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Ginzburg-Landau equations on non-compact Riemann surfaces

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\textbf{Abstract}

We study the Ginzburg-Landau equations on line bundles over non-compact Riemann surfaces with constant negative curvature. We prove existence of solutions with energy strictly less than that of the constant curvature (magnetic field) one. These solutions are the non-commutative generalizations of the Abrikosov vortex lattice of superconductivity. Conjecturally, they are (local) minimizers of the Ginzburg-Landau energy. We obtain precise asymptotic expansions of these solutions and their energies in terms of the curvature of the underlying Riemann surface. Among other things, our result shows the spontaneous breaking of the gauge-translational symmetry of the Ginzburg-Landau equations.

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1. Introduction

We consider the Ginzburg-Landau equations on a line bundle $E$ over a Riemann surface $\Sigma$ with a Hermitian metric $h$:

\[-\Delta_a \psi = \kappa^2 \left(1 - \left|\psi\right|^2\right) \psi,\]
\[d^* da = \text{Im} \left(\bar{\psi} \nabla_a \psi\right).\]  

(1.1)

Here $\kappa > 0$ is a fixed parameter, $\psi$ and $a$ are respectively a section of and a connection 1-form on the line bundle $E$. $\nabla_a$ is the covariant derivative induced by the 1-form $a$, and $-\Delta_a = \nabla^*_a \nabla_a$ is the covariant Laplacian, both acting on sections of $E$. $d$ denotes the exterior derivative on $\Sigma$. Note that the adjoint $\nabla^*_a$ depends on the metric $h$. See Appendix A for detailed definitions.

In the standard Ginzburg-Landau equations, $\kappa$ is the dimensionless Ginzburg-Landau material parameter, $\psi$ the complex order parameter for the electronic condensate on $\Sigma$, and $a$ the vector potential, with the 2-form $da$ giving the magnetic field, $|\psi|^2$ the local density of superconducting electrons, and $J(\psi, a) := \text{Im} \left(\bar{\psi} \nabla_a \psi\right)$ the supercurrent density.

(1.1) are the Euler-Lagrange equations for the Ginzburg-Landau energy,

\[\mathcal{E}(\psi, a, h) = \int_{\Sigma} \left(\frac{1}{2} |\nabla_a \psi|^2 + \frac{1}{2} |da|^2 + \frac{\kappa^2}{4} \left(|\psi|^2 - 1\right)^2\right) \omega.\]  

(1.1)

The Hermitian metric $h$ enters (1.1) through the area 2-form $\omega$ induced by $h$.

Physically, (1.1) corresponds to the Ginzburg-Landau Helmholtz free energy. By the Chern-Weil correspondence (see Section 2.3 below), $\mathcal{E}(\psi, a, h)$ can be parametrized by the average magnetic field in the sample. It is related to the Ginzburg-Landau Gibbs free energy, depending on applied magnetic field through the Legendre transform. For more discussions, see [31].

One can think of solutions to (GL) as non-commutative versions of the Abrikosov vortex lattices, with commutative lattice $\mathcal{L}$ acting on $\mathcal{C}$ by translations replaced by a non-commutative one – a Fuchsian group $\Gamma$ acting on the Poincaré half-plane $\mathbb{H}$. See Section 2 for details.

One can also connect (GL) to the Ginzburg-Landau equations on a thin superconducting membrane, and we conjecture that the mathematical techniques developed in this paper can be applied to this latter model. See [24] for a review of the physics problem.

(GL) is the first and arguably the simplest gauge theory. Indeed, (GL) is invariant under local $U(1)$-gauge transforms

\[(\psi, a) \mapsto (g \psi, a + g^{-1} dg),\]  

(1.2)
where $g$ is a $U(1)$-valued isomorphism\(^1\) and $a$ is a gauge field related to the standard connection on the principal $U(1)$-bundle.

In this paper, we construct nontrivial solutions to the Ginzburg-Landau equations (GL) defined on a unitary line bundle $E$ over a non-compact Riemann surface $\Sigma$ of finite volume. Our existence theory holds on the arithmetic surfaces

$$\Sigma \cong \mathbb{H}/\Gamma(N), \quad N = 1, 2, \ldots,$$

(1.3)

where $\mathbb{H}$ is the Poincaré half-plane, equipped with a hyperbolic metric with constant negative curvature, and $\Gamma(N)$ is the principal congruence subgroup of level $N$, acting on $\mathbb{H}$ by Möbius transform. (GL) on a non-compact Riemann surface $\Sigma$ of the form (1.3) could serve as a toy model of a superconducting circuit with several narrow open ends [26]. The precise results are given in Theorems 1.1–1.2.

We also obtain asymptotics for the energy of the our solutions and we show that under condition (1.28), the energy of the our solutions is lower than that of the constant curvature solution (see (1.6) below). The precise energy estimate is given in Corollary 1.6.

Throughout the paper, we fix some non-compact Riemann surface $(\Sigma, h)$, together with a unitary line bundle $E \to \Sigma$, and then seek a solution pair $(\psi, a)$ consisting of a section of and a connection on $E$. The parameter $\kappa > 0$ in (GL) is a fixed number. Unless otherwise stated, dependence of various quantities on $\kappa$ is not displayed but always understood.

1.1. Main results

To state our main result, we introduce some definitions. To fix the ideas, we consider the family of hyperbolic metrics on $\Sigma$ given by

$$h_r = \frac{r}{(|\text{Im}\ z|)^2} dz \otimes d\bar{z} \quad (r > 0).$$

(1.4)

The corresponding area 2-form is given by

$$\omega_r = \frac{r i}{2(|\text{Im}\ z|)^2} dz \wedge d\bar{z}.$$  

(1.5)

Each $h_r$ turns $\Sigma$ into a surface with constant curvature $-1/r$. Denote $|\Sigma|$ the area of $\Sigma$ w.r.t. the standard hyperbolic 2-form $\omega_1 = \frac{i}{2(|\text{Im}\ z|)^2} dz \wedge d\bar{z}$. Then the surface $(\Sigma, h_r)$ has area $|\Sigma|_r = r |\Sigma|$.

For fixed $\Sigma, E$, and each hyperbolic $h_r$, (GL) has the following constant curvature (or magnetic field) solutions:

---

\(^1\) The fact that (1.2) is indeed a symmetry for (GL) follows from the relations $\nabla_a (g\psi) = g\nabla_a \psi + (dg)\psi = g(\nabla_a + g^{-1}dg)\psi = g\nabla_{a+g^{-1}dg}\psi$, and $\text{Im}(g\psi \nabla_a (g\psi)) = \text{Im}(g\psi \nabla_{a+g^{-1}dg}\psi) = \text{Im}(\psi g\nabla_{a+g^{-1}dg}\psi)$. 

ψ ≡ 0, \quad a = a^{br}, \quad (1.6)

where ψ is the zero-section on the line bundle E, and a^{br} is a constant curvature connection satisfying

\[ da^{br} = b_r \omega_r, \quad (1.7) \]

and

\[ b_r \equiv b(r, \Sigma, E) := \frac{2\pi \deg E}{|\Sigma|} = \frac{b}{r}, \quad (1.8) \]

where deg E is the degree (or the first Chern number) of the bundle E (see Section A.2 for definitions), and

\[ b := \frac{2\pi \deg E}{|\Sigma|}. \quad (1.9) \]

The value of \( b_r \) in (1.8) is determined by the Chern-Weil correspondence, see (2.17) below. Once Σ, E are fixed, the value \( b_r \) can be computed explicitly using the Gauss-Bonnet formula (2.9) and the curvature parameter \( r \) in the background metric (1.4).

In the standard Ginzburg-Landau equations, solutions of the form (1.6) correspond to normal, non-superconducting states.

For fixed Σ, E, let \( b_r = b/r \) with \( r > 0 \). The value \( b_r \) turns out to be the smallest eigenvalue of \(-\Delta_{a^{br}}\) (see Section 3.2 for discussions). We denote by

\[ K(r) = \text{Null}(-\Delta_{a^{br}} - b_r) \quad (1.10) \]

the finite dimensional null space of \(-\Delta_{a^{br}} - b_r\) acting on the space of square integrable sections on \( E \to \Sigma \). Now, we define the function

\[ \beta(r) := \min \left\{ \langle |\xi|^4 \rangle : \xi \in K(r), \langle |\xi|^2 \rangle = 1 \right\}. \quad (1.11) \]

Here and below, \( \langle f \rangle := \frac{1}{|\Sigma|} \int f \). Note that, by Hölder’s inequality, \( \beta(r) > 1 \). Eq. (1.11) contains information about the energy of the solutions as a function of \( r \) (see Corollary 1.6).

We define another function of threshold Ginzburg-Landau parameter, \( \kappa_c(r) \), as

\[ \kappa_c(r) := \sqrt{\frac{1}{2} \left( 1 - \frac{1}{\beta(r)} \right)} \quad (1.12) \]

Finally, let \( H^k \) and \( \tilde{H}^k \) be the Sobolev spaces of order \( k \) of sections and weakly co-closed 1-forms (i.e., \( d^*\alpha = 0 \) in the distributional sense) on the line bundle \( E \to \Sigma \); \( O_H^k \) and \( O_{\tilde{H}}^k \) stand for error terms in the sense of the norms in \( H^k \) and \( \tilde{H}^k \), and let
\[ X^k := \mathcal{H}^k \times \tilde{\mathcal{H}}^k. \] (1.13)

The definitions of these spaces are standard. See Appendix A for details.

Now, we are ready to formulate our main results:

**Theorem 1.1 (Existence and uniqueness).** Let \((\Sigma, h_r), r > 0\), be a non-compact Riemann surface equipped with the hyperbolic metric \((1.4)\) with finite area. Let \(E \to \Sigma\) be a unitary line bundle, with \(b := 2\pi \deg E/|\Sigma| > 0\). Suppose

\[ b \neq 1/2, \] (1.14)

and \(r > 0\) satisfies

\[ 0 < |\kappa^2 r - b| \ll 1, \quad (\kappa - \sqrt{b_r})(\kappa - \kappa_c(r)) > 0, \] (1.15)
\[ \dim_{\mathbb{C}} K(r) = 1. \] (1.16)

Then, for each \(r\) as above, there exists a solution

\[ (\psi(r), a(r)) \] (1.17)

to \((\text{GL})\) in a neighborhood of \(U \subset X^k, k \geq 2\), around \((0, a^{b_r})\).

Moreover, the solution \((1.17)\) is unique in \(U \subset X^k\) up to a gauge symmetry (see (1.2)).

We remark that our results in Section 7 enable us to establish the first, existence part of Theorem 1.1 without imposing the non-degeneracy condition \((1.16)\).

Theorem 1.1 follows from the following:

**Theorem 1.2 (Parametrization and asymptotics).** Let \((1.14)\) hold and assume \(|\kappa^2 r - b| \ll 1\). Then there exists \(\epsilon > 0\) s.t. \((\text{GL})\) with metric \((1.4)\) has a \(C^2\) branch of solutions \((\psi_s, a_s, r_s), s \in \mathbb{C}, |s| \leq \epsilon\), satisfying

\[ \psi_s = s\xi + O_{\mathcal{H}^k}(|s|^3), \] (1.18)
\[ a_s = a^{b_{rs}} + |s|^2 \alpha + O_{\mathcal{H}^k}(|s|^4), \] (1.19)

where \(\xi = O_{\mathcal{H}^k}(1)\) is gauge-equivalent to a holomorphic section of \(E\) corresponding to \(a^{b_{rs}}, \alpha = O_{\mathcal{H}^k}(1)\) is a co-closed 1-form and satisfies, with * denoting the Hodge operator,

\[ d\alpha = \frac{1}{2} * \left(1 - |\xi|^2\right). \] (1.20)

Moreover, if \((1.15)\) and \((1.16)\) hold, then we can take \(s \in \mathbb{R}_{\geq 0}\), and the solution \((\psi_s, a_s, r_s)\) is unique, and the equation \(r = r_s\) can be solved for \(s\) to obtain
\[ s^2 = \frac{\kappa^2 - b_r}{(\kappa^2 - \frac{1}{2})\beta(r) + \frac{1}{2}} + O((\kappa^2 - b_r)^2). \quad (1.21) \]

Furthermore, writing \((1.21)\) as \(s = s(r)\) gives, for any \(r > 0\) as above, the solution

\[ (\psi(r), a(r)) = (\psi_{s(r)}, a_{s(r)}), \quad (1.22) \]

as in Theorem 1.1 (see \((1.17)\)).

This theorem is proved in Sects. 4–6.

**Remark 1.** Condition \((1.15)\) guarantees the r.h.s. of \((1.21)\) is positive, and was first isolated as a criterion for the existence of the Abrikosov vortex lattice in \([32–35]\). Through \((1.8)\), condition \((1.15)\) gives rise to the critical magnetic field \(b_r = \kappa^2\).

**Remark 2.** The reason of condition \((1.14)\) is explained in Section 4, where we also show that an explicit bundle \(E \to \Sigma\) satisfying condition \((1.14)\) is

\[ \Sigma = \mathbb{H}/\Gamma(6), \quad \deg E = 12. \quad (1.23) \]

**Remark 3.** As we explain in Sections 2.2 and 2.3, the number \(\kappa^2r\) corresponds to the average magnetic field in superconductors. Hence, condition \((1.15)\) gives an estimate of the neighborhood of the critical average field strength we work in, in terms of the topological degree \(\deg E\), and the hyperbolic area of \(\Sigma\) through the quantization relation \((1.8)\). In this sense \((1.15)\) can be compared to Bradlow’s condition for the existence of magnetic vortex on compact Riemann surfaces \([3]\).

**Remark 4.** Similar results as ours have been obtained for compact Riemann surfaces in \([5,22,23,25,27]\). It seems that ours is the first rigorous existence theory for \((GL)\) on non-compact Riemann surfaces.

**Remark 5.** It is possible to extend Theorems 1.1–1.2 by dropping the second condition in \((1.14)\), as we explain in Section 7.

The proof of Theorem 1.2 implies also the following proposition, whose proof can be found at the end of Section 3:

**Proposition 1.3.** The constant curvature solution \((\psi \equiv 0, a = a^{b_r})\) (see \((1.6)\)) is a minimizer of the energy \(\mathcal{E}(\psi, a, h_r)\) if and only if

\[ b_r > \kappa^2. \quad (1.24) \]

We make two conjectures concerning the energy of the solution constructed in Theorem 1.2.
Conjecture 1.4. Under the assumption of Theorem 1.2, solution (1.22) is a local minimizer of the energy $\mathcal{E}(\psi, a, h_r)$ if and only if

$$b_r < \kappa^2.$$  \hfill (1.25)

We expect that conjecture above can be proven similarly to the corresponding result for the original Ginzburg-Landau equations proven in [32, Thm. 4].

The significance of such a result is that it would show that by decreasing the curvature $b$, one passes from the (dynamically) stable constant curvature (normal) solution (1.6) to the (dynamically) stable variable curvature solution (1.22).

In fact, we expect stronger statements to be true:

Conjecture 1.5. The constant curvature solution $(\psi \equiv 0, a = a^{b_r})$ (see (1.6)) is a global minimizer of the energy $\mathcal{E}(\psi, a, h_r)$ if (1.24) holds. If (1.25) holds, then solution (1.22) is a global minimizer of the energy $\mathcal{E}(\psi, a, h_r)$.

The energy associated to the constant curvature solution (1.6) is

$$\mathcal{E}(0, a^{b_r}, h_r) = \frac{1}{2} \left( b_r^2 + \frac{\kappa^2}{2} \right) |\Sigma|_r.$$ \hfill (1.26)

As a corollary of Theorem 1.2, we obtain the following energy expansion:

Corollary 1.6. Let conditions (1.14), (1.15), (1.16) hold as in Theorem 1.2. Then, for the solution $(\psi_{s(r)}, a_{s(r)})$ constructed in (1.22), we have,

$$\mathcal{E}(\psi_{s(r)}, a_{s(r)}, h_r) = \mathcal{E}(0, a^{b_r}, h_r) - \frac{|\Sigma|_r}{4} \left( \frac{|\kappa^2 - b_r|^2}{(\kappa^2 - \frac{1}{2}) \beta(r) + \frac{1}{2}} + O \left( |\kappa^2 - b_r|^3 \right) \right).$$ \hfill (1.27)

This corollary is proved at the end of Section 6.

Remark 6. Since $(\kappa^2 - \frac{1}{2}) \beta(r) + \frac{1}{2} = \beta(r)(\kappa^2 - \kappa_c^2(r))$, expansion (1.27) ensures that $\mathcal{E}(\psi_{s(r)}, a_{s(r)}, h_r) < \mathcal{E}(0, a^{b_r}, h_r)$, provided

$$\kappa > \kappa_c(r).$$ \hfill (1.28)

Therefore, if (1.28) holds, then the solutions constructed in Theorem 1.2 are energetically favorable compared to the constant curvature one.

Moreover, Corollary 1.6 shows that there is an energy crossover between the trivial and nontrivial solutions at $\kappa_c(r) = \kappa$. Thus, at $\kappa = \kappa_c(r)$, the gauge-translational symmetry (ensuring that $a$ has a constant curvature) is broken and a non-gauge-translational invariant “ground state” emerges at this point.

Finally, the function $\beta(r) \equiv \beta(r, \Sigma, E)$ yields the asymptotics of the GL energy of bundles over Riemann surfaces.
Remark 7. As was already indicated above, in physics, $\psi$ is called, depending on the area, either the order parameter or the Higgs field. In our context, it is represented by the following two equivalent objects:

1. (Geometry) Sections of the unitary line bundle $E$ with $n = \deg E \neq 0$ over the surface $(\Sigma, h \equiv h_r)$ (as in this section);
2. (Number theory) $\Gamma$-automorphic functions with weight $k = 4\pi n/|\Sigma|$ and trivial multiplier system (see Appendix A.3 and [29]).

Similar parallel can be drawn for the constant curvature connections on $E$, weighted Maass operators on $\Sigma$, and gauge potentials with constant magnetic fields with strength $b$. See Section 2.3 for details.

Remark 8. Conceptually, if one drops the second part in condition (1.14), it is useful to introduce also the extended Abrikosov function

$$\beta : \text{Null}(-\Delta_{a^b} - b_r) \times \mathbb{R}_{>0} \times \{ \text{Fuchsian groups} \} \to \mathbb{R}_{\geq 0}$$

$$(\xi, r, \Gamma) \mapsto \|\xi\|^4_{L^4}/\|\xi\|^2_{L^2}.$$ 

Then we have $\beta(r, \Gamma) = \inf \{ \beta(\xi, r, \Gamma) : \xi \in \text{Null}(-\Delta_{a^b} - b_r) \}$, cf. (1.11).

1.2. Organization of the paper

In Section 2, after giving some preliminary definitions, we show that (GL) on $(\Sigma, h_r)$ with $h_r$ as in (1.4) is equivalent to the rescaled Ginzburg-Landau equations, (2.15), posed on $(\Sigma, h \equiv h_1)$. Then the proof of Theorem 1.2 consists of two parts of analysis on the rescaled equation (2.15).

First, in Section 3, we study the linearized problem associated to (GL), which reduces to understanding the spectral properties of the Laplacian $-\Delta_{a^b}$ associated to a constant curvature connection $a^b$, viewed as an operator acting on square-integrable sections of the unitary line bundle $E \to \Sigma$. We show that the essential spectrum of $-\Delta_{a^b}$ is given by the half-line $[\frac{1}{4} + b^2, \infty)$, and the lowest eigenvalue of this operator equals $b$ whenever the space of cusp forms on $\Sigma$ is non-trivial, in which case we give explicit description of $\text{Null}(-\Delta_{a^b} - b)$. For precise statements, see Theorem 3.2.

The operator $-\Delta_{a^b}$ is known in the physics literature as the magnetic Laplacian at constant field strength $b$, and is studied in e.g. [2,7,8,26]. For $b = 0$, $-\Delta_{a^b}$ reduces to the Laplace-Beltrami operator acting on the Poincaré half-plane, whose spectral properties are well studied in [21]. For $b \neq 0$, eigenfunctions of $-\Delta_{a^b}$ are precisely the weighted Maass forms in number theory. See e.g. [4, Sec. 2], [29], and the references therein.

Next, in Section 4, we use Lyapunov-Schmidt reduction to show that a non-trivial branch of solution of the form (1.18)–(1.19) bifurcates from the constant curvature solutions (1.6), provided the metric on $\Sigma$ satisfies condition (1.15). In Section 5, we solve the
key bifurcation equation (4.30), which, by results from the previous section, amounts to solving (GL). In Section 6, we derive precise asymptotics for the solutions constructed in Sects. 4–5. This proves Theorem 1.2 and Corollary 1.6.

Lastly, in Section 7, we explain how to drop the non-degenerate condition in (1.14) and extend the main results above to \( \dim K > 1 \).

2. Preliminaries

In this section, we explain the geometric setting for the results and proofs in this paper.

In the remainder of this paper, the following geometric assumptions are always understood:

(1) The underlying surface \( \Sigma \) is of the form (2.1), with finite area, \( g \) genus, \( m \) cusp, and no elliptic points (e.g. the principal congruence subgroup \( \Sigma = \mathbb{H}/\Gamma(N) \) with \( N \geq 2 \), defined in (2.4) below);
(2) \( b > 0 \) in (1.8) (which can always be achieved by changing orientation so that \( \deg E = 1, 2, \ldots \)).

2.1. Non-compact Riemann surfaces

Let \( \Sigma \) be a connected Riemann surface. The Uniformization Theorem states that if \( \Sigma \) is non-compact and not flat, then

\[
\Sigma \cong \mathbb{H}/\Gamma,
\]

(2.1)

where \( \mathbb{H} \) is the Poincaré half-plane,

\[
\mathbb{H} := \{ z : z \in \mathbb{C}, \operatorname{Im} z > 0 \},
\]

and \( \Gamma \) is a Fuchsian group, i.e. a discrete subgroup of

\[
PSL(2, \mathbb{R}) := SL(2, \mathbb{R})/\{ \pm 1 \}
\]

acting freely on \( \mathbb{H} \). Here, the action of \( SL(2, \mathbb{R}) \) on \( \mathbb{H} \) is by Möbius transform,

\[
\gamma z = \frac{az + b}{cz + d} \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{R}))
\]

(2.2)

By convention, we define \( \gamma \infty = a/c \).

An important class of examples are the Riemann surfaces

\[
\Sigma := \mathbb{H}/\Gamma(N), \quad N = 1, 2, \ldots,
\]

(2.3)
Fig. 1. A fundamental domain of $\Gamma(2)$ in $\mathbb{H}$ with three cusps $c_1 = 1, c_2 = 0, c_3 = \infty$.

where $\Gamma(N)$ is the principal congruence subgroup of level $N$,

\[
\Gamma(N) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) : a \equiv d \equiv 1, b \equiv c \equiv 0 \mod N \right\}. \tag{2.4}
\]

By definition, $\Gamma(N)$ is a normal subgroup of the modular group $SL(2, \mathbb{Z})$ for each $N$.

**Definition 2.1 (cusp).** Let $\Gamma$ be a Fuchsian group. A point $c \in \mathbb{R} \cup \{\infty\}$ is called a cusp of $\Gamma$ if and only if there is an element $\gamma \in \Gamma$ that is conjugate-equivalent to some horizontal translation $z \mapsto z + h$, $h \in \mathbb{R}$, s.th. $\gamma c = c$.

For example, if $\Gamma = SL(2, \mathbb{Z})$, then the only cusp of $\Gamma$ is $c = \infty$, as every integral translation $z \mapsto z + n$, $n \in \mathbb{Z}$ fixes $c$. If $\Gamma = \Gamma(2)$, then there are three cusps with $c_1 = 0, c_2 = 1, c_3 = \infty$. See Fig. 1. (Note that $-1$ is equivalent to $c_2$ through translation $z \mapsto z + 2$.)

Denote by $m$ the number of distinct cusps and $g$ the number of genera of $\Gamma$. For the principal congruence subgroup $\Gamma = \Gamma(N)$ defined in (2.4), the classical results from [30, Sec. 1.6] show that

\[
m_N = \begin{cases} 3 & (N = 2), \\
\frac{1}{2}N^2 \prod_{p|N} \left(1 - \frac{1}{p^2}\right) & (N \geq 3),
\end{cases} \tag{2.5}
\]

\[
g_N = 1 + \frac{N - 6}{12} m_N. \tag{2.6}
\]

2.2. Metric and rescaling

Let $r > 0$. As in Section 1, we equip the space $\mathbb{H}$ with the following families of hyperbolic metrics and induced area 2-forms:

\[
h_r = \frac{r}{(\text{Im } z)^2} dz \otimes d\bar{z}, \tag{2.7}
\]
\[ \omega_r = \frac{r i}{2 (\text{Im} z)^2} dz \wedge d\bar{z}. \] (2.8)

These carry over to Riemann surfaces of the form \( \Sigma = \mathbb{H}/\Gamma \). The surface \( (\Sigma, h_r) \) has constant curvature \(-1/r\) and surface area \(|\Sigma|_r = r |\Sigma|\), where \(|\Sigma|\) denotes the area of \( \Sigma \) w.r.t. the standard hyperbolic \( \omega \equiv \omega_r|_{r=1} \).

Suppose \( (\Sigma, h \equiv h_1) \) has finite area, \( g \) genus, \( m \) cusps, and no elliptic points. Then Gauss-Bonnet formula gives

\[ |\Sigma| = 2\pi(2g - 2 + m). \] (2.9)

See e.g. [21, p.43, Eqn. (2.7)]. For example, if \( \Sigma_N = \mathbb{H}/\Gamma(N) \) with \( N \geq 2 \), then \( \Sigma_N \) has no elliptic point. Hence, by (2.5)–(2.6) and Gauss-Bonnet formula, we have

\[ |\Sigma_N| = \frac{\pi N m_N}{3}. \] (2.10)

For \( h = h_r \) given by (2.7), the Ginzburg-Landau energy functional (1.1) reduces to

\[ E_r(\psi, a) = \int_{\Sigma} \left( \frac{1}{2} |\nabla_a \psi|^2 + \frac{1}{2} |da|^2 + \frac{\kappa^2}{4} \left( |\psi|^2 - r \right)^2 \right) \omega_1, \] (2.11)

defined on the base surface \( (\Sigma, h \equiv h_1) \). To see this, we distinguish the quantities related to the metric \( h_r \) by tildes and consider

\[ \tilde{\psi} = r^{-1/2} \psi, \quad \tilde{a} = r^{-1/2} a. \] (2.12)

Using the relations \( \tilde{\nabla} = r^{-1/2} \nabla \) and \( \tilde{d} = r^{-1/2} d \) (see e.g. [5, Sect. B.1]), we have

\[ \nabla_{\tilde{a}} \tilde{\psi} = \frac{1}{r} \nabla_a \psi, \quad \tilde{d} \tilde{a} = \frac{1}{r} da, \quad |\tilde{\psi}|^2 = \frac{1}{r} |\psi|^2. \] (2.13)

Plugging (2.13) into (1.1) and using \( \omega_r = r \omega_1 \) (see (2.8)), we find

\[
\begin{align*}
    rE(\tilde{\psi}, \tilde{a}, h_r) &= \int_{\Sigma} \left( \frac{1}{2} |\nabla_{\tilde{a}} \tilde{\psi}|^2 + \frac{1}{2} |\tilde{d} \tilde{a}|^2 + \frac{\kappa^2}{4} \left( |\tilde{\psi}|^2 - 1 \right)^2 \right) r^2 \omega_1 \\
    &= \int_{\Sigma} \left( \frac{1}{2} |\nabla_a \psi|^2 + \frac{1}{2} |da|^2 + \frac{\kappa^2}{4} \left( |\psi|^2 - r \right)^2 \right) \omega_1 = E_r(\psi, a),
\end{align*}
\] (2.14)

which gives (2.11).

The Euler-Lagrange equations for \( E_r(\psi, a) \) are the rescaled (GL), given by

\[ -\Delta_a \psi = \kappa^2 \left( r - |\psi|^2 \right) \psi, \]
\[ d^* da = \text{Im}(\psi \nabla_a \psi), \] (2.15)
in the space $X^k$, $k \geq 2$ defined in (1.13). For suitable values of $r > 0$, we seek solution pair $(\psi, a)$ to (2.15) on the unscaled surface $(\Sigma, h)$. In Section 2.3, we show that the parameter $r > 0$ corresponds to the average field (magnetic flux) strength. In Type II superconductors, the variation of the latter triggers second-order phase transition. Hence, in what follows we will use $r$ as the bifurcation parameter.

The rescaled Ginzburg-Landau equations (2.15) are the central objects of the subsequent sections.

2.3. Quantization of magnetic flux

In this subsection, we state the extension of the Chern-Weil correspondence (known in the physics literature as magnetic flux quantization) to non-compact Riemann surfaces.

Recall that $\deg E$ denotes the topological degree, or the first Chern number, of the unitary line bundle $E$, see Appendix A.

**Theorem 2.2** (Chern-Weil correspondence for non-compact Riemann surfaces). Let $\Sigma$ be a non-compact Riemann surface, and $E \to \Sigma$ be a unitary line bundle. For every connection $a$ on $E \to \Sigma$ with $|f_\Sigma da| < \infty$, there holds

$$\frac{1}{2\pi} \int_\Sigma da = \deg E. \quad (2.16)$$

The proof of this theorem is given in Appendix C.

Since $\deg E \in \mathbb{Z}$, equation (2.16) implies quantization of the average magnetic field (magnetic flux).

The Chern-Weil correspondence imposes a direct constrain on constant curvature connections. Indeed, let $a^\beta$ be a constant curvature connection on $(\Sigma, h_r)$ satisfying $da^\beta = \beta \omega_r$ for some $\beta \in \mathbb{R}$ (e.g. (1.7)). Then, by relation (2.16), we have

$$\beta = \frac{1}{|\Sigma|_r} \int_\Sigma da = \frac{2\pi \deg E}{|\Sigma|_r}, \quad (2.17)$$

which, together with the relation $|\Sigma|_r = r |\Sigma|$, gives $\beta = b_r$ as in (1.8).

(2.17) relates the average magnetic field (curvature) on $\Sigma$ to the geometry of $\Sigma$, and the topology of the line bundle $E \to \Sigma$. Indeed, (2.17) shows that varying the metric on $\Sigma$ as in (2.7)–(2.8) amounts to varying the constant curvature solution.

3. Linearized Ginzburg-Landau equations

From now on, the central object of study will be the rescaled GL equations (2.15). As explained in Section 2.2, these equations are posed on the unscaled surface $(\Sigma, h \equiv h_1)$. In this section, we consider the linear problem associated to (2.15).
The Ginzburg-Landau equations (2.15) have constant curvature solutions

\[ (\psi, a) = (0, a^b), \text{ where } a^b \text{ is any constant curvature connection with } da^b = b\omega. \]  

(3.1)

As a consequence of the Chern-Weil correspondence (2.16), since \( a^b \) is a constant curvature connection on \((\Sigma, h \equiv h_1)\), the number \( b \) is given by (1.9).

Linearizing (2.15) at the solution (3.1), we get a decoupled system

\[ (-\Delta a^b - \kappa^2 r)\phi = 0, \]  

(3.2)

\[ d^*d\alpha = 0. \]  

(3.3)

These are the main objects of this section. At this point, the unknown in (3.2)–(3.3) here is a (section, connection)-pair on a unitary line bundle \( E \to (\Sigma, h_1) \). The curvature parameter \( r \) from the unscaled GL equations now enters only through the second term in the l.h.s. of (3.2).

Our goal now is to obtain an explicit description of the solutions to (3.2)–(3.3) in the Sobolev space \( X^s \) defined in (1.13).

3.1. Solving the Maxwell equation

First, we solve the homogeneous equation (3.3) (the free Maxwell equation) for the connection \( a \).

By [5, Lem. 3.2], \( \alpha \) solves (3.3) if and only if \( \alpha \) is a constant curvature connection. Thus \( a^b \) is a solution. By the Chern-Weil correspondence, the linearization, (3.3), around \( (\psi, a) = (0, a^b) \) must be satisfied by a 1-form \( \alpha \) of degree 0. So again, by [5, Lem. 3.2], \( \alpha \) must be a flat connection: In the distributional sense,

\[ d^*d\alpha = 0 \iff \alpha \text{ is flat} \iff d\alpha = 0. \]  

(3.4)

So \( \alpha \) must be closed; but by the definition of (1.13) it must also be co-closed and hence harmonic. This establishes the following.

Proposition 3.1. \( d^*d \geq 0 \) and the solution space to (3.3) in \( \tilde{H}^2 \) is

\[ \Omega := \text{Null } d^*d|_{\tilde{H}^2} = \{\text{harmonic 1-forms on } \Sigma\}. \]  

(3.5)

3.2. The magnetic Laplacian

Recall that \( b = 2\pi \deg E/|\Sigma| \), and we choose the orientation of \( E \) that makes \( b > 0 \). By the classification of constant curvature connections, Theorem B.2, an one-form \( \beta \) on \( \Sigma \) satisfies \( d\beta = b\omega \) and the equivariant condition if and only if \( \beta \) is gauge-equivalent to

\[ a^b := by^{-1}dx. \]  

(3.6)
Thus in what follows we fix the canonical choice $a^b$ as in (3.6).

In the remaining subsections, we study the spectral properties of the magnetic Laplacian $-\Delta_{a^b}$ in the l.h.s. of (3.2). Locally, in the rectangular coordinate $z = x + iy$, we have

$$-\Delta_{a^b} = -y^2(\partial_{xx} + \partial_{yy}) + 2iby\partial_x + b^2$$

acting on $F_\Sigma \subset \mathbb{H}$. (3.7)

This follows from direct computation using the standard definitions in Section A. We prove the following.

**Theorem 3.2.** Let $\Sigma = \mathbb{H}/\Gamma$ be a non-compact Riemann surface with $m$ cusps and no elliptic points.

(a) $-\Delta_{a^b}$ is self-adjoint;
(b) $-\Delta_{a^b} \geq 0$;
(c) Let $S(\Sigma) \equiv S_k(\Sigma)$ denote the space of cusp forms on $\Sigma$ with weight $k = 2b = 4\pi n/|\Sigma|$. Then $b$ is an eigenvalue of $-\Delta_{a^b}$ if and only if $S(\Sigma) \neq \emptyset$, and the multiplicity of $b$ equals to $\dim S(\Sigma)$;
(d) The essential spectrum of $-\Delta_{a^b}$ consists of $m$ branches each of which filling in the semi-axis $[1/4 + b^2, \infty)$. Hence,

$$\sigma_{\text{ess}}(-\Delta_{a^b}) = [1/4 + b^2, \infty).$$

(3.8)

**Remark 9.** If the field strength $b = 0$, then $E$ is the trivial line bundle, and $-\Delta_{a^b}$ reduces to the hyperbolic Laplacian $-\Delta$ on $\Sigma$. The spectral properties of this operator have important bearings for number theory. See e.g. [21] and the references therein.

For $b > 0$, some of the results above have been obtained in [2,7,8,26] for specific choices of $\Sigma$ in relevant physics context. For the connection to weighted Maass forms in number theory, see [4, Sec. 2].

**Remark 10.** For $b = 1/2$, the eigenvalue $b$ is embedded at the bottom of the essential spectrum of $-\Delta_{a^b}$.

**Proof of Theorem 3.2.** The proof of Part (a) is standard and can be found in [4, Sec. 2.1]. Parts (b)-(c) follow from the Weitzenböck formula given in Section 3.3. Part (d) follows from the results in Section 3.4. □

We also obtain a description of the solution space to the static Schrödinger equation (3.2). Before we state this result, we need some preliminary definitions.

Let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group. Let $c_i$ be a cusp of $\Gamma$. Then the stabilizer of $c_i$ is an infinite cyclic group generated by some parabolic transform. In symbols,

$$\text{Stab}(c_i, \Gamma) \equiv \{ \gamma \in \Gamma : \gamma c_i = c_i \} = \left\{ \begin{pmatrix} 1 & r_i \\ 0 & 1 \end{pmatrix} \right\} \text{ for some } r_i \in \mathbb{R}.$$
We call \( \gamma_i \in SL(2, \mathbb{R}) \) a scaling matrix of \( c_i \) if \( \gamma_i c_i = \infty \) and \( \gamma_i \frac{1}{r_i} \gamma_i^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). Such \( \gamma_i \)'s exist, and are unique up to translation. See [21, Chapt. 2.2] for details.

### 3.3. Weitzenböck-type formula

In this subsection, let \( a \) be an arbitrary unitary connection on a smooth complex line bundle \( E \to \Sigma \) (see Appendix A for the definitions).

We decompose the covariant derivative \( \nabla_a \) into \((1, 0)\) and \((0, 1)\) parts as \( \nabla_a = \partial'_a + \partial''_a \), where

\[
\partial'_a := \partial + a_c, \quad \partial''_a := \overline{\partial} + \bar{a}_c. \quad (3.9)
\]

Here \( \partial := \frac{\partial}{\partial z} \otimes dz \) and \( \overline{\partial} := \frac{\partial}{\partial \bar{z}} \otimes d\bar{z} \), where, as usual, \( \frac{\partial}{\partial z} \equiv \partial_z := (\partial_{x_1} - i\partial_{x_2})/2 \), \( \frac{\partial}{\partial \bar{z}} \equiv \partial_{\bar{z}} := (\partial_{x_1} + i\partial_{x_2})/2 \), and

\[
a_c := \frac{1}{2}(a_1 - ia_2) \otimes dz, \quad \bar{a}_c := \frac{1}{2}(a_1 + ia_2) \otimes d\bar{z}. \quad (3.10)
\]

With the definitions above, we can rewrite (3.9) as

\[
\partial''_a = \bar{\partial}'_a \otimes d\bar{z}, \quad \bar{\partial}'_a := \bar{\partial}_z + \bar{a}^c, \quad a^c := a_1 - ia_2. \quad (3.11)
\]

Throughout this subsection, \( a^c \) denotes the complexification of a real valued 1-form, and is not to be confused with the constant curvature connection \( a^b \).

In the reverse direction, we have \( a = 2 \text{Re} a_c \) and

\[
\bar{\partial}'_a := \frac{1}{2}(\nabla_1 + i\nabla_2). \quad (3.12)
\]

In terms of \( a_c \), the curvature is given by \( F_a = 2 \text{Re} \bar{\partial} a_c \). Now we prove the following relations:

**Proposition 3.3.** We have

\[
* F_a = \frac{1}{2} [\bar{\partial}'_a, \bar{\partial}''_a], \quad (3.13)
\]

\[
- \Delta_a = \partial''_a \partial''_a + * F_a, \quad (3.14)
\]

\[
- \Delta_a \geq * F_a. \quad (3.15)
\]

**Proof.** We compute in local coordinates. Using the relations \( \nabla_1 = \frac{1}{2}(-\bar{\partial}''_a + \bar{\partial}''_a^*) \), \( \nabla_2 = \frac{1}{2\pi}(\bar{\partial}'_a + \bar{\partial}'_a^*) \), together with the expression (A.3) for the curvature \( F_a \), we compute

\[
F_a = -\frac{1}{4} [\bar{\partial}'_a - \bar{\partial}'_a^*, \bar{\partial}''_a + \bar{\partial}''_a^*] = \frac{1}{2} [\bar{\partial}'_a, \bar{\partial}''_a^*], \quad (3.16)
\]
which gives (3.13).

To find the expression for $\Delta_a$, we use the relation $\Delta = \nabla_i \nabla_i$ to compute

$$\Delta_a = \frac{1}{4}(\tilde{\partial}' '\tilde{\partial}' ' - \tilde{\partial}' ' \tilde{\partial}' ')^2 - \frac{1}{4}(\tilde{\partial}' \tilde{\partial}' ' + \tilde{\partial}' ' \tilde{\partial}' ')
= -\frac{1}{2}(\tilde{\partial}' \tilde{\partial}' ' + \tilde{\partial}' ' \tilde{\partial}' ')
= \tilde{\partial}' ' \tilde{\partial}' ' - \frac{1}{2}[\tilde{\partial}' ', \tilde{\partial}' '],$$

which gives (3.14).

Since $\partial'' \tilde{\partial}'' \geq 0$, eq. (3.14) implies (3.15).

Proposition 3.3 and the fact that $\ast F = b$ imply the next two corresponding relations in the constant curvature case:

**Corollary 3.4.** Let $a^b$ be a constant curvature connection s.th. $\ast F = b$ Then

$$-\Delta_{a^b} \geq b,$$

$$K := \text{Null}(-\Delta_{a^b} - b) = \text{Null} \partial''_{a^b}.$$ (3.19)

Hence, $b$ is an eigenvalue of $-\Delta_{a^b}$ if and only if $\text{Null} \partial''_{a^b} \neq \{0\}$.

**Proof of Theorem 3.2, Parts (b), (c).** Part (b) follows from (3.18). To complete the proof of Part (c), we now claim

$$\text{Null} \partial''_{a^b}|_{\Omega^0(E)} = H^0(E),$$ (3.20)

where $\Omega^0(E)$ denotes the space of sections of the line bundle $E$, and $H^0(E)$ the space of holomorphic sections.

Indeed, suppose (3.20) holds. By the equivalence described at the end of Section 1.1, the space $H^0(E)$ of holomorphic sections of a unitary line bundle $E$ are isomorphic (as a complex vector space) to the space $M_k(\Sigma)$ of modular forms on $\Sigma$, with weight $k = 2b = 4\pi \deg E/|\Sigma|$. Since $\Sigma$ is non-compact, the intersection $L^2(\Sigma) \cap M_k(\Sigma)$ equals to the space of cusp forms $S_k(\Sigma)$ with weight $k$. This fact follows from the results in [21, Sect. 3] and [10, Sects. 3-4]. Since we seek $H^s$ solution with $s \geq 0$, the desired result follows from here.

It remains to prove (3.20). On bundles over Riemann surfaces, we have that

$$F^{0,2}_a = \partial''_a \circ \partial''_a = 0.$$ (3.21)

For higher dimensional manifolds, condition (3.21) provides the means to construct a canonical holomorphization of $E$. The precise result is the theorem below, which gives (3.20), whereby completing the proof of Theorem 3.2(c).
The following result was obtained in [5,11]. [5] develops a more constructive approach when \( M \) is a closed Riemann surface.

**Theorem 3.5.** Given a \( C^\infty \) complex vector bundle \( E \) over \( M \) with connection \( \nabla_a \) such that \( \partial_a'' \circ \partial''_a = 0 \), there is a unique holomorphic vector bundle structure on \( E \) such that \( \partial''_a = \partial'' \).

### 3.4. Essential spectrum of the magnetic Laplacian

In this subsection, we compute the essential spectrum of the linearized operator \( -\Delta_a b \). Here we use a direct method of geometric decomposition, and we will use freely the standard definitions from Appendix A and results from Appendix B.

First, we identify \( \Sigma \) with a fundamental domain \( F_\Sigma \subset \mathbb{H} \) of \( \Gamma \), with the sides identified as in the proof of Theorem 2.2. Let

\[
\gamma_i := \begin{cases}
(0, -1) & (c_i \neq \infty), \\
(1, 0) & (c_i = \infty).
\end{cases}
\]  

(3.22)

Then \( \gamma_i \in SL(2, \mathbb{R}) \) and \( \gamma_i c_i = \infty \), where the action of \( \gamma_i \) on \( F_\Sigma \subset \mathbb{H} \) is the Möbius transform (2.2). Denote by \( \varphi_i \) the action of \( \gamma_i \) on \( F_\Sigma \subset \mathbb{H} \). Then

\[
\varphi_i : \begin{cases}
z \mapsto -\frac{1}{z - c_i} & (c_i \neq \infty), \\
z \mapsto z & (c_i = \infty).
\end{cases}
\]  

(3.23)

See Fig. 3.

Now, we decompose \( F_\Sigma \) into a compact connected set, \( U_0 \), and neighborhoods \( U_i \) of the cusps \( c_i \), in such a way that

1. \( U_i \cap U_j = \emptyset \) for \( 1 \leq i \neq j \leq m \), and \( U_0 := F_\Sigma \setminus \bigcup_{i=1}^m U_i \) is compact;
2. For \( i = 1, \ldots, m \), the map \( \varphi_i \) maps the domains \( U_i \) isometrically onto the half-cylinder

\[
Z_i := \{ z \in \mathbb{C} : \text{Im} z > s_i \}/\mathbb{Z},
\]  

(3.24)

for some \( s_i \gg 1 \).

So long as \( s_i \gg 1 \), such decomposition of \( F_\Sigma \) is easy to construct, see Fig. 2.

In the next two lemmas, we analyze the spectral property of \( -\Delta_a b \) in each domain \( U_i \) separately.
Fig. 2. Schematic diagram for the decomposition of a fundamental domain of $\Gamma(2)$ in $\mathbb{H}$ with three cusps $c_1 = 1, c_2 = 0, c_3 = \infty$.

Fig. 3. Schematic diagram illustrating the map $\varphi_2$ from (3.23) associated to cusps $c_2 = 0$.

Lemma 3.6. Let $-\Delta_a^b|_{U_i}$ be the restriction of $-\Delta_a^b$ on $L^2(U_i)$ with the Dirichlet boundary conditions on $\partial U_i$. Then

$$
\sigma_{\text{ess}}(-\Delta_a^b) = \bigcup_{i=1}^m \sigma_{\text{ess}}(-\Delta_a^b|_{U_i}).
$$

(3.25)

Proof. 1. First, note that the compact domain $U_0$ and the cuts do not contribute to the essential spectrum. Next, let $\{U'_0, \ldots, U'_m\}$ be an open covering $\Sigma$, such that

(1) $U'_i \supset U_i$ for every $i = 0, \ldots, m$;
(2) $U'_i \setminus U_i$ is bounded in $F_{\Sigma}$ for every $i = 0, \ldots, m$;
(3) $U'_i \cap U'_j = \emptyset$ for $1 \leq i \neq j \leq m$.

For Item 3 above, we note that for sufficiently small $U_i$, $i = 1, \ldots, m$, we can always choose a slightly larger $U'_i \supset U_i$ s.th. $U'_i$ are still mutually disjoint for $1 \leq i \neq j \leq m$ (see Fig. 2).

Let $\chi_i$ be a partition of unity on $\Sigma$ adapted to $U'_i$, such that

$$
\sum_{i=0}^m \chi_i^2 = 1,
$$

$$
\chi_i|_{U_i} \equiv 1, \quad \text{supp} \chi_i \subset U'_i \quad (i = 0, \ldots, m).
$$

(3.26)
Then the IMS formula (see [9]) gives

\[ -\Delta_{a^b} = \sum_{i=1}^{m} T_i + R, \quad T_i := \chi_i(-\Delta_{a^b})\chi_i, \]  
(3.28)

\[ R := \chi_0(-\Delta_{a^b})\chi_0 - \sum_{i=0}^{m} |\nabla\chi_i|^2. \]  
(3.29)

2. Since \( R \) is localized in a compact domain

\[ \text{supp } \chi_0 \cup \text{supp } \nabla\chi_1 \cup \cdots \cup \text{supp } \nabla\chi_m, \]

we expect that \(-\Delta_{a^b}\) and \(\sum T_i\) have the same essential spectrum. To show this, we will repeatedly use the fact that compact operators form a two-sided ideal among the bounded operators.

By Theorem 3.2, Part (b), the spectrum \(\sigma(-\Delta_{a^b}) \subset \mathbb{R}_{\geq 0}\). Hence \((-\Delta_{a^b} + 1)^{-1}\) is well-defined and bounded on \(L^2\).

Claim: \(R(-\Delta_{a^b} + 1)^{-2}\) is a compact operator.

Proof of the claim: Since for each \(i = 1, \ldots, m\), the set \(\text{supp } |\nabla\chi_i|\) is bounded in \(F_{\Sigma}\), it follows that the operators \(|\nabla\chi_i|^2(-\Delta_{a^b} + 1)^{-2}\) are all compact. Now, write

\[ \Delta_{a^b}\chi_0(-\Delta_{a^b} + 1)^{-2} = BK, \]
\[ B := \Delta_{a^b}(-\Delta_{a^b} + 1)^{-1}, \]
\[ K := \chi_0(-\Delta_{a^b} + 1)^{-1} - [\chi_0, \Delta_{a^b}](\Delta_{a^b} + 1)^{-2}. \]

The operator \(B\) is bounded on \(L^2\). Since \(\chi_0\) has its support bounded \(F_{\Sigma}\), the operator \(K\) is a compact operator. It follows that \(\Delta_{a^b}\chi_0(-\Delta_{a^b} + 1)^{-2} = BK\) is compact. This proves the claim.

3. We will use a modified version of Weyl’s Theorem for relative compact perturbations:

**Proposition 3.7** ([28] Chapt. XIII.4, Cor. 3). Let \(H\) and \(W\) be two self-adjoint operators s.th. \(W(H + 1)^{-n}\) is compact for some \(n \geq 1\). Then

\[ \sigma_{\text{ess}}(H + W) = \sigma_{\text{ess}}(H). \]

We use this proposition with \(W = R\) from (3.29) and \(H = -\Delta_{a^b}\). Then it follows from the claim proved in Step 2 and Proposition 3.7 that

\[ \sigma_{\text{ess}}(-\Delta_{a^b}) = \sigma_{\text{ess}}\left(\sum_{i=1}^{m} T_i\right). \]  
(3.30)
4. Next, for \( i = 1, \ldots, m \), we denote by \( -\Delta a^b \big|_{U'_i} \) the restriction on \( U'_i \) with Dirichlet boundary condition on \( \partial U'_i \).

Claim:

\[
\sigma_{\text{ess}} \left( \sum_{i=1}^{m} T_i \right) \subset \bigcup_{i=1}^{m} \sigma_{\text{ess}} ( -\Delta a^b \big|_{U'_i} ).
\] (3.31)

Proof of the claim: By construction, \( U'_i \cap U'_j = \emptyset \) for \( 1 \leq i \neq j \leq m \). Hence, one can show using Weyl’s criterion that

\[
\sigma_{\text{ess}} \left( \sum_{i=1}^{m} T_i \right) = \bigcup_{i=1}^{m} \sigma_{\text{ess}} ( T_i ).
\] (3.32)

Next, we show that for each \( i = 1, \ldots, m \),

\[
\sigma_{\text{ess}} ( T_i ) = \sigma_{\text{ess}} ( -\Delta a^b \big|_{U'_i} ).
\] (3.33)

Indeed, for each \( i = 1, \ldots, m \), we write

\[
T_i = -\Delta a^b \big|_{U'_i} - \Delta a^b \big|_{U'_i \setminus U_i} + K_i \quad \text{defined on the domain } D(\Delta a^b \big|_{U'_i}),
\] (3.34)

where \( K_i \)'s are defined by this relation. Explicitly,

\[
K_i = (-1 + \chi_i) \Delta a^b \chi_i - \Delta a^b (1 - \chi_i) \quad \text{acting on } U'_i.
\] (3.35)

Take \( \lambda \in \sigma_{\text{ess}}(T_i) \), and let \( \{u_n\} \) be a Weyl sequence for \( T_i \) and \( \lambda \), i.e.

\[
\|u_n\|_{L^2} = 1, \quad u_n \to 0 \quad \text{weakly in } L^2, \quad \|(T_i - \lambda)u_n\|_{L^2} \to 0.
\] (3.36)

Since \( \text{supp} \ u_n \subset U'_i \), each element in the sequence \( \{u_n\} \) satisfies the Dirichlet boundary condition on \( U'_i \).

By the choice of \( \chi_i \), see (3.27), the factor \( 1 - \chi_i \) vanishes away from the bounded set \( U'_i \setminus U_i \). Hence, it follows from (3.35)–(3.36) that

\[
K_i u_n \to 0 \quad \text{strongly in } L^2 \text{ as } n \to \infty \text{ for every } i = 1 \ldots, m.
\]

Similarly, we have

\[
-\Delta a^b \big|_{U'_i \setminus U_i} u_n \to 0 \quad \text{strongly in } L^2 \text{ as } n \to \infty \text{ for every } i = 1 \ldots, m.
\]

Hence, we conclude from expansion (3.34) and assumption (3.36) that

\[
(-\Delta a^b \big|_{U_i} - \lambda)u_n = (T_i - \lambda)u_n - (K_i - \Delta a^b \big|_{U'_i \setminus U_i})u_n \to 0 \text{ strongly for each } i = 1, \ldots, m.
\] (3.37)
By Weyl’s criterion, this implies \( \lambda \in \sigma_{\text{ess}}(-\Delta_{a^b}|_{U_i}) \). Thus (3.33) holds.

Eqs. (3.32)–(3.33) imply (3.31). This proves the claim.

5. Running the argument from Step 4 backwards gives the other inclusion, i.e.

\[
\sigma_{\text{ess}}\left( \sum T_i \right) = \sigma_{\text{ess}}\left( \sum -\Delta_{a^b}|_{U_i} \right).
\]  

(3.38)

Relations (3.30) and (3.38) together imply (3.25). \( \square \)

**Lemma 3.8.** For every \( i = 1, \ldots, m \), we have

\[
\sigma_{\text{ess}}(-\Delta_{a^b}|_{U_i}) \subset [1/4 + b^2, \infty).
\]

**Proof.** By construction, the map \( \varphi_i \) in (3.23) maps \( U_i \) isometrically onto the half-cylinder \( Z_i \) in (3.24). This transformation maps \( -\Delta_{a^b}|_{U_i} \) unitarily to another operator, say \( h_i \), acting on \( L^2(Z_i) \) with the Dirichlet boundary conditions on \( \partial Z_i = \{ z \in \mathbb{C} : \text{Im } z = s_i \}/\mathbb{Z} \). Thus, we have

\[
\sigma_{\text{ess}}(-\Delta_{a^b}|_{U_i}) \subset \sigma_{\text{ess}}(h_i).
\]  

(3.39)

It remains to compute the spectra of \( h_i \).

First, we note that by the invariance of the magnetic Laplacian, see (A.18), the spectral properties of \( -\Delta_{a^b} \) are not affected under any isometric transform \( \gamma \in SL(2, \mathbb{R}) \) acting on \( \mathbb{H} \). Hence, up to an initial gauge transform, we can assume \( h_i \) is of the form (3.7), now acting on the half-cylinder \( Z_i \).

Next, we pass from \( h_i \) to another unitary equivalent operator

\[
p := y^{-1}h_i y = -y(\partial_x^2 + \partial_y^2)y + 2iby\partial_x + b^2
\]

acting on \( L^2(Z_i, dx\,dy) \), with the Dirichlet boundary conditions on \( \partial Z_i \). Then

\[
\sigma_{\text{ess}}(h_i) = \sigma_{\text{ess}}(p).
\]  

(3.40)

Now, we apply the Fourier transform in \( x \) for the operator \( p \) to obtain the decomposition

\[
p = \bigoplus_{k \in \mathbb{Z}} p^b_k,
\]

where \( p^b_k \) is the operator

\[
p^b_k := y(-\partial_y^2 + k^2)y - 2byk + b^2
\]  

(3.41)

acting on \( L^2(\mathbb{R}_s) \), with \( \mathbb{R}_s := \{ y \in \mathbb{R} : y > s \} \).
For $k \neq 0$, (3.41) can be written as
\[ p_k^b := -y \partial_y^2 y + k^2 y^2 - 2byk + b^2 = -y \partial_y^2 y + k^2 (y - b/k)^2, \]
and therefore we have the estimate
\[ p_k^b \geq p_0^b \quad (k \neq 0). \tag{3.42} \]
By (3.42), we have
\[ \sigma(p_k^b) \subset [\sigma^b, \infty), \quad \sigma^b := \inf \sigma(p_0^b|_{R_s}), \]
and therefore
\[ \sigma(p) = \bigcup_{k \in \mathbb{Z}} \sigma(p_k^b|_{R_s}) \subset [\sigma^b, \infty). \tag{3.43} \]
Furthermore, by Hardy’s inequality
\[ -\partial_y^2|_{R_s} - \frac{1}{4} \frac{1}{y^2} \geq 0, \]
we have the lower bound
\[ p_0^b := -y \partial_y^2 y + b^2 \geq \frac{1}{4} + b^2. \]
This shows that $\sigma^b \geq 1/4 + b^2$, and therefore by (3.43), it follows that $\sigma(p) \subset [1/4 + b^2, \infty)$. This, together with relations (3.39)–(3.40), proves the lemma. \(\square\)

**Proof of Theorem 3.2, Part (d).** Lemmas 3.6 and 3.8 together imply
\[ \sigma_{\text{ess}}(-\Delta_{ab}) \subset [1/4 + b^2, \infty). \tag{3.44} \]
Here we note that as far as the bifurcation argument in Section 4 is concerned, a lower bound for the essential spectrum such as (3.44) suffices for our purposes.

To prove the inclusion
\[ \sigma_{\text{ess}}(-\Delta_{ab}) \supset [1/4 + b^2, \infty), \tag{3.45} \]
we compute the generalized eigenfunctions of $-\Delta_{ab}$ in the space of square-integrable equivariant functions. The details are delegated to the end of Appendix A.3.

Combining (3.44)–(3.45) proves Part (d) of Theorem 3.2. \(\square\)
Proof of Proposition 1.3. Consider the Ginzburg-Landau energy (1.1). The Hessian of $\mathcal{E}(\cdot, h_r)$ at $(0, a^{b_r})$ is given by

$$L := \text{diag}(-\Delta_{a^{b_r}} - \kappa^2, d^*d) : X^s \to X^{s-2},$$

which can be computed as in (4.13). By Theorem 3.2(b) and rescaling, $-\Delta_{a^{b_r}} \geq b_r$. Thus,

$$-\Delta_{a^{b_r}} - \kappa^2 \geq 0 \iff b_r - \kappa^2 \geq 0 \iff b_r \geq \kappa^2. \quad (3.47)$$

Next, by Proposition 3.1, $d^*d \geq 0$ on $\mathcal{H}^s$, $s \geq 2$. This, together with (3.47) and formula (3.46), implies

$$L \geq 0 \text{ if and only if } b_r \geq \kappa^2. \quad (3.48)$$

This implies Proposition 1.3. □

4. Bifurcation analysis

In the previous sections, we have found that, for the constant curvature connection $a^b$ on the unitary line bundle $E$ over a Riemann surface $\Sigma$, the ground state energy of the magnetic Laplacian $-\Delta_{a^b}$ equals to $b$. Moreover, the parameter $b$ is determined by the degree of $E$ and the signature of $\Sigma$.

Let $n = \text{deg } E$. Recall that $|\Sigma|$ is the area of $\Sigma$ w.r.t. the standard hyperbolic metric, and $b = 2\pi n/|\Sigma|$ is the critical value of the average field strength. In this section, we construct solutions to the rescaled GL equation, (2.15), for scaling parameter $r$ close to $b/\kappa^2$. This emerging solution corresponds to a second order phase transition as the applied field strength is lowered past the critical value $b = 2\pi n/|\Sigma|$.

We follow the general approach of [5]. Here we note two additional difficulties specific to our situation:

(a) If $b = 1/2$, e.g. when $\text{deg } E = 1$ and $\Sigma = \mathbb{H}/\Gamma(3)$ (in which case $|\Sigma| = 4\pi$ by the Gauss-Bonnet formula (2.10)), then ground state energy $b$ is embedded in the essential spectrum of $-\Delta_{a^b}$.

(b) In general, by Theorem 3.2, Part (c), the lowest eigenvalue of $-\Delta_{a^b}$ is not simple.

These lead to subtle technical issues due to bifurcations with higher multiplicity and bifurcation from essential spectrum. We sidestep those issues here by assuming that $\dim(-\Delta_{a^b} - b) = 1$ and $b \neq 1/2$ (see (1.14)), and explain how to overcome the second one in Section 7.

An explicit example satisfying (1.16) is given in (1.23). To see this, we note that by Theorem 3.2, condition (1.16) holds if $b \neq 1/2$ and the complex vector space
\(S_k(\Sigma)\) of cusp forms on \(\Sigma\) with weight \(k = 2b = 4\pi \deg E / |\Sigma|\) is one-dimensional. Let \(\Sigma = \mathbb{H}/\Gamma(N), N \geq 2\) be a non-compact Riemann surface, where \(\Gamma(N)\) is the principal congruence subgroup of level \(N\) (see Section 2.1 for definitions). Using classical dimension formulae found in [30, Secs. 1.6, 2.6], we find that if \(k\) is even, i.e. \(b \in \mathbb{Z}\), then

\[
\dim(S_k(\Sigma)) = \begin{cases} 
(k-1)(g_N - 1) + \frac{km_N}{2} & (k = 4, 6, \ldots), \\
g_N & (k = 2),
\end{cases}
\]

where \(g_N\) (resp. \(m_N\)) is the number of genera (resp. distinct cusps) of \(\Sigma\), given by (2.5)–(2.6). We seek integer solution \((k, N)\) to the equation

\[
\dim(S_k(\Sigma)) = 1. 
\]

For \(N \geq 3, k = 2\), (4.2) reduces to

\[
\frac{N - 6}{12} m_N = 0. 
\]

The only solution to (4.3) is \(N = 6\) and, for \(k = 2\), this gives \(\deg E = 12\).

### 4.1. Setup

Recall \(X^s = \mathcal{H}^s \times \tilde{\mathcal{H}}^s\) is the space of (section, connection)-pairs of order \(s\), defined in (1.13), and by convention \(X^0 = \mathcal{L}^2 \times \tilde{\mathcal{L}}^2\).

Define a nonlinear map \(F\) as

\[
F : X^s \times \mathbb{R} \rightarrow X^{s-2}, \quad (\psi, \alpha, r) \mapsto \left(-\Delta_{a^b+\alpha} \psi + \kappa^2 (|\psi|^2 - r) \psi, d^* d\alpha - PJ(\psi, \alpha)\right),
\]

where

\[
J(\psi, \alpha) := \text{Im}(\bar{\psi} \nabla_{a^b+\alpha} \psi) \text{ is the r.h.s. of the second equation in (GL)},
\]

\[
P : \tilde{\mathcal{H}}^s \rightarrow \tilde{\mathcal{H}}^s \text{ is the projection onto the space of co-closed 1-forms}.
\]

The map \(F\) is the central object in the remaining sections.

**Remark 11.** By definition (4.4) and direct computation, one finds that the rescaled equation (2.15) has solution \((\psi, a^b + \alpha, r)\) if and only if

\[
F(\psi, \alpha, r) = 0,
\]

and \(J(\psi, \alpha)\) is co-closed. The latter holds so long as the first GL equation in (2.15) holds. A proof of this fact is found in [5, Prop.5.1]. Hence it suffices to consider (4.7) only.
Next, by (3.1), eq. (4.7) has the trivial solution \((\psi, \alpha, r) = (0, 0, b/\kappa^2)\):

\[
F(0, 0, b/\kappa^2) = 0. \tag{4.8}
\]

Below, we seek non-trivial solutions to (4.7), i.e. solutions to (4.7) with variable curvature, in a small neighborhood around the trivial one. We write

\[
u := (\psi, \alpha) \in X^s. \tag{4.9}
\]

Recall also that \(K = \text{Null}(-\Delta_a - b)\), cf. (3.19), and the critical threshold \(\kappa_c(r)\) is defined in (1.12).

The main result of this section is the following:

**Proposition 4.1.** Suppose \(b \neq 1/2\), \(\dim K = 1\), and \(r > 0\) satisfies (cf. (1.15))

\[
0 < |\kappa^2 r - b| \ll 1. \tag{4.10}
\]

Then there exists a family of non-trivial solutions

\[
(u_s, r_s), \quad s \in \mathbb{R}, \quad 0 < |s| \ll 1,
\]

to (4.7) in a small neighborhood of \((0, b/\kappa^2)\) in \(X^k \times \mathbb{R}_{>0}, k \geq 2\).

The solution \(u_s\) is unique, up to a gauge symmetry, in a small neighborhood \(U \subset X^k\) around 0.

Furthermore, \(r_s\) has the following expansion,

\[
r_s = b/\kappa^2 + O(|s|^2), \tag{4.11}
\]

and similarly for derivatives.

The remainder of this section is devoted to the proof of Proposition 4.1.

First, we summarize the key properties of the map \(F(u, r)\) from (4.4) in the following proposition. Below, we identify \(H^s\) with a real Banach space through \(\psi \leftrightarrow (\text{Re} \psi, \text{Im} \psi)\) and view \(F : X^s \times \mathbb{R} \to X^{s-2}\) as a map between real Banach spaces. From now till the end of this paper, we assume \(s \geq 2\) in (4.4).

**Lemma 4.2.** We have

(1) The map \(F\) is \(C^2\) from the real Banach spaces \(X^s \times \mathbb{R}\) to \(X^{s-2}\), \(s \geq 2\);

(2) \(F\) has gauge symmetry in the sense that for every \(\theta \in \mathbb{R}\), we have

\[
[F(e^{i\theta}s, t, r)]_\psi = e^{i\theta}[F(s, t, r)]_\psi, \quad [F(e^{i\theta}s, t, r)]_\alpha = [F(s, t, r)]_\alpha; \tag{4.12}
\]

(3) \(F(u, r) = 0\) has the trivial branch of solution \((0, r), r > 0\).
Proof. The first claim follows from the fact that the map $F$ is a polynomial in $\psi, \bar{\psi}, \alpha$ and their derivatives up to the second order, together with the Sobolev inequalities and properties of fractional derivatives. The second claim follows directly from definition (4.4). The last claim is a rephrasing of (4.8). □

Next, consider the linearized operator of $F$ at the trivial branch $(u, r) = (0, r)$, given explicitly by (cf. (3.2)–(3.3))

$$d_u F(0, r) = \text{diag}(-\Delta_{ab} - \kappa^2 r, d^*d) : X^s \to X^{s-2}.$$  \hspace{1cm} (4.13)

For fixed $r$, this map is well-defined as the partial Fréchet derivative of $F$ at $u = 0$. Moreover, $d_u F(0, r)$ is continuous from $X^s \to X^{s-2}$ for $s \geq 2$.

The operator (4.13) enters the l.h.s. of the linearized equations (3.2)–(3.3). Put

$$N_r := \text{Null} \ d_u F(0, r). \hspace{1cm} (4.14)$$

By Proposition 3.1 and Corollary 3.4, we have

$$N \equiv N_{b/\kappa^2} = K \times \Omega, \hspace{1cm} (4.15)$$

where $\Omega$ is given in (3.5) and $K$ in (3.19).

The goal now is to show that a non-trivial branch of solution to (4.7) bifurcate from the space $N$ if $0 < |\kappa^2 r - b| \ll 1$.

4.2. Lyapunov-Schmidt reduction

In this subsection, we use the Lyapunov-Schmidt reduction to reduce the infinite-dimensional problem (4.7) to a finite-dimensional one.

Define a linear operator $Q : X^s \to X^s$ by

$$Q := \frac{1}{2\pi i} \oint R(z) \ dz \oplus Q'. \hspace{1cm} (4.16)$$

Here, the operator $Q' : \bar{\mathcal{H}}^s \to \bar{\mathcal{H}}^s$ is the orthogonal projection onto $\Omega$ defined in (3.5), which can be identified with the space of equivariant harmonic 1-forms. $R(z)$ is the resolvent of $-\Delta_{ab} - b$ at 0, which is well-defined if $b \neq 1/2$.

By construction, $Q$ given in (4.16) is an orthogonal projection onto the space $N \equiv N_{b/\kappa^2}$ from (4.15).

Define $v = Qu$, $w = Q^\perp u$, where $Q^\perp = 1 - Q$ is the projection onto $N^\perp \subset X^s$. Then the key equation (4.7) is equivalent to the following two equations,

$$QF(v + w, r) = 0, \hspace{1cm} (4.17)$$

$$Q^\perp F(v + w, r) = 0. \hspace{1cm} (4.18)$$
Lemma 4.3. Suppose $b \neq 1/2$. For every $(v, r) \in N \times \mathbb{R}_{>0}$ with

$$\|v\|_{X^s} + |\kappa^2 r - b| \ll 1,$$

eq. (4.18) has a unique solution $w = w(v, r) \in \text{ran } Q^\perp \subset X^s$ which satisfies

$$w = O(\|v\|_{X^s}^2), \quad (4.19)$$

and similarly for the derivatives of $w$.

Proof. 1. We first prove the existence and uniqueness of solution to (4.18). By Lemma 4.2 Parts (1), (3), and the Implicit Function Theorem, eq. (4.18) has a unique solution $w = w(v, r) \in N$ in a small neighborhood of $(v, r) = (0, b/\kappa^2)$ provided the partial Fréchet derivative $\frac{d_w Q^\perp}{d v} F(0, b/\kappa^2) : N^\perp \subset X^s \to X^{s-2}$ is invertible.

The partial Fréchet derivative $\frac{d_w Q^\perp}{d v} F$ evaluated at $(0, b/\kappa^2)$ is given by the diagonal operator

$$\frac{d_w Q^\perp}{d v} F(0, b/\kappa^2) = Q^\perp \text{diag}(-\Delta_{ab} - b, d*d). \quad (4.20)$$

We first consider the $\psi$-component of this operator.

By assumption, the ground state energy $b \neq 1/2$. Then by Theorem 3.2, Parts (c)-(d), $b$ is an isolated eigenvalue of $-\Delta_{ab}$ (see Remark 10). In this case, by elementary spectral theory (e.g. [19, Thm. 6.7]), the operator $-\Delta_{ab} - b$ is invertible on $K^\perp$.

It remains to consider the $\alpha$-component of the diagonal operator (4.20), namely $d^*d$. On the space of co-closed 1-forms, $d^*d$ equals to the Hodge Laplacian. By the lower bound on the essential spectrum of the Hodge Laplacian acting on 1-forms, proved in e.g. [12, Prop. 5.1]), we have $\inf \sigma_{es}(d^*d|_{\Omega^\perp}) \geq \frac{1}{4}$. Thus $d^*d|_{\Omega^\perp}$ is invertible.

It follows that $\frac{d_w Q^\perp}{d v} F(0, b/\kappa^2)$ is invertible on $N^\perp$ as desired.

2. To prove estimate (4.19), consider the equation (4.18) satisfied by $w$, which we rewrite as

$$Q^\perp F(v, r) + \tilde{L}_{v,r} w + N_{v,r}(w) = 0, \quad (4.21)$$

where $\tilde{L}_{v,r} := Q^\perp d_w F(v, r) Q^\perp$, and $N_{v,r}(w)$ is the nonlinearity defined through this relation.

The explicit expansion for $\tilde{L}_{0,r}$ follows from (4.13). By Proposition 3.1 and Theorem 3.2, the operator $\tilde{L}_{0,b/\kappa^2} \geq \delta Q^\perp$ for some $\delta > 0$ and is therefore invertible. By this fact and elementary perturbation theory, we find that $\tilde{L}_{v,r}$ is invertible for

$$\|v\|_{X^s} + |\kappa^2 r - b| \ll 1, \quad (4.22)$$

with
\[
\left\| L^{-1}_{v,r} \right\|_{X^{s-2} \to X^s} \lesssim 1. \tag{4.23}
\]

Direct computation using the definition (4.4) of \( F \) and the fact \( v \in N \) (see (4.14)–(4.15)) shows that

\[
\left\| F(v,r) \right\|_{\mathcal{H}^{s-2}} \lesssim \left\| v \right\|_{\mathcal{H}^s} (\left\| v \right\|_{\mathcal{H}^s} + \left\| v \right\|_{\mathcal{H}^s}^2), \tag{4.24}
\]

\[
\left\| F(v,r) \right\|_{\tilde{\mathcal{H}^{s-2}}} \lesssim \left\| v \right\|_{\mathcal{H}^s}^2 (1 + \left\| v \right\|_{\mathcal{H}^s}), \tag{4.25}
\]

\[
\left\| N_{v,r}(w) \right\|_{X^{s-2}} \lesssim \left\| w \right\|_{X^s}^2. \tag{4.26}
\]

Here and below, for a vector \( u \in X^s \) we write \( u = ([u]_\psi, [u]_\alpha) \).

Now we rewrite (4.21) as the fixed point equation

\[
w = -L^{-1}_{v,r}(Q \frac{1}{2} F(v,r) + N_{v,r}(w)). \tag{4.27}
\]

Applying (4.24)–(4.26) to (4.27) and using triangle inequality, we find

\[
\left\| [w]_\psi \right\|_{\mathcal{H}^s} \lesssim \left\| v \right\|_{\mathcal{H}^s} (\left\| v \right\|_{\mathcal{H}^s} + \left\| v \right\|_{\mathcal{H}^s}^2), \tag{4.28}
\]

\[
\left\| [w]_\alpha \right\|_{\tilde{\mathcal{H}^s}} \lesssim \left\| v \right\|_{\mathcal{H}^s}^2 (1 + \left\| v \right\|_{\mathcal{H}^s}), \tag{4.29}
\]

provided \( \left\| v \right\|_{X^s} + |\kappa^2 r - b| \ll 1 \). Estimates (4.28)–(4.29) give (4.19). To obtain estimates on the derivatives of \( w \), we differentiate (4.21) and then proceed with the resulting equation as above using estimates on \( N_{v,r}(w) \) and its derivatives. \( \square \)

4.3. The bifurcation equation

In this section we prove Proposition 4.1, by solving the bifurcation equation (4.17).

**Proof of Proposition 4.1.** 1. Using Lemma 4.2, we can now plug the solution \( w = w(v,r) \) to (4.18) back into (4.17), and get the bifurcation equation

\[
QF(v + w(v,r), r) = 0. \tag{4.30}
\]

Recall here \( Q \) is the orthogonal projection onto \( N := \text{Null} d_u F(0, b/\kappa^2) \) defined in (4.16).

Solving (4.30) in \( v \) and \( r \) amounts to solving (4.17)–(4.18), and therefore gives a solution \( u = v + w(v,r) \) to (4.7).

2. Let

\[
\xi \quad \text{and} \quad \eta_k, 1 \leq k \leq \dim \Omega \tag{4.31}
\]

be some orthogonal bases of \( K \) and \( \Omega \), defined respectively in (3.19) and (3.5), with \( \langle |\xi|^2 \rangle = \frac{1}{|S|} \int |\xi|^2 = 1 \).
Here $K \subset \mathcal{H}^s$ is a complex vector subspace with dimension 1 by assumption, while $\Omega \subset \mathcal{H}^s$ is real vector space with finite dimensions [12].

Let $s$ and $t = (t_1, \ldots, t_{\dim \Omega})$ respectively be the complex and real coefficients of vectors in $K$ and $\Omega$ w.r.t. the bases (4.31). For $v = (\phi, \gamma) \in K \times \Omega$, we will use the parametrization

$$\phi = \phi(s) \equiv s\xi \in K, \quad \gamma = \gamma(t) \equiv \sum_{k=1}^{\dim \Omega} t_k \eta_k \in \Omega. \quad (4.32)$$

3. Next, let $V$ be a sufficiently small neighborhood of $(s, t, r) = (0, 0, b/\kappa^2)$ in $\mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0}$ (note that hereafter, $s$ is real-valued in $V$). For every $(s, t, r) \in V$, we consider

$$u_{str} := (\psi_{str}, \alpha_{str}) = v_{st} + w_{str}, \quad (4.33)$$

where $v_{st} = (\phi(s), \gamma(t))$ is parametrized as (4.32) and $w_{str} = w(v_{st}, r)$ is the solution found in Lemma 4.3, satisfying, by estimates (4.28)–(4.29),

$$w_{str} = (O(|s|^3) + O(|s| |t|), O(|s|^2)), \quad (4.34)$$

and similarly for its derivatives in $s, t, r$ on $V$.

By Lemma 4.2, Part (1), we can expand

$$F(s, t, r) \equiv F(u_{str}, r)$$

as a $C^2$ map from $V \subset \mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0}$ to the real Banach space $X^{s-2}$, $s \geq 2$.

By the gauge symmetry (4.12) of $F$, it is not hard to check that any solution to the equation

$$F(s, t, r) = 0 \quad (4.35)$$

in $V$ gives rise to a circle of solutions to the equation $F(u, r) = 0$, with $u = v_{e^{i\theta}s,t} + w_{e^{i\theta}s,t,r}$ and any $\theta \in \mathbb{R}$. Thus, our goal now is to solve (4.35) in $V$.

To begin with, we derive an explicit expression of the map $F(s, t, r)$. Estimate (4.34), together with definition (4.33), yields

$$\psi_{str} = \phi(s) + O(|s|^3) + O(|s| |t|), \quad (4.36)$$
$$\alpha_{str} = \gamma(t) + O(|s|^2), \quad (4.37)$$

with corresponding estimates for derivatives in $s, t, r$ on $V$. Plugging (4.36)–(4.37) into (4.4), observing that the nonlinear term in the first component of (4.4) is cubic in $\psi$, and using that

$$(\Delta_{a^b + \alpha_{str}} - \Delta_{a^b})\xi = O(|t|) + O(|s|^2), \quad (4.38)$$
we obtain
\[ [F(s, t, r)]_\psi = s(-\Delta_{ab} - \kappa^2 r)\xi + O(|s| |t|) + O(|s|^3), \] (4.39)
with suitable estimates on the error terms.

Next, since \( J(\psi, \alpha) = \text{Im}(\psi \nabla_{\alpha^\ast + \alpha} \psi) \) (see (4.5)) is quadratic in \( \psi \), we find, with \( \psi_{\text{str}} \) as in (4.36),
\[ J(\psi_{\text{str}}, \alpha) = |s|^2 J(\xi, 0) - |s|^2 |\xi|^2 \alpha + O(|s|^2 |t|^2) + O(|s|^4 |t|) + O(|s|^6). \] (4.40)
For \( \alpha = \alpha_{\text{str}} \) as in (4.37), expansion (4.40) becomes
\[ J(\psi_{\text{str}}, \alpha_{\text{str}}) = |s|^2 J(\xi, 0) - \sum_k t_k |s|^2 |\xi|^2 \eta_k + O(|s|^2 |t|^2) + O(|s|^4). \] (4.41)
Plugging (4.41) into (4.4) yields
\[ [F(s, t, r)]_\alpha = d^* d\alpha_{\text{str}} - |s|^2 J(\xi, 0) \]
\[ - \sum_k t_k |s|^2 |\xi|^2 \eta_k + O(|s|^2 |t|^2) + O(|s|^4), \] (4.42)
with corresponding estimates for derivatives in \( s, t, r \) on \( V \). Moreover, in addition to using (4.36)–(4.37), we can also eliminate the even order terms in expansion (4.39) and odd order terms in (4.42), counting \( |s| \) and \( |t| \) as of the orders 1 and 2, respectively, by the gauge invariance (4.12).

4. Next, let \( G(s, t, r) = QF(u_{\text{str}}, r) \), where \( u_{\text{str}} \) is defined in (4.33) and write \( G = ([G]_\psi, [G]_\alpha) \). Our goal now is to derive a more explicit formula for \( G \) in terms of the parametrization (4.32).

Firstly, by Lemma 4.2 part (1) and the chain rule for Fréchet derivatives, \( G(s, t, r) \) is a \( C^2 \) map from \( V \subset \mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0} \) to the finite-dimensional real vector space \( N \subset X^s_{-2}, s \geq 2 \). Secondly, by definition, we have \(-\Delta_{ab} - b|_{K} = 0 \). Lastly, so long as \( \xi \) solves (3.2), i.e. \( \xi \in K \), by [5, Props. 5.1, 5.3], the supercurrent \( J(\xi, 0) \) is both co-closed and co-exact. By the characterization of the space \( \Omega \) in Proposition 3.1, this implies \( Q'd^* d\alpha_{\text{str}} = Q' J(\xi, 0) = 0 \).

Hence, using these facts, we can rewrite (4.39) and (4.42) as
\[ [G(s, t, r)]_\psi = s (\kappa^2 r - b) \xi + O(|s| |t|) + O(|s|^3), \] (4.43)
\[ [G(s, t, r)]_\alpha = |s|^2 \sum_k t_k Q'(|\xi|^2 \eta_k) + O(|s|^2 |t|^2) + O(|s|^4), \] (4.44)
with corresponding estimates for derivatives in \( s, t, r \).
5. Now we investigate the equations

\begin{align}
\frac{1}{s} [G(s, t, r)]_\psi &= 0, \quad (4.45) \\
\frac{1}{|s|^2} [G(s, t, r)]_\alpha &= 0. \quad (4.46)
\end{align}

Clearly, any solution to (4.45)–(4.46) is also a solution to the bifurcation equation (4.30). In view of the parametrization (4.31), taking inner product of (4.45)–(4.46) w.r.t. \( \xi \) and \( \eta_k \), we can write these equations as

\begin{align}
\kappa^2 r - b + R_\psi(s, t, r) &= 0, \quad (4.47) \\
\sum_{i=1}^{\dim \Omega} B_{kli} t_i + R_{\alpha,k}(s, t, r) &= 0, \quad k = 1, \ldots, \dim \Omega. \quad (4.48)
\end{align}

Here, the remainders \( R_\psi \) and \( R_{\alpha,k} \) are C² functions from \( V \) to \( \mathbb{C} \) and \( \mathbb{R}^{\dim \Omega} \), respectively, since \( G \) is a C² map from \( V \) to \( N \subset X^{s-2}, s \geq 2 \). The remainders satisfy the estimates

\begin{align}
R_\psi(s, t, r) &= O(|t|) + O(|s|^2), \quad (4.49) \\
R_{\alpha,k}(s, t, r) &= O(|t|^2) + O(|s|^2), \quad (4.50)
\end{align}

and similarly for their derivatives, and

\begin{equation}
B_{kli} := \langle \eta_k, \xi \eta_l \rangle_{L^2}. \quad (4.51)
\end{equation}

To obtain (4.48), we use self-adjointness of the orthogonal projection \( Q' : \mathcal{H}^s \to \mathcal{H}^s \), and the fact that \( \eta_k \in \Omega = \text{ran} Q' \) for every \( k \).

To summarize, we are now left with solving (4.47)–(4.48) in \( V \subset \mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0} \), which amounts to solving the bifurcation equation (4.30).

Now, we view \( s \) as a parameter and solve the algebraic system (4.47)–(4.48) for \( (t, r) \in \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0} \).

**Lemma 4.4.** For every \( s \in \mathbb{R} \) with \( |s| \ll 1 \), there exists a unique C² solution \( (t, r) \) to (4.47)–(4.48) in a small neighborhood of \( (0, b/\kappa^2) \in \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0} \).

Moreover, this solution satisfies

\begin{equation}
t = O(|s|^2), \quad r = b \frac{b}{\kappa^2} + O(|s|^2), \quad \text{as } |s| \to 0, \quad (4.52)
\end{equation}

and similarly for their derivatives.

This proposition is proved in Section 5.
6. Lemma 4.4, together with Lemma 4.3, implies that there exists a family
\[(u_s, r_s), \quad s \in \mathbb{R}, \quad |s| \ll 1,\] (4.53)
with \(r_s\) satisfying (4.52), that uniquely solves (4.7) in a small neighborhood of the trivial solution \((u, r) = (0, b/\kappa^2)\).

This completes the proof of Proposition 4.1. \(\Box\)

5. Solvability of the bifurcation equation

In this section, we solve the bifurcation equations (4.47)–(4.48) by proving Lemma 4.4.

**Proof of Lemma 4.4.** 1. We first solve (4.48). This equation has the trivial solution \((s, t, r) = (0, 0, b/\kappa^2)\) (see (4.8)), and the l.h.s. of (4.48) is \(C^2\) in a small neighborhood \(V \subset \mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}_{>0}\) around this zero.

We differentiate w.r.t. \(t\) and use and the remainder estimate (4.50) to find that the Jacobian matrix of the l.h.s. of (4.48) at this zero is given by

\[B = [B_{kl}],\] (5.1)

where \(B_{kl}\) is as in (4.51) and \(1 \leq k, l \leq \dim \Omega\).

Let \(t = (t_1, \ldots, t_{\dim \Omega})\) be an arbitrary non-zero real vector. We compute, for any \(\omega = \sum_{i=1}^{\dim \Omega} t_i \eta_i \in \Omega \setminus \{0\},\)

\[\langle Bt, t \rangle = \sum_{l,k=1}^{\dim \Omega} \langle \eta_k, |\xi|^2 \eta_l \rangle_{L^2} t_l t_k = \langle \omega, |\xi|^2 \omega \rangle_{L^2} > 0.\] (5.2)

Thus the matrix \(B\) in (5.1) is positive-definite and therefore invertible.

By construction, (4.48) is a real system of \(C^2\) equations posed on a real vector space, \(\mathbb{R} \times \mathbb{R}^{\dim \Omega} \times \mathbb{R}\). By the gauge symmetry (4.12), the functions \(R_{\alpha,k}\) depend only on \(|s|\) for all \(k\). Hence, by the invertibility of \(B\) and the Implicit Function Theorem, eq. (4.48) has a unique \(C^2\) solution \(t = (t_1, \ldots, t_{\dim \Omega})\) in a small neighborhood around \((s, r) = (0, b/\kappa^2)\) in \(\mathbb{R} \times \mathbb{R}_{>0}\), satisfying

\[t(s, r) = O(|s|^2),\] (5.3)

and similarly for its derivatives.

We note that, by the second relation in (4.12), the solution \(t\) depends only on \(|s|\) and \(r\).

2. Next, we define the l.h.s. of (4.47) after plugging the solution \(t\) found in Step 1 to (4.48) as
\[
\tilde{G}(s, r) = \kappa^2 r - b + \tilde{R}_\psi(s, r),
\]  
(5.4)

where \( \tilde{R}_\psi(s, r) := R_\psi(s, t(s, r), r) \) satisfies, by (4.49) and (5.3),
\[
\tilde{R}_\psi(s, r) = O(|s|^2),
\]  
(5.5)

and similarly for its derivatives.

We look for solution in a small neighborhood of \((s, r) = (0, b/\kappa^2)\) to the equation
\[
\tilde{G}(s, r) = 0.
\]  
(5.6)

By (4.8) and definition (5.4), eq. (5.6) has the trivial solution \((s, r) = (0, b/\kappa^2)\). Moreover, by the \(C^2\) regularity of the solution \(t = t(s, r)\) found in Step 1 above, the map \(\tilde{G}\) is \(C^2\) around this zero, and \(\partial_r \tilde{G}(0, b/\kappa^2) = \kappa^2 > 0\) by estimate (5.3) and similarly for \(\partial_s \tilde{R}_\psi\).

Thus, by the Implicit Function Theorem, we obtain a unique solution \(r = r(s)\) to (5.6) around \(s = 0\), satisfying
\[
r(s) = \frac{b}{\kappa^2} + O(|s|^2),
\]  
(5.7)

and similarly for its derivatives. This proves Lemma 4.4. \(\Box\)

6. Precise asymptotics of the non-trivial branch

In this section we finish our proof of Theorem 1.2.

Proposition 6.1. Let the conditions of Proposition 4.1 hold and let
\[
(u_s, r_s), \quad |s| \ll 1
\]
be the nontrivial branch of solution to (4.7), constructed in Proposition 4.1. Then, for \(u_s = (\psi_s, \alpha_s)\), we have
\[
\psi_s = \phi(s) + O(|s|^3),
\]  
(6.1)
\[
\alpha_s = \gamma(t(s)) + O(|s|^4).
\]  
(6.2)

Here \(\phi\) and \(\gamma\) are as in (4.32) and satisfy
\[
(-\Delta_u - b)\phi = 0,
\]  
(6.3)
\[
d\gamma = \frac{1}{2} \left( 1 - |\phi|^2 \right).
\]  
(6.4)

Moreover, we can take \(s \in \mathbb{R}_{\geq 0}\) and if, in addition, \(r\) satisfies (cf. (1.15))
\[
(\kappa - \sqrt{b/r})(\kappa - \kappa_c) > 0
\]  
(6.5)
where $\kappa_c = \kappa_c(1)$ is given by (1.12), then, the equation $r = r(s)$ can be solved for $s$ to obtain $s = s(r)$ with

$$s^2 = \frac{\kappa^2 r - b}{(\kappa^2 - \frac{1}{2})\beta + \frac{1}{2}} + O\left(\left(\frac{\kappa^2 r - b}{(\kappa^2 - \frac{1}{2})\beta + \frac{1}{2}}\right)^2\right), \quad (6.6)$$

where $\beta = \beta(1)$ is the Abrikosov function given by (1.11).

**Proof.** The solution branch $u_s$ from (4.53) is given by

$$u_s = v_s + w(v_s, r_s), \quad (6.7)$$

where

$v_s = (\phi(s), \gamma(t(s)))$ with $\phi$ and $\gamma$ given in (4.32), $t(s) = t(s, r(s))$ and $r_s = r(s)$ given in (5.3) and (5.7), and with $w$ satisfying (4.34). This implies (6.1)–(6.2). Eqs. (6.3)–(6.4) are implicit in the proof of Proposition 4.1. These parts are analogous to the proofs of [5, Prop.5.5, eqs. (5.26)–(5.27)].

It remains to prove (6.6). Observe that by gauge symmetry (4.12), $\tilde{R}_\psi(s, r)$ in (5.4) depends on $s$ through $|s|$. Thus, we can assume $s \in \mathbb{R}$. Again, by the reflection symmetry, which follows from gauge symmetry (4.12), the expansion of $r$ in $s$ contains only even powers. Thus, we have

$$r = \frac{b}{\kappa^2} + Rs^2 + O(s^4), \quad (6.8)$$

for some $R \in \mathbb{R}$, and with corresponding estimates on derivatives in the remainder.

Take $\xi \in K$ as in (4.31). Using the expansion (6.1)–(6.2), together with the (6.8), we find

$$\langle \xi, (-\Delta_{a^b + \alpha} - \kappa^2 r) \psi \rangle = \langle \xi, (-\Delta_{a^b} - b) \psi \rangle + s^2(\langle \xi, 2i\eta \cdot \nabla_{a^b} \psi \rangle - \kappa^2 R \langle \xi, \psi \rangle) + O(s^4) \quad (6.9)$$

Here, $\eta := s^{-2}(t(s))$, and we move to the last line using the fact that $-\Delta_{a^b}$ is self-adjoint, and the equation (6.3) satisfied by $\xi$.

Since, by construction, the pair $(\psi, a^b + \alpha)$ solves the rescaled Ginzburg-Landau equations (2.15), we have

$$(-\Delta_{a^b + \alpha} - \kappa^2 r) \psi + \kappa^2 \psi |^2 + \kappa^2 |\psi|^2 = 0. \quad (6.10)$$

Then taking the inner product w.r.t. $s^{-2} \xi$ on both sides of (6.10), using the expansion (6.9), and then taking $s \to 0$, we find

$$\langle \xi, 2i\eta \cdot \nabla_{a^b} \xi \rangle - \kappa^2 R \|\xi\|^2_{L^2} + \kappa^2 \|\xi\|^4_{L^4} = 0. \quad (6.11)$$
Next, recalling the definition $\langle f \rangle = \frac{1}{|\Sigma|} \int_{\Sigma} f$, we have the identity (see the equation after [5, eq. (5.32)])

$$\frac{1}{|\Sigma|} \langle \xi, 2i\eta \cdot \nabla_{a^b} \xi \rangle = \frac{1}{2} \left( \langle |\xi|^2 \rangle^2 - \langle |\xi|^4 \rangle \right). \quad (6.12)$$

Substituting (6.12) into (6.11) and solving for $R$, we find

$$R = \frac{1}{2\kappa^2} + \left( 1 - \frac{1}{2\kappa^2} \right) \langle |\xi|^4 \rangle \text{ since } \langle |\xi|^2 \rangle = 1. \quad (6.13)$$

Now, rearranging (6.8), we find

$$s^2 = \frac{\kappa^2 r - b}{\kappa^2 R} + O(s^4), \quad (6.14)$$

which, together with (6.13) and the fact that $\beta = \langle |\xi|^4 \rangle$ in the non-degenerate case with $\dim K = 1$, yields

$$s^2 = \frac{\kappa^2 r - b}{(\kappa^2 - \frac{1}{2}) \langle |\xi|^4 \rangle + \frac{1}{2}} + O(s^4). \quad (6.15)$$

With this, the equation $r = r(s^2)$ can be solved for $s^2$ to yield $s^2 = s^2(r)$, with (6.6).

Note that, since $(\kappa^2 - \frac{1}{2})\beta + \frac{1}{2} = \beta(\kappa^2 - \kappa^2)$ by definitions (1.11)–(1.12), condition (6.5) ensures that the leading term in (6.15) is strictly positive and so (6.6) is self-consistent for small $s$ (cf. Remark 1).

This completes the proof of Proposition 6.1. \Box

**Remark 12.** It follows from (6.6) and condition (1.15) that $r(s)$ is locally parabolic; i.e., $r \sim Cs^2$. Hence there can be at most two branches, $\pm s(r)$. However, since these would be related by a gauge transformation, up to gauge change, there will locally near $(r, s) = (b/\kappa^2, 0)$ be only one branch $s(r)$.

**Proof of Theorem 1.2.** For every $s = s(r)$ with $r$ satisfying (1.15), the solution $u_s \equiv (\psi_s, a_s)$ constructed in Proposition 4.1 gives a solution to the rescaled Ginzburg-Landau equation (2.15) on $(\Sigma, h_1)$ as $(\psi_s, a^b + \alpha_s)$. By relation (3.19), eq. (6.3) implies that $\xi$ is a holomorphic section of $E$ corresponding to $a^b$.

Undoing the rescaling (2.12), the asymptotics (1.18)–(1.20) follow from (5.3) and (6.1)–(6.4), and the expansion (1.21) follows from (6.6). This completes the proof of Theorem 1.2. \Box

**Proof of Corollary 1.6.** We first prove that, with $(\psi, a) \equiv (\psi_{s(r)}, a_{s(r)})$ solving (2.15) and rescaled GL energy functional $E_r(\psi, a)$ from (2.11), we have
\[ \mathcal{E}_r(\psi, a) = \mathcal{E}_r(0, a^b) - \frac{|\Sigma|}{4} \frac{|\kappa^2 r - b|^2}{(k^2 - \frac{1}{2}) \langle |\xi|^2 \rangle + \frac{1}{2}} + O(|\kappa^2 r - b|^3), \]  

(6.16)

for \( 0 < |\kappa^2 r - b| \ll 1 \). Here \( \xi \in K \equiv \text{Null}(\Delta_{a^b} - b) \), \( \langle |\xi|^2 \rangle = 1 \) is as in (4.31).

First, taking the inner product of the first equation in (2.15) with \( \psi \), and then integrating by parts, we find

\[ \int |\nabla_\omega \psi| = \int \kappa^2 (|\psi|^2 - |\psi|^4). \]

Substituting this into the rescaled GL energy (2.11), we get

\[ \mathcal{E}_r(\psi, a) = \frac{1}{2} \left( \frac{\kappa^2 r^2}{2} |\Sigma| + \|da\|_{L^2}^2 - \frac{\kappa^2}{2} \|\psi\|_{L^4}^4 \right). \]

(6.17)

This gives

\[ \mathcal{E}_r(0, a^b) = \frac{1}{2} \left( \frac{\kappa^2 r^2}{2} + b^2 \right) |\Sigma| \quad \text{with} \quad b = \frac{2\pi n}{|\Sigma|}. \]

(6.18)

Next, we choose \( \eta := |s|^{-2} \gamma(t(s)) \) according to expansion (6.2). Then \( \eta \in \Omega \) by construction and therefore \( \langle dq, d\beta \rangle = 0 \) for any 1-form \( \beta \), see Proposition 3.1. Using this, the expansion \( a = a^b + |s|^2 \eta + O(|s|^4) \) (see (6.2)), together with the fact that \( da^b = b\omega \), we find

\[ \|da\|_{L^2}^2 = \|da^b\|_{L^2}^2 + 2 |s|^2 \langle da^b, d\eta \rangle + |s|^4 \|d\eta\|_{L^2}^2 + O(|s|^6) \]

\[ = b^2 |\Sigma| + |s|^4 \|d\eta\|_{L^2}^2 + O(|s|^6). \]

(6.19)

With the choice (4.31) for \( \xi \), expansion (6.1) becomes \( \psi = s\xi + O(|s|^3) \). Plugging this into formula

\[ \frac{1}{|\Sigma|} \|d\eta\|_{L^2}^2 = - \frac{1}{4} (\langle |\xi|^2 \rangle^2 - \langle |\xi|^4 \rangle), \]

(6.20)

proved in [5, Prop. 5.5, eq. (5.32)], and using that \( \langle |\xi|^2 \rangle \equiv \frac{1}{|\Sigma|} \int |\xi|^2 = 1 \), we find

\[ |s|^4 \|d\eta\|_{L^2}^2 - \frac{\kappa^2}{2} \|\psi\|_{L^4}^4 = \frac{|\Sigma|}{2} |s|^4 \left( \frac{1}{2} - \kappa^2 \right) \langle |\xi|^4 \rangle - \frac{1}{2} + O(|s|^6). \]

(6.21)

Plugging (6.19)–(6.21) into (6.17), we conclude that

\[ \mathcal{E}_r(\psi, a) = \mathcal{E}_r(0, a^b) - \frac{|\Sigma|}{4} |s|^4 \left( \frac{\kappa^2 - 1}{2} \langle |\xi|^4 \rangle + \frac{1}{2} \right) + O(|s|^6). \]

(6.22)

This, together with expansion (6.15) for \( s \), gives (6.16).

By definition (1.11) and the assumption \( \dim K = 1 \), we have \( \beta = \langle |\xi|^4 \rangle \). This, together with (6.16), the rescaling (2.14), and the relation \( |\Sigma| = r^{-1} |\Sigma| \), gives (1.27). \( \square \)
7. Extension to the case dim $K > 1$

In this section, we illustrate how to drop the second condition in (1.14) and to extend the main existence results in Section 1 to the degenerate case, with $D = \dim K > 1$. We also show how to extend energy estimates to this more general setting (in Section 7.2).

7.1. Existence theory

We now make some remarks on the possibility of solving the bifurcation equations (4.7) in the case when $D > 1$ where $D$ is the complex dimension of $K$. The Lyapunov-Schmidt reduction detailed in section 4.2 carries over mutatis mutandis for $D > 1$. The essential change comes in section 4.3, as we will now choose a hermitian orthonormal basis $\xi_j \leq j \leq D$ for $K$. Define $v = v_{st} = (\phi(s), \gamma(t))$, where $s = (s_1, \ldots, s_D)$, $t = (t_1, \ldots, t_{\dim \Omega})$, $\phi(s) = \sum_{j=1}^{D} s_j \xi_j$, and $\gamma(t) = \sum_{k=1}^{\dim \Omega} t_k \eta_k$ as in (4.32).

Taking inner products of the bifurcation equation (4.30) respectively with $\xi_j$ and $\eta_k$, this equation can be rewritten in terms of the coordinates $s, t, r$ as

\begin{equation}
(k^2 r - b)s_j + H_{\psi,j}(s,t,r) = 0, \quad j = 1, \ldots, D, \tag{7.1}
\end{equation}

\begin{equation}
\sum_{l=1}^{\dim \Omega} B_{kl}(s)t_l + H_{\alpha,k}(s,t,r) = 0, \quad k = 1, \ldots, \dim \Omega, \tag{7.2}
\end{equation}

where $B_{k,l}(s) = (\eta_k, |\phi(s)|^2 \eta_l)_{L^2}$. The remainders satisfy the estimates

\begin{equation}
H_{\psi,j}(s,t,r) = O(|s| |t|) + O(|s|^3), \tag{7.3}
\end{equation}

\begin{equation}
H_{\alpha,k}(s,t,r) = O(|s|^2 |t|^2) + O(|s|^4), \tag{7.4}
\end{equation}

and similarly for their derivatives.

The solution of (7.1) is based on the following classical result of Krasnoselski that reduces, under appropriate conditions, a vector bifurcation problem to a scalar one.

**Theorem 7.1 ([6, Chapt. 4]).** Let $E : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be $C^2$ of the form

\begin{equation}
E(\ell, s) = \ell s + H(\ell, s) \tag{7.5}
\end{equation}

where $s \in \mathbb{R}^n$, $\ell \in \mathbb{R}$ and $H(\ell, s) = O(|s|^2)$ as $|s| \to 0$ uniformly in a neighborhood of $\ell = 0$. Assume that $n$ is odd; then, $(\ell, s) = (0, 0)$ is a bifurcation point of $E(\ell, s)$; i.e., there are solutions $(\ell, s)$ with $s \neq 0$ of the equation $E(\ell, s) = 0$ in every neighborhood of $(\ell, s) = (0, 0)$.

**Remark 13.** We note that (7.5) is the standard normal form for a nonlinear problem whose linearization has $\ell = 0$ as an isolated eigenvalue of multiplicity $n \geq 1$. 

We defer the elegant proof of Theorem 7.1 to an Appendix D, and turn to applying it to our problem.

The first step is to consider solutions of (7.2). We find that

**Lemma 7.2.** Eq. (7.2) has a unique solution \( t = t(r, s) \) in a small neighborhood of \( (b/\kappa^2, 0) \), such that \( st(s, r) \) is \( C^2 \).

We momentarily defer the proof of this lemma in order to apply it to solving the full bifurcation problem. As a consequence of the lemma, insertion of \( t(s, r) \) into the higher order terms (7.3) of (7.1) results in a bifurcation equation that is \( C^2 \) which will enable us to satisfy regularity conditions of Theorem 7.1. We now want to rewrite (7.1) in terms of real variables by setting \( s_j = x_j + iy_j \). We also apply a constant gauge transformation to reduce \( s_1 \) to \( s_1 = x_1 \). With this gauge fix in place, we take \( s \) to denote the coordinates, \( s = (x_1, x_2, y_2, \ldots, x_D, y_D) \), which coordinatizes \( \mathbb{R}^{2D-1} \). So in our application \( n = 2D - 1 \) which is odd so that condition of the theorem will be satisfied. With this gauge fix in place, (7.1) is precisely in the normal form (7.5). Indeed, setting \( \ell = \kappa^2 r - b \) and substituting the solution \( t = t(r, s) \) found in Lemma 7.2 to (7.1), the equation takes the vector form

\[
\tilde{G}(\ell, s) = \ell s + H(s, r) = 0 \tag{7.6}
\]

where \( H(s, r) = O(|s|^2) \). Since \( \tilde{G}(\ell, s) \) is \( C^2 \), all conditions of Theorem 7.1 are satisfied. This establishes the existence of a supercritical zero. Given that, an application of the vector implicit function theorem yields the existence of a smooth branch of zeroes extending from the bifurcation point to the supercritical zero. Hence, Proposition 4.1 extends to be true for \( D > 1 \).

As a brief illustration we note how Theorem 7.1, for the case \( D = 1 \) aligns with our earlier proof of Lemma 4.4. From the proof of Theorem 7.1 given in Appendix D, when \( n = 1 \), the constraint simply reduces to \( s \tilde{E}(s) = 0 \). So for each \( s \neq 0 \), \( \tilde{E}(s) = 0 \). It follows that the unique \( C^2 \) curve \( \ell(s) \) found in the proof of Theorem 7.1 determines the branch of supercritical solutions to the bifurcation equation. Applying this to (7.6), this is the branch \( \kappa^2 r(s) - b \), determining a unique \( r(s) \) that satisfies

\[
r(s) = \frac{b}{\kappa^2} + O(|s|^2). \tag{7.7}
\]

We now turn to the proof of the Lemma 7.2

**Proof of Lemma 7.2.** We consider the quadratic form associated to the leading terms of (7.2),

\[
\Phi(s, t) = \langle B(s)t, t \rangle
\]
\[
\begin{align*}
\dim \Omega &= \sum_{k, \ell = 1}^{\dim \Omega} t_k t_\ell B_{k\ell}(s) \\
&= \langle \omega, |\phi(s)|^2 \omega \rangle_{L^2}
\end{align*}
\] (7.8) (7.9)

It is manifest from this that \( \Phi \) is real valued and, away from \( s = 0 \), defined, differentiable and positive definite as a quadratic form in \( t \).

To understand \( \Phi \) in a neighborhood of \( s = 0 \) we do a (real) algebro-geometric blow-up of \( s \)-space at the origin. This construction replaces the origin by a compact hypersurface diffeomorphic to the projective space of all local directions through the origin. More precisely, in our situation, the blow-up is defined as follows. Let \( n = 2D - 1 \) and consider the product space \( \mathbb{R}^n \times \mathbb{P}^{n-1} \) with \( \mathbb{R}^n \) coordinatized by the affine coordinates \( \tilde{w} = (\tilde{x}, \tilde{y}) \) and \( \mathbb{P}^{n-1} \) coordinatized by the homogeneous coordinates \( \tilde{z} = [z_1, \ldots, z_n] \). The blow-up of \( \mathbb{R}^n \) at the origin, which we’ll denote by \( M \), is the closed subset of \( \mathbb{R}^n \times \mathbb{P}^{n-1} \) defined by the equations \( w_i z_j - w_j z_i = 0 \) \( i, j = 1, \ldots, n \). One has a natural map \( \pi : M \to \mathbb{R}^n \) induced by projection onto the first factor. It is fairly straightforward to check that away from \( \tilde{w} = 0 \), \( \pi \) is a diffeomorphism. However, \( \pi^{-1}(0) \simeq \mathbb{P}^{n-1} \); it consists of all points of the form \( 0 \times [z_1, \ldots, z_n] \). Now to see that points of \( \pi^{-1}(0) \) are in 1:1 correspondence with \( \mathbb{R}^n \), note that a line \( L \) through the origin is parametrically given by \( w_i = c_i \sigma \) where the \( c_i \) are not all zero. Then consider the lift of this line to \( \tilde{L} = \pi^{-1}(L - 0) \) in \( M - \pi^{-1}(0) \) whose parametrization is \( w_i = c_i \sigma, z_i = c_i \sigma \). However, since \( z_i \) are homogeneous coordinates, one may as well take this parametrization to be \( w_i = c_i \sigma, z_i = c_i \). These equations are well-defined for \( \sigma = 0 \), defining the closure of \( \tilde{L} \) in \( M \) which meets \( \pi^{-1}(0) \) in the point \( [c_1, \ldots, c_n] \in \mathbb{P}^{n-1} \). This defines the mapping \( L \to [c_1, \ldots, c_n] \) that gives the 1:1 correspondence between lines through the origin and points of \( \pi^{-1}(0) \). For further explanation and details we refer the reader to [18] on which the above is based.

Applying this to \( \Phi \), we set \( A_{jm} = \int_{\xi \in \mathbb{C}^2} \xi_m \bar{\xi}_j \omega \wedge *\omega = \mu_{jm} + i \nu_{jm} \). Then one has a representation of \( \Phi \) as
\[
\Phi(s, t) = \sum_{1 \leq j \leq m \leq D} \left( A_{jm} s_j \bar{s}_m + \bar{A}_{jm} \bar{s}_j s_m \right)
= 2 \mu_{jm}(x_j x_m + y_j y_m) - 2 \nu_{jm}(y_j x_m - x_j y_m).
\] (7.10)

To lift \( \Phi \) to \( M \) we set \( \bar{c} = [a_1, a_2, b_2, \ldots, a_D, b_D] \). Applying the corresponding parametrization in \( \Phi \) one has
\[
\Phi(s, t) = 2 \sigma^2 \left( \mu_{jm}(a_j a_m + b_j b_m) - \nu_{jm}(b_j a_m - a_j b_m) \right).
\] (7.11)

Under the blow-up change of coordinates to \((\bar{c}, \sigma, r)\), the higher order terms scale similarly but involve powers of \( \sigma \) greater than 2. Consequently, the terms in \( 1/\sigma^2 \) times the quadratic form associated to (7.2) are polynomial and so these equations are clearly \( C^2 \) in the blow-up coordinates. They have a leading term that is independent of \( \sigma \). Hence the
positivity seen in (7.9) is inherited by \((1/\sigma^2)\Phi(s,t)\) from (7.11). But the latter is defined just in terms of \(\hat{c}\) which are coordinates on a compact projective space. So by continuity, 
\((1/\sigma^2)\Phi(s,t)\) realizes its infimum at some point in \(\mathbb{P}^{n-1}\). But this infimum is bounded away from zero. So \(B(s)\) is globally invertible and, by the implicit function theorem, \(t\) is smoothly defined in the blow-up coordinates and \(t\) is \(O(\sigma^2)\) in these coordinates and so vanishes in a continuous fashion as \(\sigma \to 0\). Hence \(t\) on the blowup \(M\) can be pushed forward under \(\pi\) to \(t(s,r)\) in the original \(s\)-space. Similarly for the first derivatives of \(t(s,r)\) and the second derivatives of \(st(s,r)\).

7.2. Energy estimates

It is possible to extend the energy estimate proved in Corollary 1.6 to the degenerate case. In general, if one drops the second part in condition (1.14), then, for the solution \((\psi_{s(r)}, a_{s(r)})\) constructed in (1.22), we have,

\[
\mathcal{E}(\psi_{s(r)}, a_{s(r)}, h_r) \geq \mathcal{E}(0, a^{br}, h_r) - \frac{|\Sigma|_r}{4} \frac{|\kappa^2 - b_r|^2}{(\kappa^2 - \frac{1}{2})\beta(r) + \frac{1}{2}} + O(|\kappa^2 - b_r|^3) \tag{7.12}
\]

for \(\kappa \geq 1/\sqrt{2}\), and the opposite inequality for \(\kappa < 1/\sqrt{2}\).

Indeed, for \(\dim K > 1\), we have \(\beta(r) = \langle |\xi|^4 \rangle \) for all \(\phi \in K\), \(\langle |\phi|^2 \rangle = 1\). This, together with (6.16), gives (7.12) depending on the sign of \(\kappa^2 - \frac{1}{2}\).

One can also introduce the ‘upper Abrikosov function’,

\[
\beta_+(r) := \max \left\{ \langle |\xi|^4 \rangle : \xi \in K(r), \langle |\xi|^2 \rangle = 1 \right\}, \tag{7.13}
\]

which leads to the upper energy bound, for \(\kappa^2 \geq 1/2\),

\[
\mathcal{E}(\psi_{s(r)}, a_{s(r)}, h_r) \leq \mathcal{E}(0, a^{br}, h_r) - \frac{|\Sigma|_r}{4} \frac{|\kappa^2 - b_r|^2}{(\kappa^2 - \frac{1}{2})\beta_+(r) + \frac{1}{2}} + O(|\kappa^2 - b_r|^3). \tag{7.14}
\]

For \(\kappa_{c+}(r) := \sqrt{\frac{1}{2}\left(1 - \frac{1}{\beta_+(r)}\right)}\), estimate (7.14) ensures that \(\mathcal{E}(\psi_{s(r)}, a_{s(r)}, h_r) < \mathcal{E}(0, a^{br}, h_r)\) for \(\kappa > \kappa_{c+}(r)\).

Declaration of competing interest

The Authors have no conflicts of interest to declare that are relevant to the content of this article.

Data availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.
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Appendix A. Standard definitions and notation

We sketch very briefly some notions and definitions relevant for us. To fix ideas, in what follows, $E$ is a line bundle over a Riemann surface $\Sigma$, with fibers isomorphic to $\mathbb{C}$ and the gauge group $G = U(1)$.

A.1. Connections

A connection (gauge field), $a$, is a real-valued one-form with certain transformation properties. For a connection $a$, defines the covariant derivative, $\nabla_a$, on sections of $E$, which can be written locally as $\nabla_a \psi = \nabla \psi - i a \psi$. Here $\nabla$ is a fixed connection (say, the Levi-Civita one).

A connection $a$ defines the curvature two-form $F_a := da$. A connection $a$ on $E$ is said to be a constant curvature connection if its curvature is of the form

$$da = b \omega,$$

for some $b \in \mathbb{R}$, where $\omega$ is the standard hyperbolic 2-form on $\Sigma$.

By [5, Lem. 3.2], a constant curvature connection, $a$, solves the (static) Maxwell equation

$$d^* da = 0.$$  \hspace{1cm} (A.2)

Let $x^i$, $i = 1, 2$ be a local coordinate on $\Sigma$. With the summation convention understood, we can write a connection (gauge field), $a$, and the covariant derivative, $\nabla_a$, as $a = a_i dx^i$ and $\nabla_a = \nabla_i dx^i$, where $\nabla_j := \partial_i - i a_j$. Let $d$ be the exterior derivative on $\Sigma$. Then, for a section $\psi$ and 1-form $B = B_i dx^i$,

$$d_a \psi = \nabla_i \psi dx^i, \quad dB = \partial_i B_j dx^i \wedge dx^j, \quad d^* B = -\partial_i B_i$$

and the curvature, $F_a = da$, of $a$ is given by

$$F_a = [\nabla_i, \nabla_j] dx^i \wedge dx^j, \quad [\nabla_i, \nabla_j] = \partial_i a_j - \partial_j a_i.$$  \hspace{1cm} (A.3)
A.2. Automorphy factor

Let \( \Sigma = \mathbb{H}/\Gamma \) be a non-compact Riemann surface of the form (2.1). Let \( \pi_1(\Sigma) \) be the first fundamental group of \( \Sigma \), which we identify with \( \Gamma \). A map

\[
\rho(\gamma, z) : \pi_1(\Sigma) \times \mathbb{H} \cong \Gamma \times \mathbb{H} \to U(1)
\]

is called an automorphy factor if it satisfies the important co-cycle condition,

\[
\rho(\gamma \gamma', z) = \rho(\gamma, \gamma' z) \rho(\gamma', z) \quad (\gamma, \gamma' \in \Gamma, z \in \mathbb{H}).
\]  

(A.4)

For fixed \( \gamma \in \Gamma \), we often write \( \rho_{\gamma}(z) \equiv \rho(\gamma, z) \) as a function from \( \mathbb{H} \) to \( U(1) \).

For every \( b \in \mathbb{R} \), we fix a canonical choice of automorphy factor with weight \( b \) as

\[
\rho_{b}(\gamma, z) \equiv \rho_{b, \gamma}(z) := \left( \frac{cz + d}{\bar{c}z + \bar{d}} \right)^b \quad \left( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) \right). \tag{A.5}
\]

(A.5)

For results in this paper, we understood that \( b \in \mathbb{Z} \), but (A.5) can also be defined for non-integer values of \( b \) with branch cuts.

The choice (A.5) is customary in the study of Maass forms in number theory, see e.g. [4, Sec.2.1]. In Theorem B.1, we prove this choice of \( \rho(\gamma, z) \) satisfies the co-cycle condition (A.4). Note that \( \rho(\gamma, z) \) remains bounded as \( z \) approaches the boundary \( \mathbb{R} \cup \{ \infty \} \) of \( \mathbb{H} \), which contains the cusps of \( \Sigma \).

Recall that the universal cover of the Riemann surface \( \Sigma \) is the Poincaré half-plane \( \mathbb{H} \). Then using the co-cycle condition (A.4), one can define a line bundle \( E_{\rho} \) as

\[
E_{\rho} := (\mathbb{H} \times \mathbb{C}) / \rho, \text{ with the action } \rho_{\gamma} : (z, \psi) \mapsto (\gamma z, \rho_{\gamma}(z) \psi). \tag{A.6}
\]

(A.6)

In fact, (A.6) defines an one-to-one correspondence, \( \rho \leftrightarrow E_{\rho} \), between (equivalence classes of) automorphy maps for \( \Sigma \) and line \( U(1) \)-bundles over \( \Sigma \). See [16] for details.

Through the correspondence (A.6), one can define a topological degree for a unitary line bundle over \( \Sigma \), which by the Chern-Weil correspondence is equal to the Chern number, or the degree of \( E \), as follows. Suppose \( E = E_{\rho} \) as in (A.6) for an automorphy factor \( \rho \) satisfying (A.4). Suppose the Fuchsian group \( \Gamma \) has genus \( g \) and \( m \) cusps, with no elliptic points. Then \( \Gamma \) is generated by \( 2g \) hyperbolic transforms, \( A_1, B_1, \ldots, A_g, B_g \), and \( m \) parabolic transforms \( S_1, \ldots, S_m \), satisfying the relation

\[
A_1 B_1 A_1^{-1} B_1^{-1} \ldots A_g B_g A_g^{-1} B_g^{-1} S_1 \ldots S_m = 1. \tag{A.7}
\]

Now, for every \( \gamma \in \Gamma \), define a map \( \sigma = \sigma_\gamma : \mathbb{H} \to \mathbb{R} \) by the relation \( e^{i\sigma_\gamma(z)} := \rho(\gamma, z) \).

The first Chern number \( c_1(\sigma) \) is defined by

\[
c_1(\sigma) := \frac{1}{2\pi} \sum_{i=1}^{g} [\sigma_A(v_i) - \sigma_B(v_i) + i\sigma_A(v_i) - \sigma_B(v_i) + i\sigma_A(v_i)], \tag{A.8}
\]
where $v_i := B_i A_i^{-1} B_i^{-1} C_{i+1} \ldots C_g z_0 = A_i^{-1} C_i \ldots C_g z_0$, with $C_i := A_i B_i A_i^{-1} B_i^{-1}$. By [17, Theorem 2A], $c_1(\sigma)$ is independent of $z_0$, and takes values in $\mathbb{Z}$.

Let $E$ and $E'$ be the line bundles corresponding to two automorphy factors $e^{i\sigma}$, $e^{i\sigma'}$ respectively. Then we say that $E$ and $E'$ are equivalent if and only if $c_1(\sigma) = c_1(\sigma')$.

Hence, by the one-to-one correspondence (A.4), $c_1(\sigma)$ is a topological invariant for a unitary line bundle $E = E_\rho$ over $\Sigma = \mathbb{H}/\Gamma$. It turns out (see [15]) that $c_1(\sigma)$ is equal to the topological degree of $E$:

$$c_1(\sigma) = \deg E_{e^{i\sigma}}. \quad (A.9)$$

(Recall that $\deg E$ is defined through multiplicity of zeroes of sections.)

### A.3. Equivariant states

**Definition A.1.** Let $\Gamma$ be a Fuchsian group. A (function, 1-form)-pair $(\Psi, A) : \mathbb{H} \to \mathbb{C} \times \mathbb{R}^2$ is called an equivariant state w.r.t. $\Gamma$ if and only if for all $\gamma \in \Gamma$, $z \in \mathbb{H}$, the following relations hold:

$$\gamma^* \Psi(z) = \rho(\gamma, z) \Psi(z), \quad (A.10)$$

$$\gamma^* A(z) = A(z) + i \rho(\gamma, z)^{-1} d\rho(\gamma, z), \quad (A.11)$$

where $\gamma^*$ is the pull-back by the Möbius transform (2.2) associated to $\gamma \in \Gamma$, and $\rho(\gamma, z)$ is an automorphy factor, satisfying the co-cycle condition (A.4)

**Remark 14.** For a standard lattice $\mathcal{L} \subset \mathbb{C} \cong \mathbb{R}^2$, equivariant solutions to (GL) are known as the Abrikosov lattices, predicted by A.A. Abrikosov in 1957 [1]. (This discovery was recognized by a Nobel prize.) These are ground state solutions to (GL) on the flat torus

$$\mathbb{T} = \mathbb{C}/\mathcal{L}, \quad (A.12)$$

which is a compact Riemann surface of genus 1. Existence and stability theory of solutions to (GL) on $\mathbb{T}$ are studied in [32–35]. Physically, these solutions correspond to regular arrays of vortices as seen in Type II superconductors.

In (A.12), the lattice $\mathcal{L}$ acts on the complex plane by translation, and the action is commutative. In comparison, with the background geometry (2.1), the action of $\Gamma$ on the Poincaré half plane $\mathbb{H}$ by Möbius transforms is in general non-commutative. In this sense, equivariant solutions to (GL) on (2.1) are non-commutative generalizations of the Abrikosov lattice.

Equivariant states $(\Psi, A)$ are in one-to-one correspondence with (section, connection)-pairs $(\psi, a)$ on the unitary complex line bundle $E \to \Sigma$ through the explicit correspondence (A.6). One can restrict $(\Psi, A)$ to a fundamental domain $F \subset \mathbb{H}$ of $\Gamma$,
with \((A.10)\)–\((A.11)\) considered as boundary conditions. Although fundamental domain is not unique, every two fundamental domains \(F, F' \subset \mathbb{H} \) of \(\Gamma\) are related by a Möbius transform \(\gamma \in \Gamma\), which sends \(F \to F'\). Hence, if \((\Psi, A)\) is an equivariant state on \(F\) satisfying \((A.10)\)–\((A.11)\), then the change of identification \(F \to F'\) amounts to a gauge transform of the form

\[
(\Psi, A) \longrightarrow (\rho \Psi, A + i \rho^{-1} d\rho).
\]  

Since \((GL)\) is invariant under \((A.13)\), such change has no consequence when one studies the solution to \((GL)\).

The important correspondence formulated above allows us to interpret problems posed in \(X^s\) in terms of equivariant states. For example, given an automorphy factor \(\rho_b\) with weight \(b\), let \(L^2(\mathbb{H}/\Gamma) \equiv L^2(\mathbb{H}/\Gamma, \rho_b)\) be the space of square integrable equivariant function satisfying \((A.10)\) with \(\rho = \rho_b\) (see e.g. [4, Sect. 2.1] for details).

The equivariant states picture is useful for explicit computations. Below, we obtain an explicit description of the null space \(K\) from \((3.19)\) in terms of equivariant functions in \(L^2(\mathbb{H}/\Gamma)\) and prove inclusion \((3.45)\), using similar symmetrization argument as in [21], which studies the same problem with \(b = 0\). This method was introduced by Selberg in 1950s, and is well-known to the number theorists.

**Proposition A.2.** Let \(\Sigma = \mathbb{H}/\Gamma\) be a non-compact Riemann surface with \(m\) cusps and no elliptic points, with \(\mathcal{S}(\Sigma) \neq \emptyset\). Then the solution space to \((3.2)\) in \(L^2(\mathbb{H}/\Gamma)\) is spanned by the vectors

\[
\xi_i(z) := \sum_{\gamma \in \Gamma/\Gamma_i} \text{Im}(\gamma_i \gamma z)^b e^{\gamma_i \gamma 2\pi i z} \rho_b(\gamma, z)^{-1}, \quad i = 1, \ldots, m
\]  

where each \(\Gamma_i := \text{Stab}(c_i, \Gamma)\) denotes the stabilizer of a distinct cusp \(c_i\) of \(\Gamma\), and \(\gamma_i \in \text{SL}(2, \mathbb{R})\) is a scaling matrix of \(c_i\).

Moreover, there holds the asymptotics

\[
|\xi_i(\gamma_i^{-1} z)| = O(e^{-2\pi y}) \quad (y \to \infty),
\]  

which implies that \(\xi_i(z)\) decays exponentially fast as \(z\) approaches the cusp \(c_i\).

**Proof.** Suppose a function \(\phi : \mathbb{H} \to \mathbb{C}\) given by \(\phi(x, y) \equiv \phi(x + iy)\) satisfies

\[
\phi(x + 1, y) = \phi(x, y) \quad \text{for every } y > 0.
\]

Then for every cusp \(c_i\), we can form the Poincaré series

\[
E_{c_i, \phi}(z) := \sum_{\gamma \in \Gamma/\Gamma_i} \phi(\gamma_i \gamma z) \rho^{-1}(\gamma, z).
\]
This series (3.1) converges absolutely if ϕ satisfies certain growth condition in y.

The Poincaré series has two important properties:

(a) By construction, $E_{c_i, \phi}$ satisfies the equivariance condition (A.10);
(b) If $L$ is an invariant operator acting on $\mathbb{H}/\Gamma$ in the sense that

$$\Delta_{a^b}(\gamma^* \phi) = \rho(\gamma, z)(-\Delta_{a^b} \phi)$$

(A.18)

for every $\gamma \in \Gamma$, $\phi \in \mathcal{H}^s$, and $L \phi = \lambda \phi$ for some $\lambda \in \mathbb{C}$, then $L E_{c_i, \phi} = \lambda E_{c_i, \phi}$.

In view of these properties, to solve the eigenvalue problem (3.2) in $L^2(\mathbb{H}/\Gamma)$, we first solve the following problem for some $\phi \in C^2(\mathbb{H}, \mathbb{C})$:

$$(-\Delta_{a^b} - b)\phi = 0,$$

(A.19)

$$\phi(x + 1, y) = \phi(x, y), \quad \lim_{y \to \infty} \phi(x, y) = 0.$$

(A.20)

In the boundary condition (A.20), the periodicity in $x$ ensures that the symmetrization is well-defined, cf. (A.16). The decay property in $y$ is required to ensure convergence of the Poincaré series (A.17).

The solution to (A.19)–(A.20) is proportional to the function

$$\phi := y^b e^{-2\pi y} e^{i2\pi x}.$$

This is calculated in e.g. [7,8]. Forming Poincaré series (A.17) w.r.t. each of the cusps of $\Sigma$ gives (A.14). By construction, (A.14) are equivariant solution to (3.2).

By the classical results for the Fourier analysis of Poincaré series, e.g. [29, Sect. 2], there holds the Fourier expansion

$$\xi_i(\gamma_i^{-1} z) = \sum_{k \neq 0} A_k W_{b \text{sign}(k), b-\frac{1}{2}} (4\pi |n| y) e^{2k\pi i x}.$$

(A.21)

Here $W_{\beta, \mu}(y) = O(e^{-y/2})$ is the Whittaker function, a decaying solution to the ODE

$$W''(y) + \left( -\frac{1}{4} + \frac{\beta}{y} + \frac{1/4 - \mu^2}{y^2} \right) W(y) = 0.$$

This $W_{\beta, \mu}(y)$ can be expressed in terms of the modified Bessel function of the second kind [14]. Expansion (A.21) implies the asymptotics (A.15).

Remark 15. Estimates for the coefficients $A_k$ in (A.21) are of significant interest in number theory, and have been obtained in [13,20].
Proof of (3.45). We use the same symmetrization method as in the proof of Proposition A.2. First, we seek $C^2(\mathbb{H}, \mathbb{C})$-solutions to the eigenvalue problem

$$\begin{align*}
(-\Delta_{ab} - \lambda)\psi &= 0, \\
\psi(x + 1, y) &= \psi(x, y).
\end{align*}$$

(A.22)

Notice here we do not require the decay condition in $y$, cf. (A.20). The problem (A.22)–(A.23) has the following family of solutions:

$$\phi_k := y^{1/2 - ik} \quad (k \geq 0).$$

(A.24)

These $\phi_k$’s are the generalized eigenfunctions, which correspond to the spectral points

$$\lambda_k := k^2 + \frac{1}{4} + b^2 \quad (k \geq 0).$$

(A.25)

Next, fix a cusp $c_i$ of $\Sigma$ and some $k \geq 0$. For $n = 1, 2, \ldots$, let $u_n := \chi_n \phi_k$, where $\chi_n \in C^2_c(\mathbb{R}_{>0}, \mathbb{R}_{\geq 0})$ is a sequence of cut-off functions with

$$\chi_n(y) \equiv \begin{cases} 
1 & (n \leq y \leq n + 1), \\
0 & (0 < n - \frac{1}{n} \leq y \text{ or } y \geq n + \frac{1}{n}).
\end{cases}$$

(A.26)

Now, we form the Poincaré series (A.17) w.r.t. $c_i$ and $u_n$. Since $u_n$ has compact support, $E_{c_i, u_n}$ converges absolutely for every $n$. Moreover, each $E_{c_i, u_n}$ is an incomplete Eisenstein series, which is bounded and satisfies the equivariant condition (A.10). (See [21, Sec. 3.2] for a discussion.) Hence $E_{c_i, u_n} \in \mathcal{L}^2$ and satisfies (A.19) except for on the two bounded strips $\{(n - 1)/n < y < n\}$ and $\{n < y < (n + 1)/n\}$.

Finally, let

$$\tilde{u}_n := \|E_{c_i, u_n}\|_{L^2}^{-1} E_{c_i, u_n}.$$

Then $\tilde{u}_n \in L^2(\mathbb{H}/\Gamma)$ forms a Weyl sequence for $-\Delta_{ab}$ and $\lambda_k$, cf. (3.36). This shows that $\lambda_k \in \sigma_{\text{ess}}(-\Delta_{ab})$. Varying $k$, we find

$$[\frac{1}{4} + b^2) \subset \sigma_{\text{ess}}(-\Delta_{ab}). \quad \Box$$

Appendix B. Classification of $U(1)$-automorphy factors and constant curvature $U(1)$-connections

In this section, we reproduce with minor modifications results of [5] on classification of automorphy factors and constant curvature connections.

Recall that a character of $\Gamma$ is a homomorphism $\chi : \Gamma \to U(1)$. For other standard definitions, see Appendix A.
Theorem B.1 (classification of automorphy factor). For any $\beta \in \mathbb{R}$, the map $\rho_\beta : PGL(2, \mathbb{R}) \times \mathbb{H} \to U(1)$, given by

$$\rho_\beta(\gamma, z) = \left[ \frac{c + d}{c\bar{z} + d} \right]^\beta, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PGL(2, \mathbb{R}),$$

satisfies the co-cycle condition

$$\rho_\beta(s \cdot t, z) = \rho_\beta(s, t \cdot z)\rho_\beta(t, z), \quad \forall s, t \in PGL(2, \mathbb{R}).$$

Consequently, for any Fuchsian group $\Gamma$ s.t. $\Sigma := \mathbb{H}/\Gamma$ has finite area and $b := \frac{2\pi n}{|\Sigma|}$, and any character $\chi : \Gamma \to U(1)$, the map $\rho_{b,\chi} : \Gamma \times \mathbb{H} \to U(1)$ given by

$$\rho_{b,\chi}(\gamma, z) = \chi(\gamma)\rho_b(\gamma, z), \quad \gamma \in \Gamma,$$

is also an automorphy factor. Two automorphy factors related as in (B.2) are said to be equivalent.

Let $n \in \mathbb{Z}$ and $b := \frac{2\pi n}{|\Sigma|}$. Then, the first Chern number of $\rho_{b,\chi}$ is $c_1(\rho_{b,\chi}) = n$. Hence any automorphy factor $\rho : \Gamma \times \mathbb{H} \to U(1)$ of degree $n$ is equivalent to $\rho_b$ as in (A.5).

Proof. Let $s = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, t = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \in \Gamma$. Using (B.1), we compute for any $\beta$,

$$\rho_\beta(s \cdot t, z) = \rho_\beta(s, t \cdot z)\rho_\beta(t, z), \quad \forall s, t \in PGL(2, \mathbb{R}).$$

Using the formula for the Chern class, $c_1(\rho)$, of a co-cycle $\rho$ (see [17], Theorem 2a), we compute $c_1(\rho_\beta) = n$, provided $\beta = \frac{2\pi n}{|\Sigma|}$, where $|\Sigma|$ is the area of $\Sigma$ w.r.t. to the standard hyperbolic metric. □

Theorem B.2 (classification of constant curvature connections). For any $b \in \mathbb{R}$, the connection

$$A^b = by^{-1}dx,$$

on the trivial line bundle $\tilde{E} := \mathbb{H} \times \mathbb{C}$

(a) has a constant curvature with respect to the standard hyperbolic area form on $\mathbb{H}$;
(b) is equivariant with respect to the automorphy factor (B.2) for any Fuchsian group \( \Gamma \) and any character \( \chi : \Gamma \to U(1) \);

(c) is unique (as a connection on \( \tilde{E} := \mathbb{H} \times \mathbb{C} \)) up to gauge transformations (1.2).

Remark 16. The description in the above theorem does not depend at all on the complex structure of the underlying Riemann surface. Hence, if \( E_{n,\chi} \) is the unitary line bundle corresponding to the automorphy factors (B.2), then the projection of \( A^b \) to \( E_{n,\chi} \) gives the distinguished connection \( a^{b,\chi} \) on \( E_{n,\chi} \).

Proof of Theorem B.2. We begin with some preliminary constructions. We consider the trivial bundle, \( \tilde{E} := \mathbb{H} \times \mathbb{C} \) with the standard complex structure on \( \mathbb{H} \) associated to the standard hyperbolic metric \( \tilde{h} = (\Im z)^{-2}|dz|^2 \).

Since we work here on a global product space, it is natural to take the fiber metric to be induced from the metric on the base. So we take the metric on the fiber \( \mathbb{C} \) over the point \( z \in \mathbb{H} \) to be \( k_z = (\Im z)^{-2}|dw|^2 \), where \( w \) is the coordinate on the fiber \( \mathbb{C}_z \).

Let the connection \( A \) be given by \( A := A_1 dx_1 + A_2 dx_2 \). We decompose the covariant derivative \( \nabla_A \) into \((1,0)\) and \((0,1)\) parts as \( \nabla_A = \partial_A' + \partial_A'' \), where \( \partial_A' \) and \( \partial_A'' \) are defined in (3.9)–(3.10).

Recall from Section 3.3 that, in terms of \( A_c \), the curvature is given by \( F_A = 2 \Re \tilde{\partial} A_c \). Moreover, if \( A_c \) satisfies the equivariance relation \( s^* \tilde{A}_c = \tilde{A}_c - i \tilde{\partial} f_s \), then \( A \) satisfies \( s^* A = A + df_s \), with \( f_s \) satisfying \( df_s := 2 \Im \tilde{\partial} f_s \).

According to (3.10), the complexification of the connection \( A^b \) given in the theorem is

\[
A^b_c = b \frac{1}{2 \Im(z)} dz.
\]

In the remaining of this proof, we omit the superindex \( b \) in \( A^b \) and \( A^b_c \).

Proof of constant curvature. Using that \( \omega = \frac{i}{2} \Im(z)^{-2} dz \wedge d\bar{z} \), we find

\[
\tilde{\partial} A_c = \frac{\partial}{\partial \bar{z}} \frac{ib}{z - \bar{z}} d\bar{z} \wedge dz = \frac{-ib}{(z - \bar{z})^2} dz \wedge d\bar{z} = \frac{ib}{4 \Im(z)^2} dz \wedge d\bar{z} = \frac{b}{2} \omega.
\]

Since \( F_A = 2 \Re \tilde{\partial} A_c \), this gives the desired result.

Proof of uniqueness. If \( A \) and \( B \), satisfy \( dB = dA \) then \( d(A - B) = 0 \). It follows from the simple connectedness of \( \mathbb{H} \) that \( A - B = df \) for some function \( f : \mathbb{H} \to \mathbb{R} \) and \( f \) is unique up to an additive constant. So we can map \( B \) to \( A \) through a suitable reparametrization. This completes the proof.

Proof of equivariance. For a generic isometry \( s(z) = \frac{az + \beta}{\gamma z + \delta} \), we have \( \frac{\partial s(z)}{\partial z} = (\gamma z + \delta)^{-2} \) and \( \Im(s(z)) = \frac{\Im(z)}{|\gamma z + \delta|^2} \), which gives

\[
s^* \tilde{A}_c = \frac{b}{2 \Im(s(z))} \frac{\partial s(z)}{\partial z} d\bar{z} = \frac{k|\gamma z + \delta|^2}{2 \Im(z)} \frac{d\bar{z}}{(\gamma z + \delta)^2}
\]
\[ f_s = b \ln(\gamma z + \delta) + c_s. \]

Now, we define \( f_s := 2 \text{Re} \tilde{f}_s \) and use that \( \text{Re}(\overline{\partial f_s}) = d(\text{Re} \tilde{f}_s) \) (as can be checked by the direct computation: \( \frac{1}{b} \text{Re}(\overline{\partial f_s}) = -\frac{\gamma^2 x_2}{|\gamma z + \delta|^4} dx_1 + \frac{\gamma(\gamma x_1 + \delta)}{|\gamma z + \delta|^2} dx_2 \)) to obtain \( s^*A = A + df_s \), with

\[ f_s = 2 \text{Re}(\tilde{f}_s) = ib \ln \left[ \frac{\gamma z + \delta}{\gamma z + \delta} \right] + c_s, \quad \rho(s, z) = e^{i f_s(z)} = e^{ic_s} \left[ \frac{\gamma z + \delta}{\gamma z + \delta} \right]^{-b}. \] (B.4)

Here note that since \( \mathbb{H} \) is the upper half plane, the complex logarithm is well defined and \( \gamma z + d \) is always non zero. \( \square \)

**Remark 17.** The function \( \tilde{f}_s(z) \) appearing above gives the character \( \tilde{\rho}(z) = e^{i \tilde{f}_s(z)} \), which is now \( \mathbb{C}^* \)-valued instead of \( U(1) \)-valued.

**Remark 18.** \( A^b \) is \( \mathbb{R} \)-linear while \( A^c \) is naturally \( \mathbb{C} \)-linear. It is natural to ask what the action of \( i \) on \( A^b \) does when we map back to \( A^c \). A simple calculation shows that \( iA^b = i b \frac{1}{2} \text{Im}(z) dz \) is mapped to \( b y^{-1} dy \), which is flat. It turns out that the complex action of \( i \) induces a rotation into the space of flat connections.

**Appendix C. Chern-Weil correspondence**

**Proof of Theorem 2.2.** The argument below, due to D. Chouchkov, is streamlined from classical results from [17, Chap. II.4].

Let \( F_\Sigma \subset \mathbb{H} \) be a fundamental domain of \( \Gamma \). Let \( A_i, B_i, i = 1, \ldots g \), and \( S_i, i = 1, \ldots m \), be the hyperbolic and parabolic generators of \( \Gamma \), respectively (no elliptic ones by assumption). Let \( \alpha_i, \alpha'_i, \beta_i, \beta'_i, i = 1, \ldots g \), and \( \delta_j, \delta'_j, j = 1, \ldots m \) be the sides of \( F_\Sigma \).

As in the beginning of Section 3, we have the following relations:

\[ A_i^* \alpha_i = -\alpha'_i, B_i^* \beta_i = -\beta'_i, i = 1, \ldots g \], and \( S_j^* \delta_j = -\delta'_j, j = 1, \ldots m, \)

as well as the decomposition

\[ \partial F_\Sigma = \sum_{i=1}^g (\alpha_i + \alpha'_i + \beta_i + \beta'_i) + \sum_{j=1}^m (\delta_j + \delta'_j). \] (C.1)
Here the plus sign denotes disjoint union.

Let $A$ be an equivariant 1-form on $F_\Sigma$ that corresponds to the connection $a$ on $E \to \Sigma$, as in Sections A.2–A.3. By definition (A.9), the theorem is proved once we establish

$$ \frac{1}{2\pi} \int_{\partial F_\Sigma} A = c_1(\sigma). $$

Since the 1-form $A$ is gauge-equivariant, we have

$$ \gamma^* A = A - i\rho_\gamma^{-1} d\rho_\gamma, \forall \gamma \in \Gamma, z \in \partial F_\Sigma. $$ (C.2)

This, together with the definition $\rho(\gamma, z) = e^{i\sigma(z)}$, implies

$$ \int_{\alpha_i + \alpha'_i} A = \int_{\alpha_i} (A - A_i^* A) = - \int_{\alpha_i} d\sigma_{A_i} = \sigma_{A_i}(v_i) - \sigma_{A_i}(v_{i+1}) \quad (i = 1, \ldots g), $$

where $v_i, v_{i+1}$ are the vertices of $F_\Sigma$ spanned by the side $\alpha_i$. Similarly we proceed with $\int_{\beta_i + \beta'_i} A$ and $\int_{\delta_j + \delta'_j} A$. In the last case, since the gauge exponent $\sigma_{S_i}(z)$ is independent of $z$, we have $\int_{\delta_j + \delta'_j} A = 0$, so these terms do not contribute.

Summing up these contributions and using (C.1), we arrive at

$$ \int_{\partial F_\Sigma} A \sim g \sum_{i=1}^{g} \left[ \sigma_{A_i}(v_i) - \sigma_{A_i}(v_{i+1}) + \sigma_{B_i}(w_i) - \sigma_{B_i}(w_{i+1}) \right], $$

where $v_i, v_{i+1}$ and $w_i, w_{i+1}$ are the vertices of $F_\Sigma$ spanned by $\alpha_i$ and $\beta_i$, respectively. Note that the vertices $v_i, v_{i+1}, w_i, w_{i+1}$ and $u_i, u_{i+1}$ are related as $w_i = v_{i+1}, w_{i+1} = A_i v_{i+1}$ and $v_{i+1} = B_i^{-1} v_i$, which gives

$$ \frac{1}{2\pi} \int_{\partial F_\Sigma} A = \frac{1}{2\pi} \sum_{i=1}^{g} \left[ \sigma_{A_i}(v_i) - \sigma_{A_i}(B_i^{-1} v_i) + \sigma_{B_i}(B_i^{-1} v_i) - \sigma_{B_i}(A_i B_i^{-1} v_i) \right]. $$ (C.3)

By construction,

$$ v_i = B_i A_i^{-1} B_i^{-1} C_{i+1} \ldots C_g z_0 = A_i^{-1} C_i \ldots C_g z_0, $$

with $C_i := A_i B_i A_i^{-1} B_i^{-1}$ and some $z_0 \in F_\Sigma$. Hence, r.h.s. of (C.3) agrees with the definition of the first Chern number $c_1$ in (A.8). This proves the theorem. \qed

Appendix D. Proof of Theorem 7.1

One reduces to a scalar bifurcation problem by considering the scalar-valued function

$$ \phi(\ell, s) = \frac{1}{|s|^2} \langle s, \ell s + H(\ell, s) \rangle. $$ (D.1)
Manifestly, $\phi(0,0) = 0$ and $\partial \phi(0,0)/\partial \ell = 1$. It then follows from the implicit function theorem that there exists a unique function $\ell(s)$, for small $s$, such that $\phi(\ell(s), s) = 0$. Set $\hat{E}(s) = E(\ell(s), s)$, where, recall, $E(\ell, s) = \ell s + H(\ell, s)$, and let $S_\epsilon := \{ s \in \mathbb{R}^n : |s| = \epsilon \}$ be the $\epsilon$-sphere in $\mathbb{R}^n$. Then $\hat{E}(s)$, regarded as a vector field on $\mathbb{R}^n$, is everywhere tangent to $S_\epsilon$. To see this, simply note that

$$0 = \phi(\ell(s), s) = \frac{1}{|s|^2} \langle s, \ell(s)s + H(\ell(s), s) \rangle = \frac{1}{|s|^2} \langle s, \hat{E}(s) \rangle,$$

which precisely states that the vector field $\hat{E}(s)$ is everywhere tangent to the sphere. Since $n$ is odd, the sphere is even dimensional. But every vector field on an even dimensional sphere has as zero vector, i.e. $\hat{E}(s) = 0$ has a solution on $S_\epsilon$. The theorem follows.

References


