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REDUCIBILITY OF COVERS OF AFT SHIFTS

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Abstract. In this paper we show that the reducibility structure of several covers of sofic shifts is a flow invariant. In addition, we prove that for an irreducible subshift of almost finite type the left Krieger cover and the past set cover are reducible. We provide an example which shows that there are non almost finite type shifts which have reducible left Krieger covers. As an application we show that the Matsumoto algebra of an irreducible, strictly sofic shift of almost finite type is not simple.

1. Introduction

The classification of shift spaces of finite type up to flow equivalence, initiated by Parry and Sullivan and Bowen and Franks (see [30, 9]) and completed by Boyle and Huang [7] is, at present, not generalized to any but a few sporadic classes of shift spaces. This state of affairs is certainly related to the scarcity of useful invariants presently known, even in the case when the shift space is irreducible. The present paper follows a general strategy, also pursued in, for example [11, 12, 28], of trying to extract such invariants from the study of C^*-algebras associated to shift spaces: since the C^*-algebra is a flow invariant, anything that derives from it must also be one. However, we can (and will) suppress C^*-algebra theory from the presentation in this paper.

The class of sofic shifts coined by Weiss [38] and subsequently studied intensively by several authors (see, for example, [5, 6, 14, 15, 16, 18, 21, 22, 29, 35]) captures, in a sense, the next level of complexity after the shifts of finite type. The purpose of this paper is to investigate the reducibility structure of various left-resolving presentations of a sofic shift space and prove that this structure is a flow invariant. The most commonly used presentations that we refer to are the left Krieger cover, the left Fischer cover and the past set cover. The precise relationships between these covers are currently unclear to us, but will be explored further in [17].

In this paper we work towards providing conditions on an irreducible sofic shift which guarantee that the left Krieger (respectively past set) cover is reducible. Theorem 4.6 and Theorem 5.8 do not characterise the irreducible sofic shifts having this property as Examples 4.1 and Example 4.9 demonstrate.

In Section 2 we recall the background theory of sofic shift spaces, left Krieger covers and shifts of almost finite type. The definition of the left Krieger cover we take is the one given in [21, 20], but we note that in some places in the literature (for example, [10, 13]) the left Krieger cover is defined to be the cover $(E_f^\infty, \mathcal{L})$ discussed in Remarks 5.9 (i). Our intention here is two-fold: first we wish to make the present paper self-contained and second we hope to clarify some inconsistencies in the terminology found in the literature which have caused us confusion.

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In Section 3 we show that the reducibility structure of the left Krieger cover is invariant under the operations that generate flow equivalence for sofic shifts. Our main contribution here is to show in Proposition 3.9 that the irreducibility structure of the left Krieger cover of a general subshift is invariant under symbol expansion. Krieger has shown in [21] that the left Krieger cover of a sofic shift is a conjugacy invariant, and this, in combination with our result, gives invariance of the reducibility structure of the left Krieger cover of a sofic shift under flow equivalence.

In Section 4 we prove our main theorem. In Theorem 4.6 we show that the left Krieger cover of an irreducible strictly sofic shift of almost finite type is reducible. As an application of Theorem 4.6 we prove that the Matsumoto algebras (see [26],[13]) of irreducible almost finite type shift spaces are not simple in Corollary 4.7. We show in Example 4.9 that the converse of Theorem 4.6 does not hold: there are examples of irreducible strictly sofic shifts which are not of almost finite type but which have reducible left Krieger covers. We also briefly discuss another left-resolving cover $(E^f_\infty, \mathcal{L})$ of a two-sided sofic shift space which is constructed using the infinite pasts of finite words. We note in Remark 4.8 that it is possible to prove a corresponding version of Theorem 4.6 for this cover.

In Section 5 we note that the construction of the above two covers does not make sense for one-sided shift spaces. We consider the past set cover of a (one- or two-sided) sofic shift space and prove in Theorem 5.8 that this cover is reducible for irreducible strictly sofic shifts of almost finite type. We note in Remarks 5.9 that there are examples of irreducible two-sided strictly sofic shifts for which the left Krieger cover and the past set cover do not coincide. Indeed, it is possible for the left Krieger cover to be irreducible when the past set cover is reducible. In Remarks 5.9 (i) we consider a cover of a (one- or two-sided) sofic shift which is constructed by considering the finite pasts of right-infinite rays in the shift space and give a corresponding version of Theorem 4.6 for this cover. We note that this cover coincides with the left Krieger cover for two-sided sofic shift spaces. As an application we are able to give a different proof of [36, Proposition 2.7 c)] which states that strictly sofic β-shifts are not of almost finite type.

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2. Background

Sofic shift spaces. For an excellent treatment of shift spaces we refer the reader to [24]. We present a few basic definitions here in order to make the present paper self-contained.

Let $\mathcal{A}$ be a finite alphabet, and let $\mathcal{A}^*$ denote the free monoid generated by $\mathcal{A}$. A *language* is a submonoid of $\mathcal{A}^*$ for some $\mathcal{A}$. The *full shift* over $\mathcal{A}$ consists of the space $\mathcal{A}^{\mathbf{Z}}$ endowed with the product topology together with the shift map $\sigma : \mathcal{A}^{\mathbf{Z}} \to \mathcal{A}^{\mathbf{Z}}$ such that for $x \in \mathcal{A}^{\mathbf{Z}}$, $\sigma(x)_i = x_{i+1}$ for all $i \in \mathbf{Z}$. A *subshift* X is a closed σ-invariant subset of $\mathcal{A}^{\mathbf{Z}}$. The language of the shift space X is denoted by $\mathcal{B}(\mathsf{X})$ and is the collection of all *words* or *blocks* which appear in the bi-infinite sequences of X; the empty word is denoted ϵ.

For each subshift there is a collection $\mathcal{F}$ of blocks which are not permitted to occur in the sequences of X. This collection of forbidden blocks uniquely describes the subshift (cf. [24, Proposition 1.3.4]). If the set of forbidden blocks for X can be chosen to be finite, then X is called a *shift of finite type* or SFT.

Let X_1 and X_2 be two subshifts over possibly different alphabets. A *factor map* π : $\mathsf{X}_1 \to \mathsf{X}_2$ is a continuous surjective function $\pi : \mathsf{X}_1 \to \mathsf{X}_2$ which commutes with the shift maps. According to Weiss [38] a shift space is *sofic* if it is the image of an SFT under a factor map.

A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are countable sets of vertices and edges respectively and $r, s : E^1 \to E^0$ are maps giving the direction of each edge. A path $\lambda = e_1 \dots e_n$ is a sequence of edges $e_i \in E^1$ such that $r(e_i) = s(e_{i+1})$ for $i = 1, \dots, n-1$. We denote the collection of all finite paths in E by E^* and extend the range and source maps to E^* in the natural way. A *circuit* in a directed graph E is a finite path $\lambda \in E^*$ satisfying $s(\lambda) = r(\lambda)$. A directed graph E is *irreducible* if for every pair (u, v) of vertices there is a path $\lambda \in E^*$ from u to v. Otherwise, the graph is said to be *reducible*.

A directed graph is *essential* if every vertex receives and emits an edge. A directed graph is *row-finite* if every vertex emits finitely many edges. We shall work exclusively with essential row-finite graphs. The edge shift (X_E, σ_E) associated to an essential directed graph E is given by:

$$\mathsf{X}_E = \{x \in (E^1)^{\mathbf{Z}} : s(x_{i+1}) = r(x_i) \text{ for all } i \in \mathbf{Z}\} \text{ and } (\sigma_E x)_i = x_{i+1} \text{ for } i \in \mathbf{Z}.$$

Evidently $\mathcal{B}(\mathsf{X}_E)\backslash\{\epsilon\} = E^*\backslash E^0$. The following definition is adapted from [24, Definition 3.1.1]:

Definition 2.1. A *labelled graph* $(E, \mathcal{L})$ over an alphabet $\mathcal{A}$ consists of a directed graph E together with a surjective labelling map $\mathcal{L} : E^1 \to \mathcal{A}$. We say that the labelled graph $(E, \mathcal{L})$ is *essential* if the directed graph E is essential.

Observe that a directed graph E is a labelled graph $(E, \mathcal{L}_t)$ over the alphabet E^1 where $\mathcal{L}_t : E^1 \to E^1$ is the identity map.

Definitions 2.2. Let E, F be directed graphs. A map $\phi = (\phi^0, \phi^1) : E \to F$ is an *isomorphism of directed graphs* if $\phi^0 : E^0 \to F^0$ and $\phi^1 : E^1 \to F^1$ are bijections such that for all $e \in E^1$ $\phi^0(s(e)) = s(\phi^1(e))$ and $\phi^0(r(e)) = r(\phi^1(e))$.

Let $(E, \mathcal{L})$ and $(F, \mathcal{L}')$ be labelled graphs over the same alphabet. A graph isomorphism $\phi : E \to F$ is a *labelled graph isomorphism* if $\mathcal{L}'(\phi(e)) = \mathcal{L}(e)$ for all $e \in E^1$

Given an essential labelled graph $(E, \mathcal{L})$ over the alphabet $\mathcal{A}$ we may define a subshift $(\mathsf{X}_{(E,\mathcal{L})}, \sigma)$ of $\mathcal{A}^{\mathbf{Z}}$ by

$$\mathsf{X}_{(E,\mathcal{L})} = \{y \in \mathcal{A}^{\mathbf{Z}} : \text{ there exists } x \in \mathsf{X}_E \text{ such that } y_i = \mathcal{L}(x_i) \text{ for all } i \in \mathbf{Z}\},$$

where σ is the shift map inherited from $\mathcal{A}^{\mathbf{Z}}$. A *representative* of a word $w \in \mathcal{B}(\mathsf{X}_{(E,\mathcal{L})})$ is a path $\lambda = e_1 \dots e_n \in E^*$ such that $\mathcal{L}(\lambda) := \mathcal{L}(e_1) \dots \mathcal{L}(e_n) = w$. The labelled graph $(E, \mathcal{L})$ is said to be a *presentation* of the shift space $\mathsf{X} = \mathsf{X}_{(E,\mathcal{L})}$. As shown in Example 2.15 a shift space may have many different presentations. We shall consider five such presentations in this paper.

Fischer proved in [15, Theorem 1] that sofic shifts are precisely those shift spaces that can be presented by labelled graphs with finite edge and vertex sets. Since, up to conjugacy, shifts of finite type are precisely the edge shifts associated to directed graphs with finite edge and vertex sets (see [24, Proposition 2.3.9]), shifts of finite type are sofic shifts as well. We say that a sofic shift is *strictly sofic* if it is not an SFT.

Definition 2.3. A shift space X is said to be *irreducible* if for every $u, w \in \mathcal{B}(\mathsf{X})$ there is a $v \in \mathcal{B}(\mathsf{X})$ such that $uvw \in \mathcal{B}(\mathsf{X})$.

Remark 2.4. It is straightforward to see that a sofic shift is irreducible if and only if it can be presented by an irreducible labelled graph i.e. a labelled graph $(E, \mathcal{L})$ with E irreducible (see [24, Section 3.1] for details).

In this paper we will be primarily concerned with irreducible strictly sofic shifts.

Let $(E,\mathcal{L})$ be a labelled graph (sometimes called a Shannon graph after [33]) with finite edge and vertex sets which presents the sofic shift $\mathsf{X}_{(E,\mathcal{L})}$. By removing the labels from the edges of $(E,\mathcal{L})$ we obtain a presentation of an SFT X_E and a factor map $\pi_{\mathcal{L}}:\mathsf{X}_E\to\mathsf{X}_{(E,\mathcal{L})}$ induced by the 1-block map $\mathcal{L}:E^1\to\mathcal{A}$. The subshift X_E is called a *cover* of the sofic shift $\mathsf{X}_{(E,\mathcal{L})}$.

Definition 2.5. Let X be a shift space. A word $w\in\mathcal{B}(\mathsf{X})$ is *intrinsically synchronising* if whenever $uw,wv\in\mathcal{B}(\mathsf{X})$ we have $uwv\in\mathcal{B}(\mathsf{X})$.

Remarks 2.6. There is confusion in the literature as to the correct terminology for the above concept. Intrinsically synchronising is called *magic* in [8], *finitary* in [21] and *synchronising* in [37]. However these words have different meanings in other places in the literature (see, for example, [24]). We shall follow the terminology used in [24].

If $m\in\mathcal{B}(\mathsf{X})$ is intrinsically synchronising then for any $u,v\in\mathcal{B}(\mathsf{X})$ with $umv\in\mathcal{B}(\mathsf{X})$ the word umv is intrinsically synchronising.

Definition 2.7. A labelled graph $(E,\mathcal{L})$ is *left-resolving* if for every $v\in E^0$, all edges ending at v carry different labels. That is, $\mathcal{L}:r^{-1}(v)\to\mathcal{A}$ is injective for all $v\in E^0$. A labelled graph is *right-resolving* if all edges leaving each vertex carry different labels.

Definition 2.8. A labelled graph $(E,\mathcal{L})$ is *right-closing with delay D* if every pair of paths in E of length $D+1$ which start at the same vertex and represent the same word must have the same initial edge. That is, whenever paths $\mu,\nu\in E^*$ of length $D+1$ satisfy $s(\mu)=s(\nu)$ and $\mathcal{L}(\mu)=\mathcal{L}(\nu)$ we must have $\mu_1=\nu_1$. A labelled graph is *right-closing* if it is right-closing with some delay $D\geq 0$. The concept of *left-closing* for a labelled graph is similarly defined.

Remark 2.9. Let $(E,\mathcal{L})$ be a labelled graph which is right-closing with delay D and let μ,ν be paths in E of length $D+n+1$ such that $s(\mu)=s(\nu)$ and $\mathcal{L}(\mu)=\mathcal{L}(\nu)$. Repeated applications of Definition 2.8 show that $\mu_1\ldots\mu_n=\nu_1\ldots\nu_n$.

Left Krieger covers. Let X be a shift space.

Definitions 2.10. (See [18, Sections I and III],[24, Exercise 3.2.8]) We write each $x\in\mathsf{X}$ as $x=x^-x^+$ where $x^-=\ldots x_{-2}x_{-1}$ is called the *left-ray of* x and $x^+=x_0x_1x_2\ldots$ is called the *right-ray of* x. We denote by X^+ the set of all right-rays of elements of X and by X^- the set of all left-rays of elements of X. For $x^+\in\mathsf{X}^+$, the *predecessor set* of x^+ is the set of all left-rays y^- which may precede x^+; that is

$$P_\infty(x^+)=\{y^-\in\mathsf{X}^-\;:\;y^-x^+\in\mathsf{X}\}.$$

Note that X is a sofic shift if and only if the number of predecessor sets $P_\infty(x^+)$ is finite, (see [21, §2]).

Definition 2.11. The *left Krieger cover* of a shift space X is the labelled graph $(E_K,\mathcal{L}_K)$ whose vertices are the predecessor sets of the elements of X^+. There is an edge labelled $a\in\mathcal{A}$ from $P_\infty(x^+)$ to $P_\infty(y^+)$ if and only if $ay^+\in\mathsf{X}^+$ and $P_\infty(x^+)=P_\infty(ay^+)$.

Remarks 2.12.
(i) If $w\in\mathcal{B}(\mathsf{X})$ then for all $x^+\in\mathsf{X}^+$ with $wx^+\in\mathsf{X}^+$ there is a path labelled w in $(E_K,\mathcal{L}_K)$ beginning at $P_\infty(wx^+)$ and ending at $P_\infty(x^+)$. In general, a word $w\in\mathcal{B}(\mathsf{X})$ can have representatives which start at several different vertices in the left Krieger cover (see Example 2.15).
(ii) The left Krieger cover of a shift space X is evidently a left-resolving presentation of X. The left Krieger cover is also known as the past state chain, and was originally defined only for sofic shifts by Krieger, but the definition applies in general.

(iii) One may similarly define a right Krieger cover (future state chain) which is right-resolving. All of the results about shift spaces stated in this paper have right-resolving analogues for which the proofs are similar to those given.

Definitions 2.13. (See also [18, Section 3]) We say that a ray $x^+ \in \mathsf{X}^+$ is *intrinsically synchronising* if it contains an intrinsically synchronising block. We say that a predecessor set $P_\infty(x^+)$ is *intrinsically synchronising* if there is an intrinsically synchronising ray $y^+ \in \mathsf{X}^+$ with $P_\infty(x^+) = P_\infty(y^+)$.

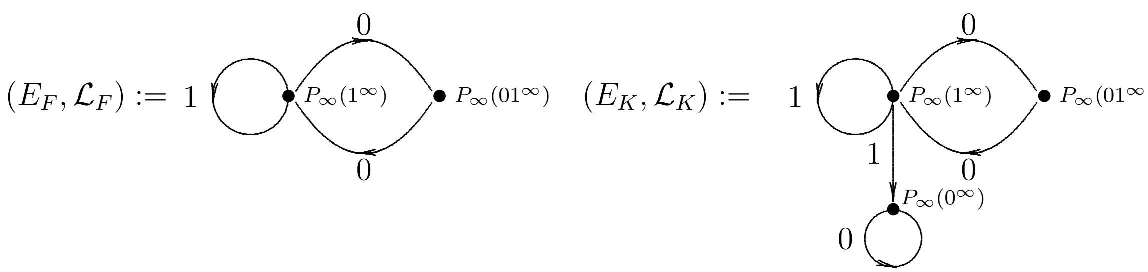
We define the *left Fischer cover* of a sofic shift X to be the minimal left-resolving labelled graph $(E_F, \mathcal{L}_F)$ which presents X.

Remarks 2.14. (i) The right Fischer cover of a sofic shift was originally described in [15] as the minimal right-resolving presentation of a sofic shift. Note that by a suitably adapted version of [24, Theorem 3.3.18] the left Fischer cover of a sofic shift is well-defined, up to labelled graph isomorphism.

(ii) The left Fischer cover $(E_F, \mathcal{L}_F)$ of an irreducible sofic shift X may be identified with the minimal irreducible subgraph of the left Krieger cover of X in which the vertices are precisely the intrinsically synchronising predecessor sets for the shift X. This result can be traced back to [21, Lemma 2.7] however the proof of Lemma 5.4 may be adapted to show this.

Example 2.15. The even shift is an irreducible strictly sofic shift which consists of the collection of all bi-infinite sequences of 0's and 1's such that the number of 0's which can occur between two 1's is an even number (including zero).

Following [3, Examples 3.3 (iii)], we see that the following graphs



are the left Fischer and left Krieger covers of the even shift, respectively. The predecessor sets $P_\infty(1^\infty)$ and $P_\infty(01^\infty)$ are intrinsically synchronising since the rays 1^∞ and 01^∞ both contain the intrinsically synchronising word 1. Note also that the left Fischer cover shares the vertices $P_\infty(1^\infty)$ and $P_\infty(01^\infty)$ with the left Krieger cover. These are the only vertices in the left Krieger cover corresponding to intrinsically synchronising predecessor sets.

AFT shifts. The original definition of an almost finite type (AFT) shift is due to Marcus [25, Definition 4].

Definition 2.16. The shift space S is said to be of *almost finite type* if there is an irreducible subshift of finite type X and a factor map $\pi : \mathsf{X} \to \mathsf{S}$ that is one-to-one on a non-trivial open set.

In [5] a hierarchy of sofic shifts is given, in terms of a certain degree. By [5, Proposition 3] the irreducible strictly sofic shifts of degree 1 are precisely the AFT shifts. Many authors (see [25, 8, 39, 18], for example) have studied AFT shifts and, as a result, we now have the following list of equivalent conditions on a strictly sofic shift.

Theorem 2.17. *Let S be a strictly sofic shift. The following are equivalent*

(1) *The shift* S *is AFT.*
(2) *The left Fischer cover of* S *is right-closing.*
(3) S *has a minimal cover (i.e. an SFT* X *and a factor map* $\pi : \mathsf{X} \to \mathsf{S}$ *such that any other factor map* $\phi : \mathsf{B} \to \mathsf{S}$ *(from an SFT* B *onto* S*) must factor through* π*).*
(4) *The right and left Fischer covers of* S *are conjugate as SFTs.*

Example 2.18. The even shift described in Example 2.15 is an example of an AFT shift since its left Fischer cover is right-closing. However, note that its left Krieger cover is reducible and it is examples of this type which inspire the following sections.

3. The reducibility structure of left Krieger cover of a shift is a flow invariant

The remarkable result of Parry and Sullivan in [30] shows that flow equivalence between shift spaces is generated by conjugacy and a certain operation that is nowadays referred to as *symbol expansion*. In this section we begin by describing the proper communication graph $PC(E)$ of a directed graph E, which describes the irreducible components of the associated shift of finite type (X_E, σ). Our main result, Theorem 3.10 shows that the proper communication graph of the left Krieger cover of a shift space is a flow invariant – hence the reducibility of the left Krieger cover of a shift is a flow invariant.

The following definition is based on [24, Section 4.4] (see also [34, §1]):

Definition 3.1. Let v, w be vertices in the directed graph E. We say that v *is connected to* w (written $v \geq w$) if there is a path $\mu \in E^*$ with $s(\mu) = v$ and $r(\mu) = w$. If $v, w \in E^0$ satisfy $v \geq w$ and $w \geq v$ then we say that v *communicates with* w.

Remark 3.2. Each vertex v communicates with itself via the empty path. It is then straightforward to check that communication is an equivalence relation.

The vertices of E are partitioned into *communicating classes*, that is, maximal sets of vertices such that each vertex communicates with all the other vertices in the class. For each communicating class C_i of E, the subgraph E_i of E whose vertices are the elements of C_i and whose edges are the elements e of E^1 with $s(e), r(e) \in C_i$ is called an *irreducible component* of E. If the number of irreducible components of E is 1 then E is irreducible. If an edge $e \in E^1$ has $s(e) \in C_i$ and $r(e) \in C_j$ where $i \neq j$ then we call e a *transitional edge for* E. We shall consider the following subset of the communication relation which is not, in general, an equivalence relation.

Definition 3.3. We say that vertices v and w *properly communicate* if there are paths $\mu, \nu \in E^*$ of length greater than or equal to one with $s(\mu) = v, s(\nu) = w$, $r(\mu) = w$ and $r(\nu) = v$.

Remark 3.4. The proper communication relation is not, in general, reflexive. A vertex v can only properly communicate with itself if it lies on some circuit $\lambda \in E^*$.

The proper communication relation may be used to define maximal disjoint subsets PC_i of vertices such that for all $v, w \in PC_i$ we have that v properly communicates with w. We call these sets *proper communication sets of vertices* and use this relation to define a directed graph which summarises the structure of the graph E in the following manner.

Definition 3.5. Let E be a directed graph. The *proper communication graph* $PC(E)$ is a directed graph constructed using the following data. For each proper communication set PC_i of E, we draw a vertex PC_i so that

$$PC(E)^0 = \{PC_i : PC_i \text{ is a proper communication set of vertices in } E\}.$$

For each $i \neq j$ we draw an edge e_{ij} from PC_i to PC_j if there are vertices $v \in PC_i$ and $w \in PC_j$ such that $v \geq w$ so that

$$PC(E)^1 = \{e_{ij} \,:\, v \geq w \text{ for some } v \in PC_i \text{ and } w \in PC_j\}$$

and we have $s(e_{ij}) = PC_i$ and $r(e_{ij}) = PC_j$.

Remarks 3.6. (1) Note that if i, j are such that there are vertices $v \in PC_i$ and $w \in PC_j$ such that $v \geq w$, then this relation holds for every pair of vertices $x \in PC_i$ and $y \in PC_j$ since all vertices in a proper communication set communicate with each other.

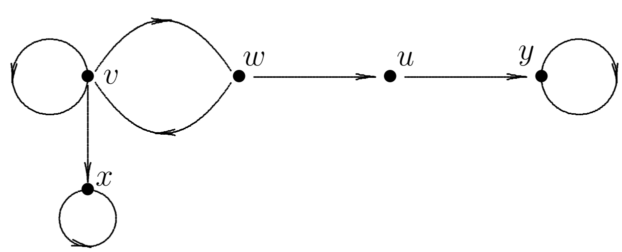
(2) The proper communication graph $PC(E)$ contains no circuits. Suppose there is a circuit $\alpha \in PC(E)^*$. Then our construction of $PC(E)$ ensures that α passes through at least two distinct vertices PC_i and PC_j. But then there are vertices $v \in PC_i$ and $w \in PC_j$ such that $v \geq w$ and vertices $x \in PC_j$ and $y \in PC_i$ with $x \geq y$. Our preceding remark then tells us that we must have $w \geq v$ so that v and w properly communicate, and so are in the same proper communication set, contradicting the maximality of PC_i and PC_j.

(3) The proper communication graph of row-finite graph E may be used to describe the lattice of gauge-invariant ideals of the graph C^*-algebra $C^*(E)$ described in [23, 2], for example. If $(E_K, \mathcal{L}_K)$ is the left Krieger cover of a sofic shift X, then by [10] this implies that the proper communication graph of E_K describes the lattice of gauge-invariant ideals of the Matsumoto algebra $\mathcal{O}_{\mathsf{X}}$ described in [26, 13], for example.

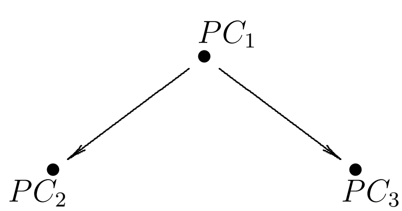
Examples 3.7. (1) If the directed graph E contains no circuits, then the proper communication graph $PC(E)$ is the empty graph.

(2) If the directed graph E contains at least one circuit and is irreducible, then the proper communication graph $PC(E)$ will consist of one vertex and no edges.

(3) We draw the proper communication graph $PC(E)$ for the following directed graph

$E :=$ 

whose proper communication sets are $PC_1 = \{v, w\}$, $PC_2 = \{x\}$ and $PC_3 = \{y\}$. The proper communication graph $PC(E)$ is as shown below



We now turn our attention to the effect of symbol expansion on the left Krieger cover of a shift space. Loosely speaking, the effect of a symbol expansion $\mathcal{E}$ on a shift space X is to form a new shift space X' by adding a new letter (which we refer to as $*$) after all occurrences of certain symbol (in this case a) in X.

More specifically, let X be a shift space over the alphabet $\mathcal{A}$ and fix $a \in \mathcal{A}$. Choose a symbol $* \notin \mathcal{A}$ and set $\mathcal{A}' = \mathcal{A} \cup \{*\}$. Let $\mathcal{E}$ be the symbol expansion map $\mathcal{E} : \mathcal{B}(\mathsf{X}) \to (\mathcal{A}')^*$ which adds the symbol $*$ after every instance of a in a word $w \in \mathcal{B}(\mathsf{X})$. The language $\mathcal{E}(\mathcal{B}(\mathsf{X}))$ is extendible, however it is not factorisable as it does not contain any word beginning with $*$. Let $\mathcal{L}$ be the smallest submonoid of $(\mathcal{A}')^*$ which contains all subwords of the elements of $\mathcal{E}(\mathcal{B}(\mathsf{X}))$, then $\mathcal{L}$ is an extendible, factorisable language and so uniquely defines a shift space X' by [24, Proposition 1.3.4]. By abuse of notation we shall also use $\mathcal{E}$ to refer to the induced maps from X to X' and X^+ to $(\mathsf{X}')^+$.

Our first result shows how the symbol expansion $\mathcal{E}$ affects the connectivity of the left Krieger covers of X and X'.

Lemma 3.8. *Let X, $\mathcal{E}$, X' be as above. Then for each $x^+, y^+ \in \mathsf{X}^+$ there is a path labelled w of length $|w| \geq 1$ from $P_\infty(x^+)$ to $P_\infty(y^+)$ in the left Krieger cover of X if and only if there is a path labelled $\mathcal{E}(w)$ of length $|\mathcal{E}(w)| \geq 1$ from $P_\infty(\mathcal{E}(x^+))$ to $P_\infty(\mathcal{E}(y^+))$ in the left Krieger cover of X'.*

Proof. Suppose there is a path labelled w of length $|w| \geq 1$ from $P_\infty(x^+)$ to $P_\infty(y^+)$ in the left Krieger cover of X. Then there is a right infinite ray $z^+ \in P_\infty(y^+)$ such that $x^+ = wz^+$. But then $\mathcal{E}(x^+) = \mathcal{E}(w)\mathcal{E}(z^+)$ and so there is a path labelled $\mathcal{E}(w)$ from $P_\infty(\mathcal{E}(x^+))$ to $P_\infty(\mathcal{E}(z^+))$ in the left Krieger cover of X'. It is straightforward to check that $P_\infty(\mathcal{E}(z^+)) = P_\infty(\mathcal{E}(y^+))$ in the left Krieger cover of X'. Moreover since $|\mathcal{E}(w)| \geq |w|$ we certainly have $|\mathcal{E}(w)| \geq 1$.

Conversely, suppose there is a path labelled v of length $|v| \geq 1$ from $P_\infty(\mathcal{E}(x^+))$ to $P_\infty(\mathcal{E}(y^+))$ in the left Krieger cover of X'. Then there is a $z^+ \in P_\infty(\mathcal{E}(y^+))$ with $\mathcal{E}(v)z^+ = x^+$. Note that z^+ cannot begin with $*$ since all infinite pasts of such a right-infinite ray must end in a. Thus $z^+ = \mathcal{E}(t^+)$ for some $t^+ \in \mathsf{X}^+$. Now v cannot begin with $*$ since $\mathcal{E}(x^+)$ cannot begin with $*$. Also, v cannot end in a since $\mathcal{E}(t^+)$ cannot begin with $*$. The construction of the left Krieger cover of X' ensures that every a which appears in v is immediately followed by $*$. Thus $v = \mathcal{E}(w)$ for some $w \in \mathcal{B}(\mathsf{X})$. Since v cannot begin in $*$ we must have $|w| \geq 1$. It is then straightforward to check that $x^+ = wt^+$ and it follows, by construction of the left Krieger cover of X, since $P_\infty(t^+) = P_\infty(y^+)$ that there is a path labelled w from $P_\infty(x^+)$ to $P_\infty(y^+)$, establishing our result. □

We now give the relationship between the proper communication graphs of the left Krieger covers of X and X'.

Proposition 3.9. *Let X, $\mathcal{E}$, X' be as above. Then there is a one-to-one correspondence between the proper communication sets of the left Krieger covers $(E_K(\mathsf{X}), \mathcal{L}_K)$ and $(E_K(\mathsf{X}'), \mathcal{L}_K)$ of X and X' respectively. Moreover, the proper communication graphs $PC(E_K(\mathsf{X}))$ and $PC(E_K(\mathsf{X}'))$ are isomorphic as directed graphs.*

Proof. We note first that

$$(\mathsf{X}')^+ = (\mathcal{E}(\mathsf{X}))^+ = \mathcal{E}(\mathsf{X}^+) \cup \sigma\left(\mathcal{E}(a\mathsf{X}^+ \cap \mathsf{X}^+)\right)$$

since $\mathcal{E}(\mathsf{X}^+)$ consists precisely of those one-sided infinite sequences in X' which do not begin with $*$. By Lemma 3.8 vertices $P_\infty(x^+)$ and $P_\infty(y^+)$ in the left Krieger cover $(E_K(\mathsf{X}), \mathcal{L}_K)$ of X properly communicate if and only if the vertices $P_\infty(\mathcal{E}(x^+))$ and $P_\infty(\mathcal{E}(y^+))$ properly communicate in the left Krieger cover $(E_K(\mathsf{X}'), \mathcal{L}_K)$ of X'.

We now show that vertices of the form $P_\infty(\sigma(\mathcal{E}(ax^+)))$ in $(E_K(\mathsf{X}'), \mathcal{L}_K)$ do not give rise to any additional proper communication sets. In $(E_K(\mathsf{X}'), \mathcal{L}_K)$ these vertices only appear in the configuration

$$P_\infty(\mathcal{E}(ax^+)) \xrightarrow{\quad a \quad} P_\infty(\sigma(\mathcal{E}(ax^+))) \xrightarrow{\quad * \quad} P_\infty(\mathcal{E}(x^+))$$

and since every past of $\sigma(\mathcal{E}(ax^+))$ necessarily ends in an a, we see that these two edges are the only ones emitted and received by $P_\infty(\sigma(\mathcal{E}(ax^+)))$.

This implies that if $P_\infty(x^+)$ and $P_\infty(ax^+)$ properly communicate in the $(E_K(\mathsf{X}), \mathcal{L}_K)$, then $P_\infty(\mathcal{E}(x^+))$, $P_\infty(\mathcal{E}(ax^+))$, and $P_\infty(\sigma(\mathcal{E}(ax^+))$ properly communicate in $(E_K(\mathsf{X}'), \mathcal{L}_K)$ and so $P_\infty(\mathcal{E}(x^+))$, $P_\infty(\mathcal{E}(ax^+)$ and $P_\infty(\mathcal{E}(\sigma(\mathcal{E}(ax^+)))$ are all elements of the same proper communication set.

On the other hand, if there is no path from $P_\infty(x^+)$ to $P_\infty(ax^+)$ in $(E_K(\mathsf{X}), \mathcal{L}_K)$ there can be no path from $P_\infty(\mathcal{E}(x^+))$ to $P_\infty(\sigma(\mathcal{E}(ax^+)))$ and so these elements do not properly communicate. Consequently, symbol expansion does not change the number of proper communication sets.

Let PC_i^{X} denote the proper communication sets of $(E_K(\mathsf{X}), \mathcal{L}_K)$ and let $PC_i^{\mathsf{X}'}$ denote the proper communication sets of $(E_K(\mathsf{X}'), \mathcal{L}_K)$. Let ϕ^0_{PC} denote the bijection between the proper communication sets of $(E_K(\mathsf{X}), \mathcal{L}_K)$ and $(E_K(\mathsf{X}'), \mathcal{L}_K)$ which satisfies the condition that for all properly communicating vertices $P_\infty(x^+)$ in $(E_K(\mathsf{X}), \mathcal{L}_K)$ we have $P_\infty(x^+) \in PC_i^{\mathsf{X}}$ if and only if $P_\infty(\mathcal{E}(x^+)) \in \phi^0_{PC}(PC_i^{\mathsf{X}})$. It follows from Lemma 3.8 that there is an edge with $s(e) = PC_i^{\mathsf{X}}$ and $r(e) = PC_j^{\mathsf{X}}$ in $PC(E_K(\mathsf{X}))$ if and only if there is an edge $\phi^1_{PC}(e)$ with $s(\phi^1_{PC}(e)) = \phi^0_{PC}(PC_i^{\mathsf{X}})$ and $r(\phi^1_{PC}(e)) = \phi^0_{PC}(PC_j^{\mathsf{X}})$ in $PC(E_K(\mathsf{X}'))$. Thus the map $\phi_{PC} = (\phi^0_{PC}, \phi^1_{PC}) : PC(E_K(\mathsf{X})) \to PC(E_K(\mathsf{X}'))$ is a directed graph isomorphism and our result is established. □

For sofic shifts we can go even further:

Theorem 3.10. *The proper communication graph $PC(E_K(\mathsf{X}))$ is a flow invariant for sofic shifts.*

Proof. By Proposition 3.9 the proper communication graph is preserved under symbol expansion. It is shown in [21] that the left Krieger cover is a conjugacy invariant for sofic shifts, and so the proper communication graph will also be preserved under conjugacy. Our result follows since symbol expansion and conjugacy are the operations which generate flow equivalence for shifts by [30]. □

Corollary 3.11. *Let X be a sofic shift whose left Krieger cover is reducible. Suppose that X' is flow equivalent to X, then X' is sofic with a reducible left Krieger cover.*

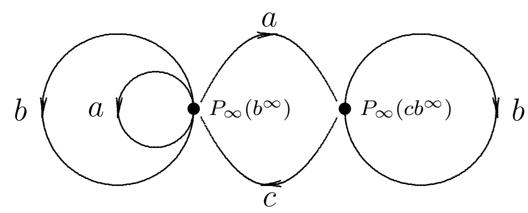
Proof. By [24, Corollary 3.2.3] any shift conjugate to a sofic shift is also sofic. Suppose that X' is obtained from X by a symbol expansion $\mathcal{E}$. The proof of Proposition 3.9 shows that the left Krieger cover of X' is obtained from that of X by adding a vertex $P_\infty(\sigma(\mathcal{E}(ax^+)))$ between $P_\infty(\mathcal{E}(ax^+))$ and $P_\infty(\mathcal{E}(x^+))$ with incoming label a and outgoing label $*$ for every vertex of the form $P_\infty(ax^+)$ in the left Krieger cover of X. The resulting labelled graph will be finite and so the shift X' will be sofic, and the result follows from [30] once again.

The remaining statement follows immediately from Theorem 3.10. □

4. Reducibility of the left Krieger cover of an AFT shift

The following example is adapted from the one found in [5, §2.3 Figure 2].

Example 4.1. Consider the sofic shift X with left Fischer cover as shown



The shift X is irreducible by Remark 2.4 and is strictly sofic since the left Fischer cover has two distinct edges with the same label. Since there are two distinct representatives of the right-ray ab^∞ starting at the same vertex, the left Fischer cover is not right-closing which implies that X is not AFT by Theorem 2.17. One checks that any right-ray x^+ which only contains the symbols a, b has predecessor set $P_\infty(x^+) = P_\infty(b^\infty) = \mathsf{X}^-$. Any right-ray of the form wcy where $y \in \mathsf{X}^+$ and w contains an a also has predecessor set $P_\infty(wcy) = \mathsf{X}^- = P_\infty(b^\infty)$. However any right-ray of the form $b^n cy$ where $y \in \mathsf{X}^+$ and $n \geq 0$ is such that $z = \ldots(ac)(ac) \in \mathsf{X}^-$ does not belong to $P_\infty(b^n cy)$. Since every representative of the right-ray $b^n cy$ must start at the same vertex we conclude that $P_\infty(b^n cy) = P_\infty(cb^\infty)$ for all $n \geq 0$ and so the left Krieger and Fischer covers coincide. In particular, the left Krieger cover is irreducible.

On the other hand, Example 2.15 shows that there are strictly sofic shifts with reducible left Krieger covers. It is then natural to try to identify the class of strictly sofic shifts for which the left Krieger cover is reducible. In this section we produce a partial answer to this question in Theorem 4.6: If the shift is AFT then the left Krieger cover is reducible. However, by Example 4.9 we see that there are irreducible non AFT shifts which have a reducible left Krieger cover.

Before giving the proof of our main result we need a few technical results concerning the left Krieger cover of an AFT shift. By Remark 2.14 (ii) we shall identify the left Fischer cover of an irreducible strictly sofic shift X with the minimal irreducible subgraph of the left Krieger cover of X.

The following result shows that the vertices in the left Krieger cover of an irreducible strictly sofic shift that correspond to non intrinsically synchronising predecessor sets do not connect to the vertices of the left Fischer cover (see Remark 2.14 (ii)).

Lemma 4.2. *Let X be an irreducible strictly sofic shift. Suppose X has a predecessor set $P_\infty(x^+)$ that is not instrinsically synchronising. Then the left Krieger cover $(E_K, \mathcal{L}_K)$ of X is reducible.*

Proof. We show that there is no intrinsically synchronising predecessor set $P_\infty(y^+)$ and path λ in E_K^* with $s(\lambda) = P_\infty(x^+)$ and $r(\lambda) = P_\infty(y^+)$. Suppose otherwise, then without loss of generality we may assume that y^+ is an intrinsically synchronising ray for X. By definition of the left Krieger cover we must have $P_\infty(\mathcal{L}_K(\lambda)y^+) = P_\infty(x^+)$. Since y^+ is an intrinsically synchronising ray, it follows that $P_\infty(x^+)$ is an intrinsically synchronising predecessor set, a contradiction. Thus the left Krieger cover of X is reducible. $\square$

The following lemma gives a characterisation of intrinsically synchronising right-rays in a sofic shift in terms of their representatives in the left Krieger cover.

Lemma 4.3. *Let X be a sofic shift, then $w \in \mathcal{B}(\mathsf{X})$ is intrinsically synchronising if and only if $P_\infty(wx^+) = P_\infty(wy^+)$ for all $x^+, y^+ \in \mathsf{X}^+$ such that $wx^+, wy^+ \in \mathsf{X}^+$. In particular, all representatives of w in the left Krieger cover of X begin at the same vertex.*

Proof. Suppose that $w \in \mathcal{B}(\mathsf{X})$ is an intrinsically synchronising word, and that $x^+, y^+ \in \mathsf{X}^+$ are such that $P_\infty(wx^+) \neq P_\infty(wy^+)$. Then, without loss of generality, we may assume that there is $x^- \in P_\infty(wx^+)$ such that $x^-wy^+ \notin \mathsf{X}$. Since $x^-wx^+ \in \mathsf{X}$, $wy^+ \in \mathsf{X}^+$ and $x^-wy^+ \notin \mathsf{X}$ it follows from [24, Corollary 1.3.5] that there are $m, n \geq 1$ such that $x^-_{-n} \dots x^-_{-1}w \in \mathcal{B}(\mathsf{X})$ and $wy_1 \dots y_m \in \mathcal{B}(\mathsf{X})$ but $x^-_{-n} \dots x^-_{-1}wy_1 \dots y_m \notin \mathcal{B}(\mathsf{X})$. But this contradicts the hypothesis that w is an intrinsically synchronising word, hence $P_\infty(wx^+) \subseteq P_\infty(wy^+)$. A symmetric argument shows that $P_\infty(wy^+) \subseteq P_\infty(wx^+)$ and hence $P_\infty(wx^+) = P_\infty(wy^+)$ as required.

Conversely, suppose that $w \in \mathcal{B}(\mathsf{X})$ is such that $P_\infty(wx^+) = P_\infty(wy^+)$ for all $x^+, y^+ \in \mathsf{X}^+$ such that $wx^+, wy^+ \in \mathsf{X}^+$. Then by Remark 2.12 (i) every path in $(E_K, \mathcal{L}_K)$ labelled w begins at $P_\infty(wx^+)$. If uw and $wv \in \mathcal{B}(\mathsf{X})$ then there are paths $\mu\nu$ and $\nu'\lambda \in E_F^*$ with $|\nu| = |\nu'| = |w|$ and $\mathcal{L}_F(\mu\nu) = uw$ and $\mathcal{L}_F(\nu'\lambda) = wv$. Since $\mathcal{L}_F(\nu) = \mathcal{L}_F(\nu') = w$, by hypothesis we must have $s(\nu) = s(\nu') = P_\infty(wx^+) = r(\mu)$ and so $\mu\nu'\lambda \in E_F^*$. Thus $\mathcal{L}_F(\mu\nu'\lambda) = uwv \in \mathcal{B}(\mathsf{X})$ and so w is an intrinsically synchronising word. □

The final statement of Lemma 4.3 is an analogue of [37, Lemma 1.1] for left Krieger covers. For completeness we include a proof of the following Lemma (cf. [21, Lemma 4.2]).

Lemma 4.4. *Let X be a strictly sofic shift. Then there is a word $w \in \mathcal{B}(\mathsf{X})$ such that $x = \dots ww \dots \in \mathsf{X}$ has more than one representative in the left Fischer cover of X. In particular there are at least two distinct circuits, $\alpha, \beta \in E_F^*$ with $\mathcal{L}_F(\alpha) = \mathcal{L}_F(\beta) = w$.*

Proof. Let X be a strictly sofic shift over $\mathcal{A}$. Then the factor map $\pi_{\mathcal{L}} : \mathsf{X}_{E_F} \to \mathsf{X}$ induced by the 1-block map $\mathcal{L}_F : E_F^1 \to \mathcal{A}$ fails to be injective at some point $x \in \mathsf{X}$. Let $y, z \in \pi_{\mathcal{L}}^{-1}(x)$ with $y \neq z$. We claim that there is an $n \in \mathbf{Z}$ such that $\{(r(y_i), r(z_i)) : i \geq n\} \subseteq E_F^0 \times E_F^0$ does not contain (v, v) for any $v \in E_F^0$. Since $y \neq z$ there is an $n \in \mathbf{Z}$ such that $y_n \neq z_n$. Let $u_n = r(y_n)$ and $v_n = r(z_n)$. Then since $\mathcal{L}(y_n) = \mathcal{L}(z_n)$ and the left Fischer cover is left-resolving we cannot have $u_n = v_n$. Now the edges y_{n+1} and z_{n+1} begin at different vertices so $y_{n+1} \neq z_{n+1}$. An inductive argument completes the proof of the claim.

As $E_F^0 \times E_F^0$ is finite there must be $(u, v) \in E_F^0 \times E_F^0$ with $u \neq v$ and $m \geq 1$ such that $(r(y_i), r(z_i)) = (u, v) = (r(y_{i+m}), r(z_{i+m}))$ where $i \geq n$. Hence $y_{i+1} \dots y_{i+m}$ and $z_{i+1} \dots z_{i+m}$ are distinct circuits α and β in E_F with $\mathcal{L}_F(\alpha) = \mathcal{L}_F(\beta) = w$, say. Hence $\dots \alpha\alpha \dots$ and $\dots \beta\beta \dots$ are the required representatives of x in the left Fischer cover. □

Lemma 4.5. *Let X be a strictly sofic shift and $w \in \mathcal{B}(\mathsf{X})$ be as in Lemma 4.4. Then w is not an intrinsically synchronising word. Moreover, for every positive integer k_0 there is an integer k with $k > k_0$ and $x^+ \in \mathsf{X}^+$ such that $P_\infty(w^{k_0}x^+) = P_\infty(w^k x^+)$.*

Proof. Now by Lemma 4.4 there are circuits $\alpha \neq \beta$ in E_F^* with $r(\alpha) = s(\alpha) \neq r(\beta) = s(\beta)$ such that $\mathcal{L}_F(\alpha) = \mathcal{L}_F(\beta) = w$. If w is intrinsically synchronising, then Lemma 4.3 implies that $s(\alpha) = s(\beta) = P_\infty(wx^+)$ for some $x^+ \in \mathsf{X}^+$, a contradiction.

Since X is a sofic shift there are a finite number of distinct predecessor sets. It follows, by the pigeonhole principle, that for each positive integer k_0 we can find an integer k with $k > k_0$ and an $x \in \mathsf{X}^+$ such that $P_\infty(w^{k_0}x^+) = P_\infty(w^k x^+)$ as required. □

We now prove that the left Krieger cover of a strictly sofic, irreducible AFT shift X is reducible.

Theorem 4.6. *Let X be an irreducible, strictly sofic AFT shift. Then the left Krieger cover $(E_K, \mathcal{L}_K)$ of X is reducible.*

Proof. Suppose, for contradiction that the left Fischer cover and the left Krieger cover of X coincide, in particular $E_K = E_F$ is irreducible by Remarks 2.14 (ii). Let w be as in Lemma 4.4, let k_0 be a positive integer and let $k > 0$ and $x^+ \in \mathsf{X}^+$ satisfy the conditions of

Lemma 4.5 with respect to w and k_0. Let α and β be distinct circuits in E_F^* representing w as in Lemma 4.4. It follows by Remark 2.12 (i) there is a path $\mu \in E_F^*$ from $P_\infty(w^k x^+)$ to $P_\infty(w^{k_0} x^+)$ labelled w^{k-k_0}. Since $P_\infty(w^k x^+) = P_\infty(w^{k_0} x^+)$, μ is a circuit. Note that since α and β are distinct representatives of w, μ must be distinct from one of α^{k-k_0} and β^{k-k_0}. Without loss of generality we may assume that μ is distinct from β^{k-k_0}.

Since the left Krieger cover and the left Fischer cover coincide the predecessor set $P_\infty(w^{k_0} x^+)$ is intrinsically synchronising, say $P_\infty(w^{k_0} x^+) = P_\infty(m y^+)$ where $m \in \mathcal{B}(\mathsf{X})$ is an intrinsically synchronising word. By Remark 2.12 (i) there is a path ν starting at $P_\infty(m y^+)$ with $\mathcal{L}_F(\nu) = m$. Since E_F is irreducible there is a path λ from $r(\nu)$ to $s(\beta)$. Let $v = \mathcal{L}_F(\lambda)$. Then mvw is represented by $\nu\lambda\beta$. Since m is intrinsically synchronising we have $P_\infty(mvw^{k_0} x^+) = P_\infty(m y^+)$ by Lemma 4.3. By Remark 2.12 (i) there is a path $\kappa \in E_F^*$ from $P_\infty(mvw^{k_0} x^+)$ to $P_\infty(w^{k_0} x^+) = P_\infty(m y^+)$ labelled by mv. Since $P_\infty(mvw^{k_0} x^+) = P_\infty(m y^+)$ it follows that κ is a circuit. Since $(E_F, \mathcal{L}_F)$ is left-resolving and $\mathcal{L}_F(\kappa) = mv = \mathcal{L}_F(\nu\lambda)$ we must have $r(\kappa) \neq r(\nu\lambda)$ and hence $\kappa \neq \nu\lambda$.

Since X is AFT, $\pi_{\mathcal{L}}$ is right-closing with delay D by Theorem 2.17 (2). By choosing n large enough so that $|mvw^{n(k-k_0)}| \geq D + |mv| + 1$ we obtain $\kappa = \nu\lambda$ by Remark 2.9, a contradiction. Hence our assumption that the left Krieger cover was equal to the left Fischer cover must have been false. In particular, the predecessor set $P_\infty(w^{k_0} x^+)$ is not intrinsically synchronising and the result then follows from Lemma 4.2. □

We are now able to deduce the following result concerning the Matsumoto algebra associated to a shift space (for more details see [13, 26]). Our argument employs graph C^*-algebra techniques. For more details about graph C^*-algebras see [2].

Corollary 4.7. *Let X be an irreducible strictly sofic AFT shift, then the Matsumoto algebra $\mathcal{O}_{\mathsf{X}}$ associated to X is not simple.*

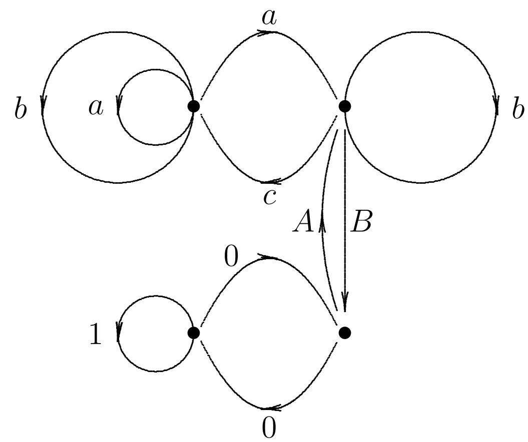
Proof. By [3, Corollary 6.8] (see also [10, Theorem 3.5], [27, Proposition 7.1]) $\mathcal{O}_{\mathsf{X}}$ is isomorphic to the graph algebra $C^*(E_K)$. By Theorem 4.6 E_K is reducible and the result then follows by [2, Proposition 5.1]. □

Remark 4.8. Another left-resolving cover of a sofic shift X over the alphabet $\mathcal{A}$ may be defined as follows. For $w \in \mathcal{B}(\mathsf{X})$ let $P_\infty^f(w) = \{x^- \in \mathsf{X}^- : x^- w \in \mathsf{X}^-\}$ be the collection of all left-rays which may precede w. The cover is given by the labelled graph $(E_\infty^f, \mathcal{L})$ where the vertices of E_∞^f are the predecessor sets $P_\infty^f(w)$ and there is an edge labelled $a \in \mathcal{A}$ from $P_\infty^f(w)$ to $P_\infty^f(v)$ if and only if $av \in \mathcal{B}(\mathsf{X})$ and $P_\infty^f(w) = P_\infty^f(av)$. One may check that the labelling $\mathcal{L}$ is well-defined.

A corresponding version of Theorem 4.6 may be proved: Let X be an irreducible, strictly sofic AFT shift. Then the cover $(E_\infty^f, \mathcal{L})$ of X is reducible. Since the sets $P_\infty^f(w)$ consist of left-infinite rays, the argument used to prove Theorem 4.6 applies *mutatis mutandis.*

The converse of Theorem 4.6 does not hold as we see in the following example which was inspired by [32, Figure 7.5.1].

Example 4.9. Consider the sofic shift Y with left Fischer cover as shown below



The shift Y is irreducible by Remark 2.4, and since the left Fischer cover has two distinct edges with the same label it follows that Y is strictly sofic. Since there are two distinct representatives of the right-ray ab^∞ starting at the same vertex, the left Fischer cover is not right-closing which implies that X is not AFT by Theorem 2.17. However Y contains the even shift as a subshift. As in Example 2.15 there are three different predecessor sets associated to right rays in the symbols 0 and 1. Hence the left Krieger cover of Y contains a subgraph isomorphic to the left Krieger cover of the even shift and so is reducible (cf. Example 2.15).

5. One-sided shifts and the past set cover

Recall from Definition 2.10, that for a shift space X, the set X^+ of right-rays consists of all sequences $x_0x_1\dots$ where $x \in \mathsf{X}$. This space is still invariant under the shift map σ, and the resulting pair (X^+, σ) is called a one-sided shift space (cf. [24, p140]). Where possible we shall indicate that we are working with a one-sided shift by adding the $^+$ superscript. Most of the concepts and definitions we have used for two-sided shifts still apply to one-sided shifts. For example, any presentation of a two-sided shift is a presentation of the corresponding one-sided shift (simply consider right-infinite paths). Hence the one-sided sofic shifts are the ones presented by finite labelled graphs (cf. [24, Definition 3.1.3]).

For one-sided shifts the left Krieger cover construction given in Definition 2.11 does not make sense as there are no left-infinite rays. In this situation it is usual to work with the past-set cover (see Definition 5.2). We give the definition for two-sided shifts, and note that it makes sense for one-sided shifts.

Definition 5.1. Let X be a shift space and w a word in $\mathcal{B}(\mathsf{X})$. The *predecessor set* $P_f(w)$ of w in X is the set of all words that can precede w in X; that is,

$$P_f(w) = \{v \in \mathcal{B}(\mathsf{X}) : vw \in \mathcal{B}(\mathsf{X})\}.$$

By suitably adapting [24, Theorem 3.2.10] one may show that X^+ is a one-sided sofic shift if and only if the set of all predecessor sets $P_f(w)$ is finite.

Definition 5.2. Suppose that X is a shift space over $\mathcal{A}$. The *past set cover of* X is the presentation $(E_P, \mathcal{L}_P)$ where the vertices of E_P are the predecessor sets $P_f(w)$. For predecessor sets $P_f(w), P_f(v)$ and $a \in \mathcal{A}$ there is an edge labelled a from $P_f(w)$ to $P_f(v)$ if and only if $P_f(av) = P_f(w)$.

As in [24, p.73] one checks that the labelling, $\mathcal{L}_P$ is well-defined and left-resolving.

One may again define the left Fischer cover of a one-sided sofic shift X^+ to be the minimal left-resolving cover of X^+.

Lemma 5.3. *Let X be a (one- or two-sided) sofic shift. Let $w \in \mathcal{B}(\mathsf{X})$. Then w is intrinsically synchronising if and only if whenever $v \in \mathcal{B}(\mathsf{X})$ is such that $wv \in \mathcal{B}(\mathsf{X})$ we have $P_f(wv) = P_f(w)$. In particular every path in $(E_P, \mathcal{L}_P)$ labelled w begins at $P_f(w)$.*

Proof. Let $w \in \mathcal{B}(\mathsf{X})$ be such that for all $v \in \mathcal{B}(\mathsf{X})$ with $wv \in \mathcal{B}(\mathsf{X})$ we have $P_f(wv) = P_f(w)$. Suppose that $u \in \mathcal{B}(\mathsf{X})$ is such that $uw \in \mathcal{B}(\mathsf{X})$. Then $u \in P_f(w) = P_f(wv)$, by assumption and so we must have $uwv \in \mathcal{B}(\mathsf{X})$. Thus w is intrinsically synchronising.

Conversely, let $m \in \mathcal{B}(\mathsf{X})$ be intrinsically synchronising and $v \in \mathcal{B}(\mathsf{X})$ be such that $mv \in \mathcal{B}(\mathsf{X})$. Suppose that $u \in \mathcal{B}(\mathsf{X})$ is such that $u \in P_f(m)$ then $umv \in \mathcal{B}(\mathsf{X})$ and so $u \in P_f(mv)$, which implies that $P_f(m) \subseteq P_f(mv)$. However, by definition of $P_f(mv)$ we have $P_f(mv) \subseteq P_f(m)$, and so $P_f(mv) = P_f(m)$ as required. □

Lemma 5.4. *Let X be a (one- or two-sided) irreducible sofic shift with past set cover $(E_P, \mathcal{L}_P)$. Let E_P^m denote the subgraph of E_P with vertices $P_f(m)$ where $m \in \mathcal{B}(\mathsf{X})$ is intrinsically synchronising and the edges in E_P connecting them. Then $(E_P^m, \mathcal{L}_P)$ is a minimal left-resolving irreducible presentation of X and so is isomorphic (as a labelled graph) to the left Fischer cover $(E_F, \mathcal{L}_F)$ of X.*

Proof. Let $m_1, m_2 \in \mathcal{B}(\mathsf{X})$ be intrinsically synchronising, then since X is irreducible there is $u \in \mathcal{B}(\mathsf{X})$ such that $m_1 u m_2 \in \mathcal{B}(\mathsf{X})$. Since all subwords of $m_1 u m_2$ which contain m_2 are also intrinsically synchronising it follows that there is a path in E_P^m labelled $m_1 u$ from $P_f(m_1 u m_2) = P_f(m_1)$ to $P_f(m_2)$. Hence E_P^m is irreducible.

The labelled graph $(E_P^m, \mathcal{L}_P)$ is left-resolving as it is a subgraph of $(E_P, \mathcal{L}_P)$. We claim that $(E_P^m, \mathcal{L}_P)$ is a presentation of X. Given $u \in \mathcal{B}(\mathsf{X})$ the irreducibility of X tells us that there are $m_1, v_1 \in \mathcal{B}(\mathsf{X})$ with $m_1 v_1 u \in \mathcal{B}(\mathsf{X})$ and m_1 intrinsically synchronising. Irreducibility again tells us that there are $m_2, v_2 \in \mathcal{B}(\mathsf{X})$ with $(m_1 v_1 u) v_2 m_2 \in \mathcal{B}(\mathsf{X})$ where m_2 is intrinsically synchronising. Hence there is a path labelled $m_1 v_1 u v_2$ starting at $P_f(m_1)$ in $(E_P^m, \mathcal{L}_P)$, and u is a finite labelled path in $(E_P^m, \mathcal{L}_P)$, which establishes our claim. If we omit the vertex $P_f(m)$ from E_F^m then Lemma 5.3 tells us that we cannot represent m in the resulting labelled graph, and so the presentation $(E_P^m, \mathcal{L}_P)$ is minimal. The final statement follows from the left-resolving analogue of [24, Theorem 3.3.18]. □

We shall henceforth identify the left Fischer cover of an irreducible sofic shift with a subgraph of the past set cover (cf. Remarks 2.14). The proof of the following Lemma is a minor modification of that given for Lemma 4.2.

Lemma 5.5. *Let X be an irreducible (one- or two-sided) strictly sofic shift. Let $w \in \mathcal{B}(\mathsf{X})$ be such that $P_f(w) \neq P_f(m)$ for any instrinsically synchronising word $m \in \mathcal{B}(\mathsf{X})$. Then the past set cover $(E_P, \mathcal{L}_P)$ of X is reducible.*

The following results for one-sided shifts can be proved in a similar way to Lemma 4.4 and Lemma 4.5.

Lemma 5.6. *Let X^+ be a one-sided strictly sofic shift. Then there is $w \in \mathcal{B}(\mathsf{X}^+)$ such that $x = ww \ldots \in \mathsf{X}^+$ has more than one representative in the left Fischer cover of X^+. More specifically, there are at least two distinct circuits, $\alpha, \beta \in E_F^*$ with $\mathcal{L}_F(\alpha) = \mathcal{L}_F(\beta) = w$.*

Lemma 5.7. *Let X^+ be a one-sided strictly sofic shift and let w be as in Lemma 5.6. Then w is not intrinsically synchronising. Moreover, for every positive integer k_0 there is a positive integer k with $k > k_0$ such that $P_f(w^{k_0}) = P_f(w^k)$.*

The proof of the following theorem is similar to the one we have given for Theorem 4.6.

Theorem 5.8. *Let X be a (one- or two-sided) irreducible, strictly sofic AFT shift. Then the past set cover $(E_P, \mathcal{L}_P)$ of X is reducible.*

Remarks 5.9. (1) Another left-resolving cover of a (one- or two-sided) sofic shift X over the alphabet $\mathcal{A}$ may be defined as follows. For $x^+ \in \mathsf{X}^+$ let

$$P_f^\infty(x^+) = \{w \in \mathcal{B}(\mathsf{X}) \ : \ wx^+ \in \mathsf{X}^+\}$$

be the collection of words which may precede x^+. The cover is given by the labelled graph $(E_f^\infty, \mathcal{L})$ where the vertices of E_f^∞ are the predecessor sets $P_f^\infty(x^+)$ and there is an edge labelled $a \in \mathcal{A}$ from $P_f^\infty(x^+)$ to $P_f^\infty(y^+)$ if and only if $ay^+ \in \mathsf{X}^+$ and $P_f^\infty(x^+) = P_f^\infty(ay^+)$. One checks that the labelling $\mathcal{L}$ is well-defined.

A corresponding version of Theorem 5.8 may be proved: Let X be an irreducible, (one- or two-sided) strictly sofic AFT shift. Then the cover $(E_f^\infty, \mathcal{L})$ of X is reducible. Since the sets $P_f^\infty(x^+)$ consist of elements of $\mathcal{B}(\mathsf{X})$, the argument used to prove Theorem 5.8 applies *mutatis mutandis*. Note that the cover $(E_f^\infty, \mathcal{L})$ is called the left Krieger cover in [10, 13] and the Perron-Frobenius cover in [32]. It is well known that this cover is isomorphic to the left Krieger cover for two-sided sofic shifts.

(2) It is worth observing that for a two-sided sofic shift the cover $(E_f^\infty, \mathcal{L})$ and the past set cover $(E_P, \mathcal{L}_P)$ are not necessarily isomorphic. For instance, the following example was given in [13, Section 4]: Let Z be the sofic shift over the alphabet $\{1,2,3,4\}$ in which the forbidden blocks are

$$\{12^k1, 32^k12, 32^k13, 42^k14 : k \geq 0\}.$$

The past set cover $(E_P, \mathcal{L}_P)$ of Z has five vertices while the cover $(E_f^\infty, \mathcal{L})$ of Z has four vertices. The past set cover contains $(E_f^\infty, \mathcal{L})$ as a subgraph. Moreover, $(E_P, \mathcal{L}_P)$ is reducible, but $(E_f^\infty, \mathcal{L})$ is irreducible.

(3) Further structural relationships between the five left-resolving covers of a sofic shift that we have described in this paper are given in [17].

(4) By recent results of Pask and Sunkara (see [31]) the past set cover of a sofic β-shift is irreducible. It follows by Theorem 5.8 that sofic β-shifts are not AFT. This provides a different proof of [36, Proposition 2.7 c)]. See also [32].

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