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The Diffeomorphism Group of the Solid Closed Torus and Hochschild Homology

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We prove that for a self-injective ribbon Grothendieck-Verdier category $\mathcal{C}$ in the sense of Boyarchenko-Drinfeld the cyclic action on the Hochschild complex of $\mathcal{C}$ extends to an action of the diffeomorphism group of the solid closed torus $S^1 \times D^2$.

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1 Introduction and summary

The Hochschild complex of any associative algebra $A$ in a suitable (higher) symmetric monoidal category can be defined as the homotopy colimit of the simplicial object

\[
\ldots \to A \otimes A \otimes A \to A \otimes A \to A \ ,
\]

whose face maps use the product of $A$ while the degeneracy maps insert units. For an algebra in vector spaces, the homology groups of the chain complex corresponding to (1.1) are the familiar Hochschild homology groups. The cyclic permutation of the tensor copies of $A$ in each degree induces a cyclic symmetry, i.e. an action of the topological group $S^1$. The homotopy orbits of this action are known as cyclic homology. The study of cyclic homology was initiated in the 1980s independently by Connes \cite{Con83} and Tsygan \cite{Tsy83}. Of course, instead of considering the Hochschild complex of an algebra, we can consider the Hochschild complex of a linear or a differential graded category.

The fact that one encounters in almost all areas of mathematics structured algebras naturally leads to the following question:
This short article offers a different and, at least at first sight, surprising answer to question (Q) that applies to an important structure in quantum algebra, namely ribbon Grothendieck-Verdier duality. Every rigid monoidal category can be seen as a Grothendieck-Verdier category whose duality functor sends an object to its dual, but not all Grothendieck-Verdier categories are of this form. This extends the theory of cyclic homology and leads to dihedral homology as introduced by Loday [Lod87]. A very systematic approach to the Hochschild homology of structured algebras is given by Wahl and Westerland in [WW16, Wah16].

Grothendieck-Verdier duality allows us to generalize the notion of a \textit{finite tensor category} [EO04] which includes \textit{rigidity} (existence of left/right duals) as part of its definition, i.e. it requires that every object \( X \) has a dual \( X^\vee \) that comes with an evaluation \( X^\vee \otimes X \to I \) and a coevaluation \( I \to X \otimes X^\vee \) subject to the so-called zigzag identities. The notion of a finite tensor category is the backbone of the approach to quantum algebra laid out in the monograph [EGNO15] by Etingof, Gelaki, Ostrik and Nikshych. Every rigid monoidal category can be seen as a Grothendieck-Verdier category whose duality functor sends an object to its dual, but not all Grothendieck-Verdier categories are of this form.

We will be interested in Grothendieck-Verdier categories in a \textit{k-linear setting}, where \( k \) is a fixed algebraically closed field, more precisely in ribbon Grothendieck-Verdier categories in the symmetric monoidal bicategory \( \text{Lex}^k \) of \( k \)-linear abelian categories (subject to some finiteness conditions), left exact functors and natural transformations. Moreover, we will require \textit{self-injectivity}, i.e. the projective objects must coincide with the injective ones.

Our main result is concerned with the Hochschild complex of a self-injective ribbon Grothendieck-Verdier category \( C \) in \( \text{Lex}^k \). The Hochschild complex of \( C \) is, as usual, the realization of the simplicial vector space

\[
\cdots \longrightarrow \bigoplus_{x_0, x_1 \in \text{Proj} C} C(x_1, x_0) \otimes C(x_0, x_1) \longrightarrow \bigoplus_{x_0 \in \text{Proj} C} C(x_0, x_0).
\]

The fact that just the projective objects are used to define the complex is standard in this context. It ensures that if \( C \) is given, as a linear category, by finite-dimensional modules over a finite-dimensional algebra \( A \), the above Hochschild complex is actually equivalent to the ‘standard one’ of \( A \) by the so-called \textit{Agreement Principle} [McC94, Kol99]. One may see the Hochschild complex as the homotopy coend \( \int^X_{x \in \text{Proj} C} C(X, X) \) over the endomorphism spaces of projective objects, and we will use this as our notation for the Hochschild complex. We may now state our main result:

**Theorem 3.6.** Let \( C \) be a self-injective ribbon Grothendieck-Verdier category in \( \text{Lex}^k \). Then its duality functor and its balancing induce on the Hochschild complex \( \int^X_{x \in \text{Proj} C} C(X, X) \) an action of the diffeomorphism group \( \text{Diff}(S^1 \times \mathbb{D}^2) \) of the solid closed torus that extends the usual cyclic agreement.
symmetry of the Hochschild complex.

Here and elsewhere in the article, diffeomorphisms and mapping classes will always be orientation-preserving. Moreover, a group action on an object in a higher category, such as chain complexes, has to be understood as a homotopy coherent action.

We prove in Corollary 3.10 that generally the $\text{Diff}(S^1 \times D^2)$-action depends on the ribbon Grothendieck-Verdier structure, i.e. in contrast to the Hochschild complex and its cyclic action, it is sensitive to more than the linear structure.

Let us highlight concrete situations to which Theorem 3.6 applies:

- We had mentioned above that the result applies in particular to finite ribbon categories in the sense of [EGNO15] — by definition these categories are always rigid. A finite ribbon category can be obtained, for example, by taking finite-dimensional modules over a finite-dimensional ribbon Hopf algebra [Kas15, XIV.6]. In that case, the Hochschild complex $\int_X \mathcal{C}(X, X)$ is equivalent to the Hochschild complex of $A$. This implies that the Hochschild complex of a finite-dimensional ribbon Hopf algebra comes with a $\text{Diff}(S^1 \times D^2)$-action, see Example 3.8 for more details.

- While the case of finite ribbon categories and in particular ribbon Hopf algebras exhibits a rich class of examples, it is important that the notion of Grothendieck-Verdier duality is designed to go beyond rigid monoidal categories. For example, Allen, Lentner, Schweigert and Wood prove in [ALSW21] that suitable choices of modules over a vertex operator algebra lead to ribbon Grothendieck-Verdier categories (this includes categories with a non-exact monoidal product which therefore cannot be rigid). To these categories, Theorem 3.6 may also be applied if they are self-injective.

Using our main result we can also exhibit at least one instance in which there is a non-trivial action of the higher homotopy groups of diffeomorphism groups on so-called differential graded conformal blocks, see Remark 3.9. To the best of our knowledge, this is a phenomenon that has not been observed so far.

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2 The diffeomorphism group of $S^1 \times D^2$ via dihedral homology

In this section, we give a specific model for the diffeomorphism group of $S^1 \times D^2$ using dihedral homology. This will in the next section allow us to write its classifying space $B\text{Diff}(S^1 \times D^2)$ in a way that is adapted to the algebraic structure of a self-injective ribbon Grothendieck-Verdier category.

Let us first recall some well-known facts about diffeomorphism groups of handlebodies. Throughout the article, handlebodies will be three-dimensional, and diffeomorphisms and their mapping classes will be orientation-preserving. Since any handlebody $H$ is a Haken manifold, the restriction $\text{Diff}_0(H) \to \text{Diff}_0(\partial H)$ is a fibration with contractible fiber by [Hat76, Theorem 2] (by $\text{Diff}_0$ we denote the identity component of $\text{Diff}$). Unless $H$ is $H_{0,0} = S^3$ or $H_{1,0} = S^1 \times D^2$ (we use here $H_{g,n}$ to denote the handlebody of genus $g$ and $n$ embedded disks), this proves that $\text{Diff}(H)$ is homotopy discrete, i.e. that the map $\text{Diff}(H) \to \text{Map}(H)$ is a homotopy equivalence (because $\text{Diff}(\partial H)$ is
homotopy discrete in these cases [EE67]). It also tells us $\text{Diff}_0(H_{1,0}) \simeq \mathbb{T}^2$ thanks to $\text{Diff}_0(\mathbb{T}^2) \simeq \mathbb{T}^2$ [Gra73, Théorème 1]. Next recall that we have an exact sequence

$$0 \to \text{Diff}_0(H_{1,0}) \to \text{Diff}(H_{1,0}) \to \text{Diff}(H_{1,0})/\text{Diff}_0(H_{1,0}) \to 0,$$

(2.1)

where $\text{Diff}(H_{1,0})/\text{Diff}_0(H_{1,0}) = \text{Map}(H_{1,0})$ is the mapping class group of $H_{1,0}$. The mapping class group $\text{Map}(H_{1,0}) \cong \mathbb{Z} \times \mathbb{Z}_2$ is generated by a Dehn twist $T$ along any properly embedded disk in $H_{1,0}$ and the rotation $R$ by $\pi$ around any axis in the plane in which the torus lies [Waj98, Theorem 14]. Under the isomorphism $\text{Map}(\mathbb{T}^2) \cong \text{SL}(2, \mathbb{Z})$ sending a mapping class to the induced automorphism on the first homology of $\mathbb{T}^2$, the inclusion $\text{Map}(H_{1,0}) \subset \text{Map}(\mathbb{T}^2)$ sends

$$T \mapsto \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad R \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$  

(2.2)

Through the matrix representation (2.2) of $\mathbb{Z} \times \mathbb{Z}_2$, we have a section of the epimorphism in (2.1). This gives us $\text{Diff}(H_{1,0}) \cong \text{Diff}_0(H_{1,0}) \times \text{Map}(H_{1,0})$. The connected component $\text{Diff}_0(H_{1,0})$ is homotopy equivalent to a torus $\mathbb{T}^2$ as just explained. The action of $\text{Map}(H_{1,0})$ on $\mathbb{T}^2$ sees $\text{Map}(H_{1,0})$ as subgroup of $\text{SL}(2, \mathbb{Z})$ which acts on $\mathbb{R}^2/\mathbb{Z}^2 = \mathbb{T}^2$.

In order to present our combinatorial model for $\text{Diff}(H_{1,0})$, we also need to recall the notion of dihedral homology: Recall that Connes’ cyclic category $\Delta$ [Con83] is the category with objects $\mathbb{N}_0$; we denote the object corresponding to $n \geq 0$ by $[n]$. A morphism $f : [n] \to [m]$ is given by an equivalence class of functions $f : \mathbb{Z} \to \mathbb{Z}$ such that $f(i + n + 1) = f(i) + m + 1$ modulo the relation $f \sim g$ if $f - g$ is a constant multiple of $m + 1$. The category $\Delta$ contains the simplex category $\Delta$ as subcategory. As generating morphisms, it has the face and degeneracy maps that we already know from the simplex category $\Delta$ and the cyclic permutations $\tau_n : [n] \to [n]$ represented by maps $\mathbb{Z} \to \mathbb{Z}$ that shift by one. The cyclic permutations fulfill, besides the obvious relation $\tau_n^{n+1} = \text{id}_{[n]}$, further compatibility relations with the face and degeneracy maps. We denote by $\Lambda \subset \Delta$ the subcategory of the cyclic category without degeneracy maps. This subcategory inclusion is homotopy initial, thereby making $\Lambda^{op} \subset \Lambda^{op}$ homotopy final. We use here the terminology of [Rie14, Chapter 8.5].

The category $\Lambda$ has a natural action of $\mathbb{Z}_2$ through the reversal functor $r : \Delta \to \Delta$ which is the identity on objects and sends any morphism $f : [n] \to [m]$ in $\Delta$ to $r(f) : [n] \to [m]$ given by $(r(f))(p) := m - f(n - p)$. We denote by $\Lambda \rtimes \mathbb{Z}_2$ the Grothendieck construction of the functor $* \otimes \mathbb{Z}_2$ and $\text{Cat}$ from the category of categories $\text{Cat}$ to the group $\text{Cat}$ of $\text{Cat}$-valued functors $* \otimes \mathbb{Z}_2$ and $\text{Cat}$ to the category $\text{Cat}$ of categories sending $*$ to $\Delta$ and the generator $-1 \in \mathbb{Z}_2$ to the reversal functor $r : \Delta \to \Delta$ (recall that the Grothendieck construction $F : \mathbb{C} \to \text{Cat}$ is the category of pairs $(c, x)$ formed by all $c \in \mathbb{C}$ and $x \in F(c)$, see e.g. [LM92, Section I.5]). The category $\Lambda \rtimes \mathbb{Z}_2$ can be identified with the dihedral category [Lod87, Spa00]. Restriction to $\Lambda$ yields a functor $\bar{r} : \Lambda \to \Lambda$.

This allows us to define $\Lambda \rtimes \mathbb{Z}_2$, the semidihedral category, also via a Grothendieck construction. Functors out of the opposite categories of $\Lambda \rtimes \mathbb{Z}_2$ and $\Lambda \rtimes \mathbb{Z}_2$ are called dihedral objects and semidihedral objects, respectively. We can see $(\Lambda \rtimes \mathbb{Z}_2)^{op}$ as the Grothendieck construction of the $\mathbb{Z}_2$-action on $\Lambda^{op}$ through the functor $\bar{r}^{op} : \Lambda^{op} \to \Lambda^{op}$ induced by $r$. In other words, $(\Lambda \rtimes \mathbb{Z}_2)^{op} = \Lambda^{op} \rtimes \mathbb{Z}_2$. Similarly, $(\Lambda \rtimes \mathbb{Z}_2)^{op} = \Lambda^{op} \rtimes \mathbb{Z}_2$.

Given an associative algebra $A$ in a symmetric monoidal $\infty$-category $S$, one can build its Hochschild object, i.e. the simplicial object in $S$ given in (1.1). If we assume that $S$ is cocomplete, we may take the homotopy colimit of (1.1) and obtain the Hochschild homology of $A$ (of course, unless the target category is chain complexes, this will not give us homology in the traditional sense of the word, but this extended meaning of the word ‘homology’ is standard). In fact, the Hochschild object is actually a cyclic object through the cyclic permutation of the tensor factors. If we take the homotopy colimit over $\Lambda^{op}$, we obtain the cyclic homology of $A$ [Con83, Tsy83]. If $A$ comes equipped with a $\mathbb{Z}_2$-action through anti algebra maps, then the Hochschild object of $A$ actually extends to a dihedral object. One defines the homotopy colimit over $\Lambda^{op} \rtimes \mathbb{Z}_2$ as
the dihedral homology of A [Lod87, Spa00]. We denote it by \(\text{DH}(A)\). Thanks to the homotopy finality statements given above, the homotopy colimits involved in the computation of Hochschild homology, cyclic homology and dihedral homology may always be computed without degeneracies.

Before stating the next result, let us introduce further notation: For a space \(X\) with \(G\)-action, we will denote by \(X_{h\!G}\) the homotopy orbits of the \(G\)-action on \(X\). Moreover, we denote by \(K(G, n)\) the \(n\)-th Eilenberg-Mac Lane space for the group \(G\) (for \(n \geq 2\), this means that \(G\) is abelian).

**Lemma 2.1.** Consider the commutative topological algebra \(S^1\) together with its trivial \(\mathbb{Z}_2\)-action. There is a homotopy equivalence

\[
\text{DH}(S^1) \simeq \text{Map}(S^1, K(\mathbb{Z}, 2))_{h\!S^1 \rtimes \mathbb{Z}_2},
\]

where \(S^1 \rtimes \mathbb{Z}_2\) (the \(\mathbb{Z}_2\)-action on \(S^1\) is by reflection) acts on the mapping space \(\text{Map}(S^1, K(\mathbb{Z}, 2))\) as follows:

(i) The \(S^1\)-action on \(\text{Map}(S^1, K(\mathbb{Z}, 2))\) comes from the natural \(S^1\)-action on \(S^1\) and precomposition.

(ii) The \(\mathbb{Z}_2\)-action on \(\text{Map}(S^1, K(\mathbb{Z}, 2))\) is given by precomposition with a reflection and postcomposition with the map \(K(\mathbb{Z}, 2) \to K(\mathbb{Z}, 2)\) induced by \(-1 : \mathbb{Z} \to \mathbb{Z}\).

**Proof.** Since the geometric realization of \(\tilde{A}^{\text{top}} \times \mathbb{Z}_2\) is equivalent to \(S^1 \times \mathbb{Z}_2\) [Lod87, Proposition 3.11], we may describe the dihedral homology of \(S^1\) as the homotopy \(S^1 \rtimes \mathbb{Z}_2\)-orbits of the Hochschild homology of \(S^1\). This follows from the \(\mathbb{Z}_2\)-equivariant analogue of [NS18, Proposition B.5]. As a consequence, \(\text{DH}(S^1) \simeq (\int_{S^1} \mathbb{S}^1)_{h\!S^1 \rtimes \mathbb{Z}_2}\), where \(\int_{S^1} \mathbb{S}^1\) is the factorization homology of the commutative algebra \(\mathbb{S}^1\) evaluated on the circle. The \(S^1 \rtimes \mathbb{Z}_2\)-action on \(\int_{S^1} \mathbb{S}^1\) can be naturally understood as follows [AF15, Example 2.11]: The space \(\mathbb{S}^1 = K(\mathbb{Z}, 1)\) is an ‘unoriented topological \(E_1\)-algebra’, i.e. an \(E_1\)-algebra with an anti algebra involution. The involution is trivial in this case (the identity of \(\mathbb{S}^1 = K(\mathbb{Z}, 1)\) is indeed an anti algebra involution since \(\mathbb{Z}\) is abelian). This allows us to compute its factorization homology over any unoriented 1-dimensional manifold. The value of factorization homology \(\int_{S^1} K(\mathbb{Z}, 1)\) on \(\mathbb{S}^1\) is the Hochschild complex. Hence, the group of not necessarily orientation preserving diffeomorphisms of \(\mathbb{S}^1\), which is homotopy equivalent to \(\text{O}(2) = S^1 \rtimes \mathbb{Z}_2\), acts on the factorization homology \(\int_{S^1} \mathbb{S}^1\), thereby giving rise to the action appearing in the computation of dihedral homology.

We have furthermore \(\mathbb{S}^1 = K(\mathbb{Z}, 1) = \Omega K(\mathbb{Z}, 2)\) as algebras (by \(\Omega\) we denote the based loop space). This also holds as algebras with anti algebra involution if we use for \(\Omega K(\mathbb{Z}, 2)\) the involution given by the reflection of based loops together with the involution on \(K(\mathbb{Z}, 2)\) described in (ii) (see also [AF15, Section 4]). With these definitions, the \(\mathbb{Z}_2\)-action on \(\Omega K(\mathbb{Z}, 2)\) is indeed trivial. This ensures that the identification \(\mathbb{S}^1 = K(\mathbb{Z}, 1) = \Omega K(\mathbb{Z}, 2)\) is really compatible with the \(\mathbb{Z}_2\)-structure. Now non-abelian Poincaré duality [AF15, Corollary 4.6] gives us \(\int_{S^1} \mathbb{S}^1 = \int_{S^1} \Omega K(\mathbb{Z}, 2) \simeq \text{Map}(\mathbb{S}^1, K(\mathbb{Z}, 2))\), and the last homotopy equivalence is in fact \(\mathbb{S}^1 \rtimes \mathbb{Z}_2\)-equivariant if \(\text{Map}(\mathbb{S}^1, K(\mathbb{Z}, 2))\) is equipped with the \(\mathbb{S}^1 \rtimes \mathbb{Z}_2\)-action described in (i) and (ii). For the \(\mathbb{Z}_2\)-action, this was just explained; for the \(\mathbb{S}^1\)-action, it is a property of the Poincaré duality map.

**Theorem 2.2.** There is a homotopy equivalence of topological spaces

\[
\text{DH}(S^1) \simeq B\text{Diff}(H_{1,0}).
\]

**Proof.** The strategy of the proof is to explicitly compute the homotopy orbits of the \(S^1 \rtimes \mathbb{Z}_2\)-action on the mapping space \(\text{Map}(S^1, K(\mathbb{Z}, 2))\) from Lemma 2.1. To this end, we will use that \(S^1\) and \(K(\mathbb{Z}, 2)\) are both realizations of 2-groupoids; in fact, \(S^1\) is the realization of \(*//\mathbb{Z}\) (one object plus morphisms given by \(\mathbb{Z}\)), and \(K(\mathbb{Z}, 2)\) is the realization of \(*//\mathbb{Z}\) (one object plus one 1-morphism
and 2-morphisms given by $Z$). The mapping space $\text{Map}(S^1, K(\mathbb{Z}, 2))$ can be described as the 2-groupoid $\ast//Z, \ast//Z$ of functors $\ast//Z \longrightarrow \ast//Z$, natural transformations and modifications. Without loss of generality, we can assume that units are preserved strictly. Up to equivalence, $\ast//Z, \ast//Z$ has only one object given by the constant functor $c_\ast$ at $\ast$. A natural transformation $\sigma : c_\ast \longrightarrow c_\ast$ consists of the unique 1-morphism $\sigma : c_\ast(\ast) \longrightarrow c_\ast(\ast)$ together with a 2-morphism $\sigma_n : c_\ast(n) \circ \sigma \longrightarrow \sigma \circ c_\ast(n)$ for every $n \in \mathbb{Z}$. This 2-morphism is an element of $Z$, and since $\sigma$ is supposed to be natural, the map $n \longmapsto \sigma_n$ is a group morphism $Z \longrightarrow Z$. Hence, the 1-morphisms of $\ast//Z, \ast//Z$ correspond each to an element in $Z$. Let $\sigma$ and $\sigma'$ be natural transformations as above. A modification $\omega : \sigma \longrightarrow \sigma'$ consists of a 2-morphism $\omega : \sigma_n \longrightarrow \sigma'_n$ such that $\sigma'_n \circ \omega = \omega \circ \sigma_n$. This implies that 2-morphisms between $\sigma$ and $\sigma'$ only exist if $\sigma' = \sigma$, and in this case, all values of $\omega \in Z$ are allowed. The composition is straightforward to compute, and we find $\ast//Z, \ast//Z \simeq (\ast//Z) \times (\ast//Z)$. As a topological space, this is of course $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2)$.

Next we describe the $S^1 \times Z_2$-action on this space:

- The $S^1$-action is described by a natural automorphism of the identity $S : \text{id}(\ast//Z) \longrightarrow \text{id}(\ast//Z) \times (\ast//Z)$. Concretely, $S$ consists of a 1-morphism $S_n : n \longrightarrow n$ which is given by $n$. This follows directly from unpacking the bicategorical definitions for the precomposition with the natural isomorphism $s : \text{id}(\ast//Z) \longrightarrow \text{id}(\ast//Z)$ whose value at the only object $\ast$ is $s_\ast = 1 \in Z$.

- The $Z_2$-action is given by the involution $(\ast//Z) \times (\ast//Z) \longrightarrow (\ast//Z) \times (\ast//Z)$ whose action on 1-morphisms is trivial and multiplies 2-morphisms by $-1$. This follows by unpacking the definition of the action by precomposition with the functor $\ast//Z \longrightarrow \ast//Z$ given by $-1$ on 1-morphisms combined with postcomposition by the functor $\ast//Z \longrightarrow \ast//Z$ given by $-1$ on 2-morphisms.

After combining this with Lemma 2.1, we arrive at

$$\mathcal{DH}(S^1) \simeq (K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2))_{S^1 \times Z_2},$$

and hence $\Omega \mathcal{DH}(S^1) \simeq (S^1 \times Z) \times (S^1 \times Z_2)$, where $S^1 \times Z_2$ acts on $S^1 \times Z$ as follows:

- $Z_2$ acts trivially on $Z$ and by reflection on $S^1$,
- and $x \in \mathbb{R}/Z = S^1$ acts trivially on $Z$ and sends $(0, n) \in S^1 \times Z$ to $(nx, n)$.

The group $(S^1 \times Z) \times (S^1 \times Z_2)$ is isomorphic to $\text{Diff}(H_1, n)$ after a straightforward rewriting. This concludes the proof.

### 3 The Hochschild complex of a self-injective ribbon Grothendieck-Verdier category

In this section, we recall the notion of Grothendieck-Verdier duality from [BD13] and prove the main result. We use here the conventions from [MW20] which are dual to the ones from [BD13].

**Definition 3.1.** A *Grothendieck-Verdier category* is a monoidal category $\mathcal{C}$ with monoidal product $\otimes$ and an object $K \in \mathcal{C}$ such that for all $X \in \mathcal{C}$ the hom functor $\mathcal{C}(K, X \otimes -)$ is representable (let us denote the representing object by $DX \in \mathcal{C}$; this means that we have $\mathcal{C}(K, X \otimes -) \cong \mathcal{C}(DX, -)$), and such that the functor $\mathcal{C} \longrightarrow \mathcal{C}^{\text{opp}}$ sending $X$ to $DX$ is an equivalence. One calls $K$ the *dualizing object* and $D$ the *duality functor*.

Building on the notion of a braiding and a balancing on a monoidal category (both notions were recalled in the introduction), we may now define:
**Definition 3.2.** A ribbon Grothendieck-Verdier category is a Grothendieck-Verdier category whose underlying monoidal category is equipped with a braiding and a balancing such that $\theta_{D_X} = D\theta_X$ for $X \in \mathcal{C}$.

We will consider ribbon Grothendieck-Verdier categories in a linear setting. To this end, let us establish some terminology: For an algebraically closed field $k$ that we fix for the rest of the article, a finite category is a $k$-linear abelian category with finite-dimensional morphism spaces, finitely many isomorphism classes of simple objects and enough projective objects; additionally, one requires every object to have finite length. One can now define a symmetric monoidal bicategory $\text{Lex}^f$ of finite linear categories, left exact functors and natural transformations. The monoidal product is the Deligne product. We refer e.g. to [FSS20] for an overview.

**Definition 3.3.** A ribbon Grothendieck-Verdier category in $\text{Lex}^f$ is an object $\mathcal{C} \in \text{Lex}^f$ equipped with a ribbon Grothendieck-Verdier structure on the underlying category and a lift of the ribbon Grothendieck-Verdier structure to structure inside $\text{Lex}^f$.

This means in particular that the monoidal product will be left exact by construction. This might seem a little confusing because monoidal products are rather right exact than left exact in practice. But note that for a Grothendieck-Verdier category in $\text{Lex}^f$, the opposite category will have a right exact monoidal product.

**Definition 3.4.** A Grothendieck-Verdier category $\mathcal{C}$ in $\text{Lex}^f$ is called self-injective if it is self-injective as linear category, i.e. if the projective objects of $\mathcal{C}$ are exactly the injective ones.

**Remark 3.5.** Self-injectivity ensures that the duality functor $D : \mathcal{C} \rightarrow \mathcal{C}^{\text{opp}}$ preserves projective objects. This can be seen as follows: For any $X \in \mathcal{C}$, the object $DX$ is always injective because $D$ is an equivalence from $\mathcal{C}$ to $\mathcal{C}^{\text{opp}}$, and by self-injectivity $DX$ is projective. For finite tensor categories (which are rigid by definition), this assumption is automatically satisfied.

We are now ready to prove the main result:

**Theorem 3.6.** Let $\mathcal{C}$ be a self-injective ribbon Grothendieck-Verdier category in $\text{Lex}^f$. Then its duality functor and its balancing induce on the Hochschild complex $\int_{X \in \text{Proj}}^\mathcal{C}(X, X)$ an action of the diffeomorphism group $\text{Diff}(H_1, 0)$ of the solid closed torus that extends the usual cyclic symmetry of the Hochschild complex.

**Proof.** For $n \geq 0$, we define the vector space

$$H[n] := \bigoplus_{X_0, \ldots, X_n \in \text{Proj} \mathcal{C}} \mathcal{C}(X_n, X_{n-1}) \otimes \cdots \otimes \mathcal{C}(X_1, X_0) \otimes \mathcal{C}(X_0, X_n)$$

(3.1)

of loops of morphisms in $\mathcal{C}$ running through $n + 1$ projective objects. We can see $H[n]$ as chain complex concentrated in degree zero. Now the precomposition with the balancing on each of the different morphisms gives us a functor

$$H[n] : (\ast / \mathbb{Z})^{(n+1)} \rightarrow \text{Ch}_k$$

(3.2)

that we denote by $H[n]$ again, by a slight abuse of notation. In more detail, it sends the only
object of \((\star // \mathbb{Z})^{n+1}\) to \(H[n]\) and the morphism \((\ell_0, \ldots, \ell_n) \in \mathbb{Z}^{n+1}\) to the map
\[
\left( X_n \xrightarrow{f_n} X_{n-1} \to \ldots X_0 \xrightarrow{f_0} X_n \right) \mapsto \left( X_n \xrightarrow{f_n \theta^{\circ}_X} X_{n-1} \to \ldots X_0 \xrightarrow{f_0 \theta^{\circ}_X} X_n \right)
\]
given by precomposition with the balancing. In fact, we might as well postcompose; this would give us the same result thanks to the naturality of the balancing.

Consider now the cyclic object used to compute cyclic homology of the topological algebra \(S^1\), but seen now as category-valued functor \(D : \Lambda^{opp} \to \text{Cat}\) sending \([n] \to (\star // \mathbb{Z})^{n+1}\). The key observation is that the functors (3.2) extend to a functor

\[
H : \int D \to \text{Ch}_k
\]

out of the Grothendieck construction \(\int D\) of the functor \(D : \Lambda^{opp} \to \text{Cat}\) by means of the usual cyclic structure of the Hochschild complex. In order to define this functor, recall that by the universal property of the Grothendieck construction it suffices to give functors \((\star // \mathbb{Z})^{n+1} \to \text{Ch}_k\) (we have these already, see (3.2)) plus natural transformations \(\alpha_f\) filling the triangles

\[
\begin{array}{ccc}
(\star // \mathbb{Z})^{n+1} & \xrightarrow{H[n]} & \text{Ch}_k \\
\downarrow D(f) & \Downarrow \alpha_f & \downarrow H[n] \\
(\star // \mathbb{Z})^{m+1}
\end{array}
\]

for every morphism \(f\) in \(\Lambda^{opp}\) such that these transformations respect the composition in \(\Lambda^{opp}\). It is clear how to define the needed transformation \(\alpha_f\) because the vector spaces (3.1) already form a cyclic object in the standard way. We only need to verify the naturality of the \(\alpha_f\) which is a direct consequence of the fact that the balancing is a natural transformation.

In the next step, we will make use of the paracyclic category \(A_{\infty}\) [NS18, Appendix B], a contractible category with \(S^1\)-action such that \(A = A_{\infty}/S^1\). Consider now the functor \(A_{\infty}^{opp} \to \text{Cat}\) sending \([n]\) to the action groupoid \(EZ^{n+1} := \mathbb{Z}^{n+1}/\mathbb{Z}^{n+1}\) of the regular free and transitive action of \(\mathbb{Z}^{n+1}\) on itself, i.e. the total space of the universal \(\mathbb{Z}^{n+1}\)-bundle. Precomposition with the quotient functor \(A_{\infty}^{opp} \to \Lambda^{opp} = A_{\infty}^{opp}/S^1\) yields a functor \(D_{\infty} : A_{\infty}^{opp} \to \text{Cat}\). The quotient functors \(A_{\infty}^{opp} \to \Lambda^{opp}\) and \(EZ^{n+1} \to \mathbb{Z}^{n+1}\) induce a functor \(Q : \int D_{\infty} \to \int D\). Since \(EZ^{n+1} \simeq *\) we have an equivalence \(A_{\infty}^{opp} \to \int D_{\infty}\).

By sending \([n]\) to \(\mathbb{Z}^{n+1}\), we obtain a cyclic group \(C\) that we may see also as paracyclic group. By realization it gives us a topological group \(G\) with \(BG = \int_{S^1} S^1\). There is a \(C\)-action on \(\int D_{\infty}\) induced by the action \(\mathbb{Z}^{n+1}\)-action on \(EZ^{n+1}\). The \(C\)-action on \(\int D_{\infty}\) and the \(S^1\)-action on the paracyclic category \(A_{\infty}\) combine into an action of \(C \times S^1\) on \(\int D_{\infty}\) (the \(S^1\)-action on \(C\) exists because it is a cyclic object) such that \((\int D_{\infty})/(C \times S^1) = \int D\). This tells us that any functor \(F : \int D \to \text{Ch}_k\) can equivalently be described as the underlying functor \(FQ : \int D_{\infty} \to \text{Ch}_k\) plus the \(C \times S^1\)-equivariance of this functor, where \(\text{Ch}_k\) is equipped with the trivial \(C \times S^1\)-action. As a result, the homotopy colimit of \(FQ\) over \(\int D_{\infty}\) carries an action of \(G \times S^1\). By [NS18, Theorem B.3] the paracyclic category comes with a homotopy final functor \(\Delta_{\infty}^{opp} \to A_{\infty}^{opp}\). Therefore, the functor \(\Delta_{\infty}^{opp} \to \Lambda_{\infty}^{opp} \cong \int D_{\infty}\) is also homotopy final. As a result, the chain complex homotopical \(FQ\) is equivalent to the realization \([F]\) of the simplicial object underlying \(F\). This implies that \([F]\) comes with an action of \(G \times S^1\).

Now let us apply this to the functor \(H : \int D \to \text{Ch}_k\) from (3.3). Its realization \([H]\) is the Hochschild complex of \(C\), which now comes with an action of \(G \times S^1 = \Omega \left((\int_{S^1} S^1)_{h S^1}\right)\). It extends the cyclic action by construction.
Finally, we observe that the symmetric monoidal bicategory of linear categories comes with a homotopy coherent action of $\mathbb{Z}_2$ that sends a category to its opposite category. A self-injective ribbon Grothendieck-Verdier category $\mathcal{C}$ does not only come with a duality functor $D: \mathcal{C} \to \mathcal{C}^{opp}$, but also a pivotal structure, i.e. an isomorphism $D^2 \cong \text{id}_\mathcal{C}$, see [BD13, Corollary 8.3]. This turns $\text{Proj}\mathcal{C}$ into a homotopy $\mathbb{Z}_2$-fixed point for the just mentioned $\mathbb{Z}_2$-action on linear categories (this crucially uses that $D$ preserves projective objects thanks to self-injectivity, see Remark 3.5). The homotopy $\mathbb{Z}_2$-fixed point structure is through balancing preserving functors thanks to $D\text{Hom}_X = \theta_{DX}$, see Definition 3.2. The Hochschild complex of $\text{Proj}\mathcal{C}$ and its opposite category $(\text{Proj}\mathcal{C})^{opp}$ can be canonically identified; this is true for any linear category. For this reason, $\int_X^{\text{Proj}\mathcal{C}} \mathcal{C}(X, X)$ inherits, in addition to the action of $\mathbb{Z}$, a pivotal structure, i.e. an isomorphism $D\mathcal{C}$ crucially uses that the symmetric monoidal bicategory of linear categories comes with a homotopy coherent action of $\mathbb{Z}_2$ that we have just established, a homotopy coherent action of $\mathbb{Z}_2$ from the $\mathbb{Z}_2$-fixed point structure (the duality $D$ acts by a chain map, and the pivotal structure gives a chain homotopy between the chain map for $D^2$ and the identity — this follows from the functoriality of the Hochschild complex).

Since the $\mathbb{Z}_2$-action is by linear balancing preserving functors, the $\mathbb{Z}_2$-action intertwines the $\Omega ((\int_S S^1)^{\mathbb{Z}_2})$-action, but in a twisted way as we explain now: The Hochschild complex of $\text{Proj}\mathcal{C}$ and $(\text{Proj}\mathcal{C})^{opp}$, even though they can be identified as chain complexes, can only be identified as cyclic objects if we precompose the cyclic object for $(\int_S S^1)^{\mathbb{Z}_2}$. As a consequence, the total action on the Hochschild complex is not an action of the product of $\Omega ((\int_S S^1)^{\mathbb{Z}_2})$ with $\mathbb{Z}_2$, but the semidirect product with $\mathbb{Z}_2$, where $\mathbb{Z}_2$-acts by reflection on both $S^1$'s in $\Omega ((\int_S S^1)^{\mathbb{Z}_2})$ that are written as subscript and trivially on the $S^1$ which is not written as subscript. In other words, the duality plays here the same role as the anti algebra involution in dihedral homology. In summary, the Hochschild complex of $\mathcal{C}$ comes with an action of the topological group $\Omega ((\int_S S^1)^{\mathbb{Z}_2}) \times \mathbb{Z}_2 = \Omega ((\int_S S^1)^{\mathbb{Z}_2 \times \mathbb{Z}_2})$. This topological group is given by the based loops of the space $(\int_S S^1)^{\mathbb{Z}_2 \times \mathbb{Z}_2}$, i.e. of the dihedral homology $\text{DH}(S^1)$ of $S^1$.

By Theorem 2.2 $\text{DH}(S^1) \simeq \text{BDiff}(H_{1,0})$. As a result, the Hochschild complex of $\mathcal{C}$ comes with an action of $\Omega ((\int_S S^1)^{\mathbb{Z}_2}) \times \mathbb{Z}_2 = \text{ODH}(S^1) \simeq \text{Diff}(H_{1,0})$.

**Remark 3.7.** By one of the main results of [MW20], ribbon Grothendieck-Verdier categories in $\text{Lex}^\dagger$ are equivalent to cyclic framed $E_2$-algebras in $\text{Lex}^\dagger$. In connection with Giansiracusa’s result [Gia11] on the relation between the modular envelope of the cyclic framed $E_2$-operad and the handlebody operad, this is used in [MW20] to build systems of handlebody group representations. These results, however, are not available for the solid closed torus and make no statements about diffeomorphism groups. For this reason, the theory of cyclic framed $E_2$-algebras in symmetric monoidal bicategories from [MW20] does not provide a shortcut to the proof of Theorem 3.6. In fact, the insight rather flows in the opposite direction: The dihedral homology computation in Theorem 2.2 will enable us to improve results on the modular envelope of the framed $E_2$ operad and help us characterize systems of handlebody group representations in [MW22].

**Example 3.8.** Any finite ribbon category $\mathcal{C}$ in the sense of [EGNO15] is in particular a self-injective ribbon Grothendieck-Verdier category. In this case, the Grothendieck-Verdier duality comes actually from rigidity. A source of finite ribbon categories are finite-dimensional ribbon Hopf algebras: If $A$ is a finite-dimensional ribbon Hopf algebra, then the category of finite-dimensional $A$-modules is a finite ribbon category [Kas15, XIV.6]. The Hochschild complex $\int_X^{\text{Proj}\mathcal{C}} \mathcal{C}(X, X)$ is equivalent to the ordinary Hochschild complex of $A$. As a consequence, the Hochschild complex of $A$ comes with an action of $\text{Diff}(H_{1,0})$ extending the cyclic action. Let us also describe the underlying action of $\text{Map}(H_{1,0}) \cong \mathbb{Z} \times \mathbb{Z}_2$ in more detail. To this end, we use the Lyubashenko coend $\mathbb{F} := \int_X^{\mathcal{C}} X \otimes X \in \mathcal{C}$ [Lyu95]. By [SW21b, Theorem 3.9], we have $\int_X^{\text{Proj}\mathcal{C}} \mathcal{C}(X, X) \simeq \mathcal{C}(I_*, \mathbb{F}_*)$, where $\mathbb{F}_*$ is a projective resolution of $\mathbb{F}$. By recapitulating the construction given in the proof of Theorem 3.6 we conclude that the action of $\text{Map}(H_{1,0}) \cong \mathbb{Z} \times \mathbb{Z}_2$
on the Hochschild complex $\int_{X \in \text{Proj}_C}^X \mathcal{C}(X, X)$ of the finite ribbon category $\mathcal{C}$ is given as follows:

(i) The $Z$-factor acts by postcomposition with the balancing. Equivalently, it can be described as the chain map $\mathcal{C}(I, F_\bullet) \to \mathcal{C}(I, F_\bullet)$ induced by the automorphism of $F = \int_{X \in \mathcal{C}} X^\vee \otimes X$ that applies the balancing to the first or second tensor factor under the coend (both give the same result).

(ii) The $Z_2$-factor acts on $\int_{X \in \text{Proj}_C}^X \mathcal{C}(X, X)$ by applying the duality functor to morphism spaces and identifying the Hochschild complexes of $\mathcal{C}$ and $\mathcal{C}^\text{opp}$. This operation is (homotopy) involutive because of the pivotal structure. (Under the equivalence $\int_{X \in \text{Proj}_C}^X \mathcal{C}(X, X) \simeq \mathcal{C}(I, F_\bullet)$, we may equivalently say that the generator of $\mathbb{Z}_2$ acts by the map induced by the inverse antipode of the Hopf algebra $F$ in $\mathcal{C}$ [Lyu95]. This follows from the definition of the antipode in [Lyu95, Section 2.5] and [MW22, Lemma 6.1].)

In the special case where $\mathcal{C}$ is given as category of modules over a ribbon Hopf algebra $A$, the coend $F$ is given by the dual $A^\text{coadj}$ of $A$ with its coadjoint $A$-action $A \otimes A^* \to A^*$ sending $a \otimes a$ to the linear form $b \mapsto (a(ba'))$ [KL01, Theorem 7.4.13]. Here $\Delta a = a' \otimes a''$ is the Sweedler notation for the coproduct $\Delta : A \to A \otimes A$, and $S : A \to A$ is the antipode. The equivalence $\int_{X \in \text{Proj}_C}^X \mathcal{C}(X, X) \simeq \mathcal{C}(I, F_\bullet)$ is well-known in the Hopf algebraic special case, see e.g. [Bic13, Section 2.2]. In that case, one obtains $CH_1(A) \simeq \text{Hom}_A(k, A^\text{coadj})$ for the Hochschild complex of $A$. The $Z$-factor acts via multiplication with the ribbon element while the $Z_2$-factor acts by postcomposition with the balancing. Equivalently, it can be described as the chain map $\mathcal{C}(I, F_\bullet) \to \mathcal{C}(I, F_\bullet)$ induced by the automorphism of $F = \int_{X \in \mathcal{C}} X^\vee \otimes X$ that applies the balancing to the first or second tensor factor under the coend (both give the same result).

Remark 3.9 (Visibility of higher homotopy groups of diffeomorphism groups on differential graded conformal blocks). Let $\mathcal{C}$ be a modular category, i.e. a finite ribbon category with non-degenerate braiding. Then by the main result of [SW21a], building on preparations in [LMSS18, SW21b, LMSS20], the category $\mathcal{C}$ gives rise to a symmetric monoidal functor $\mathfrak{C}_\mathcal{C} : \mathcal{C}\text{-Surf}^C \to \mathcal{C}_{\mathcal{C}}$ from a category of $\mathcal{C}$-labeled surfaces (morphisms are sewing operations and mapping classes of surfaces up to a specific central extension coming from the framing anomaly) to chain complexes. The functor satisfies an excision property phrased in terms of homotopy coends. The chain complex assigned to the torus is the Hochschild complex of $\mathcal{C}$, involving because of the pivotal structure. (Under the equivalence $\int_{X \in \text{Proj}_C}^X \mathcal{C}(X, X) \simeq \mathcal{C}(I, F_\bullet)$, we may equivalently say that the generator of $\mathbb{Z}_2$ acts by the map induced by the inverse antipode of the Hopf algebra $F$ in $\mathcal{C}$ [Lyu95]. This follows from the definition of the antipode in [Lyu95, Section 2.5] and [MW22, Lemma 6.1].)

For $T$, this is exactly the automorphism described in point (i) of Example 3.8. The image of $S$ is an automorphism of $\mathcal{C}(I, F_\bullet)$ induced by an automorphism $F \to F$ which by definition in [LMSS18, SW21b] is the rather complicated ‘S-transformation’ $F \to F$ [Lyu95, Section 6]. We can restrict the $\text{SL}(2, \mathbb{Z})$-action along the inclusion $\mathbb{Z} \times \mathbb{Z} \subset \text{SL}(2, \mathbb{Z})$ given by (2.2); it sends the generator of $\mathbb{Z}$ to $T$ and the generator of $\mathbb{Z}$ to $S^2$. Because the $S$-transformation $F \to F$ squares to the inverse of the antipode of the Hopf algebra $F$ [Lyu95, Section 6, eq. (5)], it follows from point (ii) in Example 3.8 that this $Z \times Z$-action on the Hochschild complex agrees with the one obtained via the section $\text{Map}(H_{1,0}) \to \text{Diff}(H_{1,0})$ and Theorem 3.6. This has the following immediate consequence: The ‘handlebody part’ of the mapping class group action on the Hochschild complex of a modular category from [LMSS18, SW21b] admits a non-trivial extension to an action of $\text{Diff}(H_{1,0})$. The non-triviality of the extension follows for example from the fact that one of the generators of $\pi_1(\text{Diff}(H_{1,0}))$ acts by the cyclic symmetry and hence is non-trivial. This is, to the best of our knowledge, the first instance of a non-trivial action of higher homotopy groups of diffeomorphism groups on conformal blocks.
Corollary 3.10. For self-injective ribbon Grothendieck-Verdier categories, the Hochschild complex equipped with its $\text{Diff}(H_{1,0})$-action is a stronger invariant than the Hochschild complex with its cyclic action. More precisely, there exist ribbon Grothendieck-Verdier categories with cyclically equivalent Hochschild complexes which are however not equivalent as $\text{Diff}(H_{1,0})$-modules.

Proof. Let $G$ be a finite group and $C$ the category of finite-dimensional $k$-linear modules over the Drinfeld double $D(G)$. This is a modular category whose differential graded modular functor is described in [SW21a, Example 3.13]. From the description given there and Remark 3.9, we may deduce that the $\mathbb{Z}$-factor of the handlebody group $\text{Map}(H_{1,0}) \cong \mathbb{Z} \times \mathbb{Z}_2$ acts non-trivially (for instance, its action on $HH_0(C)$ is non-trivial). But now observe that $C$, as a linear category, is equivalent to modules over the action groupoid $G//G$ of the adjoint action of $G$ on itself (this is because $D(G)$-modules are equivalent to Yetter-Drinfeld modules over $k[G]$, see [Kas15, Theorem IX.5.2]). The category of finite-dimensional $G//G$-modules can be endowed with a symmetric monoidal product (the level-wise monoidal product that one can define on any category of functors to a symmetric monoidal category; this is not the monoidal product coming from the Hopf algebra structure of the Drinfeld double) and hence with a trivial balancing (that is also a ribbon structure). We denote the resulting finite ribbon category as $C_0$. Then $C = C_0$ holds linearly which means that their Hochschild complexes agree (and, of course, so does the $S^1$-action on them). But in contrast to the Hochschild complex of $C$, the action of $\mathbb{Z} \subset \text{Map}(H_{1,0})$ is trivial because the balancing is the identity.

Remark 3.11 (Ansular homology). The cyclic action on the Hochschild chain complex naturally leads to cyclic homology by passing to homotopy orbits of the $S^1$-action. Since on the Hochschild complex of a self-injective ribbon Grothendieck-Verdier category in $\text{Lex}^I$, the cyclic action extends to a $\text{Diff}(H_{1,0})$-action, it is natural to consider the homotopy orbits of this action that one might call ansular homology (ansa is Latin for handle). This yields an appropriate replacement of cyclic homology sensitive to the ribbon structure. The computation of these homotopy orbits lies beyond the scope of this article.

References


