( \eta(\xi) \) is a \( C^\infty \)-function that equals 1 for \( |\xi| \leq \frac{1}{2} \) and 0 for \( |\xi| \geq 1 \); the second term is smoothing.)

Symbols with finite smoothness in \( x \) are defined as follows: The symbol space \( C^\tau S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \) for \( \tau > 0, m \in \mathbb{R} \), consists of functions \( p: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C} \) that are continuous w.r.t. \((x, \xi)\) \( \mathbb{R}^n \times \mathbb{R}^n \) and \( C^\infty \) with respect to \( \xi \in \mathbb{R}^n \), such that for every \( \alpha \in \mathbb{N}_0^n \) we have: \( \partial_\xi^\alpha p(x, \xi) \) is in \( C^\tau(\mathbb{R}^n) \) with respect to \( x \) and satisfies for all \( \xi \in \mathbb{R}^n \), \( \alpha \in \mathbb{N}_0^n \),

\[ \|\partial_\xi^\alpha p(\cdot, \xi)\|_{C^\tau(\mathbb{R}^n)} \leq C_\alpha (\xi)^{m-|\alpha|}, \] (2.9)

with \( C_\alpha > 0 \). The symbol space is a Fréchet space with the semi-norms.
\[ |p|_{k,C^rS^m_{\alpha,0}(\mathbb{R}^n \times \mathbb{R}^n)} := \max_{|\alpha| \leq k} \sup_{\xi \in \mathbb{R}^n} (\xi)^{-m+|\alpha|} \| \partial_\xi^\alpha p(\cdot, \xi) \|_{C^r(\mathbb{R}^n)} \quad \text{for } k \in \mathbb{N}_0. \] (2.10)

For such symbols there holds when \( \tau > 0 \):

\[ \text{Op}(p) : H^{s+m}_q(\mathbb{R}^n) \rightarrow H^s_q(\mathbb{R}^n) \quad \text{for all } |s| < \tau, \] (2.11)

where the operator norm for each \( s \) is estimated by a finite system of symbol seminorms (depending on \( s \)).

As explained in detail in [17, Sect. 2.3], the operators can be approximated by operators with smooth symbols: When \( p \in C^rS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \), it is approximated in the seminorms of \( C^rS^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \), any \( \tau' < \tau \), by the convolutions in \( x \) with an approximate unit: \( q_k(x) = k^n q(kx) \) for a \( q \in C^\infty_0(\mathbb{R}^n) \) with \( \|q\|_{L_1} = 1 \); here \( p_k = q_k \ast p \in S^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n) \). Hence, taking \( \tau' > |s| \), \( P_k = \text{Op}(q_k \ast p) \),

\[ \|P - P_k\|_{\mathcal{L}(H^{s+m}_q(\mathbb{R}^n),H^s_q(\mathbb{R}^n))} \rightarrow 0 \quad \text{for } k \rightarrow \infty, \quad \text{when } |s| < \tau' = \tau - \varepsilon, \] (2.12)

where \( \varepsilon > 0 \) can be taken arbitrarily small.

The subspace of classical symbols \( C^rS^m(\mathbb{R}^n \times \mathbb{R}^n) \) consists of those functions that moreover have expansions into terms \( p_j \) homogeneous in \( \xi \) of degree \( m - j \) for \( |\xi| \geq 1 \), all \( j \), such that for all \( \xi \in \mathbb{R}^n, \alpha \in \mathbb{N}_0^n, J \in \mathbb{N}_0 \),

\[ \|\partial_\xi^\alpha (p(\cdot, \xi) - \sum_{j < J} p_j(\cdot, \xi)) \|_{C^r(\mathbb{R}^n)} \leq C_{\alpha,J}(\xi)^{m-J-|\alpha|}. \] (2.13)

A classical symbol \( p(x,\xi) \) (and the associated operator \( P \)) is said to be strongly elliptic when \( \text{Re} p_0(x, \xi) \geq c|\xi|^m \) for \( |\xi| \geq 1 \), with \( c > 0 \). Moreover, a classical \( \psi \)-do \( P = \text{Op}(p(x, \xi)) \) of order \( m \in \mathbb{R} \) is said to be even, when the terms in the symbol expansion \( p \sim \sum_{j \in \mathbb{N}_0} p_j \) satisfy

\[ p_j(x,-\xi) = (-1)^j p_j(x,\xi) \quad \text{for all } x \in \mathbb{R}^n, |\xi| \geq 1, j \in \mathbb{N}_0. \] (2.14)

(The word “even” is short for even-to-even parity, meaning that the terms with even \( j \) are even in \( \xi \), the terms with odd \( j \) are odd in \( \xi \).)

2.3. \( \mu \)-Transmission spaces

The following is a rapid introduction to \( \mu \)-transmission spaces, which were presented in full detail in [19], and extended to nonsmooth domains \( \Omega \) in [2].

For some types of \( \psi \)-do’s, namely those of integer order belonging to the Boutet de Monvel calculus (as initiated in [5], see e.g. [13], [14], [15, Ch. 10–11]), results on boundary value problems can be adequately formulated within the scales of \( H^s_q \)- and \( B^s_q \)-spaces (or just \( H^s \)-spaces) over \( \Omega \) and \( \partial \Omega \). For fractional-order pseudodifferential operators — where the prominent example is the fractional Laplacian \( (-\Delta)^a \), \( 0 < a < 1 \) — we also need to
introduce the $\mu$-transmission spaces $H_{\mu}^q(s)(\mathbb{R}_+^n)$, since they are the exact solution spaces for Dirichlet problems. The definition of these spaces involves a special type of $\psi$do’s called order-reducing operators.

The simplest examples of such operators (relevant for $\Omega = \mathbb{R}_+^n$) are, with $t \in \mathbb{R}$,

$$
\Xi_{\pm}^t = \text{Op}(\chi_{\pm}^t), \quad \chi_{\pm}^t(\xi) = (\langle \xi' \rangle \pm i \xi_n)\xi^t; \quad (2.15)
$$

they preserve support in $\mathbb{R}_n\pm$, respectively, because the symbols extend as holomorphic functions of $\xi_n$ into $\mathbb{C}_\pm$, respectively; here $\mathbb{C}_\pm = \{ z \in \mathbb{C} : \text{Im} z \gtrless 0 \}$. (The functions $(\langle \xi' \rangle \pm i \xi_n)\xi^t$ satisfy only part of the estimates (2.9) (with $m = t, \tau \in \mathbb{N}_0$), but the $\psi$do definition can be applied anyway.) There is a more refined choice $\Lambda_{\pm}^t$, cf. [13,19], with symbols $\Lambda_{\pm}^t(\xi)$ that do satisfy all the required $\psi$do estimates, and where $\overline{\Lambda}_{\pm}^t = \Lambda_{\pm}^t$. These symbols likewise have holomorphic extensions in $\xi_n$ to the complex halfspaces $\mathbb{C}_\pm$, so that the operators preserve support in $\mathbb{R}_n\pm$, respectively. Operators with that property are called “plus” resp. “minus” operators. There is also a pseudodifferential definition $\Lambda_{\pm}^t$ adapted to the situation of a bounded smooth domain $\Omega$, by [13,19].

The operators define homeomorphisms $\Xi_{\pm}^t : H_q^s(\mathbb{R}^n) \sim H_q^{s-t}(\mathbb{R}^n)$, for all $s \in \mathbb{R}$. The special interest is that the “plus”/“minus” operators also define homeomorphisms related to $\mathbb{R}_n^+\pm$ and $\overline{\Omega}$, for all $s \in \mathbb{R}$:

$$
\Xi_{\pm}^t : H_q^s(\mathbb{R}_+^n) \sim H_q^{s-t}(\mathbb{R}_+^n), \quad r^+ \Xi_{\pm}^t e^+ : \overline{H}_q^s(\mathbb{R}_+^n) \sim \overline{H}_q^{s-t}(\mathbb{R}_+^n),
$$

$$
\Lambda_{\pm}^t : H_q^s(\overline{\Omega}) \sim H_q^{s-t}(\overline{\Omega}), \quad r^+ \Lambda_{\pm}^t e^+ : \overline{H}_q^s(\Omega) \sim \overline{H}_q^{s-t}(\Omega), \quad (2.16)
$$

with similar rules for $\Lambda_{\pm}^t$. Moreover, the operators $\Xi_{\pm}^t$ and $r^+ \Xi_{\pm}^t e^+$ identify with each other’s adjoints over $\mathbb{R}_n^+$, because of the support preserving properties. There is a similar statement for $\Lambda_{\pm}^t$ and $r^+ \Lambda_{\pm}^t e^+$ relative to $\mathbb{R}_n^+$, and for $\Lambda_{\pm}^t$ and $r^+ \Lambda_{\pm}^t e^+$ relative to the set $\Omega$. (The exponent $t$, and the value $\mu$ considered below, can also be allowed to be complex, as in [19,25], but we shall not need this in the present paper.) There is an abbreviation in $\psi$do-notation $P_+ = r^+ Pe^+$ that is often used, e.g. replacing $r^+ \Xi_{\pm}^t e^+$ by $\Xi_{\pm}^t e^+$.

Let $\mu > -1$. Then the $\mu$-transmission spaces $H_{\mu}^q(s)(\mathbb{R}_+^n)$ are defined for $s \mu > 1/q'$ by

$$
H_{\mu}^q(s)(\mathbb{R}_+^n) = \Xi_{\pm}^{\mu} e^+ \overline{H}_q^{s-\mu}(\mathbb{R}_+^n). \quad (2.17)
$$

(Equivalently, $\Xi_{\pm}^{\mu}$ can be replaced by $\Lambda_{\pm}^{\mu}$.) For $\mu = a > 0$, the interest is that this is the solution space for the homogeneous Dirichlet problem

$$(1 - \Delta)^a u = f \text{ on } \mathbb{R}_+^n, \quad \text{supp } u \subset \mathbb{R}_+^n,$$

when $f$ is given in $\overline{H}_q^{s-2a}(\mathbb{R}_+^n)$ for some $s a - 1/q'$ and $u$ is sought in $\hat{H}_q^s(\mathbb{R}_+^n)$ with $\sigma > a - 1/q'$. By [19], the result holds also for suitable variable-coefficient $\psi$do
generalizations of $(1 - \Delta)^{\alpha}$. A pedestrian introduction to transmission spaces is given in [27].

The symbol $(\langle \xi' \rangle + i\xi_n)^{-\mu}$ is connected to expressions with a factor $x_n^\mu$ on $\mathbb{R}^n_+$ by the formula

$$
\mathcal{F}^{-1}_{\xi_n \to x_n} \frac{1}{\langle \langle \xi' \rangle \rangle + i\xi_n)^{\mu+1}} = \frac{1}{\Gamma(\mu+1)} e^{+r^+ x_n^\mu e^{-\langle \xi' \rangle} x_n},
$$

(2.18)

which allows to show

$$
H^\mu(s)(\mathbb{R}^n_+) = \begin{cases}
\hat{H}^s_q(\mathbb{R}^n_+) & \text{if } \mu - 1/q' < s < \mu + 1/q, \\
\hat{H}^s_q(-\varepsilon)(\mathbb{R}^n_+) + e^+ x_n^\mu \mathcal{H}^{s-\mu}_q(\mathbb{R}^n_+) & \text{if } s > \mu + 1/q
\end{cases}
$$

(2.19)

(with $(-\varepsilon)$ active if $s - \mu - 1/q \in \mathbb{N}$).

For smooth bounded domains $\Omega$, the $\mu$-transmission spaces $H^\mu(s)(\overline{\Omega})$ are defined in a similar way as in (2.17) with the operator family $\Lambda^{(\mu)}$ used instead of $\Xi^{(\mu)}$; then there is a similar inclusion as in (2.19) with $x_n^\mu$ replaced by $d_0^\mu$, and these spaces are the solution spaces for homogeneous Dirichlet problems for a large class of pseudo’s [19, Th. 4.4] (containing the case $(-\Delta)^{\alpha}$, $\mu = a$). There is an analysis describing the spaces with further precision in [24].

For $C^{1+\tau}$-domains $\Omega$, the $\mu$-transmission spaces $H^\mu(s)(\overline{\Omega})$ are defined in [2, Def. 4.2], when $\tau > 0$, $\mu > -1$ and $1/q' < s < 1 + \tau$, by localization, i.e., a reduction to local coordinates in a family of open sets covering the boundary. When $\tau \geq 1$, one then has with $\varepsilon > 0$ [2, Th. 4.5]:

$$
H^\mu(s)(\overline{\Omega}) = \begin{cases}
\hat{H}^s_q(\overline{\Omega}) & \text{when } s < \mu + \frac{1}{q}, \\
\hat{H}^{s-\varepsilon}_q(\overline{\Omega}) & \text{when } s = \mu + \frac{1}{q}, \\
\hat{H}^s_q(\overline{\Omega}) + d_0^\mu e^+ \hat{H}^{s-\mu}_q(\overline{\Omega}) & \text{when } s - \mu - \frac{1}{q} \in \mathbb{R}^n_+ \setminus \mathbb{N}, \\
\hat{H}^{s-\varepsilon}_q(\overline{\Omega}) + d_0^\mu e^+ \hat{H}^{s-\mu}_q(\overline{\Omega}) & \text{when } s - \mu - \frac{1}{q} \in \mathbb{N};
\end{cases}
$$

(2.20)

it also holds with $d_0$ replaced by $d$ (cf. (2.2)). When $\tau < 1$, there is a version of (2.20) with $d_0$ replaced by local choices of $d$, cf. [2, Rem. 4.6]; for convenience, we recall the explanation here:

**Remark 2.1.** When $\Omega$ is bounded $C^{1+\tau}$-domain, each point $x_0 \in \partial \Omega$ has a bounded open neighborhood $U \subset \mathbb{R}^n$ and a function $\zeta \in C^{1+\tau}(\mathbb{R}^{n-1})$, such that (after a suitable rotation) $\Omega \cap U = \mathbb{R}^n_\zeta \cap U$ (cf. (2.1)). For $\mathbb{R}^n_\zeta$, the space $H^\mu(s)(\mathbb{R}^n_\zeta)$ is defined from $H^\mu(s)(\mathbb{R}^n_+)$ by use of the $C^{1+\tau}$-diffeomorphism $F_\zeta(x) = (x', x_n - \zeta(x'))$ (all $x \in \mathbb{R}^n$), with the notation $F_\zeta^{s,-1}(u) \equiv u \circ F_\zeta^{-1}$. Then $H^\mu(s)(\mathbb{R}^n_\zeta) = F_\zeta^*(H^\mu(s)(\mathbb{R}^n_+))$, provided with the inherited norm.

Now $H^\mu(s)(\overline{\Omega})$ is defined as the set of all $u \in H^s_{q,loc}(\Omega)$ such that for each $x_0 \in \partial \Omega$, with a $\varphi \in C_0^\infty(U)$ with $\varphi \equiv 1$ in a neighborhood of $x_0$, we have $F_\zeta^{s,-1}(\varphi u) \in H^\mu(s)(\mathbb{R}^n_+)$,
in the rotated situation. The analysis of properties of \( H_q^{\mu(s)}(\Omega) \) is then carried over to
an analysis of properties of \( H_q^{\mu(s)}(\mathbb{R}^n) \). The choices of \( \{x_0, U, \zeta, \varphi\} \) can be reduced to
a finite system \( \{x_0, U_i, \zeta_i, \varphi_i\}_{i=1, \ldots, \ell} \), where \( \bigcup U_i \) covers \( \partial \Omega \). The definition of the spaces
\( H_q^{\mu(s)}(\mathbb{R}^n) \) is then used in each of the sets \( U_i \); and they may be pieced together by a
partition of unity. One has a version of (2.20) with \( \Omega \) replaced by \( \mathbb{R}^n \), \( d_0 \) replaced by
\( d_i(x) = x_n - \zeta_i(x') \) for each \( i \). When \( \tau \geq 1 \), \( d_i \) can be replaced by the \( C^{1+\tau} \)-function
\( d_0 \) for each \( i \), and the pieces sum up to give the function \( d_0 \) entering in (2.20). When
\( \tau \in ]0, 1[ \), we still have the version of (2.20) with \( d_i \) in each localized piece \( U_i \), where \( d_i \)
is related to \( d_0 \) by (2.2), but \( d_0 \) is only \( C^\tau \).

For \( s \geq \mu + \frac{1}{\bar{q}} \) there is also a weaker result than the second line in (2.19), namely,
\[
H_q^{\mu(s)}(\mathbb{R}^n_+) = \mathbb{E}^{-\mu} + H_q^{s-\mu}(\mathbb{R}^n_+) \subset \mathbb{E}^{-\mu} \dot{H}_q^{1+\frac{1}{\bar{q}}-\epsilon}(\mathbb{R}^n_+) = \dot{H}_q^{\mu+\frac{1}{\bar{q}}-\epsilon}(\mathbb{R}^n_+), \quad \text{when } s \geq \mu + \frac{1}{\bar{q}}.
\]

In the case of \( C^{1+\tau} \)-domains, the corresponding rule
\[
H_q^{\mu(s)}(\Omega) \subset \dot{H}_q^{\mu+\frac{1}{\bar{q}}-\epsilon}(\Omega), \quad \text{for } s \geq \mu + \frac{1}{\bar{q}}, \ s < 1 + \tau, \quad (2.21)
\]
follows by use of the localization described in Remark 2.1.

2.4. Trace mappings, additional properties

The weighted trace mapping
\[
\gamma^\mu_0 : u \mapsto \Gamma(\mu + 1)(u/d^\mu)|_{\partial \Omega}, \quad (2.22)
\]
is defined on \( H_q^{\mu(s)}(\Omega) \) for \( C^{1+\tau} \)-domains in [2, Sect. 4.2]. Since we shall in the present
paper deal with nonhomogenous boundary values, we need a more elaborate version
of some results from there. The statement on \( \gamma^\mu_0 \) in [2, Prop.4.3] for a curved halfspace (2.1)
extends as follows:

**Proposition 2.2.** Let \( \mu > -1 \), \( \tau > 0 \), and \( \mu + \frac{1}{\bar{q}} < s < 1 + \tau \) with \( s - \mu < 1 + \tau \),
and let \( \mathbb{R}^n_\zeta \) be defined by \( \zeta \in C^{1+\tau}(\mathbb{R}^{n-1}) \), \( d(x) = x_n - \zeta(x') \) near \( \partial \mathbb{R}^n_\zeta \). The mapping
\( \gamma^\mu_0 : u \mapsto \Gamma(\mu + 1)(u/d^\mu)|_{\partial \mathbb{R}^n_\zeta} \) is continuous and surjective:
\[
\gamma^\mu_0 : H_q^{\mu(s)}(\mathbb{R}^n_\zeta) \to B_q^{s-\mu-\frac{1}{\bar{q}}}(\partial \mathbb{R}^n_\zeta), \quad (2.23)
\]
having a continuous right inverse. Moreover, the space \( H_q^{\mu(s)}(\mathbb{R}^n_\zeta) \) is a closed subspace
of \( H_q^{\mu(s)}(\mathbb{R}^n_\zeta) \), equal to the kernel of the mapping (2.23).

**Proof.** The properties are known from [19, Sect. 5] to hold when \( \zeta(x') \equiv 0 \) (the flat case),
as recalled e.g. in [2, Sect. 4.1, (4.7)]. They carry over to the case of general \( \zeta(x') \) in view
of the mapping properties of the diffeomorphism \( F_\zeta : (x', x_n) \mapsto (x', x_n - \zeta(x')) \) and the definitions listed in [2, Sect. 4.2]. Note that the space \( H_q^{\mu+1}(s)(\mathbb{R}^n_+) \) is well-defined with the parameter \( \mu + 1 \), since the hypotheses assure that \( s > (\mu + 1) - 1/q' \). □

When \( \tau \geq 1 \) one can, as stated in [2, Prop 4.3], replace \( d \) by \( d_0 \). We then also have the following elaborated version of the statements on \( \gamma_0^\mu \) for bounded domains \( \Omega \) in [2, Th. 4.5]:

**Theorem 2.3.** Let \( \mu > -1, \, \tau \geq 1 \) and \( \mu + \frac{1}{q} < s < \tau \) with \( s - \mu < \tau \), and let \( \Omega \subset \mathbb{R}^n_+ \) be a bounded \( C^{1+\tau} \)-domain. The mapping \( \gamma_0^\mu : u \mapsto \Gamma(\mu + 1)(u/d_0^\mu)|_{\Omega} \) is continuous and surjective:

\[
\gamma_0^\mu : H_q^{\mu}(s)(\bar{\Omega}) \to B_q^{s-\mu-\frac{1}{q}}(\partial\Omega),
\]

having a continuous right inverse \( K_{(0)}^\mu \). Moreover, the space \( H_q^{\mu+1}(s)(\bar{\Omega}) \) is a closed subspace of \( H_q^{\mu}(s)(\bar{\Omega}) \), and equals the kernel of the mapping (2.24):

\[
\{ u \in H_q^{\mu}(s)(\bar{\Omega}) \mid \gamma_0^\mu u = 0 \} = H_q^{\mu+1}(s)(\bar{\Omega}).
\]

**Proof.** The continuity of the mapping \( \gamma_0^\mu \) is established in [2, Th. 4.5] by use of a cover \( \bigcup_{i=0,1,...,t} U'_i \) and an associated partition of unity \( \{g_i\}_{i=0,1,...,t} \) such that the \( U'_i \) with \( i \geq 1 \) cover \( \partial\Omega \) and for each such \( U'_i \) there is a function \( \zeta_i \) (called \( \gamma_i \) in [2]) such that, after a rotation and translation depending on \( i \), \( \Omega \cap U'_i = \mathbb{R}^n_+ \cap U'_i \). In each such neighborhood \( U'_i \), the facts known for \( \mathbb{R}^n_+ \), can be applied to \( g_0 u \) and collected to a statement on \( u \) by summation. This goes for all the properties listed in Proposition 2.2, when we moreover observe that \( \gamma_0^\mu \) acts locally as a trace operator \((\gamma_0^\mu(\varphi u)) = \gamma_0(\varphi \gamma_0^\mu u) \) when \( \varphi \in C_c^\infty(\mathbb{R}^n) \), and \( d_0 \) is defined near the boundary consistently in the different local charts. □

**Remark 2.4.** For a description of \( K_{(0)}^\mu \), we note that the analysis in [24, Sect. 3] for smooth \( \Omega \) shows that in local coordinates reduced to the case of \( \mathbb{R}^n_+ \), \( K_{(0)}^\mu \) can be taken proportional to \( x_n^\mu K_0 \), where \( K_0 \) is the standard Poisson operator for the Laplacian (solving the problem \( (1 - \Delta)v = 0 \) on \( \mathbb{R}^n_+ \), \( \gamma_0 v = \varphi \)). This explains the factor \( d^\mu \) appearing when the result is carried over to \( \Omega \).

One can also define higher-order traces \( \gamma_k^\mu u \):

\[
\gamma_k^\mu u = \Gamma(\mu + k + 1)\gamma_k(u/d^\mu),
\]

when \( \mu + \frac{1}{q} + k < s < \tau \). Since they are not used in this paper, we leave out details.

Let us finally recall the scale of spaces built over the Hölder-Zygmund spaces \( C^\alpha_*(\mathbb{R}^n) \), with completely parallel properties, as accounted for in [2]. With the associated scales over \( \overline{\mathbb{R}}^n_+ \) and \( C^\alpha_*(\mathbb{R}^n_+) \) defined by (2.6), one defines \( C^\alpha_*(\mathbb{R}^n_+) \) (when \( \mu > -1 \)) similarly to (2.17) by
\[ C^\mu_\ast(\mathbb{R}^n_+) = \Xi^\mu_\ast e^+\overline{C}^{s-\mu,\ast}_\ast(\mathbb{R}^n_+), \quad s > \mu - 1. \quad (2.27) \]

For bounded \( C^{1+\tau}\)-domains, corresponding spaces are defined by localization when \( \tau > 0 \), \( \mu - 1 < s < 1 + \tau \), and have the properties when \( \tau \geq 1 \):

\[
C^\mu_\ast(\overline{\Omega}) \begin{cases} = \mathcal{C}^s_\ast(\overline{\Omega}), & \text{for } s < \mu, \\
\subset \mathcal{C}^{s-\epsilon}_\ast(\overline{\Omega}) + d_0^\mu e^+\overline{C}^{s-\mu}_\ast(\Omega), & \text{for } s > \mu 
\end{cases} \quad (2.28)
\]

(with \( -\epsilon \) active if \( s - \mu \in \mathbb{N} \)), cf. [2, Def. 4.2, Th. 4.5]. This also holds with \( d_0 \) replaced by \( d \) (cf. (2.2)). When \( \tau < 1 \), there is a version of (2.28) with \( d_0 \) replaced by local choices of \( d \), cf. Remark 2.1. We also have:

\[ C^\mu_\ast(\overline{\Omega}) \subset \mathcal{C}^{s-\epsilon}_\ast(\overline{\Omega}), \quad s > \mu > 0, \quad s < 1 + \tau. \quad (2.29) \]

The trace mapping \( \gamma_0^\mu : u : \Gamma(\mu + 1)(u/d_0^\mu)|_{\partial\Omega} \)

\[ \gamma_0^\mu : C^\mu_\ast(\overline{\Omega}) \to C^{s-\mu}_\ast(\partial\Omega), \quad (2.30) \]

is well-defined when \( s > \mu \). Since it plays an important role in the study of nonhomogeneous boundary problems, we include an elaborated version of the statements on \( \gamma_0^\mu \) in [2], along the lines of Theorem 2.3:

**Theorem 2.5.** Let \( \mu > -1, \tau \geq 1 \) and \( \mu < s < \tau \) with \( s - \mu < \tau \), and let \( \Omega \subset \mathbb{R}^n \) be a bounded \( C^{1+\tau} \)-domain. The mapping \( \gamma_0^\mu : u : \Gamma(\mu + 1)(u/d_0^\mu)|_{\partial\Omega} \) in (2.30) is continuous and surjective, having a continuous right inverse. Moreover, the space \( C^\mu_{\ast(\mu+1)}(\overline{\Omega}) \) is a closed subspace of \( C^\mu_\ast(\overline{\Omega}) \), and equals the kernel of the mapping (2.30).

The proof goes exactly as in the proof of Theorem 2.3.

As noted in [2, Cor. 6.11], the well-known embedding property \( H^q_\ast(\mathbb{R}^n) \subset C^{t-n/q}_{s-\epsilon}(\mathbb{R}^n) \) (any \( \epsilon > 0 \)) implies

\[ H^\mu_\ast(\mathbb{R}^n_+) \subset C^\mu_{s-n/q-\epsilon}(\mathbb{R}^n_+) \]

in view of the definitions (2.17) and (2.27); this leads to embeddings

\[ H^\mu_\ast(\overline{\Omega}) \subset C^\mu_{s-n/q-\epsilon}(\overline{\Omega}), \quad (2.31) \]

when \( \mu - 1 < s - n/q - \epsilon < s < 1 + \tau \) and \( \Omega \) is a bounded \( C^{1+\tau} \)-domain, by following the localization procedure. Letting \( q \to \infty \), we find for \( \mu - 1 < s < 1 + \tau \):

\[ \bigcap_{q>1} H^\mu_\ast(\overline{\Omega}) \subset C^\mu_{s-\epsilon}(\overline{\Omega}), \quad (2.32) \]

for small \( \epsilon > 0 \). In the other direction, one has e.g. for \( 0 < t < 1 + \tau \),
\[ C^{1+\varepsilon}(\Omega) \subset \mathcal{H}_q^t(\Omega). \] (2.33)

These observations are useful for drawing consequences on regularity in Hölder spaces.

3. Mapping properties and regularity

3.1. The homogeneous Dirichlet problem

In the rest of this paper, we consider a classical pseudodifferential operator \( P \) of order \( 2a \), \( 0 < a < 1 \), with even symbol in \( C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n) \), cf. (2.14). As accounted for in [2], the so-called \( a \)-transmission condition for an operator of order \( 2a \), with respect to a \( C^{1+\tau} \)-domain \( \Omega \), means that (2.14) is satisfied at each \( x \in \partial \Omega \) for \( \xi \) equal to the interior normal \( \nu(x) \) at \( x \). The evenness implies that \( P \) satisfies the \( a \)-transmission condition with respect to any domain, and it is shown how the spaces \( H^a_q(\Omega) \) enter in discussions of mapping properties. We remark that \( P \) also satisfies the \( (a+k) \)-transmission condition for \( k \in \mathbb{Z} \), since a \( \mu \)-transmission condition only depends on \( \mu \) modulo 1.

Recall that we assume \( 1 < q < \infty \) throughout. Our assumptions for the treatment of the homogeneous Dirichlet problem are as follows:

**Hypothesis 3.1.** 1° There are given constants \( a, \tau, q \) with \( 0 < a < 1 \), \( \tau > 2a \), and \( 1 < q < \infty \). \( \Omega \) is a bounded \( C^{1+\tau} \)-domain in \( \mathbb{R}^n \), and \( P \) is a classical \( \psi \)do of order \( 2a \), with even symbol in \( C^\tau S^{2a}(\mathbb{R}^n \times \mathbb{R}^n) \).

2° Assumptions as in 1°, and in addition \( P \) is strongly elliptic.

The following results for the homogeneous Dirichlet problem were shown in [2] (Theorems 6.4 and 6.9, Cor. 6.10):

**Theorem 3.2.** Assume Hypothesis 3.1 1°, and let \( s \) satisfy \(-a \leq s < \tau - 2a \). Then \( r^+P \) maps continuously

\[ r^+ P : H^a_q(s+2a)(\Omega) \to \mathcal{H}^r_q(\Omega). \] (3.1)

Assume moreover that Hypothesis 3.1 2° holds and \( s \geq 0 \). If \( u \in \mathcal{H}^a_q(\Omega) \) solves

\[ Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \Omega, \] (3.2)

for some \( f \in \mathcal{H}^r_q(\Omega) \), then \( u \in H^a_q(s+2a)(\Omega) \).

In other words, we have found that

\[ \{ u \in \mathcal{H}^a_q(\Omega) \mid r^+ Pu \in \mathcal{H}^r_q(\Omega) \} = H^a_q(s+2a)(\Omega). \] (3.3)

Note the remarkable fact that we have not only shown a regularity property (a conclusion from \( Pu \) to \( u \)) as the aim is in most studies of these operators, but we have found
the exact solution space for the homogeneous Dirichlet problem with data in \( H^s_q(\Omega) \); the Dirichlet domain. Moreover, \( H^a_q(s+2a)(\Omega) \) has this role universally, in the sense that it is independent of the choice of \( P \) satisfying Hypothesis 3.1.

Let us derive some consequences of the theorem, particularly concerning the range of \( r^+P \):

**Proposition 3.3.** Assume Hypothesis 3.1 and let \( 0 \leq s < \tau - 2a \). The solutions of (3.2) described in Theorem 3.2 satisfy an estimate

\[
\|u\|_{H^a_q(s+2a)(\Omega)} \leq C(\|r^+Pu\|_{\mathcal{P}^r_q(\Omega)} + \|u\|_{\dot{H}^a_q(\Omega)}).
\]

The range of \( r^+P \) in (3.1) is closed.

**Proof.** The estimate (3.4) is seen as follows: By the continuity statement in Theorem 3.2, \( \|u\| \equiv \|r^+Pu\|_{\mathcal{P}^r_q(\Omega)} + \|u\|_{\dot{H}^a_q(\Omega)} \) is a norm on \( H^a_q(s+2a)(\Omega) \) that satisfies \( \|u\| \leq C\|u\|_{H^a_q(s+2a)(\Omega)} \). By use of the regularity statement, \( H^a_q(s+2a)(\Omega) \) is seen to be complete under this norm. Then since it is a Banach space, the norms are equivalent.

The closed range property follows from a classical argument for how an a priori inequality (3.4) leads to the existence of an approximate inverse, found e.g. in the proof of [29, Th. 10.5.1]. Denote \( \overline{H}^a_q(\Omega) = X \), \( H^a_q(s+2a)(\Omega) = Y \), \( \dot{H}^a_q(\Omega) = Z \), they are all reflexive Banach spaces. Here \( Y \) is compactly injected in \( Z \), since \( H^a_q(s+2a)(\Omega) \subset \dot{H}^{a+b}_q(\Omega) \subset H^a_q(\Omega) \) for any \( 0 < b < \min\{a, \frac{1}{2}\} \) (cf. (2.20)–(2.21)), where the last injection is compact. Write \( r^+P \) as \( P \) for short. For \( u \in Y \), the inequality (3.4) reads

\[
\|u\|_Y \leq C(\|Pu\|_X + \|u\|_Z).
\]

Let \( N \) denote the nullspace of \( P : Y \to X \); it is a closed subspace of \( Y \), and there is a closed complement \( Y' \) of \( N \) in \( Y \) such that \( P : Y' \to X \) is injective (and bounded). Let \( R \) denote the range of \( P \) in \( X \). Our goal will be achieved if we show that

\[
\|u\|_{Y'} \leq C\|Pu\|_X \quad \text{when } u \in Y';
\]

for then \( P : Y' \to X \) has a bounded partial inverse \( Q : R \to Y' \); and since \( Q \) is closed, \( R \) is a closed subspace of \( X \).

Assume that (3.6) does not hold. Then there is a sequence of functions \( u_k \in Y' \) with \( \|u_k\|_Y = 1 \) such that \( \|Pu_k\|_X \to 0 \). By the weak compactness of the unit sphere in \( Y \), there is a subsequence converging weakly to an element \( u_0 \). By the compactness of the injection \( Y \subset Z \), we can take a further subsequence (call it \( u_k \) again) such that \( \|u_k - u_0\|_Z \to 0 \). Now (3.5) implies

\[
\|u_k - u_0\|_Y \leq C(\|P(u_k - u_0)\|_X + \|u_k - u_0\|_Z).
\]

\[ (3.7) \]
Since $P$ is a fortiori weakly continuous from $Y$ to $X$, $Pu_k$ goes weakly to $Pu_0$, which must be 0 since $Pu_k$ goes to 0 in norm. Then both terms in the right-hand side of (3.7) go to 0. If $u_0 = 0$, this gives a contradiction since $u_k$ has norm 1. If $u_0 \neq 0$, we have found a nontrivial null-element in $Y'$, contradicting the definition of $Y'$. Thus (3.6) must hold. □

We shall later (in Section 4) show a bijectiveness or Fredholm property of the mapping (3.1), leading to existence and uniqueness results for the homogeneous Dirichlet problem. It is important for that study to observe that not only the regularity parameter $s$, but also the integral parameter $q$ for $u$ can be lifted when the data are in a space with a higher parameter:

**Theorem 3.4.** Assume Hypothesis 3.1.

If $u \in \dot{H}_q^s(\Omega)$ solves (3.2) for some $f \in L_p(\Omega)$ with $p \geq q$, then $u \in H_p^{\alpha(2a)}(\Omega)$.

In fact, there is a sequence $q_1 = q < q_2 < q_3 < \cdots$ with $q_j \to \infty$ for $j \to \infty$, such that for all $j$, $u \in \dot{H}_q^s(\Omega)$ with $r^+Pu \in L_{q_{j+1}}(\Omega)$ imply $u \in \dot{H}_q^s(\Omega)$.

**Proof.** The integral parameter $q$ will be lifted by use of the well-known embedding rule

$$H_{p_1}^{s_1}(\mathbb{R}^n) \subset H_{p_2}^{s_2}(\mathbb{R}^n) \text{ when } s_1 \geq s_2, s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}, 1 < p_1 \leq p_2 < \infty.$$  (3.8)

It holds also for $\dot{H}_p^s(\Omega)$-spaces with $\Omega$ open $\subset \mathbb{R}^n$, by their definition as closed subspaces of $H_p^s(\mathbb{R}^n)$, cf. (2.5).

In view of (2.20) and (2.21) we have for $1 < p < \infty$,

$$H_p^{\alpha(2a)}(\Omega) \begin{cases} = \dot{H}_p^{2a}(\Omega) \text{ if } a < \frac{1}{p} \\ \subset \dot{H}_p^{\alpha + \frac{1}{p} - \varepsilon}(\Omega) \text{ if } a \geq \frac{1}{p}. \end{cases}$$  (3.9)

To reduce this to one statement, define

$$m(p) = \min\{ap, 1\} \text{ for } p \geq q;$$

it is 1 when $p \geq 1/a$, and if $a < \frac{1}{q}$ so that the interval between $q$ and $1/a$ is nontrivial, $m(p)$ takes values in $[aq, 1]$ on that interval. Altogether, $m(p)$ satisfies

$$\min\{aq, 1\} \leq m(p) \leq 1 \text{ for all } p \geq q.$$  (3.10)

Then (3.9) implies

$$H_p^{\alpha(2a)}(\Omega) \subset \dot{H}_p^{\alpha + \frac{1}{p}m(p) - \varepsilon}(\Omega) \text{ when } p \geq q.$$  

Now we define an increasing sequence $q_1 < q_2 < \cdots < q_j < \ldots$, where $q_1 = q$, and the next values are defined successively such that
\[ H_{q_j}^{a+\frac{1}{q}m(q_j)-\varepsilon}(\Omega) \subset H_{q_{j+1}}^{a}(\Omega). \]

By (3.8), this is satisfied if \( a + \frac{1}{q_j}m(q_j) - \varepsilon = \frac{n}{q_j} \geq a - \frac{n}{q_{j+1}} \), which may be rewritten as

\[ \frac{n}{q_{j+1}} > \frac{n-m(q_j)}{q_j}, \text{ i.e.,} \]

\[ q_{j+1} < \frac{n}{n-m(q_j)q_j}. \tag{3.11} \]

In view of (3.10), the factor \( n/(n - m(p)) \) is for all \( p \geq q \) greater than a fixed constant \( c > 1 \), so the inequality (3.11) allows defining \( q_j \) as a sequence going to \( \infty \) (it holds e.g. if we take \( q_j = c_1^{j-1}q_1 \) with a \( c_1 \in ]1, c[ \), for \( j \in \mathbb{N} \)). Note that we have obtained

\[ H_{q_j}^{a(2a)}(\Omega) \subset H_{q_{j+1}}^{a}(\Omega), \text{ for all } j \in \mathbb{N}. \tag{3.12} \]

The sequence satisfies the assertion in the theorem.

For the given \( u \in \dot{H}_{q}^{a}(\Omega) \) and \( f \in L_p(\Omega) \), we use this in a successive passage from \( q_j \) to \( q_{j+1} \) for \( j = 1, 2, \ldots \), until \( p \) is reached: Starting with the information \( r^+Pu = f \in L_p(\Omega) \subset L_q(\Omega), q_1 = q \), we have from Theorem 3.2 that \( u \in H_{q_1}^{a(2a)}(\Omega) \), hence is in \( \dot{H}_{q_2}(\Omega) \) by (3.12). If \( q_2 \geq p \), this ends the proof since \( u \in \dot{H}_{q}^{a}(\Omega) \) then, so that by Theorem 3.2, \( u \in H_{q_1}^{a(2a)}(\Omega) \). Otherwise, we repeat the argument. The general step, as long as \( q_j < p \), is that the information \( r^+Pu = f \in L_p(\Omega) \subset L_{q_j}(\Omega) \) gives by Theorem 3.2 that \( u \in H_{q_j}^{a(2a)}(\Omega) \), which is in \( \dot{H}_{q_{j+1}}^{a}(\Omega) \) by (3.12). If \( q_{j+1} \leq p \), Theorem 3.2 implies that \( u \in H_{q_{j+1}}^{a(2a)}(\Omega) \). When \( q_{j+1} > p \), Theorem 3.2 is applied with the parameter \( p \). \( \square \)

### 3.2. The nonhomogeneous local Dirichlet problem

We shall now include nontrivial Dirichlet boundary values, and therefore go out in the larger spaces \( H_{q}^{a-1}(s+2a)(\Omega) \) to define solutions. These spaces are defined as in Section 2.3 with \( \mu = a-1 \) (which is \( > -1 \)). To the basic assumptions listed in Hypothesis 3.1, we add the condition \( \tau > 2a + 1 \).

There is the following forward mapping property based on [2, Cor. 5.14]:

**Theorem 3.5.** Assume Hypothesis 3.1 \(^1\) with \( \tau > 2a + 1 \), and let \( s \) satisfy \(-a - 1 \leq s < \tau - 2a - 1\). Then for curved halfspaces \( \mathbb{R}^n_\zeta \) with \( \zeta \in C^{1+\tau}(\mathbb{R}^{n-1}) \), and for bounded \( C^{1+\tau} \)-domains \( \Omega \), \( r^+P \) maps continuously

\[ r^+P: H_{q}^{a-1}(s+2a)(\mathbb{R}^n_\zeta) \rightarrow \overline{H}_{q}^{s}(\mathbb{R}^n_\zeta), \]

\[ r^+P: H_{q}^{a-1}(s+2a)(\Omega) \rightarrow \overline{H}_{q}^{s}(\Omega). \tag{3.13} \]

**Proof.** By [2, Cor. 5.14], the first statement holds in the flat case \( \zeta \equiv 0 \), since \( P \) satisfies the global \((a-1)\)-transmission condition. It carries over to general \( \zeta \) exactly as in the proofs of [2, Th. 6.1, Cor. 6.2], with \( a \) replaced by \( a - 1 \) in the definitions of transmission spaces.
For the second statement, this is carried over to bounded $C^{1+\tau}$-domains analogously to the proof of [2, Th. 6.4]. □

Combining this with the mapping properties of $\gamma_0^{a-1}$ shown in Proposition 2.2 and Theorem 2.3, we have as a corollary:

Corollary 3.6. Assume Hypothesis 3.1 \(1^\circ\) with $\tau > 2a + 1$, and let $s$ satisfy $-a - 1/q' < s < \tau - 2a - 1$. The following maps are continuous:

\[
\begin{align*}
\{r^+ P, \gamma_0^{a-1}\} : H_q^{(a)(s+2a)}(\mathbb{R}^n) &\rightarrow \overline{H}_q^s(\mathbb{R}^n) \times B_q^{s+a+1/q'}(\partial \mathbb{R}^n), \\
\{r^+ P, \gamma_0^{a-1}\} : H_q^{(a)(s+2a)}(\Omega) &\rightarrow \overline{H}_q^s(\Omega) \times B_q^{s+a+1/q'}(\partial \Omega),
\end{align*}
\]  

(3.14)

These results allow a development of regularity results for the nonhomogeneous Dirichlet problem:

\[
P u = f \text{ in } \Omega,
\]

\[
u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\]

\[
\gamma_0^{a-1} u = \varphi \text{ on } \partial \Omega.
\]

(3.15)

Theorem 3.7. Assume Hypothesis 3.1 with $\tau > 2a + 1$, and let $s$ satisfy $0 \leq s < \tau - 2a - 1$.

When $u \in H_q^{(a)(1-a)}(\Omega)$, it satisfies (3.15) for some $f \in \overline{H}_q^{-a}(\Omega)$ and $\varphi \in B_q^{1/q'}(\partial \Omega)$.

Let $u \in H_q^{(a)(1-a)}(\Omega)$, and let $p \geq q$. If $u$ solves (3.15) with $f \in \overline{H}_p^{-a}(\Omega)$ and $\varphi \in B_q^{s+a+1/p'}(\partial \Omega)$, then $u \in H_p^{(a)(s+2a)}(\Omega)$.

In other words,

\[
\{u \in H_q^{(a)(1-a)}(\Omega) \mid r^+ P u \in \overline{H}_q^s(\Omega), \gamma_0^{a-1} u \in B_q^{s+a+1/p'}(\partial \Omega)\} = H_p^{(a)(s+2a)}(\Omega).
\]

(3.16)

Proof. The first statement on $u$ follows from the mapping property in Corollary 3.6 for $s = -a$.

Now for the improvement of regularity and integral parameters: Assume $f \in \overline{H}_p^s(\Omega)$ and $\varphi \in B_q^{s+a+1/p'}(\partial \Omega)$. By the surjectiveness of $\gamma_0^{a-1}$ in Theorem 2.3, there is a $v \in H_p^{(a)(s+2a)}(\Omega)$ with $\gamma_0^{a-1} v = \varphi$. By Theorem 3.5, $g = r^+ P v \in \overline{H}_p^s(\Omega)$. Thus $w = u - v \in H_q^{(a)(1-a)}(\Omega)$ and solves

\[
Pw = f - g \text{ in } \Omega,
\]

\[
w = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\]

\[
\gamma_0^{a-1} w = 0 \text{ on } \partial \Omega,
\]

(3.17)

where $f - g \in \overline{H}_p^s(\Omega)$. This is a homogeneous Dirichlet problem, and since $w$ is in the kernel of $\gamma_0^{a-1}$, (2.25) shows that $w \in H_q^{a(a)}(\Omega) = \hat{H}_q^a(\Omega)$. Then Theorems 3.2 and 3.4
apply to show that since \( f - g \in \overline{H}^s_p(\Omega) \), \( w \) is in \( H^{(a+2a)}_p(\Omega) \), and hence in the larger space \( H^{(a-1)(s+2a)}_p(\Omega) \). Finally, \( u = v + w \in H^{(a-1)(s+2a)}_p(\Omega) \). \( \square \)

The existence and uniqueness of solutions will be taken up in Section 5.

**Remark 3.8.** It is possible to describe the spaces \( H_q^{(a)}(\Omega) \) and \( H^{(a-1)(s)}_p(\Omega) \) more precisely. We already know that \( H_q^{(a)}(\Omega) = \dot{H}_q^s(\Omega) \) when \( s < a + \frac{1}{q} \), and \( H^{(a-1)(s)}_p(\Omega) = \dot{H}_q^s(\Omega) \) when \( s < a - 1 + \frac{1}{q} = a - \frac{1}{q} \), by \( (2.20) \) for \( \mu = a \) resp. \( a - 1 \).

For higher \( s \) (up to \( \min\{\tau, \tau + \mu\} \)), let \( K^{(\mu)}_0 \) be a right inverse of \( \gamma^{(\mu)}_0 \) as in Theorem 2.3. The last statement in Theorem 2.3 implies

\[
H^{(\mu)}_q(\Omega) = H^{(\mu+1)(s)}_q(\Omega) + K^{(\mu)}_0 B^{s+\mu/q}(\partial \Omega), \quad \text{when } s > \mu + \frac{1}{q},
\]

(3.18)

In particular, we have with \( \mu = a \):

\[
H^{(a)}_q(\Omega) = \dot{H}_q^a(\Omega) + K^{(a)}_0 B^{s-a-1/q}(\partial \Omega) \quad \text{for } s - a \in \left[ \frac{1}{q}, 1 + \frac{1}{q} \right],
\]

(3.19)

and with \( \mu = a - 1 \):

\[
H^{(a-1)(s)}_p(\Omega) = \dot{H}_q^s(\Omega) + K^{(a-1)}_0 B^{s-a+1/q}(\partial \Omega) \quad \text{for } s - a \in \left[ \frac{1}{q}, 1 + \frac{1}{q} \right].\]

(3.20)

The operators \( K^{(a)}_0 \) and \( K^{(a-1)}_0 \) provide coefficients \( d^a \) resp. \( d^{a-1} \), cf. Remark 2.4. In particular, when \( a < \frac{1}{q} \), \( H_q^{(2a)}(\Omega) = \dot{H}_q^{2a}(\Omega) \) and \( H_q^{(a-1)(2a)}(\Omega) = \dot{H}_q^{2a}(\Omega) + K^{(a-1)}_0 B^{a+1/q}(\partial \Omega) \); and when \( a > \frac{1}{q} \), \( H_q^{(2a)}(\Omega) = \dot{H}_q^{2a}(\Omega) + K^{(a)}_0 B^{a-1/q}(\partial \Omega) \) and \( H_q^{(a-1)(2a)}(\Omega) = \dot{H}_q^{2a}(\Omega) + K^{(a-1)}_0 B^{a+1/q}(\partial \Omega) \).

By playing on the possibility to take \( p \) very large in Theorem 3.7, we can draw a conclusion on problems with data in Hölder spaces:

**Corollary 3.9.** Assume Hypothesis 3.1 with \( \tau > 2a+1 \), and let \( s \) satisfy \( 0 \leq s < \tau - 2a-1 \). Let \( u \in H^{(a-1)(s)}_q(\Omega) \) for some \( 1 < q < \infty \). If \( u \) solves \((3.15)\) with \( f \in C^s(\Omega) \) and \( \varphi \in C^{s+a+1} \) \( \partial \Omega \), then

\[
u \in C^{(a-1)(s+2a-\varepsilon)}(\Omega) \subset C^{(a-1)(s+2a-\varepsilon)}(\Omega) + d^{a-1} C^{s+a+1-\varepsilon}(\partial \Omega),
\]

for small \( \varepsilon > 0 \) (with noninteger \( s + 2a - \varepsilon \) and \( s + a + 1 - \varepsilon \)).
Proof. In view of (2.33), we have for large $p \in [q, \infty[$ and small $\varepsilon' > 0$ that $f \in \mathcal{H}_{p}^{s-\varepsilon'}(\Omega)$ and $\varphi \in B_{p}^{s+a+1/p'-\varepsilon'}(\partial \Omega)$. Then Theorem 3.7 implies that $u \in H_{p}^{(a-1)(s-\varepsilon'+2a)}(\Omega) \subset H_{p}^{s-\varepsilon'+2a}(\Omega) + d^{a-1}e^{+H_{p}^{s-\varepsilon'+a+1}}(\Omega)$. The result follows in view of (2.32) by letting $p \to \infty$. □

4. Eigenfunctions and Fredholm properties

We shall now derive existence-and-uniqueness (or Fredholm) properties of $r^+ P$ acting as in (3.1). This will follow from an analysis of eigenfunctions, also for $r^+ P^*$, of the Dirichlet realizations they define in $L_q(\Omega)$.

4.1. The Dirichlet realization in $L_2(\Omega)$

In order to define the Dirichlet realization in $L_2(\Omega)$ of a strongly elliptic pseudodifferential operator $P$, we first recall the Gårding inequality, adapting it to the present symbols.

Lemma 4.1. When $\tau > a > 0$, and $P = \text{Op}(p)$ with symbol $p(x, \xi)$ in $C^{\tau}S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ is strongly elliptic, i.e., the principal symbol satisfies

$$\text{Re} p_0(x, \xi) \geq c_0|\xi|^{2a} \text{ for all } x \in \mathbb{R}^n, |\xi| \geq 1,$$

(4.1)

then the operator satisfies the Gårding inequality:

$$\text{Re}(Pu, u) \geq c\|u\|^2_{H_\alpha(\mathbb{R}^n)} - \beta\|u\|^2_{L_2(\mathbb{R}^n)}, \text{ for } u \in C_0^\infty(\mathbb{R}^n),$$

(4.2)

for some $c > 0$ and $\beta \in \mathbb{R}$. One can take $c = c_0 - \delta$, any $\delta \in ]0, c_0[$.

Proof. The Gårding inequality for smooth symbols is an old and well-established fact. The reader who wants to see a proof on $\mathbb{R}^n$ can find it e.g. in [16, Lemma 5.1], which carries over verbatim to the case where $A$ is replaced by a $\psi$do $P$ with symbol $p$ in $S^{2a}(\mathbb{R}^n \times \mathbb{R}^n)$ ($p_0$ taken smooth near $\xi = 0$ with $\text{Re} p_0 > 0$), and $m$ is replaced by $a$.

Note here that since $P - (c_0 - \delta)\text{Op}(|\xi|^{2a})$ is likewise strongly elliptic when $\delta \in ]0, c_0[$,

$$\text{Re}((P - (c_0 - \delta)\text{Op}(|\xi|^{2a}))u, u) = \text{Re}(Pu, u) - (c_0 - \delta)\|u\|^2_{H_\alpha(\mathbb{R}^n)} \geq c'|u|^2_{H_\alpha(\mathbb{R}^n)} - \beta'|u|^2_{L_2(\mathbb{R}^n)},$$

(4.3)

with $c' > 0$, implying that (4.2) holds with $c = c_0 - \delta$.

The given nonsmooth symbol $p$ can be approximated by smooth symbols $p_k$ as described around (2.12). For small $\varepsilon_1, \varepsilon_2 \in ]0, c_0[$, we let $P_k = \text{Op}(p_k)$ with symbol $p_k = \varrho_k * p$, choosing $k$ so large that $|p_0(x, \xi) - p_{k,0}(x, \xi)| \leq \varepsilon_1|\xi|^{2a}$ for $|\xi| \geq 1$, and
\[\|P - P_k\|_{\mathcal{L}(H^s, H^{-s})} \leq \varepsilon_1,\] and we apply (4.3) to \(P_k\) with \(c_0\) replaced by \(c_0 - \varepsilon_1, \delta\) replaced by \(\varepsilon_2\). This gives

\[
\text{Re}(Pu, u) = \text{Re}(Pku, u) + \text{Re}((P - P_k)u, u) \\
\geq ((c_0 - \varepsilon_1) - \varepsilon_2)|u|_{H^s}^2 - \beta'|u|_{L^2}^2 - \|(P - P_k)u\|_{H^{-s}}|u|_{H^s} \\
\geq (c_0 - (2\varepsilon_1 + \varepsilon_2))|u|_{H^s}^2 - \beta'|u|_{L^2}^2.
\]

Since any \(\delta \in [0, c_0]\) can be written as \(2\varepsilon_1 + \varepsilon_2\), this shows the assertion.

Since \(\dot{H}^a(\Omega) \subseteq H^a(\mathbb{R}^n)\), \(r^+P\) defines in particular a continuous operator \(\mathcal{P}\) from \(\dot{H}^a(\Omega)\) to \(\overline{H}^{-a}(\Omega)\). Define the Dirichlet realization \(PD_2\) as the operator in \(L_2(\Omega)\) acting like \(r^+P\) with domain

\[D(PD_2) = \{u \in \dot{H}^a(\Omega) \mid r^+Pu \in L_2(\Omega)\}. \tag{4.4}\]

This can be viewed in a variational framework: Define the sesquilinear form

\[s(u, v) = \int_\Omega Pu \bar{v} \, dx, \tag{4.5}\]

first for \(u, v \in C_0^\infty(\Omega)\), then extended by closure to a continuous sesquilinear form on \(\dot{H}^a(\Omega)\). Since \(P\) is strongly elliptic, the form is coercive in view of Lemma 4.1:

\[\text{Re} s(u, u) \geq c|u|_{\dot{H}^a(\Omega)}^2 - \beta|u|_{L_2(\Omega)}^2, \text{ with } c > 0 \text{ and } \beta \in \mathbb{R}. \tag{4.6}\]

Then the Lax-Milgram lemma (as recalled in e.g. [15, Sect. 12.4]) applies. For one thing \(s(u, v)\) equals \(\langle \mathcal{P}u, v \rangle_{\overline{\dot{H}^{-a}(\Omega)}, \dot{H}^a(\Omega)}\), where \(\mathcal{P}: \dot{H}^a(\Omega) \rightarrow \overline{\dot{H}^{-a}(\Omega)}\) acts like \(r^+P\); moreover this induces the operator \(PD_2\) with domain \(\{u \in \dot{H}^a(\Omega) \mid \mathcal{P}u \in L_2(\Omega)\}\), the same as the domain described in (4.4). The inequality (4.6) holds for \(u\) in the domain with \(s(u, u)\) replaced by \(\langle Pu, u \rangle_{L_2(\Omega)}\).

When \(P\) moreover has even symbol and \(\tau > 2a\), we have from Theorem 3.2 that

\[D(PD_2) = H^{a(2a)}(\Omega), \tag{4.7}\]

further described in (2.20)ff. and Remark 3.8.

When \(\beta = 0\) in (4.6), \(\mathcal{P}: \dot{H}^a(\Omega) \rightarrow \overline{\dot{H}^{-a}(\Omega)}\) is a homeomorphism with

\[\|\mathcal{P}u\|_{\overline{\dot{H}^{-a}(\Omega)}} \geq c\|u\|_{\dot{H}^a(\Omega)}, \tag{4.8}\]

and \(PD_2\) has lower bound \(\inf\{\text{Re}(PD_2u, u)/\|u\|_{L_2}^2 \mid u \in D(PD_2) \setminus 0\} \geq c\) and is bijective from \(D(PD_2)\) to \(L_2(\Omega)\). In general this holds for \(P + \beta I\) instead of \(P\). Moreover, elementary estimates of the numerical range \(\nu(PD_2)\) gives

\[\{(PD_2u, u)/\|u\|_{L_2}^2 \mid u \in D(PD_2) \setminus 0\}\]
give (as in [15, Cor. 12.21]) that the spectrum and the numerical range are contained in a sectorial region

\[ M = \{ \lambda \in \mathbb{C} \mid \text{Re} \lambda \geq c - \beta, |\text{Im} \lambda| \leq Cc^{-1}(\text{Re} \lambda + \beta) \}, \quad (4.9) \]

where \( C \) is a positive constant for which \(|s(u, u)| \leq C\|u\|^2_{H^a(\Omega)}\) on \( \dot{H}^a(\Omega) \). In particular, the resolvent \((P_{D,2} - \lambda)^{-1}\) exists for \( \lambda \) outside \( M \), and its operator norm satisfies

\[ \| (P_{D,2} - \lambda)^{-1} \|_{\mathcal{L}(L^2(\Omega))} \leq C(\lambda)^{-1} \text{ for } \text{Re} \lambda \leq -\beta. \quad (4.10) \]

Since the injection of \( \dot{H}^a(\Omega) \) into \( L^2(\Omega) \) is compact, the spectrum of \( P_{D,2} \) is discrete, lying in \( M \). We shall denote

\[ \Sigma = \text{the spectrum of } P_{D,2}. \quad (4.11) \]

For \( \lambda \in \Sigma \), \( P_{D,2} - \lambda \) is a Fredholm operator, with index 0 since the index depends continuously on \( \lambda \).

Let us collect the outcome in a theorem:

**Theorem 4.2.** Assume Hypothesis 3.1.

The Dirichlet realization \( P_{D,2} \) in \( L^2(\Omega) \) has domain \((4.7)\). This equals \( \dot{H}^{2a}(\Omega) \) when \( a \in ]0, \frac{1}{2}[ ; \) it is contained in \( \dot{H}^{1-\varepsilon}(\Omega) \) when \( a = \frac{1}{2} \); and it is contained in \( \dot{H}^{2a}(\Omega) + d^a\dot{H}^{a}(\Omega) \) and in \( \dot{H}^{a+\frac{1}{2}-\varepsilon}(\Omega) \) when \( a \in ]\frac{1}{2}, 1[ \) (locally if \( \tau < 1 \)).

With \( c_0 \) satisfying \((4.1)\), there is for any \( c \in ]0, c_0[ \) a number \( \beta \in \mathbb{R} \) such that

\[ \text{Re}(P_{D,2}u, u)_{L^2(\Omega)} \geq c\|u\|^2_{H^a(\Omega)} - \beta\|u\|^2_{L^2(\Omega)} \text{ for } u \in D(P_{D,2}). \quad (4.12) \]

The spectrum \( \Sigma \) of \( P_{D,2} \) is discrete and lies in \( M \) \((4.9)\), which also contains the numerical range. If \( \beta = 0 \), \( P_{D,2} \) is a homeomorphism of \( H^{a(2a)}(\Omega) \) onto \( L^2(\Omega) \); more generally, \( P_{D,2} - \lambda I \) has this homeomorphism property when \( \lambda \in \mathbb{C} \setminus \Sigma \), and there is a resolvent estimate \((4.10)\). For \( \lambda \in \Sigma \), \( P_{D,2} - \lambda I \) defines a Fredholm operator with index zero from \( H^{a(2a)}(\Omega) \) to \( L^2(\Omega) \).

The resolvent set of \( P_{D,2} \) is \( \mathbb{C} \setminus \Sigma \). We denote by \( N_\lambda \) the kernel of \( P_{D,2} - \lambda I \); it is nontrivial only when \( \lambda \in \Sigma \). The \( L^2 \)-adjoint \((P_{D,2})^*\) has as its spectrum the conjugated set \( \Sigma^* \), and we denote the kernel of \((P_{D,2})^* - \lambda I \) by \( N_\lambda^* \). It is a cokernel of \( P_{D,2} - \lambda I \), in the sense that the range \((P_{D,2} - \lambda I)D(P_{D,2}) \) equals the orthogonal complement of \( N_\lambda^* \) in \( L^2(\Omega) \); this is the set of functions \( f \) satisfying

\[ \int_{\Omega} f\overline{\psi} \, dx = 0 \text{ for } \psi \in N_\lambda^*. \quad (4.13) \]

Here \( \dim N_\lambda^* = \dim N_\lambda \).
4.2. The regularity of eigenfunctions

We shall now study the structure of the eigenfunctions of the $L_q$-Dirichlet realizations of $P$; i.e. the nontrivial solutions of

$$ r^+ Pu = \lambda u, \ u \in H^a_q(\Omega), $$

for $\lambda \in \mathbb{C}$; here we use the regularity results for the homogeneous Dirichlet problem.

Smoothness properties of eigenfunctions were found earlier in the $C^\infty$-setting in [20], and we shall employ a similar strategy, as far as it goes when the limited Hölderness of the symbol and the domain are taken into account.

In the analysis we need an observation on the comparison of Hölder spaces, when powers of the distance to the boundary $d_0(x)$ enter into the picture: When $\Omega$ is a $C^{1+\tau}$-domain with $\tau > 0$, then for $a, b > 0$ with $a + b < 1 + \tau$, and $a, b, a + b \notin \mathbb{N},$

$$ C^{a+b}_t(\Omega) \subset d_0^a C^b_t(\Omega). $$

This is undoubtedly very well known (and enters in some form in many papers), but since we have not been able to find an elementary reference, we include a proof in the Appendix, see Lemma A.5. The result is extended to more general distance functions $d(x)$ in Lemma A.6. For the $a$-transmission spaces, this implies:

**Lemma 4.3.** Let $\Omega$ be a $C^{1+\tau}$-domain, $\tau > 0$, and let $0 < a < 1$. There holds, for $a < t < \tau + a$,

$$ C^a_s(t)(\Omega) \subset C^t_s(\Omega) + d^a e^+ C^{t-a}(\Omega) \subset d^a e^+ C^{t-a}(\Omega), \text{ when } t, t - a \notin \mathbb{N}. $$

When $\tau \geq 1$, the general distance function $d$ can here be replaced by the more precise function $d_0(x) = \text{dist}(x, \partial \Omega)$ (near $\partial \Omega$). When $\tau < 1$, the inclusions hold in a local sense, as in Remark 2.1.

**Proof.** When $\tau \geq 1$, the first inclusion in (4.17) holds globally, with distance function $d$ or $d_0$ at convenience; both are $C^{1+\tau}$-functions. Then the second inclusion follows by application of Lemmas A.5 resp. A.6 to $C^t_s(\Omega)$, showing that
Thus sequence in When Proof. Theorem describe for satisfaction: for any \( p \geq q \), any \( s < \tau - 2a \), hence also in \( C^{a(\tau-\varepsilon)}(\Omega) \subset d^a C^{\tau-a-\varepsilon}(\Omega) \) for \( \varepsilon > 0 \) (with \( \tau - a - \varepsilon, \tau - \varepsilon \notin \mathbb{N} \)).

2° For nonzero eigenvalues \( \lambda \) there holds: The eigenfunctions \( u_\lambda \) of \( P_{D,q} \) are in \( H^{a(t)}_p(\Omega) \) for all \( p \geq q \) and all \( t \leq 3a \) with \( t < \tau \). Hence they are in \( C^{a(t)}_*(\Omega) \subset d^a C^{\tau-a}(\Omega) \) for \( t < \min\{3a,\tau\} \) (with \( t - a, t \notin \mathbb{N} \)), and

\[
u_\lambda \in d^a C^{\min\{2a,\tau-a\}-\varepsilon}(\Omega).
\] (4.18)

The inclusions hold in a local sense (cf. Remark 2.1) when \( \tau < 1 \). In all cases, \( u_\lambda \in \hat{C}^a(\Omega) \).

Proof. When \( \lambda \) is an eigenvalue, the associated eigenfunctions \( u_\lambda \) are the nontrivial solutions of (4.15).

1°. If \( \lambda = 0 \), \( r^+ Pu_0 = 0 \in L_p(\Omega) \) for all \( p \geq q \), so Theorem 3.4 gives that \( u_0 \in \hat{H}^a_{q_1}(\Omega) \) for all \( p \geq q \). Then furthermore, Theorem 3.2 gives that \( u_0 \in H^{a(t)}_p(\Omega) \) for any \( t < \tau \).

In view of (2.32), we then also have that \( u_0 \in C^{a(\tau-\varepsilon)}(\Omega) \), any \( \varepsilon > 0 \). By Lemma 4.3,

\[
u_0 \in d^a C^{\tau-a-\varepsilon}(\Omega),
\] (4.19)
in a local form if \( \tau < 1 \). This is applicable since \( \tau - \varepsilon < \tau \), a fortiori \( \tau - a - \varepsilon < \tau \).

2°. Now consider a nonzero eigenvalue with eigenfunction \( u_\lambda \). Since \( u_\lambda = \frac{1}{\lambda} r^+ Pu_\lambda \), we have from Theorem 3.2 that \( u_\lambda \in H^{a(\min\{3a,\tau\}-\varepsilon)}_{q_1}(\Omega) \subset H^{a(2a)}_{q_1}(\Omega) \), \( q_1 = q \). Using the sequence constructed in Theorem 3.4, we find successively for \( j = 1,2,\ldots \) that \( u_\lambda \in \hat{H}^a_{q_j}(\Omega) \), hence \( r^+ Pu_\lambda = \lambda u_\lambda \in \hat{H}^a_{q_j}(\Omega) \subset \hat{H}^a_{q_j}(\Omega) \), imply \( u \in H^{a(2a)}_{q_j}(\Omega) \subset \hat{H}^a_{q_{j+1}}(\Omega) \). Thus \( u_\lambda \in \hat{H}^a_p(\Omega) \) for all \( p \geq q \).

Since \( r^+ Pu_\lambda = \lambda u_\lambda \in \hat{H}^a_p(\Omega) \subset \hat{H}^a_p(\Omega) \), we get more precisely from Theorem 3.2 that \( u_\lambda \in H^{a(t)}_p(\Omega) \) for all \( p \geq q \) and all \( t \leq 3a \) with \( t < \tau \), and consequently by (2.32) and Lemma 4.3,
\[ u_\lambda \in C^{a(t)}(\Omega) \subset d^a C^{1-a}(\Omega), \text{ for } t < \min\{3a, \tau\}, t, t - a \in \mathbb{R}_+ \setminus \mathbb{N}; \quad (4.20) \]

the inclusion holds in a local sense when \( \tau < 1 \). Hence \( u_\lambda \in d^a C^{\min\{2a, \tau-a\}-\varepsilon(\Omega)} \).

For smooth domains and symbols, it was shown in [20] that \( u_0 \in \mathcal{E}_a(\Omega) \) for \( \lambda = 0 \), and \( u_\lambda \in d^a C^{2a-\varepsilon(\Omega)} \) for \( \lambda \neq 0 \); this can be improved to \( d^a C^{2a}(\Omega) \) when \( a \neq \frac{1}{2} \) by use of the precise formulas in [18] for how the operators act in \( C^*_a \)-spaces. As shown in [24], the regularity of \( u_\lambda \) in the case \( \lambda \neq 0 \) cannot in general be lifted to \( d^a C^{2a+\delta}(\Omega) \) with \( \delta > 0 \) (by comparison of Taylor expansions at \( \partial\Omega \)). We expect the same to be the case in situations with finite smoothness.

**Corollary 4.5.** Assume Hypothesis 3.1.

The Dirichlet realizations \( P_{D,q}, 1 < q < \infty \), have the same discrete set of eigenvalues and eigenfunctions as \( P_{D,2} \) for all \( q \in ]1, \infty[ \).

**Proof.** When \( \lambda \) is an eigenvalue and \( u_\lambda \) an associated eigenfunction for some \( q \), it also so for any \( p < q \), since \( L_q(\Omega) \subset L_p(\Omega) \). For \( p > q \), the regularity shown in Theorem 3.4 implies that \( u_\lambda \) is an eigenfunction with the same \( \lambda \). We know that the set of eigenvalues in case \( q = 2 \) is discrete with finite-dimensional eigenspaces. These eigenvalues and eigenfunctions have the same role for all other \( q \).

These considerations show in particular that the operator in \( L_q(\Omega) \), \( r^+ P - \lambda: H^{a(2a)}_q(\Omega) \rightarrow L_q(\Omega) \), has the same finite-dimensional nullspace (kernel) \( N_\lambda \) as in the case \( q = 2 \); it lies in \( d^a C^{\min\{2a, \tau-a\}-\varepsilon(\Omega)} \).

For \( q \geq 2 \), this leads immediately to Fredholm properties of \( P_{D,q} - \lambda \):

**Corollary 4.6.** Assume Hypothesis 3.1, and let \( q \geq 2 \). Consider

\[ P_{D,q} - \lambda: H^{a(2a)}_q(\Omega) \rightarrow L_q(\Omega). \]

For \( \lambda \in \mathbb{C} \setminus \Sigma \), this is a homeomorphism. For \( \lambda \in \Sigma \), this is a Fredholm operator with kernel \( N_\lambda \) and cokernel \( N'_\lambda \), in the sense that the range consists of the function \( f \in L_q(\Omega) \) satisfying (4.13). In particular, the spectrum of \( P_{D,q} \) is \( \Sigma \).

**Proof.** Let \( \lambda \in \Sigma \). Since \( L_q(\Omega) \subset L_2(\Omega) \), and \( N_\lambda \subset \dot{C}^a(\Omega) \subset L_q(\Omega) \) the nullspace of \( P_{D,q} - \lambda \) equals \( N_\lambda \), as already noted. Moreover, \( N'_\lambda \subset L_2(\Omega) \subset L_q'(\Omega) \). Since \( P_{D,2} - \lambda \) is surjective from \( H^{a(2a)}(\Omega) \) onto the functions in \( L_2(\Omega) \) satisfying (4.13), its restriction \( P_{D,q} - \lambda \) is surjective from \( H^{a(2a)}_q(\Omega) \) onto the functions in \( L_q(\Omega) \) satisfying (4.13).

When \( \lambda \in \mathbb{C} \setminus \Sigma \), we can argue in the same way, replacing \( N_\lambda \) and \( N'_\lambda \) by zero spaces.

The mapping properties can be lifted to \( H^s_q \)-spaces with higher \( s \), but when \( \lambda \neq 0 \), we must here take into account that the multiplication by \( \lambda \) is limited by the possibility to embed \( H^{a(s+2a)}_q(\Omega) \) in \( \mathcal{T}^s(\Omega) \).
Proposition 4.7. Assume Hypothesis 3.1 1°, and let \( \lambda \in \mathbb{C} \).
Let \(-a \leq s < \tau - 2a\). Then \( r^+P - \lambda \) maps continuously

\[
r^+P - \lambda : H^a(s+2a)(\Omega) \to \overline{H}^{s'}_q(\Omega), \text{ where } s' = \min\{s, a + \frac{1}{q} - \varepsilon\}, \text{ for small } \varepsilon > 0. \tag{4.21}
\]

Assume moreover that Hypothesis 3.1 2° holds. Let \( u \in \dot{H}^a_q(\Omega) \) satisfy \( (r^+P - \lambda)u \in \overline{H}^s_q(\Omega) \). If \( s < a + \frac{1}{q} \), then \( u \in H^a(s+2a)(\Omega) \). If \( s \geq a + \frac{1}{q} \), then \( u \in H^a(3a+\frac{1}{q} - \varepsilon)(\Omega) \), any \( \varepsilon > 0 \). In particular, \( u \in H^a(3a)(\Omega) \) if \( s \geq a \).

Proof. We have (3.1) for \( r^+P \) alone. As for the multiplication by \( \lambda \), note that in view of (2.20) and (2.21),

\[
H^a(s+2a)(\Omega) \begin{cases} 
\dot{H}^{s+2a}_q(\Omega) \text{ if } s + a < \frac{1}{q} \\
\subset H^{a+\frac{1}{q} - \varepsilon}_q(\Omega) \text{ if } s + a \geq \frac{1}{q}.
\end{cases} \tag{4.22}
\]

Therefore the multiplication by \( \lambda \) satisfies:

\[
\lambda : H^a(s+2a)(\Omega) \to \begin{cases} 
\dot{H}^{s+2a}_q(\Omega) \subset \overline{H}^s_q(\Omega) \text{ if } s + a < \frac{1}{q} \\
\dot{H}^{a+\frac{1}{q} - \varepsilon}_q(\Omega) \text{ if } s + a \geq \frac{1}{q}.
\end{cases} \tag{4.23}
\]

The combined operator \( r^+P - \lambda \) maps into the space with the smallest exponent (4.21).

Now assume moreover that \( P \) is strongly elliptic and \( s \geq 0 \). Since \( u \in \dot{H}^a_q(\Omega) \), \( \lambda u \in \dot{H}^a_q(\Omega) \), so \( r^+Pu \in \overline{H}^s_q(\Omega) + \dot{H}^a_q(\Omega) \). Then we conclude from Theorem 3.2 that \( u \in H^a_q(t+2a)(\Omega) \) with \( t = \min\{s, a\} \). If \( s \leq a \), the proof is complete. If \( s > a \), we iterate the argument. Note first that for \( s > a + \frac{1}{q} - \varepsilon \), (4.23) at best gives \( r^+Pu - \lambda u \in \overline{H}^{a+\frac{1}{q} - \varepsilon}(\Omega) \) and hence \( u \in H^a(a+\frac{1}{q}+2a-\varepsilon)(\Omega) = H^a(3a+\frac{1}{q} - \varepsilon)(\Omega) \); it cannot be lifted further in this way.

Let \( a < s < a + \frac{1}{q} - \varepsilon \), i.e., \( s = a + \delta \) with \( 0 < \delta < \frac{1}{q} - \varepsilon \). An application of Theorem 3.2 gives

\[
u \in H^a(3a+\delta)(\Omega) \subset \begin{cases} 
\dot{H}^{3a+\delta}_q(\Omega) \text{ if } 2a + \delta < \frac{1}{q} \\
\dot{H}^{a+\frac{1}{q} - \varepsilon}_q(\Omega) \text{ if } 2a + \delta \geq \frac{1}{q}.
\end{cases}
\]

If \( 2a + \delta < \frac{1}{q} \), we iterate the argument to find

\[
u \in H^a(5a+\delta)(\Omega) \subset \begin{cases} 
\dot{H}^{5a+\delta}_q(\Omega) \text{ if } 4a + \delta < \frac{1}{q} \\
\dot{H}^{a+\frac{1}{q} - \varepsilon}_q(\Omega) \text{ if } 4a + \delta \geq \frac{1}{q}.
\end{cases}
\]
For very small values of $a$, further iterations may be needed, each providing a lift by $2a$. Eventually after $N$ steps, $2Na + \delta$ will surpass $\frac{1}{q}$, so that we can conclude $u \in H_{q}^{\alpha(3a+\frac{1}{q}−\varepsilon)}(\Omega)$. □

For a small $\varepsilon > 0$, define

$$r_{q} = \min\{2a, a + \frac{1}{q} - \varepsilon\}, \quad (4.24)$$

then the outcome of Proposition 4.7 is that $r^{+}P - \lambda: H_{q}^{\alpha(s+2a)}(\Omega) \rightarrow \overline{H}_{q}^{\alpha}(\Omega)$ has the same regularity properties as $r^{+}P$ when $s \in [0, r_{q}]$. Then Corollary 4.6 extends easily to $H_{q}^{\alpha}$-spaces:

**Theorem 4.8.** Assume Hypothesis 3.1, let $s < \tau - 2a$, $q \geq 2$, $\lambda \in \mathbb{C}$, and if $\lambda \neq 0$ let $s \in [0, r_{q}]$. Consider

$$r^{+}P - \lambda: H_{q}^{\alpha(s+2a)}(\Omega) \rightarrow \overline{H}_{q}^{\alpha}(\Omega).$$

For $\lambda \in \mathbb{C} \setminus \Sigma$, this is a homeomorphism. For $\lambda \in \Sigma$, this is a Fredholm operator with kernel $N_{\lambda}$, and cokernel $N_{\lambda}^{\perp}$ in the sense that the range consist of the function $f \in \overline{H}_{q}^{\alpha}(\Omega)$ which satisfy (4.13).

**Proof.** This follows from Corollary 4.6 by restriction of $P_{D,q} - \lambda$ to $H_{q}^{\alpha(s+2a)}(\Omega)$. □

Corollary 4.6 and Theorem 4.8 will be extended to all $q \in ]1, \infty[$ in Theorems 4.16–4.18 below.

Note that the theorem shows existence and uniqueness, resp. Fredholm solvability, for the Dirichlet problem

$$Pu - \lambda u = f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^{n} \setminus \Omega,$$

with $f \in \overline{H}_{q}^{\alpha}(\Omega)$ with $s \geq 0$, $q \geq 2$, $u$ given in $\hat{H}^{\alpha}(\Omega)$. This has the following corollary for $f \in C^{s}(\Omega)$ when $\lambda = 0$:

**Corollary 4.9.** Assume Hypothesis 3.1, and let $0 \leq s < \tau - 2a$. Consider the homogeneous Dirichlet problem (3.2), with $f \in C^{s}(\Omega)$, and $u$ a priori assumed to be in $\hat{H}^{\alpha}(\Omega)$.

If $0 \notin \Sigma$, there is a unique solution $u$, which satisfies

$$u \in C^{s+2a-\varepsilon}_{\ast}(\Omega) \subset d^{a}C^{s+a-\varepsilon}(\Omega), \quad (4.25)$$

for small $\varepsilon > 0$ (with noninteger $s + a - \varepsilon$).

If $0 \in \Sigma$, there is a solution $u$, unique modulo $N_{0}$, when $f$ satisfies (4.13); here $u$ satisfies (4.25).
Proof. By (2.31), $f \in \mathcal{H}_q^{s-\varepsilon'}(\Omega)$ for any $q \in ]1, \infty[, \text{ any } \varepsilon' > 0$. We know from Corollary 4.6 that there is unique resp. Fredholm solvability, and $u$ lies in $\mathcal{H}_q^{(s-\varepsilon'+2\alpha)}(\Omega)$. Letting $q \to \infty$, we find (4.25) (with a local interpretation when $\tau < 1$).

4.3. Realizations of the adjoint operator

When $q < 2$, we need some additional information on the adjoint $P^*$, to get useful regularity properties of $N^\perp_N$.

The formal adjoint of $P$ on $\mathbb{R}^n$ (the pseudodifferential operator $P^*$ in $\mathcal{S}'(\mathbb{R}^n)$ that satisfies $\langle P^*u, \varphi \rangle = \langle u, P\varphi \rangle$ for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$) is the operator $P^* = \text{Op}(p(y, \xi))$ in $y$-form (in the general concept of operators defined from symbols $a(x, y, \xi)$ explained in [2]). We here work with a sesquilinear duality for consistency with the $L_2$-duality. $P^*$ is the sum of the $x$-form operator $\overline{P} = \text{Op}(\overline{p}(x, \xi))$ and a difference operator $P'$

$$P^* = \overline{P} + P', \tag{4.26}$$

where $\overline{P}$ has similar properties as $P$, and $P'$ is the remainder. Its mapping properties were described in Marschall [35, Cor. 3.6] which has the following consequence in our setting:

**Proposition 4.10.** Let $A = \text{Op}(a(x, \xi))$, where $a(x, \xi) \in C^\tau \mathcal{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ for a $\tau > 0$ and an $m \geq 0$, let $\overline{A} = \text{Op}(\overline{a}(x, \xi))$ and let $A' = A^* - \overline{A}$. Then $A'$ maps continuously, for $\theta \in [0,1]$ with $\theta < \tau$,

$$A': H_q^{s+m-\theta}(\mathbb{R}^n) \to H_q^s(\mathbb{R}^n), \text{ when } -\tau + \theta < s < \tau. \tag{4.27}$$

**Proof.** Corollary 3.6 in [35] states (with some relabeling of parameters) that when $a(x, \xi)$ is in $H_Q^m \mathcal{S}^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, namely the symbol space with estimates

$$|a(x, \xi)| \leq C(|\xi|^m, \|\partial_\xi^\alpha a(\cdot, \xi)\|_{H_Q^m(\mathbb{R}^n)} \leq C_\alpha (|\xi|^{m-|\alpha|}, \text{ all } \alpha,$$

then for $0 < p \leq \infty$, $0 \leq \theta \leq 1$, $n/Q + \theta < \sigma$, $0 < Q \leq \infty$ and

$$n(\frac{1}{Q} + \frac{1}{p} - 1) - \sigma + \theta < s < \sigma - n(\frac{1}{Q} - \frac{1}{p}) +,$$

the mapping $A': H_p^{s+m-\theta}(\mathbb{R}^n) \to H_p^s(\mathbb{R}^n)$ is bounded. For a given $p$, we shall take $Q \in \mathbb{R}_+$ so large that $\left(\frac{1}{Q} + \frac{1}{p} - 1\right)_+ = \left(\frac{1}{Q} - \frac{1}{p}\right)_+$ are 0. Since $C^\tau(\mathbb{R}^n) \subset H_Q^{s-\varepsilon}(\mathbb{R}^n)$ (any small $\varepsilon > 0$), the given symbol is in the space $H_Q^m \mathcal{S}^m_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$ with $\sigma = \tau - \varepsilon$ and any $Q$. Then the asserted continuity follows when $\sigma = \tau - \varepsilon$, $0 \leq \theta < \sigma$ and $\theta \leq 1$, and $-\sigma + \theta < s < \sigma$. This reduces to the mentioned conditions since $\varepsilon$ can be taken arbitrarily small. \(\square\)

For our operator $P'$ we conclude
**Corollary 4.11.** Assume Hypothesis 3.1 1°. The operator $P' = P^* - \overline{P}$ maps continuously

$$P': H^s_q(\mathbb{R}^n) \rightarrow H^s_q(\mathbb{R}^n), \text{ when } 2a \leq 1, -\tau + 2a < s < \tau; \quad (4.28)$$

$$P': H^{s+2a-1}_q(\mathbb{R}^n) \rightarrow H^s_q(\mathbb{R}^n), \text{ when } 2a > 1, -\tau + 1 < s < \tau; \quad (4.29)$$

in particular, the mapping properties hold for $0 \leq s < \tau$.

**Proof.** Let $m = 2a$. If $2a \leq 1$, we can take $\theta = 2a$ (which is $< \tau$) in (4.27); this shows (4.28). If $2a > 1$, hence $\tau > 1$, (4.27) holds with $\theta = 1$, this shows (4.29). \( \square \)

A first consequence is that $P'$ maps $H^s_q(\mathbb{R}^n) \rightarrow L^s_q(\mathbb{R}^n)$. This follows obviously from (4.28) when $2a \leq 1$. When $2a > 1$, $a = s + 2a - 1$ when $s = 1 - a$, so $P'$ maps $H^s_q(\mathbb{R}^n)$ to $H^{1-a}_q(\mathbb{R}^n) \subset L^s_q(\mathbb{R}^n)$ by (4.29). A fortiori, $P'$ maps $H^a_q(\mathbb{R}^n)$ to $H^{-a}_q(\mathbb{R}^n)$. Then since $\overline{P}$ has the same mapping properties as $P$, $r^+P^*$ is continuous from $\dot{H}^a_q(\mathbb{R})$ to $\overline{H}^{-a}_q(\mathbb{R})$, and there holds, by extension by continuity from $u, v \in C^\infty_0(\Omega)$,

$$\langle r^+P^*u, v \rangle_{\mathbb{P}^{-a}_q(\Omega), \dot{H}^a_q(\mathbb{R})} = \langle u, r^+P^*v \rangle_{\dot{H}^a_q(\mathbb{R}), \mathbb{P}^{-a}_q(\Omega)}, \text{ for } u \in \dot{H}^a_q(\mathbb{R}), v \in \dot{H}^{-a}_q(\mathbb{R}), \quad (4.30)$$

again with sesquilinear dualities.

The adjoint of $P_{D,2}$ in $L^2(\Omega)$ by the Lax-Milgram Lemma (cf. e.g. [15, Sect. 12.4]) is the realization $(P^*)_{D,2}$ of $r^+P^*$ in $L^2(\Omega)$ with domain

$$D((P^*)_{D,2}) = \{ u \in \dot{H}^a_q(\mathbb{R}) \mid r^+P^*u \in L^2(\Omega) \}.$$ 

Here $r^+\overline{P}u \in L^2(\Omega) \iff u \in H^{a(2a)}(\mathbb{R})$; and since $r^+P'$ maps $\dot{H}^a(\mathbb{R})$ to $L^2(\Omega)$ as seen above, hence maps also the subset $H^{a(2a)}(\mathbb{R})$ to $L^2(\Omega)$, it follows that $D((P^*)_{D,2}) = H^{a(2a)}(\mathbb{R})$.

We can henceforth drop the parentheses in the notation, noting that the adjoint of $P_{D,2}$ in $L^2(\Omega)$ is the operator $P^*_{D,2}$ acting like $r^+P^*$ with domain

$$D(P_{D,2}^*) = H^{a(2a)}(\mathbb{R}) = \{ u \in \dot{H}^a_q(\mathbb{R}) \mid r^+P^*u \in L^2(\Omega) \}. \quad (4.31)$$

**Remark 4.12.** If $P$ is $x$-independent, $P^*$ equals $\overline{P}$, which behaves in exactly the same way as $P$; typical examples of $x$-independent operators are $(-\Delta)^a$ and $(m^2 - \Delta)^a$. Note also that if an $x$-dependent $P$ is known to be formally selfadjoint, i.e. $P^* = P$, no further analysis of $P^*$ is needed (this holds for instance for fractional powers of a formally selfadjoint strongly elliptic differential operator). But in general, a nontrivial difference operator $P'$ will give some limitations in the analysis of regularity properties for $P^*$.

We can now show the following restricted variant of Theorem 3.2 for $P^*$:

**Theorem 4.13.** Assume Hypothesis 3.1 1°, and let $0 \leq s < \tau - 2a$. 

1° Let $2a \leq 1$, and set $t_0 = a + \frac{1}{q}$. Then $r^+ P^*$ maps continuously, for any small $\varepsilon > 0$,

$$r^+ P^* : H^a_q(s+2a)(\Omega) \rightarrow \begin{cases} H^s_q(\Omega), & \text{when } s < t_0, \\ H^{t_0 - \varepsilon}_q(\Omega), & \text{when } s \geq t_0. \end{cases} \tag{4.32}$$

If moreover Hypothesis 3.1 2° holds, one has when $u \in \dot{H}^a_q(\Omega)$:

$$r^+ P^* u \in \dot{H}^s_q(\Omega) \implies \begin{cases} u \in H^a_q(s+2a)(\Omega) & \text{when } s < t_0, \\ u \in H^a_q(t_0 - \varepsilon + 2a)(\Omega) & \text{when } s \geq t_0. \end{cases} \tag{4.33}$$

In particular, $r^+ P^* u \in \dot{H}^a_q(\Omega)$ implies $u \in H^a_q(3a)(\Omega)$.

2° Let $2a > 1$, and set $t_1 = 1 - a + \frac{1}{q}$. Then $r^+ P^*$ maps continuously, for any small $\varepsilon > 0$,

$$r^+ P^* : H^a_q(s+2a)(\Omega) \rightarrow \begin{cases} H^s_q(\Omega), & \text{when } s < t_1, \\ H^{t_1 - \varepsilon}_q(\Omega), & \text{when } s \geq t_1. \end{cases} \tag{4.34}$$

If moreover Hypothesis 3.1 2° holds, one has when $u \in \dot{H}^a_q(\Omega)$:

$$r^+ P^* u \in \dot{H}^s_q(\Omega) \implies \begin{cases} u \in H^a_q(s+2a)(\Omega) & \text{when } s < t_1, \\ u \in H^a_q(t_1 - \varepsilon + 2a)(\Omega) & \text{when } s \geq t_1. \end{cases} \tag{4.35}$$

In particular, $r^+ P^* u \in \dot{H}^{1-a}_q(\Omega)$ implies $u \in H^a_q(1+a)(\Omega)$.

**Proof.** 1°. The case $2a \leq 1$. By (4.28), $P'$ preserves $H^s(\mathbb{R}^n)$, so $r^+ P' : \dot{H}^s_q(\Omega) \rightarrow \dot{H}^s_q(\Omega)$ for $0 \leq s < \tau$, acting similarly to the multiplication by $\lambda$ considered in Proposition 4.7. Since Theorem 3.2 applies to $\overline{P}$, we can apply the proof of Proposition 4.7 to $\overline{P} + P'$, concluding that under Hypothesis 3.1 1°,

$$r^+(\overline{P} + P') : H^a_q(s+2a)(\Omega) \rightarrow \begin{cases} H^s_q(\Omega), & \text{when } s < a + \frac{1}{q}, \\ H^{a + \frac{1}{q} - \varepsilon}_q(\Omega), & \text{when } s \geq a + \frac{1}{q}, \end{cases}$$

for small $\varepsilon > 0$; this shows (4.32).

When moreover Hypothesis 3.1 2° holds, we find as in Proposition 4.7 that for small $\varepsilon > 0$,

$$u \in \dot{H}^a_q(\Omega), r^+ P^* u \in \dot{H}^s_q(\Omega) \implies \begin{cases} u \in H^a_q(s+2a)(\Omega) & \text{when } s < a + \frac{1}{q}, \\ u \in H^a_q(3a + \frac{1}{q} - \varepsilon)(\Omega) & \text{when } s \geq a + \frac{1}{q}, \end{cases}$$

this shows (4.33).
2°. The case $2a > 1$. Here $r^+ P' : \dot{H}^{s+2a-1}_q(\Omega) \to \overline{H}^s_q(\Omega)$ for $0 \leq s < \tau$. Now (cf. (2.20)–(2.21))

$$
H^a_q(s+2a)(\Omega) \subset H^a_q(s-1+2a)(\Omega) \begin{cases} 
\dot{H}^{s-1+2a}_q(\Omega) \text{ if } s-1+a < \frac{1}{q}, \\
H^{a+\frac{1}{q}-\varepsilon}_q(\Omega) \text{ if } s-1+a \geq \frac{1}{q}.
\end{cases}
$$

In the first case, the space is in view of (4.29) mapped by $r^+ P'$ into $\overline{H}^s_q(\Omega)$; in the second case into $\overline{H}^{1-a+\frac{1}{q}-\varepsilon}_q(\Omega)$. This shows the forward mapping property (4.34).

Now assume moreover that $P$ is strongly elliptic, so that the full Theorem 3.2 is valid for $\overline{P}$. Let $u \in \dot{H}^a_q(\Omega)$ with $r^+ P^u \in \overline{H}^a_q(\Omega)$. Since $r^+ P' : \dot{H}^a_q(\Omega) \to \overline{H}^{1-a}_q(\Omega)$ by (4.29), we have that

$$
r^+ \overline{P} u = r^+ P^u - r^+ P' u \in \overline{H}^a_q(\Omega) + \overline{H}^{1-a}_q(\Omega). \tag{4.36}
$$

If $s \leq 1-a$, we conclude $u \in H^a_q(s+2a)(\Omega)$ from the regularity property of $\overline{P}$. If $s > 1-a$, this argument gives that $u \in H^a_q(1+a)(\Omega)$. There is a small improvement of the latter property: We have that $H^a_q(1+a)(\Omega) \subset H^a_q(1+a-\varepsilon)(\Omega)$ by (2.21), hence $r^+ P' u \in \overline{H}^{1-a-\varepsilon}_q(\Omega)$. Insertion of this information in (4.36) shows that if $s < \frac{1}{q} + 1-a$, the regularity property of $r^+ \overline{P}$ gives that $u \in H^a_q(s+2a)(\Omega)$, and when $s \geq \frac{1}{q} + 1-a$, it gives that $u \in H^a_q(1+1+a-\varepsilon)(\Omega)$. This shows (4.35). \qed

For these results, recall that $s$ is also subject to the condition $s < \tau - 2a$; if $t_0$ or $t_1$ is larger, the range of possible $s$ may not include $t_0$ resp. $t_1$.

But $s = 0$ is always included, and gives as a special case:

**Corollary 4.14.** Assume Hypotheses 3.1 1°.

$r^+ P^*$ maps $H^a_q(2a)(\Omega)$ continuously into $L_q(\Omega)$. When moreover Hypotheses 3.1 2° holds, we have: When $u \in \dot{H}^a_q(\Omega)$ satisfies $r^+ P^u \in L_q(\Omega)$, then $u \in H^a_q(2a)(\Omega)$.

In the following, indications of Hölder spaces with an $\varepsilon$, it is understood that $\varepsilon$ is chosen so that integer exponents are avoided.

For general $q$ we define the Dirichlet realization $P^*_{D,q}$ of $P^*$ as the operator acting like $r^+ P^*$ with domain

$$
D(P^*_{D,q}) = \{ u \in \dot{H}^a_q(\Omega) \mid r^+ P^u \in L_q(\Omega) \} = H^a_q(2a)(\Omega); \tag{4.37}
$$

the last equality holds in view of Corollary 4.14.

Again, we can show a regularity of eigenfunctions:

**Theorem 4.15.** Assume Hypothesis 3.1. Let $u_\lambda \in \dot{H}^a_q(\Omega)$ be an eigenfunction of $r^+ P^*$, i.e. a nontrivial solution of $r^+ P^* u_\lambda = \lambda u_\lambda$. Then
\[ u_\lambda \in C^s_{a\min\{2a,1,\tau-a\} + 2a - \varepsilon}(\Omega) \subset d^a C^{\min\{2a,1,\tau-a\} - \varepsilon}(\Omega); \quad (4.38) \]

in a local sense if \( \tau < 1 \).

As a consequence, the Dirichlet realizations \( P_{D,q}^{s} \), \( 1 < q < \infty \), have the same discrete set of eigenvalues and eigenfunctions as \( P_{D,2}^{s} \) for all \( q \in [1, \infty[. \)

**Proof.** Note first that in view of Corollary 4.14, the lifting of the \( q \)-parameter in Theorem 3.4 can be performed also with \( P \) replaced by the present \( P^s \). Therefore we can conclude that \( u_\lambda \in \mathcal{H}_p^a(\Omega) \) for all \( p < \infty \). To use Theorem 4.13, we must combine the information given there with the restriction \( s < \tau - 2a \).

In the case \( 2a \leq 1 \), we find from Theorem 4.13 1° that \( r^+ P^s u_\lambda \in \mathcal{H}_p^a(\Omega) \) implies

\[ u_\lambda \in \mathcal{H}_p^{a\min\{a,\tau-2a-\varepsilon,a+\frac{1}{p}-\varepsilon\}+2a}(\Omega). \]

for small \( \varepsilon > 0 \). In view of (2.32) and Lemma 4.3, this implies

\[ u_\lambda \in C^{a\min\{a,\tau-2a\}+2a-\varepsilon}(\Omega) \subset d^a C^{\min\{2a,\tau-a\} - \varepsilon}(\Omega), \quad (4.39) \]

for small \( \varepsilon > 0 \).

A better estimate for \( \lambda = 0 \) cannot be expected (with the present strategy), because of the presence of \( t_0 - \varepsilon = a + \frac{1}{p} - \varepsilon \) in the estimate in (4.33), no matter how large \( s \) is.

In the case \( 2a > 1 \), we find from Theorem 4.13 2° that \( r^+ P^s u_\lambda \in \mathcal{H}_p^a(\Omega) \) implies

\[ u_\lambda \in \mathcal{H}_p^{a\min\{1-a,\tau-2a-\varepsilon,a+\frac{1}{p}-\varepsilon\}+2a}(\Omega). \]

for small \( \varepsilon > 0 \). In view of (2.32) and Lemma 4.3, this implies

\[ u_\lambda \in C^{a\min\{1-a,\tau-2a,a\}+2a-\varepsilon}(\Omega) = C^{a\min\{1,\tau-a\}+a-\varepsilon}(\Omega) \subset d^a C^{\min\{1,\tau-a\} - \varepsilon}(\Omega), \quad (4.40) \]

for small \( \varepsilon > 0 \). Now (4.39) and (4.40) together imply (4.38).

The last statement follows as in Corollary 4.5. \( \square \)

The theorem shows in particular that the cokernels \( N_\lambda^L \) for \( P_{D,2} - \lambda, \lambda \in \Sigma \), have the regularity in (4.38).

4.4. Fredholm operators in \( L_q \) with a spectral parameter

We now have the tools to deduce some spectral properties like those of \( P_{D,2} \) for the realizations \( P_{D,q} \), besides what was shown for \( q \geq 2 \) in Corollary 4.6 and Theorem 4.8.

It is well-known that \( L_q(\Omega) \) is a reflexive Banach space, the dual space identifying with \( L_q'(\Omega), \frac{1}{q'} = 1 - \frac{1}{q}, \) with a sesquilinear duality \( \langle f, g \rangle_{L_q,L_q'} = \int_\Omega f \bar{g} \, dx \). The adjoint
$T^*$ in $L_{q'}(\Omega)$ of a closed, densely defined (unbounded) operator $T$ in $L_q(\Omega)$, has the domain consisting of the functions $v$ for which there exist $v^*$ such that

$$
(Tu,v)_{L_q,L_{q'}} = (u,v^*)_{L_q,L_{q'}}, \text{ all } u \in D(T);
$$

then $v \in D(T^*)$ with $T^*v = v^*$ (uniquely). $T^*$ is likewise closed and densely defined, and $T^{**} = T$. This is usually deduced by a consideration of the graphs, where one also finds that $T$ is bijective if and only if $T^*$ is so. (Fredholm theory in reflexive Banach spaces is presented e.g. in Schechter [39].)

For $q \neq 2$, the Dirichlet realizations of $r^+P$ and $r^+P^*$, as unbounded operators in the reflexive Banach spaces $L_q(\Omega)$ resp. $L_q'(\Omega)$, will now be shown to be adjoints:

**Theorem 4.16.** Assume Hypothesis 3.1.

1° For each $1 < q < \infty$, the adjoint of $P_{D,q}$ in $L_q(\Omega)$ is the operator $P_{D,q'}^*$ in $L_{q'}(\Omega)$.

2° For all $\lambda$ in the resolvent set $\mathbb{C} \setminus \Sigma$, $P_{D,q} - \lambda I$ is bijective from $D(P_{D,q})$ to $L_q(\Omega)$.

**Proof.** In view of (4.30), there holds

$$
(P_{D,q} u, v)_{L_q,L_{q'}} = (u, P_{D,q'}^* v)_{L_q,L_{q'}}, \text{ for all } u \in D(P_{D,q}), v \in D(P_{D,q'}^*),
$$

so $D(P_{D,q'}) \subset D((P_{D,q})^*)$, and $P_{D,q'}^* \subset (P_{D,q})^*$ there.

Consider first the case where $0$ is not an eigenvalue of $P_{D,2}$, hence not of $P_{D,2}^*$, nor of $P_{D,q}$ and $P_{D,q'}^*$ by Corollary 4.5 and Theorem 4.15.

Let $q < 2$ so that $q' > 2$. We know that $P_{D,q}$ is injective, and since $P_{D,q}$ extends $P_{D,2}$, its range contains $L_2(\Omega)$. By Proposition 3.3, the range is closed, so since $L_2(\Omega)$ is dense in $L_q(\Omega)$, $P_{D,q}$ is surjective from its domain to $L_q(\Omega)$; altogether it is bijective. Then the adjoint $(P_{D,q})^*$ is likewise bijective, from its domain to $L_{q'}(\Omega)$.

For $P_{D,q'}^*$, we know that it is injective (by Theorem 4.15), and since $q' > 2$ and $P_{D,2}^*$ is surjective, $P_{D,q'}^*$ is surjective onto $L_{q'}(\Omega)$. Then the inclusion $P_{D,q'}^* \subset (P_{D,q})^*$ holds for two bijective operators, which implies that they are equal.

For $q > 2$, hence $q' < 2$, there is a similar proof where the roles of $P$ and $P^*$ are reversed. This shows 1°.

In the case where $0$ is an eigenvalue, we can choose an arbitrary $\lambda_0 \in \mathbb{C} \setminus \Sigma$ and apply the above argumentation to the $L_q$ resp. $L_{q'}$ Dirichlet realizations of $P - \lambda_0$ and $(P - \lambda_0)^* = P^* - \lambda_0I$, showing that they are adjoints and bijective. Here we use the regularity properties of the shifted operator $P - \lambda_0I$ established in Proposition 4.7; there is a similar rule for $P^* - \lambda_0I$.

Statement 2° formulates the bijectiveness property of $P_{D,q}$ resp. $P_{D,q} - \lambda_0$ obtained in the above proof. □

Moreover, the Fredholm properties of operators $P_{D,q} - \lambda$ shown in Corollary 4.6 for $q \geq 2$ can now be extended to $q < 2$:

For \( \lambda \in \Sigma \), \( P_{D,q} - \lambda \) is a Fredholm operator from \( H^a_q(\Omega) \) to \( L_q(\Omega) \) with index 0. The kernel of \( P_{D,q} - \lambda \) equals \( N_\lambda \) (the kernel of \( P_{D,2} - \lambda \)), and a cokernel of \( P_{D,q} - \lambda \) is \( N'_\lambda \) (the cokernel of \( P_{D,2} - \lambda \) equal to the kernel of \( P_{D,q}^*, \lambda \)), in the sense that the homogeneous Dirichlet problem for \( P - \lambda \),

\[
(P - \lambda)u = f \text{ in } \Omega, \quad u = 0 \text{ in } \mathbb{R}^n \setminus \Omega,
\]

is solvable with \( u \in H^a_q(\Omega) \) when \( f \in L_q(\Omega) \) satisfies \( (4.13) \) (i.e., \( f \) is in the annihilator of \( N'_\lambda \) in \( L_q(\Omega) \)).

Here \( N_0 \subset d^a C^{-a-\varepsilon}(\Omega) \), and \( N_\lambda \subset d^a C^{\min\{2a,\tau-a\} - \varepsilon}(\Omega) \), \( N'_\lambda \subset d^a C^{\min\{2a,1,\tau-a\} - \varepsilon}(\Omega) \) for general \( \lambda \in \Sigma \) (locally when \( \tau < 1 \)); they are all contained in \( \hat{C}^a(\Omega) \). For all \( \lambda \in \Sigma \),

\[ \dim N_\lambda = \dim N'_\lambda. \]

Proof. The last paragraph in the theorem repeats results from Theorems 4.2, 4.4 and 4.15. Since \( \min\{2a,1,\tau-a\} > a \) and \( d^a \in \hat{C}^a(\Omega) \), all the eigenspaces are contained in \( \hat{C}^a(\Omega) \).

For the main statement, consider for clarity first the case where \( \lambda = 0 \), i.e., \( 0 \) is an eigenvalue of \( P_{D,2} \) with eigenspace \( N_0 \), and \( P_{D,2} \) has a cokernel \( N'_0 \) which is the eigenspace of \( P_{D,2}^* \). Their regularity is recalled above. By Corollary 4.5, \( N_0 \) is also the eigenspace of \( P_{D,q} \), and by Theorem 4.15, \( N'_0 \) is the eigenspace of \( P_{D,q}^* \). Since \( P_{D,q} \) and \( P_{D,q}^* \) are adjoints, \( N'_0 \) is a cokernel of \( P_{D,q} \). Thus \( P_{D,q} \) is Fredholm with kernel \( N_0 \) and cokernel \( N'_0 \).

If \( \lambda \neq 0 \), we apply the same arguments to \( (P - \lambda)_{D,q} = P_{D,q} - \lambda \) and its adjoint \( (P^* - \lambda)_{D,q} = P_{D,q}^* - \lambda \), which in view of Proposition 4.7 and a similar rule for \( P^* - \lambda \) have sufficient regularity properties to allow this. \( \square \)

The resolvent problem \( (4.41) \) was discussed for selfadjoint operators like \( P = (-\Delta)^a \) by Chan, Gomez-Castro and Vazquez [6] with \( f \) given (primarily) in a weighted \( L_1 \)-space; and a Fredholm property was deduced from the knowledge of the solution operator for \( \lambda = 0 \), as an integral operator with kernel \( G(x,y) \) (Green’s kernel) satisfying explicit estimates. The above results develop such knowledge for a large class of operators (containing \( (-\Delta)^a \)) which can be \( \varepsilon \)-dependent and nonselfadjoint, with precise mapping properties in \( L_q \) Sobolev spaces for \( 1 < q < \infty \).

We can also draw consequences for data in spaces with higher regularity:


Consider the problem \( (4.41) \) with \( f \) given in \( H_q^s(\Omega) \) for some \( 0 \leq s < \tau - 2a \). For a small \( \varepsilon > 0 \), let \( r_q = \min\{2a, a + \frac{1}{q} - \varepsilon\} \).

If \( \lambda \notin \Sigma \), and \( s \leq r_q \) if \( \lambda \neq 0 \), \( (4.41) \) is uniquely solvable with solution in \( H_q^{a(s+2a)}(\overline{\Omega}) \), and the solution operator, a restriction of \( (P_{D,q} - \lambda)^{-1} \), is continuous from \( H_q^s(\Omega) \) to \( H_q^{a(s+2a)}(\overline{\Omega}) \).
If $\lambda \in \Sigma$, and $s \leq \min\{a, r_q\}$ if $\lambda \neq 0$, then the problem (4.41) is Fredholm solvable from $\mathcal{H}_q^a(\Omega)$ to $H_q^{a+2a}(\overline{\Omega})$, in the sense that a solution $u \in H_q^{a+2a}(\overline{\Omega})$ exists for $f \in \mathcal{H}_q^a(\Omega)$ satisfying (4.13), and is unique modulo $N_\lambda$.

**Proof.** For $\lambda = 0$, the statements follow from Theorem 3.2 together with the added knowledge on bijectiveness or Fredholm solvability shown in Theorems 4.16 and 4.17.

Now let $\lambda \neq 0$. We have from (4.22)

$$H_q^{a}(\Omega) \begin{cases} = \mathcal{H}_q^{2a}(\overline{\Omega}) & \text{if } a < \frac{1}{q} \\ \subset \mathcal{H}_q^{a+\frac{1}{q}-\varepsilon}(\overline{\Omega}) & \text{if } a \geq \frac{1}{q}, \end{cases} \quad (4.42)$$

so that $H_q^{a+2a}(\overline{\Omega}) \subset \mathcal{H}_q^{a}(\Omega) \subset \mathcal{H}_q^{a+\frac{1}{q}-\varepsilon}(\overline{\Omega})$ for $s \in [0, r_q]$. When $u$ solves (4.41) and $f \in \mathcal{H}_q^{a}(\Omega)$ for some $s \in [0, r_q]$, then since $u \in H_q^{a+2a}(\overline{\Omega})$ by Theorem 4.17,

$$\tau^+ Pu = f + \lambda u \in \mathcal{H}_q^{a}(\Omega),$$

and it follows from Theorem 3.2 that $u \in H_q^{a+2a}(\overline{\Omega})$.

When $\lambda \in \mathbb{C} \setminus \Sigma$, the mapping from $f$ to $u$ is bijective as a consequence of Theorem 4.16, hence continuous (by the closed graph theorem) since the inverse is so by Theorem 3.2.

When $\lambda \in \Sigma$, we need the condition $s \leq a$ to have $N_\lambda \subset H_q^{a+2a}(\overline{\Omega})$, so that the Fredholm solvability can be inferred from the property known from Theorem 4.17 for $s = 0$. \hfill \Box

**Remark 4.19.** In continuation of Remark 4.12, let us underline that in cases where $P^*$ is as smooth as $P$, the results hold with as good estimates for $P^*$ as for $P$. This goes for $x$-independent operators (as for example $(m^2 - \Delta)^a$, $m \in \mathbb{R}$) and for selfadjoint cases as mentioned in Remark 4.12.

**Example 4.20.** In the case where $P$ has an $x$-independent symbol $p(\xi)$ that is homogeneous of degree $2a$, even and positive for $\xi \neq 0$, there is for any bounded set $\Omega \subset \mathbb{R}^n$ a Poincaré inequality $(Pu, u) \geq c\|u\|_{L_2(\Omega)}^2$ for $u \in H^a(\overline{\Omega})$, cf. e.g. the survey of Ros-Oton [36, (3.4)ff.]. Then $P_{D_2}$ is bijective. Moreover, the $L_q$-realizations $P_{D, q}$ are bijective from $D(P_{D, q}) = H_q^{a}(\overline{\Omega})$ to $L_q(\Omega)$ when $\Omega$ is $C^{1+\tau}$, for all $1 < q < \infty$, by Corollary 4.5.

5. **Solvability of the nonhomogeneous Dirichlet problem, spectral analysis of “large” solutions**

For the nonhomogenous Dirichlet problem, we have to go out in the larger spaces $H_q^{a-1+2a}(\overline{\Omega})$; here we get solvability results by use of the results in the homogeneous case for $\lambda = 0$, combined with Theorem 2.3.

**Theorem 5.1.** Assume Hypothesis 3.1 with $\tau > 2a + 1$. Let $0 \leq s < \tau - 2a - 1$. 
Consider the nonhomogeneous Dirichlet problem (3.15), recalled here:

\[ Pu = f \text{ in } \Omega, \]
\[ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \]
\[ \gamma_{0}^{a-1}u = \varphi \text{ on } \partial \Omega; \]

with \( f \) given in \( \overline{H}^{s}(\Omega) \), \( \varphi \) given in \( B^{s+a+1/q'}_{q'}(\partial \Omega) \), and the solution being sought in \( H^{(a-1)(s+2a)}_{q}(\overline{\Omega}) \).

If \( 0 \notin \Sigma \), it is uniquely solvable, and the solution operator, given by the formula (5.3) below, is continuous from \( \overline{H}^{s}(\Omega) \times B^{s+a+1/q'}_{q'}(\partial \Omega) \) to \( H^{(a-1)(s+2a)}_{q}(\overline{\Omega}) \).

If \( 0 \in \Sigma \), it is Fredholm solvable, in the sense that a solution \( u \) exists, unique modulo \( N_0 \), when \( f' = f - r^{+}PK_{(0)}^{a-1}\varphi \) plays the role of \( f \) in (4.13) for \( \psi \in N_0' \).

**Proof.** Note that since \( \tau > 2a + 1 \), the case \( \tau < 1 \) does not occur here.

According to Theorem 2.3, there exists a right inverse \( K_{(0)}^{a-1} \) of \( \gamma_{0}^{a-1} \), mapping

\[ K_{(0)}^{a-1}: B^{s+a+1/q'}_{q'}(\partial \Omega) \to H^{(a-1)(s+2a)}_{q}(\overline{\Omega}), \]

for \(-a - 1/q' < s < \tau - 2a - 1\).

Set \( v = K_{(0)}^{a-1}\varphi \), then \( u \in H^{(a-1)(s+2a)}_{q}(\overline{\Omega}) \) solves the given problem if and only if \( w = u - v \) solves

\[ Pw = f - Pv \text{ in } \Omega, \]
\[ w = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \]
\[ \gamma_{0}^{a-1}w = 0 \text{ on } \partial \Omega. \]

Here \( r^{+}Pv \in \overline{H}^{s}(\Omega) \), and by the last statement in Theorem 2.3,

\[ w \in \{ u \in H^{(a-1)(s+2a)}_{q}(\overline{\Omega}) \mid \gamma_{0}^{a-1}u = 0 \} = H^{a(s+2a)}_{q}(\overline{\Omega}), \]

so (5.1) is a homogeneous Dirichlet problem with right-hand side \( f - r^{+}Pv \in \overline{H}^{s}(\Omega) \); \( w \) being sought in \( H^{a(s+2a)}_{q}(\overline{\Omega}) \). Here Theorems 4.16 and 4.17 apply:

If \( 0 \notin \Sigma \), the problem is uniquely solvable, with

\[ u = P_{D,q}^{-1}(f - r^{+}Pv) = P_{D,q}^{-1}f - P_{D,q}^{-1}r^{+}PK_{(0)}^{a-1}\varphi. \]

If \( 0 \in \Sigma \), the problem is Fredholm solvable, in the sense that a solution \( w \) exists if and only if \( f' = f - Pv \) plays the role of \( f \) in (4.13) with \( \psi \in N_0' \), and it is unique modulo \( N_0 \). This implies the statement in the theorem for \( u = w + v \). \( \Box \)
Observe that this is an existence-and-uniqueness theorem (resp. Fredholm theorem), which completes the regularity result Theorem 3.7 shown in Section 3. We can also complete Corollary 3.9 with solvability in Hölder spaces:

**Corollary 5.2.** Assume Hypothesis 3.1 with \( \tau > 2a + 1 \), and let \( 0 \leq s < \tau - 2a - 1 \). Let \( f \in C^s(\Omega) \) and \( \varphi \in C^{s+a+1}(\partial \Omega) \).

With \( u \) a priori assumed to be in \( H_q^{(a-1)(a)}(\Omega) \) for some \( q \), problem (3.15) is uniquely solvable if \( 0 \notin \Sigma \), and uniquely solvable modulo \( N_0 \) when \( f' = f - \lambda u \) satisfies (4.13); and the solution satisfies \( u \in C^{(a-1)(s+2a-\varepsilon)}(\Omega) \subset C^{s+2a-\varepsilon}(\Omega) + d^{a-1}C^{s+a+1-\varepsilon}(\Omega) \) as in Corollary 3.9.

**Proof.** The statement follows by embedding the Hölder spaces in \( H_q^s \)-spaces and applying Theorem 5.1, letting \( q \to \infty \) as in the proof of Corollary 3.9. \( \square \)

Next, we shall consider nonhomogeneous problems with a spectral parameter \( \lambda \) subtracted from \( P \). A study of such problems was initiated by Chan, Gomez-Castro and Vazquez in [6]. Since the solutions generally blow up at the boundary (when \( u/d^{a-1} \) has a nonzero boundary value), it is more demanding than in the homogeneous case to have \( r^\lambda Pu \) and \( \lambda u \) lying in the same space. [6] handles this (for operators like \( P = (-\Delta)^a \)) by considering \( P \) in a weighted \( L_1 \)-space. In our treatment, we have \( P \) defined on \( H_q^{(a-1)(s+2a)}(\Omega) \) and can get results when this is contained in \( L_q(\Omega) \).

**Lemma 5.3.** When \( q < (1 - a)^{-1} \), then

\[
H_q^{(a-1)(s)}(\Omega) \subset L_q(\Omega) \quad \text{for} \quad s \geq 0.
\]

**Proof.** Denote \( a - 1 + 1/q = t; \) the hypothesis means that \( t > 0 \). Since the spaces \( H_q^{(a-1)(s)}(\Omega) \) decrease with increasing \( s \), it suffices to show the statement for \( 0 \leq s < t \). Working in local coordinates, we have for such \( s \) (cf. (2.17))

\[
H_q^{(a-1)(s)}(\mathbb{R}^n_+) = \Xi^{-a+1}e^t H_q^{s-a+1}(\mathbb{R}^n_+) = \Xi^{-a+1}e^t H_q^{s+t+1/q}(\mathbb{R}^n_+)
\]

\[
= \Xi^{-a+1} H_q^{s-t+1/q}(\mathbb{R}^n_+)
\]

\[
= H_q^{s-t+1/q-a+1}(\mathbb{R}^n_+) = H_q^{s-a+1}(\mathbb{R}^n_+) \subset L_q(\mathbb{R}^n_+). \quad \square
\]

Thus for any given \( a \in ]0, 1[ \), the inclusion (5.4) holds when \( q \) is sufficiently low. Note that for \( q = 2 \),

\[
H^{(a-1)(s)}(\Omega) \subset L_2(\Omega), \text{ when } a > \frac{1}{2}, s \geq 0.
\]

**Theorem 5.4.** Assume Hypothesis 3.1 with \( \tau > 2a + 1 \). Assume moreover \( q < (1 - a)^{-1} \). Consider the problem
\[ Pu - \lambda u = f \text{ in } \Omega, \]
\[ u = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \]
\[ \gamma_0^{a-1} u = \varphi \text{ on } \partial \Omega, \]

(5.6)

with \( f \) given in \( L_q(\Omega) \), \( \varphi \) given in \( B_q^{a+1/q'}(\partial \Omega) \), and the solution being sought in \( H_q^{(a-1)(2a)}(\Omega) \).

If \( \lambda \not\in \Sigma \), it is uniquely solvable, and the solution operator, given by the formula (5.8) below, is continuous from \( L_q(\Omega) \times B_q^{a+1/q'}(\partial \Omega) \) to \( H_q^{(a-1)(2a)}(\Omega) \).

If \( \lambda \in \Sigma \), it is Fredholm solvable, in the sense that there is a solution \( u \in H_q^{(a-1)(2a)}(\Omega) \), unique modulo \( N_\lambda \), when \( f' = f - (r^+P - \lambda)K_0^{a-1}\varphi \) satisfies (4.13). The solution operator is continuous from the closed subset of \( L_q(\Omega) \times B_q^{a+1/q'}(\partial \Omega) \) of pairs \( \{f, \varphi\} \) such that \( f' = f - (r^+P - \lambda)K_0^{a-1}\varphi \) satisfies (4.13), to \( H_q^{(a-1)(2a)}(\Omega) \).

**Proof.** Using the right inverse \( K_0^{a-1} \) of \( \gamma_0^{a-1} \) recalled in the preceding proof, we set \( v = K_0^{a-1}\varphi \); then \( u \) solves the problem (5.6) if and only if \( w = u - v \) solves

\[ Pw - \lambda w = f - (P - \lambda)v \text{ in } \Omega, \]
\[ w = 0 \text{ in } \mathbb{R}^n \setminus \Omega, \]
\[ \gamma_0^{a-1} w = 0 \text{ on } \partial \Omega. \]

(5.7)

Here \( \lambda v \in L_q(\Omega) \) by Lemma 5.3, so \( f - (r^+P - \lambda)v \in L_q(\Omega) \). Problem (5.7) is in fact a homogeneous Dirichlet problem, so by Theorem 4.16 it has the unique solution \( w = (P_{D,q} - \lambda)^{-1}(f - (r^+P - \lambda)v) \), when \( \lambda \not\in \Sigma \). The solution of (5.6) is then

\[ u = (P_{D,q} - \lambda)^{-1}(f - (r^+P - \lambda)v) + v \]
\[ = (P_{D,q} - \lambda)^{-1}f + (1 - (P_{D,q} - \lambda)^{-1}(r^+P - \lambda))K_0^{a-1}\varphi. \]

(5.8)

When \( \lambda \in \Sigma \), we apply the Fredholm solvability of the homogeneous Dirichlet problem (5.7) shown in Theorem 4.17. \( \square \)

The theorem can be extended to slightly more smooth data with \( s > 0 \), subject to the condition \( a - \frac{1}{q'} > s \).

In the discussion of the problem (5.6) in [6] (for operators like \( P = (-\Delta)^a \)), it is called an eigenvalue problem. For \( f \neq 0 \), we see it more as a resolvent problem (where the solution operator may be useful e.g. in associated evolution problems).

However for \( f = 0 \), (5.6) can certainly be regarded as an eigenvalue problem, where solutions are sought that satisfy a fixed nonhomogeneous boundary condition. Here [6] proved the existence of a nontrivial solution when \( \varphi \) is a continuous function \( \neq 0 \), by use of a resolvent for the homogeneous Dirichlet problem (corresponding to our \( (P_{D,q} - \lambda)^{-1} \)), acting in weighted \( L_1 \)-spaces.

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It is a major point in [6] that these “eigenfunctions” blow up at the boundary (like \(d^{a-1}\)), being “large solutions”.

We agree of course that it is striking, that these solutions are generally unbounded at \(\partial \Omega\), but we also find that it is natural, in view of the systematic point of view on how to define a nonhomogeneous local Dirichlet condition, as presented at the start of this paper.

**Example 5.5.** Here are some more details on the occurring spaces in the case \(q = 2, a > \frac{1}{2}\), cf. (5.5). The boundary space is here \(H^{a + \frac{1}{2}}(\partial \Omega)\) and the solution space is \(H^{(a - 1)(2a)}(\Omega)\), which is a certain subspace of \(L_2(\Omega)\). More technically, it is described in Remark 3.8 by:

\[
H^{(a - 1)(2a)}(\Omega) = H^{(2a)}(\Omega) + K^{a - 1}_1 H^{a + \frac{1}{2}}(\partial \Omega),
\]

(5.9)

the operators \(K^{a - 1}_1\) and \(K^a_0\) provide factors \(d^{a-1}\) resp. \(d^a\).

### 6. Resolvent estimates and evolution problems

Solutions of evolution problems (parabolic problems) for \(P\) with homogeneous boundary conditions were constructed in [22,23] in smooth settings, and we can now extend those results to the present cases of operators \(P\) and domains \(\Omega\) with limited smoothness. Moreover, we can introduce completely new results on evolution problems with nonhomogeneous boundary conditions.

#### 6.1. Results for \(q = 2\)

Denote, for \(\lambda \in \mathbb{C} \setminus \Sigma\),

\[
(P_{D,2} - \lambda)^{-1} = R_\lambda;
\]

(6.1)

it is the solution operator for the homogeneous Dirichlet problem for \(r^+ P - \lambda\) in \(L_2(\Omega)\). By Theorem 4.2, it is defined in particular for \(\lambda\) in the complement of \(M\) (4.9).

As a special case of (4.22),

\[
H^{(2a)}(\Omega) = H^{2a}(\Omega) \quad \text{when} \quad 0 < a < \frac{1}{2},
\]

\[
\subset H^{a + \frac{1}{2} - \varepsilon}(\Omega) \quad \text{when} \quad \frac{1}{2} \leq a < 1,
\]

(6.2)

any \(\varepsilon > 0\). Define

\[
r = \min\{2a, a + \frac{1}{2} - \varepsilon\},
\]

(6.3)

for a small \(\varepsilon \in [0, a + \frac{1}{2}]\); then \(D(P_{D,2}) \subset \dot{H}^r(\Omega) \subset \overline{\mathcal{H}}^r(\Omega)\). (Here \(r = r_2\) in (4.24).)
We know from Theorem 4.8 that $R_\lambda$ by restriction defines a homeomorphism from $\overline{H}^s(\Omega)$ to $H^{a(s+2a)}(\Omega)$ for $\lambda$ in the resolvent set $\mathbb{C} \setminus \Sigma$ including $\mathbb{C} \setminus M$, when $s \in [0, r]$, $0 \leq s < \tau - 2a$.

For the treatment of evolution problems we need norm estimates of $R_\lambda$ that are uniform in $\lambda$. To start with, there is the estimate (4.10); by the proof of [23, Th. 5.8] it can be supplied with estimates in spaces of higher regularity. Recall the notation from [23] for a general operator $A$ in $L_2(\Omega)$:

$$D_s(A) = \{u \in D(A) \mid Au \in \overline{H}^s(\Omega)\} \text{ for } s \geq 0;$$ (6.4)

it will be applied to $A = P_{D,2}$. The space equals $H^{a(2a+s)}(\Omega)$ when $s$ is as in Theorem 3.2.

Theorem 6.1. Assume Hypothesis 3.1, and let $s \in [0, r]$, $s < \tau - 2a$. Then for $\lambda \notin M$, the resolvent $R_\lambda = (P_{D,2} - \lambda)^{-1}$ maps continuously

$$R_\lambda : \overline{H}^s(\Omega) \simarrow H^{a(s+2a)}(\Omega),$$ (6.5)

satisfying the estimates when $\Re \lambda \leq -\beta$:

$$\|R_\lambda f\|_{D_0(P_{D,2})} + \langle \lambda \rangle \|R_\lambda f\|_{L_2(\Omega)} \leq C_0 \|f\|_{L_2(\Omega)},$$

$$\|R_\lambda f\|_{D_s(P_{D,2})} + \langle \lambda \rangle^{j+1} \|R_\lambda f\|_{L_2(\Omega)} \leq C_j(\|f\|_{\overline{H}^s(\Omega)} + \langle \lambda \rangle \|f\|_{L_2(\Omega)}) \text{ for } j \in \mathbb{N}. \quad (6.6)$$

Proof. The mapping property (6.5) is known from Theorem 4.8 (as noted above).

The first (well-known) estimate in (6.6) is obtained by writing (4.10) as

$$\langle \lambda \rangle \|R_\lambda f\|_{L_2(\Omega)} \leq C' \|f\|_{L_2(\Omega)},$$

and supplying it with the observation using that $P_{D,2} = (P_{D,2} - \lambda) + \lambda$,

$$\|R_\lambda f\|_{D_0(P_{D,2})} \leq c_1 \|P_{D,2} R_\lambda f\|_{L_2} \leq c_1(\|f\|_{L_2} + \|\lambda R_\lambda f\|_{L_2}) \leq c_1(\|f\|_{L_2} + C' \|f\|_{L_2}).$$

The second estimate in (6.6) is shown in [23], proof of Th. 6.8 (see in particular formula (5.33) there), where it is seen how the result follows from a combination of the regularity results and numerical range estimates for the homogeneous Dirichlet problem. Since these prerequisites hold for $P_{D,2}$ under the present hypotheses on $\tau$ and $s$, the conclusion follows. The constant denoted $\xi_0$ there equals $\beta$ in Theorem 4.2 here. □

One can possibly extend the second estimate in (6.6) to allow replacement of $j \in \mathbb{N}$ by $j \in \mathbb{R}_+$, but we think a nonzero $j$ is needed if one wants to have estimates with $s > 0$.

The estimates were used in [23] to get regularity estimates for the solutions of the evolution problem (for some $T \in \mathbb{R}_+$)
\[ Pu + \partial_t u = f \text{ on } \Omega \times I, \quad I = ]0, T[, \]
\[ u = 0 \text{ on } (\mathbb{R}^n \setminus \Omega) \times I, \quad u|_{t=0} = 0; \tag{6.7} \]

with zero initial value and homogeneous boundary condition. Namely, as accounted for in the proofs of [23, Th. 5.6 and 5.8], there holds:

**Theorem 6.2.** Assume Hypothesis 3.1, and let \( s \in [0, r] \), \( s < \tau - 2a \). Solutions of \eqref{6.7} are searched for \( u \) in \( L_2(I; H^{a(2a)}(\Omega)) \).

1° For \( f \) given in \( L_2(\Omega \times I) \), there is a unique solution \( u \) of \eqref{6.7} satisfying

\[ u \in L_2(I; H^{a(2a)}(\Omega)) \cap \overline{H}^1(I; L_2(\Omega)); \tag{6.8} \]

moreover, \( u \in \overline{C}^0(I; L_2(\Omega)) \).

2° Let \( 0 \leq s \leq r \). If \( f \in L_2(I; \overline{H}^s(\Omega)) \cap \overline{H}^1(I; L_2(\Omega)) \), with \( f|_{t=0} = 0 \), then the solution of \eqref{6.7} satisfies

\[ u \in L_2(I; H^{a(2a+s)}(\Omega)) \cap \overline{H}^2(I; L_2(\Omega)). \tag{6.9} \]

3° For any integer \( j \geq 2 \), if \( f \in L_2(I; \overline{H}^s(\Omega)) \cap \overline{H}^j(I; L_2(\Omega)) \) with \( \partial_t^j f|_{t=0} = 0 \) for \( l < j \), then

\[ u \in L_2(I; H^{a(2a+s)}(\Omega)) \cap \overline{H}^{j+1}(I; L_2(\Omega)). \tag{6.10} \]

It follows in particular that

\[ f \in \bigcap_l \overline{H}^j(I; \overline{H}^s(\Omega)), \quad \partial_t^j f|_{t=0} = 0 \text{ for } l \in \mathbb{N}_0 \implies u \in \bigcap_l \overline{H}^j(I; H^{a(2a+s)}(\Omega)). \tag{6.11} \]

**Proof.** Statement 1° follows as in [23, Th. 5.6]. Statements 2° and 3° are shown in [23, Th. 5.8] to follow from the second estimate in \eqref{6.6} by use of the abstract result of Lions and Magenes [34] quoted as [23, Th. 5.7]. \( \square \)

Observe that the results allow high regularity in \( t \), but that the regularity in \( x \) is at most achieved up to \( s < \frac{3}{2} \) (since \( r < \frac{3}{2} \) for all \( a \in [0, 1] \)). We think that this is not just due to the method; there will in general be upper bounds on the \( x \)-regularity for general data, as studied in detail in Hölder spaces in [24].

We can now also derive some results where nonhomogeneous boundary conditions are included. Assume \( \tau > 1 + 2a \). Recall from Theorem 2.3 that \( \gamma_0^{a-1} \) has a continuous right inverse \( K_{(0)}^{a-1} \), mapping

\[ K_{(0)}^{a-1}: H^{s-a+\frac{1}{2}}(\partial \Omega) \to H^{(a-1)s}(\Omega), \quad \text{for } a - \frac{1}{2} < s < \tau + a - 1; \]

in particular, \( K_{(0)}^{a-1}: H^{a+\frac{1}{2}}(\partial \Omega) \to H^{(a-1)(2a)}(\Omega) \). \( \tag{6.12} \)
Now we need to assume \( a > \frac{1}{2} \) to end up in an \( L_2 \)-space, simply because \( d^{a-1} \) is only in \( L_2(\Omega) \) then; the factor \( d^{a-1} \) is provided by \( K^{a-1}_0 \) no matter how large \( s \) may be. Recall also the observation in (5.5) that for \( a > \frac{1}{2} \), \( H^{(a-1)(2a)}(\overline{\Omega}) \subset L_2(\Omega) \), and the description in Example 5.5 of how factors \( d^{a-1} \) come in. Then we can show the following theorem:

**Theorem 6.3.** Assume Hypothesis 3.1, For \( a > \frac{1}{2} \) and \( \tau > 2a + 1 \), consider the evolution problem

\[
Pu + \partial_t u = f \quad \text{on} \quad \Omega \times I,
\]

\[
u = 0 \quad \text{on} \quad (\mathbb{R}^n \setminus \Omega) \times I,
\]

\[
\gamma_0^{a-1} u = \psi \quad \text{on} \quad \partial\Omega \times I,
\]

\[
u|_{t=0} = 0.
\]  

(6.13)

For \( f(x,t) \) given in \( L_2(\Omega \times I) \), and \( \psi(x,t) \) given in \( L_2(I; H^{a+\frac{1}{2}}(\partial\Omega)) \cap \overline{H}^1 \) with \( \psi(x,0) = 0 \) (some \( \varepsilon > 0 \)), there is a unique solution \( u(x,t) \) of (6.13) satisfying

\[
u \in L_2(I; H^{(a-1)(2a)}(\overline{\Omega})) \cap \overline{H}^1(I; L_2(\Omega)).
\]  

(6.14)

**Proof.** Let \( v(x,t) = K^{a-1}_0 \psi(x,t) \); it lies in \( L_2(I; H^{(a-1)(2a)}(\overline{\Omega})) \cap H^1(\Omega) \) and in view of (6.12), contained in \( \overline{H}^1(I; L_2(\Omega)) \) by Lemma 5.3. It satisfies

\[
\gamma_0^{a-1} v = \psi, \quad v|_{t=0} = 0, \quad r^+ P v \in L_2(\Omega \times I), \quad \partial_t v \in L_2(\Omega \times I).
\]

Then \( w = u - v \) is in \( L_2(I; H^{(a-1)(2a)}(\overline{\Omega})) \) with \( \gamma_0^{a-1} w = 0 \), hence in \( L_2(I; H^{(a-2a)}(\overline{\Omega})) \) by Theorem 2.3. Moreover, \( (r^+ P + \partial_t)(u - v) \in L_2(\Omega \times I) \). Thus in order for \( u \) to solve (6.13), \( w \) must solve a problem (6.7) with homogeneous boundary condition and \( f \) replaced by \( f - (r^+ P + \partial_t) v \). Here Theorem 6.2 i° assures that there is a unique solution \( w \in L_2(I; H^{(2a)}(\overline{\Omega})) \cap \overline{H}^1(I; L_2(\Omega)) \). Then \( u = v + w \) is the unique solution of (6.13), satisfying (6.14). \( \square \)

Also the statements in Theorem 6.2 with higher derivatives in \( t \) can be extended to nonhomogeneous problems; we leave this to the interested reader. As for a lifting of the \( x \)-regularity, there is very little leeway (\( 0 \leq s < a - \frac{1}{2} \)), so we leave out details.

6.2. General \( q \in ]1,\infty[ \)

For general \( q \), the case of translation-invariant operators with real even homogeneous symbol has been treated in [22,23] for smooth domains \( \Omega \). We shall present some straightforward consequences for our types of nonsmooth domains, and then supply this with new results for nonhomogeneous boundary conditions.
Theorem 6.4. Assume Hypothesis 3.1, and assume moreover that the symbol \( p \) is independent of \( x \), and is real, even and homogeneous of degree \( 2a \).

1° The evolution problem (6.7) for \( P \) with homogeneous Dirichlet condition has for every \( f \in L_q(\Omega \times I) \) a unique solution

\[
    u \in L_q(I; H_q^{a(2a)}(\overline{\Omega})) \cap \overline{H_q^1}(I; L_q(\Omega)); \tag{6.15}
\]

moreover,

\[
    u \in \mathring{C}^0(I; L_q(\Omega)). \tag{6.16}
\]

2° Let \( s \in \mathbb{R}_+ \setminus \mathbb{N} \). Then when \( u \) solves (6.7) with \( I \) replaced by \( \mathbb{R}_+ \),

\[
    f \in \mathring{C}^s(\mathbb{R}_+; L_p(\Omega)) \iff u \in \mathring{C}^s(\mathbb{R}_+; H_q^{a(2a)}(\overline{\Omega})) \cap \mathring{C}^{s+1}(\mathbb{R}_+; L_q(\Omega)). \tag{6.17}
\]

Proof. 1° was proved for \( C^\infty \)-domains in [22, Th. 4.3], also recalled in [23, Th. 5.9]. Since the symbol is \( x \)-independent, the only new aspect in the nonsmooth case is that \( \Omega \) is allowed to be \( C^{1+\tau} \). The details of proof given in [22] are still valid in that case: Let \( k(y) = \mathcal{F}^{-1}p(\xi) \), it is homogeneous of degree \( -2a - n \), \( C^\infty \), positive and even for \( y \neq 0 \), and the sesquilinear form defining \( P_{D, 2} \) can be written as

\[
    Q(u, v) = \frac{1}{2} \int_{\mathbb{R}^n} (u(x) - u(y))(v(x) - v(y)) k(x - y) \, dx \, dy \text{ for } u, v \in \dot{H}^a(\overline{\Omega})
\]

(cf. also Ros-Oton [36]). As noted in Example 4.20, the operator \( P_{D, 2} \) it defines, as well as the operators \( P_{D, q} \), are bijective for all \( 1 < q < \infty \). Moreover, the quadratic form \( E(u) = Q(u, u) \) with domain \( D(E) = \dot{H}^a(\overline{\Omega}) \) has the Markovian property: When \( u_0 \) is defined from a real function \( u \in D(E) \) by \( u_0 = \min\{\max\{u, 0\}, 1\} \), then \( u_0 \in D(E) \) and \( E(u_0) \leq E(u) \). It is a so-called Dirichlet form, as explained in Fukushima, Oshima and Takeda [12], pages 4–5 and Example 1.2.1, and Davies [8]. (Such a Markovian property enters e.g. in [4] for the regional Dirichlet problem.) Then, by [12] Th. 1.4.1 and [8] Th. 1.4.1–1.4.2, \( -P_{D, q} \) generates a strongly continuous contraction semigroup \( T_q(t) \) not only in \( L_2(\Omega) \) for \( q = 2 \) but also in \( L_q(\Omega) \) for any \( 1 < q < \infty \), and \( T_q(t) \) is bounded holomorphic. Hereby we have the prerequisites to apply the theorem of Lambertson [33], which shows that (6.7) is solvable with \( u \in L_q(I; D(P_{D, q})) \cap \overline{H_q^1}(I; L_q(\Omega)) \).

Now we know moreover from (4.14) (based on Theorem 3.2) that \( D(P_{D, q}) = H_q^{a(2a)}(\overline{\Omega}) \), so 1° follows by insertion of this fact. Since \( D(P_{D, q}) \subset L_q(\Omega) \), the statement \( u \in \mathring{C}^0(I; L_q(\Omega)) \) follows from the continuity in \( t \in \mathcal{T} \) of functions in \( \overline{H_q^1}(I; X) \) valued in a Banach space \( X \).

2°. The details for this extension were given in [23, Sect. 5.3], where it is proved (by use of Hille and Phillips [28, Th. 17.5.1], also in Kato [32, Th. IX.1.23]) that the operator properties shown for \( P_{D, q} \) moreover imply a resolvent estimate

\[
    \langle \lambda \rangle \| (P_{D, q} - \lambda)^{-1} \|_{\mathcal{L}(L_q(\Omega))} \leq C, \tag{6.18}
\]
for \( \lambda \) outside a sectorial region like \( M \) (4.9). Then (6.17) follows as in [23, Th. 5.14] by use of a theorem of Amann [3] cited as [23, Th. 5.13]. \( \square \)

The analysis shows in particular that \( P_{D,q} \) has maximal \( L_q \)-regularity in \( I \) as defined e.g. in Denk and Seiler [9]. Also the other results of [23, Sect. 5.3] extend to nonsmooth \( \Omega \), with \( I = \mathbb{R}_+ \) or \([-0, T]\).

We can now moreover show results for the problem (6.13) with a nonhomogeneous local Dirichlet condition, when \( q < (1-a)^{-1} \):

**Theorem 6.5.** **Assumptions of Theorem 6.4.** If in addition \( \tau > 2a + 1 \) and \( q < (1-a)^{-1} \), the evolution problem (6.13) for \( P \) with nonhomogeneous Dirichlet condition has for every \( f \in L_q(\Omega \times I) \), \( \psi \in L_q(I; B_{q+1/q}'(\partial \Omega)) \cap \overline{H^1_q(I; B_{q}'(\partial \Omega))} \) with \( \psi|_{t=0} = 0 \) (some \( \varepsilon > 0 \)), a unique solution

\[
 u \in L_q(I; H_q^{(a-1)(2a)}(\Omega)) \cap \overline{H^1_q(I; L_q(\Omega))).
\] (6.19)

**Proof.** One proceeds exactly as in the proof of Theorem 6.3, eliminating the boundary condition by subtracting a lifting of the boundary value (using Theorem 2.3), and applying the result for the homogeneous case. \( \square \)

Consequences can be drawn as in Theorem 6.4 2° concerning higher time-derivatives.

Hereby all values of \( a \in ]0, 1[ \) can be included in the treatment of evolution problems with nonhomogeneous boundary conditions.

Theorems 6.4 and 6.5 apply to \( P = (-\Delta)^a \), and to all operators with symbols \( p(\xi) = g(\xi/|\xi|)|\xi|^{2a} \), where \( g(\eta) \) is a real positive even \( C^\infty \)-function on \( S^{n-1} \). (Note the explanation after (2.8) about how the nonsmoothness at \( \xi = 0 \) can be handled.)

**Data availability**

No data was used for the research described in the article.

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**Appendix A**

A.1. **Identification of weighted boundary maps**

In [1], Abatangelo showed an integral formula, that we recall here for functions \( u \) vanishing on \( \mathbb{R}^n \setminus \overline{\Omega} \), \( u \) and \( v \) being real:
Proposition A.1. [1] Let $\Omega$ be open, bounded and $C^{1,1}$. Let $u$ be a function such that $u \in C^{2a+\varepsilon}_{\text{loc}}(\Omega)$, $u/d^{a-1} \in C(\Omega)$ and $u = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$. Let $v \in C^a(\mathbb{R}^n)$ be such that $v = 0$ on $\mathbb{R}^n \setminus \overline{\Omega}$ and $r^+(-\Delta)^a v \in C_0^\infty(\Omega)$.

Define a boundary value of $u$ in the following way: With $G_{\Omega}(x, y)$ denoting the Green’s kernel for the restricted fractional Laplacian on $\Omega$ with homogeneous Dirichlet condition (i.e., the kernel of the operator solving (1.7) with $P = (-\Delta)^a$), let

$$M_{\Omega}(x, \theta) = \lim_{y \to \theta, y \in \Omega} \frac{G_{\Omega}(x, y)}{d(y)^a} \text{ for } x \in \Omega, \theta \in \partial \Omega, \quad m(x) = \int_{\partial \Omega} M_{\Omega}(x, \theta') d\sigma(\theta'),$$

and define the trace operator $E$ by

$$Eu(\theta) = \lim_{x \to \theta, x \in \Omega} \frac{u(x)}{d^{a-1}(x)m(x)}, \quad \theta \in \partial \Omega.$$

Then (cf. [1, (9)]):

$$\int_{\Omega} u(-\Delta)^a v \, dx - \int_{\Omega} (-\Delta)^a u v \, dx = \int_{\partial \Omega} Eu \gamma_0(\frac{\partial}{\partial \nu}) \, d\sigma. \quad (A.1)$$

It is stated in [1, Sect. 3.1], that $Eu \in C(\partial \Omega)$.

In [21] we showed a Green’s formula, which implies the following “halfway Green’s formula” (cf. Cor. 4.5 (4.34) there):

Proposition A.2. [21] Let $\Omega \subset \mathbb{R}^n$, open, bounded and smooth, and let $P$ be a classical pseudodifferential operator of order $2a > 0$ satisfying the $\alpha$-transmission condition at $\partial \Omega$ (it suffices for this that $P$ is even). If $u \in H^{(a-1)(s)}(\overline{\Omega})$, $v \in H^{a(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$, and $s \geq 2a$,

$$\int_{\Omega} u P^* v \, dx - \int_{\Omega} P u v \, dx = \Gamma(a)\Gamma(a + 1) \int_{\partial \Omega} s_0 \gamma_0(\frac{\partial}{\partial \nu}) \gamma_0(\frac{\partial}{\partial \nu}) \, d\sigma. \quad (A.2)$$

Here $s_0(x)$ is a function defined from the principal symbol of $P$; it is 1 when $P = (-\Delta)^a$. The spaces $H^{(a-1)(s)}(\overline{\Omega})$ and $H^{a(s)}(\overline{\Omega})$ are the solution spaces for the nonhomogeneous, resp. homogeneous, Dirichlet problem for $P$ in the scale of $L_2$-Sobolev spaces.

The formula holds a fortiori for functions in Hölder spaces

$$u \in C^s_{a-1(s+\varepsilon)}(\overline{\Omega}) \subset C^{s+\varepsilon}(\overline{\Omega}) + d^{a-1}e^+ C^{s-a+1+\varepsilon}(\overline{\Omega}),$$

$$v \in C^s_{a(s+\varepsilon)}(\overline{\Omega}) \subset C^{s+\varepsilon}(\overline{\Omega}) + d^{a}e^+ C^{s-a+\varepsilon}(\overline{\Omega}) \subset d^{a}e^+ C^{s-a+\varepsilon}(\overline{\Omega}); \quad (A.3)$$

here $d^{a-1}$ is $L_1$-integrable over $\Omega$, and we take small $\varepsilon > 0$ such that the indexations avoid integers.
The formula (A.2) can be extended to suitable nonsmooth domains by use of the solvability results in Section 3 above, but we shall not take up space here with details.

Applying (A.1) and (A.2) to $P = (-\Delta)^a = P^*$, we find that

$$
\int_{\partial \Omega} E u \gamma_0(\frac{u}{\delta^a}) d\sigma = \Gamma(a) \Gamma(a + 1) \int_{\partial \Omega} \gamma_0(\frac{u}{\delta^{a-\varepsilon}}) \gamma_0(\frac{u}{\delta^a}) d\sigma \quad (A.4)
$$

holds for real functions $u$ and $v$ satisfying the hypotheses for both propositions.

In view of (A.3) and the fact that $C^a_\phi(\Omega) \subset C^a_{\text{loc}}(\Omega)$, the requirements on $u$ in Proposition A.1 are satisfied when $u \in C^a_\phi(\Omega)$. As for $v$, we know from [19] that $v \in C^a_\phi(\Omega)$ with $(-\Delta)^a v \in C^\infty(\Omega)$ implies that $v \in \mathcal{E}_a(\Omega)$ (cf. (1.1)), and $v \in \mathcal{E}_a(\Omega)$ implies $s^+(\Delta)^a v \in C^\infty(\Omega)$, so the functions $v$ satisfying the requirement of Proposition A.1 are a subset of $\mathcal{E}_a(\Omega)$. We claim that one can conclude

$$
E u = c \gamma_0(\frac{u}{\delta^{a-\varepsilon}}), \quad c = \Gamma(a) \Gamma(a + 1). \quad (A.5)
$$

To show this, we still have to prove that $\gamma_0(v/\delta^a)$ assumes enough values to allow a passage from the weak identity (A.4) to the identity (A.5).

**Lemma A.3.** For the functions $v$ satisfying the hypotheses of Proposition A.1, $\gamma_0(v/\delta^a)$ runs through a dense subset of $L_2(\partial \Omega)$.

**Proof.** For small $\varepsilon' > 0$, $\overline{H}^{\frac{1}{2} - \varepsilon'}(\Omega)$ identifies with $\dot{H}^{\frac{1}{2} - \varepsilon'}(\Omega)$, and $C^\infty(\Omega)$ is dense in the space. The solution operator for the homogeneous Dirichlet problem maps this space bijectively onto $H^{a(2a + \frac{1}{2} - \varepsilon')}(\Omega)$. So when $(-\Delta)^a v$ runs through the dense subset $C^\infty(\Omega)$ of $\overline{H}^{\frac{1}{2} - \varepsilon'}(\Omega)$, $v$ runs through a dense subset of $H^{a(2a + \frac{1}{2} - \varepsilon')}(\Omega)$.

Here we have from [24] Th. 3.4 the precise statement (defining a direct sum) when $2a + \frac{1}{2} - \varepsilon' - a \in [\frac{1}{2}, \frac{3}{2})$, i.e., $0 < a - \varepsilon' < 1$:

$$
v \in H^{a(2a + \frac{1}{2} - \varepsilon')}(\Omega) \iff v = w + \delta^a K(0) \varphi, \ w \in \dot{H}^{a(2a + \frac{1}{2} - \varepsilon')}(\Omega), \ \varphi \in H^{a-\varepsilon'}(\partial \Omega),
$$

where $\varphi = \gamma_0(v/\delta^a)$, and $K(0)$ is a certain Poisson operator with $\gamma_0 K(0) = I$ (this was first shown in local coordinates in [19], Th. 5.4, that we could point to instead). Thus when $v$ runs through a dense subset of $H^{a(2a + \frac{1}{2} - \varepsilon')}(\Omega)$, $\varphi$ runs through a dense subset of $H^{a-\varepsilon'}(\partial \Omega)$. A fortiori, for the considered $v$, $\varphi = \gamma_0(v/\delta^a)$ runs through a dense subset of $L_2(\partial \Omega)$. \(\square\)

Then the conclusion of (A.5) from (A.4) follows, and we have obtained:

**Theorem A.4.** When $\Omega$ is bounded smooth, the boundary trace $E u$ introduced in [1] equals the constant $c = \Gamma(a) \Gamma(a + 1)$ times $\gamma_0(u/\delta^{a-1})$, cf. (A.5).
This holds for functions \( u \in C^{2a+\varepsilon}_0(\Omega) \) satisfying (A.3). More precisely, they are the solutions of the nonhomogeneous Dirichlet problem (1.10) with \( f \in C^\varepsilon(\Omega) \), \( \varphi \in C^{a+\varepsilon}(\partial \Omega) \).

A different proof is given in [6], referring to the Pohozaev formula shown by Ros-Oton and Serra in [37]. In [6], the constant \( c \) is stated to be equal to \( \Gamma(a+1)^2 \) (same constant as in the Pohozaev formula); it seems that a factor \( a^{-1} \) has been overlooked in the application of the boundary mapping to a derivative of \( u \). (This is confirmed in an arXiv posting 22.4.2022, arXiv:2004.04579v2.)

A.2. An embedding property in Hölder spaces

For a \( C^{1+\tau} \)-domain, the distance function \( d_0(x) \), equal to \( \text{dist}(x, \partial \Omega) \) near \( \partial \Omega \) and extended smoothly and positively to \( \Omega \), is a \( C^\tau \)-function on \( \overline{\Omega} \). If \( \tau \geq 1 \), \( d_0 \) is \( C^{\tau+1} \) (as recalled in Section 2).

**Lemma A.5.** Let \( \Omega \) be a \( C^{1+\tau} \)-domain, \( \tau > 0 \). Let \( a \) and \( b > 0 \) with \( a + b < 1 + \tau \). For \( a, b, a + b \notin \mathbb{N} \), there holds

\[
\dot{C}^{a+b}(\overline{\Omega}) \subset d_0(x)^a \dot{C}^b(\overline{\Omega}). \tag{A.6}
\]

**Proof.** The following calculations take place in a neighborhood of \( \partial \Omega \) where \( d_0(x) = \text{dist}(x, \partial \Omega) \). On the interior, \( v = u/d_0^a \) is \( C^b \) simply because \( u \) and \( d_0^{-a} \) are so.

1°. First consider the basic case where \( a + b < 1 \). Let \( u \in \dot{C}^{a+b}(\overline{\Omega}) \), and let \( v(x) = u(x)/d_0(x)^a \). For \( u \) we have

\[
|u(x) - u(y)| \leq C|x - y|^{a+b}, \quad |u(x)| \leq Cd_0(x)^{a+b}. \tag{A.7}
\]

For the various positions of \( x \) and \( y \), we distinguish the following two cases: a) \( |x - y| > \frac{1}{3}d_0(y) \), b) \( |x - y| \leq \frac{1}{3}d_0(y) \). We can assume \( 0 < d_0(x) \leq d_0(y) \) (and of course \( x \neq y \)). Denote by \( x_0 \) a point on \( \partial \Omega \) where \( |x - x_0| = d_0(x) \); then

\[
d_0(y) \leq |y - x_0| \leq |y - x| + |x - x_0| = |y - x| + d_0(x), \tag{A.8}
\]

hence \( |d_0(x) - d_0(y)| \leq |x - y| \).

a) \( |x - y| > \frac{1}{3}d_0(y) \). Here

\[
\left| \frac{v(x) - v(y)}{|x - y|^b} \right| = \left| \frac{u(x)/d_0(x)^a - u(y)/d_0(y)^a}{|x - y|^b} \right| \leq \frac{Cd_0(x)^b}{|x - y|^b} + \frac{Cd_0(y)^b}{|x - y|^b} \leq Cd_0(x)^b(\frac{1}{3}d_0(x))^{-b} + Cd_0(y)^b(\frac{1}{3}d_0(y))^{-b} = C'. \tag{A.9}
\]

b) \( |x - y| \leq \frac{1}{3}d_0(y) \). In view of (A.8) we have

\[
d_0(x) \geq d_0(y) - |x - y| \geq \frac{2}{3}d_0(y) \geq 2|x - y|.
\]
Now
\[
\frac{|v(x) - v(y)|}{|x - y|^b} = \frac{|u(x)/d_0(x)^a - u(y)/d_0(x)^a + u(y)/d_0(x)^a - u(y)/d_0(y)^a|}{|x - y|^b} \leq I + II,
\]
\[I = \frac{|u(x) - u(y)|d_0(x)^{-a}}{|x - y|^b}, \quad II = \frac{|u(y)||d_0(x)^{-a} - d_0(y)^{-a}|}{|x - y|^b}.
\]
(A.10)

For \(I\),
\[I \leq C|x - y|^a d_0(x)^{-a} \leq C\left(\frac{1}{2} d_0(x)^a\right) d_0(x)^{-a} = C''.
\]

For \(II\), denote \(d_0(x) = s, d_0(y) = t\) and \(r = 1 - s/t\). If \(d_0(x) = d_0(y)\), \(II = 0\), so we can assume \(s \neq t\). Here
\[0 < r = \frac{t - s}{t} \leq \frac{|x - y|}{3|x - y|} = \frac{1}{3}.
\]

Then in view of (A.7) and (A.8),
\[II \leq C t^{a+b} \frac{|s^{-a} - t^{-a}|}{|s - t|^b} = C \frac{|1 - (s/t)^{-a}|}{|1 - s/t|^b} = C \frac{|1 - (1 - r)^{-a}|}{r^b}
\]
\[= C \frac{|ar + O(r^2)|}{r^b} \leq C''', \quad (A.11)
\]
since \(r \in [0, \frac{1}{3}]\) and \(b < 1\) (we have used a Taylor expansion of \((1 - r)^{-a}\)). This shows that \(v \in C^b(\Omega)\). Since \(v(x)\) is \(O(d_0(x)^b)\) with \(b > 0\), \(v\) vanishes at \(\partial \Omega\), so in fact, \(v \in C^b(\Omega)\).

2°. Next, consider cases where \(a\) is free, but \(b\) is still assumed to be \(< 1\). So \(a + b \in [k, k + 1]\), where the case \(k = 0\) was treated above. Let \(k = 1\). In (A.7), the estimates are replaced by
\[|u(x) - u(y)| \leq C|x - y|, \quad |\nabla u(x) - \nabla u(y)| \leq C|x - y|^{a+b-1},
\]
\[|u(x)| \leq C d_0(x)^{a+b}, \quad |\nabla u(x)| \leq C d_0(x)^{a+b-1}.
\]
(A.12)

The calculation (A.9) carries over verbatim, and so does the treatment of \(II\) defined in (A.10). Only \(I\) needs a modified argument: There holds for \(\theta \in [0, 1]\):
\[|d_0(x + \theta(y - x)) - d_0(x)| \leq |x + \theta(y - x) - x| \leq |x - y|,
\]
hence since \(|x - y| \leq \frac{1}{2} d_0(x)\),
\[|u(x) - u(y)| = |(x - y) \cdot \int_0^1 \nabla u(x + \theta(y - x)) d\theta| \leq C|x - y| \sup_\theta (d_0(x + \theta(y - x))^{a+b-1}
\]
\[\leq C|x - y|(d_0(x) + |x - y|)^{a+b-1} \leq C'\|x - y\|d_0(x)^{a+b-1}.
\]
Thus

\[ I \leq C'|x - y|d_0(x)^{a+b-1}d_0(x)^{-a}|x - y|^{-b} = C'|x - y|^{1-b}d_0(x)^{b-1} \leq C'(1)^{b-1}. \]

For higher \( k \), one similarly uses Taylor’s formula on the \( k \)'th level.

3°. Finally, let also \( b > 1 \). In this case, \( \tau > 1 \), so \( d_0(x) \in C^{1+\tau}(\Omega) \). When \( b \in |k,k+1| \), it is the \( k \)'th derivatives \( \partial^\alpha v \) with \(|\alpha| = k\) that we have to estimate. E.g., for \( k = 1 \),

\[ \partial_j v = \partial_j(ud^{-a}) = \partial_j u d^{-a} + u(-a)d^{-a-1} \partial_j d. \]

For \( \partial_j u d^{-a} \), the result follows by application of \( 2^a \) to \( \partial_j u \in \dot{C}^{a+b-1} \), this gives an element of \( \dot{C}^{b-1} \). For \( u d^{-a} \partial_j d \), we likewise apply \( 2^a \) to find an element of \( \dot{C}^{b-1}(\Omega) \), since higher positive \( a \) are allowed, and multiplication by \( \partial_j d \in C^{\tau}(\Omega) \) preserves being in \( \dot{C}^{b-1}(\Omega) \). Altogether, \( \partial_j v \in \dot{C}^{b-1}(\Omega) \), and hence \( v \in \dot{C}^b(\Omega) \).

Passing to \( k = 2 \), we use the information from the cases \( k = 0 \) and \( 1 \). Then we go on successively to all the relevant larger \( k \); by the Leibniz formula there are more and more elements to account for, but the principle is the same. □

The lemma can be extended to the distance function \( d(x) = x_n - \zeta(x') \) defined for the set \( \mathbb{R}^n_\zeta = \{ x = (x',x_n) \mid x_n > \zeta(x') \} \), \( \zeta \in C^{1+\tau}(\mathbb{R}^{n-1}) \); this is useful when \( \tau < 1 \), since the regularity results in the main text are primarily formulated with \( d \) rather than \( d_0 \) then.

**Lemma A.6.** Let \( \tau > 0 \) and consider the curved halfspace \( \mathbb{R}^n_\zeta = \{ x = (x',x_n) \mid x_n > \zeta(x') \} \), defined from a function \( \zeta \in C^{1+\tau}(\mathbb{R}^{n-1}) \), and provided with the distance function \( d(x) = x_n - \zeta(x') \) for \( x_n \leq K + 1 \) (some \( K \geq \sup |\zeta| \)).

Let \( a \) and \( b > 0 \) with \( a + b < 1 + \tau \). For \( a, b, a + b \notin \mathbb{N} \), one has near \( \partial \mathbb{R}^n_\zeta \) that

\[ u \in \dot{C}^{a+b}(\mathbb{R}^n_\zeta) \implies u(x)/d(x)^a \in \dot{C}^b(\mathbb{R}^n_\zeta). \quad (A.13) \]

When \( b < \tau \), the same result holds if \( d \) is replaced by another distance function \( d' \) in \( C^{1+\tau}(\mathbb{R}^n_\zeta) \) bounded above and below by \( d \).

**Proof.** We shall show how the proof of Lemma A.5 is generalized to this case. Assume \( a + b < 1 \). Let \( u \in \dot{C}^{a+b}(\mathbb{R}^n_\zeta) \), and let \( v(x) = u(x)/d(x)^a \) (where \( d(x',x_n) = x_n - \zeta(x') > 0 \)). For \( u \) we have

\[ |u(x) - u(y)| \leq C|x - y|^{a+b}, \quad |u(x)| \leq C d(x)^{a+b}. \quad (A.14) \]

Let \( x \) and \( y \) be points with \( d(y) > d(x) > 0 \). Since \( \zeta \in C^{1+\tau}(\mathbb{R}^{n-1}) \), there is a \( c_1 \) such that \(|\zeta(y') - \zeta(x')| \leq c_1|y' - x'| \). For the given \( x = (x',x_n) \), denote \((x',\zeta(x')) = x_0 \), with a similar notation for \( y \). \( x_0 \) is the “footpoint” on \( \partial \mathbb{R}^n_\zeta \), i.e. the point with the same \( x' \)-value as \( x \), so that \( d(x) = |x - x_0| \).
We then have
\[ |y - x_0| \leq |y - x| + |x - x_0| = |y - x| + d(x). \]
Moreover,
\[ |y - x_0| = |y - y_0 + y_0 - x_0| \geq d(y) - |y_0 - x_0| \geq d(y) - (1 + c_1)|x' - y'|. \]
Then if we assume \(|x - y| \leq \frac{1}{3}(2 + c_1)^{-1}d(y)|\), then
\[ d(x) \geq |y - x_0| - |y - x| \geq d(y) - (2 + c_1)|y - x| \geq \frac{2}{3}d(y). \tag{A.15} \]

Now proceed as in the proof of Lemma A.5:
(a) The case \(|x - y| > \frac{1}{3}(2 + c_1)^{-1}d(y)|\). Here the desired estimate is obtained as in a) of Lemma A.5.
(b) The case \(|x - y| \leq \frac{1}{3}(2 + c_1)^{-1}d(y)|\). In view of (A.15) we have
\[ d(x) \geq \frac{2}{3}d(y) \geq \frac{2}{3}(2 + c_1)|x - y|. \]

With these inequalities, the proof of b) in Lemma A.5 goes through, just with other constants.
In this way the desired result is obtained for \(a + b < 1\); and it extends to more general values in the same way as in Lemma A.5.

The last statement follows from the fact that \(d/d'\) and \(d'/d\) are \(C^\tau\); it is included to allow general distance functions as introduced around (2.2). \(\square\)

References
