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Anomalous Josephson current through a driven double quantum dot

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Abstract
Josephson junctions based on quantum dots offer a convenient tunability by means of local gates. Here we analyze a Josephson junction based on a serial double quantum dot in which the two dots are individually gated by phase-shifted microwave tones of equal frequency. We calculate the time-averaged current across the junction and determine how the phase shift between the drives modifies the current-phase relation of the junction. Breaking particle-hole symmetry on the dots is found to give rise to a finite average anomalous Josephson current with phase bias between the superconductors fixed to zero. This microwave gated weak link thus realizes a tunable “Floquet $\phi_0$ junction” with maximum critical current achieved for driving frequencies slightly off resonance with the subgap excitation energy. We provide numerical results supported by an analytical analysis for infinite superconducting gap and weak interdot coupling. We identify an interaction-driven $0-\pi$ transition of anomalous Josephson current as a function of driving phase difference. Finally, we show that this junction can be tuned so as to provide for complete rectification of the time-averaged Josephson current-phase relation.

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I. INTRODUCTION

The Josephson junction (JJ) has become a ubiquitous device serving in a wide range of applications, including the superconducting qubits which have led to impressive advances in quantum computing during the past two decades [1–5]. The weak link coupling the two superconductors can either be a plain insulating tunnel barrier, or it may exhibit internal structure such as a normal region, a quantum point contact, a magnetic tunnel barrier, or a quantum dot (QD), which all host subgap states which may strongly influence the current-phase relation (CPR) of the junction [6–12]. In this way, electrically gatable links such as quantum dots or semiconductors offer a certain tunability of the JJ characteristics [13–16], a feature which has been employed in the design of a hybrid gate mon [17–19], adding gate control to the superconducting transmon qubit [20,21], which has already demonstrated its efficiency in solid state quantum computing [22–24].

Whereas normal Josephson junctions carry no current at zero phase bias, $\varphi_{sc} = \varphi_L - \varphi_R$, a weak link which breaks both time-reversal and chiral symmetry may carry an anomalous Josephson current between two superconductors maintained at zero phase bias [25]. A number of proposals [25–38] have been made for such $\phi_0$ junctions with an anomalous Josephson current, $I(\varphi_{sc}) = I_C \sin(\varphi_{sc} + \varphi_0)$, at least two of which have already been realized experimentally [39,40]. Of particular relevance to the present work is the proposal by Zazunov et al. [25] to use a multi-orbital QD with interorbital (spin-orbit) tunneling and an external field. With such a link in the JJ, traversing electrons pick up different phases, depending on the tunneling direction, giving rise to an anomalous Josephson current. This proposal has since been realized in an experiment by Szombati et al. [40], using an InSb-wire QD contacted by superconducting NbTiN leads. Here, we propose a nonequilibrium version of the multi-orbital QD considered in Ref. [25], based on the device illustrated in Fig. 1. In this Josephson junction, the two superconductors are coupled by a serial double quantum dot (DQD) where the two dots are driven by individual ac gate voltages with a common

![Fig. 1. Sketch of a Josephson junction with a structured weak link (gray region) based on a driven double quantum dot. The superconductors (blue) are maintained at a fixed phase bias $\varphi_{sc} = \varphi_L - \varphi_R$, and the weak link is driven by two microwave gates with the same amplitude and frequency, $A$, $\Omega$, shifted in phase by $\theta_L = \theta_L - \theta_R$. The internal and the two external tunneling amplitudes are denoted by $t_d$, $t_L$, and $t_R$, respectively.](image-url)
amplitude, $A$, and microwave frequency, $\Omega$. This endows each of the QDs with Floquet sidebands, which play the roles of the additional spin-orbit coupled orbitals in Ref. [25]. As we demonstrate below, the phase difference between the two drive voltages, $\theta_L = \theta_L - \theta_R$, can have a strong influence on the time-averaged JJ CPR, and with QD levels tuned away from particle-hole symmetry it gives rise to anomalous current, which in the limit of weak tunnel couplings reduces to a simple $\phi_0$ junction, with $\phi_0 = \theta_L$. Since the time-averaged critical current is maximized when the microwave frequency is close to the energy for exciting both of the subgap states induced in the two proximitized quantum dots, this device comprises a nonadiabatic Cooper pair pump, or more aptly a “Floquet $\phi_0$ junction.”

The undriven DQD Josephson junction with individual gating of the two dots has already been realized experimentally [16,41–43], and understood to constitute a strongly correlated two-site system with on-site energies modulated by individual ac gate voltages (cf. Fig. 1). The Hamiltonian reads

$$H(t) = \sum_{\alpha=L,R} H_{sc,\alpha} + H_d(t) + H_t,$$

with superconducting leads described by BCS Hamiltonians

$$H_{sc,\alpha} = \sum_{\mathbf{k},\sigma} \left[ \xi_{\mathbf{k}\sigma} c_{\mathbf{k}\alpha\sigma}^\dagger c_{\mathbf{k}\alpha\sigma} + (\Delta e^{i\phi_0} c_{\mathbf{k}\alpha\uparrow} c_{\mathbf{k}-\mathbf{q}\downarrow} + \text{H.c.}) \right],$$

for $\alpha = L, R$. The two leads are kept at the same chemical potential and are assumed to have the same gap magnitude, $\Delta > 0$, with different phases, $\varphi_{L,R} = \pm \varphi_{\pi}/2$. Both leads are represented by a featureless band structure near a common chemical potential, i.e., $\xi_{\mathbf{k}\alpha} = \epsilon_{\mathbf{k}\alpha} - \mu$, corresponding to a common density of states, $\nu_F$, near the Fermi level. The time-dependent Hamiltonian of the double quantum dot system reads

$$H_d(t) = \sum_{\sigma,\mathbf{a},\mathbf{a}' \in L,R} d_{\mathbf{a}\sigma}^\dagger \left[ \epsilon_{\mathbf{d}\sigma}(t) \tau_{\mathbf{a}\sigma}^0 + i \tau_{\mathbf{a}\sigma}^x d_{\mathbf{a}'\sigma} \right],$$

with individual ac gate voltages given as $\epsilon_{\mathbf{d}\sigma}(t) = \epsilon_d + A \cos(\Omega t + \theta_\sigma)$, in terms of common (time) average energies, $\epsilon_d$, driving amplitudes, $A$, frequencies, $\Omega$, and two independent phase constants, $\theta_\sigma$. Here, $\tau^\dagger$ denotes the $i$th Pauli matrix, $\tau^0$ the Kronecker delta, and $d_d$ is the interdot tunneling
amplitude. The tunneling Hamiltonian reads

$$H = \sum_{k, \sigma; \alpha = L, R} t_{\alpha k} c_{\alpha k \sigma}^\dagger d_{\alpha k}^\sigma + \text{H.c.} \quad (4)$$

Written in terms of Nambu spinors, $\psi_{\alpha k}^\dagger = (c_{\alpha k \uparrow}^\dagger, c_{\alpha k \downarrow})$ and $\phi_{\alpha}^\dagger = (d_{\alpha \uparrow}^\dagger, d_{\alpha \downarrow})$, the full Hamiltonian reads

$$H(t) = \sum_{\alpha k} \psi_{\alpha k}^\dagger (\tilde{\Sigma}_{\alpha k \sigma} - \Delta \sigma) \psi_{\alpha k}$$

$$+ \sum_{\alpha \sigma} \phi_{\alpha}^\dagger (\varepsilon_{d \sigma}(t) \sigma_{\alpha \sigma}^\dagger + t_d \tau_{\sigma \alpha}) \phi_{\alpha}$$

$$+ \sum_{\alpha k} (\psi_{\alpha k}^\dagger \tau_{\alpha} \phi_{\alpha} + \phi_{\alpha}^\dagger \tau_{\alpha}^\dagger \psi_{\alpha k}), \quad (5)$$

where the phase of the superconducting leads has been gauged into the tunneling matrix, $\tilde{\Sigma}_{\alpha k \sigma} = t_d \sigma \delta \phi_{\alpha}^\dagger / \Sigma_{1}$. For simplicity, we assume below that tunneling amplitudes to the leads are real and equal, i.e., $t_d = t_l \equiv t$.

As discussed in the Introduction, we neglect the charging energies of both quantum dots, except for the limiting case of infinite gap considered in the Appendix, and consider this noninteracting resonant level model as an effective model for a proximitized quantum dot.

### III. KELDYSY FLOQUET GREEN’S FUNCTIONS

To calculate the current through the ac-driven device, we employ the nonequilibrium Green’s function technique [71–73]. Dealing with a harmonic drive, it is convenient to use Floquet-Keldysh Green’s functions [74,75], which offer a representation of the two-time Green’s functions, which, besides being convenient for numerical calculations, allows for some degree of physical interpretation of the elementary transport process in terms of Floquet sidebands. The time-dependent charge current from dot $\alpha$ to lead $\alpha$ for this driven junction is found as [73]

$$I_{\alpha}(t) = 2(\epsilon) \text{Tr} \left\{ \sigma_z \text{Re} \left[ \int dt' \left( G_{d,\alpha \sigma}^R(t, t') \Sigma_{\alpha}(t', t) \right) \right. \right.$$}

$$+ \left. G_{d,\alpha \sigma}^A(t, t') \Sigma_{\alpha}(t', t) \right\}, \quad (6)$$

with $\epsilon = |\epsilon|$, and where the trace is taken in Nambu space with Nambu/lead ($\eta/\alpha$) matrix Green’s functions for the quantum dots defined as

$$G^{R, A}_{\alpha \eta, \alpha' \eta'}(t, t') = \mp i \theta(\pm t \mp t') \langle \phi_{\alpha \eta}(t), \phi_{\alpha' \eta'}(t') \rangle, \quad (7)$$

$$G^{<, A}_{\alpha \eta, \alpha' \eta'}(t, t') = i \phi_{\alpha \eta}^\dagger(t') \phi_{\alpha' \eta}(t),$$

$$G^{<, A}_{\alpha \eta, \alpha' \eta'}(t, t') = - i \phi_{\alpha \eta}(t) \phi_{\alpha' \eta'}^\dagger(t'),$$

with self-energies, which are exact to second order in dot-lead tunneling,

$$\Sigma_{\alpha}^{R, A, <}(t) = \sum_k T^A_{\alpha \sigma} g_{\alpha k}^{R, A, <} \tau_{\alpha} \Sigma_{\alpha}(t), \quad (8)$$

where $g_{\alpha k}$ denotes the Nambu Green’s function in lead $\alpha$.

From this self-energy, the dot Green’s functions can be found by solving the steady-state Dyson equations,

$$G^{R, A}_{\alpha}(t, t') = G^{R, A}_{\alpha}(0)(t, t') + \int dt_1 dt_2 G^{R, A}_{\alpha}(t, t_1)$$

$$\times \Sigma^{R, A}_{\alpha}(t_1 - t_2) G^{R, A}_{\alpha}(t_2, t'), \quad (10)$$

$$G^{<}(t, t') = \int dt_1 dt_2 G^{R}(t, t_1) \Sigma^{<}(t_1 - t_2)$$

$$\times G^{A}(t_2, t'), \quad (11)$$

with matrix products between Green’s functions implied.

With a periodic drive, it is convenient to transform these two-time Green’s functions into Floquet matrices [75]

$$O_{nm}(\omega) = \int_{-\infty}^{\infty} dt \frac{1}{T} \int_{0}^{T} dt' e^{i(\omega + \Omega \nu - i\sigma - mn\Omega') t'} O(t, t'), \quad (12)$$

defined with $\omega \in [-\Omega/2, \Omega/2]$. This transformation preserves that the Green’s functions are periodic in both time arguments, with the driving period $T = 2\pi / \Omega$, and thereby rests on the assumption that the system has reached a nonequilibrium steady state (NESS). In this way, the time-averaged current, $J = J_L = -J_R$, may be found from the zeroth Floquet components,

$$J_{\alpha} = \frac{1}{T} \int_{0}^{T} dt I_{\alpha}(t)$$

$$= 2(\epsilon) \text{Tr} \left\{ \sigma_z \text{Re} \left[ \int_{-\Omega/2}^{\Omega/2} d\omega \left( G_{d,\alpha \sigma}^R(\omega) \Sigma_{\alpha}^{<}(\omega) \right. \right. \right.$$

$$+ \left. \left. G_{d,\alpha \sigma}^A(\omega) \Sigma_{\alpha}^{A}(\omega) \right] \right\}, \quad (13)$$

where the momentum-summed lead Nambu Green’s functions are given explicitly as

$$g_{\alpha}^{R, A}(\omega) = \pi \nu F \left( \frac{-(\omega \pm i0_+)}{\sqrt{\Delta^2 - (\omega \pm i0_+)^2}} \right)$$

$$+ \Delta \sigma^\dagger, \quad (15)$$

$$g_{\alpha}^{<}(\omega) = n_F(\omega) g_{\alpha}^{R}(\omega) - g_{\alpha}^{A}(\omega),$$

where $n_F$ denotes the Fermi function. Henceforth, temperature is assumed to be zero.

Finally, using the Dyson equation (10), the retarded double-dot Green’s function is found by inverting the following infinite-dimensional Floquet matrix of $4 \times 4$ matrices in Nambu-dot space:

$$\left( G_{d,\alpha}(\omega) \right)_{\alpha \sigma', \alpha \sigma}^{-1} = \left\{ -t_d \sigma^\dagger \tau_{\alpha \sigma'} + \left[ (\omega + n\Omega) \sigma^0 \right. \right.$$}

$$- \left. T^A_{\alpha \sigma} g_{\alpha k}^{R, A, <} \tau_{\alpha} \right\} \delta_{\sigma \sigma'} - \frac{A}{2} (e^{-i\theta_0 / 2} \delta_{n, m-1} + e^{i\theta_0 / 2} \delta_{m, n-1}) \sigma^z \tau_{\alpha \sigma'}. \quad (17)$$
From the resulting retarded and advanced Green’s functions, the lesser function is found from Eq. (11) by simple matrix multiplication.

**IV. INFINITE-GAP LIMIT WITH WEAK INTERDOT TUNNEL COUPLING**

It is instructive to first consider the analytically tractable limit of an infinite superconducting gap. This limit prohibits quasiparticle tunneling altogether and transport therefore takes place only via Cooper pairs. It captures much of the physics of the bound states, including a singlet to doublet ground state transition in the presence of interactions [45,51]. In the infinite-gap limit, the retarded QD self-energy becomes

\[ \Gamma^R_{\sigma}(\omega + n\Omega) \approx -\Gamma e^{-i\theta_{\sigma}}\sigma_x, \]  

(18)

with \( \Gamma = \pi v_F |t|^2 \), corresponding to an effective Hamiltonian describing a proximitized quantum dot with an induced superconducting gap of \( G \):

\[ H_{\infty}(t) = \sum_{\sigma=L,R} \phi_{\sigma}(t) \sigma^z - \Gamma \sigma^x \phi_{\sigma}, \]

(19)

with a matrix of tunneling amplitudes given by \( T_{\sigma,\nu} = |t| \sigma_\nu \exp(i\sigma_\nu \phi_{\sigma}/2) \), where \( \phi_{\sigma} = \phi_{L\sigma} - \phi_{R\sigma} \).

In order to illustrate the basic microwave-assisted Cooper pair transport mechanism within this infinite-gap model, we calculate here the weak-coupling tunneling charge current from right, to left dot, to second order in the interdot coupling \( t_d \), given by the perturbative expression [76]

\[ I(t) = 2(-\pi)|t_d|^2 \text{Re} \left[ \int_0^t dt' \sigma_\nu^z \sigma_\eta^z \left( \sigma_\nu^{-1} \sigma_{\eta}^{-1} \right)^{\sigma_\nu/2} \times (G_{L\sigma,\eta\nu}^z(t',t) G_{R\sigma,\eta\nu}^z(t',t) - G_{L\sigma,\eta\nu}^z(t',t) G_{R\sigma,\eta\nu}^z(t',t)) \right]. \]

(20)

The driving enters this expression through the time-dependent correlation functions, \( G_{\sigma\nu,\eta\mu}^z(t,t') \), describing the dynamics of the QD proximitized by lead \( \alpha = L, R \).

The perturbative expression for the current requires the Green’s functions for \( t_d = 0 \), and in this case the Hamiltonian (19) describes two independent quantum dots. It is readily diagonalized by the time-dependent Bogoliubov transformation (suppressing the QD index, \( \alpha = L, R \)):

\[ \chi_\nu = U_{\eta\nu} \phi_\eta, \quad \chi^*_\nu = \phi_\eta^* U_{\eta\nu}^{-1}, \]

(21)

with Nambu spinors

\[ \chi = \left( \begin{array}{c} \gamma^\dagger_l \\ \gamma^\dagger_u \end{array} \right), \quad \phi = \left( \begin{array}{c} d^\dagger_l \\ d^\dagger_u \end{array} \right), \]

(22)

and time-dependent unitary transformation matrix

\[ U(t) = \left( \begin{array}{cc} u(t) & -v(t) \\ v(t) & u(t) \end{array} \right), \quad U^{-1}(t) = U^T(t), \]

(23)

with \( E_d(t) = \sqrt{\frac{\varepsilon_d^2(t) + \Gamma^2}{2}} \) and real coherence factors given by

\[ u(t) = \sqrt{\frac{1 + E_d(t)/E_d(t)}{2}}, \quad v(t) = \sqrt{\frac{1 - E_d(t)/E_d(t)}{2}}. \]

(24)

Notice that we omit the \( \alpha = L, R \) subscript for clarity since it only enters in the two different phase shifts, \( \theta_\alpha \), and can readily be reinstalled. This transformation diagonalizes the Hamiltonian for each of the two different proximitized levels,

\[ H^0_{\infty}(t) = \phi^\dagger [E_d(t) \sigma^z - \Gamma \sigma^x] \phi = \chi^\dagger E_d(t) \sigma^z \chi, \]

(25)

and endows the quasiparticles with dynamics governed by the equation of motion,

\[ i \frac{d}{dt} \chi_\nu(t) = \left[ E_d(t) \sigma^z \right] u_{\nu\nu} + \frac{A\Omega}{2E_d(t)} \sigma_{\nu\nu} \chi_\nu(t), \]

(26)

where the last term has been obtained as

\[ -iU_{\nu\nu}(t) \left( \frac{d}{dt} U_{\nu\eta}^{-1}(t) \right) = \frac{A\Omega}{2E_d(t)} \sigma_{\nu\nu}. \]

(27)

The corresponding transformation of the correlation functions reads

\[ G_{\sigma\nu,\eta\mu}^z(t,t') = iU_{\nu\eta}^{-1}(t)U_{\nu\nu}(t) \chi_\mu^+(t') \chi_\nu(t), \]

\[ G_{\sigma\nu,\eta\mu}^z(t,t') = -iU_{\nu\eta}^{-1}(t)U_{\nu\nu}(t) \chi_\mu^-(t') \chi_\nu(t). \]

(28)

The many-body eigenstates of the uncoupled and undriven QD are the empty QD, \( |0\rangle \), the single-electron doublet, \( |\sigma\rangle = d^\dagger_\sigma |0\rangle \), and the doubly occupied QD, \( |2\rangle = d^\dagger_\uparrow d^\dagger_\downarrow |0\rangle \), with energies \( 0, E_d, E_d, \) and \( 2E_d \), respectively. For the proximitized QD, the BCS-like ground state becomes \( |0\rangle = u(|0\rangle + |v\rangle) \), the excited doublet, \( |\sigma\rangle \), remains unchanged, and the highest excited state becomes \( |2\rangle = u^2 - v^2 \rangle \), with energies \( 0, E_d, E_d, \) and \( 2E_d \), respectively (cf. Fig. 2(a)).

When the system is driven with low amplitude, \( A \ll E_d \), close to resonance, i.e., \( \Omega \approx 2E_d \), transport can take place as indicated in Fig. 2(b) to second order in \( t_d \). Since the mixing term (27) is already proportional to driving amplitude, \( A \), we shall neglect the time dependence of \( E_d(t) \) in its denominator and in coherence factors \( u \) and \( v \), assuming that \( A \ll \max(e_d, \Gamma) \). This allows us to include the mixing term (27) within a rotating-wave approximation (RWA), which leads to the following equation of motion:

\[ i \frac{d}{dt} \chi_\nu(t) \approx \left[ E_d \sigma^z \right] u_{\nu\nu} + ge^{-i(\Omega t + \theta_{\nu\nu})} \sigma_{\nu\nu} \chi_\nu(t), \]

(29)

where \( g = A\Omega/2E_d^2 \). This equation is readily solved by

\[ \chi_\nu(t) = e^{-i\sigma_{\nu\nu}(\Omega t + \theta_{\nu\nu})/2} U_{\nu\nu}^{-1} \chi_\nu(t), \]

(30)

with a secondary unitary transformation as

\[ U_{\nu\nu}^{-1} = \left( \begin{array}{cc} \tilde{u} & \tilde{v} \\ -\tilde{v} & \tilde{u} \end{array} \right), \]

(31)

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Reinstating the lead index to the driven DQD junction in progression from panels 1–5. Driving the microwave gates with the excited doublet has energy $E_{d}$ and the two-quasiparticle state has energy $2E_{d}$. (b) Diagram illustrating the path of a Cooper pair through the driven DQD junction in progression from panels 1–5, followed by a two-step excitation transfer to the right QD via $\nu_{d}$ (2–4), which finally decays via its own microwave gate (4 and 5).

defined in terms of

$$\bar{u} = \sqrt{1 + \delta/E}/2, \quad \bar{v} = \sqrt{1 - \delta/E}/2.$$  \hspace{1cm} (32)

Here, $\delta = E_d - \Omega/2$ is the detuning, and the Rabi energy $E = \sqrt{\delta^{2} + g^{2}}$ captures the slow time evolution of the corotating Nambu spinor

$$\xi_{\mu}(t) = \xi_{\mu}(0)e^{-it\sigma_{\mu}^{z}},$$  \hspace{1cm} (33)

with initial condition $\xi_{\mu}(0) = e^{i\sigma_{0}\phi/2}U_{\mu\nu}\chi_{\nu}(0)$.

For concreteness, we assume both proximitized quantum dots to be in their ground state, $|0\rangle$, at time $t = 0$. Using the relations

$$\chi_{\nu}(0)|\bar{0}\rangle = \delta_{\nu,2}|-\downarrow\rangle, \quad \chi_{\nu}(0)|\bar{0}\rangle = \delta_{\nu,1}|\uparrow\rangle,$$  \hspace{1cm} (34)

the time-evolved states are found as

$$\chi_{\nu}(t)|\bar{0}\rangle = X_{\nu}(t)|\downarrow\rangle, \quad \chi_{\nu}(t)|\bar{0}\rangle = i\sigma_{\nu}\chi_{\nu}(t)|\uparrow\rangle,$$  \hspace{1cm} (35)

with

$$X(t) = \left(e^{-i2\delta t/2} e^{-it\delta} (g/E) \sin(\tilde{E}t)\right).$$  \hspace{1cm} (36)

Reinstating the lead index $\alpha$ on $t$ and inserting this into Eq. (28), one finally arrives at the correlation functions

$$G_{\alpha\alpha',\alpha''\gamma'}(t, t') = i(h_{\alpha}(t), a_{\alpha'}(t'), a_{\alpha''}(t'))_{\eta},$$

$$G_{\alpha\alpha',\alpha''\gamma'}(t, t') = -i(h_{\alpha}(t), -b_{\alpha'}(t'), a_{\alpha''}(t'), -b_{\alpha''}(t'))_{\eta},$$  \hspace{1cm} (37)

with generalized time-dependent coherence factors,

$$a_{\alpha}(t) = e^{it\theta_{\alpha}/2}(-v, u)_{\alpha}X_{\alpha}(t), \quad b_{\alpha}(t) = e^{it\theta_{\alpha}/2}(u, v)_{\alpha}X_{\alpha}(t),$$  \hspace{1cm} (38)

satisfying $|a_{\alpha}(t)|^2 + |b_{\alpha}(t)|^2 = 1$. The time-dependent current in Eq. (20) may now be expressed as

$$I(t) = 4(-\epsilon)|d_{\alpha}|^2 \int_{0}^{t} dt' (a_{\alpha}(t')b_{\alpha}(t')$$

$$+ b_{\alpha}(t')a_{\alpha}(t') e^{-it\delta} - (L \leftrightarrow R, \varphi_{sc} \leftrightarrow -\varphi_{sc})],$$  \hspace{1cm} (39)

involving time-local interdot pair amplitudes like

$$a_{\alpha}(t) b_{\alpha}(t) = i(u g^{\dagger}/2 - v^{\dagger} e^{-i\theta_{\alpha}/2})(g/E) \sin(\tilde{E}t)$$

$$\times [\cos(\tilde{E}t) + i(\delta/E) \sin(\tilde{E}t)]$$

$$+ uv e^{-i(2\delta t + \pi)} (g/E)^{2} \sin^{2}(\tilde{E}t)$$

$$+ c^{\dagger} \sin(\tilde{E}t) + i(\delta/E) \sin(\tilde{E}t)]^2.$$  \hspace{1cm} (40)

with $\theta_{d} = \theta_{L} - \theta_{R}$ and $\bar{\theta} = (\theta_{L} + \theta_{R})/2$.

This result relies on the weak-amplitude assumption of neglecting the time dependence of $E_{d}(t)$ in Eq. (26) and the subsequent RWA, which is expected to hold only close to resonance, i.e., for $\delta, g \ll \Omega/2$. In the undriven limit, $g \rightarrow 0$, the product in Eq. (40) reduces to

$$a_{\alpha}(t) b_{\alpha}(t) = uv e^{i(2E_{d} + \bar{\theta})},$$

which leads to the time-dependent current

$$I(t) = \frac{|d_{\alpha}|^2 \Gamma^{2}}{\epsilon_{d}^{2} + \Gamma^{2}} \sin(\varphi_{sc}) R e^{i2E_{d}t} \int_{0}^{t} dt' e^{i2E_{d}t'}$$

$$= \frac{|d_{\alpha}|^2 \Gamma^{2}}{(\epsilon_{d}^{2} + \Gamma^{2})^{3/2}} \sin(\varphi_{sc}) [1 - \cos(2E_{d}t)].$$  \hspace{1cm} (41)

The time-dependent part, which derives from the $t = 0$ limit of the integral, vanishes under long-time averaging leaving only the equilibrium supercurrent

$$J = \lim_{t_{0} \rightarrow \infty} \frac{1}{t_{0}} \int_{0}^{t_{0}} dt I(t) = \frac{|d_{\alpha}|^2 \Gamma^{2}}{(\epsilon_{d}^{2} + \Gamma^{2})^{3/2}} \sin(\varphi_{sc}).$$  \hspace{1cm} (42)

In the other limit, where $\delta = 0$, the system is driven exactly at resonance, and one finds that all terms in $I(t)$ become sinusoidal and the long-time average of the current vanishes altogether. Notice that this is regardless of $\varphi_{sc}$, meaning that to leading order in $t_{d}$ the resonant drive washes out the equilibrium supercurrent carried by the undriven system.

In the general case of finite coupling, $g$, and finite detuning, $\delta$, the last two terms in Eq. (40) will generally be suppressed by the fast oscillating phase factors, and we may therefore retain only the first slowly oscillating term, which carries no information about $\theta$ and only depends on the phase difference, $\theta_{d}$. These products reduce to

$$a_{\alpha}(t) b_{\alpha}(t) \approx -f(\theta_{d}) h(t), \quad a_{\alpha}(t) b_{\alpha}(t) \approx -f(-\theta_{d}) h(t),$$  \hspace{1cm} (43)

with

$$f(\theta_{d}) = \frac{E_{d}}{\delta \epsilon_{d}} \cos(\theta_{d}/2) + i \sin(\theta_{d}/2),$$

$$h(t) = \frac{\delta g}{E^{2}} \sin^{2}(\tilde{E}t) - i \frac{g}{2E} \sin(2\tilde{E}t).$$  \hspace{1cm} (44)
Finally, introducing \( \kappa(\theta_d, \varphi_{sc}) = f(\theta_d)e^{i\varphi_{sc}/2} = \kappa' + i\kappa'' \), the current takes the following form:

\[
I(t) \approx 8 \langle e \rangle |t_d|^2 \text{Re} \left[ \kappa' + i\kappa''(\theta_d, \varphi_{sc})h(t) \int_0^t dt' h(t') \right] \kappa'(\theta_d, \varphi_{sc})
- (\theta_d \leftrightarrow -\theta_d, \varphi_{sc} \leftrightarrow -\varphi_{sc})
- 2e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
(2\dot{\mathcal{E}}t \sin(2\dot{\mathcal{E}}t) - \sin^2(2\dot{\mathcal{E}}t) - 4 \sin^4(\dot{\mathcal{E}}t)).
\] (45)

The long-time average becomes

\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t dt (\sin^2(2\dot{\mathcal{E}}t) + 4 \sin^4(\dot{\mathcal{E}}t) - 2\dot{\mathcal{E}}t \sin(2\dot{\mathcal{E}}t))
= 2 + 4(\mathcal{E}t_0),
\] (46)

which still depends on the integration time, \( t_0 \), but with a well-defined long-time average deriving from the first term, which leads to

\[
J = 4e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
\approx 4e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
\frac{2E_d - \Omega}{(2E_d - \Omega)^2 + (\alpha G/E_d)^2}
\times \left[ \left( \frac{\delta g}{E_d} \cos^2(\theta_d/2) - \sin^2(\theta_d/2) \right) \sin(\varphi_{sc}) \right]
+ \left( \frac{\delta g}{E_d} \sin(\theta_d) \right) \cos(\varphi_{sc})
\right]
= 4e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
\left[ \frac{2\delta g E_d \tan(\theta_d/2) + \pi \theta(\delta g E_d \tan(\theta_d/2)) - e_d, \varphi_{sc} = \arctan(\frac{2\delta g E_d \tan(\theta_d/2) + \pi \theta(\delta g E_d \tan(\theta_d/2)) - e_d, \varphi_{sc}}{\delta g E_d \tan(\theta_d/2) - e_d)}.\right. (48)

This expression is valid to leading order in \( t_d \), close to resonance, \( |\Omega - 2E_d| \ll |\Omega| \), and for weak drive amplitude \( \alpha G \ll \Omega^2 \). The fast rotating terms which we have neglected in this expression are formally smaller by factors \( \delta/\Omega \) and \( g/\Omega \) and provide for the following correction to the long-time average current (47):

\[
\delta J = \frac{4e|t_d|^2\delta}{16E_d^2(g^2 + \delta^2)^2[4(g^2 + \delta^2)^2 - \Omega^2]}
\times \left( \left( 16g^2(\cos(\theta_d) + \sin(\varphi_{sc})/2) \sin(\theta_d/2) \right)
+ \left( \delta g^2 \sin(\varphi_{sc})/2 \cos(\theta_d/2) \right)
+ \left( \right) \left( 2g^2 \cos(\theta_d) + \sin(\theta_d/2) \right)
+ \left( \delta^2 \sin(\varphi_{sc}) \right) \left( 2g^2 \delta (3\delta - 4\Omega) \right)
- 9g^4 + \delta^3(15\delta - 8\Omega))
\right]
\] (49)

which tends towards the undriven result, Eq. (42), in the limit of \( g \to 0 \) with \( g \ll \delta \ll \Omega \).

The time-averaged current in Eq. (47) switches sign, when tuning the drive across resonance at \( \Omega = 2E_d \) from red (\( \delta > 0 \)), to blue (\( \delta < 0 \)) detuning. This can be traced back to the fact that each of the driven quantum dots Rabi oscillates between states \( |0⟩ \) and \( |2⟩ \), with time-averaged probabilities of finding the QD in either state given by \( P_{0,2} = (1 \pm \delta^2/E_d^2)/2 \).

Since the time-dependent current in Eq. (39) has opposite sign when choosing the initial state to be \( |2⟩ \), this implies that the total current averages to zero when the system is driven at resonance, \( \delta = 0 \). Likewise, the long-time average of the interdot pair amplitude (40) comprising the current in Eq. (41) reduces to

\[
\langle a_d b_d \rangle = \lim_{t_0 \to \infty} \frac{1}{t_0} \int_0^t dt \, a_d(t)b_d(t)
= - \frac{g\delta}{2E_d^2} f(\theta_d), \] (50)

which is linear in the detuning and vanishes at resonance. The average current attains its maximum for \( \Omega = 2E_d \pm A\Gamma/(\sqrt{3}E_d) \) with

\[
\langle I \rangle_{\text{max}} \approx \frac{3\sqrt{3}e|t_d|^2E_d}{4A\Gamma} \left[ \frac{\delta g^2}{E_d} \cos^2(\theta_d/2) + \sin^2(\theta_d/2) \right]
\times \sin(\varphi_{sc} + \varphi_0). \] (51)

One should keep in mind that the counter-rotating terms neglected within the RWA will lead to a Bloch-Siegert shift [54] of the resonance frequency of the order of \( A^2/E_d \ll 1 \), and for strong enough drive amplitudes the RWA breaks down altogether.

Tuning the levels away from the Fermi level, i.e., for \( |\varepsilon_d| \gg \Gamma \), we have \( E_d \approx |\varepsilon_d| \) and the current becomes

\[
J \approx 4e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
\frac{2E_d - \Omega}{(2E_d - \Omega)^2 + (\alpha G/E_d)^2}
\times \left( \right) \left( 1 - \frac{\delta^2}{E_d^2} \cos^2(\theta_d/2) \right) \sin(\varphi_{sc} + \varphi_0), \] (52)

which is a Floquet \( \varphi_0 \) function with

\[
\varphi_0(\theta_d) = \text{sgn}(\varepsilon_d/\theta_d) + \pi \theta(-\cos(\theta_d)), \] (53)

exhibiting a sharp sign change in current when \( \cos(\theta_d) \) passes through zero.

In the opposite limit, where the two levels are close to the Fermi levels of the two superconducting leads, i.e., \( |\varepsilon_d| \ll \Gamma \), we have \( E_d \approx \Gamma \) and arrive at

\[
J \approx 4e|t_d|^2\frac{\delta g^2}{E_d} \kappa'(\theta_d, \varphi_{sc}) \kappa'(\theta_d, \varphi_{sc})
\frac{2(\Omega - \delta) \sin^2(\theta_d/2)}{(2|\varepsilon_d| - \Omega)^2 + (\alpha G/\varepsilon_d)^2}
\times \sin(\varphi_{sc}). \] (54)

which corresponds to a normal Josephson 0 junction below resonance (\( \Omega < 2\Gamma \)), and a \( \pi \) junction above resonance (\( \Omega > 2\Gamma \)). In this limit, the phase shift of the two drives, \( \theta_{1,2} \), serves only to modulate the amplitude, attaining maximum critical time-averaged current when the drives are shifted by \( \theta_{1,2} = \pi \), and blocking it altogether for \( \theta_{1,2} = 0 \).

This average current was calculated under the assumption of an even number of electrons occupying each of the two levels, with the specific initial condition that the system is in its lowest energy state at time zero. In a real system, however, quasiparticle poisoning and relaxation will cause occasional switching of the parity of each of the two levels, limiting the accessible integration time, \( t_i \). With typical parity flip
times of the order of 20–200 \mu s \cite{65,77–79}, a resonant drive frequency, \Omega \sim 2E_d, of the order of 10 GHz, say, will take the system through some 10^6 drive cycles before the parity is flipped. The long-time average in Eq. (46) makes the current in Eq. (45) resemble the undriven result in Eq. (41), but with frequency down-converted from \Ed to the much slower Rabi frequency \E and with amplitude given by Eq. (47). Within the validity of the RWA, \E/\Omega \sim 10^{-3}, say, this down-converted current would still oscillate through some 10^3 Rabi periods between subsequent parity flips, leaving sufficient integration time to define a long-time average. The full problem incorporating the stochastic parity switching dynamics poses an interesting problem in itself, which we shall not pursue further in this work. Instead, we shall analyze the steady-state Dyson equation (11), in which the parity is relaxed in the infinite gap limit by a weak coupling to a normal metallic reservoir. For a finite gap, the Floquet sidebands of the continuum provide the same effect and the normal metallic reservoir is no longer needed.

Notice that the full lesser component of the Dyson equation has a second contribution \cite{73}, (1 + \Sigma^A G^\theta (1 + \Sigma^A G^\theta)), referring to the initial lesser function, and that this term has been omitted altogether in Eq. (11). This omission rests on the tacit assumption that \Sigma^\infty contains relaxation mechanisms, which will wash out the initial conditions, i.e., that \Sigma^\infty \gg G_0^{-,1}G_0^{-,1} = (G_0^{-,1} - G_0^{-,1})f_0, where f_0 denotes an initial distribution function. In the present tunneling problem, \Sigma^\infty refers to quasiparticle tunneling to and from the superconducting leads and to the weak tunneling of electrons directly between the dots and a normal metal reservoir. The former contribution vanishes altogether in the infinite-gap limit, and the steady-state Dyson equation (11) as well as the Floquet-Keldysh transformation (12) are therefore justified in the infinite gap limit by the normal metal tunneling rate, \Gamma_m, which is large enough to dominate the finite \eta = (G_0^{-,1} - G_0^{-,1})/(2i) used in our numerical implementation of the bare Green’s functions of the leads, yet small enough not to affect the result.

V. NUMERICAL RESULTS

In this section we present the numerical results obtained with the Floquet-Keldysh Green’s functions introduced in Sec. III. We shall focus entirely on the time-averaged quantities, which may be found as the zeroth Floquet components, and we shall narrow down the rather large parameter space to illustrate some of the most interesting time-averaged current-phase relations realized by this driven junction.

In practice, the inversion of the Nambu-Floquet matrix (17) is carried out by truncating to the \n_{\text{max}} lowest Floquet modes, i.e., working with square matrices of dimension 4(1 + 2\n_{\text{max}}). For all numerical results presented below, we ensure that \n_{\text{max}} is large enough that increasing it further does not affect the results. Furthermore, we use a finite broadening in the lead Green’s functions, replacing \delta_0 by \eta = 10^{-3} in Eq. (15), which, like all energy and frequency (\bar{\Omega} \equiv 1) parameters used below (except for the infinite-gap limit), is specified in units of \Delta. In order to facilitate the numerical integration over the sharp subgap states in the infinite-gap limit, both levels are weakly coupled to a normal metallic lead with chemical potential aligned with the two superconducting leads, \mu_m = 0. This gives rise to a finite imaginary part, \Gamma_m, of the d-electron self-energies (14), which is chosen to be smaller than any other scale in the problem, yet resolved by the discretized numerical integrations. In practice, this corresponds to a finite parity relaxation time, which is longer than any other timescale in the problem. As discussed in the previous section, this also constitutes the formal justification of the steady-state Dyson equation (11). For a finite gap, the continuum of the superconducting leads provides the necessary broadening for the numerical calculations, and the normal metallic lead is not needed.

A. Infinite- and large-gap results

In order to connect to the results of the previous section, we first consider the large-gap limit, \Delta \gg \Gamma, in which all current is carried by Cooper pairs, at weak tunnel coupling and close to resonance. The resulting time-averaged current [cf. Eq. (6)] is shown in Fig. 3 as a function of the driving frequency in a narrow range around resonance. It is seen to match the perturbative results very well. Figure 4 shows the time-averaged current as a function of the two phase differences, \varphi_{sc} and \theta_{sc}, for a fixed driving frequency slightly below resonance. We show this together with two cuts illustrating a good match to Eq. (47).

Increasing the amplitude of the drive and fixing the driving phase shift at \theta_{sc} = \pi/2, gives rise to highly nontrivial CPRs, of which a few examples are shown in Fig. 5. For a small driving amplitude (A = 0.1\Gamma shown) the CPR is modified by narrow dips of the current, occurring at values of \varphi_{sc}, for which an integer multiple of the driving frequency, \Omega, becomes...
a weak normal metal tunneling rate $\Gamma_1$ coupled by $t_d$ obtained by numerical evaluation of Eq. (6). The current vanishes per conductor phase difference and phase shift of the two drives at the black solid lines. (b) Solid lines correspond to cuts along the dashed blue ($\varphi_{sc} = 0$) lines indicated in the upper panel, together with the corresponding analytical infinite-gap weak-coupling current from Eq. (47) limit (dashed). In both panels, parameters are $\Gamma_d = \Gamma/100$, $A = \Gamma_1 = \Gamma/10$, and $\Omega = 2\Gamma$. In the numerical evaluation the infinite gap was replaced by $\Delta = 10^4$, while $\eta = 10^{-4}$ and $\Gamma_m = \Gamma/500$.

FIG. 4. (a) Density plot of the weak-coupling current vs superconductor phase difference and phase shift of the two drives obtained by numerical evaluation of Eq. (6). The current vanishes at the black solid lines. (b) Solid lines correspond to cuts along the dashed blue ($\theta_d = \pi$), and green ($\varphi_{sc} = 0$) lines indicated in the upper panel, together with the corresponding analytical infinite-gap weak-coupling current from Eq. (47) limit (dashed). In both panels, parameters are $\Gamma_d = \Gamma/100$, $A = \Gamma_1 = \Gamma/10$, and $\Omega = 2\Gamma$. In the numerical evaluation the infinite gap was replaced by $\Delta = 10^4$, while $\eta = 10^{-4}$ and $\Gamma_m = \Gamma/500$.

FIG. 5. Current-phase relation in the infinite-gap limit ($\Delta = 10^4\Gamma$) with and without drives of amplitude $A$, frequency $\Omega = 2\Gamma$, and phase shift $\theta_d = \pi/2$. Both levels have energy $\varepsilon_{d} = 0.8\Gamma$ with a weak normal metal tunneling rate $\Gamma_{sc} = \Gamma/500$, and are tunnel coupled by $t_d = 2\Gamma$.

resonant with a subgap transition energy. This is similar to what is predicted for superconducting junctions with only a single drive [56,57], the main difference being that in the present case (like in Ref. [61]) the CPRs are not symmetric around $\varphi_{sc} = \pi$ and the resonant dips are not current nodes. For higher driving amplitudes ($A = 0.8\Gamma$ shown) the current is reduced, as for the junctions with only a single drive, but now the CPR is severely modified with no special significance of either $\varphi_{sc} = 0$ or $\varphi_{sc} = \pi$, both exhibiting finite supercurrent.

For comparison, in the Appendix we calculate the current using the same parameters as for the blue curve ($A = 0.1\Gamma$) in Fig. 5, but now using Floquet states to determine the time evolution starting from the nondriven even-parity ground state. This is done in the infinite-gap limit and with no coupling to a normal metal ($\Gamma_m = 0$), and the long-time average of the resulting current shows excellent correspondence with the steady-state current in Fig. 5 (cf. Fig. 10). Furthermore, intradot Coulomb interactions are straightforwardly included in this approach and are shown in Fig. 10 to remove the sharp dips in the current when the interaction strength becomes of the order of the driving frequency $\Omega$. The systematic behavior relies on many parameters, but the main effect of the interactions is to change the resonance condition for the drive. In a real system with parity relaxation, increasing the interaction strength will of course stabilize an odd-parity ground state for the undriven system serving as a $\pi$ junction [16,45].

B. Finite-gap results

Turning to the case of a finite BCS gap, i.e., the more realistic case where $\Delta$ is no longer much larger than all other energy scales, we first fix the superconductor phase difference to zero ($\varphi_{sc} = 0$) and plot the time-averaged current as a function of the drive frequency in the upper panel (a) of Fig. 6. The lower panel (b) shows the corresponding time-averaged density of states on the proximitized double quantum dot, exhibiting pronounced peaks at two slightly different ABS energies, split by the interdot tunnel coupling $t_d$, together with their weaker first, and even weaker second Floquet sidebands. From this plot, one may now understand the various features in the current.

Coming from large $\Omega$, the small bump in the current near the vertical grid line labeled $c$ corresponds to a resonance between the first sideband of the two ABS and the BCS quasiparticle continuum near $\Omega = \Delta + E_{ABS} \simeq 1.5\Delta$. At lower frequencies the current attains its largest magnitude slightly off resonance, and a node right at resonance, $\Omega \simeq 2E_{ABS} \simeq 1.2\Delta$, near the vertical grid line labeled $b$. Here a positive ABS energy matches the first sideband of a negative ABS energy and vice versa, as illustrated in Fig. 2 for weak $t_d$. For lower frequencies near the vertical grid line labeled $a$, the current exhibits another bump, corresponding to crossings of ABS sidebands with each other or with the continuum. Apart from this additional structure arising from the finite gap or from a substantial driving amplitude, the overall frequency dependence of the current clearly resembles the resonant structure found in the infinite-gap limit in Fig. 3. For the rest of the paper, we fix the drive frequency to be slightly off the main resonance at $\Omega \simeq 2E_{ABS}$ where this anomalous supercurrent attains its maximum.
To further investigate the effect of the continuum in the anomalous current we show in Fig. 7 how the current varies with \( \Omega/\Delta \) evaluated with the time-averaged density of states on the DQD. Both panels are evaluated with \( \Delta = 3\Gamma/2 \), \( t_d = \Gamma \), \( A = 0.4\Gamma \), \( \varepsilon_d = 0.1\Gamma \), \( \varphi_{sc} = 0 \), \( \theta_d = \pi/2 \), and \( n_{\text{max}} = 7 \).

The time-averaged current can be tuned in a number of ways. In the previous sections, we have focused on the resonant aspect by tuning the external driving frequency, \( \Omega \), and the BCS gap, \( \Delta \). In this section, we keep these parameters fixed and study instead how the CPR of this driven DQD Josephson junction is modified by the level position, \( \varepsilon_d \), and the driving phase shift, \( \theta_d \), respectively. As established in the Appendix for single-state time evolution in the infinite-gap limit, the symmetries of the Floquet Hamiltonian guarantee the following symmetries of the time-averaged current:

\[
J(\varphi_{sc}, \theta_d, \varepsilon_d) = -J(-\varphi_{sc}, \theta_d, -\varepsilon_d),
\]

\[
= -J(-\varphi_{sc}, -\theta_d, \varepsilon_d).
\]

As we shall see below, these symmetries are also respected when the current is calculated from steady-state Green’s functions and with a finite BCS gap.

In Fig. 8(a), we show the time-averaged current as a function of superconducting phase difference, \( \varphi_{sc} \), and level position, \( \varepsilon_d \), with a driving phase shift of \( \theta_d = \pi/2 \). The current is calculated using the numerical steady-state Green’s function approach and is observed to respect the symmetry relation expressed by Eq. (55). Three different cuts are shown in the lower panel (b). For \( \varepsilon_d = 0 \), the vertical black dashed cut illustrates the usual antisymmetry around \( \varphi_{sc} = \pi \), and zero anomalous current at \( \varphi_{sc} = 0 \). This symmetry breaks down for \( \varepsilon_d \neq 0 \) and, as indicated by the vertical green dashed cut, may even lead to a unidirectional supercurrent, corresponding to complete rectification. The horizontal blue dashed cut, on the other hand, illustrates the antisymmetry of the current under
FIG. 8. (a) Density plot of the time-averaged current vs $\varepsilon_d$ and $\varphi_{sc}$. The current vanishes at the black solid lines. (b) Cuts in panel (a) for, respectively, $\varphi_{sc} = 0$ (blue), $\varepsilon_d = 0$ (black), and $\varepsilon_d = 0.2\Delta$. The green cut at $\varepsilon_d = 0.2\Delta$ shows a completely rectified current, which remains positive for all values of $\varphi_{sc}$. In both panels, parameters are $2\Gamma = t_d = 0.7\Delta$, $A = 0.8\Delta$, $\Omega = 0.9\Delta$, $\theta_d = \pi/2$, and $n_{\text{max}} = 7$.

inversion of $\varepsilon_d$ for $\varphi_{sc} = 0$. In Fig. 9(a), we show instead the time-averaged current as a function of superconducting phase difference, $\varphi_{sc}$, and level position, $\varepsilon_d$, with a driving phase shift of $\theta_d = 0.8\Delta$. This plot is observed to respect the symmetry relation expressed by Eq. (56). Again, three different cuts are shown in the lower panel (b). The vertical red dashed cut shows the anomalous relation between time-averaged current and the driving phase shift, $\theta_d$, with the superconductor phase difference fixed at $\varphi_{sc} = 0$. Akin to an ordinary $\pi$ junction, the anomalous current attains its maximum near, although not right at, $\theta_d = \pi/2$. The three horizontal (black, blue, and green) dashed cuts illustrate the strongly modified CPRs for fixed $\theta_d$. Switching from $\theta_d = 0$ to $\theta_d = \pi$, the driven Josephson junction is seen to switch the current-phase relation from a $\pi$ to a 0 junction, as seen in the black and the blue curves, respectively, up to a slight anharmonicity in both. Once again, the green cut realizes a rectified time-averaged current. Since the BCS gap is finite, there is no guarantee that this completely rectified pump current is exclusively a current of Cooper pairs. Nevertheless, as we show in Fig. 12 in the Appendix a nearly completely rectified current can also be obtained in the infinite-gap limit where all current must be carried by Cooper pairs, indicating that there is no fundamental obstacle to attaining a unidirectional time-averaged supercurrent for all $\varphi_{sc}$.

VI. CONCLUSIONS

The microwave-enabled DQD Josephson junction studied here offers a highly tunable superconducting circuit element, in which the traversing supercurrent is controlled by two phase-shifted microwave tones applied to the individual gate voltages of each quantum dot. This driven device comprises an effective Josephson junction with a highly nontrivial CPR which can be tuned electronically. More specifically, the phase-shifted microwave drives induce an alternating tunneling current between the two superconducting leads, whose long-time average exhibits an anomalous and often highly anharmonic relation to the superconductor phase difference.

The supercurrent response to the driving relies on nonadiabatic resonant photon-assisted tunneling. This was established in the infinite-gap limit by means of perturbation theory and by time evolution of the nondriven ground state using Floquet theory (cf. Appendix ). For a finite BCS gap, the steady-state
time-averaged current was calculated numerically by means of Floquet-Keldysh Green's functions. Whereas the general finite-gap current may include some fraction of normal current carried by BCS quasiparticles, the main resonant pump current arising when the drive frequency is slightly off resonance with the energy difference between the two subgap ABS was argued to be carried mainly by Cooper pairs.

For clarity, we have restricted our analysis to a symmetrically coupled device, where only the microwave phase shift breaks the $L/R$-inversion symmetry. None of the salient features demonstrated in this case rely critically on this symmetry, as can readily be assessed in the infinite-gap limit using the Floquet theory employed in the Appendix . Likewise, local Coulomb interactions, reflecting the finite charging energies of the quantum dots, are readily included within the infinite-gap Floquet theory and was shown in the Appendix to alter the resonance conditions and thereby affect the time-averaged current. Nevertheless, the anomalous Josephson effect (and the rectification) persisted, and was found to exhibit a $0-\pi$ transition in $\theta_d$, as the interaction strength increased past a critical value. Furthermore, the symmetry relations, Eqs. (55) and (56), were generalized to the interacting case, and used to infer that the anomalous current vanishes more generally at

\[
\epsilon_d = -\frac{U}{2}.
\]

Note added. Recently, we became aware of a paper by Soori [85] pointing out a Josephson diode effect present in a nonadiabatically driven two-site SNS junction explored recently arising when the drive frequency is slightly off resonance with the energy difference between the two subgap ABS was argued to be carried mainly by Cooper pairs.

In light of the recent interest in Josephson diodes [81–84], we emphasize the fact that this driven Josephson junction offers complete rectification of the time-averaged supercurrent.

Figure added. Recently, we became aware of a paper by Soori [85] pointing out a Josephson diode effect present in a nonadiabatically driven two-site SNS junction explored also in Ref. [70]. With this work we have demonstrated that complete rectification persists throughout a finite range of parameters. The more detailed requirements for rectification in driven junctions are relegated to future work.

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APPENDIX: FLOQUET ANALYSIS OF THE INTERACTING INFINITE-GAP LIMIT

The infinite-gap limit offers relatively easy access to the symmetries of the problem, which are also revealed by the steady-state numerical calculations presented in the main text. In this Appendix, we employ Floquet theory to provide a brief supplementary analysis of this more tractable limit, in which local Coulomb interactions on the quantum dots can readily be included. Furthermore, since no quasiparticle excitations are involved in the infinite-gap limit, all currents calculated below are carried exclusively by Cooper pairs. We choose to consider only the even-parity sector, but a similar analysis is straightforwardly made for the odd-parity sector.

In the even-parity sector, the Hilbert space is spanned by the basis $\{|00\rangle, |20\rangle, |02\rangle, |22\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle\}$, where the left (right) index indicates the many-body states of the left (right) dot. In this basis the first quantized Hamiltonian reads

\[
\hat{H}_{e,\infty} = \begin{pmatrix}
0 & -\Gamma & -\Gamma & 0 & 0 & 0 \\
-\Gamma & 2\epsilon_d + U & \Gamma & 0 & -\Gamma & z^* \\
-\Gamma & 0 & 2\epsilon_d + U & \Gamma & -\Gamma & -z^* \\
0 & -\Gamma & -\Gamma & 4\epsilon_d + 2U & 0 & 0 \\
0 & z^* & z & 0 & 2\epsilon_d & 0 \\
0 & -z^* & -z & 0 & 0 & 2\epsilon_d
\end{pmatrix},
\]

(A1)

with tunneling matrix elements $z = t_d e^{i\phi_d/2}$, and with the local intradot Coulomb interaction, $U$, now included. From this, one may construct the even-parity Floquet Hamiltonian, $\hat{H}_F^e$, corresponding to the harmonic driving term, $A \cos(\Omega t)$, from the matrix elements [86–88]

\[
\hat{H}_{e,\infty} = (\hat{H}_{e,\infty} - n\Omega \hat{I}) \delta_{m\bar{m}} + \hat{V} \delta_{m\bar{m}+1} + \hat{V}^\dagger \delta_{m\bar{m}-1},
\]

(A2)

where $\hat{I}$ denotes the $6 \times 6$ unit matrix, and $\hat{V}$ is defined as the $6 \times 6$ matrix with diagonal elements

\[
A \left\{ 0, e^{i\phi_d}, e^{-i\phi_d}, e^{i\phi_d} + e^{-i\phi_d}, \frac{e^{i\phi_d} + e^{-i\phi_d}}{2} \right\},
\]

(A3)

and zeros elsewhere. Truncating this infinite-dimensional matrix and solving the $6(2n_{\text{max}} + 1)$-dimensional eigenvalue problem

\[
\sum_{n=-n_{\text{max}}}^{n_{\text{max}}} \hat{H}_{e,\infty,n}^F |u_n^e\rangle = \epsilon_n |u_n^e\rangle,
\]

(A4)

the time-dependent Schrödinger equation is solved by the six Floquet states,

\[
|\psi(t)\rangle = e^{-i\epsilon_n t} \sum_{n=-n_{\text{max}}}^{n_{\text{max}}} e^{-i\Omega t} |u_n^e\rangle,
\]

(A5)

corresponding to the six quasienergies in the first Floquet Brillouin zone, $-\Omega/2 < \epsilon_n < \Omega/2$, for $n = 1, 2, \ldots, 6$. Expressing these six eigenstates in the original six-dimensional even-parity basis, $|u_n^e\rangle = \sum_i u_n^e(i)|i\rangle$, a given initial state may...
now be expressed as

$$|\Psi(0)\rangle = \sum_{v,i=1}^{6} c_{v} \sum_{n=-n_{\text{max}}}^{n_{\text{max}}} u_{n}^{v}(i)|i\rangle,$$

(A6)

from where the coefficients $c_{v}$ are found by inverting the square $(nv)$ matrices $u_{n}^{v}(i)$. Finally, the solution for the full time evolution of the state can be expressed as

$$|\Psi(t)\rangle = \sum_{v,i=1}^{6} c_{v} e^{-i\varepsilon_{v}t} \sum_{n=-n_{\text{max}}}^{n_{\text{max}}} e^{-i\Omega t} u_{n}^{v}(i)|i\rangle.$$

(A7)

1. Time-averaged current

From this time-evolved state, the time-dependent expectation value of the charge current operator, $\hat{I} = (2e)\hat{\varphi}_{\varphi_{0}} \hat{H}_{c,\infty}$, is determined as

$$I(t) = \langle \Psi(t)|\hat{I}|\Psi(t)\rangle,$$

(A8)

$$I(t) = 2e \sum_{v,\mu,\nu,\mu_{n},\mu_{m}} c_{v}^{*} c_{\mu} [u_{n}^{\mu}(j)]^{*} u_{n_{\mu}}^{\nu}(i)$$

$$\times e^{i\varepsilon_{\mu}t - i\varepsilon_{\nu} + (m-n)\Omega t} (j|\hat{I}|i),$$

which leads to the long-time average

$$J = \lim_{t_{f} \to \infty} \frac{1}{t_{f}} \int_{0}^{t_{f}} dt I(t)$$

$$= 2e \sum_{v,\mu,\nu,\mu_{n},\mu_{m}} [c_{v}^{*}]^{2} [u_{n}^{\mu}(j)]^{*} u_{n_{\mu}}^{\nu}(i)(j|\hat{I}|i).$$

(A9)

Using the same parameters as in Fig. 5 and choosing the ground state of the undriven system as the initial state, one may now calculate the matrices, $u_{n}^{v}(i)$, together with the corresponding coefficients, $c_{v}$, and evaluate the time-averaged current using formula (A9). The result is shown in Fig. 10, with full blue (green, red) lines corresponding to $U = 0$ ($U = \Gamma$, $U = 2\Gamma$). For comparison, the Green’s function result shown in Fig. 5 for $A = 0.1\Gamma$ is included here as the black dashed line. The two methods are in excellent agreement, and capture the same resonances, shown here to lie on top of the grid lines placed at the values of $\varphi_{\varphi_{0}}$ at which an integer multiple of $\Omega$ matches a bound state transition energy. As $U$ is increased, the effects of driving are diminished and at $U = 3\Gamma$ they are completely gone.

The red curve in Fig. 5, corresponding to $A = 0.7\Gamma$, displays a finite anomalous Josephson current at $\varphi_{\varphi_{0}} = 0$. In Fig. 11, we use the same parameters to show that this anomalous Josephson current depends strongly on the phase difference of the two drives, $\theta_{d}$, as found also with a finite BCS gap in the red curve of the right panel of Fig. 9. Here, however, one observes also a sign change of the anomalous Josephson current, corresponding to a transition from a $0$-to $\pi$-junction behavior in $\theta_{d}$, when increasing the interaction strength. For the chosen parameters, this takes place at a critical interaction strength, $U_{c} \sim \Omega$, but the more detailed parametric dependence of $U_{c}$ is beyond the scope of this paper.

Finally, with Fig. 12, we demonstrate that nearly complete rectification of the time-averaged current is possible also in the infinite-gap limit, where all current is carried by Cooper pairs. Unlike the finite-gap results shown in Figs. 8 and 9, parameters have been fine-tuned so as to make the current positive for all phase differences, $\varphi_{\varphi_{0}}$. Figure 12 also demonstrates an explicit dependence of the current on the initial conditions as a spread in curves obtained for different Floquet gauges [89], corresponding to different values of $\theta_{R}$. This is indicated by a set of some 63 gray curves, corresponding to evenly spaced values of $\theta_{R}$ between 0 and $2\pi$, which are averaged to obtain the blue curve. A similar spread will be obtained for the curves in Fig. 11 (not shown for clarity), whereas in Fig. 10, the driving amplitude is low enough that the results depend only on the phase difference, $\theta_{d}$. This spread increases with driving amplitude and gives a rough indication of sensitivity of the long-time average of the Floquet time-evolved
current on initial conditions, and thereby whether they can be expected to be valid also within a driven steady state.

2. Symmetries of the current

The time-dependent current and thereby its long-time average obeys a few basic symmetries, which are most easily revealed by reverting to the time-dependent infinite-gap Hamiltonian for the even sector obtained by replacing \(\varepsilon_d\) by \(\varepsilon_d(t)\) in \(\hat{H}_{\infty}\). The corresponding time-dependent infinite gap Hamiltonian, \(\hat{H}(\varepsilon_d, \varphi_{sc}, \theta_L, \theta_R, t) = \hat{H}_{\infty}|_{\varepsilon_d \rightarrow \varepsilon_d(t)}\), and the current operator obey the transformation properties

\[
\hat{T}\hat{H}(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\hat{T}^\dagger = \hat{H}(-\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t),
\]

\[
\hat{C}\hat{H}(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\hat{C}^\dagger = \hat{H}(-\varphi_{sc}, \theta_L + \pi, \theta_R + \pi, U, -\varepsilon_d - U, t) + \delta\hat{H},
\]

and

\[
\hat{T}\hat{I}(\varphi_{sc})\hat{T} = -\hat{I}(-\varphi_{sc}), \quad \hat{C}\hat{I}(\varphi_{sc})\hat{C}^\dagger = -\hat{I}(-\varphi_{sc}),
\]

with orthogonal matrices given by

\[
\hat{T} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & -1
\end{pmatrix},
\]

\[
\hat{C} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix},
\]

with \(\hat{T}\) corresponding to inversion, while \(\hat{C}\) is related to charge conjugation, but defined here without the complex conjugation operator. The correction term induced by \(\hat{C}\) has matrix elements

\[
\delta\hat{H}_{ij} = \left[4\varepsilon_d + 2U + 2A \sum_{a=L,R} \cos(\theta_a + \Omega t)\right] \delta_{ij},
\]

which merely shifts the diagonal terms, and amounts simply to a multiplicative phase factor between the transformation partner states. From the transformation properties (A10), one finds the transformation of a given solution to the time-dependent Schrödinger equation to be itself a solution with different parameters, namely,

\[
\hat{T}\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t) = |\Psi(-\varphi_{sc}, \theta_R, \theta_L, U, \varepsilon_d, t)\rangle,
\]

\[
\hat{C}\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t) = e^{-i(\theta(t) - \theta(0))}\Psi(-\varphi_{sc}, \theta_L + \pi, \theta_R + \pi, U, -\varepsilon_d - U, t),
\]

where the common time-dependent phase factor has been introduced as

\[
\Theta(t) = 4(\varepsilon_d + U/2)t + 4(A/\Omega) \sum_{a=L,R} \sin(\theta_a + \Omega t).
\]

Together with the transformation properties of the current operator, this implies that

\[
I(-\varphi_{sc}, \theta_R, \theta_L, U, \varepsilon_d, t) = (\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\hat{T}\hat{I}\Psi(\varphi_{sc}))\hat{T}^\dagger
\]

\[
\times |\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\rangle = -I(-\varphi_{sc}, \theta_R, \theta_L, U, \varepsilon_d, t),
\]

and

\[
I(-\varphi_{sc}, \theta_L + \pi, \theta_R + \pi, U, -\varepsilon_d - U, t) = (\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\hat{C}\hat{I}\Psi(\varphi_{sc}))\hat{C}^\dagger
\]

\[
\times |\Psi(\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t)\rangle = -I(-\varphi_{sc}, \theta_L, \theta_R, U, \varepsilon_d, t).
\]
From these instantaneous symmetries one may infer the symmetries (55) and (56) of the time-averaged currents,

\[ J(\psi_{sc}, \theta_d, U, \varepsilon_d) = -J(-\psi_{sc}, \theta_d, U, -\varepsilon_d - U), \]  
\[ J(\psi_{sc}, -\theta_d, U, \varepsilon_d) = J(\psi_{sc}, -\theta_d, U, -\varepsilon_d - U), \]  
\[ J(\psi_{sc}, \theta_d, U, -\varepsilon_d - U) = -J(\psi_{sc}, \theta_d, U, \varepsilon_d), \]

which are observed also in the noninteracting finite-gap numerical results shown in Figs. 8 and 9. The inversion symmetry relation (A19) alone dictates that the anomalous Josephson current must vanish at \( \theta_d = \pi = 2\pi - \pi \), as observed in Fig. 11. The particle-hole symmetry relation (A18), and thereby (A20), holds only when the average phase of the two drives plays no role, i.e., when either the driving amplitude is sufficiently small or when all transients have been erased by relaxation via the quasiparticle continuum available for finite BCS gaps or weak tunneling to normal metals as modeled by \( \Gamma_m \) in the NESS Floquet-Keldysh Green’s function method employed in the main text.

From these symmetries, the anomalous Josephson current at \( \psi_{sc} = 0 \) is seen to satisfy the symmetries

\[ J(0, \theta_d, U, \varepsilon_d) = -J(0, -\theta_d, U, \varepsilon_d) \]  
\[ J(0, \theta_d, U, -\varepsilon_d - U) = J(0, -\theta_d, U, \varepsilon_d). \]  

This implies that the anomalous Josephson current must vanish for quantum dots tuned to the particle-hole symmetric point, \( \varepsilon_d = -U/2 \). Within the Floquet picture, this vanishing of the anomalous Josephson current at the particle-hole symmetric point can be understood as a destructive interference between paths through respectively positive and negative Floquet sidebands. This is illustrated in Fig. 13, in which the blue and red paths need to be offset from particle-hole symmetry in order not to interfere destructively.
