Distance covariance for random fields

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1. Introduction to Model

1.1. Distance covariance in Euclidean space and literature review. It is well known that two \( q \)- and \( r \)-dimensional random vectors \( X \) and \( Y \), respectively, are independent if and only if their joint characteristic function factorizes, i.e.,

\[
\varphi_{X,Y}(s, t) = \mathbb{E} \left[ \exp(i s^\top X + it^\top Y) \right] = \mathbb{E} \left[ \exp(i s^\top X) \right] \mathbb{E} \left[ \exp(it^\top Y) \right] = \varphi_X(s) \varphi_Y(t), \quad s \in \mathbb{R}^q, t \in \mathbb{R}^r.
\]

However, this identity is difficult to check if one has data at the disposal; a replacement of the corresponding characteristic functions by empirical versions does not lead to powerful statistical tools for detecting independence between \( X \) and \( Y \). In the univariate case, Klebanov and Zinger [21], Feuerverger [14], and later Székely et al. \cite{33,30,31,32} in the general multivariate case recommended to use a weighted \( L^2 \)-distance between \( \varphi_{X,Y} \) and \( \varphi_X \varphi_Y \): for \( \beta \in (0, 2) \), the distance covariance between \( X \) and \( Y \) is given by

\[
T_\beta(X, Y) = c_q c_r \int_{\mathbb{R}^{q+r}} |\varphi_{X,Y}(s, t) - \varphi_X(s)\varphi_Y(t)|^2 |s|^{-(q+\beta)} |t|^{-(r+\beta)} \, ds \, dt,
\]

where the constants \( c_d \) for \( d \geq 1 \) are chosen such that

\[
(1.1) \quad c_d \int_{\mathbb{R}^d} (1 - \cos(s^\top x)) |x|^{-(d+\beta)} \, dx = |s|^\beta.
\]

Here and in what follows, we suppress the dependence of the Euclidean norm \( | \cdot | \) on the dimension; it will always be clear from the context what the dimension is.

The quantity \( T_\beta(X, Y) \) is finite under suitable moment conditions on \( X, Y \). The corresponding distance correlation is given by

\[
R_\beta(X, Y) = \frac{T_\beta(X, Y)}{\sqrt{T_\beta(X, X)T_\beta(Y, Y)}}.
\]

Of course, \( X \) and \( Y \) are independent if and only if \( R_\beta(X, Y) = T_\beta(X, Y) = 0 \).

Thanks to the choice of the weight function \( |s|^{-(q+\beta)} |t|^{-(r+\beta)} \) and \((1.1)\), \( T_\beta(X, Y) \) has an explicit form: assuming that \( (X_i, Y_i), i = 1, 2, \ldots \), are iid copies of \( (X, Y) \), we have

\[
T_\beta(X, Y) = \mathbb{E} |X_1 - X_2|^\beta |Y_1 - Y_2|^\beta + \mathbb{E} |X_1 - X_2|^\beta \mathbb{E} |Y_1 - Y_2|^\beta.
\]
(1.3) \[ -2 \mathbb{E}[|X_1 - X_2|^\beta |Y_1 - Y_3|^\beta], \]

and \( R_\beta(cX,cY) = R_\beta(X,Y) \) for \( c \in \mathbb{R} \), i.e., \( R_\beta \) is scale-invariant.

The definition already points at some nice properties of the distance covariance/correlation. It vanishes if and only if \( X, Y \) are independent, hence one can capture arbitrary dependence structures such as non-linear and non-monotone ones. By a handy choice of the weight function as in (1.1) an explicit interpretation through moments is available. It translates to the empirical moments (cf. (1.9), (1.10)) which can be easily calculated. Moreover, the empirical distance covariance is approximated by a (degenerate) \( U \)-statistic whose theoretical properties have been studied in great detail in the literature. Due to the aforementioned advantages the empirical distance correlation has found applications in a rather wide area where dependence modeling is crucial: time series \([36, 8, 16]\), functional time series \([17, 24]\), continuous-time stochastic processes \([23, 9]\). Moreover, taking advantage of the arbitrariness of dimensions, applications in high-dimensional data have been studied intensively in recent years; see \([31, 34, 15, 35]\).

There are numerous other applications of distance correlation in addition to those mentioned above. However, to the best of our knowledge, applications to continuous-parameter random fields (which have infinite dimension) have not been considered. Some work has been conducted on independence tests for continuous-parameter random fields, for example \([18]\), where a high-dimensional case is tractable, but a dimension reduction approach was used. In this paper we consider an independence test based on distance correlation for random fields, exploiting the arbitrariness of dimensions and making use of techniques developed for tests on stochastic processes. Our approach will be clarified in the following text.

1.2. Distance covariance for random fields on a lattice in \([0, 1]^d\). Székely et al. \([31]\) showed that distance correlation fails for high-dimensional vectors \( X, Y \) if their components are independent. Therefore Matsui et al. \([23]\) and Dehling et al. \([9]\) required some dependence structure on the components. They extended distance covariance and distance correlation to stochastic processes. In particular, \([9]\) studied discretizations of stochastic processes \( X,Y \) on \([0,1] \). Instead of using the vectors \((X(t_i))_{i=1,\ldots,p} \) and \((Y(t_i))_{i=1,\ldots,p} \) for partitions \( t_0 = 0 < t_1 < \cdots < t_p = 1, \Delta_i = (t_{i-1},t_i) \) such that \( p = p_n \to \infty \) and \( \delta_n = \max_i |\Delta_i| = \max_i(t_i-t_{i-1}) \to 0 \) as \( n \to \infty \), one introduces the weighted vectors

\[
(1.4) \quad X^{(p)} = (\sqrt{t_i-t_{i-1}}X(t_i))_{i=1,\ldots,p} \quad \text{and} \quad Y^{(p)} = (\sqrt{t_i-t_{i-1}}Y(t_i))_{i=1,\ldots,p}. 
\]

Here we introduce a direct analog of this approach to \([0,1]^d\) for \( d > 1 \). We assume that any random field \( Z = (Z(u))_{u \in [0,1]^d} \) of interest is observed on a lattice with constant mesh size. We start by partitioning \([0,1] \) into equidistant points \( t_i = i/q, \ i = 0,1,\ldots,q, \) for some positive integer \( q \). From them we construct a lattice in \([0,1]^d\) via the points

\[
t_i = (t_{i_1}, \ldots, t_{i_d}), \quad i = (i_1, \ldots, i_d) \in \{0,1,\ldots,q\}^d.
\]

We have \( p := q^d \) lattice points in \((0,1)^d\). We discretize \( Z \) on the cells \( \Delta_i = (t_{i-1},t_i) \) with volume

\[
(1.5) \quad |\Delta| = |\Delta_i| = p^{-1}.
\]

Consider the vector

\[
Z_p = (|\Delta|^{1/2}Z(t_i)), \quad i \in \Pi_d = \{1,\ldots,q\}^d,
\]

and a step-function approximation to \( Z \):

\[
(1.6) \quad Z^{(p)}(t) = \sum_{i \in \Pi_d} Z(t_i)1(t \in \Delta_i), \quad t \in [0,1]^d.
\]
For a measurable bounded square-integrable field $Z$ on $[0,1]^d$ we have for a.e. sample path

$$|Z_p|^2 = \frac{1}{p} \sum_{i \in \mathbb{N}_d} Z^2(t_i) = \sum_{i \in \mathbb{N}_d} Z^2(t_i) |\Delta_i| = \|Z(p)\|_2^2$$

$$\to \int_{[0,1]^d} Z^2(t) \, dt = \|Z\|_2^2, \quad p \to \infty,$$

where $\| \cdot \|_2$ denotes the $L_2$-norm on $[0,1]^d$. Motivated by (1.3) and this approximation, we define the distance covariance $T_\beta(X,Y)$, $\beta \in (0,2)$, between two random fields $X,Y$ on some bounded Borel set $B \subset \mathbb{R}^d$ of finite positive Lebesgue measure by

$$T_\beta(X,Y) = \mathbb{E}[\|X_1 - X_2\|_2^\beta \|Y_1 - Y_2\|_2^\beta] + \mathbb{E}[\|X_1 - X_2\|_2^\beta \mathbb{E}[\|Y_1 - Y_2\|_2^\beta]] - 2 \mathbb{E}[\|X_1 - X_2\|_2^\beta \|Y_1 - Y_2\|_2^\beta],$$

(1.8)

where $(X_i,Y_i)$, $i = 1,2,\ldots$, are iid copies of $(X,Y)$, and the distance correlation $R_\beta(X,Y)$ is defined correspondingly. (The $L_2$-norm $\| \cdot \|_2$ is to be understood on $B$.) A further motivation is the following fact:

**Lemma 1.1.** Assume that $\beta \in (0,2)$, $X,Y$ are random fields on $B$ which are square-integrable, bounded, stochastically continuous and $\mathbb{E}[\|X\|_2^2 + \|Y\|_2^\beta + \|X\|_2^\beta \|Y\|_2^\beta] < \infty$. Then $T_\beta(X,Y) = 0$ if and only if $X,Y$ are independent.

For $\beta \leq 1$ the statement follows from Lyons [22], who also addressed the more general problem of distance covariance in metric spaces, and for $\beta \in (0,2)$ a straightforward modification of the proofs in Dehling et al. [9] for random fields on general Borel sets $B$ yields the result.

Sample versions of $T_\beta$ and $R_\beta$ are given by

$$T_{n,\beta}(X,Y) = \frac{1}{n^2} \sum_{k,l=1}^n \|X_k - X_l\|_2^\beta \|Y_k - Y_l\|_2^\beta + \frac{1}{n^2} \sum_{k,l=1}^n \|X_k - X_l\|_2^\beta \frac{1}{n} \sum_{k,l=1}^n \|Y_k - Y_l\|_2^\beta$$

$$- 2 \frac{1}{n^3} \sum_{k,l,m=1}^n \|X_k - X_l\|_2^\beta \|Y_k - Y_m\|_2^\beta,$$

(1.9)

$$R_{n,\beta}(X,Y) = \frac{T_{n,\beta}(X,Y)}{\sqrt{T_{n,\beta}(X,X) T_{n,\beta}(Y,Y)}}.$$  

(1.10)

Since $T_{n,\beta}$ is a V-statistic, under suitable moment conditions $T_{n,\beta}(X,Y)$ and $R_{n,\beta}(X,Y)$ are consistent estimators of $T_\beta(X,Y)$ and $R_\beta(X,Y)$, respectively, and if $X,Y$ are independent, one also has

$$n R_{n,\beta}(X,Y) \xrightarrow{d} \sum_{i=1}^\infty \lambda_i (N_i^2 - 1) + \text{const},$$

for a square-summable real sequence $(\lambda_i)$, iid $N(0,1)$ random variables $(N_i)$. These asymptotic results in combination with the fact that $T_\beta(X,Y) = 0$ if and only if $X,Y$ are independent encourage one to build a statistical test about independence of $X,Y$ on the quantities $T_{n,\beta}(X,Y)$.

Unfortunately, a sample of paths of $(X,Y)$ is rarely at our disposal, and so one has to think about discretizations of $(X,Y)$. Motivated by the Riemann sum approximation (1.7) possible choices are the lattice discretizations $(X(p),Y(p))$ in (1.6) and, for an iid sample $(X_i,Y_i)$, $i = 1,\ldots,n$, the sample version $T_{n,\beta}(X(p),Y(p))$ as replacement of the test statistic $T_{n,\beta}(X,Y)$. In agreement with [9] in the case $d = 1$ we will show that this idea can be made to work for $d > 1$ if $p = p_n \to \infty$. 
1.3. Distance covariance for random fields at random locations. In this paper, we follow a second path of research.

- We introduce distance covariance $T_\beta(X,Y)$ and $R_\beta(X,Y)$ for random fields $X,Y$ on some Borel set $B \subset \mathbb{R}^d$ of positive Lebesgue measure.
- We define $T_\beta(X^{(p)},Y^{(p)})$ and $R_\beta(X^{(p)},Y^{(p)})$ for non-lattice based discretizations $X^{(p)},Y^{(p)}$ of $X,Y$ on $B$. In contrast to (1.4), we choose a random number $N_p$ of random locations $(U_i)$ where the processes $X,Y$ are observed. Typically, these locations are uniformly distributed on $B$ and $N_p \overset{a.s.}\to \infty$ as $p$ increases with the sample size $n$ to infinity.

The second idea has already been advocated in Matsui et al. [23]. There it was assumed that $(U_i)$ are the points of a Poisson process with constant intensity $p$. For statistical purposes this means that one would have to observe a sample of iid copies $(X_i^{(p)},Y_i^{(p)})$ of $(X^{(p)},Y^{(p)})$ at random locations that change across $i$ and their number would change as well. The asymptotic analysis of this setting is not very elegant and does not lead beyond consistency of $T_{n,\beta}(X^{(p)},Y^{(p)}).

To be precise, we consider bounded stochastically continuous square-integrable random fields $X,Y$ on a bounded Borel set $B \subset \mathbb{R}^d$ of positive Lebesgue measure. We define the distance covariance $T_\beta(X,Y)$ between $X,Y$ as in (1.8) but the norm is now given by the (standardized) $L^2$-norm on $B$:

$$\|f\|_2 = \left( |B|^{-1} \int_B f^2(u) \, du \right)^{1/2}. $$

For ease of notation, we assume without loss of generality that the Lebesgue measure $|B|$ of $B$ is one. The distance correlation $R_\beta(X,Y)$ is defined correspondingly.

Consider the point process

$$(1.12) \quad N^{(p)}(\cdot) = \sum_{i=1}^{N_p} \varepsilon_{U_i}(\cdot) = \# \{ i \leq N_p : U_i \in \cdot \}, \quad p > 0,$$

on the state space $B$ where $(U_i)$ is iid uniform on $B$ independent of the counting number $N_p \overset{a.s.}\to \infty$ as $p \to \infty$. If $N_p$ is Poisson distributed and $p$ is fixed then $N^{(p)}$ constitutes a homogeneous Poisson process on $B$; see p. 132 in Resnick [29]. In view of the order statistics property of any homogeneous Poisson process $N$ on $B$ we also know that the points of $N$, conditionally on $N(B)$, are iid uniform on $B$. However, the representation (1.12) goes beyond the order statistics property since we require the point sequence $(U_i)$ to be the same for all $p > 0$.

Recycling the notation $X^{(p)},Y^{(p)}$, the discretizations of $X,Y$ are now given by

$$X^{(p)}(u) = \sum_{i=1}^{N_p} X(u)\mathbf{1}_{U_i}(u) \quad \text{and} \quad Y^{(p)}(u) = \sum_{i=1}^{N_p} Y(u)\mathbf{1}_{U_i}(u), \quad u \in B,$$

where $(N^{(p)})_{p>0}$ and $X,Y$ are independent. If $N_p = 0$, $X^{(p)} = Y^{(p)} = 0$ by convention. Again recycling the symbol $\| \cdot \|_2$, we write

$$\|X^{(p)} - Y^{(p)}\|^2 := \frac{1}{N_p} \sum_{i : U_i \in B} (X - Y)^2(U_i) = \frac{1}{N_p} \int_B (X - Y)^2 \, dN^{(p)},$$

with the convention that the right-hand side is zero if $N_p = 0$. We notice that for a.e. realization of $(N^{(p)})_{p>0}$, $U_1, \ldots, U_{N_p}$ are getting arbitrarily tight in $B$ since we assume $N_p \overset{a.s.}\to \infty$ as $p \to \infty$. If we assume that $X,Y$ are path-wise square-integrable on $B$ and we condition on $(N^{(p)})_{p>0}$ then for a.e. realization of $X,Y$ we have the Riemann sum approximation $\|X^{(p)} - Y^{(p)}\|^2 \Rightarrow \|X - Y\|^2$.

In what follows, we define the sample versions $T_{n,\beta}(X,Y), R_{n,\beta}(X,Y)$ in the natural way by applying the $L^2$-norm (1.11) on $B$. We assume that the iid sequence $((X_i,Y_i))$ is independent of $(N^{(p)})_{p>0}$. Then the random locations $(U_i)$ are the same for any sample size $n$, and $((X_i^{(p)},Y_i^{(p)}))$
are iid, conditional on \((N_p)_{p>0}\). Moreover, we define \(T_{n,\beta}(X^{(p)}, Y^{(p)}), R_{n,\beta}(X^{(p)}, Y^{(p)})\) accordingly by using the notation in (1.13).

The random discretizations of \(X, Y\) have some advantages over the lattice case:
- they can be defined on quite general sets \(B\),
- the random fields can be observed on irregularly spaced locations,
- the smoothness of the field does not play a significant role for the asymptotic theory of the sample distance covariance.

Of course, one needs preliminary confirmation about the Poisson property of the counting variable \(N_p\) and the uniformity of the locations \((U_i)\). There exists a large literature on tests for Poisson point process models (see e.g. [27, 25, 11] and references therein). This research topic is still progressing, and tests on arbitrary multi-dimensional subsets of \(\mathbb{R}^d\) or even on sets of unknown support have been developed e.g. in [6, 7, 12].

1.4. **Organization of the paper.** We provide the necessary asymptotic theory for the two aforementioned methods:

(i) in Section 2 for random locations and increasing intensity \(p\) on a bounded Borel set \(B \subset \mathbb{R}^d\) of positive Lebesgue measure,

(ii) in Section 3 for the lattice case on \(B = [0,1]^d\) for increasing \(p\) which is the total (deterministic) number of grid points.

(iii) in Section 4 we combine the two methods by averaging the observations in each cell of the regular lattice on \(B = [0,1]^d\).

Since the (non-Gaussian) limit distributions of the considered discretized sample distance covariances are not tractable, in Section 5 we consider some modifications of these quantities. Roughly speaking, we find approximating degenerate \(U\)-statistics to the sample distance covariances and apply bootstrap techniques tailored for these \(U\)-statistics. We show the consistency of the bootstrap. In the case of random locations, this part is quite delicate and rather different from the theory developed in Dehling et al. [9] in the lattice case for \(d = 1\). Indeed, we show bootstrap consistency for these \(U\)-statistics *conditional* on the random locations and their number. In Section 6 we illustrate how the convergence of the sample distance correlation depends on the sample size and the number of discretization points. We also show how the bootstrap performs for the sample distance correlation on selected discretized random fields, in particular for fractional Brownian and infinite variance stable Lévy sheets. In Section 7 we apply the sample distance correlation for testing the independence of Japanese meteorological data observed at stations all over Japan. We interpret their number as random and their location as uniformly distributed.

In the remaining sections we provide the proofs of the main results.

2. Random field at random locations

2.1. **Technical conditions.** We use the notation and assumptions of Section 1.3. In particular, \(X, Y\) are measurable bounded stochastically continuous random fields on a bounded Borel set \(B \subset \mathbb{R}^d\) of positive Lebesgue measure with Riemann square-integrable sample paths on \(B\). Also recall that \(p = p_n \to \infty\) and \(N_p \overset{a.s.}{\to} \infty\) as \(n \to \infty\).

In what follows, \(c\) denotes any positive constant whose value is not of interest.

The following result gives some asymptotic results for the sample distance covariance. They are the basis for proving Theorem 2.6.

**Proposition 2.1.** Choose \(\beta \in (0, 2)\) and assume that \(X, Y\) are independent. Consider the following conditions:

1a) \(X, Y\) have finite second moment and \(\int_B \mathbb{E}[X^2(u) + Y^2(u)] \, du < \infty\).

1b) \(X, Y\) have infinite second moment but \(\mathbb{E}[\sup_{u \in B} |X|^{\beta}(u) + \sup_{u \in B} |Y|^{\beta}(u)] < \infty\).
(2a) $X$ has finite second moment, $[\beta \in (0, 1] \text{ and } \int_B \mathbb{E}[X^2(u)] \, du < \infty]$ or $[\beta \in (1, 2) \text{ and } \int_B \mathbb{E}[|X|^{2\beta}(u)] \, du < \infty]$.\\
(2b) $X$ has infinite second moment, $\beta \in (0, 1)$ and $\mathbb{E}[\sup_{u \in B} |X|^{2\beta}(u)] < \infty$.

If (1a) or (1b) hold then

$$\mathbb{E} \left[ |T_{n, \beta}(X^{(p)}, Y^{(p)}) - T_{n, \beta}(X, Y)| \, | N^{(p)} \right] \overset{a.s.}{\to} 0, \quad n \to \infty. \tag{2.1}$$

If (2a) or (2b) hold then

$$\mathbb{P} \left( |T_{n, \beta}(X^{(p)}, X^{(p)}) - T_{n, \beta}(X, X)| > \varepsilon \, | N^{(p)} \right) \overset{a.s.}{\to} 0, \quad n \to \infty. \tag{2.2}$$

The proof of this proposition is given in Section 8.

2.2. Some examples. We consider some examples of random fields and discuss the fulfillment of the conditions in Proposition 2.1.

Example 2.2. Let $X(u) = B^H(u)$, $u \in [0, 1]^d$, be a fractional Brownian sheet with Hurst parameter $H = (H_1, \ldots, H_d) \in (0, 1)^d$ introduced by Kamont [19]; see also Ayache and Xiao [5]. It is a centered Gaussian random field with continuous sample paths and covariance function

$$\text{cov}(B^H(s), B^H(t)) = \prod_{i=1}^d \left( \frac{1}{2} |s_i|^{2H_i} + |t_i|^{2H_i} - |s_i - t_i|^{2H_i} \right), \quad s, t \in [0, 1]^d. \tag{2.3}$$

Moreover, for all $\gamma > 0$, $\mathbb{E}[[B^H(u)]^\gamma]$ is a continuous function of $u$. Therefore all conditions on $X$ are satisfied.

Example 2.3. Let $H \in (0, 1)$ and $X^H$ be a centered Gaussian random field on $[0, 1]^d$ with covariance function

$$\text{cov}(X^H(s), X^H(t)) = \frac{1}{2} (|s|^{2H} + |t|^{2H} - |s - t|^{2H}), \quad s, t \in [0, 1]^d.$$

This process is called Lévy fractional Brownian field; see Samorodnitsky and Taqqu [28], p. 393. It has stationary increments and continuous sample paths and moments of any order.

Example 2.4. Let $X$ be a centered Lévy sheet on $[0, 1]^d$ with Lévy-Khintchine triplet $(\mu, \sigma^2, \nu)$; see Khoshnevisan and Xiao [20], Example 2.1. This means that, for disjoint intervals $A_i \in [0, 1]^d$, $i = 1, \ldots, k$, and $k \geq 1$, the increments $X(A_1), \ldots, X(A_k)$ are independent and for each interval $A = (s, t]$, the characteristic function of the increment

$$X(A) = \sum_{k_1 \in \{0, 1\}} \cdots \sum_{k_d \in \{0, 1\}} (-1)^{d - \sum_{i=1}^d k_i} X(s_1 + k_1(t_1 - s_1), \ldots, s_d + k_d(t_d - s_d)),$$

is given by

$$\mathbb{E}[e^{iuX(A)}] = \exp(-|A| \Psi(u)), \tag{2.4}$$

where

$$\Psi(u) = i\mu u + u^2\sigma^2/2 + \int_{\mathbb{R}} (e^{iux} - 1 - ux1_{|x|\leq 1}) \nu(dx).$$

If $X$ has finite $\gamma$th moment for some $\gamma > 0$ then $\mathbb{E}[[X(u)]^\gamma]$ is a continuous function of $u$ and the function $\mathbb{E}[[\sup_{u \in [0, 1]^d} X(u)]^\gamma]$ is finite as well. The field $X$ has càdlàg sample paths which are bounded and square-integrable on $[0, 1]^d$.

Example 2.5. Consider a symmetric $\alpha$-stable Lévy sheet on $[0, 1]^d$ for some $\alpha \in (0, 2)$; see for example Ehm [13], Samorodnitsky and Taqqu [28]. This field has infinite variance and is a Lévy random sheet whose increments have a symmetric $\alpha$-stable distribution, i.e. the characteristic function of $X(A)$ is given by (2.4) with $\Psi(u) = c|u|^\alpha$ for some $c > 0$. It has moments of order $\gamma \in (0, \alpha)$. 

2.3. Main result. The following statement is the main result in the case of random locations.

**Theorem 2.6.** Choose \( \beta \in (0, 2) \) and assume that \( X, Y \) are independent,

1. If (1a) or (1b) of Proposition 2.1 hold then for all \( \varepsilon > 0 \) along a.e. sample path of \( (N^{(p)})_{p>0} \), as \( n \to \infty \) and \( p = p_n \to \infty \),
   \[
P\left( |T_{n,\beta}(X^{(p)}, Y^{(p)})| > \varepsilon \mid N^{(p)} \right) \to 0.
\]

2. If [(1a) and (2a)] both for \( X, Y \) or [(1b) and (2b) both for \( X, Y \)] of Proposition 2.1 hold then we also have for all \( \varepsilon > 0 \) along a.e. sample path of \( (N^{(p)})_{p>0} \), as \( n \to \infty \) and \( p = p_n \to \infty \),
   \[
P\left( |R_{n,\beta}(X^{(p)}, Y^{(p)})| > \varepsilon \mid N^{(p)} \right) \to 0.
\]

**Proof.** 1. From Proposition 2.1, in particular from (2.1), we conclude that \( T_{n,\beta}(X^{(p)}, Y^{(p)}) \to X_{\beta}(X, Y) \) as \( n \to \infty \) in probability conditional on \( N^{(p)} \). The rest is analogous to the derivations in Dehling et al. [9]: \( T_{n,\beta}(X, Y) \) is a V-statistic and satisfies the strong law of large numbers \( T_{n,\beta}(X, Y) \overset{a.s.}{\to} T_{\beta}(X, Y) = 0 \) under the required moment conditions on \( X, Y \).

2. Under the additional conditions (2a), (2b) both for \( X, Y \) we also have \( T_{n,\beta}(X^{(p)}, X^{(p)}) \to X_{\beta}(X, X) \to 0 \) and \( T_{n,\beta}(Y^{(p)}, Y^{(p)}) \to Y_{\beta}(Y, Y) \to 0 \) as \( n \to \infty \) in probability conditional on \( N^{(p)} \). Moreover, by the strong law of large numbers \( T_{n,\beta}(X, X) \overset{a.s.}{\to} T_{\beta}(X, X) \) and \( T_{n,\beta}(Y, Y) \overset{a.s.}{\to} T_{\beta}(Y, Y) \), hence \( R_{n,\beta}(X, Y) \to R_{\beta}(X, Y) = 0 \) in probability conditional on \( N^{(p)} \). \( \square \)

In Section 5 we provide much stronger asymptotic results under stronger conditions. In particular, we show that \( nT_{n,\beta}(X^{(p)}, Y^{(p)}) \) conditional on \( (N^{(p)})_{p>0} \) has the same weak limit as \( nT_{n,\beta}(X^{(p)}, Y^{(p)}) \), and we also show consistency of a suitable bootstrap procedure.

3. Random field at a lattice

In the present and next sections we consider sampling schemes of \( (X, Y) \) on a deterministic lattice on \( B = [0, 1]^d \). We will assume conditions on the smoothness of \( (X, Y) \) from which we can derive convergence rates of \( T_{n,\beta}(X^{(p)}, Y^{(p)}) \) to \( T_{\beta}(X, Y) \). Here we closely follow Dehling et al. [9] who dealt with the lattice case for \( d = 1 \). The following conditions and results are adaptations of those in [9].

3.1. Technical conditions. We use the notation and assumptions of Section 1.2. In what follows, we introduce moment and smoothness conditions on the fields \( X, Y \) and require certain rates for \( p = p_n \to \infty \) as \( n \to \infty \). The conditions are separated in two parts depending on whether \( (X, Y) \) have finite second moments or not.

If \( X, Y \) have finite second moments we will work under the following conditions.

**Condition (A): finite second moments**

(A1) Smoothness of increments. There exist \( \gamma_X, \gamma_Y > 0 \) and \( c > 0 \) such that
   \[
   \text{var} (X(t) - X(s)) \leq c|t - s|^{\gamma_X} \quad \text{and} \quad \text{var} (Y(t) - Y(s)) \leq c|t - s|^{\gamma_Y}.
   \]

(A2) Additional moment conditions. If \( \beta \in (1, 2) \) we have
   \[
   \max_{t \in [0, 1]^d} E[|X(t)|^{2(2\beta - 1)}] + \max_{t \in [0, 1]^d} E[|Y(t)|^{2(2\beta - 1)}] < \infty.
   \]

(A3) Growth condition on \( p = p_n \to \infty \). We have
   \[
   p^{-1}n^2 \left( \frac{1}{d - 1}(\gamma_X \wedge \gamma_Y) (\beta \wedge 1) \right) \to 0, \quad n \to \infty.
   \]

If \( X, Y \) possibly have infinite second moments we will work under the following conditions:

**Condition (B): infinite second moments** Assume \( \beta \in (0, 2) \).
(B1) **Finite \( \beta \)th moment.**

\[
\mathbb{E} \left[ \max_{t \in [0,1]^d} |X(t)|^\beta \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \max_{t \in [0,1]^d} |Y(t)|^\beta \right] < \infty.
\]

(B2) **Smoothness of increments.** There exist \( \gamma_X, \gamma_Y > 0 \) and \( c > 0 \) such that

\[
\max_{i \in \mathbb{N}_d} \mathbb{E} \left[ \max_{t \in \Delta_i} |X(t) - X(t_i)|^\beta \right] \leq cp^{-\gamma_X/d} \quad \text{and} \quad \max_{i \in \mathbb{N}_d} \mathbb{E} \left[ \max_{t \in \Delta_i} |Y(t) - Y(t_i)|^\beta \right] \leq cp^{-\gamma_Y/d}.
\]

(B3) **Additional moment and smoothness conditions.**

\[
\mathbb{E} \left[ \max_{t \in [0,1]^d} |X(t)|^{2\beta} \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \max_{t \in [0,1]^d} |Y(t)|^{2\beta} \right] < \infty.
\]

If \( \beta \in (0,1) \) there also exist \( \gamma'_X, \gamma'_Y > 0 \) and \( c > 0 \) such that

\[
\max_{i \in \mathbb{N}_d} \mathbb{E} \left[ \max_{t \in \Delta_i} |X(t) - X(t_i)|^{2\beta} \right] \leq cp^{-\gamma'_X/d} \quad \text{and} \quad \max_{i \in \mathbb{N}_d} \mathbb{E} \left[ \max_{t \in \Delta_i} |Y(t) - Y(t_i)|^{2\beta} \right] \leq cp^{-\gamma'_Y/d}.
\]

(B4) **Growth condition on \( p = p_n \to \infty \).** We have \( \beta/2 + (\gamma_X \wedge \gamma_Y)/d > 1 \) and

\[
p^{-1} n^{(\beta/(\beta \wedge 1))} / ((\beta/2 + (\gamma_X \wedge \gamma_Y)/d - 1) \to 0, \quad n \to \infty.
\]

In the lattice case the following result is the analog of Proposition 2.1.

**Proposition 3.1.** Assume the following conditions.

1. \( X, Y \) are independent stochastically continuous bounded processes on \([0,1]^d\) defined on the same probability space.
2. If \( X, Y \) have finite expectations, then these are assumed to be equal to 0.
3. \( \beta \in (0,2) \).
4. \( p = p_n \to \infty \) as \( n \to \infty \).

Then the following statements hold.

1. If also (A1) holds then there is a constant \( c \) such that for any \( n \geq 1 \),

\[
\mathbb{E} \left[ |T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X, Y)| \right] \leq cp^{-(\beta \wedge 1)/2}.
\]

2. If also (B1),(B2) hold then there is a constant \( c \) such that for any \( n \geq 1 \),

\[
\mathbb{E} \left[ |T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X, Y)| \right] \leq c p^{-(\beta/2 + (\gamma_X \wedge \gamma_Y)/d)} / (\beta \wedge 1)^{\beta}.
\]

3. If also (A1), (A2) hold in the finite variance case and \( \beta \in (0,1) \), (B3) and \( 1 < \beta + (\gamma_X \wedge \gamma_Y)/d \) in the infinite variance case, then

\[
T_{n,\beta}(X^{(p)}, X^{(p)}) - T_{n,\beta}(X, X) \overset{p}{\to} 0, \quad n \to \infty,
\]

and the analogous result holds for \( Y \).

The proof of this result is completely analogous to the case \( d = 1 \) given in [9] and therefore omitted.

A comparison of Propositions 2.1 and 3.1 shows that in the former result one does not need conditions on the smoothness of the sample paths such as (A1), (B2) or (B3). In the latter result, the smoothness parameters \( \gamma_X, \gamma_Y \) appear explicitly in the approximation rates in (3.1) and (3.2). For example, consider two iid Brownian sheets \( X, Y \). According to Example 3.2 below, \( \gamma_X = \gamma_Y = 1 \) since \( H_t = 0.5 \). Then the right-hand side of (3.1) turn into \( cp^{-(\beta \wedge 1)/(2d)} \). In Proposition 2.1 the parameter \( p \) has a distinct meaning. Assume that \( \{N_p\}_{p \geq 0} \) are Poisson variables, e.g. \( N_p = M(pB) \) for a unit rate homogeneous Poisson process \( M \) on \( \mathbb{R}^d \). When restricted to \( B, M(p \cdot) \) has intensity \( p|B| \) and we have \( N_p/p \overset{p}{\to} |B| \) and therefore \( p|B| \) is a rough approximation of \( N_p \). We observe that (3.1) is the analog of (2.1). In the latter case, we do not have a convergence rate. This is perhaps not surprising since knowledge of \( X, Y \) on the grid points of a lattice in combination with the smoothness of the sample paths of \( X, Y \) often give us additional information about the sample
path inside a cell $\Delta_i$. In Section 4 we will moderate between the two sampling schemes: we will average the observations at random locations in each cell of the lattice. This property allows one to use smoothness properties of the random field also for observations at random locations.

Thus, if the observations are given on a lattice it is preferable to use distance covariance on it. However, the random location case has the advantage that smoothness does not matter and it works on quite general bounded sets $B$.

**Example 3.2.** Let $X(u) = B^H(u), u \in [0, 1]^d$, be a fractional Brownian sheet with Hurst parameter $H = (H_1, \ldots, H_d) \in (0, 1)^d$; see Example 2.2. From (2.3) we have

$$\text{var}(B^H(t) - B^H(s)) = \left( \prod_{i=1}^d |t_i|^{2H_i} - \prod_{i=1}^d 0.5(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}) \right)$$

$$+ \left( \prod_{i=1}^d |s_i|^{2H_i} - \prod_{i=1}^d 0.5(|t_i|^{2H_i} + |s_i|^{2H_i} - |t_i - s_i|^{2H_i}) \right)$$

$$= \sum_{j=1}^{i-1} \left( \prod_{i=1}^d |t_j|^{2H_i} \frac{1}{2}(|t_i|^{2H_i} - |s_i|^{2H_i} + |t_i - s_i|^{2H_i}) \right) \prod_{k=i+1}^d \frac{1}{2}(|t_k|^{2H_k} + |s_k|^{2H_k} - |t_k - s_k|^{2H_k}).$$

Thus we have for some constant $c > 0$,

$$\text{var}(B^H(t) - B^H(s)) \leq c \sum_{i=1}^d |t_i - s_i|^{(2H_i)\wedge 1} \leq c|t - s|^{(2\min_i H_i)\wedge 1}.$$

Hence (A1) is satisfied with $\gamma_X = (2\min_{i=1,\ldots,d} H_i) \wedge 1$. Moreover, for any $\gamma > 0$, $\mathbb{E}[|B^H(u)|^\gamma]$ is continuous. Hence (A2) holds.

**Example 3.3.** Let $X^H(u), u \in [0, 1]^d$, for some $H \in (0, 1)$ be a Lévy fractional Brownian field; see Example 2.3. It has all moments and stationary increments. Hence

$$\text{var}(X^H(t) - X^H(s)) = |t - s|^{2H}.$$

Hence (A1) is satisfied with $\gamma_X = 2H$. Moreover, for any $\gamma > 0$, $\mathbb{E}[|X^H(u)|^\gamma]$ is continuous. Hence (A2) holds.

**Example 3.4.** Let $X$ be a Lévy sheet on $[0, 1]^d$; see Example 2.4. Recall that $X(t) = X([0, t])$. Therefore

$$X(t) - X(s) = X([0, t] \setminus (0, s] \cap [0, t]) - X(0, s] \setminus (0, s] \cap [0, t]),$$

where the two random variables on the right-hand side are defined on disjoint sets, hence they are independent. We observe that the set $((0, s] \cup (0, t]) \setminus (0, s] \cap [0, t])$ for $s, t \in [0, 1]^d$ is the union of a finite number (only depending on $d$) of disjoint sets $(a, b]$ such that for some $1 \leq j \leq d$ and a constant $c > 0$, $|a, b| \leq c|t_j - s_j|$. Thus, if $X$ has finite second moment, we have for some constants $c_1$ and $c_2$,

$$\text{var}(X(t) - X(s)) \leq c_1 \max_{i=1,\ldots,d} |t_i - s_i| \leq c_2|t - s|.$$

Hence (A1) holds with $\gamma_X = 1$. If the moment condition $\mathbb{E}[|X(u)|^{2(2\beta - 1)}] < \infty$ holds for some $u$, it is finite for all $u$ and also continuous. Hence (A2) holds for $X$.  

Example 3.5. Consider a symmetric $\alpha$-stable Lévy sheet on $[0,1]^d$ for some $\alpha \in (0,2)$; see Example 2.5. It has moments of order $\beta \in (0, \alpha)$. Since $X$ is symmetric and has independent increments an application of Levy’s maximal inequality yields

$$
P \left( \max_{t \in \Delta} |X(t) - X(t_1)| > x \right) \leq 2 P \left( |X(t_1) - X(t_{1-1})| > x \right), \quad x > 0.
$$

By definition of a symmetric $\alpha$-stable sheet

$$
E \left[ |X(t_1) - X(t_{1-1})|^\beta \right] = c \left[ [0,t_1] \setminus [0,t_{1-1}] \right] \beta/\alpha \leq c p^{-\beta/(\alpha d)}.
$$

Thus (B2) holds with $\gamma = \beta/\alpha$. The same argument yields (B3) with $\gamma' = 2\beta/\alpha$ for $\beta \in (0, 0.5\alpha)$.

3.2. Main result in the lattice case. Now we proceed with the main theorem. In the case $d = 1$ it corresponds to Theorem 3.1 in Dehling et al. \[9\].

Theorem 3.6. Assume the following conditions:

1. $X, Y$ are independent stochastically continuous bounded processes on $[0,1]^d$.
2. If $X, Y$ have finite expectations, then they are centered.
3. $\beta \in (0,2)$.
4. $p = p_n \to \infty$ as $n \to \infty$.

Then the following statements hold.

1. If either (A1) or [(B1),(B2)] and $1 < \beta/2 + (\gamma_X \wedge \gamma_Y)/d$ are satisfied then

$$
T_{n,\beta}(X^{(p)},Y^{(p)}) \xrightarrow{P} 0
$$

holds.

2. If either (A1),(A3) or (B1),(B2),(B4) hold then

$$
n T_{n,\beta}(X^{(p)},Y^{(p)}) \xrightarrow{d} \sum_{i=1}^{\infty} \lambda_i (N_i^2 - 1) + c,
$$

for an iid sequence of standard normal random variables $(N_i)$, a constant $c$, and a square summable sequence $(\lambda_i)$.

3. If either (A1),(A2) or $[\beta \in (0,1)$, (B1)-(B3) and $1 < \beta + (\gamma'_X \wedge \gamma'_Y)/d]$ hold then

$$
R_{n,\beta}(X^{(p)},Y^{(p)}) \xrightarrow{P} 0.
$$

4. If either (A1)-(A3) or $[\beta \in (0,1)$, (B1)-(B4) and $1 < \beta + (\gamma'_X \wedge \gamma'_Y)/d]$ hold then $n R_{n,\beta}(X^{(p)},Y^{(p)})$ converges to a scaled version of the limit in (3.4).

Proof. 1. Under the assumptions, Proposition 3.1 yields that $T_{n,\beta}(X,Y) - T_{n,\beta}(X^{(p)},Y^{(p)}) \xrightarrow{P} 0$ while $T_{n,\beta}(X,Y) \xrightarrow{a.s.} T_{\beta}(X,Y) = 0$ by the strong law of large numbers for $V$-statistics. The statement follows.

2. Under the additional conditions, where $1 < \beta/2 + (\gamma_X \wedge \gamma_Y)/d$ is implied by (B4), we have from Proposition 3.1 that $n |T_{n,\beta}(X,Y) - T_{n,\beta}(X^{(p)},Y^{(p)})| \xrightarrow{P} 0$. By degeneracy of the $V$-statistics $n T_{n,\beta}(X,Y)$ converges in distribution to a sum of independent weighted $\chi^2$ random variables, so does $n T_{n,\beta}(X^{(p)},Y^{(p)})$.

3. and 4. They follow by combining (1) and (2) with Proposition 3.1 which allows one to switch from $T_{n,\beta}$ to $R_{n,\beta}$. We omit further details. \[\Box\]
4. Random field at random locations grouped in lattice cells

In this section we consider a combination of the two previous sampling schemes for \((X,Y)\). For the sake of argument we restrict ourselves to fields on \(B = [0,1]^d\). We assume that the fields are observed at the locations \((U_i)_{i=1,\ldots,p}\) which are iid uniformly distributed on \(B\) and \(p = p_n \to \infty\) is a deterministic integer sequence. This time we average the randomly scattered \(U_i\) in each cell of a regular lattice grid.

Similarly to Section 1.2 we partition \(B\) into \(\tilde{p} = [p/\log p]\) disjoint cells \(\Delta_i, i = 1,\ldots,\tilde{p}\) with side length \(\tilde{p}^{-1/d}\) and volume \(\tilde{p}^{-1}\). The main idea of this approach is to average the observations \((X(U_j), Y(U_j))\) in a given cell. Since \(p\) increases slightly faster than the number of cells \(\tilde{p}\) the probability that there is no observation in a given cell decreases at a certain rate.

For any random field \(Z\) on \(B\) we consider the discretization (again abusing notation),

\[
Z^{(p)}(u) = \sum' 1_{\Delta_j}(u) Z_j,
\]

where \(\sum'\) denotes summation over those \(j = 1,\ldots,\tilde{p}\) such that \(#\Delta_j = \#\{i \leq p : U_i \in \Delta_j\} \neq 0\) and

\[
Z_j = \sum_{k:U_k \in \Delta_j} \frac{Z(U_k)}{\#\Delta_j}.
\]

We also recycle the notation

\[
\|Z^{(p)}\|_2^2 := \int_{[0,1]^d} \sum' 1_{\Delta_j}(u) Z_j^2\,du = \tilde{p}^{-1} \sum' Z_j^2.
\]

4.1. Technical conditions. The results in this section parallel those in the lattice case. The conditions are similar to \((A)\) and \((B)\); we also use the notation from \((A)\) and \((B)\) in the sequel.

If \(X,Y\) have finite second moments we will need the following condition.

\((A3')\) Growth condition on \(p = p_n \to \infty\).

\[
(p^{-1} \lor \tilde{p}^{-1})^{\gamma_X \lor \gamma_Y} n^{2/(\beta \land 1)} \to 0, \quad n \to \infty.\]

If \(X,Y\) possibly have infinite second moments we will need the following condition.

Assume \(\beta \in (0,2)\) and

\((B4')\) Growth condition on \(p = p_n \to \infty\). We have

\[
(p^{-1} \lor \tilde{p}^{-1})^{\gamma_X \lor \gamma_Y} p^{1-\beta/2} n^{3/(1 \land \beta)} \to 0, \quad n \to \infty.\]

If we remove \(p^{-1}\) in \((A3')\) and \((B4')\) and replace \(\tilde{p}\) by \(p\) then we recover \((A3)\) and \((B4)\).

The following result is an analog of Proposition 2.1 for the discretizations \(X^{(p)}, Y^{(p)}\) of \(X,Y\) defined via \((4.1)\). In contrast to the latter case, we have explicit rates of convergence for \(\mathbb{E}[|T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X,Y)|] \to 0\). These rates are achieved due to the averaging of values in the cells \(\Delta_j\). Then one can also exploit the smoothness of the field over these small cells.

**Proposition 4.1.** Assume the following conditions.

1. \(X,Y\) are independent stochastically continuous bounded processes on \([0,1]^d\) defined on the same probability space.
2. If \(X,Y\) have finite expectations, then these are assumed to be equal to 0.
3. \(\beta \in (0,2)\).

Then the following statements hold.

1. If also \((A1)\) holds then there is a constant \(c\) such that for all \(n \geq 1\),

\[
\mathbb{E}[|T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X,Y)|] \leq c (\tilde{p}^{-1})^{(\beta \land 1)/2}.
\]
2. If also (B1),(B2) hold then there is a constant $c$ such that for all $n \geq 1,$
\[
E \left[ |T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X, Y)| \right] \leq c \left( p^{1-\beta/2}(\tilde{p}^{-(\gamma_X \wedge \gamma_Y)/d} + p^{-1}) \right)^{(\beta \wedge 1)/\beta}.
\]
3. Under the additional conditions (A1), (A2) in the finite variance case and under $[\beta \in (0, 1), (B1)-(B3) \text{ and } 1 < \beta + \gamma_X \wedge \gamma_Y/d]$ in the infinite variance case,
\[
T_{n,\beta}(X^{(p)}, Y^{(p)}) - T_{n,\beta}(X, X) \overset{\mathbb{P}}{\to} 0, \quad n \to \infty.
\]

The proof of Proposition 4.1 is rather technical and given in Section 9.

4.2. Main result. The following is the main result of this section.

**Theorem 4.2.** Assume the following conditions:

1. $X, Y$ are independent stochastically continuous bounded processes on $[0, 1]^d$.
2. $X, Y$ have finite expectations, then they are centered.
3. $\beta \in (0, 2)$ and $p_n \to \infty$ as $n \to \infty$.

Then the following statements hold.

1. If either (A1) or $[(B1),(B2) \text{ and } 1 < \beta/2 + (\gamma_X \wedge \gamma_Y)/d]$ are satisfied then
   \[
   T_{n,\beta}(X^{(p)}, Y^{(p)}) \overset{\mathbb{P}}{\to} 0
   \]
   hold.
2. If either (A1),(A3') or (B1),(B2),(B4') hold then
   \[
   n T_{n,\beta}(X^{(p)}, Y^{(p)}) \overset{d}{\to} \sum_{i=1}^{\infty} \lambda_i(N_i^2 - 1) + c
   \]
   for an iid sequence of standard normal random variables $(N_i)$, a constant $c$, and a square sumable sequence $(\lambda_i)$.
3. If either (A1),(A2) or $[\beta \in (0, 1), (B1)-(B3) \text{ and } 1 < \min(\beta + (\gamma_X \wedge \gamma_Y)/d, \beta/2 + (\gamma_X \wedge \gamma_Y)/d)]$ hold then
   \[
   R_{n,\beta}(X^{(p)}, Y^{(p)}) \overset{\mathbb{P}}{\to} 0.
   \]
4. If either (A1),(A2),(A3') or $[\beta \in (0, 1), (B1)-(B3),(B4') \text{ and } 1 < \beta + (d^{-1}(\gamma_X \wedge \gamma_Y) \wedge 1)]$ hold then $n R_{n,\beta}(X^{(p)}, Y^{(p)})$ converges to a scaled version of the limit in (4.3).

One can follow the lines of the proof of Theorem 3.6.

5. The bootstrap for $T_{n,\beta}$

In this section we introduce a bootstrap procedure for $T_{n,\beta}(X, Y)$ and $T_{n,\beta}(X^{(p)}, Y^{(p)})$.

5.1. $T_{n,\beta}(X, Y)$ as a degenerate $V$-statistic. We recall some facts from the Appendix in Dehling et al. [9] (see also Lyons [22]) and adapt them to the situation of a random field on $B$. We assume that $Z_i = (X_i, Y_i), \ i = 1, 2, \ldots$, is an iid sequence with generic element $Z = (X, Y)$ whose components are bounded Riemann square-integrable random fields on $B$, and $\mathbb{E}[\|X\|^\beta_2 + \|Y\|^\beta_2 + \|X\|^\beta_2 \|Y\|^\beta_2] < \infty$ for some $\beta \in (0, 2)$. Under these assumptions, $T_{n,\beta}(X, Y)$ has representation as a $V$-statistic of order 4 with symmetric degenerate kernel of order 1.

We start with the kernel
\[
f((x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)) = f(z_1, z_2, z_3, z_4)
\]
\[
= \|x_1 - x_2\|^\beta_2 \|y_1 - y_2\|^\beta_2 + \|x_1 - x_2\|^\beta_2 \|y_3 - y_4\|^\beta_2 - 2\|x_1 - x_2\|^\beta_2 \|y_1 - y_3\|^\beta_2.
\]
From this representation,
\[ T_{n,\beta}(X, Y) = \frac{1}{n^4} \sum_{1 \leq i,j,k,l \leq n} f(Z_i, Z_j, Z_k, Z_l). \]

Then one can define the corresponding symmetric kernel via the usual symmetrization as
\[ h(z_1, z_2, z_3, z_4) = \frac{1}{24} \sum_{(i_1, i_2, i_3, i_4) \text{ permutation of } (1,2,3,4)} f(z_{i_1}, z_{i_2}, z_{i_3}, z_{i_4}). \]

The kernel \( h \) is at least 1-degenerate: under the null hypothesis of independence between \( X \) and \( Y \),
\[ \mathbb{E}[f(z_1, Z_2, Z_3, Z_4)] + \mathbb{E}[f(Z_2, z_1, Z_3, Z_4)] + \mathbb{E}[f(Z_2, Z_3, z_1, Z_4)] 
+ \mathbb{E}[f(Z_2, Z_3, Z_4, z_1)] = 0. \]

Still under the null hypothesis of independence between \( X \) and \( Y \),
\[ \mathbb{E}[h(z_1, z_2, (X_3, Y_3), (X_4, Y_4))] \]
\[ = \frac{1}{6} \left( \|x_1 - x_2\|_2^4 \right. 
+ \mathbb{E}[\|X_1 - X_2\|_2^4] - \mathbb{E}[\|x_1 - X\|_2^4] - \mathbb{E}[\|x_2 - X\|_2^4] ) 
\times \left. \left( \|y_1 - y_2\|_2^4 + \mathbb{E}[\|Y_1 - Y_2\|_2^4] - \mathbb{E}[\|y_1 - Y\|_2^4] - \mathbb{E}[\|y_2 - Y\|_2^4] \right) , \right. \]
and the right-hand side is not constant. Hence, the kernel \( h \) is precisely 1-degenerate. One can follow the lines of the proof in the Appendix in [3] to show the following result (the only necessary change is the replacement of \([0, 1]\) by \(B\)).

**Lemma 5.1.** If \( X, Y \) are independent and \( \mathbb{E}[\|X\|_2^\beta + \|Y\|_2^\beta] < \infty \) for some \( \beta \in (0, 2) \) then \( T_{n,\beta}(X, Y) \) has representation as a \( V \)-statistic of order 4 with symmetric 1-degenerate kernel \( h \). The corresponding \( U \)-statistic \( \tilde{T}_{n,\beta}(X, Y) \) is obtained from \( T_{n,\beta}(X, Y) \) by restricting the summation to indices \((i_1, i_2, i_3, i_4)\) with mutually distinct components. Then
\[ n \left( T_{n,\beta}(X, Y) - \tilde{T}_{n,\beta}(X, Y) \right) \xrightarrow{p} \mathbb{E}[\|X_1 - X_2\|_2^2] \mathbb{E}[\|Y_1 - Y_2\|_2^2], \quad n \to \infty. \]

An immediate consequence of this result is that, up to an additive constant, \((n T_{n,\beta}(X, Y))\) and \((n \tilde{T}_{n,\beta}(X, Y))\) have the same limit distribution which is indicated in (1.4). In what follows, we will focus on \( \tilde{T}_{n,\beta}(X, Y) \).

5.2. **Bootstrapping** \( T_n(X, Y) \). For the sake illustration of the method we restrict ourselves to \( \beta = 1 \) and suppress \( \beta \) in the notation. We also assume that
\[ \int_B \mathbb{E}[X^2(u) + Y^2(u)] du < \infty. \]

A generic element \( Z = (X, Y) \) has trajectory \( z = (x, y) \) on \( B \subset \mathbb{R}^d \).

Under the assumptions of Lemma 5.1, \( \tilde{T}_n(X, Y) \) has representation as \( U \)-statistic of order 4 with a 1-degenerate symmetric kernel \( h(x_1, x_2, x_3, x_4) \). Applying the Hoeffding decomposition to \( \tilde{T}_n(X, Y) \), the limit distributions (modulo a change of location/scale) of \( nT_n(X, Y) \) and the following normalized version of \( \tilde{T}_n(X, Y) \) coincide:
\[ U_n(Z) = \frac{1}{n} \sum_{1 \leq i \neq j \leq n} h_2(Z_i, Z_j; F_Z) \]
where \( F_Z = F_X \times F_Y \) and \( h_2 \) is defined by
\[ h_2(z_1, z_2; F_Z) = \mathbb{E}[h(z_1, z_2, Z_3, Z_4)] - \mathbb{E}[h(z_1, Z_2, Z_3, Z_4)] 
- \mathbb{E}[h(Z_1, Z_2, z_3, Z_4)] + \mathbb{E}[h(Z_1, Z_2, Z_3, Z_4)]. \]
We write $F_n$ for the empirical distribution of the sample $Z_1, \ldots, Z_n$ and $(Z_{n1}^*, \ldots, Z_n^*)$ for a bootstrap sample, i.e., given $Z_1, \ldots, Z_n$ it is iid with distribution $F_n$. Arcones and Giné [4] proved that the correct bootstrap version of $U_n(Z)$ is

$$U_n(Z^*) = \frac{1}{n} \sum_{1 \leq i \neq j \leq n} h_2(Z_{ni}^*, Z_{nj}^*; F_n).$$

The fact that the limiting distributions of $U_n(Z)$ and $U_n(Z^*)$ coincide follows from Dehling and Mikosch [10]; see also Corollary 5.2 in [9] in the case $B = [0, 1]$.

**Proposition 5.2.** Under the aforementioned conditions, and if also $E[|h(Z_{i1}, \ldots, Z_{i4})|^2] < \infty$ for all indices $1 \leq i_1 \leq \cdots \leq i_4 \leq 4$, we have

$$d_2(\mathcal{L}(U_n(Z)), \mathcal{L}(U_n(Z^*))) \to 0, \quad n \to \infty,$$

for almost all realizations of $(Z_i)$. Here $d_2$ denotes the Wasserstein distance of order 2.

The additional second moment assumption on $h$ is satisfied for our kernel. Note that it suffices to consider the non-symmetric kernel $f$, and to show that $E[(f(Z_{i1}, Z_{i2}, Z_{i3}, Z_{i4}))^2] < \infty$ for all indices $1 \leq i_1, \ldots, i_4 \leq 4$. For our specific kernel, this condition reads as

$$E\left[\left(\|X_{i1} - X_{i2}\| + \|Y_{i1} - Y_{i2}\| + \|Y_{i3} - Y_{i4}\| - 2\|Y_{i1} - Y_{i3}\|\right)^2 \right] < \infty,$$

and this holds under the moment conditions in this paper.

The Wasserstein distance $d_2$ metrizes weak convergence and moment convergence up to the second order. Therefore Proposition 5.2 proves bootstrap consistency for $U_n(Z^*)$. In particular, $U_n(Z^*)$ and $nT_n(X, Y)$ have the same limit distribution up to some change of scale/location. From now, we will work with $U_n(Z)$ as a surrogate of the normalized sample distance covariance $nT_n(X, Y)$ and with its bootstrap version $U_n(Z^*)$.

### 5.3. Bootstrapping $U_n(Z^{(p)})$

Our goal is to show that we are allowed to replace $Z = (X, Y)$ in $U_n(Z)$ by the corresponding discretizations $Z^{(p)} = (X^{(p)}, Y^{(p)})$ as well as the corresponding result for $U_n(Z^*)$. We restrict ourselves to the case of random locations studied in Section 2. The corresponding bootstrap consistency results in the lattice case of Section 3 follow by a straightforward adaptation of the results in Dehling et al. [9] who considered the case $d = 1$ and $B = [0, 1]$.

We start by showing that $U_n(Z)$ and $U_n(Z^{(p)})$ are close conditionally on $N^{(p)}$.

**Lemma 5.3.** Assume that $X, Y$ are independent, [5,3] holds, and $N_p \to \infty$ a.s.as $p = p_n \to \infty$. Then

$$E[(U_n(Z) - U_n(Z^{(p)}))^2 \mid N^{(p)}] \to 0, \quad n \to \infty,$$

for a.e. realization of $(N^{(p)})$.

**Proof.** We consider

$$U_n(Z) - U_n(Z^{(p)}) \mid N^{(p)} := \frac{1}{n} \sum_{1 \leq i \neq j \leq n} \left( h_2(Z_i, Z_j) - h_2(Z_i^{(p)}, Z_j^{(p)} \mid N^{(p)}) \right),$$

where

$$h_2(z_1, z_2; F_Z^{(p)} \mid N^{(p)})$$

$$:= E[h(z_1, z_2, Z_3^{(p)}, Z_4^{(p)}) \mid N^{(p)}] - E[h(z_1, Z_2^{(p)}, Z_3^{(p)}, Z_4^{(p)}) \mid N^{(p)}]$$

$$- E[h(Z_1^{(p)}, Z_2^{(p)}, Z_3^{(p)}, Z_4^{(p)}) \mid N^{(p)}] + E[h(Z_1^{(p)}, Z_2^{(p)}, Z_3^{(p)}, Z_4^{(p)}) \mid N^{(p)}].$$
By construction, given \( N^{(p)} \), \( U_n(Z) - U_n(Z^{(p)} \mid N^{(p)}) \) only depends on the sample \( Z_1, \ldots, Z_n \) and is a \( U \)-statistic of order 2 with symmetric 1-degenerate kernel. By Lemma A in Serfling [29, p. 183], we have

\[
\begin{align*}
\mathbb{E}[(U_n(Z) - U_n(Z^{(p)} \mid N^{(p)})^2 \mid N^{(p)}] & \\
\leq c & \mathbb{E}[(h(Z_1, Z_2) - h(Z^{(p)}_1, Z^{(p)}_2 \mid N^{(p)})^2 \mid N^{(p)}] \\
\leq c & \mathbb{E}[(h(Z_1, Z_2, Z_3, Z_4) - h(Z^{(p)}_1, Z^{(p)}_2, Z^{(p)}_3, Z^{(p)}_4)^2 \mid N^{(p)}] \\
\leq c & \mathbb{E}[(f(Z_1, Z_2, Z_3, Z_4) - f(Z^{(p)}_1, Z^{(p)}_2, Z^{(p)}_3, Z^{(p)}_4)^2 \mid N^{(p)}] \\
& \leq c\mathbb{E}[(\|X_1 - X_2\|_2^2\|Y_1 - Y_2\|_2^2 - \|X_1^{(p)} - X_2^{(p)}\|_2^2\|Y_1^{(p)} - Y_2^{(p)}\|_2^2)^2 \mid N^{(p)}]

+(\|X_1 - X_2\|_2\|Y_3 - Y_4\|_2 - \|X_1^{(p)} - X_2^{(p)}\|_2\|Y_3^{(p)} - Y_4^{(p)}\|_2)^2 \mid N^{(p)}] \tag{5.6}
\end{align*}
\]

Now we can proceed as in the proof of Proposition 2.1 to show that the right-hand side converges to zero as \( n \to \infty \) along a.e. sample path of \( (N^{(p)})_{p>0} \). We illustrate this for the first term on the right-hand side. We have by independence of \((X_i)\) and \((Y_i)\),

\[
\begin{align*}
\mathbb{E}[(\|X_1 - X_2\|_2^2\|Y_1 - Y_2\|_2^2 - \|X_1^{(p)} - X_2^{(p)}\|_2^2\|Y_1^{(p)} - Y_2^{(p)}\|_2^2)^2 \mid N^{(p)}] & \\
\leq c & \mathbb{E}[(\|X_1 - X_2\|_2^2 - \|X_1^{(p)} - X_2^{(p)}\|_2^2)^2 \mid N^{(p)}] \mathbb{E}[\|Y_1 - Y_2\|_2^2] \\
+c & \mathbb{E}[\|X_1^{(p)} - X_2^{(p)}\|_2^2 \mid N^{(p)}] \mathbb{E}[\|Y_1 - Y_2\|_2^2 - \|Y_1^{(p)} - Y_2^{(p)}\|_2^2)^2 \mid N^{(p)}] \tag{5.7}
\end{align*}
\]

The expectation \( \mathbb{E}[\|Y_1 - Y_2\|_2^2] \) is finite by (5.3). For the same reason and by Riemann-sum approximation along a.e. sample path of \( (N^{(p)})_{p>0} \),

\[
\begin{align*}
\mathbb{E}[\|X_1^{(p)} - X_2^{(p)}\|_2^2 \mid N^{(p)}] & = \frac{1}{N_p} \sum_{i=1}^{N_p} \mathbb{E}[(X_1 - X_2)^2(U_i) \mid U_i] \\
& \to \int_{B} \mathbb{E}[(X_1 - X_2)^2(u)] \, du .
\end{align*}
\]

We have by dominated convergence

\[
\begin{align*}
\mathbb{E}[(\|X_1 - X_2\|_2^2 - \|X_1^{(p)} - X_2^{(p)}\|_2^2)^2 \mid (N^{(q)})_{q>0}] & \\
\leq c & \mathbb{E}[(\int_{B} (X_1 - X_2)^2(u) \, du - \|X_1^{(p)} - X_2^{(p)}\|_2^2 \mid (N^{(q)})_{q>0}] \\
& \to 0 , \quad p \to \infty ,
\end{align*}
\]

and the corresponding result holds if we replace \((X_1, X_2)\) by \((Y_1, Y_2)\). Hence the right-hand side in (5.7) converges to zero along a.e. sample path of \( (N^{(p)}) \).}

Our next goal is to show that \( U_n(Z^*) \) and \( U_n(Z^{(p)^*} \mid N^{(p)}) \) are asymptotically close where the latter quantity is defined as in (5.5) if we replace the distribution of \( Z \) by \( F_n \). We write \( \text{var}^* \) for the variance with respect to bootstrap probability measure.

We will need some stronger assumptions to achieve this goal.

**Lemma 5.4.** We assume that \( (N_p) \) are Poisson variables such that \( \mathbb{E}[N_p] = p = p_n \) and \( N_p \to \infty \) a.s. as \( n \to \infty \), \( \sum_n p_n^{-1/2} < \infty \) and

\[
\int_{B} \mathbb{E}[X^4(u) + Y^4(u)] \, du < \infty .
\]

Then for a.e. sample paths of \( (N^{(p)}) \) and \( (Z_i) \),

\[
\mathbb{P}(U_n(Z^*) - U_n(Z^{(p)*} \mid N^{(p)}) \to 0 , n \to \infty \mid (Z_i), (N^{(p)}) = 1 .
\]
Proof. We observe that given \( N^{(p)} \),
\[
U_n(Z^*) - U_n(Z^{*(p)} | N^{(p)}) = \frac{1}{n} \sum_{1 \leq i \neq j \leq n} (h_2(Z_i^*, Z_j^*) - h_2(Z_i^{*(p)}, Z_j^{*(p)} | N^{(p)}))
\]
is a \( U \)-statistic of order 2 with a 1-degenerate kernel. Again applying the variance formula in
Lemma A of [29], p. 183, we obtain
\[
\text{var}^*(U_n(Z^*) - U_n(Z^{*(p)} | N^{(p)}) | N^{(p)})
\]
\[
\leq c \text{var}^*(h_2(Z_1^*, Z_2^*) - h_2(Z_1^{*(p)}, Z_2^{*(p)} | N^{(p)}) | N^{(p)})
\]
\[
= c \frac{1}{n^2} \sum_{k,\ell=1}^{n} (h_2(Z_k, Z_\ell) - h_2(Z_k^{(p)}, Z_\ell^{(p)} | N^{(p)}))^2.
\]
Taking expectations on both sides, we have
\[
\mathbb{E} \left[ \text{var}^*(U_n(Z^*) - U_n(Z^{*(p)} | N^{(p)}) | N^{(p)}) \right] \leq c \mathbb{E} \left[ (h_2(Z_1, Z_2) - h_2(Z_1^{(p)}, Z_2^{(p)} | N^{(p)}))^2 \right].
\]
Now appeal to (5.6) and take expectations on both sides. Keeping in mind (5.7), we have to bound
expressions of the following type:
\[
c \mathbb{E} \left[ \left( \|X_1 - X_2\|_2 - \|X_1^{(p)} - X_2^{(p)}\|_2 \right)^2 | N^{(p)} \right] \mathbb{E} \left[ \|Y_1 - Y_2\|_2^2 \right]
\]
\[
+c \mathbb{E} \left[ \left( \|X_1 - X_2^{(p)}\|_2 \right)^2 \left( \|Y_1 - Y_2\|_2 - \|Y_1^{(p)} - Y_2^{(p)}\|_2 \right)^2 | N^{(p)} \right]
\]
\[
\leq c \mathbb{E} \mathbb{E} \left[ \|X_1 - X_2 - X_1^{(p)} + X_2^{(p)}\|_2^2 \right]
\]
\[
+c \left( \mathbb{E} \left[ \|X_1 - X_2^{(p)}\|_2^2 \right] \right)^{1/2} \left( \mathbb{E} \left[ \left( \|Y_1 - Y_2\|_2 - \|Y_1^{(p)} - Y_2^{(p)}\|_2 \right)^4 \right] \right)^{1/2}.
\]
We write \( A = X_1 - X_2 \). Then we have for the first expression on the right-hand side
\[
c \mathbb{E} \left[ \left( \|A\|_2 - \|A^{(p)}\|_2 \right)^2 \right] \leq c \mathbb{E} \left[ \mathbb{E} \left[ N_p^{-1} \sum_{i=1}^{N_p} A^2(U_i) - \int_B A^2(u)du \right] | A, N_p \right]
\]
\[
\leq c \mathbb{E} \mathbb{E} \left[ \left( \mathbb{E} \left[ N_p^{-1} \sum_{i=1}^{N_p} A^2(U_i) - \int_B A^2(u)du \right] \right)^2 | A, N_p \right]^{1/2}
\]
\[
= c \mathbb{E} \left[ N_p^{-1/2} 1(N_p > 0) \right] \mathbb{E} \left[ \left( \mathbb{E} \left[ \text{var}(A^2(U_1) | A) \right] \right)^{1/2} \right]
\]
\[
\leq c \mathbb{E} \left[ N_p^{-1/2} 1(N_p > 0) \right] \left( \int_B \mathbb{E} \left[ (X_1 - X_2)^4(u)d\mathbf{u} \right] \right)^{1/2}.
\]
The second factor is finite by the moment assumptions on \( X \). Using the Poisson structure of \( N_p \),
we can apply Lemma 4.1 to the first factor and conclude that it has the asymptotic order \( O(p^{-1/2}) \).
Now we consider the first expectation in (5.8):
\[
\mathbb{E} \left[ \|A^{(p)}\|_2^4 \right] = \mathbb{E} \left[ \left( N_p^{-1} \sum_{i=1}^{N_p} A^2(U_i) \right)^2 \right] \leq \mathbb{E} \left[ N_p^{-1} \sum_{i=1}^{N_p} A^4(U_i) \right]
\]
\[
= \mathbb{E} \left[ 1(N_p > 0) \right] \int_B \mathbb{E} \left[ (X_1 - X_2)^4(u)d\mathbf{u} \right].
\]
The right-hand side is finite in view of the moment conditions on \( X \). Next we turn to the second expression in (5.8). Write \( C = Y_1 - Y_2 \). It remains to bound
\[
(\mathbb{E} \left[ \|C\|_2 - \|C^{(p)}\|_2 \right]^4)^{1/2} \leq c \left( \mathbb{E} \mathbb{E} \left[ \left( N_p^{-1} \sum_{i=1}^{N_p} \left( C^2(U_i) - \int_B C^2(u)du \right) \right)^2 | C, N_p \right] \right)^{1/2}
\]
Proof. We have the decomposition
\[
\begin{align*}
\sum_{i,j=1}^{n} \epsilon_i \epsilon_j &= \frac{1}{n^2} \sum_{i,j=1}^{n} \left( \|X_i - X_j\|_2^2 - \|X_i^{(p)} - X_j^{(p)}\|_2^2 \right)
\end{align*}
\]
Therefore the Borel-Cantelli lemma implies that
\[
U_n(Z^*) - U_n(Z^{(p)}) \rightarrow 0, \quad n \rightarrow \infty.
\]
for a.e. sample path of \((Z_i)\) and \((N^{(p)})_{p>0}\). \(\square\)

**Corollary 5.5.** Under the conditions of Lemma 5.4, \((nT_n(X,Y))\) and \((U_n(Z^{(p)})\) (conditional on \((N^{(p)})\)) have the same limit distribution in \((3.4)\) (up to changes of scale/location).

This result means that the modified bootstrap for \((nT_n(X^{(p)}, Y^{(p)}))\) is consistent.

Proof. Following the discussion in Sections 5.1 and 5.3, \((nT_n(X,Y))\), \((U_n(Z))\) and \((U_n(Z^*))\) (given the data) have the same limit distribution (up to possible changes of scale/location). In this section we proved that \((U_n(Z))\) and \((U_n(Z^{(p)}))\) (given \((N^{(p)})\)) and \((U_n(Z^*))\) and \((U_n(Z^{(p)}))\) (given the data and \((N^{(p)})\)) have the same limit distribution. This finishes the proof. \(\square\)

It remains to show the bootstrap consistency for \((nR_n(X^{(p)}, Y^{(p)}))\). We consider the following modified bootstrap version of the latter sequence:
\[
\frac{n U_n(Z^{(p)})}{\sqrt{T_n(X^{(p)}, X^{(p)}) T_n(Y^{(p)}, Y^{(p)})}}.
\]
Since under our moment conditions, \(T_n(X, X) \overset{a.s.}{\rightarrow} T(X, X)\) and \(T_n(Y, Y) \overset{a.s.}{\rightarrow} T(Y, Y)\) by the strong law of large numbers for V-statistics it remains to show that
\[
(5.10) \quad T_n(X^{(p)}, X^{(p)}) - T_n(X, X) \overset{a.s.}{\rightarrow} 0, \quad n \rightarrow \infty,
\]
and the corresponding result for \((T_n(Y^{(p)}, Y^{(p)}))\). This is the content of the following lemma.

**Lemma 5.6.** Under the conditions of Lemma 5.4 and summability of \((p_n^{-1/4})\), \((5.10)\) and the corresponding result for \((T_n(Y^{(p)}, Y^{(p)}))\) hold.

Proof. We have the decomposition
\[
T_n(X, X) - T_n(X^{(p)}, X^{(p)})
= \frac{1}{n^2} \sum_{i,j=1}^{n} \left( \|X_i - X_j\|_2^2 - \|X_i^{(p)} - X_j^{(p)}\|_2^2 \right)
+ \left[ \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \|X_i - X_j\|_2 \right)^2 - \left( \frac{1}{n^2} \sum_{i,j=1}^{n} \|X_i^{(p)} - X_j^{(p)}\|_2 \right)^2 \right]
- 2 \frac{1}{n^2} \sum_{i,j,k=1}^{n} \left( \|X_i - X_j\|_2 - \|X_i^{(p)} - X_j^{(p)}\|_2 \right) \|X_i - X_k\|_2
\]
\[ + \|X_i^{(p)} - X_j^{(p)}\|_2 (\|X_i - X_k\|_2 - \|X_k^{(p)} - X_k^{(p)}\|_2) \]

\[ =: I_1 + I_2 - 2(I_{31} + I_{32}). \]

Writing \(A = X_1 - X_2\), we see that

\[ \mathbb{E}[|I_1|] \leq \mathbb{E}[\|A\|_2^2 - \|A^{(p)}\|_2^2] = \mathbb{E}\left[ N_p^{-1} \sum_{i=1}^{N_p} (A^2(U_i) - \int_B A^2(u)du) \mid A, N_p \right]. \]

Now we can proceed as for (5.9) and conclude that the right-hand side is \(O(p^{-1/2})\). Since we assume summability of \((p_n^{-1/4})\) the Borel-Cantelli lemma yields \(I_1 \Rightarrow 0\).

For \(I_2 \Rightarrow 0\) it suffices to prove that

\[ I_4 := \frac{1}{n^2} \sum_{i,j=1}^{n} (\|X_i - X_j\|_2 - \|X_i^{(p)} - X_j^{(p)}\|_2) \Rightarrow 0, \quad n \to \infty \]

since, by the strong law of large numbers for \(U\)-statistics, \(n^{-2} \sum_{i,j=1}^{n} \|X_i - X_j\|_2^2 \Rightarrow \mathbb{E}[\|X_1 - X_2\|_2^2]\). We have

\[ \mathbb{E}[|I_4|] \leq \mathbb{E}[\|A\|_2^2 - \|A^{(p)}\|_2^2] \leq \left( \mathbb{E}[\|A\|_2^2 - \|A^{(p)}\|_2^2] \right)^{1/2}. \]

Following the proof in (5.9), the right-hand side is bounded by \(c (\mathbb{E}[N_p^{-1/2} 1(N_p > 0)])^{1/2} = O(p^{-1/4})\). The summability of \((p_n^{-1/4})\) and the Borel-Cantelli lemma prove that \(I_4 \Rightarrow 0\).

We focus on showing \(I_{32} \Rightarrow 0\); in the case \(I_{31} \Rightarrow 0\) is analogous. We have

\[ \mathbb{E}[|I_{32}|] \leq \mathbb{E}\left[ \|X_1^{(p)} - X_2^{(p)}\|_2 \|X_1 - X_3\|_2 - \|X_1^{(p)} - X_3^{(p)}\|_2 \right] \leq \left( \mathbb{E}[\|A^{(p)}\|_2^2] \right)^{1/2} \left( \mathbb{E}[\|A\|_2^2 - \|A^{(p)}\|_2^2] \right)^{1/2}. \]

The first expected value is bounded uniformly for \(p\). Another application of (5.9) shows that the right-hand side is of the order \(O(p_n^{-1/4})\). Another application of the Borel-Cantelli lemma proves \(I_{32} \Rightarrow 0\). This finishes the proof. \(\square\)

6. A Monte Carlo study

6.1. Finite sample study of the convergence. In this section we conduct a Monte Carlo study of the finite sample behavior of the sample distance correlation for \(\beta = 1\); in what follows we suppress \(\beta\) in the notation. We consider a fractional Brownian sheet (fB for short) with Hurst parameter \(H = (H_1, H_2) \in (0, 1)^2\), in particular, \(H_i \in \{1/4, 1/2, 3/4\}\). As a heavy-tailed alternative we choose symmetric 1.8-stable Lévy sheets. The observations are given either on a lattice or at random locations in \([0, 1]^2\). In the lattice case we take \(q\) equidistant grid points on each side of \([0, 1]^2\), resulting in a lattice of \(p = q \times q\) points. We choose sample sizes \(n \in \{100, 200, 300\}\) and repeat the Monte Carlo simulations 500 times for each process.

For the simulation of fB sheets we calculate their covariance function

\[ \text{cov}(B^H(s), B^H(t)) = \prod_{i=1}^{d} \frac{1}{2} (|s_i|^{2H_i} + |t_i|^{2H_i} - |s_i - t_i|^{2H_i}), \quad s, t \in [0, 1]^d, \]

at the lattice points and use the multivariate Gaussian random generator in R (mvfast) while for Brownian (this is the case \(H_1 = H_2 = 0.5\)) and stable sheets we use the independent and stationary increment property: for each cell in \([0, 1]^2\) we draw properly scaled independent Gaussian or stable random variables and sum up, starting from the origin until we reach the boundary of \([0, 1]^2\). (We generated Brownian sheets with both methods, leading to very similar results.)

Figure 6.1 shows boxplots for \(R_n(X^{(p)}, Y^{(p)})\) based on simulations from independent fields \(X, Y\) on the described lattice. In the top and middle rows, \(q = 30\), hence \(p = 900\), and \(X, Y\) are fB
Figure 6.1. Boxplots for $R_n(X^{(p)}, Y^{(p)})$ based on 500 simulations of independent sheets $X, Y$. The sheets are simulated on a $p = q \times q$ equidistant lattice on $[0, 1]^2$ for $q = 30$. **Top and middle row:** fB sheets for different choices of Hurst coefficients $H_i \in \{1/4, 1/2, 3/4\}$, $i = 1, 2$ and increasing sample size $n$. **Bottom row:** Effects of increasing $n$ (for fixed $q = 100$) and increasing $q$ (for fixed $n = 300$). The left (right) graphs are based on Brownian (1.8-stable) sheets.

Sheets for different choices of $H_1$ and $H_2$. We see the influence of the smoothness of the sample paths: the larger $H_i$, the smoother the sample paths and the closer $R_n(X^{(p)}, Y^{(p)})$ to zero. In the bottom row we simulate from Brownian and 1.8-stable sheets and examine the effect of increasing $n \in \{100, 200, 300\}$ for fixed $q = 100$ and of increasing $q \in \{100, 200, 300\}$ for fixed $n = 300$. An increase of $n$ apparently improves the performance of $R_n(X^{(p)}, Y^{(p)})$: the larger $n$ the closer it is to zero. On the other hand, if we fix $n$ one hardly sees a change for different values of $p = q \times q$. It is surprising for us that the sample distance correlation for independent 1.8-stable sheets outperforms...
The bottom row graphs are based on dependent Brownian/1.8-stable sheets for the same choices of $H$.

We present boxplots for the same choices of $H_1$ and $H_2$ as in the top and middle rows of Figure 6.1. The construction of $X, Y$ is based on (6.1) and (6.2) for fB and 1.8-stable sheets for the same $n$, $p$ as in the top and middle rows of Figure 6.1.

In Figure 6.2 we confront the results of Figure 6.1 with the sample distance correlation for some dependent $X, Y$, again on the lattice. We consider iid fB sheets $X, X'$ and define $Y \overset{d}{=} X$ by

$$Y = \rho X + (1 - \rho^2)^{1/2} X', \quad \rho \in (0, 1).$$

This choice of $X, Y$ leads to the correlation $\rho$ between $X$ and $Y$. In Figure 6.2 we choose $\rho = 0.5$. We present boxplots for the same choices of $H_1$ and $H_2$ as in the top and middle rows of Figure 6.1.

The corresponding sample distance correlation for independent fB sheets: one detects independence between $X$ and $Y$ already for medium sample sizes.

Figure 6.2. Boxplots for $R_n(X^{(p)}, Y^{(p)})$ based on 500 simulations of dependent fB and 1.8-stable sheets $X, Y$ in the lattice case. The parameters $H_1$ and $H_2$ and the values $n, p$ are the same as for the corresponding graph at the same location in Figure 6.1. The construction of $X, Y$ is based on (6.1) and (6.2) for fB and 1.8-stable sheets for $\rho = 0.5$, respectively.
as at the bottom of Figure 6.1. We consider iid 1.8-stable symmetric Lévy sheets $X, X'$ and define $Y$ by

$$Y = \rho X + (1 - \rho^{1.8})^{1/1.8} X', \quad \rho \in (0, 1).$$

Then in particular, $X \overset{d}{=} Y$. In Figure 6.2 we choose $\rho = 0.5$. The graphs in Figure 6.2 are in stark contrast to those in Figure 6.1; they clearly point at the dependence of $X, Y$. Again, the value of $p$ seems irrelevant.

We also examine boxplots in the random location setting. In the lattice case in the top and middle graphs of Figures 6.1 and 6.2 we had $p = 900$ points in $[0, 1]^2$. In the random location case we choose uniform locations $U_i$ on $[0, 1]^2$ whose number $N_p$ is Poisson with parameter $p = 1000$, i.e., on average there are 1000 points. In Figures 6.3 and 6.4 we keep the same parameters and
sample sizes as in the previous two figures. In Figure 6.3 we illustrate the case of independent $X, Y$. Comparing Figures 6.3 and 6.1, we observe similar finite sample behavior: the sample size $n$ is more relevant for convergence to zero than the number $N_p$ and the random locations. The bottom graphs in Figure 6.3 look less convincing (the median of the boxplot is higher than in the lattice case) but this is due to the fact that we do not have the information from $100 \times 100 = 10000$ locations but only from $N_p \approx 1000$. (The choice of a Poisson variable $N_p$ with parameter $p = 10000$ leads to a complexity which we could not handle on our laptops.) Figure 6.4 corresponds to the setting of dependent $X, Y$ in Figure 6.2 with random locations. The graphs in both figures show quite convincingly the difference between dependence and independence of $X$ and $Y$. There is one significant difference to Figures 6.1 and 6.2: the distribution of $R_n(X^{(p)}, Y^{(p)})$ in Figures 6.3 and 6.4 is more spread than in the lattice case. This is due to the additional uncertainty of the random locations.
6.2. Size and power of the distance correlation test. We illustrate the performance of the bootstrap procedure for the test for independence based on distance correlation in the cases of fixed locations on a lattice and of randomly scattered locations. We focus on independent pairs \( X, Y \), Brownian or 1.8-stable sheets. Given a sample \((X_1^{(p)}, Y_1^{(p)}), \ldots, (X_n^{(p)}, Y_n^{(p)})\), we draw 500 bootstrap samples. From each bootstrap sample we calculate the sample distance correlation and from the corresponding bootstrap distribution the \((1 - \xi)\)-quantile \( q^*_{1-\xi} \). Finally, we verify whether
\[
R_n(X^{(p)}, Y^{(p)}) \geq q^*_{1-\xi}.
\]
Then we repeat this procedure \( M \in \{500, 1000\} \) times and count the successes of (6.3).

In Table 6.5 we choose \( q = 100 \) in the lattice case and observe the empirical rejection rates for independent Brownian and 1.8-stable sheets \( X, Y \): each cell of the table corresponds to a given sample size \( n \), test level \( \xi \) and iteration number \( M \). In Table 6.6 we consider the simulation results in the lattice case for dependent standard Brownian sheets with correlation \( \rho \in (0, 1) \) and \( M = 500 \), where the correlation \( \rho \) is accomplished through (6.1). We also consider dependent 1.8-stable Lévy sheets \( X, Y \) with dependence parameter \( \rho \) introduced in (6.2). In agreement with the theory, the rejection rates increase as \( n \) and \( \rho \) increase.

| \( n \) \( \backslash \) \( \xi \) | \[ \frac{n}{\rho} \] Brownian sheets \( 500 \) | \( 1000 \) | \( 1000 \) | \( 1000 \) | \[
\]
| \( 100 \) | 12.8 | 6.2 | 11.6 | 6.2 | 11.4 | 3.6 | 9.2 | 2.6 |
| \( 200 \) | 9.4 | 4.8 | 9.6 | 4.8 | 8.2 | 3.4 | 8.4 | 4.8 |
| \( 300 \) | 8.8 | 4.4 | 11.6 | 6.0 | 9.2 | 3.8 | 7.0 | 3.2 |
| \( 400 \) | 12.8 | 7.0 | 12.2 | 7.0 | 8.6 | 4.2 | 9.6 | 4.2 |

Table 6.5. Bootstrap size of \( nR_n(X^{(p)}, Y^{(p)}) \): lattice case. Empirical rejection rates of bootstrap test based on \( nR_n(X^{(p)}, Y^{(p)}) \) for independent Brownian and 1.8-stable sheets \( X, Y \) with \( M = 500, 1000 \) iterations. We choose \( q = 100 \).

Table 6.6 shows the empirical rejection rates for independent Brownian and 1.8-stable sheets \( X, Y \) in the random observation setting. We generate the Poisson number \( N_p \) and iid uniform locations \((U_j)_{j \leq N_p}\). To generate the discretized Brownian sheets we calculate the correlation at the given random lattice and use the multivariate Gaussian random generator in R (mvfast) and choose \( p = 1000 \). For generating a stable sheet at random locations, we first proceed as in the lattice case, generating independent stable random variables for each cell of the random lattice, where the lattice is constructed by cutting \([0, 1]^2\) at all marginal points of \((U_j)\). Then we calculate the sheet at \( U_j \) by summing the cells, starting from the origin. The computational complexity for doing this is high and therefore we restrict ourselves to the smaller Poisson parameter \( p = 500 \). As discussed in Section 4 the bootstrap of \( Z^{(p)} = (X^{(p)}, Y^{(p)}) \) is conducted conditionally on \( N_p \). In Tables 6.7 and 6.8 we present results for \( N_{500} = 492 \) and \( N_{1000} = 982 \). We have examined the bootstrap procedure for several realizations of \( N_p \) with \( p = 500, 1000 \), but we did not find differences in the performance.

In Table 6.8 we illustrate the power of the test in the random observation case. The dependence parameter \( \rho \) and the sample size \( n \) are the same as in Table 6.6. Again, \( p = 1000 \) \((p = 500)\) are the Poisson parameters of \( N_p \) for Brownian (stable) sheets. When comparing Tables 6.7 and 6.8, the empirical powers are quite similar, despite the different models for \((X^{(p)}, Y^{(p)})\).

7. An application to Japanese meteorological data

We apply our results to Japanese meteorological data. We choose the 3 most fundamental factors: temperature (temp), precipitation (prec) and wind speed (wind) which have been observed for a long time and
Table 6.6. Bootstrap power of $nR_n(X^{(p)}, Y^{(p)})$ with dependent stable sheets: lattice case. Empirical rejection rates of bootstrap test based on $nR_n(X^{(p)}, Y^{(p)})$ for dependent Brownian sheets and 1.8-stable sheets. We choose $q = 100$. In each cell of the table the rejection rate is calculated from $M = 500$ iterations.

<table>
<thead>
<tr>
<th>$\rho \setminus \xi$</th>
<th>Dependent Brownian sheets</th>
<th>Dependent 1.8-stable Lévy sheets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0.2</td>
<td>0.50</td>
<td>0.65</td>
</tr>
<tr>
<td>0.3</td>
<td>0.85</td>
<td>0.93</td>
</tr>
<tr>
<td>0.4</td>
<td>0.98</td>
<td>0.99</td>
</tr>
<tr>
<td>0.5</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 6.7. Bootstrap size of $nR_n(X^{(p)}, Y^{(p)})$: random location case. Empirical rejection rates of bootstrap test based on $nR_n(X^{(p)}, Y^{(p)})$ for independent Brownian and 1.8-stable sheets $X, Y$ with $M = 500$ iterations. There is a Poisson number $N_p$ of iid uniform locations on $(0,1)^2$, the Poisson parameter is $p = 1000$ ($p = 500$) for Brownian (stable) sheets.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$\xi = 0.1$</th>
<th>$\xi = 0.05$</th>
<th>$\xi = 0.1$</th>
<th>$\xi = 0.05$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>12.2</td>
<td>5.4</td>
<td>10.6</td>
<td>4.0</td>
</tr>
<tr>
<td>200</td>
<td>11.4</td>
<td>6.0</td>
<td>8.6</td>
<td>4.0</td>
</tr>
<tr>
<td>300</td>
<td>10.6</td>
<td>3.6</td>
<td>8.5</td>
<td>4.4</td>
</tr>
<tr>
<td>400</td>
<td>8.8</td>
<td>5.6</td>
<td>9.0</td>
<td>3.8</td>
</tr>
</tbody>
</table>

Table 6.8. Bootstrap power of $nR_n(X^{(p)}, Y^{(p)})$: random location case. Empirical rejection rates of bootstrap test based on $nR_n(X^{(p)}, Y^{(p)})$ for dependent Brownian and 1.8-stable sheets. The number $N_p$ is Poisson distributed with $p = 1000$ ($p = 500$) for Brownian (stable) sheets. In each cell of the table the rejection rate corresponds to $M = 500$ iterations.

<table>
<thead>
<tr>
<th>$\rho \setminus \xi$</th>
<th>Dependent Brownian sheets</th>
<th>Dependent 1.8-stable sheets</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>100</td>
<td>200</td>
</tr>
<tr>
<td>0.1</td>
<td>0.05</td>
<td>0.05</td>
</tr>
<tr>
<td>0.2</td>
<td>0.52</td>
<td>0.67</td>
</tr>
<tr>
<td>0.3</td>
<td>0.88</td>
<td>0.93</td>
</tr>
<tr>
<td>0.4</td>
<td>0.99</td>
<td>0.96</td>
</tr>
<tr>
<td>0.5</td>
<td>1.00</td>
<td>1.00</td>
</tr>
</tbody>
</table>

over a wide range of Japan. The data are available in various formats at the web-page of the Japanese Meteorological Agency [https://www.data.jma.go.jp/gmd/risk/obsdl]. Since daily data include many zeros and can be sparse, especially for precipitation, monthly average data are taken. From January 1980 to January 2021 we have 493 monthly data at 783 observation stations. There exist more such points in Japan. They are, however, subject to problems such as change of position, many missing data, or only precipitation is observed. We removed these points, but still have plenty of points left; see the red dots in the map of Japan in Figure 7.1. Missing values are observed at less than 30 stations, corresponding to less than 10 out of 493 months in total. A missing value at a station is replaced by the average value at the station.

In our analysis we focus on the distance correlations for the pairs (prec & temp), (prec & wind), (temp & wind) denoted by $R_{pt}$, $R_{pw}$, $R_{wt}$. In a preliminary analysis we conduct pair-wise independence tests between temp, prec and wind at each station. Here we use the classical sample distance correlation/correlation coefficients (Pearsen, Spearman, and Kendall) between two components of a random vector. At each station we use the time series of 493 monthly observations and conduct bootstrap tests for pair-wise independence
Based on 1000 resamples. We follow the procedure described in Section 6 with significance levels $\xi \in \{0.05, 0.01\}$. For other correlation coefficient tests, we use the “corr.test” R package. The results are reported in Table 7.3. Each cell contains the number of rejections of the hypothesis of pair-wise independence between (prec & temp), (prec & wind), (temp & wind). The hypothesis of independence is overwhelmingly rejected at the majority of stations while the distance correlation based test tends to detect dependence more often than other correlation tests.

<table>
<thead>
<tr>
<th>rejection number</th>
<th>prec &amp; temp</th>
<th>prec &amp; wind</th>
<th>temp &amp; wind</th>
</tr>
</thead>
<tbody>
<tr>
<td>methods $\setminus \xi$</td>
<td>0.05</td>
<td>0.01</td>
<td>0.05</td>
</tr>
<tr>
<td>Pearson Corr.</td>
<td>734</td>
<td>720</td>
<td>599</td>
</tr>
<tr>
<td>Spearman Corr.</td>
<td>731</td>
<td>710</td>
<td>599</td>
</tr>
<tr>
<td>Kendall Corr.</td>
<td>730</td>
<td>710</td>
<td>599</td>
</tr>
<tr>
<td>Dist. Corr.</td>
<td>782</td>
<td>773</td>
<td>698</td>
</tr>
</tbody>
</table>

Table 7.3. Rejection numbers of pair-wise independence for the 783 stations.

We interpret the monthly meteorological observations as realizations of a random field on the area of Japan. We regard the monthly data as iid random field observations and conduct independence tests on them. Figure 7.1 shows quite convincingly that the stations are scattered randomly all over Japan, so the assumption of uniformly distributed locations is plausible. For further confirmation of the uniformity of the locations we use some exploratory statistical tools. For the marginal uniformity of all observations on Japan we considered QQ-plots of the (re-scaled to $(0,1)$) latitude and longitude of all observations against the uniform distribution; see Figure 7.4. The fit is not good due to the distorted diagonal shape of the Japanese islands, and there are large mountain areas and big lakes with observation points. The marginal fit improves when considering QQ-plots in particular regions; see Figure 7.5. For checking bivariate uniformity we applied Ripley’s K-functions [27] (see also [3]) which are a convenient tool for measuring clustering and randomness of spatial data; more sophisticated methods were described in Section 1.3. We chose rectangular areas on the Japanese islands as wide as possible such that they contain as many as possible points. Then we re-scaled the rectangles to unit square size. Three such regions are selected; see Figure 7.6. In view of the plots and shapes of the K-function we may accept the assumption of uniformly distributed locations. Therefore, we follow the approach of random field at random locations from Sections 1.3 and 2. We define distance as in (1.13) and rely on Theorem 2.6 for the theoretical background on the independence test. The tests are based on the bootstrap procedure introduced in the simulation study (Section 6); the theory for the bootstrap was developed in Section 5.
Figure 7.4. QQ-plots of uniform quantiles on (0, 1) against the (re-scaled) latitude and longitude of all observation points in Japan. The red dotted lines indicate 95% asymptotic confidence bands.

Figure 7.5. QQ-plots of uniform quantiles on (0, 1) against the (re-scaled) latitude and longitude of observation points in 8 Japanese regions. The red dotted lines indicate 95% asymptotic confidence interval.
Figure 7.6. Ripley’s K-function for three retangular regions on the Japanese islands, rescaled to the unit square. Top: plots of observation points from specific ranges of latitude and longitude (indicated on the left and bottom sides of the graphs). Bottom: Ripley’s K-functions of the corresponding regions. If data are randomly scattered the curves must be close to their limit $r^2$ (light blue curves) as the number of points increases. The other tree curves, isotropic, translate and border are edge corrected versions. Notice that the estimation of K is affected by edge effects since points outside the window are unobservable (see [3]).

The first line in Table 7.7 yields the pair-wise sample distance correlations for the whole of Japan. They clearly hint at dependence between the 3 factors, and this is also confirmed by bootstrap tests of the sample distance correlation at different significance levels; see the right part of the table. Next we study dependence locally in the distinct regions of Japan and at different seasons of the year.

Most parts of Japan are classified as humid subtropical climate area. However, the country stretches out from the far north to the far south, and the weather is strongly influenced by the seasonal monsoon wind. The changes caused by the 4 seasons are significant; cf. [1]. Therefore we also test for independence at regional level in each season. The seasonal samples consist of $4 \times 123 = 492$ months, discarding the last month. Seasons are defined as common in the Northern hemisphere: March–May, June–August, September–November, December–February.

Table 7.7 provides the pair-wise (distance) correlations for annual and quarterly meteorological data for the whole of Japan (Ja, 783 stations) as well as 8 main regions (indicated in Figure 7.2): Hokkaido (Ho, 159), Tohoku (To, 137), Kanto (Ka, 73), Chubu (Chb, 140), Kinki (Ki,65), Chugoku (Chg, 71), Shikoku (Shi, 40) and Kyusyu (Kyu, 98). After the names of the regions and seasons the pair-wise distance correlations $R$ and the results of the independence tests based on the bootstrap for $R$ are presented for 1000 resamples, following the previous section. We can extract several implications from the test results for the distance correlation and its magnitude which are also in agreement with the research of meteorologists; cf. [2]. The pair-wise distance correlations of the 3 factors in the whole of Japan clearly imply dependence of all weather...
patterns through the 4 seasons. Especially, the values in autumn are rather high. The situation is somewhat different if one looks at each region. The 3 factors still keep stable pair-wise dependence in autumn and winter in most regions (an exception is the winter of Chugoku area), though the values of $R_{pt}$ in winter are rather small in the southern region.

The autumn values are relatively high in most regions, in agreement with the values for the whole country. Possible causes include the typhoon (tropical cyclones) attacks in August and October, yielding a lot of precipitation and strong wind speed together with changes in temperature. Rain fronts in autumn can be another reason. While (prec & temp) and (temp & wind) indicate dependence in all areas through the 4 seasons, the relation between (prec & wind) shows independence in several regions, especially in spring and summer. One reason is that the baiu front, causing the rainy season, is stationary and moves very slowly around in May–June. This fact may violate the dependence relations between the 3 factors.

8. **Proof of Proposition 2.1**

Write for any Riemann square-integrable random field $Z$ on $B$ independent of $(N^{(p)})_{p > 0}$, and any $\beta \in (0, 2]$,

$$D_p^{(\beta)}(Z) = \|Z^{(p)}\|_2 - \|Z\|_2^\beta = \left(N_p^{-1} \int_B \|Z^{(p)}\|_2 - \|Z\|_2^\beta \right)^{\beta/2} - \left(\int_B \|Z^{(p)}(u)\|_2^\beta du\right)^{\beta/2}.$$ 

Then we have

$$|D_p^{(\beta)}(Z)| \leq |D_p^{(2)}(Z)|^{\beta/2}.$$ 

We start by proving [2.1]. We have

$$T_{n, \beta}(X^{(p)}, Y^{(p)}) = T_{n, \beta}(X, Y) = I_1 + I_2 - 2I_3,$$

where

$$I_1 = \frac{1}{n^2} \sum_{k,l=1}^n \left(\|X_k - X_l\|_2 \|Y_k - Y_l\|_2 - \|X_k - X_l\|_2 \|Y_k - Y_l\|_2^\beta\right),$$

$$I_2 = \frac{1}{n^2} \sum_{k,l=1}^n \left(\|X_k - X_l\|_2 \frac{1}{n^2} \sum_{k,l=1}^n \|Y_k - Y_l\|_2 - \frac{1}{n^2} \sum_{k,l=1}^n \|X_k - X_l\|_2 \sum_{k,l=1}^n \|Y_k - Y_l\|_2\right)^{\beta/2},$$

$$I_3 = \frac{1}{n^2} \sum_{k,l,m=1}^n \left(\|X_k - X_l\|_2 \frac{1}{n^2} \sum_{k,l=1}^n \|Y_k - Y_l\|_2 - \|X_k - X_l\|_2 \|Y_m\|_2\right).$$

We have

$$|I_1| \leq \frac{1}{n^2} \sum_{k,l=1}^n |D_p^{(\beta)}(X_k - X_l)| \|Y_k - Y_l\|_2 + \frac{1}{n^2} \sum_{k,l=1}^n |D_p^{(\beta)}(Y_k - Y_l)| \|X_k - X_l\|_2 \|Y_k - Y_l\|_2$$

$$=: I_{11} + I_{12}.$$

Since the conditions on $X$ and $Y$ are symmetric it suffices to consider $I_{11}$. It will be convenient to set $Z = X_1 - X_2$. Since $((X_i, Y_i))$ are iid and independent of $N^{(p)}$ in view of [8.1], we get the bound

$$\mathbb{E}[I_{11} \mid N^{(p)}] \leq \mathbb{E}[\|D_p^{(\beta)}(Z)\| \mid N^{(p)}] \mathbb{E}[\|Y_1^{(p)} - Y_2^{(p)}\|_2 \mid N^{(p)}]$$

$$\leq \mathbb{E}[\|D_p^{(2)}(Z)\|^{\beta/2} \mid N^{(p)}] \mathbb{E}[\|Y_1^{(p)} - Y_2^{(p)}\|_2 \mid N^{(p)}].$$

Assume that $X, Y$ have finite variance. Then, by the strong law of large numbers,

$$\mathbb{E}[\|D_p^{(2)}(Z)\|^{\beta/2} \mid N^{(p)}] \leq \left(\frac{N_p}{n} \sum_{i=1}^n \mathbb{E}[\|Z^{(p)}(U_i)\|_2] + \int_B \mathbb{E}[\|Z^{(p)}(u)\|_2] du\right)^{\beta/2}$$

$$\overset{a.s.}{\to} \left(2 \int_B \mathbb{E}[\|Z^{(p)}(u)\|_2] du\right)^{\beta/2}, \quad p \to \infty.$$ 

The right-hand side is finite by assumption (1a). If $X, Y$ have infinite variance and $\beta \in (0, 2)$ then

$$\mathbb{E}[\|D_p^{(2)}(Z)\|^{\beta/2} \mid N_p] \leq 2 \mathbb{E}\left[\sup_{u \in B} \|Z^{(\beta)}(u)\|_2\right].$$

The right-hand side is finite by assumption (1b). Conditional on \((N^{(p)})_{p>0}\) we have the convergence
\[
|D_p^{(2)}(Z)|^{\beta/2} \xrightarrow{a.s.} 0, \quad p \to \infty,
\]
due the convergence of the Riemann sums \(\|Z^{(p)}\|_2 \to |Z|_2\) for a.e. realization of \(Z\). Then an application of dominated convergence yields
\[
E[|D_p^{(2)}(Z)|^{\beta/2} | N^{(p)}] \xrightarrow{a.s.} 0.
\]
A similar argument shows that \((E[|Y_1^{(p)} - Y_2^{(p)}|^{\beta/2} | N^{(p)}])\) is bounded a.s. We conclude that \(E[|I_1| | N^{(p)}] \xrightarrow{a.s.} 0, p \to \infty\). We can deal with \(I_2, I_3\) in the same way by observing that
\[
|I_2| \leq \frac{1}{n^2} \sum_{k,l=1}^n |D_p^{(\beta)}(X_k - X_l)| \frac{1}{n^2} \sum_{k,l=1}^n \|Y_k^{(p)} - Y_l^{(p)}\|^\beta_2
\]
\[
+ \frac{1}{n^2} \sum_{k,l=1}^n \|X_k - X_l\|^\beta_2 \frac{1}{n^2} \sum_{k,l=1}^n |D_p^{(\beta)}(Y_k - Y_l)|,
\]
and
\[
|I_3| \leq \frac{1}{n^2} \sum_{k,l,m=1}^n |D_p^{(\beta)}(X_k - X_l)| \|Y_k^{(p)} - Y_m^{(p)}\|^\beta_2 + \frac{1}{n^3} \sum_{k,l,m=1}^n |D_p^{(\beta)}(Y_k - Y_m)| \|X_k - X_l\|^\beta_2.
\]
We omit further details.

Next we deal with the case \(X = Y\). Again using the decomposition (8.2) and writing \(Z = X_1 - X_2\), we observe that
\[
E[|I_1| | N^{(p)}] \leq E[|D_p^{(\beta)}(Z)| (\|Z\|_2^{\beta} + \|Z\|_2^\beta) | N^{(p)}]
\]
\[
\leq c E[\|Z\|_2^{2\beta} + \|Z^{(p)}\|_2^{2\beta} | N^{(p)}].
\]
Since we already know that \(D_p^{(\beta)}(Z) \xrightarrow{a.s.} 0\) conditionally on \((N^{(p)})_{p>0}\) we intend to use dominated convergence to show that the left-hand side converges to zero a.s. Therefore we will bound the right-hand terms. First assume that \(X\) has finite variance. We observe that
\[
E[\|Z\|_2^{2\beta} | N^{(p)}] = E\left[\left(\int_B Z^2(u) \, du\right)^{\beta}\right] \leq \begin{cases} \left(\int_B E[Z^2(u) \, du]\right)^{\beta} & \text{for } \beta \leq 1, \\ \int_B E[|Z|^{2\beta}(u) \, du] \, du & \text{for } \beta > 1. \end{cases}
\]
The right-hand side is finite by assumption (2a). If \(X\) has infinite variance and \(\beta \in (0, 1)\) we have \(E[\|Z\|_2^{2\beta}] \leq E[\sup_{u \in B} |Z|^{2\beta}(u)]\), and the right-hand side is finite by assumption (2b). The same bounds apply to
\[
E[\|Z\|_2^{2\beta} | N^{(p)}] = E[(N_{-1}^{-1} \sum_{i=1}^{N_p} Z^2(U_i))^\beta | N^{(p)}].
\]
Thus for any \(\varepsilon > 0\), \(P(|I_1| > \varepsilon | N^{(p)}) \to 0\) as \(n \to \infty\) along a.e. sample path of \((N^{(p)})\).

We also have
\[
|I_2| \leq \frac{1}{n^2} \sum_{k,l=1}^n |D_p^{(\beta)}(X_k - X_l)| \frac{1}{n^2} \sum_{k,l=1}^n \left(\|X_k^{(p)} - X_l^{(p)}\|_2^\beta + \|X_k - X_l\|_2^\beta\right) =: I_{21} I_{22}.
\]
We have already proved
\[
E[I_{21}^{(\beta)} | N^{(p)}] \leq E[|D_p^{(\beta)}(Z)| | N^{(p)}] \to 0.
\]
Moreover,
\[
E[I_{22} | N^{(p)}] \leq E[\|Z^{(p)}\|_2^\beta | N^{(p)}] + E[\|Z\|_2^\beta].
\]
We have already shown that the right-hand side is bounded for a.e. sample path of \((N^{(p)})_{p>0}\). Therefore we have for any \(\varepsilon > 0\) and a.e. sample path of \((N^{(p)})_{p>0}\),
\[
P(|I_2| > \varepsilon | N^{(p)}) \to 0, \quad n \to \infty.
\]
We have

\[ |I_3| \leq \frac{1}{n^3} \sum_{k,l,m=1}^{n} |D_p^{(\beta)}(X_k - X_l)| \left( \|X^{(p)}_k - X^{(p)}_l\|_2 + \|X_k - X_l\|_2^2 \right), \]

hence by Cauchy-Schwarz,

\[ \mathbb{E}[|I_3| \mid N^{(p)}] \leq \mathbb{E}[|D_p^{(\beta)}(X_1 - X_2)| \left( \|X^{(p)}_1 - X^{(p)}_2\|_2 + \|X_1 - X_2\|_2^2 \right) \mid N^{(p)}] \]

\[ \leq \mathbb{E}[|D_p^{(2)}(X_1 - X_2)|^{\gamma/2} \left( \|X^{(p)}_1 - X^{(p)}_2\|_2 + \|X_1 - X_2\|_2^2 \right) \mid N^{(p)}] \]

\[ \leq \left( \mathbb{E}[|D_p^{(2)}(Z)|^{\beta} \mid N^{(p)}] \right)^{1/2} \left( \mathbb{E}[\|Z^{(p)}_1\|_2^2 \mid N^{(p)}] \right)^{1/2} + \left( \mathbb{E}[\|Z^{(p)}_2\|_2^2] \right)^{1/2}. \]

The right-hand side converges to zero by arguments similar to those above. We finally conclude that

\[ \mathbb{P}(|I_3| > \varepsilon \mid N^{(p)}) \to 0, \quad n \to \infty. \]

This finishes the proof of (2.2).

9. Proof of Proposition 4.1

Lemma 9.1. (1) Assume \( \beta \in (0,2) \) and, if \( X \) has finite second or \( \beta \)th moment, we assume (A1) or (B1), respectively. Then there exists a constant \( c > 0 \) such that

\[ \sup_p \mathbb{E}[\|X^{(p)}_1\|_2^2 \mid U^{(p)}] \leq c \quad \text{a.s.} \]

where \( U^{(p)} = (U_1, \ldots, U_p) \).

(2) If \( \beta \in (1,2) \) and \( \max_t \mathbb{E}[|X(t)|^{2(2\beta-1)}] < \infty \) then we have \( \mathbb{E}[\|X\|_2^{2(2\beta-1)}] < \infty \), and there exists a constant \( c > 0 \) such that

\[ \sup_p \mathbb{E}[\|X^{(p)}_1\|_2^{2(2\beta-1)} \mid U^{(p)}] \leq c \quad \text{a.s.} \]

Proof. (1) Assume (A1). Then \( \sup_t \mathbb{E}[X^2(t)] = c_0 < \infty \). In view of (4.2) and by Jensen’s inequality we have

\[ \mathbb{E}[\|X^{(p)}_1\|_2^2 \mid U^{(p)}] = \mathbb{E} \left[ \left( \tilde{p}^{-1} \sum_j X_j^2 \right)^{\beta/2} \mid U^{(p)} \right] \]

\[ \leq \mathbb{E} \left[ \left( \tilde{p}^{-1} \sum_j \sum_{\ell \in U_j \Delta_j} X^2(\ell) \mid U^{(p)} \right)^{\beta/2} \right] \]

\[ \leq \left( \tilde{p}^{-1} \sum_j \sum_{\ell \in U_j \Delta_j} \mathbb{E}[X^2(\ell) \mid U^{(p)}] \right)^{\beta/2} \leq c_0^{\beta/2} \quad \text{a.s.} \]

Assume (B1). Then by similar arguments

\[ \mathbb{E}[\|X^{(p)}_1\|_2^2 \mid U^{(p)}] \leq c \mathbb{E} \left[ \left( \sup_t X^2(t) \right)^{\beta/2} \right] \leq c \mathbb{E} \left[ \left( \sup_t |X(t)|^\beta \right) \right] < \infty \quad \text{a.s.} \]

(2) The first statement is immediate. Using Jensen’s inequality, we also have

\[ \mathbb{E}[\|X^{(p)}_1\|_2^{2(2\beta-1)} \mid U^{(p)}] \leq \tilde{p}^{-1} \sum_j \sum_{k: U_k \Delta_j} \mathbb{E}[\|X(U_k)|^{2(2\beta-1)} \mid U^{(p)}] \leq c \max_t \mathbb{E}[|X(t)|^{2(2\beta-1)}] \leq \infty \quad \text{a.s.} \]

Now we proceed with the proof of the proposition. In what follows, it will be convenient to write \( Z = X_1 - X_2 \). We use the decomposition (8.2) adjusted to the present situation.

1. We assume (A1).

First we study \( I_1 \). By a symmetry argument it suffices to consider \( I_{11} \).

Assume \( \beta \in (0,1] \). Using the independence of \( X, Y \) and [Lemma 9.1(1)], we have

\[ \mathbb{E}[I_{11}] = \mathbb{E}[\mathbb{E}[I_{11} \mid U^{(p)}]] \]

\[ \leq \mathbb{E}[\mathbb{E}[|D_p^{(\beta)}(Z)| \mid U^{(p)}] \mathbb{E}[|Y_1^{(p)} - Y_2^{(p)}|_2^\beta \mid U^{(p)}]] \]
(9.1) \[ \leq c \mathbb{E}[\|D_p^{(\beta)}(Z)\|] \leq c \mathbb{E}[\|Z^{(p)} - Z\|^2], \]

where we used \( \beta \in (0,1) \) in the last step. We have

(9.2) \[ \mathbb{E} ][Z^{(p)} - Z|^2] = \mathbb{E} \left[ \left( \sum_{j=1}^{\hat{p}} \int_{\Delta_j} (Z_j - Z(t))^2 \, dt \right)^{\beta/2} \right] \]

Then an application of Jensen’s inequality yields

\[ J_2 \leq \left( \mathbb{E} \left[ \sum_{j=1}^{\hat{p}} \int_{\Delta_j} Z^2(t) \, dt \right] \right)^{\beta/2} \]

Now we turn to \( J_1 \). Then applications of Jensen’s inequality and (A1) yield

\[ J_1 \leq \left( \mathbb{E} \left[ \sum_{j=1}^{\hat{p}} \int_{\Delta_j} (Z_j - Z(t))^2 \, dt \right] \right)^{\beta/2} \]

Assume \( \beta \in (1,2) \). Applying \( |x^\gamma - y^\gamma| \leq (x \vee y)^{\gamma-1} |x - y| \) for positive \( x, y \) and Hölder’s inequality, we obtain

(9.3) \[ \mathbb{E}[|I_{11}|] \leq c \left( J_1 + J_2 \right) \leq c (p^{-1} + \hat{p}^{-\gamma x / d})^{\beta/2}, \]

and in turn

\[ \mathbb{E}[|I_1|] \leq c (p^{-1} + \hat{p}^{-\gamma x / d})^{\beta/2}. \]

Assume \( \beta \in (1,2) \). Applying \( |x^\gamma - y^\gamma| \leq (x \vee y)^{\gamma-1} |x - y| \) for positive \( x, y \) and Hölder’s inequality, we obtain

(9.4) \[ \mathbb{E}[D_p^{(\beta)}(Z)] \leq c \mathbb{E}[\|Z^{(p)}\|^2 \vee \|Z\|^{2(1-\beta)}] \]

Since we assume finite second moments via (A1) we have \( P_1 < \infty \). Moreover,

\[ \frac{2}{\gamma x / d} = \mathbb{E}[Z^{(p)} - Z_2^{(3-\beta)}] = \mathbb{E} \left[ \left( \sum_{j=1}^{\hat{p}} \int_{\Delta_j} (Z_j - Z(t))^2 \, dt \right) \right]^{1/(3-\beta)}. \]

Observing that \( (3 - \beta)^{-1} < 1 \) we can use the same arguments as for \( \beta \in (0,1) \) to obtain the bound

\[ P_2^{2/(3-\beta)} \leq c (p^{-1} + \hat{p}^{-\gamma x / d})^{1/(3-\beta)}. \]
Under (A1), we conclude that we have the following upper bound for $\beta \in (1, 2)$:

$$\mathbb{E}||I_1|| \leq c\left(\hat{p}^{-\gamma X/d} + p^{-1}\right)^{1/2}.$$  

Combining this bound with [9,3], we finally have for any $\beta \in (0, 2)$,

$$\mathbb{E}||I_1|| \leq c\left(\hat{p}^{-\gamma X/d} + p^{-1}\right)^{(1\wedge \beta)/2}.$$  

Now we turn to $I_2, I_3$. We have

$$|I_2| \leq \frac{1}{n^2} \sum_{k,l=1}^{n} |D_p^{(2)}(X_k - X_l)| \frac{1}{n^2} \sum_{k,l=1}^{n} \|Y_k^{(p)} - Y_l^{(p)}\|_2^\beta + \frac{1}{n^2} \sum_{k,l=1}^{n} \|X_k - X_l\|_2 \frac{1}{n^2} \sum_{k,l=1}^{n} |D_p^{(2)}(Y_k - Y_l)|$$

and a similar bound holds for $|I_3|$. The same arguments as for $\mathbb{E}||I_1||$ yield

$$\mathbb{E}||I_2 + I_3|| \leq c\left(\hat{p}^{-\gamma X/d} + p^{-1}\right)^{(1\wedge \beta)/2}.$$  

We omit further details.

2. Now we assume that $X, Y$ have finite $\beta$th moment for some $\beta \in (0, 2)$ and (B1), (B2) hold. 

First we study $I_1$ and focus on $I_{11}$. We follow the patterns of the proof under condition (A1) in the finite variance case.

Assume $\beta \in (0, 1]$. We again write $Z = X_1 - X_2$. In view of [9,1] and [Lemma 9.1 (1)] it suffices to bound $J_1, J_2$. If (B1) holds we can proceed as in part 1.:

$$J_2 \leq \left(\mathbb{E}\left[\hat{p}^{-1} \sum_{j=1}^{\hat{p}} 1(\#\Delta_j = 0)\right]\right)^{\beta/2} \mathbb{E}\left[\max_{i \in B} |Z(i)|^{\beta}\right] \leq c\hat{p}^{-\beta/2}.$$  

We also have by (B2)

$$J_1 \leq c\hat{p}^{-\beta/2} \sum_{j=1}^{\hat{p}} \mathbb{E}\left[\max_{v, t \in \Delta_j} |X(v) - X(t)|^{\beta}\right] \leq c\hat{p}^{1-\gamma X/d - \beta/2}.$$  

We conclude that

$$(9.5) \quad \mathbb{E}||I_1|| \leq c\hat{p}^{-\beta/2}(\hat{p}^{-\gamma X/d} + p^{-1}).$$  

Similar bounds hold for $I_2, I_3$. We omit details.

Assume $\beta \in (1, 2)$. We start from [0,4]:

$$\mathbb{E}[\mathbb{E}[D_p^{(2)}(Z)]] \leq c\mathbb{E}\left[\left(\|Z(p)\|_2^{\beta - 1} \vee \|Z\|_2^{\beta - 1}\right)D_p^{(1)}(Z)\right] \leq c\left(\mathbb{E}[[Z(p)]^{\beta} \vee \|Z\|_2^{\beta}]\right)^{(\beta - 1)/\beta} \mathbb{E}\left[\left(\|Z(p) - Z\|_2^{\beta}\right)^{1/\beta}\right] =: c\hat{P}_1\hat{P}_2.$$  

The quantity $\hat{P}_1$ is bounded by a constant for all $p$. Proceeding as for $\beta \in (0, 1]$, we obtain

$$\hat{P}_2 \leq c\left(p^{1-\beta/2}(\hat{p}^{-\gamma X/d} + p^{-1})\right)^{1/\beta},$$  

and we finally have

$$\mathbb{E}[||I_1||] \leq c\left(p^{1-\beta/2}(\hat{p}^{-\gamma X/d} + p^{-1})\right)^{1/\beta}.$$  

The quantities $\mathbb{E}[||I_1||], i = 2, 3$ can be bounded in a similar way. This proves part 2.

3. We will show that $\mathbb{E}[|T_n, \beta(X, X) - T_n, \beta(X(p), X(p))|] \to 0$ as $n \to \infty$. We again use the decomposition [8,2] for $X = Y$.

The finite variance case. Assume (A1), (A2).

Assume $\beta \in (0, 1]$. We observe that

$$(9.6) \quad \mathbb{E}[||I_1||] \leq \mathbb{E}[[Z(p) - Z]^{\beta}(\|Z(p)\|^2_2 + \|Z\|^2_2)] \leq \left(\mathbb{E}[\|Z(p) - Z\|_2^{2\beta}]\right)^{1/2} \left(\mathbb{E}[[Z(p)]^{2\beta}]\right)^{1/2}.$$  

We can re-use the argument for bounding $\mathbb{E}[\|Z(p) - Z\|_2^{\beta}]$, replacing $\beta/2$ by $\beta$ in the derivation. Thus the first expectation in (9.6) is bounded by $c(\hat{p}^{-\gamma X/d} + p^{-1})^{\beta}$. The remaining two expectations are finite thanks to [Lemma 9.1 (1)]. Thus we have

$$\mathbb{E}[||I_1||] \leq c(\hat{p}^{-\gamma X/d} + p^{-1})^{\beta/2} \to 0, \quad n \to \infty.$$  

Assume $\beta \in (1, 2)$. We may proceed similarly as for the distance covariance. We have
\[
\mathbb{E}[|I_1|] \leq \mathbb{E}[\|Z(p)\|^2_2 - \|Z\|^2_2] \\
\leq c\mathbb{E}[\|(Z(p))^{2\beta-1} \vee \|Z\|^{2\beta-1}) \|Z(p) - Z\|_2] \\
\leq c(\mathbb{E}[\|Z(p)\|^{2(\beta-1)}] \vee \|Z\|^{2(\beta-1)})^{1/2}\mathbb{E}[\|Z(p) - Z\|^2_2]^{1/2} \\
\leq c(\mathbb{E}[\|Z(p) - Z\|^2_2])^{1/2}.
\]
In the last step we used [Lemma 9.1 (2)] and (A2). Now we may proceed as for the bound of (9.2) to obtain
\[
\mathbb{E}[|I_1|] \leq c(\bar{p}^{-\gamma_1/d} + p^{-1})^{1/2} \to 0, \quad n \to \infty.
\]
We can deal with $\mathbb{E}[|I_2|], \mathbb{E}[|I_3|]$ in the same way by observing that
\[
|I_2| \leq \frac{1}{n^2} \sum_{j,k,l,m=1}^n |D_p^{(\beta)}(X_j - X_k)||X_l(p) - X_m(p)||^{\beta}_2 + \frac{1}{n^2} \sum_{j,k,l,m=1}^n \|X_j - X_l||D_p^{(\beta)}(X_l - X_m)|
\]
\[=: I_{21} + I_{22}\]
and
\[
|I_3| \leq \frac{1}{n^2} \sum_{k,l,m=1}^n |D_p^{(\beta)}(X_k - X_l)||X_l(p) - X_m(p)||^{\beta}_2 + \frac{1}{n^2} \sum_{k,l,m=1}^n \|X_k - X_m||D_p^{(\beta)}(X_l - X_m)|
\]
\[=: I_{31} + I_{32}.
\]
This means that $\mathbb{E}[I_{21}], \mathbb{E}[I_{22}], \mathbb{E}[I_{31}], \mathbb{E}[I_{32}]$ can be bounded in a similar way and these expectations converge to zero as $n \to \infty$. We illustrate this for $I_{32}$.

Assume $\beta \in (0, 1]$. By the Cauchy-Schwarz inequality and using similar bounds as above, we have
\[
\mathbb{E}[I_{32}] \leq (\mathbb{E}[\|X_1 - X_2\|^2_2])^{1/2}(\mathbb{E}[|D_p^{(\beta)}|^2])^{1/2} \\
\leq (\mathbb{E}[\|X_1 - X_2\|^2_2])^{1/2}(\mathbb{E}[\|X_l(p) - X_m(p) - (X_l - X_m)\|^{2\beta}_2])^{1/2} \\
\leq c(\bar{p}^{-\gamma_1/d} + p^{-1})^{\beta/2} \to 0, \quad n \to \infty.
\]
For $\beta \in (1, 2)$, multiple use of Hölder’s inequality yields
\[
\mathbb{E}[I_{32}] \leq c\mathbb{E}[\|X_1(p) - X_2(p) - (X_1 - X_2)\|_2(\|X_1(p) - X_2(p)\|^{\beta-1}_2 \vee \|X_1 - X_2\|^{\beta-1}_2)\|X_1 - X_3\|^2_2] \\
\leq c(\mathbb{E}[\|Z(p) - Z\|^2_2])^{1/2}(\mathbb{E}[\|Z\|^{2(\beta-1)}])^{\beta-1/2} \\
\leq c(\mathbb{E}[\|Z(p) - Z\|^2_2])^{1/2}(\mathbb{E}[\|Z\|^{2(\beta-1)}])^{\beta-1/2} \\
\times \left(\mathbb{E}[\|Z\|^{2(\beta-1)}])^{\beta-1/2} + \mathbb{E}[\|Z\|^{2(\beta-1)}])^{\beta-1/2}\right).
\]
The first quantity is bounded by $c(\bar{p}^{-\gamma_1/d} + p^{-1})^{1/2}$ and the other quantities are finite by virtue of [Lemma 9.1 (2)]. Hence $\mathbb{E}[I_{32}] \to 0, n \to \infty$.

The infinite variance case. We assume a finite $2\beta$th moment of $X$, $\beta \in (0, 1)$, and (B3).

Let $2\beta \in (0, 1]$. First we follow the inequality [9.6] of distance variance and then we use the bound of $\mathbb{E}[\|Z(p) - Z\|^{2\beta}_2]$ for the distance covariance in the infinite variance case; see [9.5] with $\beta$ replaced by $2\beta$. By [Lemma 9.1 (1)], $\mathbb{E}[\|Z(p)\|^2_2]$ and $\mathbb{E}[\|Z\|^2_2]$ are bounded by a constant. As a result we have by (B3),
\[
\mathbb{E}[|I_1|] \leq c(\mathbb{E}[\|Z(p) - Z\|^{2\beta}_2])^{1/2} \leq c(1^{-\beta}(\bar{p}^{-\gamma_1/d} + p^{-1})^{1/2}.
\]
The right-hand side converges to zero as $n \to \infty$ provided $1 < \beta + \gamma_1/d$. For $2\beta \in (1, 2)$ we have by Hölder’s inequality
\[
\mathbb{E}[|I_1|] \leq \mathbb{E}[\{\|Z(p)\|^{2\beta}_2 \vee \|Z\|^{2\beta}_2\} \|Z(p) - Z\|_2] \\
\leq c(\mathbb{E}[\|Z(p)\|^{2\beta}_2 \vee \|Z\|^{2\beta}_2])^{(2\beta-1)/(2\beta)}(\mathbb{E}[\|Z - Z(p)\|^2_2])^{1/(2\beta)} \\
= cP_1 P_2.
\]
We can bound $\hat{P}_2$ in the same way as for $\beta \in (0, 1/2)$:

$$\hat{P}_2 \leq c(p^{1-\beta}(\tilde{p}^{-\gamma}/d + p^{-1}))^{1/(2\beta)}.$$ 

The quantity $\hat{P}_1$ is finite by virtue of [Lemma 9.1 (1)] and (B3). Thus we have the bound for general $\beta \in (0, 1)$:

$$\mathbb{E}[\|I_1\|] \leq c(p^{1-\beta}(\tilde{p}^{-\gamma}/d + p^{-1}))^{(1+\beta^{-1})/2} \to 0, \quad n \to \infty.$$ 

For $I_2$ and $I_3$, recall the bounds $|I_2| \leq I_{21} + I_{22}$ and $|I_3| \leq I_{31} + I_{32}$, and take expectations on both sides of the inequalities. We illustrate how to deal with $I_{21}$ and $I_{22}$. Since $2\beta \in (0, 2)$ inequality (9.7) holds. Then for $2\beta \in (0, 1)$ it follows from (9.8) that $\mathbb{E}[I_{32}] \leq c(p^{1-\beta}(\tilde{p}^{-\gamma}/d + p^{-1}))^{1/2}$, and for $2\beta \in (1, 2)$ we obtain the same bound as for $\hat{P}_2$ by similar calculations. This finishes the proof.

**Appendix A. Negative moments of a Poisson random variable**

We consider a family $(N_p)_{p>0}$ of Poisson random variables with $\mathbb{E}[N_p] = p$ and are interested in the asymptotic behavior of $\mathbb{E}[N_p^{-\gamma}1(N_p > 0)]$ for $\gamma > 0$ as $p \to \infty$.

**Lemma A.1.** We consider a family $(N_p)_{p>0}$ of Poisson random variables with $\mathbb{E}[N_p] = p$. Then for any $\gamma > 0$, $p^\gamma \mathbb{E}[N_p^{-\gamma}1(N_p > 0)] \to 1$.

**Proof.** We observe that

$$\mathbb{E}[N_p^{-\gamma}1(N_p > 0)] = \left(\sum_{k \in A_p} + \sum_{k \in A'_p}\right) k^{-\gamma}e^{-p\frac{p^k}{k!}},$$

where $A_p = \{k \geq 1 : |k - p| > \varepsilon_p p\}$ and $A'_p = \{k \geq 1 : |k - p| \leq \varepsilon_p p\}$ and $\varepsilon_p = \sqrt{(C\log p)/p}$ for some constant $C = C(\gamma) > 0$. We will show that

$$p^\gamma \mathbb{P}(|N_p - p| > \varepsilon_p p) \to 0, \quad p \to \infty.$$ 

Then $\mathbb{P}(|N_p - p| \leq \varepsilon_p p) \to 1$ and

$$p^\gamma \sum_{k \in A'_p} k^{-\gamma}e^{-p\frac{p^k}{k!}} = \sum_{k \in A'_p} (p/k)^\gamma e^{-p\frac{p^k}{k!}} = (1 + o(1)) \mathbb{P}(|N_p - p| \leq \varepsilon_p p) \to 1.$$

Next we turn to the proof of (A.1). By Markov’s inequality and a Taylor expansion we have for $h = \varepsilon_p$,

$$p^\gamma \mathbb{P}(N_p > (1 + \varepsilon_p) p) \leq p^\gamma e^{-h(N_p)} \mathbb{E}[e^{hN_p}] = p^\gamma \exp(-h(1 + \varepsilon_p) p - p(1 - e^h)) = p^\gamma \exp(-\varepsilon_p p + \frac{\varepsilon_p^2 p}{2}) + (\varepsilon_p p + 0.5\varepsilon_p^2 p(1 + o(1)))) = \exp(-0.5\varepsilon_p^2 p(1 + o(1))) + \gamma \log p = \exp(-0.5C\log p(1 + o(1)) + \gamma \log p) \to 0, \quad p \to \infty,$$

provided $C > 2\gamma$. On the other hand, by monotonicity of the Poisson probabilities for $k \leq p$ we have

$$e^{-p\frac{p^k}{k!}} \leq e^{-p\frac{p^{[p(1-\varepsilon_p)]}}{[p(1-\varepsilon_p)]!}}, \quad k \leq [p(1-\varepsilon_p)].$$

Hence by Stirling’s formula, $(\sqrt{2\pi n}(n/e)^n \leq n!)$,

$$p^\gamma \mathbb{P}(p - N_p > \varepsilon_p p) = p^\gamma \mathbb{P}(N_p < p(1-\varepsilon_p)) \leq p^\gamma + e^{-p\frac{p^{[p(1-\varepsilon_p)]}}{[p(1-\varepsilon_p)]!}} \leq c p^{\gamma+0.5} e^{-p[p(1-\varepsilon_p)]} \left(\frac{p}{[p(1-\varepsilon_p)]}ight)^{[p(1-\varepsilon_p)]} \leq c \exp \left((\gamma + 0.5) \log p - p\varepsilon_p + [p(1-\varepsilon_p)] \log \frac{p}{[p(1-\varepsilon_p)]}\right) = : c e^{v(p)}.$$
We have by a Taylor expansion, for large $p$, 
\[
v(p) - (\gamma + 0.5) \log p = \frac{p}{p(1 - \varepsilon_p)} \left(1 - \frac{1}{2} \left(\frac{p}{p(1 - \varepsilon_p)} - 1\right)^2 (1 + o(1))\right)
\leq -0.25 p \varepsilon_p^2 = -0.25 C \log p.
\]
Choosing $C > 4(\gamma + 0.5)$, we have $e^{v(p)} \to 0$. Combining these bounds, we proved the lemma. \hfill \Box

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### Table 7.7. Correlation and distance correlation of Japanese meteorological data.

Pair-wise (distance) correlations for annual and quarterly meteorological data of the whole of Japan and 8 main regions. Each row presents the corresponding region and seasons, followed by 3 cells with the sample distance correlations and 3 cells with the results of the independence tests. The symbols ⃝, △, × stand for rejection of the independence hypothesis at both significance levels $\xi = 0.05, 0.01$, only at $\xi = 0.05$, no rejection, respectively.

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