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Abstract
Given a couple of subspaces $\mathcal{Y} \subset \mathcal{X}$ of the complex plane $\mathbb{C}$ satisfying some mild conditions (a “nice couple”), and given a PMQ-pair $(\mathcal{Q}, G)$, consisting of a partially multiplicative quandle (PMQ) $\mathcal{Q}$ and a group $G$, we introduce a “Hurwitz–Ran” space $\text{Hur}(\mathcal{X}, \mathcal{Y}; \mathcal{Q}, G)$, containing configurations of points in $\mathcal{X} \setminus \mathcal{Y}$ and in $\mathcal{Y}$ with monodromies in $\mathcal{Q}$ and in $G$, respectively. We further introduce a notion of morphisms between nice couples, and prove that Hurwitz–Ran spaces are functorial both in the nice couple and in the PMQ-group pair. For a locally finite PMQ $\mathcal{Q}$ we prove a homeomorphism between $\text{Hur}((0, 1)^2; \mathcal{Q}_+)$ and the simplicial Hurwitz space $\text{Hur}^\Delta(\mathcal{Q})$, introduced in previous work of the author: this provides in particular $\text{Hur}((0, 1)^2; \mathcal{Q}_+)$ with a cell stratification in the spirit of Fox–Neuwirth and Fuchs.

Keywords Quandle · Partial monoid · Hurwitz space · Ran space · Group actions · Cell decompositions · Homology manifolds

Mathematics Subject Classification 18F60 · 54B15 · 55R80

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1 Introduction

In [3] we introduced the notion of partially multiplicative quandle (shortly, PMQ) and for a PMQ $Q$ we defined a simplicial Hurwitz space $\text{Hur}^\Delta(Q)$: the space $\text{Hur}^\Delta(Q)$ is the difference of the geometric realisations of two bisimplicial sets, and is thus equipped with a cell stratification. The construction requires $Q$ to be augmented as a PMQ (see [3, Definition 4.9]).

As discussed in [3, Sect. 6], a point in $\text{Hur}^\Delta(Q)$ can be regarded as the datum of a finite subset $P$ of the open unit square $(0, 1)^2 \subset \mathbb{C}$, together with a monodromy $\psi$, defined on certain loops of $\mathbb{C} \setminus P$ and taking values in $Q$.

In this article we introduce, for a semi-algebraic subspace $X \subset \mathbb{H}$ of the closed upper half-plane $\mathbb{H} = \{ \Re t \geq 0 \} \subset \mathbb{C}$, and for a PMQ $Q$, a “Hurwitz–Ran” space $\text{Hur}(X; Q)$. More generally we introduce, for a nice couple $(X, Y)$ of subspaces $Y \subseteq X$ of $\mathbb{H}$ (see Definition 2.3) and for a PMQ-group pair $(Q, G)$ (see [3, Definition 2.15]), a Hurwitz–Ran space $\text{Hur}(X, Y; Q, G)$. The part “Hurwitz” in the name suggests that elements of $\text{Hur}(X, Y; Q, G)$ are configurations of points in the plane with monodromy data, like in classical Hurwitz spaces; the part “Ran” refers to the topology, defined in such a way that “collisions” between points are allowed (at least under certain circumstances), like in classical Ran spaces.

One can think of $\text{Hur}(X, Y; Q, G)$ as a relative version of $\text{Hur}(X; Q)$. In the classical theory of configuration spaces, points of $P$ disappear when they move inside the relative subspace $Y$; in our setting they do not disappear, but are “downgraded” to points around which only a $G$-valued monodromy, instead of a $Q$-valued monodromy, is defined. The Hurwitz–Ran construction has the advantage of being natural not only in the PMQ-group pair $(Q, G)$, but also in the nice couple $(X, Y)$: for nice couples $(X', Y')$ and $(X'', Y'')$, and for suitable maps $\xi : \mathbb{C} \to \mathbb{C}$ sending $X' \to X''$ and $Y \to Y''$, we are able to define a corresponding map $\xi_* : \text{Hur}(X', Y'; Q, G) \to \text{Hur}(X'', Y''; Q, G)$; most notably, we can do this in certain cases in which $\xi$ is not injective, in particular it is not a homeomorphism of $\mathbb{C}$.

1.1 Statement of results

We prove that the construction of $\text{Hur}(X; Q)$ is functorial both in $Q$ and in $X$; as special cases we have the following:

- a morphism of PMQs $\psi : Q \to Q'$ induces a continuous map $\Psi_* : \text{Hur}(X; Q) \to \text{Hur}(X; Q')$;
- a semi-algebraic homeomorphism $\xi : \mathbb{C} \to \mathbb{C}$ with compact support in $\mathbb{H}$, sending $X$ inside $X'$, induces a continuous map $\xi_* : \text{Hur}(X; Q) \to \text{Hur}(X'; Q)$.

In Definitions 4.2 and 4.3 we introduce morphisms and lax morphisms of nice couples: both types of morphisms arise as certain continuous maps $\mathbb{C} \to \mathbb{C}$. The first, main result of the article is the following theorem, combining Theorems 4.1, 4.4 and 4.7.

**Theorem A** The Hurwitz–Ran spaces $\text{Hur}(X, Y; Q, G)$ are functorial both in the nice couple $(X, Y)$, with respect to morphisms ofnice couples, and in the PMQ-group pair. If we restrict to complete PMQs, then functoriality holds also with respect to lax morphisms of nice couples.

We recall that the simplicial Hurwitz space $\text{Hur}^\Delta(Q)$ was defined in [3] only when $Q$ is an augmented PMQ. For such a PMQ, we can set $X = (0, 1)^2 \subset \mathbb{C}$ and consider the Hurwitz–Ran space $\text{Hur}((0, 1)^2; Q)$. Inside the latter, we let $\text{Hur}((0, 1)^2; Q_+)$, be the subspace of
configurations $(P, \psi)$ such that the local monodromy around each point of $P$ lies in $Q_+ := Q \setminus \{1\}$. The second main result of the article is the following theorem, which is Theorem 9.1.

**Theorem B** If $Q$ is augmented and locally finite (see [3, Definition 4.12]), then the simplicial Hurwitz space $\text{Hur}^\Delta(Q)$ is homeomorphic to $\text{Hur}((0, 1)^2; Q_+)$. 

Thus Hurwitz–Ran spaces generalise simplicial Hurwitz spaces from [3].

In [3, Definition 6.19] we also introduced the notion of Poincaré PMQ: an augmented and locally finite PMQ $Q$ is Poincaré if each connected component of $\text{Hur}^\Delta(Q)$ is a topological manifold. More generally, for a commutative ring $R$, we introduce the notion of $R$-Poincaré PMQ: each component of $\text{Hur}^\Delta(Q)$ is required to be an $R$-homology manifold (see Definition 9.4). An $R$-Poincaré PMQ $Q$ has the following advantage: the $R$-homology groups of a component of $\text{Hur}^\Delta(Q)$ are isomorphic by Poincaré–Lefschetz duality to the reduced $R$-cohomology groups of the one point compactification of that component; this one point compactification is endowed with a finite cell decomposition à la Fox–Neuwirth–Fuchs [3, Sect. 6] which can in principle be used for actual cohomology computations.

The third main result of the article is the following, combining Theorems 9.3 and 9.6, and giving a criterion to recognise Poincaré and $R$-Poincaré PMQs. Recall that a PMQ $Q$ embeds into its completion $\hat{Q}$, and $\hat{Q}$ is strictly larger than $Q$ unless $Q$ is already complete. Recall also from [3, Theorem 6.14] that the connected components of $\text{Hur}^\Delta(Q)$ are in bijection with $\hat{Q}$.

**Theorem C** In order to prove that $Q$ is Poincaré (respectively, $R$-Poincaré), it suffices to check that for all $a \in Q$ the corresponding component $\text{Hur}^\Delta(Q)(a)$ of $\text{Hur}^\Delta(Q)$ is a topological manifold (respectively, an $R$-homology manifold).

### 1.2 Outline of the article

In Sect. 2 we introduce the notion of nice couple $\mathfrak{C} = (\mathcal{X}, \mathcal{Y})$ of subspaces $\mathcal{Y} \subset \mathcal{X}$ of the closed upper half-plane $\mathbb{H}$. For each finite subset $P \subset \mathcal{X}$, we introduce the fundamental PMQ $\mathcal{Q} \subset (P)$, arising as a subset of $\pi_1(\mathbb{C} \setminus P)$, which will allow us to define configurations in $\text{Hur}(\mathfrak{C}; Q, G)$ supported on the set $P$. In order to define a convenient topology on the set $\text{Hur}(\mathfrak{C}; Q, G)$, we introduce also the notion of covering $U$ of a finite subset $P \subset \mathcal{X}$, and associate several PMQs also with the datum of a finite set and a covering of it.

In Sect. 3 we introduce the Hurwitz–Ran space $\text{Hur}(\mathfrak{C}; Q, G)$, for each nice couple $\mathfrak{C}$ and each PMQ-group pair $(Q, G)$, first as a set and then as a Hausdorff topological space (see Proposition 3.8). We also discuss a variation of the definition, using a contractible subspace $\mathbb{T} \subset \mathbb{C}$ as “ambient space” instead of the entire $\mathbb{C}$.

In Sect. 4 we introduce morphisms and lax morphisms of nice couples, and we prove Theorem A.

In Sect. 5 we give some applications of functoriality of Hurwitz–Ran spaces, in particular we study some local properties of their topology. We also specialise Hurwitz–Ran spaces $\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)$ to the case $\mathcal{Y} = \emptyset$: in this case the resulting space only depends on $Q$ and not on $G$, and we denote it $\text{Hur}(\mathcal{X}; Q)$.

In Sect. 6 we introduce the total monodromy, which is a discrete, continuous invariant of configurations in Hurwitz–Ran spaces. The total monodromy is a map $\omega: \text{Hur}(\mathfrak{C}; Q, G) \to G$ in the relative case, and $\hat{\omega}: \text{Hur}(\mathcal{X}; Q) \to \hat{Q}$ in the absolute case, where $\hat{Q}$ is the completion of $Q$. We also define three actions of $G$ on Hurwitz–Ran spaces: the first is the action
on $\text{Hur}(\mathbb{C}; Q, G)$ by global conjugation; the other two actions are only defined on certain subspaces of $\text{Hur}(\mathbb{C}; Q, G)$, and leverage the possibility of changing the local monodromy around a single point $z$ in the support of a configuration, provided that $z$ is the leftmost or rightmost point in the support.

In Sect. 7 we introduce, in the hypothesis that $Q$ is augmented, a subspace $\text{Hur}(\mathbb{C}; Q_+, G)$ of $\text{Hur}(\mathbb{C}; Q, G)$; using the notion of explosion we prove that the inclusion $\text{Hur}(\mathbb{C}; Q_+, G) \subseteq \text{Hur}(\mathbb{C}; Q, G)$ is in several cases a homotopy equivalence.

In Sect. 8, for an augmented PMQ $Q$, we construct a continuous bijection $\nu: |\text{Arr}(Q)| \to \text{Hur}((0, 1)^2; \hat{Q}_+)$, and prove Theorem B: here $\hat{Q}$ is the completion of $Q$, and $\text{Arr}(Q)$ is the bisimplicial set from [3, Definition 6.6], whose geometric realisation contains $\text{Hur}^\Delta(Q)$ as an open subspace.

In Sect. 9 we prove that $\nu: \text{Hur}^\Delta(Q) \to \text{Hur}((0, 1)^2; Q_+)$ is a homeomorphism under the additional hypothesis that $Q$ is a locally finite PMQ. We then move our focus to Poincaré PMQs and prove Theorem C.

Finally, Appendix A contains the proofs of the most technical lemmas and propositions of the article; these proofs have been deferred to help the reader focus on the general framework. Throughout the article we make heavy use of the results of [3, Sects. 2–6]: we cite every time which specific fact we are employing, so that the reader does not need to be familiar with the details of [3].

1.3 Motivation

This is the second article in a series about Hurwitz spaces. Our motivation to define Hurwitz–Ran spaces is two-fold.

- The standard setting for Hurwitz spaces takes as input an integer $k \geq 0$ and conjugacy-invariant subset $Q$ in a group $G$, and gives a space $\text{hur}_k(Q)$ of moduli of branched $G$-covers of the unit square $(0, 1)^2$ with exactly $k$ branch points and local monodromies in $Q$; see for instance [6]. The disjoint union $\text{hur}(Q) := \bigsqcup_{k \geq 0} \text{hur}_k(Q)$ carries a natural $E_1$-algebra structure, and one can try to extract “stable” information about a particular space $\text{hur}_k(Q)$ from the group-completion $\Omega BH\text{ur}(Q)$. As we will see in a subsequent article [4], Hurwitz–Ran spaces are well-suited for modeling the homotopy type of $BH\text{ur}(Q)$, and provide, in a certain sense, even a double delooping “$B^2(\text{hur}(Q))$” of $\text{hur}(Q)$, as if the latter were an $E_2$-algebra.

- The connection between Hurwitz spaces and moduli spaces of Riemann surfaces is well-known, and by using the family of PMQs $\mathcal{G}_d^{\text{geo}}$ from [3, Sect. 7], we will see in a subsequent article [5] that one can in fact model the homotopy type of moduli spaces $\mathcal{M}_{g,n}$ as components of Hurwitz–Ran spaces; here $\mathcal{M}_{g,n}$ denotes the moduli space of Riemann surfaces of genus $g \geq 0$ with $n \geq 1$ ordered and parametrised boundary components. This connection leads to an alternative proof of the Mumford conjecture on the stable rational cohomology of moduli spaces $\mathcal{M}_{g,n}$, originally proved by Madsen and Weiss [8].

2 Groups and PMQs from configurations in the plane

In this section we introduce nice couples of subspaces of $\mathbb{C}$ and use them to associate several PMQs to a finite configuration $P$ of points in $\mathbb{C}$.
2.1 Nice couples

Notation 2.1 We endow the complex plane \( \mathbb{C} \) with the basepoint \( * \) corresponding to the complex number \( -\sqrt{-1} \), which is contained in the lower half-plane. The closed, upper half-plane is \( \mathbb{H} := \{ z \in \mathbb{C} \mid \Im(z) \geq 0 \} \).

Definition 2.2 A subset \( J \subseteq \mathbb{C} \) is semi-algebraic if it can be expressed as a finite union of subsets \( J_1, \ldots, J_r \subseteq \mathbb{C} \), such that each \( J_i \) is defined by a finite system of polynomial equalities and (weak or strict) inequalities in the real, affine coordinates \( \Re(z) \) and \( \Im(z) \) of \( \mathbb{C} \). Given two semi-algebraic sets \( J, J' \subseteq \mathbb{C} \), a continuous map \( \xi : J \to J' \) is semi-algebraic if \( J \) can be expressed as a finite union of semi-algebraic subsets \( J_1, \ldots, J_r \subseteq J \), and the coordinates of \( \xi|_{J_i} \) are expressed, in the real affine coordinates of \( J_i \), by fractions of polynomials with real coefficients.

Note that a semi-algebraic subset \( J \subseteq \mathbb{C} \) can be written locally as a finite union of subsets of \( \mathbb{C} \) that are diffeomorphic to points, open segments and open triangles; finite unions and intersections of semi-algebraic subsets are again semi-algebraic.

Definition 2.3 A nice couple \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) is a couple of subspaces \( \emptyset \subseteq \mathcal{Y} \subseteq \mathcal{X} \subseteq \mathbb{H} \), such that the following properties hold:

- \( \mathcal{X} \) and \( \mathcal{Y} \) are semi-algebraic;
- \( \mathcal{Y} \) is closed in \( \mathcal{X} \).

By abuse of notation, for \( \mathcal{X} \subset \mathbb{H} \) we will denote by \( \mathcal{X} \) also the nice couple \( (\mathcal{X}, \emptyset) \).

2.2 Configurations and coverings

We fix a nice couple \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) for the rest of the section, and consider finite configurations of points in \( \mathcal{X} \subset \mathbb{C} \).

Notation 2.4 We will often denote by \( P = \{z_1, \ldots, z_k\} \subset \mathcal{X} \) a finite collection of distinct points, for some \( k \geq 0 \). We will assume that there is \( 0 \leq l \leq k \) such that \( z_1, \ldots, z_l \) are precisely the points of \( P \) lying in \( \mathcal{X} \setminus \mathcal{Y} \).

Definition 2.5 Let \( P \) as in Notation 2.4. A covering of \( P \) is a sequence \( U = (U_1, \ldots, U_\kappa) \) of convex, semi-algebraic, disjoint open subsets of \( \mathbb{C} \setminus \{*\} \), satisfying the following conditions:

- \( P \) is contained in the union \( U_1 \cup \cdots \cup U_\kappa \);
- each \( U_i \) intersects \( P \) at least in one point;
- the closures \( \overline{U_i} \subseteq \mathbb{C} \) of the sets \( U_i \) are disjoint, compact and do not contain \( * \).

A covering of \( P \) is adapted (to \( P \)) if the following additional properties hold:

- \( \kappa = k \), and each \( U_i \) contains exactly one point of \( P \);
- for all \( 1 \leq i \leq l \), i.e. for all \( i \) such that \( z_i \notin \mathcal{Y} \), if \( z_i \in U_j \) then the closure \( \overline{U_j} \) is disjoint from \( \mathcal{Y} \).

Since \( \mathcal{Y} \) is closed in \( \mathcal{X} \), each finite \( P \subset \mathcal{X} \) admits some adapted covering. Note that if \( U \) is an adapted covering of \( P \), then the inclusion \( \mathbb{C} \setminus U \hookrightarrow \mathbb{C} \setminus P \) is a homotopy equivalence. See Fig. 1.
Fig. 1 On left, a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$, and a covering $U'$ of a configuration $P \subset \mathcal{X}$; on right, an adapted covering $U$ of $P$.

Notation 2.6 Let $U = (U_1, \ldots, U_\kappa)$ be a covering of $P$ as in Definition 2.5. By abuse of notation we denote also by $U$ the union $U_1 \cup \cdots \cup U_\kappa \subset \mathbb{C}$. We assume that there is $0 \leq \lambda \leq \kappa$ such that $U_1, \ldots, U_\lambda$ are precisely the open sets of $U$ with $\bar{U}_i$ disjoint from $\mathcal{Y}$. If $U$ is an adapted covering of $P$, using Notation 2.4, we also assume $z_i \in U_i$ for $1 \leq i \leq \kappa$.

The notion of covering is classically used to give a topology to the Ran space $\text{Ran}(\mathcal{X})$, as we also recall in Sect. 3.1. Intuitively, a perturbation of a configuration $P$ will be a new configuration $P'$ obtained by slightly moving the points of $P$ and by splitting some points $z_i \in P$ in two or more points of $P'$; all these splittings occur inside an adapted covering $U$, which is also a covering of $P'$.

2.3 Fundamental group and admissible generating sets

Definition 2.7 Let $P$ be as in Notation 2.4. The fundamental group of $P$, denoted $\mathfrak{G}(P)$, is by definition the fundamental group of the complement of $P$ in the plane:

$$\mathfrak{G}(P) := \pi_1(\mathbb{C} \setminus P, \ast).$$

For $P$ as in Notation 2.4, the group $\mathfrak{G}(P)$ is a free group on $k$ generators: in the following we construct an explicit set of free generators for $\mathfrak{G}(P)$. See Fig. 2.

We choose an adapted covering $U$ of $P$, and use Notation 2.6. The boundary curves of $\bar{U}_1, \ldots, \bar{U}_\kappa$ are denoted by $\partial U_1, \ldots, \partial U_\kappa$ respectively, and are oriented clockwise. We also choose embedded arcs $\zeta_1, \ldots, \zeta_\kappa$ joining the basepoint $\ast$ with the curves $\partial U_1, \ldots, \partial U_\kappa$. We assume the following:

- for all $1 \leq i \leq \kappa$, the arc $\zeta_i$ has an endpoint at $\ast$, and the other endpoint on $\partial U_i$;
- there is no other intersection point between two arcs $\zeta_i, \zeta_j$ or between an arc $\zeta_i$ and a boundary curve $\partial U_j$, for $1 \leq i, j \leq \kappa$.

For $1 \leq i \leq \kappa$ let $f_i \in \mathfrak{G}(P)$ be the element represented by a loop that begins at $\ast$, runs along $\zeta_i$ until it reaches the intersection with $\partial U_i$, spins clockwise around $\partial U_i$ and runs back to $\ast$ along $\zeta_i$. Then $f_1, \ldots, f_\kappa$ exhibit $\mathfrak{G}(P)$ as a free group on $k$ generators.

Definition 2.8 Let $P$ be as in Notation 2.4. A set of generators $f_1, \ldots, f_\kappa$ of $\mathfrak{G}(P)$ obtained as described above is called an admissible generating set.
2.4 Fundamental PMQ

In the following we introduce several PMQs arising as subsets of $\mathfrak{G}(P)$, for $P$ as in Notation 2.4. Recall that a conjugacy class in $\mathfrak{G}(P)$ corresponds to a free (i.e. unbased) homotopy class of maps $S^1 \to \mathbb{C} \setminus P$.

Definition 2.9 Let $P$ be as in Notation 2.4. For all $1 \leq i \leq l$ we denote by $\Omega(P, z_i) \subset \mathfrak{G}(P)$ the conjugacy class corresponding to a small (unbased) simple closed curve that spins once, clockwise, around $z_i$; we define

$$
\Omega(P) = \Omega_{\mathcal{C}}(P) := \{1\} \cup \bigcup_{1 \leq i \leq l} \Omega(P, z_i) \subset \mathfrak{G}(P),
$$

and call it the fundamental PMQ of $P$ relative to the nice couple $\mathcal{C}$. We consider on $\Omega(P)$ the PMQ structure inherited from $\mathfrak{G}(P)$ (see [3, Definition 2.8]).

Note that a based loop representing a class in $\Omega(P)$ may intersect essentially $\mathcal{Y}$.

Let $P$ be as in Notation 2.4, and fix an admissible generating set $f_1, \ldots, f_k$ of $\mathfrak{G}(P)$ (see Definition 2.8): then the elements of $\Omega(P)$ are precisely $1$ and all conjugates in $\mathfrak{G}(P)$ of the elements $f_1, \ldots, f_l$. In particular the group isomorphism $\mathfrak{G}(P) \cong \mathbb{F}^k$, given by choosing an...
admissible generating set, restricts to a bijection $\mathcal{Q}(P) \cong FQ^k$ (see [3, Definition 3.2]), and the hypotheses required by [3, Definition 2.8] are fulfilled. The partial product of $\mathcal{Q}(P)$ is trivial, and $(\mathcal{Q}(P), \mathcal{G}(P))$ is a PMQ-group pair. See Fig. 3, left, for examples of elements in $\mathcal{Q}(P)$.

### 2.5 Extended fundamental PMQ

We extend Definition 2.9 by considering more general simple closed curves.

**Definition 2.10** Let $P$ be as in Notation 2.4. We denote by $\mathcal{Q}^{ext}(P) = \mathcal{Q}^{ext}_C(P) \subset \mathcal{G}(P)$ the union of all conjugacy classes corresponding to (unbased) oriented simple closed curves $\beta \subset C \setminus \mathcal{Y}$, such that $\beta$ spins clockwise and $\beta$ bounds a disc contained in $C \setminus \mathcal{Y}$. We consider on $\mathcal{Q}^{ext}(P)$ the PMQ structure inherited from $\mathcal{G}(P)$, and call it the *extended fundamental PMQ* of $P$ relative to the nice couple $\mathcal{C}$.

Note that there is an inclusion of sets $\mathcal{Q}(P) \subseteq \mathcal{Q}^{ext}(P)$. See Fig. 3, right, for a non-trivial example of an element in $\mathcal{Q}^{ext}(P)$. The fact that the hypotheses of [3, Definition 2.8] are fulfilled by $\mathcal{Q}^{ext}(P) \subseteq \mathcal{G}(P)$ needs some explanation: this is contained in the following two propositions. In the following, recall from [3, Definition 3.5] that a *decomposition* of an
element \( g \in \Omega^{\text{ext}}(P) \) with respect to \( \Omega(P) \) is a sequence \((g_1, \ldots, g_r)\) of elements of \( \Omega(P) \) whose product \( g_1 \ldots g_r \), computed in \( \mathcal{S}(P) \), is equal to \( g \).

**Proposition 2.11** Let \( P \) be as in Notation 2.4. The set \( \Omega^{\text{ext}}(P) \) is generated under partial product by \( \Omega(P) \), i.e., every element \( g \) of \( \Omega^{\text{ext}}(P) \) admits a decomposition \((g_1, \ldots, g_r)\) with respect to \( \Omega(P) \).

**Proposition 2.12** Let \( P \) be as in Notation 2.4. Then \( \Omega^{\text{ext}}(P) \subset \mathcal{S}(P) \) satisfies the hypotheses of [3, Definition 2.8], and hence inherits a structure of PMQ: as a consequence \((\Omega^{\text{ext}}(P), \mathcal{S}(P))\) is a PMQ-group pair.

The proofs of Propositions 2.11 and 2.12 are in Sects. A.1 and A.2 of the appendix, respectively.

It follows that the inclusion \( \Omega(P) \subseteq \Omega^{\text{ext}}(P) \) is a map of PMQs, and the inclusion \((\Omega(P), \mathcal{S}(P)) \subset (\Omega^{\text{ext}}(P), \mathcal{S}(P))\) is a map of PMQ-group pairs. In the following we study the problem of extending over \( \Omega^{\text{ext}}(P) \) maps of PMQs defined over \( \Omega(P) \).

**Definition 2.13** Let \( P \) be as in Notation 2.4, let \( \Omega \) be a PMQ and let \( \psi : \Omega(P) \to \Omega \) be a map of PMQs. Let \( g \in \Omega^{\text{ext}}(P) \) and let \((g_1, \ldots, g_r)\) be a decomposition of \( g \) with respect to \( \Omega(P) \). We say that \( \psi \) can be extended over \( g \) if the product \( \psi(g_1) \ldots \psi(g_r) \) is defined in \( \Omega \). We denote by \( \Omega^{\text{ext}}(P)_\psi = \Omega^{\text{ext}}(P)|_{\psi} \subseteq \Omega^{\text{ext}}(P) \) the subset containing all elements \( g \) over which \( \psi \) can be extended.

Some comments on Definition 2.13 are needed. For \( g \in \Omega^{\text{ext}}(P) \), the existence of a decomposition \((g_1, \ldots, g_r)\) of \( g \) with respect to \( \Omega(P) \) is granted by Proposition 2.11. This decomposition is in general not unique; nevertheless, by [3, Proposition 3.7], if \((g_1', \ldots, g_r')\) is another decomposition of \( g \) with respect to \( \Omega(P) \), then the two decompositions are connected by a sequence of standard moves (see [3, Definition 3.6]). Since \( \psi \) is a map of PMQs, we obtain that the sequence \((\psi(g_1), \ldots, \psi(g_r))\) of elements of \( \Omega \) can be transformed into the sequence \((\psi(g_1'), \ldots, \psi(g_r'))\) by a sequence of standard moves; it is then a direct consequence of the definition of PMQ, that the product \( \psi(g_1) \ldots \psi(g_r) \) is defined if and only if the product \( \psi(g_1') \ldots \psi(g_r') \) is defined, and if both products are defined then they are equal to each other. This shows that, whether \( \psi \) can be extended over \( g \), only depends on the element \( g \) but not on the decomposition \((g_1, \ldots, g_r)\) of \( g \) with respect to \( \Omega(P) \); moreover the assignment \( g \mapsto \psi(g_1) \ldots \psi(g_r) \) gives a well-defined map of sets \( \psi^{\text{ext}} : \Omega^{\text{ext}}(P)_\psi \to \Omega \), which extends the map \( \psi : \Omega(P) \to \Omega \).

**Proposition 2.14** The subset \( \Omega^{\text{ext}}(P)_\psi \subseteq \mathcal{S}(P) \) satisfies the requirements of [3, Definition 2.8], and therefore \( \Omega^{\text{ext}}(P)_\psi \) inherits a structure of PMQ. The map \( \psi^{\text{ext}} : \Omega^{\text{ext}}(P)_\psi \to \Omega \) is the unique map of PMQs \( \Omega^{\text{ext}}(P)_\psi \to \Omega \) restricting to \( \psi : \Omega(P) \to \Omega \) on \( \Omega(P) \).

The proof of Proposition 2.14 is in Sect. A.3 of the appendix. As a consequence of Proposition 2.14, \((\Omega^{\text{ext}}(P)_\psi, \mathcal{S}(P))\) is naturally a PMQ-group pair.

### 2.6 PMQs from Coverings

We extend the previous definitions by replacing a configuration of points \( P \) with a configuration of open, convex sets \( \overline{U} \) in \( \mathbb{C} \).

**Definition 2.15** Let \( \overline{U} \) be a covering of \( P \) (see Definition 2.5). We define \( \mathcal{G}(\overline{U}) \) as the group \( \pi_1(\mathbb{C} \setminus \overline{U}, *) \), and call it the fundamental group of \( \overline{U} \).
We use Notation 2.6 and define $\Omega(U) \subset \mathcal{G}(U)$ as the union of $\{1\}$ and the $\lambda$ conjugacy classes corresponding to the simple closed curves $\partial U_1, \ldots, \partial U_\lambda$, oriented clockwise. Similarly as in Definition 2.9, we consider on $\Omega(U)$ the PMQ structure inherited from $\mathcal{G}(U)$. The PMQ $\Omega(U)$ is called the fundamental PMQ of $U$.

Finally, we define $\Omega(P, U) = \Omega_c(P, U) \subset \Omega_{\text{ext}}(P)$ as the union of all conjugacy classes in $\mathcal{G}(P)$ represented by simple closed curves $\beta$ which are oriented clockwise and are contained in one of the regions $U_i \setminus P \subset \mathcal{C} \setminus \mathcal{Y}$, for some $1 \leq i \leq \lambda$. The set $\Omega(P, U)$ is called the relative fundamental PMQ of $P$ with respect to $U$.

Note that, for $U$ as in Notation 2.6, $\mathcal{G}(U)$ is a free group on $\kappa$ generators, and an admissible generating set $f_1, \ldots, f_\kappa$ can be constructed in the same way as in Sect. 2.3 to give an isomorphism $\mathcal{G}(U) \cong \mathbb{F}_\kappa$. By the same arguments used in Sect. 2.4, the previous identification restricts to an identification $\Omega(U) \cong \mathbb{F}_\kappa^\times$, and therefore, analogously as in the case of $\Omega(P) \subset \mathcal{G}(P)$, the set $\Omega(U)$ inherits from $\mathcal{G}(U)$ a structure of PMQ, and $(\Omega(U), \mathcal{G}(U))$ is a PMQ-group pair.

**Lemma 2.16** Using the notation above, the set $\Omega(P, U) \subset \mathcal{G}(P)$ inherits a structure of PMQ from $\mathcal{G}(P)$ in the sense of [3, Definition 2.8], and thus $(\Omega(P, U), \mathcal{G}(P))$ is a PMQ-group pair.

**Proof** We have to check that if $g = g_1 \ldots g_r$ is a decomposition of $g \in \Omega(P, U)$ with all factors $g_j \in \Omega(P, U)$, then for each $1 \leq j \leq j' \leq r$ the product $g_j \ldots g_{j'}$ also lies in $\Omega_c(P, U)$. Let $f_1, \ldots, f_k$ be an admissible generating set for $\mathcal{G}(P)$, and consider the abelianisation map $ab \colon \mathcal{G}(P) \to \mathcal{G}(P)^{ab} \cong \mathbb{Z}^k$, where the latter isomorphism is given by the basis $ab(f_1), \ldots, ab(f_k)$. For each $1 \leq i \leq \lambda$ consider the nice couple $\mathcal{C}_i = (\mathbb{H}, \mathbb{H} \setminus U_i)$. Then $\Omega_c(P, U) \subset \mathcal{G}(P)$ can be identified with the subset $\bigcup_{i=1}^{\lambda} \Omega_{\text{ext}}(\mathcal{C}_i)(P) \subset \mathcal{G}(P)$. Moreover $ab(g)$ is a vector with all entries equal to 0 or 1, as $g$ is in the conjugacy class represented by a simple closed curve spinning clockwise; if $g = 1$, then all entries are zero, and if $g \in \Omega_{\text{ext}}(\mathcal{C}_i) \setminus \{1\}$ for some $1 \leq i \leq \lambda$ then at least one entry is equal to 1, and all entries equal to 1 correspond to generators $f_j$ spinning around points of $P \cap U_i$. Since $ab(g) = ab(g_1) + \ldots + ab(g_r)$ is a sum with no cancelation in any coordinate, we conclude that there is a unique $1 \leq i \leq \lambda$ such that $g_1, \ldots, g_r \in \Omega_{\text{ext}}(\mathcal{C}_i)$ (such $\lambda$ is unique unless all $g_j = 1$). We can now apply Proposition 2.12 to $\Omega_{\text{ext}}(\mathcal{C}_i)$ to show that, for all $1 \leq j \leq j' \leq r$ the product $g_j \ldots g_{j'}$ also lies in $\Omega_{\text{ext}}(\mathcal{C}_i)(P)$, and hence in $\Omega_c(P, U)$.

We conclude the subsection by analysing which inclusions hold between the groups and PMQs introduced so far. The inclusion $\mathbb{C} \setminus U \subset \mathbb{C} \setminus P$ induces an injection of PMQ-group pairs $(\Omega_c(U), \mathcal{G}(U)) \subseteq (\Omega_c(P, U), \mathcal{G}(P))$.

On the other hand we have a chain of inclusions $\Omega_c(P, U) \subset \Omega_{\text{ext}}(P) \subset \mathcal{G}(P)$; observe that for a generic covering $U$ we do not have an inclusion $\Omega_c(P) \subset \Omega_c(P, U)$; yet Proposition 2.11 specialises to the fact that $\Omega(P, U)$ is generated by $\Omega(P) \cap \Omega(P, U)$ under partial multiplication.

### 2.7 Maps induced by forgetting points

**Notation 2.17** Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple and let $P \subseteq P' \subset \mathcal{X}$. We denote by $i_P^P : \mathcal{G}(P') \to \mathcal{G}(P)$ the map induced by the inclusion $\mathcal{C} \setminus P' \subset \mathcal{C} \setminus P$.

Note that $i_P^P$ restricts to maps $\Omega(P') \to \Omega(P)$ and $\Omega_{\text{ext}}(P') \to \Omega_{\text{ext}}(P)$. To see this, let $[\gamma] \in \Omega(P')$ (respectively $[\gamma] \in \Omega_{\text{ext}}(P')$) be represented by a loop $\gamma$ which is freely
homotopic in \( C \setminus P' \) to a simple curve \( \beta \subset C \setminus (P' \cup \mathcal{V}) \) spinning clockwise around at most one point (respectively, some points) of \( P' \setminus \mathcal{V} \); then the same properties hold for \( \iota_p' (|\gamma|) \) in \( \Omega(P) \) (respectively, in \( \Omega^{\text{ext}}(P) \)).

### 3 Hurwitz–Ran spaces with monodromies in a PMQ-group pair

In this section we define, for a PMQ-group pair \((\mathcal{Q}, G)\) and a nice couple \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) as in Definition 2.3, the Hurwitz–Ran space \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\), containing configurations of points in \(\mathcal{C}\) with monodromies in \((\mathcal{Q}, G)\).

Throughout the section we fix a PMQ-group pair \((\mathcal{Q}, G) = (\mathcal{Q}, G, e, r)\) as in [3, Definition 2.15] and a nice couple \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) as in Definition 2.3.

#### 3.1 Ran spaces

We recall the definition and the main properties of the Ran space \(\text{Ran}(\mathcal{X})\), focusing on the case of a connected subspace \(\mathcal{X} \subset \mathbb{H}\). We use [7, Sect. 5.5.1] as main reference.

**Definition 3.1** Let \(\mathcal{X} \subset \mathbb{H}\) be a subspace. We define \(\text{Ran}(\mathcal{X})\) as the set of all finite subsets \(P \subset \mathcal{X}\), including \(\emptyset\); we denote by \(\text{Ran}_+(\mathcal{X})\) the set \(\text{Ran}(\mathcal{X}) \setminus \{\emptyset\}\).

We define a topology on \(\text{Ran}(\mathcal{X})\). For \(P \in \text{Ran}(\mathcal{X})\) and \(U\) an adapted covering of \(P\) with respect to the nice couple \((\mathcal{X}, \emptyset)\) (see Definition 2.5), we let \(U(P, U) = \cup_{\mathcal{X}}(P, U) \subset \text{Ran}(\mathcal{X})\) be the subset of all \(P' \in \text{Ran}(\mathcal{X})\) satisfying the following:

- \(P' \subset U\);
- \(P' \cap U_i \neq \emptyset\) for all \(1 \leq i \leq \kappa\), using Notation 2.6.

A subset of the form \(U(P, U)\) is called a normal neighbourhood of \(P\) in \(\text{Ran}(\mathcal{X})\). Normal neighbourhoods form the basis of a Hausdorff topology on \(\text{Ran}(\mathcal{X})\).

For \(\emptyset \neq P_0 \subset \mathcal{X}\) we denote by \(\text{Ran}(\mathcal{X})_{P_0} \subset \text{Ran}_+(\mathcal{X})\) the subspace containing all \(P \subset \mathcal{X}\) with \(P_0 \subseteq P\). Similarly, for \(z_0 \in \mathcal{X}\) we denote \(\text{Ran}(\mathcal{X})_{z_0} = \text{Ran}(\mathcal{X})_{\{z_0\}}\).

Our definition of \(\text{Ran}(\mathcal{X})\) differs from the usual one in the literature (e.g. [7, Definition 5.5.1.2]) because we allow also \(\emptyset\) as a point in \(\text{Ran}(\mathcal{X})\). Note however that our \(\text{Ran}(\mathcal{X})\) is the topological disjoint union of the singleton \(\{\emptyset\}\) and \(\text{Ran}_+(\mathcal{X})\).

The following results are originally due to Beilinson and Drinfeld [1]. See also [7, Lemma 5.5.1.8, Theorem 5.5.1.6].

**Lemma 3.2** Let \(\mathcal{X} \subset \mathbb{H}\) be path connected and let \(P_0 \subset \mathcal{X}\) be a finite non-empty subset. Then \(\text{Ran}(\mathcal{X})_{P_0}\) is weakly contractible.

**Lemma 3.3** Let \(\mathcal{X} \subset \mathbb{H}\) be path connected. Then \(\text{Ran}_+(\mathcal{X})\) is weakly contractible.

**Notation 3.4** For a nice couple \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) we write \(\text{Ran}(\mathcal{C}) = \text{Ran}(\mathcal{X})\) and similarly for the subspaces introduced in Definition 3.1.

#### 3.2 Hurwitz sets

We first define \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) as a set, and discuss later in Sect. 3.3 its Ran topology, mimicking the topology on \(\text{Ran}(\mathcal{C})\).
Definition 3.5 Let \((\mathcal{Q}, G)\) be a PMQ-group pair and let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) a nice couple. An element of the Hurwitz set \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) is a configuration \(c = (P, \psi, \varphi)\), where

- \(P = \{z_1, \ldots, z_k\}\) is a finite subset of \(\mathcal{X}\), i.e. \(P \in \text{Ran}(\mathcal{X})\);
- \((\psi, \varphi) : (\Omega(P), \mathfrak{S}(P)) \rightarrow (\mathcal{Q}, G)\) is a map of PMQ-group pairs (see Definitions 2.7 and 2.9 for the PMQ-group pair \((\Omega(P), \mathfrak{S}(P))\)).

If \(c = (P, \psi, \varphi)\), we say that \(c\) is supported on \(P\); if \(S\) is a subspace of \(\mathcal{X}\) and \(P \subset S\), we say that \(c\) is supported in \(S\). The maps \(\psi\) and \(\varphi\) are the \(\mathcal{Q}\)-valued and \(G\)-valued monodromies of \(c\).

Roughly speaking, the monodromy \(\varphi\), with values in \(G\), is defined around every point of \(P\), whereas the monodromy \(\psi\), with values in \(\mathcal{Q}\), is defined only around points of \(P\) which lie in \(\mathcal{X} \setminus \mathcal{Y}\). We can think of \(\psi\) as a refinement of \(\varphi\) away from \(\mathcal{Y}\): indeed the composition \(\psi \circ \varphi : \Omega(P) \rightarrow G\) is equal to \(\varphi|_{\Omega(P)}\), where the map of PMQs \(\psi : \mathcal{Q} \rightarrow G\) is part of the structure of PMQ-group pair of \((\mathcal{Q}, G)\).

Notation 3.6 We usually present a configuration \(c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) as \(c = (P, \psi, \varphi)\) and use Notation 2.4 for \(P\). Similarly we present another configuration \(c'\) as \((P', \psi', \varphi')\), and write \(P' = \{z'_1, \ldots, z'_k\}\).

3.3 The topology on Hurwitz–Ran spaces

We introduce a topology on the set \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\), in the spirit of the topology of the Ran space \(\text{Ran}(\mathcal{C})\).

Definition 3.7 We use Notations 3.6 and 2.6. Let \(c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) and let \(U\) be an adapted covering of \(P\); we denote by \(\text{U}(c; U) = \text{U}_c(c; U)\) the subset of \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\) containing all configurations \(c'\) satisfying the following conditions:

- \(P' \in \text{U}(P, U)\) (see Definition 3.1); as a consequence there is a natural inclusion of PMQ-group pairs \((\Omega(U), \mathfrak{S}(U)) \subseteq (\Omega(P', U), \mathfrak{S}(P'))\) (see Definition 2.15);
- \(\Omega(P', U)\) is contained in \(\Omega^\text{ext}(P')_{\psi'}\) (see Definition 2.13), and the following composition of maps of PMQ-group pairs is equal to \((\psi, \varphi)\):

\[
(\Omega(P), \mathfrak{S}(P)) \xrightarrow{\sim} (\Omega(U), \mathfrak{S}(U)) \xleftarrow{\subseteq} (\Omega(P', U), \mathfrak{S}(P'))
\]

\[
(\Omega^\text{ext}(P')_{\psi'}, \mathfrak{S}(P')) \xrightarrow{((\psi')^\text{ext}, \varphi')} (\mathcal{Q}, G),
\]

where we use the isomorphism discussed in the remark after Definition 2.15, and the map \((\psi')^\text{ext} : \Omega^\text{ext}(P')_{\psi'} \rightarrow \mathcal{Q}\) from Proposition 2.14.

Each subset \(\text{U}(c; U)\) is called a normal neighbourhood of \(c\) in \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\).

Roughly speaking, if \(c' \in \text{U}(c; U)\), then \(P'\) is obtained from \(P\) by splitting each \(z_i\) into \(r_i \geq 1\) points \(z'_1, \ldots, z'_{r_i}\) inside the neighbourhood \(U_i\) of \(z_i\). For all \(1 \leq i \leq k\) the \(G\)-valued monodromy around \(z_i\) is decomposed as a product of the \(G\)-valued monodromies \(\varphi'\) around the points \(z'_1, \ldots, z'_{r_i}\); similarly, for \(1 \leq i \leq l\) the \(\mathcal{Q}\)-valued monodromy around \(z_i\) is decomposed as a product of the \(\mathcal{Q}\)-valued monodromies around the points \(z'_1, \ldots, z'_{r_i}\). See Fig. 4 for an example of two configuration \(c\) and \(c'\) with \(c'\) in a normal neighbourhood of \(c\).

Proposition 3.8 The subsets \(\text{U}(c; U)\) for varying \(c\) and \(U\) form the basis of a Hausdorff topology on the set \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)\).
Fig. 4 On left, a configuration $c = (P, \psi, \varphi)$ in the space Hur($X, Y, Q, G$) and an adapted covering $U$ of $P$; on right, another configuration $c'$ in the normal neighbourhood $U(c, U)$. The drawn loops are labelled with their $Q$-valued monodromy if they belong to $\Omega(P)$ and $\Omega(P')$ respectively, and are labelled with their $G$-valued monodromy otherwise.

**Proof** Let $c_1 = (P_1, \psi_1, \varphi_1)$ and $c_2 = (P_2, \psi_2, \varphi_2)$ denote two configurations in Hur($C; Q, G$), let $k_1 = |P_1|$ and $k_2 = |P_2|$, and let $U_1 = (U_{1,1}, \ldots, U_{1,k_1})$ and $U_2 = (U_{2,1}, \ldots, U_{2,k_2})$ be adapted coverings of $P_1$ and $P_2$ respectively; finally, let $U(c_1; U_1)$ and $U(c_2; U_2)$ be the corresponding normal neighbourhoods.

Suppose that $c' = (P', \psi', \varphi')$ lies in $U(c_1; U_1) \cap U(c_2; U_2)$. Then we can define $U' = (U_1', \ldots, U_{k_1}')$ as the family of all convex open sets of the form $U_{1,i} \cap U_{2,j}$ that contain at least one point of $P'$. By construction $U'$ is a covering of $P'$; we can then find a covering $U'' = (U_1'', \ldots, U_{k_1}'')$ of $P'$ which is adapted to $P'$ and is finer than $U'$, i.e. each $U_i''$ is contained in some $U_j'$. It follows from Definition 3.7 that

$$\Omega(c'; U'') \subseteq \Omega(c_1; U_1) \cap \Omega(c_2; U_2).$$

Hence normal neighbourhoods are the basis of a topology on Hur($C; Q, G$).

To see that this topology is Hausdorff, let $c, c' \in$ Hur($C; Q, G$) and use Notation 3.6. If $P = P'$, then for any adapted covering $U$ of $P$ the two normal neighbourhoods $\Omega(c; U)$ and $\Omega(c'; U)$ are disjoint. If $P \neq P'$, without loss of generality we can assume that there is a point $z \in P \setminus P'$; let $U$ and $U'$ be adapted coverings of $P$ and $P'$ respectively, such that the connected component of $U$ containing $z$ is disjoint from $U'$; then $\Omega(c; U)$ and $\Omega(c'; U')$ are again disjoint. 

$\square$
Definition 3.10 We define \( \varepsilon: \text{Hur}(\mathcal{C}; Q, G) \to \text{Ran}(\mathcal{C}) \) as the map given by the assignment \( \varepsilon: (P, \psi, \varphi) \mapsto P \), i.e. a configuration is sent to its support.

Note that the preimage of \( \Omega(P, U) \subset \text{Ran}(\mathcal{C}) \) along \( \varepsilon \) is the disjoint union of all normal neighbourhoods \( \Omega(c, U) \) for \( c \) varying in the configurations of \( \text{Hur}(\mathcal{C}; Q, G) \) supported on \( P \); this shows continuity of \( \varepsilon \). Note also that \( \varepsilon: \text{Hur}(\mathcal{C}; Q, G) \to \text{Ran}(\mathcal{C}) \) is a homeomorphism if \( (Q, G) = (1, 1) \), where we use the following notation.

Notation 3.11 We denote by \( (1, 1) \) the initial and terminal PMQ-group pair, consisting of the trivial PMQ \{1\} and of the trivial group \{1\}.

Notation 3.12 For all nice couples \( \mathcal{C} \) we denote by \( (\emptyset, 1, 1) \in \text{Hur}(\mathcal{C}; Q, G) \) the unique configuration \( (P, \psi, \varphi) \) with \( P = \emptyset \); note that in this case the maps \( \psi \) and \( \varphi \) are defined on the trivial PMQ and on the trivial group respectively, so they have as images \{1\} \subset Q and \{1\} \subset G respectively.

Notation 3.13 Let \( P_0 \subset X \) be a finite non-empty subset. We denote by \( \text{Hur}(\mathcal{C}; Q, G)_{P_0} \subset \text{Hur}_+(\mathcal{C}; Q, G) \) the preimage of \( \text{Ran}(\mathcal{C})_{P_0} \) along \( \varepsilon \). For \( c \in \text{Hur}(\mathcal{C}; Q, G)_{P_0} \) and an adapted covering \( U \) of \( \varepsilon(c) \) we denote

\[
\Omega(c, U)_{P_0} = \Omega(c, U) \cap \text{Hur}(\mathcal{C}; Q, G)_{P_0}.
\]

3.4 Change of ambient space

Let \( T \subset C \) be a contractible space containing \(*\), and denote by \( \hat{T} \) the interior of \( T \), i.e. the set of all \( z \in T \) for which there is an open disc \( z \in U \subset T \). Let \( \mathcal{C} = (X, Y) \) be a nice couple with \( Y \subset X \subset \hat{T} \); then for all finite subsets \( P \subset X \) the inclusion \( T \setminus P \to \hat{T} \setminus P \) induces an identification \( \pi_1(T \setminus P, *) \cong \mathfrak{S}(P) \). Similarly we identify \( \Omega_{\mathcal{C}}(P) \) with the subset of \( \pi_1(T \setminus P, *) \) consisting of \{1\} and all conjugacy classes corresponding to small simple closed curves in \( T \setminus P \) spinning clockwise around one of the points of \( P \setminus Y \). We thus have an alternative construction of the Hurwitz set \( \text{Hur}(\mathcal{C}; Q, G) \), in which we use as ambient space the subspace \( T \) instead of the entire \( C \). The topology on \( \text{Hur}(\mathcal{C}; Q, G) \) from Proposition 3.8 can be obtained by only considering coverings \( \mathcal{U} \subset T \).

Definition 3.14 We denote by \( \text{Hur}^T(\mathcal{C}; Q, G) \) the Hurwitz–Ran space constructed using \( T \) as ambient space. For \( P \subset X \) we let \( \mathfrak{S}^T(P) := \pi_1(T \setminus P, *) \cong \mathfrak{S}(P) \), and we denote by \( \Omega^T_{\mathcal{C}}(P) \subset \mathfrak{S}^T(P) \) the subset corresponding to \( \Omega_{\mathcal{C}}(P) \subset \mathfrak{S}(P) \).

Suppose now that we have a nice couple \( \mathcal{C} = (X, Y) \) and two contractible subspaces \( T, T_1 \subset C \) satisfying the following properties:

- \(* \in T_1 \subset T;\)
- \( X \subset \hat{T};\)
- \( X \) splits as a disjoint union \( X_1 \sqcup X_2 \), with \( X_1 \subset \hat{T}_1 \) and \( X_2 \) contained in the interior of \( T \setminus T_1 \).
Denote by $Y_1 = Y \cap X_1$ and by $C_1$ the nice couple $(X_1, Y_1)$; then every finite subset $P \subset X_1$ decomposes as a union of $P_1 = P \cap X_1$ and $P_2 = P \cap X_2$; moreover the inclusion $T_1 \setminus P_1 \hookrightarrow T \setminus P$ induces an inclusion of PMQ-group pairs $\iota_{T_1}^T(P_1, P): (\Omega_{C_1}^{T_1}(P_1), \Theta_{C_1}^{T_1}(P_1)) \hookrightarrow (\Omega_{C}^{T}(P), \Theta_{C}^{T}(P))$, and if $(\varphi, \psi): (\Omega_{C}^{T}(P), \Theta_{C}^{T}(P)) \rightarrow (Q, G)$ is a map of PMQ-group pairs, we can consider the restriction $(\varphi, \psi) \circ \iota_{T_1}^T(P_1, P): (\Omega_{C_1}^{T_1}(P_1), \Theta_{C_1}^{T_1}(P_1)) \rightarrow (Q, G)$.

**Definition 3.15** The above construction gives a map of sets $\iota_{T_1}^T: \text{Hur}^T_C(Q, G) \rightarrow \text{Hur}_{T_1}^T(C_1; Q, G)$, defined by sending $c = (P, \psi, \varphi)$ to $c' = (P', \psi', \varphi')$, where $P' = P_1 = P \cap T_1$ and $(\psi', \varphi') = (\varphi, \psi) \circ \iota_{T_1}^T(P_1, P)$. See Fig. 5.

To prove that $\iota_{T_1}^T$ is continuous, let $c$ and $c'$ be as in Definition 3.15 and choose an adapted covering $U' \subset T$ of $P'$ with respect to the nice couple $C_1$; since $P_2$ is contained in the interior of $T \setminus T_1$, we can extend $U'$ to an adapted covering of $P$ with respect to the nice couple $C$, by adjoining open sets contained in $T \setminus T_1 \subset C$ and covering $P_2$. We then have that $\iota_{T_1}^T$ sends $\iota(c; U) \subset \text{Hur}^T_C(Q, G)$ inside $\iota(c'; U') \subset \text{Hur}_{T_1}^T(C_1; Q, G)$.

Note also that the canonical homeomorphism from Definition 3.14 can be rewritten as $\iota_{T_1}^T: \text{Hur}(C; Q, G) \cong \text{Hur}_{T_1}^T(C_1; Q, G)$.

---

**Fig. 5** On left: a contractible subspace $T_1 \subset C$; a nice couple $C = (X, Y)$ decomposing as a disjoint union $C_1 \sqcup C_2$, with $C_1 \subset \mathring{T_1}$ and $C_2$ contained in the interior of $C \setminus T_1$; and a configuration $c \in \text{Hur}(C; Q, G)$. On right, the image of $\iota_{T_1}^T(c)$ in $\text{Hur}_{T_1}^T(C_1; Q, G)$.
Consider now the following setting. Let $\mathbb{T}_1, \mathbb{T}_2$ be contractible subspaces of $\mathbb{C}$ containing $\ast$, such that $\mathbb{T}_1 \cap \mathbb{T}_2$ and $\mathbb{T} := \mathbb{T}_1 \cup \mathbb{T}_2$, are contractible. Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple of subspaces of $\mathbb{T}$, and assume that we have a splitting $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 = (\mathcal{X}_1, \mathcal{Y}_1) \cup (\mathcal{X}_2, \mathcal{Y}_2)$, with $\mathcal{X}_1 \subseteq \mathbb{T}_1 \setminus \mathbb{T}_2$ and $\mathcal{X}_2 \subseteq \mathbb{T}_2 \setminus \mathbb{T}_1$. Given $c_1 = (P_1, \psi_1, \varphi_1) \in \text{Hur}^{\mathbb{T}_1}(\mathcal{C}_1; Q, G)$ and $c_2 = (P_2, \psi_2, \varphi_2) \in \text{Hur}^{\mathbb{T}_2}(\mathcal{C}_2; Q, G)$, we can define a new configuration $(P, \varphi, \psi) \in \text{Hur}^\mathbb{T}(\mathcal{C}; Q, G)$ as follows:

- $P = P_1 \cup P_2$;
- by the theorem of Seifert and van Kampen the group $\mathcal{G}(P)$ decomposes naturally as a free product $\mathcal{G}(P_1) * \mathcal{G}(P_2)$; we define $\varphi : \mathcal{G}(P) \to G$ as $\varphi_1 * \varphi_2$;
- the inclusions $\mathcal{G}(P_1) \subset \mathcal{G}(P)$ and $\mathcal{G}(P_2) \subset \mathcal{G}(P)$ restrict to inclusions $\Omega^{\mathcal{C}_1}(P_1) \subset \Omega^{\mathcal{C}_2}(P)$ and $\Omega^{\mathcal{C}_2}(P_2) \subset \Omega^{\mathcal{C}_2}(P)$; using [3, Theorem 3.3] we define $\psi : \Omega^{\mathcal{C}_2}(P) \to G$ by imposing that it restricts to $\psi_1$ on $\Omega^{\mathcal{C}_1}(P_1)$ and to $\psi_2$ on $\Omega^{\mathcal{C}_2}(P_2)$, and that $(\psi, \varphi) : (\mathcal{G}(P), \Omega^{\mathcal{C}_2}(P)) \to (Q, G)$ is a map of PMQ-group pairs.

**Definition 3.16** The above construction gives a map of sets

$$\square : \text{Hur}^{\mathbb{T}_1}(\mathcal{C}_1; Q, G) \times \text{Hur}^{\mathbb{T}_2}(\mathcal{C}_2; Q, G) \to \text{Hur}^{\mathbb{T}}(\mathcal{C}; Q, G).$$

To prove that $\square$ is continuous, note that if $U_1 \subset \mathbb{T}_1$ is an adapted covering of $P_1$ with respect to $\mathcal{C}_1$, and $U_2 \subset \mathbb{T}_2$ is an adapted covering of $P_2$ with respect to $\mathcal{C}_2$, then $\square$ restricts to a bijection between $\Omega(c, U_1) \times \Omega(c, U_2)$ and $\Omega(c, U)$, where $U = U_1 \cup U_2 \subset \mathbb{T}$ is also an adapted covering of $P$. This argument shows in fact that $\square$ is a homeomorphism, with inverse given by the map

$$\left(\iota_1^\mathbb{T}, \iota_2^\mathbb{T}\right) : \text{Hur}^{\mathbb{T}}(\mathcal{C}; Q, G) \to \text{Hur}^{\mathbb{T}_1}(\mathcal{C}_1; Q, G) \times \text{Hur}^{\mathbb{T}_2}(\mathcal{C}_2; Q, G).$$

### 4.1 Functoriality in the PMQ-group pair

In this subsection we fix a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ as in Definition 2.3 and prove the following theorem.

**Theorem 4.1** The assignment $(Q, G) \mapsto \text{Hur}(\mathcal{C}; Q, G)$ extends to a functor from the category of PMQ-group pairs to the category of topological spaces.

**Proof** Let $(Q, G)$ and $(Q', G')$ be two PMQ-group pairs, and let $(\Psi, \Phi) : (Q, G) \to (Q', G')$ be a morphism of PMQ-group pairs. In the following we define an induced map $(\Psi, \Phi)_* : \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q', G')$.

Given a configuration $c = (P, \psi, \varphi)$ in the set $\text{Hur}(\mathcal{C}; Q, G)$, we associate with it the configuration $c' = (P, \psi', \varphi')$ in the set $\text{Hur}(\mathcal{C}; Q', G')$, where the map of PMQ-group pairs $(\psi', \varphi') : (\mathcal{G}(P), \Omega(P)) \to (Q', G')$ is the composition $(\Psi, \Phi) \circ (\psi, \varphi)$:

$$(\psi', \varphi') : (\mathcal{G}(P), \Omega(P)) \xrightarrow{(\psi, \varphi)} (Q, G) \xrightarrow{(\Psi, \Phi)} (Q', G').$$
We obtain a map of sets \((\Psi, \Phi)_* = \text{Hur}(\mathcal{C}; \Psi, \Phi) : \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q', G')\). To show that \((\Psi, \Phi)_*\) is continuous, let \(c' = (P', \psi', \varphi') \in \text{Hur}(\mathcal{C}; Q', G')\) and let \(\mathcal{U}(c', U') \subset \text{Hur}(\mathcal{C}; Q', G')\) be the normal neighbourhood associated with an adapted covering \(U'\) of \(P'\) (see Definition 3.7). Then the preimage of \(\mathcal{U}(c', U')\) along \((\Psi, \Phi)_*\) is the disjoint union \(\bigsqcup_{c} \mathcal{U}(c, U')\), where \(c\) ranges over all configurations in the fibre \((\Psi, \Phi)_*^{-1}(c')\); in particular it is an open set.

The assignment \((\Psi, \Phi) \mapsto (\Psi, \Phi)_*\) makes \(\text{Hur}(\mathcal{C}; -)\) into a functor from the category \text{PMQGrp}\) of PMQ-group pairs to the category \text{Top}\) of topological spaces.

Note also that if \((\Psi, \Phi)\) is an injective map of PMQ-group pairs, then it induces an inclusion of spaces \((\Psi, \Phi)_* : \text{Hur}(\mathcal{C}; Q, G) \hookrightarrow \text{Hur}(\mathcal{C}; Q', G')\), i.e. \((\Psi, \Phi)_*\) is a homeomorphism onto its image. In particular we can take \(Q' = \hat{Q}\) to be the completion of \(Q\), and consider the inclusion of PMQ-group pairs \((Q, G) \subset (\hat{Q}, G)\), yielding an inclusion of \(\text{Hur}(\mathcal{C}; Q, G)\) into the Hurwitz–Ran space \(\text{Hur}(\mathcal{C}; \hat{Q}, G)\), associated with a PMQ-group pair consisting of a complete PMQ and a group. We further notice that the inclusion \(\text{Hur}(\mathcal{C}; Q, G) \subset \text{Hur}(\mathcal{C}; \hat{Q}, G)\) is open: given \(c \in \text{Hur}(\mathcal{C}; Q, G)\) and an adapted covering \(U\) of \(\varepsilon(c)\), the normal neighbourhood \(\mathcal{U}(c; U) \subset \text{Hur}(\mathcal{C}; Q, G)\) is mapped bijectively onto the corresponding normal neighbourhood \(\mathcal{U}'(c; U) \subset \text{Hur}(\mathcal{C}; \hat{Q}, G)\).

Another consequence of the functoriality in the PMQ-group pair is the following. For all PMQ-group pairs \((Q, G)\) there is a unique inclusion of PMQ-group pairs \((1, 1) \hookrightarrow (Q, G)\) (see Notation 3.11). This induces an inclusion \(\text{Hur}(\mathcal{C}; 1, 1) \to \text{Hur}(\mathcal{C}; Q, G)\), and using the homeomorphism \(\varepsilon : \text{Hur}(\mathcal{C}; 1, 1) \cong \text{Ran}(\mathcal{C})\) (see Definition 3.10), we obtain a natural inclusion \(\text{Ran}(\mathcal{C}) \subset \text{Hur}(\mathcal{C}; Q, G)\), for all nice couples \(\mathcal{C}\) and all PMQ-group pairs \((Q, G)\). Vice versa, we can consider the map \(\varepsilon : \text{Hur}(\mathcal{C}; Q, G) \to \text{Ran}(\mathcal{C})\) as the map \(\text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; 1, 1)\) induced by the unique map of PMQ-group pairs \((Q, G) \to (1, 1)\).

### 4.2 Two categories of nice couples

We now fix a PMQ-group pair \((Q, G)\) throughout the rest of the section. To discuss functoriality of Hurwitz–Ran spaces in the nice couple \(\mathcal{C}\), we first need a good notion of map between nice couples.

**Definition 4.2** Recall Definition 2.3, and let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) and \(\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')\) be two nice couples. A **morphism of nice couples** \(\xi : \mathcal{C} \to \mathcal{C}'\) is a continuous, pointed map \(\xi : (\mathcal{C}, \ast) \to (\mathcal{C}', \ast)\) such that the following properties hold:

1. \(\xi\) is semi-algebraic and proper from \(\mathcal{C}\) to \(\mathcal{C}'\);
2. \(\xi\) is **orientation-preserving** in the sense that the induced map \(\xi^* : H^2_\mathcal{C}(\mathcal{C}) \to H^2_\mathcal{C}(\mathcal{C}')\) in cohomology with compact support is the identity;
3. \(\xi\) restricts to maps \(\mathcal{X} \to \mathcal{X}'\) and \(\mathcal{Y} \to \mathcal{Y}'\);
4. for all \(z \in \mathcal{C}\) the fibre \(\xi^{-1}(z) \subset \mathcal{C}\) is non-empty, compact and contractible;
5. for all \(z \in \mathcal{X}' \setminus \mathcal{Y}'\) the fibre \(\xi^{-1}(z)\) contains at most one point of \(\mathcal{X}' \setminus \mathcal{Y}'\).

The composition of two morphisms \(\xi : \mathcal{C} \to \mathcal{C}' \text{ and } \xi' : \mathcal{C}' \to \mathcal{C}''\) is defined as the composition of maps \(\xi' \circ \xi : (\mathcal{C}, \ast) \to (\mathcal{C}, \ast)\). We denote by \(\text{NC}\) the category of nice couples and morphisms of nice couples.

Property (4) ensures that a morphism \(\xi\) of nice couples is in particular a **local homotopy equivalence** in the following sense: if \(\mathcal{J} \subset \mathcal{C}\) is a semi-algebraic set, then the restriction
\(\xi: \xi^{-1}(\mathcal{J}) \to \mathcal{J}\) is a homotopy equivalence; more generally, if \(\mathcal{J} \subset \mathcal{J}' \subset \mathcal{C}\) are two semi-algebraic sets, then \(\xi: (\xi^{-1}(\mathcal{J}), \xi^{-1}(\mathcal{J}')) \to (\mathcal{J}, \mathcal{J}')\) is a homotopy equivalence of couples. This follows from the main theorem of [9].

The previous remark holds in particular when \(\mathcal{J} = (\xi')^{-1}(\mathcal{C})\) is a fibre of another morphism of nice couples \(\xi'\), over some point \(z \in \mathcal{C}\); thus the composition \(\xi \circ \xi'\) also satisfies property (4) of Definition 4.2; properties (1),(2),(3) and (5) are also automatically satisfied by the composition \(\xi \circ \xi'\). This proves in particular that Definition 4.2 is a good definition. We will sometimes need to relax condition (5) in Definition 4.2, hence we give the following definition.

**Definition 4.3** Let \(\mathcal{C}\) and \(\mathcal{C}'\) be nice couples as in Definition 4.2. A lax morphism of nice couples is a map \(\xi: (\mathcal{C}, \ast) \to (\mathcal{C}, \ast)\) satisfying all conditions in Definition 4.2 except, possibly, condition (5). We denote by \(\mathbf{LNC}\) the category of nice couples and lax morphisms of nice couples.

Note that \(\mathbf{NC}\) is a subcategory of \(\mathbf{LNC}\) containing all objects, but not all morphisms. Whenever we refer to a morphism of nice couples without specifying the word “lax”, we will assume that condition (5) in Definition 4.2 holds.

We conclude the subsection with the following remark: if \(D' \subset \mathcal{C}\) is a subspace homeomorphic to a disc and \(\xi: (\mathcal{C}, \ast) \to (\mathcal{C}, \ast)\) satisfies properties (1),(2) and (4) from Definition 4.2, then also \(\xi^{-1}(D')\) is homeomorphic to a disc.

### 4.3 Functoriality in \(\mathbf{NC}\)

In this subsection we prove the following theorem

**Theorem 4.4** The assignment \(\mathcal{C} \mapsto \operatorname{Hur}(\mathcal{C}; Q, G)\) extends to a functor from the category \(\mathbf{NC}\) to the category of topological spaces.

Fix two nice couples \(\mathcal{C}\) and \(\mathcal{C}'\) and let \(\xi: \mathcal{C} \to \mathcal{C}'\) be a morphism of nice couples. In the following we construct an induced map \(\xi_*: \operatorname{Hur}(\mathcal{C}; Q, G) \to \operatorname{Hur}(\mathcal{C}'; Q, G)\).

Given a configuration \(c = (P, \psi, \varphi)\) in the set \(\operatorname{Hur}(\mathcal{C}; Q, G)\), we associate with it a configuration \(c' = (P', \psi', \varphi')\) in the set \(\operatorname{Hur}(\mathcal{C}'; Q, G)\) as follows. First, we define \(P' := \xi(P)\) \(\in \operatorname{Ran}(\lambda')\). To define \(\psi'\), note that \(\xi\) restricts to a homotopy equivalence \(\mathcal{C} \setminus \xi^{-1}(P') \to \mathcal{C} \setminus P\); in particular we obtain an isomorphism of groups \(\mathcal{G}(P') \cong \pi_1(\mathcal{C} \setminus \xi^{-1}(P'), \ast)\). Moreover the inclusion \(\mathcal{C} \setminus \xi^{-1}(P') \hookrightarrow \mathcal{C} \setminus P\) induces a map of groups \(\pi_1(\mathcal{C} \setminus \xi^{-1}(P'), \ast) \to \mathcal{G}(P)\).

We denote by \(\xi^*: \mathcal{G}(P') \to \mathcal{G}(P)\) the composition \(\mathcal{G}(P') \cong \pi_1(\mathcal{C} \setminus \xi^{-1}(P'), \ast) \to \mathcal{G}(P)\), and define \(\psi': \mathcal{G}(P') \to G\) as the composition \(\varphi \circ \xi^*\).

**Lemma 4.5** The map of groups \(\xi^*: \mathcal{G}(P') \to \mathcal{G}(P)\) restricts to a map of PMQs \(\Omega_{\mathcal{C}'}(P') \to \Omega_{\mathcal{C}}(P)\).

The proof of Lemma 4.5 is in Sect. A.4 of the appendix. We can now define \(\psi' := \psi \circ \xi^*: \Omega_{\mathcal{C}'}(P') \to Q\). Since \(\xi^*: (\Omega_{\mathcal{C}'}(P'), \mathcal{G}(P')) \to (\Omega_{\mathcal{C}}(P), \mathcal{G}(P))\) is a map of PMQ-group pairs, also \((\psi', \varphi') = (\psi, \varphi) \circ \xi^*\) is a map of PMQ-group pairs; hence \(c' = (P', \psi', \varphi')\) is a well-defined configuration in \(\operatorname{Hur}(\mathcal{C}'; Q, G)\). This construction gives a map of sets \(\xi_*: \operatorname{Hur}(\mathcal{C}; Q, G) \to \operatorname{Hur}(\mathcal{C}'; Q, G)\); see Fig. 6.

**Proof of Theorem 4.4** It is left to check that the map \(\xi_*\) constructed above is continuous. Let \(c \in \operatorname{Hur}(\mathcal{C}; Q, G)\), denote \(c' = \xi_*(c)\), and use Notation 3.6; let \(U(\mathcal{C}', U') \subset \operatorname{Hur}(\mathcal{C}'; Q, G)\) be a normal neighbourhood associated with an adapted covering \(U'\) of \(P'\). We have \(\xi(P) \subset U'\), hence we can find an adapted covering \(U\) of \(P\), such that \(\xi(U) \subset U'\). Then the entire normal
Fig. 6  On left, a configuration $c \in \text{Hur}(\mathcal{X}, \mathcal{Y}, Q, G)$; on right, its image $c' \in \text{Hur}(\mathcal{X}', \mathcal{Y}', Q, G)$ along the map $\xi_*$ induced by a morphism of nice couples $\xi$. The morphism $\xi$ has the effect of collapsing horizontally a rectangular region of $\mathcal{X}$ onto the vertical segment $\mathcal{Y}'$, and of expanding horizontally the complement of this rectangular region. The thick horizontal segment is the preimage along $\xi$ of $\xi(z_2) = \xi(z_3)$. The dashed loop on left is the image of the dashed loop on right along $\xi^*$. The thick horizontal segment is the preimage along $\xi$ of $\xi(z_2) = \xi(z_3)$. The dashed loop on left is the image of the dashed loop on right along $\xi^*$.

neighbourhood $\mathcal{U}(c; U) \subseteq \text{Hur}(\mathcal{C}; Q, G)$ is mapped along $\xi_*$ inside $\mathcal{U}(\xi(c'), U')$. Thus sending $\xi \mapsto \xi_*$ makes $\text{Hur}(\mathcal{C}; Q, G)$ a functor from $\text{NC}$ to $\text{Top}$. \hfill $\square$

Note that if $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ are two nice couples, and if $\mathcal{X} \subseteq \mathcal{X}'$, $\mathcal{Y} \subseteq \mathcal{Y}'$ and $\mathcal{Y} = \mathcal{X} \cap \mathcal{Y}'$, then $\text{Id}_\mathcal{C}$ is a morphism of nice couples $\mathcal{C} \rightarrow \mathcal{C}'$. The induced map $(\text{Id}_\mathcal{C})_* : \text{Hur}(\mathcal{C}; Q, G) \rightarrow \text{Hur}(\mathcal{C}'; Q, G)$ is an inclusion of spaces: more precisely, $\text{Hur}(\mathcal{C}; Q, G)$ contains all configurations of $\text{Hur}(\mathcal{C}'; Q, G)$ supported in $\mathcal{X}$.

In particular, given a nice couple $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$, by Definition 2.3 the space $\mathcal{Y}$ is closed in $\mathcal{X} \subseteq \mathbb{H}$: this means that the closure $\bar{\mathcal{Y}}$ of $\mathcal{Y}$ in $\mathbb{H}$ is contained in $\bar{\mathcal{X}}$, and $\mathcal{Y} = \mathcal{X} \cap \bar{\mathcal{Y}}$. Then $\text{Id}_\mathcal{C}$ is a morphism of nice couples $\mathcal{C} = (\mathcal{X}, \mathcal{Y}) \rightarrow \mathcal{C} := (\bar{\mathcal{X}}, \bar{\mathcal{Y}})$. Thus every Hurwitz–Ran space $\text{Hur}(\mathcal{C}; Q, G)$ can be regarded as a subspace of a Hurwitz–Ran space $\text{Hur}(\mathcal{C}; Q, G)$ associated with a nice couple of closed subspaces of $\mathbb{H}$.

4.4 A weak form of enriched functoriality

One can consider $\text{NC}$ as a category enriched in topological spaces: for all nice couples $\mathcal{C}$ and $\mathcal{C}'$ one can consider the compact-open topology on the set of morphisms $\xi : \mathcal{C} \rightarrow \mathcal{C}'$, considered as a subset of all continuous maps $\xi : \mathcal{C} \rightarrow \mathcal{C}$. The functor $\text{Hur}(\mathcal{C}; Q, G)$ is
then likely to be a Top-enriched functor from NC to Top. We will not attempt to prove this property in general, and we will restrict our attention to the following proposition.

**Proposition 4.6** Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ be nice couples and let $(Q, G)$ be a PMQ-group pair. Let $S$ be a topological space, and let $\mathcal{H} : C \times S \rightarrow C$ be a continuous map, such that for all $s \in S$ the map $\mathcal{H}(\cdot, s) : C \rightarrow C$ is a morphism of nice couples $\mathcal{C} \rightarrow \mathcal{C}'$ (see Definition 4.2). Let $H_{s} : \text{Hur}(C; Q, G) \times S \rightarrow \text{Hur}(C'; Q, G)$ be the map of sets defined by $H_{s}((c, s)) = (\mathcal{H}(c, s))_{s}$. Then $H_{s}$ is continuous.

**Proof** Fix $(c, s) \in \text{Hur}(C; Q, G) \times S$, let $c' = H_{s}(c, s)$ and use Notation 3.6. Let $U'$ be an adapted covering of $P'$ and let $\mathcal{U}'(c', U') \subset \text{Hur}(C'; Q, G)$ be the corresponding normal neighbourhood. By continuity of $\mathcal{H}$ we can find a neighbourhood $V \subset S$ of $s$ and an adapted covering $U$ of $P$ such that $\mathcal{H}$ sends $U \times V$ inside $U'$; here we regard $U$ and $U'$ as subsets of $C$. Then $H_{s}$ sends the product neighbourhood $\mathcal{U}(c, U) \times V \subset \text{Hur}(C; Q, G) \times S$ inside $\mathcal{U}(c', U')$.

### 4.5 Functoriality in LNC

In this subsection we assume that $Q$ is a complete PMQ (all products are defined), and write $\hat{Q} = Q$ to stress this choice; hence we work with the PMQ-group pair $(\hat{Q}, G)$. We prove the following theorem.

**Theorem 4.7** The assignment $\mathcal{C} \mapsto \text{Hur}(C; \hat{Q}, G)$ extends to a functor from the category LNC to the category of topological spaces.

Let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be a nice couple, and let $\psi = (P, \psi, \varphi) \in \text{Hur}(C; \hat{Q}, G)$. By Definition 3.5, $\psi$ is a map of PMQs defined on $\Omega(P)$; using the completeness of $\hat{Q}$, Proposition 2.11 implies the equality $\Omega^{\text{ext}}(P) = \Omega^{\text{ext}}(P)_{\psi}$ (see also Definitions 2.10 and 2.13). Proposition 2.14 yields a map of PMQ-group pairs

$$(\psi^{\text{ext}}, \varphi) : (\Omega^{\text{ext}}(P), \mathfrak{S}(P)) \rightarrow (\hat{Q}, G),$$

extending $(\psi, \varphi) : (\Omega(P), \mathfrak{S}(P)) \rightarrow (\hat{Q}, G)$.

**Definition 4.8** We define a set $\text{Hur}^{\text{ext}}(\mathcal{C}; \hat{Q}, G)$: it contains triples $\psi = (P, \psi, \varphi)$, where $P \in \text{Ran}(\mathcal{X})$ is a finite subset of $\mathcal{X}$, and $(\psi, \varphi) : (\Omega^{\text{ext}}(P), \mathfrak{S}(P)) \rightarrow (\hat{Q}, G)$ is a map of PMQ-group pairs.

The previous discussion implies that the sets $\text{Hur}^{\text{ext}}(\mathcal{C}; \hat{Q}, G)$ and $\text{Hur}(\mathcal{C}; \hat{Q}, G)$ are in natural bijection. We can use this bijection to transfer the topology of $\text{Hur}(\mathcal{C}; \hat{Q}, G)$ to $\text{Hur}^{\text{ext}}(\mathcal{C}; \hat{Q}, G)$; in particular, for a configuration $\psi = (P, \psi, \varphi) \in \text{Hur}^{\text{ext}}(\mathcal{C}; \hat{Q}, G)$ and for an adapted covering $U$ of $P$, the normal neighbourhood $\mathfrak{U}(\psi, U) = \Omega(U, \mathfrak{S}(U)) \subset \text{Hur}^{\text{ext}}(\mathcal{C}; \hat{Q}, G)$ contains all configurations $(P', \psi', \varphi')$ such that

- $P' \subset U$; as a consequence there is a natural inclusion of PMQ-group pairs $(\Omega(U), \mathfrak{S}(U)) \subset (\Omega(P', U), \mathfrak{S}(P'))$ (see Definition 2.15);
the following composition of maps of PMQ-group pairs is equal to the restriction of $(ψ, φ)$ on the PMQ-group pair $(Ω(P), Σ(P))$:

$$(Ω(P), Σ(P)) \overset{=} \rightarrow (Ω(U), Σ(U)) \subseteq (Ω(P', U), Σ(P'))$$

$$(Ω^\text{ext}(P'), Σ(P')) \overset{((ψ' \circ ξ)^\text{ext}, φ')}{\rightarrow} (\hat{Q}, G).$$

Given a lax morphism of nice couples $ξ : C \rightarrow C'$, we can now follow the same procedure used in Sect. 4.3 and define a continuous map $ξ^* : Hurext_C(C; ˆQ, G) \rightarrow Hurext_{C'}(C; ˆQ, G)$.

The only difference is that Lemma 4.5 is replaced by the following lemma, whose proof is in Sect. A.5 of the appendix.

**Lemma 4.9** Let $ξ : C \rightarrow C'$ be a lax morphism of nice couples, let $P \subset X$ and let $P' = ξ(P) \subset X'$. Then the map of groups $ξ^* : G(P') \rightarrow G(P)$ restricts to a map $Q^\text{ext}_C(P') \rightarrow Q^\text{ext}_C(P)$ of PMQs.

Continuity of $ξ^* : Hurext_C(C; ˆQ, G) \rightarrow Hurext_{C'}(C; ˆQ, G)$ is proved in the same way as in the case of a (non-lax) morphism of nice couples; similarly one can generalise Proposition 4.6 to the following.

**Proposition 4.10** Let $C = (X, Y)$ and $C' = (X', Y')$ be nice couples and let $(ˆQ, G)$ be a PMQ-group pair with $ˆQ$ complete. Let $S$ be a topological space, and let $H : C \times S \rightarrow C$ be a continuous map, such that for all $s \in S$ the map $H(−, s) : C \rightarrow C$ is a lax morphism of nice couples $C \rightarrow C'$ (see Definition 4.2). Let

$${\mathcal{H}}_s : Hurext(C; ˆQ, G) \times S \rightarrow Hurext(C'; ˆQ, G)$$

be the map of sets defined by $H_s(c, s) = (H(−, s))^*(c)$. Then $H_s$ is continuous.

## 5 Applications of functoriality

In this section we apply the results from Sect. 4 to obtain basic information about Hurwitz–Ran spaces; moreover we introduce the operation of external product.

### 5.1 Product structure for normal neighbourhoods

The first application combines the discussion of Sect. 3.4 with the functoriality with respect to inclusions of nice couples.

**Proposition 5.1** Let $c \in Hurext(C; Q, G)$, use Notations 3.6 and 2.6, and let $U$ be an adapted covering of $P$. Then there exist configurations $c'_i \in Hurext(C; Q, G)$ supported on $\{z_i\}$ and a homeomorphism

$${\mathcal{U}}(c, U) \cong \prod_{i=1}^k {\mathcal{U}}(c'_i, U_i).$$

**Proof** We can fix arcs $ζ_1, \ldots, ζ_k$ as in Definition 2.8: the arc $ζ_i$ joins $*$ with a point on $∂U_i$. For all $1 \leq i \leq k$ we define $T_i = ζ_i \cup U_i$, $X_i = X \cap U_i$ and $Y_i = Y \cap U_i$. Let moreover

$${\mathcal{U}}(c, U).$$
The homeomorphism of Proposition 5.1 depends in general on the arcs \( \zeta_i \).

5.2 Three useful homeomorphisms

In this subsection we prove three homeomorphisms between Hurwitz–Ran spaces. First, let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) be a nice couple and let \( (Q, e, \tau) \) be a PMQ-group pair. The map of PMQs \( e : Q \to G \) has an adjoint map of groups \( G(e) : G(Q) \to G \), so that the couple of maps \( \text{Id}_Q : Q \to Q \) and \( G(e) : G(Q) \to G \) yields a map of PMQ-group pairs \( (\text{Id}_Q, G(e)) : (Q, G(Q)) \to (Q, G) \). By functoriality we obtain a continuous map

\[
(\text{Id}_Q, G(e))_* : \text{Hur}(\mathcal{C}, Q, G(Q)) \to \text{Hur}(\mathcal{C}, Q, G).
\]

Lemma 5.2 Let \( \mathcal{C} \) be a nice couple of the form \( (\mathcal{X}, \emptyset) \); then the above map \( (\text{Id}_Q, G(e))_* \) is a homeomorphism.

The proof of Lemma 5.2 is in Sect. A.6 of the appendix. Roughly speaking, Lemma 5.2 says that if we consider a nice couple of the form \( (\mathcal{X}, \emptyset) \), then the space \( \text{Hur}(\mathcal{C}, Q, G(Q)) \) only depends on \( Q \): the monodromy \( \psi \) uniquely determines the monodromy \( \phi \). This motivates the following notation, which can be thought of as an absolute definition of Hurwitz–Ran spaces, whereas the general one, given in Definition 3.5 and depending on a nice couple and a PMQ-group pair, can be considered as the general, relative definition.

Notation 5.3 For a subspace \( \mathcal{X} \subset \mathbb{H} \) and a PMQ \( Q \) we denote by \( \text{Hur}(\mathcal{X}; Q) \) the space \( \text{Hur}(\mathcal{X}, \emptyset; Q, G(Q)) \). A configuration \( c \in \text{Hur}(\mathcal{X}; Q) \) is usually presented as \( (P, \psi, \phi) \) instead of \( (P, \psi, \phi) \) as in Notation 3.6, since \( \phi \) is uniquely determined by \( \psi \).
For the second homeomorphism, let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ be any nice couple, and consider the PMQ-group pair $(G, G)$, where $G$ is considered as a PMQ with full product, the first $G$ maps to the second $G$ by $\text{Id}_G$ and the second $G$ acts on the first $G$ by right conjugation. Then $\text{Id}_G$ is a morphism of nice couples $(\mathcal{X}, \mathcal{Y}) \rightarrow (\mathcal{X}, \mathcal{X})$. By functoriality we obtain a continuous map

$$(\text{Id}_G)_\ast : \text{Hur}(\mathcal{C}; G, G) \rightarrow \text{Hur}(\mathcal{X}, \mathcal{X}; G, G).$$

**Lemma 5.4** The above map $(\text{Id}_G)_\ast$ is a homeomorphism.

The proof of Lemma 5.4 is in Sect. A.7 of the appendix.

For the third homeomorphism, let $(Q, G)$ be any PMQ-group pair and let $\mathcal{C}$ be a nice couple of the form $(\mathcal{X}, \mathcal{X})$; then for all finite subset $P \subset X$ we have $Q_{\mathcal{C}}(P) = \{1\}$; in particular for all $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; Q, G)$ we have that $\psi : Q_{\mathcal{C}}(P) \rightarrow Q$ is the trivial map of PMQs: roughly speaking, this means that we can replace $Q$ by another PMQ fitting with $G$ into a PMQ-group pair, without changing the topology of $\text{Hur}(\mathcal{C}; Q, G)$. For instance we can consider the map of PMQ-group pairs $(\epsilon, \text{Id}_G) : (Q, G) \rightarrow (G, G)$, thus replacing $Q$ by $G$. We obtain the following lemma.

**Lemma 5.5** For all $\mathcal{X} \subset \mathbb{H}$ and all PMQ-group pair $(Q, G)$ the following map is a homeomorphism

$$(\epsilon, \text{Id}_G)_\ast : \text{Hur}(\mathcal{X}, \mathcal{X}; Q, G) \rightarrow \text{Hur}(\mathcal{X}, \mathcal{X}; G, G).$$

Using Lemmas 5.2, 5.4 and 5.5, we can simplify our notation for Hurwitz–Ran spaces $\text{Hur}(\mathcal{X}, \mathcal{Y}; Q, G)$ whenever one of the following conditions is satisfied:

- $\mathcal{Y} = \emptyset$, then we identify $\text{Hur}(\mathcal{X}, \emptyset; Q, G) \cong \text{Hur}(\mathcal{X}; Q)$;
- $\mathcal{Y} = \mathcal{X}$, then we identify $\text{Hur}(\mathcal{X}, \mathcal{X}; Q, G) \cong \text{Hur}(\mathcal{X}, \mathcal{X}; G, G) \cong \text{Hur}(\mathcal{X}; G)$;
- $Q = G$, then we identify $\text{Hur}(\mathcal{X}, \mathcal{Y}; G, G) \cong \text{Hur}(\mathcal{X}, \mathcal{X}; G, G) \cong \text{Hur}(\mathcal{X}; G)$.

### 5.3 Functoriality and change of ambient space

Recall Definition 3.14, let $\mathcal{C} = (\mathcal{X}, \mathcal{Y})$ and $\mathcal{C}' = (\mathcal{X}', \mathcal{Y}')$ be two nice couples, and let $\mathbb{T}$ and $\mathbb{T}'$ be two contractible semi-algebraic subspaces of $\mathbb{C}$ containing $\ast$, such that $\mathcal{X} \subset \mathbb{T}$ and $\mathcal{X}' \subset \mathbb{T}'$.

Let $\xi : (\mathbb{T}, \ast) \rightarrow (\mathbb{T}, \ast)$ be a semi-algebraic homeomorphism restricting to an orientation-preserving homeomorphism $\xi : \mathbb{T} \rightarrow \mathbb{T}'$, and to maps $\xi : \mathcal{X} \rightarrow \mathcal{X}'$ and $\xi : \mathcal{Y} \rightarrow \mathcal{Y}'$ (we restrict to the case of a homeomorphism for simplicity, but any map $\xi$ satisfying a suitable analogue of conditions (1)-(5) in Definition 4.2 may be used).

We define an induced map $\xi_\ast : \text{Hur}^\mathbb{T}(\mathcal{C}; Q, G) \rightarrow \text{Hur}^{\mathbb{T}'}(\mathcal{C}'; Q, G)$. Given a configuration $c = (P, \psi, \varphi) \in \text{Hur}^\mathbb{T}(\mathcal{C}; Q, G)$, we define $c' = \xi_\ast(c) = (P', \psi', \varphi') \in \text{Hur}^{\mathbb{T}'}(\mathcal{C}'; Q, G)$ as follows:

- $P' = \xi(P)$; note that $\xi$ restricts to a homeomorphism $\mathbb{T} \setminus P \rightarrow \mathbb{T}' \setminus P'$;
- $(\psi', \varphi') : (\mathcal{Q}_{\mathcal{C}'}^\mathbb{T}(P') \rightarrow (Q, G)$ is the following composition of maps of PMQ-group pairs

$$(\mathcal{Q}_{\mathcal{C}'}^\mathbb{T}(P'), \mathcal{C}'(P')) \xrightarrow{\xi^{-1}} \mathcal{Q}_{\mathcal{C}}^\mathbb{T}(P), \mathcal{C}(P)) \xrightarrow{\psi, \varphi} (Q, G).$$
The same arguments used in Sect. 4.3 show that $\xi_\ast$ is continuous. In the next articles of this series we will use this fact in the particular case in which $\xi$ restricts also to homeomorphisms $X \to X'$ and $Y \to Y'$: then we can use the inverse homeomorphism $\xi^{-1} : T' \to T$ to define a map $(\xi^{-1})_\ast : \text{Hur}_T^T(C, Q, G) \to \text{Hur}_T^T(C, Q, G)$. The maps $\xi_\ast$ and $(\xi^{-1})_\ast$ are inverse homeomorphism, and we obtain in particular the following proposition.

**Proposition 5.6** Let $\mathcal{C} = (X, Y)$ and $\mathcal{C}' = (X', Y')$ be nice couples, and let $T, T' \subseteq C$ be contractible semi-algebraic subspaces containing $\ast$, with $X \subseteq T$ and $X' \subseteq T'$. Let $\xi : C \to C$ be a map inducing a morphism of nice couples $\mathcal{C} \to \mathcal{C}'$ and restricting to homeomorphisms $T \cong T', X \cong X'$ and $Y \cong Y'$. Then the map $\xi_\ast : \text{Hur}(\mathcal{C}, Q, G) \to \text{Hur}(\mathcal{C}', Q, G)$ is a homeomorphism.

### 5.4 External products of Hurwitz–Ran spaces

In this subsection we fix a nice couple $\mathcal{C} = (X, Y)$ and two PMQ-group pairs $(Q, G)$ and $(Q', G')$.

**Definition 5.7** Recall from [3, Definition 2.16] the explicit description of the (categorical) product of two PMQ-group pairs. We define an *external product*

$$- \times - : \text{Hur}(\mathcal{C}, Q, G) \times \text{Hur}(\mathcal{C}', Q', G') \to \text{Hur}(\mathcal{C}, Q, G) \times (Q', G')).$$

Let $(c, c') \in \text{Hur}(\mathcal{C}, Q, G) \times \text{Hur}(\mathcal{C}', Q', G')$, and use Notation 5.6. We define $c \times c'$ as the configuration $(P'', \psi'', \psi'') \in \text{Hur}(\mathcal{C}, (Q, G) \times (Q', G'))$, where:

- $P'' = P \cup P' \subseteq X$;
- $(\psi'', \psi') : (\Omega(P''), \mathfrak{G}(P'')) \to (Q, G) \times (Q', G')$ is the map of PMQ-group pairs given by $(\psi, \varphi) \circ i_{P''}, (\psi', \varphi') \circ i_{P'}$ (see Notation 2.17).

**Proposition 5.8** The external product $- \times -$ from Definition 5.7 is continuous. Denoting $p : (Q, G) \times (Q', G') \to (Q, G)$ and $p' : (Q, G) \times (Q', G') \to (Q', G')$ the projections, we have that $- \times -$ is a retraction of the map

$$(p_\ast, p'_\ast) : \text{Hur}(\mathcal{C}, (Q, G) \times (Q', G')) \to \text{Hur}(\mathcal{C}, Q, G) \times \text{Hur}(\mathcal{C}, Q', G').$$

**Proof** Let $c, c', c''$ be as in Definition 5.7, and let $U''$ be an adapted covering of $P''$. Then we can obtain an adapted covering $U$ of $P$ (respectively, $U'$ of $P'$) by selecting the components of $U''$ containing one point of $P$ (respectively, of $P'$). We note that the product of normal neighbourhoods $\mathfrak{U}(c, U) \times \mathfrak{U}(c', U')$ is mapped by the external product inside $\mathfrak{U}(c'', U'')$: this shows continuity of the external product.

For the second statement, let $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}, (Q, G) \times (Q', G'))$: then both $p_\ast(c) = (P, p \circ (\psi, \varphi))$ and $p'_\ast(c) = (P, p' \circ (\psi, \varphi))$ are supported on the set $P \subseteq X$, so that by Definition 5.7 also $p_\ast(c) \times p'_\ast(c)$ is supported on $P \cup P = P$. It now follows directly from Definition 5.7 that $p_\ast(c) \times p'_\ast(c)$ is equal to $c$. $\square$

**Notation 5.9** We will mainly use the external product in the case $(Q', G') = (I, I)$. By abuse of notation we will denote by $- \times -$ also the composition

$$\begin{align*}
\text{Hur}(\mathcal{C}, Q, G) \times \text{Ran}(\mathcal{C}) &\xrightarrow{\cong} \text{Hur}(\mathcal{C}, Q, G) \times \text{Hur}(\mathcal{C}, I, I) \\
\text{Hur}(\mathcal{C}, (Q, G) \times (I, I)) &\xrightarrow{(p_{(Q,G)})_\ast} \text{Hur}(\mathcal{C}, Q, G).
\end{align*}$$
5.5 Contractible normal neighbourhoods

The following lemma gives an effective way to prove contractibility of normal neighbourhoods in concrete situations.

**Lemma 5.10** Let \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) be a nice couple and \( (\mathcal{Q}, G) \) a \( \mathsf{PMQ} \)-group pair. Let \( c \in \text{Hur}(\mathcal{X}, \mathcal{Q}) \), use Notations 3.6 and 2.6, and let \( U \) be an adapted covering of \( P \). Assume that there is a homotopy \( \mathcal{H}^U : \mathcal{C} \times [0, 1] \to \mathcal{C} \) satisfying the following:

1. \( \mathcal{H}^U(-, t) \) is a lax morphism of nice couples \( \mathcal{C} \to \mathcal{C} \) for all \( 0 \leq t \leq 1 \);
2. \( \mathcal{H}^U(-, 0) = \text{Id}_\mathcal{C} \);
3. \( \mathcal{H}^U(-, t) \) restricts to a map \( U_i \to U_i \), for all \( 1 \leq i \leq k \) and all \( 0 \leq t \leq 1 \);
4. \( \mathcal{H}^U(-, 1) \) maps \( U_i \) constantly to \( z_i \) for all \( 1 \leq i \leq k \).

Then the normal neighbourhood \( \mathfrak{N}(c, U) \) is contractible.

**Proof** We include \( \mathcal{Q} \) into its completion \( \hat{\mathcal{Q}} \), and consequently include \( (\mathcal{Q}, G) \) into \( (\hat{\mathcal{Q}}, G) \). Recall that the inclusion \( \text{Hur}(\mathcal{X}, \mathcal{Q}) \subset \text{Hur}(\mathcal{X}, \hat{\mathcal{Q}}, G) \) is open, and more precisely it maps normal neighbourhoods bijectively onto normal neighbourhoods. Thus suffices to prove that \( \mathfrak{N}(c, U) \) is contractible when considered as a normal neighbourhood in \( \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \). Proposition 4.10 and property (1) give a homotopy

\[
\mathcal{H}^U_{\hat{\mathcal{Q}}} : \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \times [0, 1] \to \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G),
\]

Consider now the union \( \bigsqcup_{\tilde{c}} \mathfrak{N}(\tilde{c}; U) \subset \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \), where \( \tilde{c} \) ranges among all configurations of \( \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \) supported on \( P \). By property (3) the map \( \mathcal{H}^U_{\hat{\mathcal{Q}}} \) restricts to a homotopy

\[
\mathcal{H}^U : \bigsqcup_{\tilde{c}} \mathfrak{N}(\tilde{c}; U) \times [0, 1] \to \bigsqcup_{\tilde{c}} \mathfrak{N}(\tilde{c}; U).
\]

The argument in the proof of Proposition 3.8 shows that \( \bigsqcup_{\tilde{c}} \mathfrak{N}(\tilde{c}; U) \) is the topological disjoint union of its open subspaces \( \mathfrak{N}(\tilde{c}; U) \). The map \( \mathcal{H}^U_{\hat{\mathcal{Q}}}(-, 0) \) is the identity of \( \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}}, G) \) by property (2), in particular \( \mathcal{H}^U_{\hat{\mathcal{Q}}}(-, 0) \) preserves each subspace \( \mathfrak{N}(\tilde{c}; U) \). It follows that \( \mathcal{H}^U_{\hat{\mathcal{Q}}} \) restricts to a homotopy

\[
\mathcal{H}^U : \mathfrak{N}(\tilde{c}, U) \times [0, 1] \to \mathfrak{N}(\tilde{c}, U)
\]

for each \( \tilde{c} \) supported on \( P \), in particular for \( \tilde{c} = c \). By property (4) the map \( \mathcal{H}^U_{\hat{\mathcal{Q}}}(-, 1) \) takes values in configurations in \( \text{Hur}(\mathcal{C}; \mathcal{Q}', G) \) supported on the set \( P = \{z_1, \ldots, z_k\} \), and the only such configuration inside \( \mathfrak{N}(c, U) \) is \( c \).

The hypothesis that the spaces \( \mathcal{X} \) and \( \mathcal{Y} \) occurring in a nice couple \( \mathcal{C} \) are semi-algebraic implies that, given \( c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) supported on a finite set \( P \), one can choose a small enough adapted covering of \( P \) for which a homotopy \( \mathcal{H}^U \) as in Lemma 5.10 exists. It follows that the space \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) is locally contractible.

6 Total monodromy and group actions

In this section we define the total monodromy of configurations in \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) and describe several actions of \( G \) on \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) and on certain subspaces of it.
6.1 Total monodromy

The total monodromy is the simplest invariant of connected components of \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \).

**Definition 6.1** Let \( \mathcal{C} \) be a nice couple, \((\mathcal{Q}, G)\) a PMQ-group pair, and let \( c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \). Let \( \gamma : [0, 1] \to \mathbb{C} \) be a simple closed loop spinning clockwise around \( P \), i.e., \( \gamma \) bounds a disc in \( \mathbb{C} \) that contains \( P \). We define \( \omega(c) = \varphi([\gamma]) \in G \), and call it the **total monodromy** of the configuration \( c \): this gives a function \( \omega : \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \to G \). See Fig. 7, left.

Note that the loop \( \gamma \) is well-defined up to homotopy, so that \( \omega \) is well-defined as a map of sets. Since for any given covering \( U \) of \( P \) we can choose \( \gamma \) spinning clockwise also around \( U \), we note that \( \omega \) is constant on the normal neighbourhood \( \mathcal{U}(c; U) \); hence the total monodromy is locally constant and therefore an invariant of connected components of \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \).

Note also that if \( \xi : \mathcal{C} \to \mathcal{C}' \) is a map of nice couples, then \( \xi(\gamma) \) is homotopic to a simple loop spinning clockwise around \( \xi(P) \) (see Definition 4.2); in particular \( \omega(c) = \omega(\xi(c)) \), i.e. the total monodromy is preserved under maps of Hurwitz–Ran spaces induced by maps of nice couples. If \((\Psi, \Phi) : (\mathcal{Q}, G) \to (\mathcal{Q}', G')\) is a morphism of PMQ-group pairs, then for all \( c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) we have \( \Phi(\omega(c)) = \omega((\Psi, \Phi)\circ(c)) \).

![Fig. 7](image-url) On left, a configuration \( c \) in \( \text{Hur}(\mathcal{X}, \mathcal{Y}; \mathcal{Q}, G) \), whose total monodromy is the \( G \)-valued monodromy of the dashed loop; on right, a configuration \( c \) in \( \text{Hur}(\mathcal{X}, \mathcal{Q}) \), whose total monodromy is the \( \hat{\mathcal{Q}} \)-valued monodromy of the dashed loop.
Notation 6.2 For a nice couple \( \mathcal{C} \), a PMQ-group pair \((\mathcal{Q}, G)\) and \( g \in G \) we denote by \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G)_g \subset \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \) the preimage of \( g \) along \( \omega \). If \( P_0 \) is as in Definition 3.13, we denote by \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0;g} \) the corresponding subspace of \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0} \).

For \( P_0 \subset \mathcal{X} \) (possibly \( P_0 = \emptyset \)) we obtain a natural decomposition

\[
\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0} = \bigsqcup_{g \in G} \text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0;g}.
\]

In the case \( \mathcal{Y} = \emptyset \), we can refine Definition 6.1 by taking the values of \( \omega \) in the completion \( \hat{\mathcal{Q}} \) of \( \mathcal{Q} \), instead of \( G \). Let \( \mathcal{X} \subset \mathbb{H} \) be a semi-algebraic subset, and let \( \epsilon = (P, \psi) \in \text{Hur}(\mathcal{X}; \mathcal{Q}) \); if \( \gamma \) is a simple loop in \( C \setminus P \) spinning clockwise around \( P \), then \([\gamma] \in \Omega^\text{ext}(P)\), and since \( \hat{\mathcal{Q}} \) is complete (and contains \( \mathcal{Q} \)), we can extend \( \psi : \Omega(P) \rightarrow \mathcal{Q} \) to a map \( \psi^\text{ext} : \Omega^\text{ext}(P) \rightarrow \hat{\mathcal{Q}} \), compare also with Sect. 4.5.

**Definition 6.3** We define a locally constant map \( \hat{\omega} : \text{Hur}(\mathcal{X}; \mathcal{Q}) \rightarrow \hat{\mathcal{Q}} \) by setting \( \hat{\omega}(\epsilon) := \psi^\text{ext}([\gamma]) \), using the notation above. See Fig. 7, right. For \( a \in \hat{\mathcal{Q}} \) and \( \emptyset \subset P_0 \subset \mathcal{X} \) we let \( \text{Hur}(\mathcal{X}; \mathcal{Q})_{P_0;a} \subset \text{Hur}(\mathcal{X}; \mathcal{Q}) \) be the preimage of \( a \) along \( \hat{\omega} \).

We obtain a decomposition

\[
\text{Hur}(\mathcal{X}; \mathcal{Q})_{P_0} = \bigsqcup_{a \in \hat{\mathcal{Q}}} \text{Hur}(\mathcal{X}; \mathcal{Q})_{P_0;a}.
\]

The maps \( \omega : \text{Hur}(\mathcal{X}, \emptyset; \mathcal{Q}, \mathcal{G}(\mathcal{Q})) \rightarrow \mathcal{G}(\mathcal{Q}) \) and \( \hat{\omega} : \text{Hur}(\mathcal{X}; \mathcal{Q}) \rightarrow \hat{\mathcal{Q}} \) are related by the equality \( \omega = \eta_\hat{\mathcal{Q}} \circ \hat{\omega} \), where we the groups \( \mathcal{G}(\mathcal{Q}) \) and \( \mathcal{G}(\hat{\mathcal{Q}}) \) are canonically identified, and \( \eta_\hat{\mathcal{Q}} : \hat{\mathcal{Q}} \rightarrow \mathcal{G}(\hat{\mathcal{Q}}) \) is the unit of the adjunction.

As a first application of the total monodromy, we prove the following proposition.

**Proposition 6.4** Let \( \mathcal{X} \) be a non-empty, semi-algebraic, convex and bounded subset of \( \mathbb{H} \), and let \( \hat{\mathcal{Q}} \) be a complete PMQ. Then the connected components of \( \text{Hur}_+(\mathcal{X}, \hat{\mathcal{Q}}) \) are contractible and there is a bijection

\[
\hat{\omega} : \pi_0(\text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})) \cong \hat{\mathcal{Q}}.
\]

**Proof** Since \( \mathcal{X} \) is bounded, we can find a bounded, convex, semi-algebraic open set \( \mathcal{X} \subset U \subset C \setminus \{\ast\} \). Fix a point \( z_0 \in \mathcal{X} \); for all \( a \in \hat{\mathcal{Q}} \) we can define a configuration \( \epsilon_a = ([z_0], \psi_a) \in \text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}}) \) by setting \( \psi_a([\gamma]) = a \) for a simple loop \( \gamma \) spinning clockwise around \( z_0 \). By Lemma 5.10 each normal neighbourhood \( \mathcal{U}(\epsilon_a; U) \) deformation retracts onto the configuration \( \epsilon_a \); the statement follows from the observation that each \( \epsilon \in \text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}}) \) is contained in one of these normal neighbourhood, namely in \( \mathcal{U}(\epsilon_a; U) \).

The reader will notice that in the proof of Proposition 6.4 we only used that \( \mathcal{X} \) is star-shaped around a point \( z_0 \in \mathcal{X} \), and not that \( \mathcal{X} \) is convex. In fact the bijection \( \hat{\omega} : \pi_0(\text{Hur}_+(\mathcal{X}; \hat{\mathcal{Q}})) \cong \hat{\mathcal{Q}} \) holds whenever \( \mathcal{X} \) is path connected; we will not use this more general fact and therefore we will leave its proof to the reader. Proposition 6.4 has the following corollary.

**Corollary 6.5** Let \( \mathcal{X} \subset \mathbb{H} \) be semi-algebraic, convex and bounded; then for all \( a \in \mathcal{Q} \) the space \( \text{Hur}_+(\mathcal{X}; \mathcal{Q})_a \) is contractible.

**Proof** The fact that \( \hat{\mathcal{Q}} \setminus \mathcal{Q} \) is an ideal of the complete PMQ \( \hat{\mathcal{Q}} \) implies that, for all \( a \in \mathcal{Q} \), the natural inclusion \( \text{Hur}(\mathcal{X}; \mathcal{Q})_a \subset \text{Hur}(\mathcal{X}; \hat{\mathcal{Q}})_a \) is in fact a homeomorphism.

\( \square \)
6.2 Action by global conjugation

Let \((Q, G)\) be a PMQ-group pair. Then the group \(G\) acts (on right) by conjugation on \(Q\) and on \(G\) itself. In particular the right action of \(G\) on \(Q\) takes the form of a map of groups \(\tau: G \to Aut_{\text{PMQ}(Q)}^{op}\), which is part of the structure of PMQ-group pair. The actions of \(G\) on \(Q\) by conjugation is compatible with respect to the map of PMQs \(\epsilon: Q \to G\), which is also part of the structure of PMQ-group pair. In the following we define a corresponding right action of \(G\) on the space Hur(\(\mathcal{C}; Q, G\)), for any nice couple \(\mathcal{C}\).

**Definition 6.6** For \(g \in G\) and \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C})\) we define \(c^g = (P, \psi^g, \varphi^g) \in \text{Hur}(\mathcal{C}; Q, G)\) as follows:

- \(\psi^g\) is the composition \(\Omega(P) \overset{\psi}{\to} Q \overset{\tau(g)}{\to} Q\) of maps of PMQs.
- \(\varphi^g\) is the composition \(\Theta(P) \overset{\varphi}{\to} G \overset{(-)^g}{\to} G\) of maps of groups, where \((-)^g: g' \mapsto g^{-1}g'\).

The maps \((-)^g: \text{Hur}(\mathcal{C}, Q, G) \to \text{Hur}(\mathcal{C}; Q, G)\) are homeomorphisms (they map normal neighbourhoods bijectively to normal neighbourhoods) and assemble into a right action of \(G\) on the space Hur(\(\mathcal{C}; Q, G\)), called action by global conjugation.

Note that the total monodromy \(\omega\) (see Definition 6.1) satisfies the formula \(\omega(c^g) = \omega(c)^g \in G\). Note also that for \(P_0 \subset X\) as in Definition 3.13 the action by global conjugation restricts to the subspace Hur(\(\mathcal{C})p_0\).

In the case \(Y = \emptyset\), we can refine Definition 6.6 and let the completion \(\hat{Q}\) of \(Q\) act on Hur(\(\mathcal{X}; Q\)). By definition, a (right) action of \(\hat{Q}\) on Hur(\(\mathcal{X}; Q\)) is a map of PMQs \(\hat{Q} \to Aut_{\text{PMQ}(\mathcal{X}; Q)}^{op}\). For \(a \in \hat{Q}\) and \(c = (P, \psi) \in \text{Hur}(\mathcal{X}; Q)\) we define \(c^a = (P, \psi^a) \in \text{Hur}(\mathcal{X}; Q)\) by setting \(\psi^a\) to be the composition \(\Omega(P) \overset{\psi}{\to} Q \overset{(-)^a}{\to} Q\). Here we use that \(Q \subset \hat{Q}\) is closed under conjugation by elements in \(\hat{Q}\).

6.3 Left and right-based nice couples

In this subsection we consider other natural actions of \(G\), defined on suitable subspaces of Hurwitz–Ran spaces.

**Definition 6.7** For \(t \in \mathbb{R}\) we define a homeomorphism \(\tau_t: (\mathbb{C}, \ast) \to (\mathbb{C}, \ast)\) by:

\[
\tau_t(z) = \begin{cases} 
  z & \text{if } \Im(z) \leq -1 \\
  z + t & \text{if } \Im(z) \geq 0 \\
  z + (\Im(z) + 1)t & \text{if } -1 \leq \Im(z) \leq 0.
\end{cases}
\]

Note that \(\tau_t(\ast) = \ast\) for all \(t \in \mathbb{R}\). Note also that the assignment \(t \mapsto \tau_t\) defines a continuous, piecewise linear action of \(\mathbb{R}\) on \(\mathbb{C}\).

**Notation 6.8** For \(t \in \mathbb{R}\) we denote by \(C_{\Re \geq t} \subset \mathbb{C}\) the subspace containing all \(z \in \mathbb{C}\) with \(\Re(z) \geq t\). Similarly we define \(C_{\Re \leq t}, C_{\Re < t}, C_{\Re = t}\) and \(C_{\Re = t}\), the latter being a vertical line. For all \(-\infty \leq t \leq t' \leq +\infty\) we define a subspace \(S_{t, t'} \subset \mathbb{C}\) by

\[S_{t, t'} = \tau_t(C_{\Re \geq 0}) \cap \tau_{t'}(C_{\Re \leq 0}).\]

where \(\tau_{-\infty}(C_{\Re \geq 0}) = \tau_{+\infty}(C_{\Re \leq 0}) = \mathbb{C}\) and \(\tau_{-\infty}(C_{\Re \leq 0}) = \tau_{+\infty}(C_{\Re \geq 0}) = \emptyset\).
Definition 6.9 A left-based nice couple is a nice couple \( \mathcal{C} = (\mathcal{X}, \mathcal{Y}) \) together with a choice of a point \( z^l \in \mathcal{X} \) such that \( \mathcal{R}(z^l) \leq \mathcal{R}(z) \) for all \( z \in \mathcal{X} \). We denote by \((z^l, \mathcal{C})\) a left-based nice couple.

Similarly, a right-based nice couple is a nice couple with a choice of a point \( z^r \in \mathcal{Y} \) such that \( \mathcal{R}(z) \leq \mathcal{R}(z^r) \) for all \( z \in \mathcal{X} \). We denote it by \((\mathcal{C}, z^r)\).

A left-right-based nice couple (shortly, lr-based) is a nice couple which is both left- and right-based, such that \( \mathcal{R}(z^l) < \mathcal{R}(z^r) \); we denote it by \((z^1, \mathcal{C}, z^r)\).

6.4 Action by left and right multiplication

We define a left action of \( G \) on the space \( \text{Hur}(\mathcal{C}; Q, G)_z \) (resp. \( z^l \), \( \mathcal{C} \)) is a left-based nice couple. Similarly, there is a right action of \( G \) on \( \text{Hur}(\mathcal{C}; Q, G)_z \) if \((\mathcal{C}, z^r)\) is a right-based nice couple. We will describe the construction focusing on the left-based case; the right-based case is analogous, and we will mention the differences in parentheses.

For the entire subsection fix a left-based (right-based) nice couple as in Definition 6.9. Choose an arc \( \xi^l \) embedded in \( S_{-\infty, \mathcal{R}(z^l)} \) and joining \( * \) with \( z^l \); assume also that the interior of \( \xi^l \) is contained in \( \tilde{S}_{-\infty, \mathcal{R}(z^l)} \). (In the right-based case, we would choose an arc \( \xi^r \) embedded in \( S_{\mathcal{R}(z^r), +\infty} \))

Definition 6.10 Let \( P \subset \mathcal{X} \) with \( z^l \in P \) (resp. \( z^r \in P \)). An admissible generating set for \( \mathcal{G}(P) \) is left-based (resp. right-based) if it can be constructed as in Definition 2.8, using \( \xi^l \) (resp. \( \xi^r \)) as the arc associated with \( z^l \) (resp. \( z^r \)), and using only arcs contained in \( S_{\mathcal{R}(z^l), +\infty} \) (in \( S_{-\infty, \mathcal{R}(z^l)} \)) for the other points of \( P \).

Notation 6.11 We denote by \( f^l \in \mathcal{G}(P) \) (resp. \( f^r \)) the generator represented by a loop spinning around \( z^l \) (resp. \( z^r \)).

Definition 6.12 Let \( g \in G \); let \( \epsilon \in \text{Hur}(\mathcal{C}; Q, G)_z \) (resp. \( \epsilon \in \text{Hur}(\mathcal{C}; Q, G)_z \)), use Notation 3.6, and let \( f_1, \ldots, f_k \) be a left-based (right-based) admissible generating set for \( \mathcal{G}(P) \). We let \( g \cdot \epsilon \) (resp. \( \epsilon \cdot g \)) be the configuration \((P, \psi', \phi')\), where:

- \( \phi' \) is defined on the free group \( \mathcal{G}(P) \) by setting \( \phi'(f^l) = g \cdot \phi(f^l) \) (by setting \( \phi'(f^r) = \phi(f^r) \cdot g \) and by setting \( \phi'(f_i) = \phi(f_i) \) for \( 1 \leq i \leq k \) such that \( f_i \neq f^l \) (respectively \( f_i \neq f^r \)).
- \( \psi' \) is defined on \( \Omega(P) \) using [3, Theorem 3.3], by setting \( \psi'(f_i) = \psi(f_i) \) for all \( 1 \leq i \leq l \) and imposing that \((\psi', \phi') : (\Omega(P), \mathcal{G}(P)) \rightarrow (\Omega, P) \) is a map of PMQ-group pairs. See Fig. 8.

Proposition 6.13 For all \( g \in \mathcal{G}(Q) \) the assignment \( \epsilon \mapsto g \cdot \epsilon \) (respectively \( \epsilon \mapsto \epsilon \cdot g \)) does not depend on the choice of the left-based (right-based) admissible generating set, and gives rise to a continuous self-map \( g \cdot - \) of \( \text{Hur}(\mathcal{C}; Q, G)_z \) (respectively a self-map \( - \cdot g \) of \( \text{Hur}(\mathcal{C}; Q, G)_z \)).

The collection of all maps \( g \cdot - \) (all maps \( - \cdot g \)) gives a left (right) action of \( G \) on \( \text{Hur}(\mathcal{C}; Q, G)_z \) (on \( \text{Hur}(\mathcal{C}; Q, G)_z \)).

The proof of Proposition 6.13 is in Sect. A.8 of the appendix.
Fig. 8 On left, a configuration $c \in \text{Hur}(C, Q, G)_{z_l}$; on right, its image under the right action of $g \in G$.

### 6.5 Compatibilities of the left and right actions

**Lemma 6.14** Let $\mathcal{C}$ be a left-based (right-based) nice couple. Then the total monodromy $\omega : \text{Hur}(\mathcal{C}; Q) \to G$ is a $G$-equivariant map, where $G$ acts on itself by left (right) multiplication.

**Proof** We focus on the left-based case. Let $c \in \text{Hur}(\mathcal{C}; Q, G)_{z_l}$, let $c' = g \cdot c$ and use Notation 3.6. Let $f_1, \ldots, f_k$ be a left-based admissible generating set for $P = P'$, suppose $f_1 = f^1$ (see Notation 6.11), and suppose, up to permuting the indices from 2 to $k$, that the product $f_1 \ldots f_k$ represents an element $[\gamma] \in \mathcal{G}(P)$ as in Definition 6.1. Let $g = f_2 \ldots f_k$, so that $[\gamma] = f^1 \cdot g$. Note that $\varphi'(g) = \varphi(g)$. Then

$$\omega(g \cdot c) = \varphi'([\gamma]) = \varphi'(f^1) \cdot \varphi'(g) = g \cdot \varphi(f^1) \cdot \varphi(g) = g \cdot \varphi([\gamma]) = g \cdot \omega(c).$$

Let now $(z', \mathcal{C}, z')$ be a lr-based nice couple: both spaces $\text{Hur}(\mathcal{C}; Q, G)_{z_l}$ and $\text{Hur}(\mathcal{C}; Q, G)_{z_r}$ contain $\text{Hur}(\mathcal{C}; Q, G)_{z_l, z_r}$ as subspace, and this subspace is preserved under both actions of $G$, on left on $\text{Hur}(\mathcal{C}; Q, G)_{z_l}$ and on right on $\text{Hur}(\mathcal{C}; Q, G)_{z_r}$.

**Lemma 6.15** Let $(z', \mathcal{C}, z')$ be a lr-based nice couple. Then the left and the right actions of $G$ on $\text{Hur}(\mathcal{C}; Q, G)_{z_l, z_r}$ commute, i.e., for every $g, h \in \mathcal{G}(Q)$ the self-maps $g \cdot -$ and $- \cdot h$ of $\text{Hur}(\mathcal{C}; Q, G)_{z_l, z_r}$ commute.
Proof Fix $c = (P, \varphi, \psi) \in \text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ and let $\xi^l$ and $\xi^r$ be arcs as in Definition 6.10. We can choose disjoint arcs $\zeta_i$ contained in $\mathcal{S}_{\text{Hur}(\zeta^l), \text{Hur}(\zeta^r)}$ completing $\xi^l$ and $\xi^r$ to a system of arcs as in Definition 2.8 yielding an admissible generating set for $\mathfrak{G}(P)$ which is both left- and right-based. The equality $g \cdot (c \cdot h) = (g \cdot c) \cdot h$ follows directly from Definition 6.12.

We thus obtain a (left) action of the group $G \times G^{\text{op}}$ on $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$, and by Lemma 6.14 the map $\omega: \text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r} \to G$ is $G \times G^{\text{op}}$-equivariant.

Lemma 6.16 In the hypotheses of Lemma 6.15, the action of $G \times G^{\text{op}}$ on the space $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ is free and properly discontinuous.

Proof Let $c = (P, \varphi, \psi) \in \text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$, let $U$ be an adapted covering of $P$, and denote by $U^l$ and $U^r$ the components of $U$ containing $\zeta^l$ and $\zeta^r$ respectively. Fix an admissible generating set for $\mathfrak{G}(P)$ which is both left- and right-based, and let $f^l$ and $f^r$ be as in Notation 6.11.

Let $(g, h)$ be a non-trivial element in $G \times G^{\text{op}}$, and denote by $c' = (P, \varphi', \psi')$ the configuration $g \cdot c \cdot h$. Then either $\varphi'(f^l) = \varphi(f^l) \neq \varphi(f^l)$, or $\varphi'(f^r) = \varphi(f^r) \neq \varphi(f^r)$, or both inequalities hold: in any case we conclude $c' \neq c$, so the action of $G \times G^{\text{op}}$ on $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ is free.

Recall Definition 3.13, and note that the normal neighbourhood $\mathfrak{N}(c, U)_{\zeta^l, \zeta^r}$ is mapped by $g \cdot c \cdot h$ to the normal neighbourhood $\mathfrak{N}(g \cdot c \cdot h, U)_{\zeta^l, \zeta^r}$; since the configurations $c$ and $c'$ are supported on the same set $P$, but $\varphi' \neq \varphi$, the argument in the proof of Proposition 3.8 shows that $\mathfrak{N}(c, U)$ and $\mathfrak{N}(g \cdot c \cdot h, U)$ intersect trivially in $\text{Hur}(\mathcal{C}; Q, G)$, and a fortiori $\mathfrak{N}(c, U)_{\zeta^l, \zeta^r}$ and $\mathfrak{N}(g \cdot c \cdot h, U)_{\zeta^l, \zeta^r}$ intersect trivially in $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$. Hence the action of $G \times G^{\text{op}}$ on $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ is properly discontinuous.

Recall Notation 6.2: we can decompose $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ as a disjoint union of subspaces $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r, s}$ according to $\omega$. If we act on $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ only on left, these subspaces will be permuted among each other, so that the quotient of $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ by the left action is homeomorphic to $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r, s}$. The same holds if we quotient $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ only by the right action of $G$; we can define a more interesting space by quotienting $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ by both actions.

Notation 6.17 We denote by $\text{Hur}(\mathcal{C}; Q, G)_{G, G^{\text{op}}}$ the quotient of $\text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r}$ by the left and right actions of $G$: the points $\zeta^l, \zeta^r$ will always be clear from the context and will thus be omitted from the notation. We denote by

\[ p_{G, G^{\text{op}}}: \text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r} \to \text{Hur}(\mathcal{C}; Q, G)_{G, G^{\text{op}}} \]

the projection map.

By Lemma 6.16 we have in particular a covering map

\[ p_{G, G^{\text{op}}}: \text{Hur}(\mathcal{C}; Q, G)_{\zeta^l, \zeta^r, s} \to \text{Hur}(\mathcal{C}; Q, G)_{G, G^{\text{op}}}, \]

with deck transformation group isomorphic to $G$ and acting transitively on fibres.

7 Hurwitz–Ran spaces with monodromies in augmented PMQs

In this section we introduce, for an augmented PMQ $Q$, a subspace $\text{Hur}(\mathcal{C}; Q_+, G)$ of $\text{Hur}(\mathcal{C}; Q, G)$; under suitable conditions on $\mathcal{C}$ the inclusion $\text{Hur}(\mathcal{C}; Q_+, G) \hookrightarrow \text{Hur}(\mathcal{C}; Q, G)$ is a weak homotopy equivalence. Recall from [3, Definition 4.9] that $Q$ is augmented if an equality $ab = 1$ in $Q$ implies $a = b = 1$. 
Definition 7.1 Let \( \mathcal{C} = (X, Y) \) be a nice couple and let \((Q, G)\) be a PMQ-group pair, with \(Q\) augmented. We define \(\text{Hur}(\mathcal{C}; Q_+, G) \subseteq \text{Hur}(\mathcal{C}; Q, G)\) as the subspace containing all configurations \(c = (P, \psi, \varphi)\) such that \(\psi : \Omega(P) \to Q\) is an augmented map of PMQs, i.e. \(\psi^{-1}(1_Q) = \{1_{\Omega(P)}\}\) (see [3, Definition 4.9]). If \(Y\) is empty we also write \(\text{Hur}(X; Q_+) \subseteq \text{Hur}(\mathcal{C}; Q, G)\) for the space \(\text{Hur}(\mathcal{C}; Q_+, G(Q))\).

Roughly speaking, a configuration \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; Q, G)\) belongs to \(\text{Hur}(\mathcal{C}; Q_+, G)\) if the monodromy \(\psi\) attains non-trivial values around each point of \(P \setminus Y\); these are also all points of \(P\) around which \(\psi\) is defined. By “non-trivial value” we mean a value different from \(1_Q\), i.e. a value in \(Q_+\), whence the notation.

Note that if \(\xi : \mathcal{C} \to \mathcal{C}'\) is a morphism of nice couples and \(Q\) is augmented, then the induced map \(\xi_* : \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}', Q, G)\) restricts to a map of spaces \(\xi_* : \text{Hur}(\mathcal{C}; Q_+, G) \to \text{Hur}(\mathcal{C}', Q_+, G)\). This is true also for a lax morphism \(\xi : \mathcal{C} \to \mathcal{C}'\) (see Definition 4.3), provided that \(Q\) is complete and augmented.

Lemma 7.2 If \(Q\) is augmented, then \(\text{Hur}(\mathcal{C}; Q_+, G)\) is closed in \(\text{Hur}(\mathcal{C}; Q, G)\).

Proof Let \(c \in \text{Hur}(\mathcal{C}; Q, G) \setminus \text{Hur}(\mathcal{C}; Q_+, G)\); then, using Notation 3.6, there is some \(1 \leq i \leq l\) such that \(\psi\) sends each element of \(\Omega(P, z_i)\) to \(1\) (see also Definition 2.9). Let \(U\) be an adapted covering of \(P\); then we claim that the entire normal neighbourhood \(\mathcal{U}(c; U)\) lies in the difference \(\text{Hur}(\mathcal{C}; Q, G) \setminus \text{Hur}(\mathcal{C}; Q_+, G)\). To see this, let \(c' = (P', \psi', \varphi') \in \mathcal{U}(c, U)\), use Notation 2.6, and let \(z'\) be a point in \(P' \cap U_i\). Then each element \([y'] \in \Omega(P', z')\) is sent by \(\psi'\) to an element \(\psi'([y']) \in Q\) which occurs as a factor of a decomposition of \(1_Q\) in the partial monoid \(Q\); since \(Q\) is augmented we have \(\psi'([y']) = 1\) and therefore \(c'\) does not lie in \(\text{Hur}(\mathcal{C}; Q_+, G)\).

7.1 Homotopy equivalences from augmented PMQs

The rest of the section is devoted to the proof of the following technical propositions.

Proposition 7.3 Let \(X \subseteq \mathbb{H}\) be a semi-algebraic, non-empty and connected subspace, and let \(Q\) be an augmented PMQ. Then the spaces \(\text{Hur}(X; Q_+)\) and \(\text{Hur}_+ (X; Q)\) are homotopy equivalent.

Proposition 7.4 Let \((Q, G)\) be a PMQ-group pair with \(Q\) augmented, and let \(\mathcal{C} = (X, Y)\) be a nice couple with both \(X\) and \(Y\) non-empty and connected. Let \(P_0 \subseteq Y\) be a finite, non-empty subset. Then the inclusion \(\text{Hur}(\mathcal{C}; Q_+, G)_{P_0} \subseteq \text{Hur}(\mathcal{C}; Q, G)_{P_0}\) is a homotopy equivalence.

In the rest of the section we fix a PMQ-group pair \((Q, G)\) with \(Q\) augmented. Let \(\mathcal{C} = (X, Y)\) be a nice couple.

Definition 7.5 Let \(c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; Q, G)\). A point \(z \in P\) is inert for \(c\) if \(z \in X \setminus Y\) and \(\psi\) maps each element of \(\Omega(P, z)\) to \(1_Q\) (see Definition 2.9).

Definition 7.6 We define a retraction of sets \(\rho : \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q_+, G)\) of the inclusion \(\text{Hur}(\mathcal{C}; Q_+, G) \subseteq \text{Hur}(\mathcal{C}; Q, G)\): for a configuration \(c \in \text{Hur}(\mathcal{C}; Q, G)\), we construct \(\rho(c)\) by “forgetting” its inert points. More precisely, using Notation 3.6, if \(P' \subseteq P\) is the subset of non-inert points for \(c\), then we note that \(\varphi : \mathcal{E}(P') \to G\) and \(\psi : \Omega(P') \to Q\) factor through maps \(\tilde{\varphi} : \mathcal{E}(P') \to G\) and \(\tilde{\psi} : \Omega(P') \to Q\) along the surjections \(i_{P'} : \mathcal{E}(P) \to \mathcal{E}(P')\) and \(i_{P'}^1 : \Omega(P) \to \Omega(P')\), (see Notation 2.17), and we define \(\rho : (P, \psi, \varphi) \mapsto (P', \tilde{\psi}, \tilde{\varphi})\).
Unfortunately, even assuming that \(Q\) is augmented, \(\rho\) is in general not continuous: for instance, if \(P\) contains a point \(z_i \in \mathcal{Y}\) whose local monodromy with respect to \(\varphi\) is \(\mathbb{I} \in G\), then \(\rho(P, \psi, \varphi)\) is a configuration supported also on the point \(z_i\); however if we perturb slightly \(z_i\) so that it “enters” in \(\mathcal{X} \setminus \mathcal{Y}\) (for this, suppose that \(z_i\) is an accumulation point for \(\mathcal{X} \setminus \mathcal{Y}\)), then in defining \(\rho(P, \psi, \varphi)\) we forget \(z_i\) and we do not replace it by any other point close to it.

### 7.2 Explosions

The previous issue can only occur when \(\mathcal{Y} \neq \emptyset\), and in fact if \(\mathcal{Y} = \emptyset\), then \(\rho: \text{Hur}(\mathcal{C}; Q, G) \to \text{Hur}(\mathcal{C}; Q^+, G)\) is continuous, as we will see in Corollary 7.9. In the general case we cannot just let an inert point \(z_i \in P \setminus \mathcal{Y}\) disappear; what we can do is to let every point \(z_i \in P\) explode (including non-inert points), by replacing \(z_i\) with one or more other points of \(\mathcal{X}\). This idea is elaborated in the following definition.

**Definition 7.7** Let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) be a nice couple. An explosion \(\mathcal{E}\) of \(\mathcal{C}\) is a continuous map \(\mathcal{E}: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{C})\) such that for all \(z \in \mathcal{Y}\) and all \(0 \leq t \leq 1\), \(z \in \mathcal{E}(z, t)\). An explosion \(\mathcal{E}\) is **standard** if \(\mathcal{E}(z, 0) = \{z\} \in \text{Ran}(\mathcal{C})\) for all \(z \in \mathcal{X}\).

Given an explosion \(\mathcal{E}\), a finite subset \(P \subset \mathcal{X}\) and a time \(0 \leq t \leq 1\), we can define a subset \(\mathcal{E}(P, t) \in \text{Ran}(\mathcal{C})\) as the union of the subsets \(\mathcal{E}(z, t)\) for \(z \in P\). Thus an explosion \(\mathcal{E}\) induces a continuous map \(\text{Ran}(\mathcal{C}) \times [0, 1] \to \text{Ran}(\mathcal{C})\), that by abuse of notation we still denote \(\mathcal{E}\). If \(\mathcal{E}\) is standard, then \(\mathcal{E}(-, 0)\) is the identity of \(\text{Ran}(\mathcal{C})\).

**Proposition 7.8** Let \(\mathcal{C} = (\mathcal{X}, \mathcal{Y})\) be a nice couple, let \(\mathcal{E}: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{C})\) be an explosion, and let \((Q, G)\) be a PMQ-group pair with \(Q\) augmented. Recall Definition 3.10 and Notation 5.9, and let \(\mathcal{E}_*: \text{Hur}(\mathcal{C}; Q, G) \times [0, 1] \to \text{Hur}(\mathcal{C}; Q, G)\) be the following composition of maps of sets:

\[
\begin{align*}
\text{Hur}(\mathcal{C}; Q, G) \times [0, 1] & \xrightarrow{(\rho, \varepsilon) \times \text{Id}} \text{Hur}(\mathcal{C}; Q^+, G) \times \text{Ran}(\mathcal{C}) \times [0, 1] \\
\text{Hur}(\mathcal{C}; Q^+, G) \times \text{Ran}(\mathcal{C}) & \xrightarrow{- \times -} \text{Hur}(\mathcal{C}; Q, G).
\end{align*}
\]

Then \(\mathcal{E}_*\) is continuous. If moreover \(\mathcal{E}\) is standard, then \(\mathcal{E}_*(-, 0)\) is the identity of \(\text{Hur}(\mathcal{C}; Q, G)\).

The proof of Proposition 7.8 is in Sect. A.9 of the appendix. A particular application of Proposition 7.8 is the following:

**Corollary 7.9** Let \(\mathcal{X} \subset \mathbb{H}\) be a semi-algebraic set and let \(Q\) be an augmented PMQ; then the map \(\rho: \text{Hur}(\mathcal{X}; Q) \to \text{Hur}(\mathcal{X}; Q^+)\) is continuous.

**Proof** Consider the explosion \(\mathcal{E}_0^\emptyset: \mathcal{X} \times [0, 1] \to \text{Ran}(\mathcal{X})\) taking the constant value \(\emptyset \in \text{Ran}(\mathcal{X})\); then \(\mathcal{E}_0^\emptyset(-, 0) = \rho\) is a continuous map. \(\square\)

### 7.3 Proof of Propositions 7.3 and 7.4

**Proof of Proposition 7.3** Recall Definition 5.7 and Notation 5.9, and fix a point \(z_0 \in \mathcal{X}\). We claim that the map \(- \times z_0: \text{Hur}(\mathcal{X}; Q^+) \to \text{Hur}^+ (\mathcal{X}; Q)\) is a homotopy equivalence, where we denote by \(z_0\) also the singleton \(\{z_0\} \in \text{Ran}(\mathcal{X})\).
Let Hur($\mathcal{X};Q_+)_z$ be the subspace of Hur$_+(\mathcal{X};Q)$ containing all configurations $\epsilon = (P, \psi)$ such that $z_0 \in P$ and all points of $P \setminus \{z_0\}$ are not inert; then $- \times z_0$ gives a homeomorphism Hur($\mathcal{X};Q_+)_z \cong$ Hur($\mathcal{X};Q_+)_z$, with inverse given by the restriction of $\rho$, which is continuous by Corollary 7.9. It suffices therefore to prove that the inclusion Hur($\mathcal{X};Q_+)_z \hookrightarrow$ Hur$_+(\mathcal{X};Q)$ is a homotopy equivalence. By Lemma 3.3 the space Ran$_+(\mathcal{X})$ is weakly contractible; since $\mathcal{X}$ is homeomorphic to a CW complex, there is a homotopy $\delta^{z_0}: \mathcal{X} \times [0, 1] \to$ Ran$_+(\mathcal{X})$ with $\delta^{z_0}(z, 0) = \{z\}$ and $\delta^{z_0}(z, 1) = \{z_0\}$ for all $z \in \mathcal{X}$; $\delta^{z_0}$ is a standard explosion and gives rise to an extended explosion $\delta^{z_0}: \text{Ran}(\mathcal{X}) \times [0, 1] \to \text{Ran}(\mathcal{X})$.

Proposition 7.8 yields a homotopy $\delta^{z_0}: \text{Hur}(\mathcal{X};Q) \times [0, 1] \to$ Hur$_+(\mathcal{X};Q)$, which restricts to a homotopy of Hur$_+(\mathcal{X};Q)$. We note the following:

- $\delta^{z_0}_x(-, 0)$ is the identity of Hur$_+(\mathcal{X};Q)$, again by Proposition 7.8;
- $\delta^{z_0}_x(-, 1)$ restricts to the identity on the subspace Hur($\mathcal{X};Q_+)_z$: indeed if $\epsilon = (P, \psi) \in$ Hur($\mathcal{X};Q_+)_z$, then $\rho(\epsilon)$ is either equal to $\epsilon$, or is obtained by forgetting $z_0 \in P$ in case $z_0$ is inert; since $\delta^{z_0}(-, 1)$ is constant on Ran$_+(\mathcal{X})$ with value $z_0$, we have anyway the equality $\rho(\epsilon) \times \delta^{z_0}(P, 1) = \rho(\epsilon) \times z_0 = \epsilon$, i.e. the point $z_0$ is added again in the further composition defining $\delta^{z_0}_x(-, 1)$;
- $\delta^{z_0}_x(-, 1)$ takes values in Hur($\mathcal{X};Q_+)_z$: this follows again from the equality $\delta^{z_0}_x(\epsilon, 1) = \rho(\epsilon) \times z_0$, holding for all $\epsilon \in$ Hur($\mathcal{X};Q_+)_z$.

The homotopy $\delta^{z_0}_x$ shows that the inclusion Hur($\mathcal{X};Q_+)_z \hookrightarrow$ Hur$_+(\mathcal{X};Q)$ is a homotopy equivalence.

Note that in the particular case $Q = \{1\}$, Proposition 7.3 implies that Ran$_+(\mathcal{X})$ is contractible; this is a mild improvement of the statement of Lemma 3.3.

**Proof of Proposition 7.4** The proof is similar to the one of Proposition 7.3. By Lemma 3.3 the spaces Ran$_+(\mathcal{Y}) \subset$ Ran$_+(\mathcal{X})$ are weakly contractible. The couple $(\mathcal{X}, \mathcal{Y})$ is homeomorphic to a couple of CW complexes, therefore we can find a homotopy $\delta^{X,Y}: \mathcal{X} \times [0, 1] \to$ Ran$_+(\mathcal{X})$ with $\delta^{X,Y}(z, t) = \{z\}$ whenever $z \in \mathcal{Y}$ or $t = 0$, and such that $\delta^{X,Y}(z, 1) \in$ Ran$_+(\mathcal{Y})$ for all $z \in \mathcal{X}$. In particular $\delta^{X,Y}$ is a standard explosion, inducing an extended explosion $\delta^{X,Y}: \text{Ran}(\mathcal{X}) \times [0, 1] \to \text{Ran}(\mathcal{X})$. By Proposition 7.8, and using that $P_0 \subset \mathcal{Y}$, we obtain a homotopy $\delta^{X,Y}_x: \text{Hur}(\mathcal{X};Q,G)_{P_0} \times [0, 1] \to$ Hur($\mathcal{X};Q,G)_{P_0}$ with the following properties:

- $\delta^{X,Y}_x(\epsilon, 0)$ is the identity of Hur($\mathcal{X};Q,G)_{P_0}$;
- $\delta^{X,Y}_x(\epsilon, 1)$ takes values in Hur($\mathcal{X};Q_+G)_{P_0}$.

It suffices now to prove that there is a homotopy of maps Hur($\mathcal{X};Q_+G)_{P_0} \to$ Hur($\mathcal{X};Q_+G)_{P_0}$ from $\delta^{Y}(\epsilon, -1)$|Hur($\mathcal{X};Q_+G)_{P_0}$ to the identity of Hur($\mathcal{X};Q_+G)_{P_0}$.

Using weak contractibility of Ran$_+(\mathcal{Y})$ (see Lemma 3.3) together with the fact that $\mathcal{X}$ is homeomorphic to a CW complex, we can find a homotopy $\delta^Y: \mathcal{X} \times [0, 1] \to$ Ran$_+(\mathcal{Y})$ satisfying the following properties:

- $\delta^Y(\epsilon, 0) = \delta^{X,Y}(\epsilon, 1)$;
- $\delta^Y(\epsilon, 1)$ is the constant map with value $P_0 \in$ Ran$_+(\mathcal{Y})$.

Denote by $\delta^Y: \text{Ran}(\mathcal{X})_{P_0} \times [0, 1] \to$ Ran$_+(\mathcal{Y})$ also the induced map on Ran spaces. Consider the homotopy $\mathcal{H}^Y: \text{Hur}(\mathcal{X};Q_+G)_{P_0} \times [0, 1] \to$ Hur($\mathcal{X};Q_+G)_{P_0}$ given by the
composition
\[
\begin{align*}
\text{Hur}(\mathcal{C}; Q_+, G)_{P_0} \times [0, 1] & \xrightarrow{(\text{Id}, \varepsilon) \times \text{Id}} \text{Hur}(\mathcal{C}; Q_+, G)_{P_0} \times \text{Ran}(X)_{P_0} \times [0, 1] \\
\text{Hur}(\mathcal{C}; Q_+, G)_{P_0} \times \text{Ran}_+(Y) & \xrightarrow{\varepsilon \times \text{Id}} \text{Hur}(\mathcal{C}; Q_+, G)_{P_0},
\end{align*}
\]
where \(\varepsilon\) is from Definition 3.10 and in the last step we use Notation 5.9. Since \(\rho\) restricts to the identity on \(\text{Hur}(\mathcal{C}; Q_+, G)_{P_0}\), the map \(\mathcal{H}^{P_0}(\cdot, 0)\) coincides with the restriction of \(\mathcal{H}^{X,Y}(\cdot, 1)\) to \(\text{Hur}(\mathcal{C}; Q_+, G)_{P_0}\). On the other hand, \(\mathcal{H}^{Y}(\cdot, 1)\) is the identity of \(\text{Hur}(\mathcal{C}; Q_+, G)_{P_0}\). \(\square\)

7.4 An application of contractibility of Ran spaces

The following proposition deals with a generic PMQ-group pair \((\mathcal{Q}, G)\), with \(\mathcal{Q}\) possibly non-augmented; its proof relies on Proposition 7.3.

Proposition 7.10 Let \(\mathcal{C} = (X, Y)\) be a nice couple with \(X\) non-empty and connected, let \(P_0 \in \text{Ran}_+(X)\), and let \((\mathcal{Q}, G)\) be a PMQ-group pair. Then the inclusion \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0} \hookrightarrow \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\) is a homotopy equivalence.

Proof By Proposition 7.3 there is a homotopy \(\mathcal{H}^{P_0}_0 : \text{Ran}_+(\mathcal{C}) \times [0, 1] \rightarrow \text{Ran}_+(\mathcal{C})\) contracting \(\text{Ran}_+(\mathcal{C})\) onto the configuration \(P_0\). We use \(\mathcal{H}^{P_0}_0\) to define a homotopy \(\mathcal{H}^{P_0} : \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] \rightarrow \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\) as the composition
\[
\begin{align*}
\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times [0, 1] & \xrightarrow{(\text{Id}, \varepsilon) \times \text{Id}} \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times \text{Ran}_+(\mathcal{C}) \times [0, 1] \\
\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G) \times \text{Ran}_+(\mathcal{C}) & \xrightarrow{\varepsilon \times \text{Id}} \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G),
\end{align*}
\]
where \(\varepsilon\) and \(\varepsilon \times \text{Id}\) are from Definition 3.10 and Notation 5.9. Note the following:

- \(\mathcal{H}^{P_0}(\cdot; 0)\) is the identity of \(\text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\);
- \(\mathcal{H}^{P_0}(\cdot; 1)\) restricts to the identity on \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\);
- for all \(0 \leq s \leq 1\) and \(\varepsilon \in \text{Hur}_+(\mathcal{C}; \mathcal{Q}, G)\), if we denote \(\varepsilon' = \mathcal{H}^{P_0}(\varepsilon; s)\) and use Notation 3.6, then \(P \subseteq P'\); if moreover we assume \(s = 1\), then \(P_0 \subseteq P'\).

In particular \(\mathcal{H}^{P_0}\) preserves \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\) at all times and \(\mathcal{H}^{P_0}(\cdot; 1)\) takes values in \(\text{Hur}(\mathcal{C}; \mathcal{Q}, G)_{P_0}\). This implies the statement. \(\square\)

8 Cell stratifications

Throughout the section we fix an augmented PMQ \(\mathcal{Q}\) with completion \(\hat{\mathcal{Q}}\).

Notation 8.1 We denote by \(\hat{\mathcal{R}}\) the open unit square \((0, 1)^2 \subset \mathbb{H}\), and by \(\mathcal{R}\) the closed unit square \([0, 1]^2 \subset \mathbb{H}\).

In this section we introduce a cell stratification on \(\text{Hur}(\hat{\mathcal{R}}; Q_+).\) More precisely, we will do the following:

1. We regard \(\text{Hur}(\hat{\mathcal{R}}; Q_+)\) as an open subspace of \(\text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+),\) by applying functoriality to the inclusions \(\hat{\mathcal{R}} \hookrightarrow \mathcal{R}\) and \(\hat{\mathcal{Q}}_+ \hookrightarrow \hat{\mathcal{Q}};\)
(2) We consider the bisimplicial complex \( \text{Arr}(Q) \) from [3, Definition 6.6]; and define a continuous bijection \( \nu: |\text{Arr}(Q)| \rightarrow \text{Hur} (R; \hat{Q}_+) \), by glueing suitable continuous maps \( e^2; \Delta^p \times \Delta^q \rightarrow \text{Hur} (R; \hat{Q}_+) \), one for every non-degenerate array \( q \in \text{Arr}_{p,q} (Q) \);

(3) The map \( \nu \) restricts to a continuous bijection with source the simplicial Hurwitz space

\[
\nu: \text{Hur}^\Delta (Q) := |\text{Arr}(Q)| \setminus |\text{NAdm}(Q)| \rightarrow \text{Hur} (R; \hat{Q}),
\]

and in the additional hypothesis that \( Q \) is a locally finite PMQ, this latter bijection is a homeomorphism.

### 8.1 A construction with simplices

We start by fixing some notation and by making some constructions with simplices and products of simplices. For \( p \geq 0 \), we regard the standard \( p \)-simplex \( \Delta^p \) as the subspace of \([0, 1]^p\) containing all \( p \)-tuples \( x = (s_1, \ldots , s_p) \) with \( 0 \leq s_1 \leq \cdots \leq s_p \leq 1 \).

**Notation 8.2** Whenever needed, we extend each \( p \)-tuple \( x = (s_1, \ldots , s_p) \) representing a point in \( \Delta^p \) to a \( p + 2 \)-tuple \( s_0, \ldots , s_{p+1} \) by setting \( s_0 = 0 \) and \( s_{p+1} = 1 \).

**Notation 8.3** For \( p \geq 0 \) we denote by \( \text{bar}^p = (\text{bar}_1^p, \ldots , \text{bar}_p^p) = (\frac{1}{p+1}, \ldots , \frac{p}{p+1}) \) the barycentre of \( \Delta^p \).

**Definition 8.4** We denote by \( \tilde{\Delta}^p \subset \Delta^p \times \Delta^p \) the subspace containing all pairs \( (x, x') \) such that the following holds: for all \( 0 \leq i \leq p \), if \( s_i = s_{i+1} \) then \( s'_i = s'_{i+1} \).

**Definition 8.5** We define a continuous map \( \mathcal{H}^p: \mathbb{R} \times \tilde{\Delta}^p \rightarrow \mathbb{R} \) by the formula

\[
\mathcal{H}^p(x; x, x') = \begin{cases} 
\frac{x-s_i}{s_{i+1}-s_i} s'_{i+1} + \frac{s_{i+1}-x}{s_{i+1}-s_i} s'_i & \text{if } x \in \mathbb{R} \setminus (0, 1); \\
\frac{s_i}{s_{i+1}-s_i} s'_{i+1} + \frac{s_{i+1}-s_i}{s_{i+1}-s_i} s'_i & \text{if } x = s_i = s_{i+1}.
\end{cases}
\]

Roughly speaking, \( \mathcal{H}^p(-; x, x'): \mathbb{R} \rightarrow \mathbb{R} \) is constructed by fixing \((-\infty, 0] \cup [1, \infty)\) pointwise, by mapping each \( s_i \mapsto s'_i \) and by extending by linear interpolation on the segments \([0, s_1], \ldots , [s_p, 1]\); some of these segments might be degenerate, in this case no extension is needed. The subspace \( \tilde{\Delta}^p \subset \Delta^p \times \Delta^p \) is essentially defined as the subspace of couples \( (x, x') \) for which \( \mathcal{H}^p(-; x, x'): \mathbb{R} \rightarrow \mathbb{R} \) is well-defined and continuous. The previous description of \( \mathcal{H}^p \) implies directly the following lemma.

**Lemma 8.6** Let \( x, x', x'' \in \Delta^p \) such that both pairs \( (x, x') \) and \( (x', x'') \) lie in \( \tilde{\Delta}^p \); then also \( (x, x'') \in \tilde{\Delta}^p \), and the map \( \mathcal{H}^p(-; x, x'')\): \( \mathbb{R} \rightarrow \mathbb{R} \) coincides with the composition \( \mathcal{H}^p(-; x, x') \circ \mathcal{H}^p(-; x, x'') \).

**Definition 8.7** For all \( p, q \geq 0 \) we define a map \( \mathcal{H}^{p,q}: \mathbb{C} \times \tilde{\Delta}^p \times \tilde{\Delta}^q \rightarrow \mathbb{C} \) by

\[
\mathcal{H}^{p,q} (x + y \sqrt{-1}; x, x', x'') = \mathcal{H}^p(x; x, x') + \mathcal{H}^q(y; x', x'').
\]

Note that for all \( (x, x'; l, l') \in \tilde{\Delta}^p \times \tilde{\Delta}^q \) the map \( \mathcal{H}^{p,q}(-; x, x'; l, l'): \mathbb{C} \rightarrow \mathbb{C} \) is a lax self morphism of the nice couple \( (\mathcal{R}, \emptyset) \) (see Definition 4.3). By Proposition 4.10 we obtain a map

\[
\mathcal{H}^{p,q}_+: \text{Hur} (R; \hat{Q}_+) \times \tilde{\Delta}^p \times \tilde{\Delta}^q \rightarrow \text{Hur} (R; \hat{Q}_+).
\]
Note also that $\Delta^p$ embeds diagonally into $\tilde{\Delta}^{p, p}$; the restricted map
\[ \mathcal{H}^{p, q}_{\ast} : \text{Hur}(\mathcal{R}; \hat{Q}_+) \times \Delta^p \times \Delta^q \to \text{Hur}(\mathcal{R}; \hat{Q}_+) \]
is just the projection on the first factor, since for all $(s, t) \in \Delta^p \times \Delta^q$ the map
\[ \mathcal{H}^{p, q}_{\ast} : \mathcal{H}_{\ast}^{p, q}(-; s, t, l, t') : \mathbb{C} \to \mathbb{C} \]
is the identity of $\mathbb{C}$.

Lemma 8.6 applied twice, together with functoriality, yields the following lemma.

Lemma 8.8 Let $(s', s''') \in \Delta^p$ such that $(s', s'') \in \tilde{\Delta}^{p, p}$, and let $t', t'' \in \Delta^q$ such that $(t', t'') \in \tilde{\Delta}^{q, q}$. Then $(s', s'') \in \tilde{\Delta}^{p, q}$, and the following equality of maps
\[ \mathcal{H}_{\ast}^{p, q}(-; s', s''; t', t'') \circ \mathcal{H}_{\ast}^{p, q}(-; s, s'; t, t') = \mathcal{H}_{\ast}^{p, q}(-; s, s''; t, t''). \]

8.2 The array filtration

The next step is to define a filtration on $\text{Hur}(\mathcal{R}; \hat{Q}_+)$ by closed subspaces $F^\text{arr}_v \text{Hur}(\mathcal{R}; \hat{Q}_+)$, for $v \geq -1$.

Definition 8.9 Let $P \in \text{ Ran}(\mathcal{R})$ be a finite, possibly empty subset of $\mathcal{R}$, and use Notation 2.4- We define the horizontal array degree of $P$, denoted $\text{arr}_{\text{hor}}(P) \geq 0$, as the cardinality of the finite set $\mathcal{H}(P) \setminus \{0, 1\} = \mathcal{H}(z_1), \ldots, \mathcal{H}(z_k) \setminus \{0, 1\}$; similarly we define the vertical array degree of $P$, denoted $\text{arr}_{\text{ver}}(P) \geq 0$, as the cardinality of the finite set $\mathcal{Z}(P) \setminus \{0, 1\} = \mathcal{Z}(z_1), \ldots, \mathcal{Z}(z_k) \setminus \{0, 1\}$. The array bidegree $\text{arr}(P)$ is defined as the couple $(\text{arr}_{\text{hor}}(P), \text{arr}_{\text{ver}}(P))$, and the total array degree is defined as $|\text{arr}|(P) = \text{arr}_{\text{hor}}(P) + \text{arr}_{\text{ver}}(P)$.

Roughly speaking, $F^\text{arr}_v \text{Hur}(\mathcal{R}; \hat{Q}_+)$ contains configurations $c = (P, \psi)$ such that the total number of horizontal and vertical lines passing through some point of $P$, excluding the sides of $\mathcal{R}$, does not exceed $v$. Similarly, $\mathcal{Z}^v \text{Hur}(\mathcal{R}; \hat{Q}_+) \mathcal{F}$ contains those configurations $c$ for which this total number of lines is equal to $v$.

Lemma 8.10 For $v \geq -1$ the subspace $F^\text{arr}_v \text{Hur}(\mathcal{R}; \hat{Q}_+) \subset \text{Hur}(\mathcal{R}; \hat{Q}_+)$ is closed.

Proof We prove that $\text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F^\text{arr}_v \text{Hur}(\mathcal{R}; \hat{Q}_+)$ is open. Let $c \in \text{Hur}(\mathcal{R}; \hat{Q}_+)$ be a configuration with $|\text{arr}(c)| \geq v + 1$, use Notations 3.6 and 2.6, and let $U$ be an adapted covering of $P$ with the following property: for all $1 \leq i \leq k$ the projection $\mathcal{H}(U_i) \subset \mathbb{R}$ intersects the finite set $\mathcal{H}(P) \cup \{0, 1\}$ only in the point $\mathcal{H}(z_i)$, and the projection $\mathcal{Z}(U_i) \subset \mathbb{R}$ intersects the finite set $\mathcal{Z}(P) \cup \{0, 1\}$ only in the point $\mathcal{Z}(z_i)$.

We claim that $\mathcal{U}(c; U)$ is contained in $\text{Hur}(\mathcal{R}; \hat{Q}_+) \setminus F^\text{arr}_v \text{Hur}(\mathcal{R}; \hat{Q}_+)$. Let $c' \in \mathcal{U}(c; U)$; then $P'$ intersects each $U_i$ in at least one point; by the choice of $U$ we have $\text{arr}_{\text{hor}}(c') \geq \text{arr}_{\text{hor}}(c)$ and $\text{arr}_{\text{ver}}(c') \geq \text{arr}_{\text{ver}}(c)$, hence $|\text{arr}|(c') \geq |\text{arr}|(c) \geq v + 1$. □

8.3 Standard generating set

We introduce, for a finite set $P \subset \mathcal{R}$, a particular admissible generating set of $\mathfrak{G}(P)$, the standard generating set. Fix $P \subset \mathcal{R}$, and let $\mathcal{H}(P) \cup \{0, 1\}$ consist of the points $0 = x_0 <
Let \( x_1 < \cdots < x_p < x_{p+1} = 1 \), where \( p = \text{arr}_{\text{hor}}(P) \); similarly let \( 0 = y_0 < y_1 < \cdots < y_q < y_{q+1} = 1 \) be the elements of \( \Im(P) \cup \{0, 1\} \), where \( q = \text{arr}_{\text{ver}}(P) \). For all \((i, j) \in \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\}\) denote by \( z_{i,j} \) the complex number \( x_i + y_j \sqrt{-1} \in \mathbb{C} \), and let \( I(P) \subset \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\} \) be the subset of pairs \((i, j)\) such that \( z_{i,j} \) is a point of \( P \).

Recall Notation 6.8. For all \((i, j) \in I(P)\) with \( 0 \leq i \leq p \) let \( \zeta_{i,j}^{P, \text{std}} \) be an arc contained in \( S_{x_i, x_{i+1}} \) and joining \(*\) with \( z_{i,j} \). Similarly, for all \((p + 1, j) \in I(P)\) let \( \zeta_{p+1,j}^{P, \text{std}} \) be an arc contained in \( S_{1, \infty} \) joining \(*\) with \( z_{p+1,j} \). Up to changing the arcs by an isotopy, we may assume that the arcs \( \zeta_{i,j}^{P, \text{std}} \) are disjoint away from \(*\). Note also that these arcs are uniquely determined up to an ambient isotopy of \( \mathbb{C} \) that fixes \( P \) pointwise and preserves each subspace \( S_{x_i, x_{i+1}} \).

**Definition 8.11** Recall Definition 2.8. We denote by \((f_{i,j}^{P, \text{std}})_{(i,j) \in I(P)}\) the admissible generating set of \( \mathfrak{G}(P) \) associated with the arcs \( \zeta_{i,j}^{P, \text{std}} \), and call it the **standard generating set** for \( \mathfrak{G}(P) \). See Fig. 9.

![Fig. 9](image_url) The standard generating set of a configuration \( P \subset \mathcal{R} \); we have \( \text{arr}_{\text{hor}}(P) = 2 \) and \( \text{arr}_{\text{ver}}(P) = 3 \). The dashed loop represents the product \( cf_{1,3} \in \mathfrak{G}(P) \).
We will also make use of the following products of standard generators, compare with [3, Notation 6.7].

**Notation 8.12** Use the notation from Definition 8.11. For $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 2$ we denote by $c_{f_{i,j}}^{P,\text{std}}$ the product

$$c_{f_{i,j}}^{P,\text{std}} = f_{i,0}^{P,\text{std}} f_{i,1}^{P,\text{std}} \cdots f_{i,j-1}^{P,\text{std}} \in \mathfrak{S}(P),$$

where we set $f_{i,j}^{P,\text{std}} = \mathbb{1} \in \mathfrak{S}(P)$ whenever $(i, j)$ does not belong to $I(P)$.

One can represent $c_{f_{i,j}}^{P,\text{std}}$ by a simple loop $\gamma \subset \mathbb{C} \setminus P$ satisfying the following:

- $\gamma \subset \mathbb{S}_{x_{i-1},x_{i+1}} \cap \{ z \in \mathbb{C} \mid \Im(z) < y_j \}$, where we use the conventions $x_{-1} = -\infty$ and $x_{q+2} = y_{q+2} = \infty$;
- $\gamma$ bounds a disc in $\mathbb{C}$ containing the points $z_{i,0}, \ldots, z_{i,j-1}$.

In particular $c_{f_{i,j}}^{P,\text{std}} \in \mathfrak{E}^\text{ext}(P)$, see Definition 2.10. For $j = 0$ we have $c_{f_{i,j}}^{P,\text{std}} = \mathbb{1}$.

**8.4 Characteristic maps of cells**

In this subsection we introduce maps

$$e^2 : \Delta^p \times \Delta^q \to \text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+)$$

depending on a non-degenerate array $a \in \text{Arr}_{p,q}(\mathcal{Q})$. As we will see, each map $e^2$ sends the interior of $\Delta^p \times \Delta^q$ injectively inside $\mathfrak{S}_{p+q}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+)$, and sends the boundary of $\Delta^p \times \Delta^q$ inside $\mathfrak{F}_{p+q-1}^{\text{arr}} \text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+)$.

Recall from [3, Definitions 5.8 and 6.6] that the bisimplicial set $\text{Arr}(\mathcal{Q})$ consists of the sets $\text{Arr}_{p,q}(\mathcal{Q}) \cong \hat{\mathcal{Q}}^{(p+2) \times (q+2)}$ for $p, q \geq 0$. An element $a \in \text{Arr}_{p,q}(\mathcal{Q})$ is an array of size $(p + 2) \times (q + 2)$ with entries in $\hat{\mathcal{Q}}$. For $p \geq 1$ and $0 \leq i \leq p$, the $i$th horizontal face map is denoted $d^\text{hor}_i : \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p-1,q}(\mathcal{Q})$, and for $q \geq 1$ and $0 \leq j \leq q$, the $j$th vertical face map is denoted $d^\text{ver}_j : \text{Arr}_{p,q}(\mathcal{Q}) \to \text{Arr}_{p,q-1}(\mathcal{Q})$. Similarly $s^\text{hor}_i$ and $s^\text{ver}_j$ denote the horizontal and vertical degeneracy maps. Formulas for face and degeneracy maps are given in [3, Lemma 6.8]. An array $a \in \text{Arr}_{p,q}(\mathcal{Q})$ is non-degenerate if and only if it is in the image of any horizontal or vertical degeneracy map of the bisimplicial set $\text{Arr}(\mathcal{Q})$.

**Notation 8.13** For $a \in \text{Arr}(p, q)$ we let $I(a) \subset \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\}$ denote the set of pairs $(i, j)$ with $a_{i,j} \neq \mathbb{1}$.

An array $a$ is non-degenerate if and only if the following conditions hold, compare with [3, Sect. 6.3]:

- for all $1 \leq i \leq p$ there is $0 \leq j \leq q + 1$ with $(i, j) \in I(a)$;
- for all $1 \leq j \leq q$ there is $0 \leq i \leq p + 1$ with $(i, j) \in I(a)$.

Our next goal is to define, for any non-degenerate array $a \in \text{Arr}(p, q)$, a configuration $c_a \in \text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+)$ with $\text{arr}_{\text{hor}}(c_a) = p$ and $\text{arr}_{\text{ver}}(c_a) = q$: it will be the “central” configuration in the image of $e^2$.

**Notation 8.14** We denote by $P^{p,q} \subset \mathcal{R}$ the set of complex numbers $z^{p,q}_{i,j} := \frac{i}{p+1} + \frac{j}{q+1} \sqrt{-1}$, with $0 \leq i \leq p + 1$ and $0 \leq j \leq q + 1$.

We let $P_a \subset P^{p,q}$ be the set containing all elements $z^{p,q}_{i,j}$ for $(i, j) \in I(a)$.
Since \( q \) is admissible we have \( \text{arr}_{\text{hor}}(P_a) = p \) and \( \text{arr}_{\text{ver}}(P_a) = q \).

**Notation 8.15** We denote by \((f^a_{i,j})_{(i,j)\in I(a)}\) the standard generating set \((f^a_{i,j})^{\text{std}}_{(i,j)}\) of \( \mathfrak{G}(P_a) \) (see Definition 8.11). For \( 0 \leq i \leq p + 1 \) and \( 0 \leq j \leq q + 2 \) we denote by \( cf^a_{i,j} \) the product \( cf^a_{i,j} \) (see Notation 8.12).

**Definition 8.16** We define \( c_a \) as the configuration \((P_a, \psi_a) \in \text{Hur}(R; \hat{Q}_+)\), where \( \psi_a \) is defined by setting \( \psi_a : f^a_{i,j} \mapsto a_{i,j} \) for all \((i, j) \in I(a)\).

Note that since \( \hat{Q} \) is complete we have an equality \( \Omega^\text{ext}(P_a) = \Omega^\text{ext}(P_a)\psi_a\), so that we can extend \( \psi_a \) to a map of PMQs \( \psi^\text{ext}_a : \Omega^\text{ext}(P_a) \to \hat{Q}; \) the element \( cf^a_{i,j} \in \Omega^\text{ext}(P_a) \) is mapped to the product \( a_{i,0} \cdots a_{i,-1} \in \hat{Q} \) along \( \psi^\text{ext}_a \).

**Definition 8.17** Recall Notation 8.3 and Definition 8.4, and note that for all \( s \in \Delta^p \) the pair \((\bar{\text{bar}}^p, s)\) belongs to \( \Delta^{p+q} \). For a non-degenerate array \( a \in \text{Arr}(p, q) \) we define a continuous map \( e^a : \Delta^p \times \Delta^q \to \text{Hur}(R; \hat{Q}_+) \) by the formula
\[
e^a(s, t) = \mathcal{H}^\text{arr}_{s, t}(c_a, \bar{\text{bar}}^p, s; \bar{\text{bar}}^q, t).
\]

**Lemma 8.18** Let \( a \in \text{Arr}_{p, q} \) be non-degenerate; then \( e^a \) satisfies the following:

- \( e^a \) sends the interior of \( \Delta^p \times \Delta^q \) injectively inside \( \mathfrak{H}^\text{arr}_{p+q} \) Hur\( (R; \hat{Q}_+) \);
- \( e^a \) sends \( \partial(\Delta^p \times \Delta^q) \) inside \( F^\text{arr}_{p+q-1} \) Hur\( (R; \hat{Q}_+) \).

**Proof** Let \((s, t) \in \Delta^p \times \Delta^q \) and let \( \mathcal{E} = (P, \psi) := e^a(s, t) \). Then \( P \) is the image of \( P_a \) along the map \( \mathcal{H}^\text{arr}_{s, t}(-; \bar{\text{bar}}^p, s; \bar{\text{bar}}^q, t) : \mathbb{C} \to \mathbb{C} \). The latter map sends \( z^a_{i,j} \mapsto s_i + t_j \sqrt{-1} \) for all \( 0 \leq i \leq p + 1 \) and \( 0 \leq j \leq q + 1 \), in particular for \((i, j) \in I(a)\). It follows that \( \mathfrak{N}(P) \setminus \{0, 1\} = \{s_1, \ldots, s_p\} \setminus \{0, 1\} \) consists of at most \( p \) points, and \( \mathfrak{N}(P) \setminus \{0, 1\} = \{t_1, \ldots, t_q\} \setminus \{0, 1\} \) consists of at most \( q \) points. More precisely, using that \( a \) is non-degenerate, we have that \( \mathfrak{N}(P) \setminus \{0, 1\} = p \) if \( 0 < s_1 \leq \cdots < s_p < 1 \), i.e. if \( s \) is in the interior of \( \Delta^p \), whereas \( \mathfrak{N}(P) \setminus \{0, 1\} = q \) if \( t \) is in the interior of \( \Delta^q \), and \( \mathfrak{N}(P) \setminus \{0, 1\} = \emptyset \) if \( t \in \partial \Delta^q \).

Hence \( \lvert \text{arr}(c) \rvert \leq p + q \); equality holds if and only if \((s, t) \notin \partial(\Delta^p \times \Delta^q) \). \( \square \)

**Lemma 8.19** Let \( \nu \geq 0 \) and let \( c \in \mathfrak{H}^\text{arr}_\nu \) Hur\( (R; \hat{Q}_+) \); then there is precisely one couple of indices \( p, q \geq 0 \) with \( p + q = \nu \), and precisely one admissible array \( a \in \text{Arr}(p, q) \), such that \( c \) is in the image of \( e^a \).

**Proof** We first show the existence of \( p, q \) and \( a \) with the required properties. Use Notation 3.6 for \( c \), and let \( \mathfrak{N}(P) \cup \{0, 1\} = \{0 < s_1 < \cdots < s_p < 1\} \) have \( p + 2 \) elements, for some \( p \geq 0 \); similarly let \( \mathfrak{N}(P) \cup \{0, 1\} = \{0 < t_1 < \cdots < t_q < 1\} \) have \( q + 2 \) elements, for some \( q \geq 0 \). By the hypothesis that \( c \in \mathfrak{H}^\text{arr}_\nu \) Hur\( (R; \hat{Q}_+) \) we have the equality \( p + q = \lvert \text{arr}(c) \rvert = \nu \).

Define an array \( a \) of size \((p + 2) \times (q + 2)\) by setting \( a_{i,j} = \psi(f^a_{i,j})^{\text{std}} \in \hat{Q} \) for all \((i, j) \in I(P)\), and \( a_{i,j} = 1 \) for all \((i, j) \in \{0, \ldots, p + 1\} \times \{0, \ldots, q + 1\} \setminus I(P)\) (see Definition 8.11). The hypothesis that \( c \) lies in Hur\( (R; \hat{Q}_+) \subseteq \text{Hur}(R; \hat{Q}) \) ensures that \( a \) is a non-degenerate array in Arr\( p, q(\mathcal{Q}) \). Moreover we have \( I(a) = I(P) \).

We claim that \( e^a \) sends \((s, t) \in \Delta^p \times \Delta^q \) to \( c \). Indeed \( \mathcal{H}^\text{arr}_{s, t}(-; \bar{\text{bar}}^p, s; \bar{\text{bar}}^q, t) \) is a homeomorphism of \( \mathbb{C} \), sending \( P_a \) bijectively to \( P \), and sending the standard generating
set \((f^\text{P, std}_{i,j})_{(i,j) \in I(P)}\) of \(\mathcal{G}(P)\) to the standard generating set \((f^\text{P, std}_{i,j})_{(i,j) \in I(P)}\) of \(\mathcal{G}(P)\). This proves the existence of \(p, q\) and \(a\) as desired.

For uniqueness, suppose that we are given two integers \(p', q' \geq 0\) and a non-degenerate array \(a' \in \text{Arr}(p, q)\), such that \(p' + q' = v\) and such that there is a point \((s', t') \in \Delta^p \times \Delta^q\) with \(e^{a'}: (s', t') \mapsto c\). Then by Lemma 8.18 we have that \((s', t')\) lies in the interior of \(p' \times q'\), since \(c\) lies in \(\mathfrak{Z}_{v}^\text{arr}\text{Hur}(\mathcal{R}; \hat{\mathcal{Q}}_+)\). Again the map \(\mathcal{H}^{p', q'}\left(\bar{\text{bar}}^p, s'; \bar{\text{bar}}^q, t'\right)\) is a homeomorphism of \(C\) mapping the set \(P_a\) bijectively to the set \(P\) by the formula

\[
e_{i,j}^{a'} \mapsto s_i^{a'} + \sqrt{-1}t_j^{a'}.
\]

It follows that \(\{0 < s_1^{a'} < \cdots < s_p^{a'} < 1\}\) is equal to \(\mathfrak{R}(P) \cup \{0, 1\}\), and similarly \(\{0 < t_1^{a'} < \cdots < t_q^{a'} < 1\}\) is equal to \(\mathfrak{S}(P) \cup \{0, 1\}\); in particular, comparing with the construction above, we have \(p = p', q = q', s = s', t = t'\) and \(I(a') = I(P)\).

Since \(\mathcal{H}^{p', q'}\left(\bar{\text{bar}}^p, s'; \bar{\text{bar}}^q, t'\right)\) gives a bijection between the standard generating sets \((f^a_{i,j})_{(i,j) \in I(a')}\) and \((f^\text{P, std}_{i,j})_{(i,j) \in I(P)}\), it also follows that

\[
a_{i,j} = \psi_{a'}(f^a_{i,j}) = \psi(f_{i,j}) = a_{i,j}
\]

for all \((i, j) \in I(P)\), where \(\psi_{a'}\) is the monodromy of the configuration \(c_{a'}\), see Definition 8.16; hence \(a' = a\).

\[\square\]

### 8.5 Face restrictions and the bijection \(\mathcal{U}\)

In the following two propositions we analyse the restriction of \(e^a\) to a face of \(\Delta^p \times \Delta^q\), and thus establish a link between the bisimplicial set \(\text{Arr}(Q)\) and the cell stratification on \(\text{HUr}(\mathcal{R}; \hat{\mathcal{Q}}_+)\).

**Notation 8.20** For \(0 \leq i \leq p\) we denote by \(d_i^{\text{hor}}\Delta^p \times \Delta^q\) the face \((d_i\Delta^p) \times \Delta^q \subset \Delta^p \times \Delta^q\); for \(0 \leq j \leq q\) we denote by \(d_j^{\text{ver}}\Delta^p \times \Delta^q\) the face \(\Delta^p \times (d_j\Delta^q) \subset \Delta^p \times \Delta^q\).

Each face \(d_i\Delta^p \subset \Delta^p\) can be identified with a the simplex \(\Delta^{p-1}\) by using either the coordinates \((s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_p)\), for \(i \neq 0\), or the coordinates \((s_1, \ldots, s_{i}, s_{i+2}, \ldots, s_p)\), for \(i \neq p\); for \(1 \leq i \leq p-1\) the two choices give rise to the same identification. Similarly, there are canonical identifications of \(d_i^{\text{hor}}\Delta^p \times \Delta^q\) with \(\Delta^{p-1} \times \Delta^q\), and of \(d_j^{\text{ver}}\Delta^p \times \Delta^q\) with \(\Delta^p \times \Delta^{q-1}\).

**Proposition 8.21** Let \(a\) be a non-degenerate array in \(\text{Arr}_{p,q}(Q)\). For some \(p \geq 1\) and \(q \geq 0\), and let \(0 \leq i \leq p\). Then the restriction of \(e^a: \Delta^p \times \Delta^q \rightarrow \text{HUr}(\mathcal{R}; \hat{\mathcal{Q}}_+)\) to the face \(d_i^{\text{hor}}\Delta^p \times \Delta^q \cong \Delta^{p-1} \times \Delta^q\) is equal to the map \(e^{a_i}\), where \(a_i = d_i^{\text{hor}}a\).

**Proposition 8.22** Let \(a\) be a non-degenerate array in \(\text{Arr}_{p,q}(Q)\), for some \(p \geq 0\) and \(q \geq 1\), and let \(0 \leq j \leq q\). Then the restriction of \(e^a: \Delta^p \times \Delta^q \rightarrow \text{HUr}(\mathcal{R}; \hat{\mathcal{Q}}_+)\) to the face \(d_j^{\text{ver}}\Delta^p \times \Delta^q \cong \Delta^p \times \Delta^{q-1}\) is equal to the map \(e^{a_j}\), where \(a_j = d_j^{\text{ver}}a\).

The expressions “\(d_i^{\text{hor}}a\)” and “\(d_j^{\text{ver}}a\)” refer to the bisimplicial set \(\text{Arr}(Q)\). The proof of Propositions 8.21 and 8.22 is in Sects. A.10 and A.11 of the appendix.

Recall from [3, Lemma 6.10] that there is a semi-bisimplicial set \(\text{Arr}^\text{std}(Q)\) containing all non-degenerate arrays of \(\text{Arr}(Q)\), and with vertical and horizontal face maps given by the
restriction of those of $\text{Arr}(\mathcal{Q})$. Consider the geometric realisation $\|\text{Arr}^{\text{ndeg}}(\mathcal{Q})\|$ of the semi-bisimplicial complex $\text{Arr}^{\text{ndeg}}(\mathcal{Q})$, and note that there is a homeomorphism $\|\text{Arr}^{\text{ndeg}}(\mathcal{Q})\| \cong |\text{Arr}(\mathcal{Q})|$. By Propositions 8.21 and 8.22 we obtain a continuous map $$\nu: |\text{Arr}(\mathcal{Q})| \to \text{Hur}(\mathcal{R}; \mathcal{Q}_+) .$$

Lemmas 8.18 and 8.19 imply that $\nu$ is bijective; if we consider on $|\text{Arr}(\mathcal{Q})|$ the skeletal filtration and on $\text{Hur}(\mathcal{R}; \mathcal{Q}_+)$ the filtration $F^{\text{arr}}_i$, then Lemma 8.18 also implies that $\nu$ is a map of filtered spaces.

We say that an entry $a_{i,j}$ of an array $\mathcal{a} \in \text{Arr}_{p,q}(\mathcal{Q})$ is in boundary position if $i \in \{0, p + 1\}$ or $j \in \{0, q + 1\}$ (or both conditions hold). Recall from [3, Definition 6.11 and Lemma 6.12] that we have a sub-bisimplicial set $\text{NAdm}(\mathcal{Q}) \subset \text{Arr}(\mathcal{Q})$ of non-admissible arrays: an array $\mathcal{a} \in \text{Arr}(p, q)$ is non-admissible if either of the following requirements is satisfied:

- there exists an entry $a_{i,j}$ lying in $\mathcal{Q} \setminus \mathcal{Q}$;
- there exists an entry $a_{i,j} \neq \mathbb{1}$ in boundary position.

**Lemma 8.23** The map $\nu$ restricts to continuous bijections $$\nu: |\text{NAdm}(\mathcal{Q})| \to \text{Hur}(\mathcal{R}; \mathcal{Q}_+) \setminus \text{Hur}(\mathcal{R}; \mathcal{Q}_-) ;$$ $$\nu: \text{Hur}^{\Delta}(\mathcal{Q}) = |\text{Arr}(\mathcal{Q})| \setminus |\text{NAdm}(\mathcal{Q})| \to \text{Hur}(\mathcal{R}; \mathcal{Q}_+) .$$

**Proof** Let $\epsilon \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$, and use Notation 3.6. In the proof of Lemma 8.19 we have given a construction, depending on $\epsilon$, of a couple of numbers $p, q \geq 0$, a point $(s, t)$ in the interior of $\Delta^p \times \Delta^q$ and a non-degenerate array $\mathcal{a} \in \text{Arr}(p, q)$ such that $\epsilon = \epsilon^\mathcal{a}(s, t)$. The data $(\mathcal{a}; s, t)$ represent a point in $|\text{Arr}(\mathcal{Q})|$, which is precisely the preimage $\nu^{-1}(\epsilon)$; we have $|\text{NAdm}(\mathcal{Q})|$ if and only if $\mathcal{a}$ is non-admissible.

The array $\mathcal{a}$ was constructed by considering the standard generating set of $\mathcal{G}(P)$, and by setting $a_{i,j} = \psi(f_{i,j})$ for $(i, j) \in I(P)$, and $a_{i,j} = \mathbb{1}$ otherwise. It follows that $\mathcal{a}$ has all entries in $\mathcal{Q}$ if and only if $\psi: \mathcal{Q}(P) \to \hat{\mathcal{Q}}$ has image in $\mathcal{Q}$, that is, $\epsilon \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$; and $\mathcal{a}$ has all entries in boundary position equal to $\mathbb{1}$ if and only if $P \subset \mathcal{R}$, that is, $\epsilon \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$. We have therefore $$\epsilon \in \text{Hur}(\mathcal{R}; \mathcal{Q}_+) = \text{Hur}(\mathcal{R}; \mathcal{Q}_+) \cap \text{Hur}(\mathcal{R}; \mathcal{Q}_+) \subset \text{Hur}(\mathcal{R}; \mathcal{Q}_+)$$ if and only if $\mathcal{a}$ is admissible. \hfill $\Box$

### 9 Locally finite and Poincaré PMQs

We consider the Hurwitz–Ran spaces $\text{Hur}(\mathcal{R}; \mathcal{Q})$ in the special cases of a locally finite PMQ and of a Poincaré PMQ $\mathcal{Q}$. Recall that a PMQ $\mathcal{Q}$ is Poincaré if each connected component of $\text{Hur}^{\Delta}(\mathcal{Q})$ is a topological manifold; this condition implies that $\hat{\mathcal{Q}}$ is endowed with an intrinsic norm $h: \hat{\mathcal{Q}} \to \mathbb{N}$ such that $\text{Hur}^{\Delta}(\mathcal{Q})(\mathcal{a})$ is an orientable manifold of dimension $2h(\mathcal{a})$ for all $\mathcal{a} \in \mathcal{Q}$, see [3, Proposition 6.23]. A Poincaré PMQ is always locally finite, and a locally finite PMQ is always augmented.
9.1 Locally finite PMQs

In this subsection we prove the following theorem.

**Theorem 9.1** Let $Q$ be a locally finite PMQ. Then the bijection $\nu : \text{Hur}^\Delta(Q) \rightarrow \text{Hur}(\widehat{R}; Q_+)$ is a homeomorphism.

We first note that both $|\text{Arr}(Q)|$ and $\text{Hur}(\widehat{R}; \widehat{Q}_+)$ decompose as topological disjoint unions of subspaces

$$|\text{Arr}| = \bigsqcup_{a \in \widehat{Q}} |\text{Arr}(Q)(a)|, \quad \text{Hur}(\widehat{R}; \widehat{Q}_+) = \bigsqcup_{a \in \widehat{Q}} \text{Hur}(\widehat{R}; \widehat{Q}_+)a.$$

Recall from [3, Definition 6.1] the category $\widehat{Q}/\widehat{Q}$ and the notion of $\widehat{Q}$-crossed object in a category. The space $|\text{Arr}(Q)(a)|$ is the geometric realisation of the bisimplicial set $\text{Arr}(Q)(a)$, which is the value at $a \in \widehat{Q}/\widehat{Q}$ of the $\widehat{Q}$-crossed bisimplicial set $\text{Arr}(Q)$: concretely, $\text{Arr}_{p,q}(Q)(a)$ contains all arrays $a$ satisfying $\prod_{i=0}^{p+1} \left( \prod_{j=0}^{q+1} a_{i,j} \right) = a \in \widehat{Q}$. For the space $\text{Hur}(\widehat{R}; \widehat{Q}_+)a$, see Notation 6.2.

The map $\nu$ restricts for all $a \in \widehat{Q}$ to a bijection $\nu_a : |\text{Arr}(Q)(a)| \rightarrow \text{Hur}(\widehat{R}; \widehat{Q}_+)a$. Consider first an element $a \in Q \subset \widehat{Q}$. The hypothesis that $Q$ is locally finite implies that $\text{Arr}(Q)(a)$ is a bisimplicial complex with finitely many non-degenerate arrays: hence the geometric realisation $|\text{Arr}(Q)(a)|$ is compact. The bijection $\nu_a$ thus has a compact space as source and a Hausdorff space as target, and is therefore a homeomorphism. Restricting to $\text{Hur}^\Delta(Q)(a)$ and $\text{Hur}(\widehat{R}; \widehat{Q}_+)a$, we have a homeomorphism $\nu_a : \text{Hur}^\Delta(Q)(a) \rightarrow \text{Hur}(\widehat{R}; \widehat{Q}_+)a$.

Consider now a generic element $a \in \widehat{Q}$, let $c \in \text{Hur}(\widehat{R}; \widehat{Q}_+)a$, use Notation 3.6 and the notation of Sect. 8.3. Let $U$ be an adapted covering of $P$, and assume that for all $(i, j) \in I(P)$ the component $U_{i,j} \subset U$ containing $z_{i,j}$ satisfies the following properties:

- $\mathfrak{H}(U_{i,j})$ intersects $\mathfrak{H}(P) \cup \{0, 1\}$ only in $\mathfrak{H}(z_{i,j})$;
- $\mathfrak{I}(U_{i,j})$ intersects $\mathfrak{I}(P) \cup \{0, 1\}$ only in $\mathfrak{I}(z_{i,j})$.

Let $\overline{U}$ denote the union $\bigcup_{(i,j) \in I(P)} \overline{U}_{i,j}$, i.e. the closure of $U$, and note that $\overline{U}$ is compact. The normal neighbourhood $\Omega(c; \overline{U})$ can be regarded as an open subspace of $\text{Hur}(\overline{U}; Q_+)a$, which by the argument of Proposition 5.1 is homeomorphic to a product of spaces

$$\text{Hur}(\overline{U}; Q_+)a \cong \prod_{(i,j) \in I(P)} \text{Hur}(\overline{U}_{i,j}; Q_+)a_{i,j},$$

for suitable elements $a_{i,j} \in Q$. Note that if Proposition 5.1 is applied using the arcs $(\xi_{i,j})_{(i,j) \in I(P)}$ yielding the standard generating set of $\mathfrak{H}(P)$, then the elements $a_{i,j}$ are precisely the entries different from 1 of the array $a$ describing the cell of $\text{Hur}^\Delta(Q)(a)$ containing $\nu^{-1}(c)$.

The previous analysis shows that each factor $\text{Hur}(\overline{U}_{i,j}; Q_+)a_{i,j}$ is compact; it follows that $\text{Hur}(\overline{U}; Q_+)a$ is compact i.e. $\text{Hur}(\widehat{R}; \widehat{Q}_+)a$ is locally compact.

Consider $\text{Hur}(\overline{U}; Q_+)a$ as a subspace of $\text{Hur}(\widehat{R}; \widehat{Q}_+)a$; the hypothesis that $Q$ is locally finite implies that the preimage $\nu_a^{-1}(\text{Hur}(\overline{U}; Q_+)a)$ intersects only finitely many cells in the cell decomposition of $|\text{Arr}(Q)(a)|$. Hence $\nu_a^{-1}(\text{Hur}(\overline{U}; Q_+)a)$ is compact, being a closed subset of a finite cell sub-complex of $|\text{Arr}(Q)(a)|$. Since $\nu_a^{-1}(\text{Hur}(\overline{U}; Q_+)a)$ contains the open neighbourhood $v_a^{-1}(\Omega(c; \overline{U}))$ of $v_a^{-1}(c)$, we obtain that $\text{Hur}^\Delta(Q)(a)$ is also locally compact.
We conclude that \( \nu : \text{Hur}^\Delta(Q)(a) \rightarrow \text{Hur}(\hat{\mathcal{R}}; Q_+)_{a} \) is a proper continuous bijection between locally compact spaces, hence it is a homeomorphism.

### 9.2 A counterexample to Theorem 9.1 for non-locally finite PMQs

For an augmented but not locally finite PMQ \( Q \), the bijection \( \nu \) may not restrict to a homeomorphism \( \text{Hur}^\Delta(Q) \rightarrow \text{Hur}(\hat{\mathcal{R}}; Q_+) \): see the following example.

**Example 9.2** Let \( \hat{\mathcal{Q}} \) be the completion of \( \mathcal{Q} := \mathbb{I}^2 \), the free group on two generators \( f_1, f_2 \) endowed with trivial multiplication, as in [3, Example 4.13].

Let \( \epsilon = (P, \psi \in \text{Hur}(\hat{\mathcal{R}}; \hat{\mathcal{Q}}_+) \) be a configuration supported on \( P := \{z_c\} = \left\{ \frac{1 + \sqrt{-1}}{2} \right\} \) with \( \psi \) defined by sending the unique element \([y] \in \Omega(P) \setminus \{1\} \) to \( w = f_1 f_2 \in \hat{\mathcal{Q}} \). For all \( 0 < \epsilon \leq 1/2 \) we have an adapted covering of \( P \) of the form \( U_\epsilon = \{z \in \mathbb{C} : |z - z_c| < \epsilon\} \); the associated normal neighbourhoods \( \Pi(c; U_\epsilon) \) form a fundamental system of neighbourhoods of \( \epsilon \in \text{Hur}(\hat{\mathcal{R}}; \hat{\mathcal{Q}}_+) \).

For \( \epsilon > 0 \), denote by \( P_\epsilon \) the set of two points \( \{z_c \pm \epsilon/2\} \), and note that \( P_\epsilon \subset U_\epsilon \). For every decomposition \( w = a \cdot b \) with respect to \( \hat{\mathcal{Q}}_1 \) we can define a configuration \( \epsilon_{\epsilon,a,b} = (P_\epsilon, \psi_{\epsilon,a,b}) \in \Pi(c; U_\epsilon) \), where \( \psi_{\epsilon,a,b} \) sends the standard generators \( f_{1,1}^{P_\epsilon,\text{std}} \) and \( f_{2,1}^{P_\epsilon,\text{std}} \) to \( a \) and \( b \) respectively. Using that \( w \) has infinitely many non-trivial decompositions with respect to \( \hat{\mathcal{Q}}_1 \), we obtain for all \( 0 < \epsilon \leq 1/2 \) an infinite family of configurations \( \epsilon_{\epsilon,a,b} \) supported on the same set \( P_\epsilon \) and contained in an arbitrary small normal neighbourhood \( \Pi(c; U_\epsilon) \).

Note that the configurations \( \nu^{-1}(\epsilon_{\epsilon,a,b}) \), for fixed \( \epsilon \) and varying \( a, b \) with \( w = ab \), belong to different open cells of the cell stratification of \( \text{Hur}^\Delta(\hat{\mathcal{Q}}) \). By a diagonal argument one can find a neighbourhood of \( \nu^{-1}(\epsilon) \) in \( \text{Hur}^\Delta(\hat{\mathcal{Q}}) \) containing, for all \( \epsilon > 0 \), only finitely many points \( (a; s, t) \) such that \( \nu(a; s, t) \) has support precisely \( P_\epsilon \). Thus \( \nu : \text{Hur}^\Delta(\hat{\mathcal{Q}}) \rightarrow \text{Hur}(\hat{\mathcal{R}}; \hat{\mathcal{Q}}_+) \) is not a homeomorphism.

In light of Example 9.2 one could argue that the topology on \( \text{Hur}(\hat{\mathcal{R}}; \hat{\mathcal{Q}}_+) \), described in Sect. 3, is not the correct topology to consider on Hurwitz–Ran spaces, and that one should rather consider the CW topology induced by \( \vert \text{Arr}(\mathcal{Q}) \vert \) along the bijection \( \nu \). This would indeed simplify the discussion in this section, by making Theorem 9.1 tautological. Nevertheless it would become much more elaborate to replace the topology on \( \text{Hur}(\mathcal{C}; \mathcal{Q}, G) \), for a generic nice couple \( \mathcal{C} \) and a generic PMQ-group pair \( (\mathcal{Q}, G) \), with the topology of a difference of CW complexes. Moreover, the functoriality of Hurwitz–Ran spaces with respect to morphisms of nice couplex, discussed Sect. 4, would also become much more complicated to prove.

### 9.3 Poincaré PMQs

In this subsection we prove the following theorem.

**Theorem 9.3** Let \( Q \) be a locally finite PMQ. If for all \( a \in \mathcal{Q} \subset \hat{\mathcal{Q}} \) the space \( \text{Hur}^\Delta(Q)(a) \) is a topological manifold of some dimension, then \( Q \) is Poincaré.

**Proof** By Theorem 9.1 the simplicial Hurwitz space \( \text{Hur}^\Delta(Q) \) is homeomorphic to \( \text{Hur}(\hat{\mathcal{R}}; Q_+) \), so it suffices to prove that for all \( b \in \hat{\mathcal{Q}} \) the space \( \text{Hur}(\hat{\mathcal{R}}; Q_+)_{b} \) is a topological manifold. In the following we fix \( b \in \hat{\mathcal{Q}} \).

Let \( \epsilon \in \text{Hur}(\hat{\mathcal{R}}; \hat{\mathcal{Q}}_+) \), use Notations 3.6 and 2.6 and let \( U \) be an adapted covering of \( P \). By Proposition 5.1 the normal neighbourhood \( \Pi(c, U) \subset \text{Hur}(\hat{\mathcal{R}}; \mathcal{Q}) \) is homeomorphic to a product of normal neighbourhoods \( \prod_{i=1}^{k} \Pi(c'_i, U_i) \) for suitable configurations \( c'_i \in \text{Hur}(\hat{\mathcal{R}}; \mathcal{Q}) \).
the argument of the proof of Proposition 5.1 shows in fact that $c'_i$ is a configuration in $\text{Hur}(\hat{R}; Q_+)$ supported on the single point $z_i$, and that there is a restricted homeomorphism

$$\cup(c, U) \cap \text{Hur}(\hat{R}; Q_+) \cong \prod_{i=1}^{k} \left( \cup(c'_i, U) \cap \text{Hur}(\hat{R}; Q_+)_b \right),$$

with $b_i = \omega(c'_i)$. A priori $b_i \in \hat{Q}$; since $c'_i$ is supported on one point we have $b_i \in Q$.

The hypothesis on $Q$ ensures that each space $\text{Hur}(\hat{R}; Q_+)_{b_i}$ is a topological manifold; thus also each open subset $\cup(c'_i, U) \cap \text{Hur}(\hat{R}; Q_+)_b$ is a topological manifold, and therefore $\cup(c, U) \cap \text{Hur}(\hat{R}; Q_+)_b$ is a topological manifold. This shows that each configuration in $\text{Hur}(\hat{R}; Q_+)_b$ has a neighbourhood which is a topological manifold, and thus the space $\text{Hur}(\hat{R}; Q_+)_b$, which is Hausdorff, is a topological manifold.

The proof of Theorem 9.3 can be generalised to homology manifolds as follows.

**Definition 9.4** Let $R$ be a commutative ring. A locally finite PMQ $Q$ is $R$-Poincaré if for all $a \in \hat{Q}$ the space $\text{Hur}(\hat{R}; Q)$ is a $R$-homology manifold of some dimension, i.e. for all $c \in \text{Hur}(\hat{R}; Q_+)$ the local homology

$$\hat{H}_s \left( \text{Hur}(\hat{R}; Q_+) \times \text{Hur}(\hat{R}; Q_+) -\{c\} \times \hat{R} \right)$$

is isomorphic to $R$ in a single degree, and vanishes in all other degrees.

**Lemma 9.5** Let $Q$ be a locally finite PMQ, let $a \in Q_+$ and let $z_0 \in \hat{R}$; then the space $\text{Hur}(\hat{R}; Q_+)_a$ is homeomorphic to the cone over the space

$$\partial \text{Hur}(\hat{R}; Q_+)_a := \text{Hur}(\hat{R}; Q_+)_a \setminus \text{Hur}(\hat{R}; Q_+)_a,$$

with vertex the unique configuration $c_{z_0,a} \in \text{Hur}(\hat{R}; Q_+)_a$ supported on $z_0$.

**Proof** Let $\hat{Q}$ be the completion of $Q$, and note that $\text{Hur}(\hat{R}; Q_+)_a$ is homeomorphic to $\text{Hur}(\hat{R}; Q_+)_a$. Without loss of generality, we may assume that $Q$ is already complete. Note that the space $\partial \text{Hur}(\hat{R}; Q_+)_a$ is a closed subspace of $\text{Hur}(\hat{R}; Q_+)_a$, containing all configurations $c \in \text{Hur}(\hat{R}; Q_+)_a$ whose support intersects $\partial \mathcal{R}$.

Fix a map $\mathcal{H}^{\mathcal{Q}_a} : \mathbb{C} \times [0,1] \to \mathcal{C}$ satisfying the following properties:

- $\mathcal{H}^{\mathcal{Q}_a}(z, s) = sz_0 + (1-s)z$ for all $z \in \mathcal{R}$ and $0 \leq s \leq 1$;
- $\mathcal{H}^{\mathcal{Q}_a}(-s, s)$ is a lax morphism of nice couples $(\mathcal{R}, \emptyset) \to (\mathcal{R}, \emptyset)$ for all $0 \leq s \leq 1$.

By Proposition 4.10 we obtain a continuous map $\mathcal{H}^{\mathcal{Q}_a}_\ast : \text{Hur}(\hat{R}; Q_+) \times [0,1] \to \text{Hur}(\hat{R}; Q_+)_a$, that we can restrict to a map

$$\partial \mathcal{H}^{\mathcal{Q}_a}_\ast : \partial \text{Hur}(\hat{R}; Q_+)_a \times [0,1] \to \text{Hur}(\hat{R}; Q_+)_a.$$

The map $\partial \mathcal{H}^{\mathcal{Q}_a}_\ast$ sends the subspace $\partial \text{Hur}(\hat{R}; Q_+)_a \times \{1\}$ constantly to the configuration $c_{z_0,a}$. The quotient map

$$\partial \mathcal{H}^{\mathcal{Q}_a}_\ast : \partial \text{Hur}(\hat{R}; Q_+)_a \times [0,1] / \partial \text{Hur}(\hat{R}; Q_+)_a \times \{1\} \to \text{Hur}(\hat{R}; Q_+)_a$$

is a continuous bijection between compact Hausdorff spaces, hence it is a homeomorphism.
In particular for all \( z_0 \in \hat{R} \) we have an isomorphism of homology groups, where \( R \)-coefficients are understood:

\[
\tilde{H}_s \left( \text{Hur}(\hat{R}; Q_+)_a, \text{Hur}(\hat{R}; Q_+)_a \setminus \{c_{z_0,a}\} \right) \cong \tilde{H}_s (|\text{Arr}(Q)(a)|, |\text{NAdm}(Q)(a)|).
\]

The argument used in the proof of Theorem 9.3, together with the Künneth isomorphism, implies directly the following theorem.

**Theorem 9.6** Let \( R \) be a commutative ring and let \( Q \) be a locally finite PMQ. Suppose that for all \( a \in Q \) the relative homology groups

\[
\tilde{H}_s (|\text{Arr}(Q)(a)|, |\text{NAdm}(Q)(a)|); R
\]

are supported in a single degree, with corresponding homology group equal to \( R \). Then \( Q \) is \( R \)-Poincaré.

Theorem 9.6 is the non-trivial arrow of an “if and only if” statement: if \( Q \) is \( R \)-Poincaré, then in particular for all \( a \in Q \) the space \( \text{Hur}^\Delta(Q)(a) \cong \text{Hur}(\hat{R}; Q_+)_a \) is a \( R \)-homology manifold; since this space is contractible (see Proposition 6.4), by Poincaré–Lefschetz duality the relative homology groups

\[
\tilde{H}_s (|\text{Arr}(Q)(a)|, |\text{NAdm}(Q)(a)|); R
\]

are supported in one degree, namely the \( R \)-homology dimension of \( \text{Hur}^\Delta(Q)(a) \), with corresponding group isomorphic to \( H^0(\text{Hur}^\Delta(Q)(a); R) \cong R \).

The proofs of [3, Proposition 6.23] and [3, Proposition 6.24] generalise to give the following Proposition.

**Proposition 9.7** Let \( R \) be a commutative ring and let \( Q \) be a \( R \)-Poincaré PMQ. Then \( Q \) is coconnected and admits an intrinsic norm \( h: Q \rightarrow \mathbb{N} \).

If we denote by \( h: \hat{Q} \rightarrow \mathbb{N} \) also the extension of the intrinsic norm to the completion \( \hat{Q} \) of \( Q \), then for all \( a \in \hat{Q} \) the space \( \text{Hur}(\hat{R}; Q_+)_a \) is a \( R \)-homology manifold of dimension \( 2h(a) \).

**Proof** Recall that for a locally finite PMQ \( Q \) the candidate for the intrinsic norm \( h: Q \rightarrow \mathbb{N} \) is the function of sets associating with \( a \in Q \) the maximum \( r \geq 0 \) for which there exist a decomposition \( a = a_1 \ldots a_r \) with \( a_i \in Q_+ \).

If \( a \in Q_+ \) is irreducible, then \( \text{Hur}(\hat{R}; Q_+)_a \) is homeomorphic to \( \hat{R} \), which is a \( R \)-homology manifold of dimension \( 2 = 2h(a) \); more generally, if \( a = a_1 \ldots a_r \) is a decomposition witnessing the equality \( h(a) = r \), then we can fix a configuration \( c = (P, \psi) \in \text{Hur}(\hat{R}; Q_+)_a \) supported on a subset \( P \subset \hat{R} \) of precisely \( r \) points. By Proposition 5.1 a normal neighbourhood of \( c \) is homeomorphic to an open subset of \( (\hat{R})^r \); it follows that the \( R \)-homology dimension of \( \text{Hur}(\hat{R}; Q_+)_a \), computed around \( c \), is equal to \( 2h(a) \).

The same argument, applied to any decomposition \( a = bc \) in \( Q \), shows that the \( R \)-homology dimension of \( \text{Hur}(\hat{R}; Q_+)_a \) is equal to the sum of the \( R \)-homology dimensions of \( \text{Hur}(\hat{R}; Q_+)_b \) and \( \text{Hur}(\hat{R}; Q_+)_c \): in fact we can find an open set of \( \text{Hur}(\hat{R}; Q_+)_a \) homeomorphic to the product of two open sets of \( \text{Hur}(\hat{R}; Q_+)_b \) and \( \text{Hur}(\hat{R}; Q_+)_c \) respectively. It follows that \( h(a) = h(b) + h(c) \), i.e. \( h \) is an intrinsic norm. The \( R \)-homology dimension of \( \text{Hur}(\hat{R}; Q_+)_a \) can be computed to be \( h(a) \) for a generic \( a \in \hat{Q} \) by the same argument, after fixing a decomposition of \( a \) as product of elements of \( Q \).

This shows that \( Q \) admits an intrinsic norm, and in particular it is maximally decomposable. To prove that \( Q \) is coconnected, let \( Q_{\leq 1} \subset Q \) be the sub-PMQ containing elements of norm \( \leq 1 \); the inclusion of augmented PMQs \( Q_{\leq 1} \subset Q \) induces, for all \( a \in \hat{Q} \) a surjective,
bismimplicial map $|\text{Arr}(Q_{\leq 1})(a)| \cong |\text{Arr}(Q)(a)|$, which is a bijection when restricted to bismimplices of dimension $2\ h(a)$ and $2\ h(a) - 1$ (see the proof of [3, Proposition 6.23]). Here we write $|\text{Arr}(Q_{\leq 1})(a)|$ for the disjoint union $\bigsqcup_{a'} |\text{Arr}(Q_{\leq 1})(a')|$, where $a'$ ranges among all elements of $Q_{\leq 1}$ which are sent to $a \in Q$ along the (surjective, but a priori not bijective) map $\overline{Q_{\leq 1}} \to \hat{Q}$.

It follows that the induced map

$$H_{2h(a)} \left( |\text{Arr}(Q_{\leq 1})(a)|, |\text{NAdm}(Q_{\leq 1})(a)| \right) \to H_{2h(a)} \left( |\text{Arr}(Q)(a)|, |\text{NAdm}(Q)(a)| \right),$$

with $R$-coefficients for homology understood, is an isomorphism of $R$-modules. The rank of the second $R$-module is 1, because $H_{2h(a)}( |\text{Arr}(Q)(a)|, |\text{NAdm}(Q)(a)|; R) \cong H_0(\text{Hur}^\Delta(Q)(a))$, and the space $\text{Hur}^\Delta(Q)(a)$ is contractible. Similarly the rank of the first $R$-module is the number of connected components of $\text{Hur}^\Delta(Q_{\leq 1})(a)$. It follows that there is exactly one element $a' \in Q_{\leq 1}$ which is mapped to $a$ along the map $\overline{Q_{\leq 1}} \to \hat{Q}$, and that $\text{Hur}^\Delta(Q_{\leq 1})(a)$ is connected. This shows that $Q$ is coconnected. □

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**Appendix A: Deferred proofs**

**A.1. Proof of Proposition 2.11**

Let $g \in \Omega^{\text{ext}}(P)$, and assume first that $g = [\gamma]$ is represented by a simple loop $\gamma$ in $\mathbb{C} \setminus P$. The loop $\gamma$ is freely isotopic to a simple closed curve in $\mathbb{C} \setminus \mathbb{C}$. In particular $\gamma$ bounds a disc $D$ in $\mathbb{C}$ which intersects $P$ only in points of $P \setminus \mathbb{C}$; without loss of generality, assume that $D \cap P$ consists of the points $z_1, \ldots, z_r$ for some $1 \leq r \leq l$.

We can then find an admissible generating set $f_1, \ldots, f_k$ of $\mathcal{G}(P)$ such that $g = f_1 \cdots f_r \in \Omega^{\text{ext}}(P)$ (see Definition 2.8): for this it suffices to choose the arcs $\zeta_1, \ldots, \zeta_r$ inside $D$ in a convenient way. This gives a decomposition $(f_1, \ldots, f_r)$ of $g$ with respect to $\Omega(P)$ as required.

If $g \in \Omega^{\text{ext}}(P)$ is not represented by a simple loop, we can still find a conjugate $g'$ of $g$ in $\Omega^{\text{ext}}(P) \subset \mathcal{G}(P)$, with $g' = [\gamma']$ represented by a simple loop $\gamma'$. By the previous argument we can decompose $g' = g'_1 \cdots g'_r$, with all $g'_i \in \Omega(P)$; we can then conjugate the previous decomposition in $\mathcal{G}(P)$ to obtain a decomposition $g = g_1 \cdots g_r$, with all $g_i$ still lying in $\Omega(P)$. 

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A.2. Proof of Proposition 2.12

Let \( g \in \Omega^{\text{ext}}(P) \) and assume first that \( g = \gamma \) is represented by a simple loop \( \gamma \) in \( \mathbb{C} \setminus P \). Let \( g = g_1 \cdots g_\rho \) be a decomposition of \( g \) in elements \( g_i \in \Omega^{\text{ext}}(P) \). Each \( g_i \) can be further decomposed, by Proposition 2.11, as \( g_{i,1} \cdots g_{i,r_i} \), with \( g_{i,j} \in \Omega(P) \); therefore we obtain a decomposition

\[
g = g_{1,1} \cdots g_{1,r_1} g_{2,1} \cdots g_{2,r_2} \cdots \cdots g_{\rho,1} \cdots g_{\rho,r_\rho}\]

of \( g \) with respect to \( \Omega(P) \). Our aim to show that, for all \( 1 \leq i < j \leq \rho \), the product \( g_i \cdots g_j \) belongs to \( \Omega^{\text{ext}}(P) \). It suffices to prove the same statement for the second, finer decomposition involving the elements \( g_{i,j} \). Hence, from now on, we assume that the elements \( g_1, \ldots, g_\rho \) already belong to \( \Omega(P) \). According to [3, Definition 3.5], \( (g_1, \ldots, g_\rho) \) is then a decomposition of \( g \) with respect to \( \Omega(P) \).

By the same argument used in the proof of Proposition 2.11, we can find an admissible generating set \( f_1, \ldots, f_r \) of \( \mathfrak{G}(P) \) such that, for some \( 1 \leq r \leq l \), we have \( g = f_1 \cdots f_r \), and such that \( f_1, \ldots, f_r \) are contained in the subgroup \( \pi_1(D \setminus P, *) \cong \mathbb{F}^r \) of \( \mathfrak{G}(P) \), where \( D \) is the disc bounded by \( \gamma \).

We note that \((f_1, \ldots, f_r)\) is also a decomposition of \( g \) with respect to \( \Omega(P) \), and a simple argument involving the projection onto the abelianisation of \( \mathfrak{G}(P) \) shows that \( \rho = r \) (see the remark after [3, Definition 3.5]). The decompositions \((f_1, \ldots, f_r)\) and \((g_1, \ldots, g_r)\) are connected by a sequence of standard moves (see [3, Definition 3.6, Proposition 3.7]).

A consequence of the previous argument is that \( g_1, \ldots, g_r \in \mathfrak{G}(P) \) can be generated using the elements \( f_1, \ldots, f_r \), and therefore \( g_1, \ldots, g_r \) also lie in the subgroup \( \pi_1(D \setminus P, *) \subseteq \mathfrak{G}(P) \).

In analogy with Definition 2.8, we say that \( f_1, \ldots, f_r \) is an admissible generating set of \( \pi_1(D \setminus P, *) \), meaning that each \( f_i \) is represented by a simple loop that spins around one of the \( r \) points of \( D \cap \mathbb{C} \setminus P \), and these loops only intersect at \( * \).

It is now a classical fact that standard moves on admissible generating sets of \( \pi_1(D \setminus P, *) \) can be implemented by homeomorphisms of \( D \). More precisely, if \( \tilde{f}_1, \ldots, \tilde{f}_r \) is an admissible generating set of \( \pi_1(D \setminus P, *) \) and the sequence \((\tilde{g}_1, \ldots, \tilde{g}_r)\) of elements of \( \pi_1(D \setminus P, *) \) is obtained from the sequence \((\tilde{f}_1, \ldots, \tilde{f}_r)\) by a standard move, then there is a homeomorphism \( \xi : D \to D \) such that

- \( \xi \) fixes \( \gamma = \partial D \) pointwise: in particular \( \xi(*) = * \);
- \( \xi \) fixes \( D \cap \mathbb{C} \) as a set: in particular, \( \xi \) restricts to a homeomorphism of \( D \) \( \setminus P \);
- the map \( \xi_* : \pi_1(D \setminus P, *) \to \pi_1(D \setminus P, *) \) sends \( \tilde{f}_i \mapsto \tilde{g}_i \) for all \( 1 \leq i \leq r \).

By applying this argument several times, we obtain that \( g_1, \ldots, g_r \) is also an admissible generating set of \( \pi_1(D \setminus P, *) \), and the fact that the product \( g = g_1 \cdots g_r \) is represented by a simple loop implies that the elements \( g_1, \ldots, g_r \) are ordered in a standard way, so that for all \( 1 \leq i < j \leq r \), the product \( g_i \cdots g_j \) is represented by a simple loop in \( D \setminus P \subset \mathbb{C} \setminus P \); thus \( g_i \cdots g_j \in \Omega^{\text{ext}}(P) \).

The case in which \( g \) is not represented by a simple loop \( \gamma \) is treated in the same way as in the proof of Proposition 2.11: we can find a conjugate \( g' \in \mathfrak{G}(P) \) represented by a simple loop \( \gamma' \); in particular \( g' \in \Omega^{\text{ext}}(P) \); we conjugate the factorisation \( g = g_1 \cdots g_\rho \) to obtain a factorisation \( g' = g'_1 \cdots g'_\rho \); by the previous argument each product \( g'_i \cdots g'_j \) lies in \( \Omega^{\text{ext}}(P) \), and therefore also its conjugate \( g_i \cdots g_j \) lies in \( \Omega^{\text{ext}}(P) \).
A.3. Proof of Proposition 2.14

Let \( g = g_1 \ldots g_\rho \) be a decomposition of \( g \in \mathcal{Q}^{\text{ext}}(P)_\psi \) with \( g_i \in \mathcal{Q}^{\text{ext}}(P)_\psi \) for all \( 1 \leq i \leq \rho \). As in the proof of Proposition 2.12, we replace each \( g_i \) by a decomposition \( g_{i_1} \ldots g_{i_r} \), with \( g_{i_j} \in \mathcal{Q}(P) \); thus we obtain a decomposition of \( g \) with respect to \( \mathcal{Q}(P) \)

\[
g = g_{1_1} \ldots g_{1_r} g_{2_1} \ldots g_{2_r} \ldots \ldots g_{\rho_1} \ldots g_{\rho_r}.
\]

Since \( g \in \mathcal{Q}^{\text{ext}}(P)_\psi \), the following product is defined in \( \mathcal{Q} \):

\[
\psi^{\text{ext}}(g) = \psi(g_{1_1}) \ldots \psi(g_{1_r}) \psi(g_{2_1}) \ldots \psi(g_{2_r}) \ldots \ldots \psi(g_{\rho_1}) \ldots \psi(g_{\rho_r});
\]

In particular for all \( 1 \leq i < j \leq \rho \), the sub-product \( \psi(g_{i_1}) \ldots \psi(g_{j_r}) \) is defined in \( \mathcal{Q} \). Together with Proposition 2.12, this shows that \( g \) lies in \( \mathcal{Q}^{\text{ext}}(P)_\psi \), hence \( \mathcal{Q}^{\text{ext}}(P)_\psi \subseteq \mathcal{Q}(P) \) satisfies the hypotheses of [3, Definition 2.8].

The same argument shows also that \( \psi^{\text{ext}}(g) = \psi^{\text{ext}}(g_{1_1}) \ldots \psi^{\text{ext}}(g_{r_r}) \) in \( \mathcal{Q} \), hence \( \psi^{\text{ext}} \) is a map of partial monoids. It is also evident that \( \psi^{\text{ext}} \) restricts to \( \psi \) on \( \mathcal{Q}(P) \). To see that \( \psi^{\text{ext}} \) also preserves conjugation, let \( g, g' \in \mathcal{Q}^{\text{ext}}(P)_\psi \) and choose decompositions \( (g_1, \ldots, g_r) \) and \( (g'_{1_1}, \ldots, g'_{r_r}) \) of \( g \) and \( g' \) respectively with respect to \( \mathcal{Q}(P) \). We have a chain of equalities

\[
\psi^{\text{ext}}(t^0) = \psi^{\text{ext}}(t_{1_1}^0 \ldots t_{r_r}^0) = \psi^{\text{ext}}(t_{1_1}^0) \ldots \psi^{\text{ext}}(t_{r_r}^0) = \psi(t_{1_1}^0) \ldots \psi(t_{r_r}^0)
\]

Thus \( \psi^{\text{ext}} : \mathcal{Q}^{\text{ext}}(P)_\psi \rightarrow \mathcal{Q} \) is a map of PMQs, restricting to the map \( \psi : \mathcal{Q}(P) \rightarrow \mathcal{Q} \). The fact that \( \psi^{\text{ext}} \) is the unique map of PMQs with these properties is a direct consequence of Proposition 2.11.

A.4. Proof of Lemma 4.5

Let \( z' \in P' \setminus \gamma' \) and let \( \gamma' \) be a based loop in \( C \setminus P' \) which is freely homotopic to a simple closed curve \( \beta' \subset C \setminus P' \) spinning clockwise around \( z' \); in particular \( \beta' \) bounds a closed disc \( D' \subset C \setminus P' \), with \( D' \cap P' = \{z'\} \).

We have that \( D = \xi^{-1}(D') \) is also a disc in \( C \setminus P \), and by property (5) in Definition 4.2 and by definition of \( P' := \xi(P) \), there is a unique \( z \in P \) with \( \xi(z) = z' \). We consider \( \beta = \partial D \) as a simple closed curve in \( C \setminus P \) spinning clockwise around \( z \); then \( \xi \) restricts to a homotopy equivalence \( \beta \rightarrow \beta' \), since:

- both spaces are homotopy equivalent to \( S^1 \), hence it suffices to prove that \( \xi \) induces a cohomology equivalence;
- the inclusions \( \beta \subset D \setminus z \) and \( \beta' \subset D' \setminus z' \) are homotopy equivalences, in particular cohomology equivalences;
the map \( \xi : D \setminus z \to D' \setminus z' \) is a cohomology equivalence: this can be seen by comparing the cohomology long exact sequences of the couples \((D, D \setminus z)\) and \((D', D' \setminus z')\), using in particular that the map \( \xi^* : H^2(D', D' \setminus z') \to H^2(D, D \setminus z) \) can be rewritten as \( \xi^* : H^2_\mathcal{C}(\mathcal{C}) \to H^2_\mathcal{C}(\mathcal{C}) \), and is thus an isomorphism.

Moreover property (2) in Definition 4.2 implies that \( \xi : \beta \to \beta' \) is orientation-preserving, if both curves are oriented clockwise.

This implies that the conjugacy class represented by \( \beta' \) is mapped along \( \xi^* \) inside the conjugacy class represented by \( \beta \), which is contained in \( \Omega e(P) \).

A.5. Proof of Lemma 4.9

Let \( \gamma' \subset \mathbb{C} \setminus P' \) be a based loop homotopic to a simple closed curve \( \beta' \), with \( \beta' \) contained in \( \mathbb{C} \setminus \gamma' \) and \( \beta' \) oriented clockwise, such that \( \beta' \) bounds a disc \( D' \subset \mathbb{C} \setminus \gamma' \).

Let \( D = \xi^{-1}(D') \), which is a topological disc contained in \( \mathbb{C} \setminus \gamma' \), and let \( \beta = \partial D \). Let \( K' \subset D' \) be a smaller, closed disc containing \( P' \cap D' \), and denote \( K = \xi^{-1}(K') \).

Then \( \xi : \beta \to \beta' \) is a homotopy equivalence, since:

- both spaces are homotopy equivalent to \( S^1 \), hence it suffices to prove that \( \xi \) induces a cohomology equivalence;
- the inclusions \( \beta \subset D \setminus K \) and \( \beta' \subset D' \setminus K' \) are homotopy equivalences, in particular cohomology equivalences;
- the map \( \xi : D \setminus K \to D' \setminus K' \) is a cohomology equivalence: this can be seen by comparing the cohomology long exact sequences of the couples \((D, D \setminus K)\) and \((D', D' \setminus K')\), using in particular that the map \( \xi^* : H^2(D', D' \setminus K') \to H^2(D, D \setminus K) \) can be rewritten as \( \xi^* : H^2_\mathcal{C}(\mathcal{C}) \to H^2_\mathcal{C}(\mathcal{C}) \), and is thus an isomorphism.

Moreover property (2) in Definition 4.2 implies that \( \xi : \beta \to \beta' \) is orientation-preserving, if both curves are oriented clockwise.

It follows that \( \xi^* \) maps the conjugacy class of \( \beta' \) inside the conjugacy class of \( \beta \), which is contained in \( \Omega e^{\text{ex}}(P) \).

A.6. Proof of Lemma 5.2

Let \( \epsilon = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G}(\mathcal{Q})) \), use Notation 2.4 and let \( f_1, \ldots, f_k \) be an admissible generating set for \( \mathcal{G}(P) \). Since we are dealing with the nice couple \((\mathcal{X}, \emptyset)\), whose second space is empty, we have that \( f_1, \ldots, f_k \in \Omega(P) \). By Definition 3.5 we have \( \varphi(f_i) = \eta_{\mathcal{Q}}(\psi(f_i)) \in \mathcal{G}(\mathcal{Q}) \); since \( f_1, \ldots, f_k \) exhibit \( \mathcal{G}(P) \) as a free group, we have that \( \varphi : \mathcal{G}(P) \to \mathcal{G}(\mathcal{Q}) \) is uniquely determined by \( \psi \). On the other hand, by [3, Theorem 3.3], given any finite subset \( P \subset \mathcal{X} \), and a map of PMQs \( \psi : \Omega(P) \to \mathcal{Q} \), one can use the assignment \( f_i \mapsto \eta_{\mathcal{Q}}(\psi(f_i)) \) to define a group homomorphism \( \varphi : \mathcal{G}(P) \to \mathcal{G}(\mathcal{Q}) \) making \( (\psi, \varphi) : (\Omega(P), \mathcal{G}(P)) \to (\mathcal{Q}, \mathcal{G}(\mathcal{Q})) \) into a map of PMQ-group pairs.

Let \( c' = (P', \psi', \varphi') \) be the image of \( c \) along \( (\text{Id}_{\mathcal{Q}}, \mathcal{G}(\mathcal{Q}))_* \); then we have \( P' = P \) and \( \psi' = \psi \); from the previous discussion it follows that \( c \) can be reconstructed from \( c' \), and this proves injectivity of \( (\text{Id}_{\mathcal{Q}}, \mathcal{G}(\mathcal{Q}))_* \).

Vice versa, let \( c' = (P, \psi, \varphi) \) be any configuration in \( \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G}) \); then the previous discussion shows that one can construct a configuration \( c \in \text{Hur}(\mathcal{C}; \mathcal{Q}, \mathcal{G}(\mathcal{Q})) \) which is sent to \( c' \) along \( (\text{Id}_{\mathcal{Q}}, \mathcal{G}(\mathcal{Q}))_* \): it suffices to take \( c = (P, \psi, \varphi) \), with \( \varphi \) defined as above by setting \( f_i \mapsto \eta_{\mathcal{Q}}(\psi(f_i)) \); this proves surjectivity of \( (\text{Id}_{\mathcal{Q}}, \mathcal{G}(\mathcal{Q}))_* \).
To conclude, note that for all adapted coverings $U$ of $P$, the map $(\text{Id}_Q, \mathcal{G}(e))_*$ restricts to a bijection from $\mathcal{U}(e; U) \subset \text{Hur}(\mathcal{C}; Q, \mathcal{G}(Q))$ to $\mathcal{U}(e'; U) \subset \text{Hur}(\mathcal{C}; Q, G)$, where again we let $e'$ be the image of $e$ along $(\text{Id}_Q, \mathcal{G}(e))_*$. This shows that $(\text{Id}_Q, \mathcal{G}(e))_*$ is a homeomorphism.

### A.7. Proof of Lemma 5.4

The proof is analogous to the one of Lemma 5.2. Let $c = (P, \psi, \varphi) \in \text{Hur}(\mathcal{C}; G, G)$, use Notation 2.4 and let $f_1, \ldots, f_k$ be an admissible generating set for $\mathcal{G}(P)$. Since we are dealing with the PMQ-group pair $(G, G)$, the composition $\Omega_G(P) \subset \mathcal{G}(P) \xrightarrow{\psi} G$ equals $\psi: \Omega_G(P) \rightarrow G$. In particular $\psi$ can be recovered from $\varphi$.

On the other hand, by [3, Theorem 3.3], given any finite subset $P \subset \mathcal{X}$ and a map of groups $\varphi: \mathcal{G}(P) \rightarrow G$, one can use the assignment $\psi: f_i \mapsto \varphi(f_i)$ for $1 \leq i \leq l$ (using Notation 2.4) to define a map of PMQs $\psi: \Omega_G(P) \rightarrow G$ making $(\psi, \varphi): (\Omega_G(P), \mathcal{G}(P)) \rightarrow (G, G)$ into a map of PMQ-group pairs.

Let $c' = (P', \psi', \varphi')$ be the image of $c$ along $(\text{Id}_C)_*$; then we have $P' = P$ and $\varphi' = \varphi$; from the previous discussion it follows that $c$ can be reconstructed from $c'$, and this proves injectivity of $(\text{Id}_C)_*$.

Vice-versa, let $c' = (P, \psi, \varphi')$ be any configuration in $\text{Hur}(\mathcal{X}, \mathcal{X}; G, G)$; then the previous discussion shows that one can construct a configuration $c \in \text{Hur}(\mathcal{C}; G, G)$ mapping to $c'$ along $(\text{Id}_C)_*$; it suffices to take $c = (P, \psi, \varphi)$, with $\psi$ defined as above by setting $\psi: f_i \mapsto \varphi(f_i)$; this proves surjectivity of $(\text{Id}_C)_*$.

To conclude, note that if $U$ is an adapted covering of $P$ with respect to $\mathcal{C}$, then $U$ is also adapted with respect to $\mathcal{X}$, $\mathcal{X}$, and the map $(\text{Id}_C)_*$ restricts to a bijection from $\mathcal{U}(c; U) \subset \text{Hur}(\mathcal{C}; G, G)$ to $\mathcal{U}(c'; U) \subset \text{Hur}(\mathcal{X}, \mathcal{X}; G, G)$, where again we let $e'$ be the image of $e$ along $(\text{Id}_C)_*$. This proves that $(\text{Id}_C)_*$ is a homeomorphism.

### A.8. Proof of Proposition 6.13

We focus on the left-based case. Let $c \in \text{Hur}(\mathcal{C}; Q, G)_{\mathcal{C}1}$, use Notations 3.6 and 2.6 and let $U$ be an adapted covering of $P$. Denote by $U^1$ the component of $U$ containing $\mathcal{C}^1$; possibly up to shrinking $U^1$, we can assume that the simple closed curve $\partial U^1$ is cut by $S_{\mathcal{R}(\mathcal{C}1), \mathcal{R}(\mathcal{C}2)}$ in two arcs. We decompose $C$ as the union of two subspaces: the first subspace is $T_1$, which is defined as the closure in $C$ of $S_{-\infty, \mathcal{R}(\mathcal{C}1)} \cup U^1$; the other subspace is $T_2 = S_{\mathcal{R}(\mathcal{C}2), \infty} \setminus U^1$. The first subspace contains $P_1 := \{z_1\}$ in its interior, the second subspace contains $P_2 := P \setminus \{z_1\}$ in its interior. The two subspaces $T_1$ and $T_2$ intersect in a contractible space containing $\ast$.

Using the theorem of Seifert and van Kampen we can write $\mathcal{G}(P)$ as the free product $\pi_1(T_1 \setminus P_1, \ast) \ast \pi_1(T_2 \setminus P_2, \ast)$; the first factor is freely generated by $f_1^1$, the second factor is freely generated by the other generators $f_i$ in a left-based admissible generating set. The map $\varphi'$ in Definition 6.12 can then be equivalently defined by setting $\varphi'(f_1^1) = g \cdot (f_1^1)$, and by imposing that $\varphi'$ and $\varphi$ agree on the second factor.

Moreover, consider the nice couple $\mathcal{C}_2 := (\mathcal{X} \setminus T_2, Y \setminus T_2)$: then $P_2$ is contained in $\mathcal{X} \setminus T_2$. The composition

$$\Omega_{\mathcal{X}}(P_2) \xrightarrow{\subset} \mathcal{G}(P_2) \xrightarrow{\pi_1(T_2 \setminus P_2, \ast)} \mathcal{G}(P)$$

has image in $\Omega_{\mathcal{X}}(P)$ and identifies $\Omega_{\mathcal{X}}(P)$ with the sub-PMQ of $\Omega_{\mathcal{X}}(P)$ containing homotopy classes which can be represented by a simple loop in $T_2 \setminus P_2$ spinning clockwise.
around one of the points $z_1, \ldots, z_l$. The map $\psi': \Omega_\mathcal{E}(P') \to Q$ from Definition 6.12 can be characterised by the following two properties:

- $\psi$ and $\psi'$ have the same restriction on $\Omega_\mathcal{E}_2(P)$, regarded as a subset of $\Omega_\mathcal{E}(P)$ as explained above.
- $(\psi', \varphi')$ is a map of PMQ-group pairs $(\Omega_\mathcal{E}(P), \mathcal{G}(P)) \to (Q, G)$.

The fact that this is a characterisation (i.e. existence and uniqueness of $\psi'$ with these properties) is shown using a choice of a left-based admissible generating set for $\mathcal{G}(P)$ and using [3, Theorem 3.3]; but the characterising properties of $\varphi'$ and $\psi'$ are now stated without reference to a left-based admissible generating set.

The fact that the collection of all maps $g \cdot -$ gives an action of $G$ on the set $\text{Hur}(\mathcal{C}; Q, G)_{z^t}$ follows directly from the formulas in Definition 6.12. To prove continuity of $g \cdot -$, note that for all adapted coverings $U$ of $P$ the map $g \cdot -$ establishes a bijection between the open subspaces $\mathcal{U}(c, U)_{z^t}$ and $\mathcal{U}(g \cdot c, U)_{z^t}$ of $\text{Hur}(\mathcal{C}; Q, G)_{z^t}$. In particular $g \cdot -$ is a homeomorphism of $\text{Hur}(\mathcal{C}; Q, G)$ with inverse $g^{-1} \cdot -$.

The right-based case is analogous; the main difference is that, in the first part, one considers the component $U^r$ of $U$ covering $z^t$, and decomposes $\mathcal{C}$ as the union of $T_1 = S_{-\infty, \infty}(z^t) \setminus U^r$ and $T_2$ being the closure in $\mathcal{C}$ of $S_{[0, \infty)} \cup U^r$.

A.9. Proof of Proposition 7.8

Fix $(c, t) \in \text{Hur}(\mathcal{C}; Q, G) \times [0, 1]$, denote $c' = \varepsilon_n(c, t)$ and $c'' = \rho(c)$, and use Notation 3.6. Without loss of generality assume that $z_1, \ldots, z_r \in P \setminus \mathcal{Y}$ are precisely the inert points of $c$, for some $0 \leq r \leq l$. Then $P'' = P \setminus \{z_1, \ldots, z_r\}$ and $P' = P'' \cup \mathcal{E}(P, t)$.

Let $U'$ be an adapted covering of $P'$. Our aim is to find a neighbourhood of $(c, t) \in \text{Hur}(\mathcal{C}; Q, G) \times [0, 1]$ which is mapped by $\varepsilon_n$ inside $\mathcal{U}(c', U')$. Let $U''(P, t) \subset U'$ denote the restriction of $U'$ to $\mathcal{E}(P, t) \subset P'$, i.e. the sequence of components of $U'$ containing a point in $\mathcal{E}(P, t)$. Then by continuity of $\varepsilon$ we can find an adapted covering $\mathcal{U}$ of $P$ and a neighbourhood $V$ of $t$ such that $\varepsilon$ maps the entire product neighbourhood $\mathcal{U}(P, U) \times V$ inside $\mathcal{U}(\mathcal{E}(P, t), U''(P, t)) \subset \text{Ran}(\varepsilon_n)$.

Use Notation 2.6: up to shrinking the components of $U$, we may assume that $U_i \subset U'_i$ whenever $z_i$ belongs to $P'' \subset P \cap P'$, that is $U_{i+1} \cup \cdots \cup U_k \subset U'_i$. We claim that $\mathcal{U}(c, U) \times V$ is mapped by $\varepsilon_n$ inside $\mathcal{U}(c', U')$; the rest of the proof is devoted to this claim.

We fix $\tilde{\epsilon} = (\tilde{P}, \tilde{\psi}, \tilde{\varphi}) \in \mathcal{U}(c, U)$ and $\tilde{t} \in V$, and let $\tilde{c'} = (\tilde{P}', \tilde{\psi}', \tilde{\varphi}') = \varepsilon_n(\tilde{c}, \tilde{t})$. First, we prove that $\tilde{P}' \subset U'$. We can partition $\tilde{P}$ into subsets $\tilde{P}_1, \ldots, \tilde{P}_k$, with $\tilde{P}_i \subset U_i$. Note that for all $1 \leq i \leq r$ and for all $\tilde{\epsilon} \in \tilde{P}_i$, the point $\tilde{\epsilon}$ is inert for $\tilde{c}$; indeed $\tilde{\epsilon} \in U_i \subset \mathcal{C} \setminus \mathcal{Y}$ because $\mathcal{U}$ is an adapted covering of $P$, hence $\tilde{\epsilon} \in \mathcal{Y} \setminus \mathcal{Y}$; moreover $\tilde{\psi}$ sends each element of $\Omega(\tilde{P}, \tilde{\epsilon})$ to a factor of $\mathcal{I}$ in the augmented PMQ $Q$, i.e. to $\mathcal{I}$. It follows that $\tilde{P}'$ is a subset of $\tilde{P}_{i+1} \cup \cdots \cup \tilde{P}_k \cup \mathcal{E}(\tilde{P}, \tilde{t})$, and the latter set is contained in $U'$ by our choice of $U'$.

Second, we prove that every component of $U'$ intersects $\tilde{P}'$ in at least one point. Let $U'_i$ be the component of $U'$ containing the point $z'_i \in P'$, for some $1 \leq i \leq k'$. There are several cases to consider:

- If $z'_i \in \mathcal{E}(P, t) \subset P'$, then there is $z_j \in P$ with $z'_i \in \mathcal{E}(z_j, t)$, and there is $\bar{z} \in \bar{P} \cap U_j$. We can restrict $U'$ to an adapted covering $U'_i(\mathcal{E}(z_j, t))$ of $\mathcal{E}(z_j, t) \subset P'$, by selecting the relevant connected components; then our hypothesis on $U$ implies that $\varepsilon$ sends $(U_1 \cap \mathcal{X}) \times V$ inside $\mathcal{U}(\mathcal{E}(z_j, t), U'_i(\mathcal{E}(z_j, t)))$, where we use Notation 2.6. In particular $\varepsilon(\bar{z}, \tilde{t}) \in \mathcal{U}(\varepsilon(z_j, t), U'_i(\mathcal{E}(z_j, t)))$, implying that $\varepsilon(\bar{z}, \tilde{t}) \tilde{P}'$ contains a point lying in $U'_i$.
If $z'_i \in \mathcal{Y}$, then $z'_i \in P$ and $z'_i \in \mathcal{E}(z'_i, t)$, so we fall in the previous case.

If $z'_i \in X \setminus \mathcal{Y}$ and $z'_i \in P \setminus \mathcal{E}(P, t)$, then $z'_i$ must be a non-inert point of $P$ for $c$. Since $Q$ is augmented, there is a point $\tilde{z} \in \tilde{P} \cap U'_i$ which is non-inert for $\tilde{c}$. This point $\tilde{z}$ also belongs to $\tilde{P}' \cap U'_i$.

The previous discussion shows that $\Upsilon(P, U) \times V$ is mapped by $\mathcal{E}$ inside $\Upsilon(P', U')$.

Let now $\gamma \subset C \setminus U'$ be a simple loop spinning clockwise around a component $U'_i$; up to slightly perturbing $\gamma$ we may assume that it is also disjoint from the finite set $P$, i.e., $\gamma$ avoids the points $z_1, \ldots, z_r$. Then we have the following chain of equalities

$$\varphi'(\{\gamma\}) = \varphi(\{\gamma\}) = \hat{\varphi}(\{\gamma\}) = \tilde{\varphi}'(\{\gamma\}).$$

If $\gamma$ represents a class in $\Omega(P')$, we also have the following chain of equalities

$$\psi'(\{\gamma\}) = \psi^{\text{ext}}(\{\gamma\}) = \tilde{\psi}^{\text{ext}}(\{\gamma\}) = (\tilde{\psi}')^{\text{ext}}(\{\gamma\}),$$

where we use that $[\gamma]$ represents elements of $\Omega^{\text{ext}}(P), \Omega^{\text{ext}}(\tilde{P})$, and $\Omega^{\text{ext}}(\tilde{P}')$, and refer to Definition 2.13 and Proposition 2.14. This shows that $\tilde{c}' \in \Upsilon'(c', U')$.

Suppose now that $\mathcal{E}$ is standard and let $c \in \text{Hur}(C; Q, G)$; then $\mathcal{E}_a(c, 0)$ is computed by first deleting all inert points of $c$ via $\rho$, and then by reading these inert points through the external product of $\rho(c)$ with $P := \varepsilon(c) = \mathcal{E}(P, 0) \in \text{ Ran}(\mathcal{E})$.

### A.10. Proof of Proposition 8.21

We introduce some notation for barycentres of faces of simplices. Recall Notation 8.3: for $p \geq 1$ and $0 \leq i \leq p$ we denote by $\text{bar}^{p-1,i} = \left(\frac{1}{p}, \frac{2}{p}, \ldots, \frac{i-1}{p}, \frac{i}{p}, \frac{i+1}{p}, \ldots, \frac{p-1}{p}\right) \in \Delta^p$ the barycentre of the face $d_i \Delta^p$.

**Lemma A.1** Recall Definition 8.16. The map $e^2$ sends $(\text{bar}^{p-1,i}, \text{bar}^q) \in \Delta^p \times \Delta^q$ to $c_q'$, where $a' = d_i^{|\text{tor}}(a)$.

Before proving Lemma A.1, we will argue how Proposition 8.21 follows from it. Let $(s, t) \in d_i^{\text{hor}}(\Delta^p \times \Delta^q) \subset \Delta^p \times \Delta^q$ (see Notation 8.20). Then the pair $(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ belongs to $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ as a composition $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ of $\mathcal{H}_{p,q}^{s,t}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ by Lemma 8.8. Assuming Lemma A.1, we have that $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ sends $c_{s,t} \mapsto c_{s,t}'$, and then by definition the second map $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ sends $c_{s,t}$ to $e^2(s, t) \in \bar{C}$ (regarding $(s, t)$ as a point in $\Delta^p \times \Delta^q$), and the composition $\mathcal{H}_{s,t}^{p,q}(\text{bar}^p, \text{bar}^q; s, t)$ sends $c_{s,t} \mapsto e^2(s, t)$.

The rest of the subsection is devoted to the proof of Lemma A.1. By definition, $e^2(\text{bar}^{p-1,i}, \text{bar}^q)$ is the image of $c_{s,t}$ under $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q; s, t)$ in the following we abbreviate by $\xi : C \to \bar{C}$ the map $\mathcal{H}_{s,t}^{p,q}(\text{bar}^{p-1,i}, \text{bar}^q)$.

Recall Notation 8.14: the map $\xi$ sends $z_{p,i}' \mapsto z_{p,i}'$ for $0 \leq i' \leq i$, and $z_{p,i}' \mapsto z_{p,i-1}'$ for $i + 1 \leq i' \leq p + 1$; it follows that the image of $P_a$ along $\xi$ is the set $P_a'$. This shows that $e^2(\text{bar}^{p-1,i}, \text{bar}^q)$ is a configuration supported on $P_a'$.

Recall Notation 8.15, and consider the standard generating sets $(f_{i,j}^a)_{(i', j') \in I(a)}$ of $\mathcal{G}(P_a')$ and $(f_{i,j}^a)_{(i', j') \in I(a)}$ of $\mathcal{G}(P_a)$, and the homomorphism $\xi^* : \mathcal{G}(P_a') \to \mathcal{G}(P_a)$ from Sect. 4.3: for all $0 \leq i' \leq p$ and $0 \leq j \leq q + 2$, the homomorphism $\xi^*$ maps the product of
standard generators \( cf_{i,j}^{a'} \) to

\[
\xi^*(cf_{i,j}^{a'}) \mapsto \begin{cases} 
  cf_{i,j}^{a'} & \text{if } i' \leq i - 1; \\
  cf_{i,j}^{a'} \cdot cf_{i+1,j}^{a'} & \text{if } i' = i; \\
  cf_{i+1,j}^{a'} & \text{if } i' \geq i + 1.
\end{cases}
\]

This follows from the description of \( cf_{i,j}^{a'} \) as the class of a simple loop supported on
\( S_{x_{i'-1},x_{i'+1}} \cap \{ z \leq y \} \) and spinning clockwise around the points \( z_{i'}^{a'}, \ldots, z_{i'+1}^{a'} \). For \( i' = i \) we have in particular that \( \xi^* (cf_{i,j}^{a'}) \) is represented by a loop spinning around the horizontal segments joining \( z_{i',j}^{a'} \) with \( z_{i'+1,j}^{a'} \), for \( 0 \leq j' \leq j \): these horizontal segments are the preimages along \( \xi \) of the points \( z_{i,j}^{a'} \) for \( 0 \leq j' \leq j \).

We can now use that \( \xi^* \) is a group homomorphism and compute \( \xi^*(f_{i,j}^{a'}) \) for all \( (i', j) \in I(a') \). In particular, for \( i' = i \) we obtain

\[
\xi^*(f_{i,j}^{a'}) = \xi^* \left( (f_{i,j}^{a'})^{-1} \cdot cf_{i,j+1}^{a'} \cdot f_{i+1,j}^{a'} \right) = \left( cf_{i,j+1}^{a'} \cdot cf_{i+1,j+1}^{a'} \right)^{-1} \cdot cf_{i,j+1}^{a'} \cdot cf_{i+1,j+1}^{a'}
\]

This follows from the description of \( cf_{i,j}^{a'} \) as the class of a simple loop supported on
\( S_{x_{i'-1},x_{i'+1}} \cap \{ z \leq y \} \) and spinning clockwise around the points \( z_{i'}^{a'}, \ldots, z_{i'+1}^{a'} \). For \( i' = i \) we have in particular that \( \xi^* (cf_{i,j}^{a'}) \) is represented by a loop spinning around the horizontal segments joining \( z_{i',j}^{a'} \) with \( z_{i'+1,j}^{a'} \), for \( 0 \leq j' \leq j \): these horizontal segments are the preimages along \( \xi \) of the points \( z_{i,j}^{a'} \) for \( 0 \leq j' \leq j \).

We can now use that \( \xi^* \) is a group homomorphism and compute \( \xi^*(f_{i,j}^{a'}) \) for all \( (i', j) \in I(a') \). In particular, for \( i' = i \) we obtain

\[
\xi^*(f_{i,j}^{a'}) = \xi^* \left( (f_{i,j}^{a'})^{-1} \cdot cf_{i,j+1}^{a'} \cdot f_{i+1,j}^{a'} \right) = \left( cf_{i,j+1}^{a'} \cdot cf_{i+1,j+1}^{a'} \right)^{-1} \cdot cf_{i,j+1}^{a'} \cdot cf_{i+1,j+1}^{a'}
\]

Similarly, for \( i' < i \) we have \( \xi^*(f_{i',j}^{a'}) = f_{i',j}^{a'} \), and for \( i' > i \) we have \( \xi^*(f_{i',j}^{a'}) = f_{i+1,j}^{a'} \).

Applying \( \varphi_{\tilde{a}} \) to \( \xi^*(f_{i,j}^{a'}) \), we obtain the equality \( a_{i',j} = \varphi_{\tilde{a}} (f_{i',j}^{a'}) \), which together with
[3, Lemma 6.8] yields \( a' = d_{i}^{\text{hor, } \tilde{a}} \).

A.11. Proof of Proposition 8.22

Similar as for Proposition 8.21, the statement of Proposition 8.22 follows directly from the following lemma.

Lemma A.2 The map \( e^{a} \) sends \( (\bar{a}^{p}, \bar{a}^{q-1}) \mapsto c\bar{a}^{q} \), where \( a' = d_{i}^{\text{hor, } \tilde{a}} \).

The rest of the subsection is devoted to the proof of Lemma A.2. By definition, \( e^{a} (\bar{a}^{p}, \bar{a}^{q-1}) \) is the image of \( c\bar{a}^{p} \) under the map \( \xi_{*} \), where for the rest of the proof we abbreviate by \( \xi : \mathbb{C} \to \mathbb{C} \) the map \( \mathcal{H}^{p,q} (-; \bar{a}^{p}, \bar{a}^{q}; \bar{a}^{q}, \bar{a}^{q-1}) \).

The map \( \xi \) sends \( z_{i,j}^{p,q} \mapsto z_{i,j}^{p,q} \) for \( 0 \leq j' \leq j \), and \( z_{i,j}^{p,q} \mapsto z_{i,j-1}^{p,q} \) for \( j+1 \leq j' \leq q+1 \); it follows that the image of \( \bar{a}^{p} \) along \( \xi \) is the set \( P_{\bar{a}^{p}} \), and as in the horizontal case we obtain that \( e^{a} (\bar{a}^{p}, \bar{a}^{q-1}) \) is a configuration supported on \( P_{\bar{a}^{p}} \).

Consider now the two standard generating sets \( (f_{i,j}^{a'})_{(i,j)\in I(a')} \) of \( \mathfrak{S}(P_{a'}^{q}) \), and \( (f_{(i,j')}^{a})_{(i,j')\in I(a)} \) of \( \mathfrak{S}(P_{a}^{q}) \), and consider the homomorphism \( \xi' : \mathfrak{S}(P_{a'}^{q}) \to \mathfrak{S}(P_{a}^{q}) \).

The key observation is that, for all \( 0 \leq i \leq p+1 \) and \( 0 \leq j' \leq q+1 \), the homomorphism \( \xi' \) maps the product of standard generators \( cf_{i,j}^{a'} \) to

\[
\xi' (cf_{i,j}^{a'}) \mapsto \begin{cases} 
  cf_{i,j}^{a'} & \text{if } j' \leq j; \\
  cf_{i,j+1}^{a'} & \text{if } j' \geq j + 1.
\end{cases}
\]
This follows from the description of $\text{cf } a'_{i,j'}$ as the class of a simple loop supported on $S_{x_{i-1},x_{i+1}} \cap \{ 3 \leq y \}$ and spinning clockwise around the points $z_{i,0}', \ldots, z_{i,j-1}', z_{i,j+2}', \ldots, z_{i,j+1}'$. For $j' \geq j + 1$, in particular, $\xi^*(\text{cf } a'_{i,j'})$ is represented by a loop spinning around the points $z_{i,0}', \ldots, z_{i,j-1}', z_{i,j+2}', \ldots, z_{i,j+1}'$ and around the vertical segment joining $z_{i,j}'$ with $z_{i,j+1}'$. Note that this segment is in the preimage along $\xi$ of the point $z_{i,j}'$.

We can now use that $\xi^*$ is a group homomorphism and compute $\xi^*(f_{i,j'})$ for all $(i, j') \in I(a')$. In particular, for $j' = j$ we obtain

$$\xi^*(f_{i,j'}) = \xi^*((cf_{i,j'})^{-1} \cdot cf_{i,j+1}) = (cf_{i,j+2})^{-1} \cdot cf_{i,j} = f_{i,j} \cdot f_{i,j+1}.$$ 

Similarly, for $j' < j$ we have $\xi^*(f_{i,j'}) = f_{i,j'}^a$, and for $j' > j$ we have $\xi^*(f_{i,j'}) = f_{i,j+1}^a$. Applying $\psi_a$ to $\xi^*(f_{i,j}')$, we obtain the equality $a'_{i,j'} = \psi_a(f_{i,j}')$, which together with [3, Lemma 6.8] yields $a' = d^\text{hor}_i a$.

References


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