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Maiani, Andrea; Geier, Max; Flensberg, Karsten

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Conductance matrix symmetries of multiterminal semiconductor-superconductor devices

Andrea Maiani, Max Geier, and Karsten Flensberg
Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen, Denmark

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Nonlocal tunneling spectroscopy of multiterminal semiconductor-superconductor hybrid devices is a powerful tool to investigate the Andreev bound states below the parent superconducting gap. We examine how to exploit both microscopic and geometrical symmetries of the system to extract information on the normal and Andreev transmission probabilities from the multiterminal electric or thermoelectric differential conductance matrix under the assumption of an electrostatic potential landscape independent of the bias voltages, as well as the absence of leakage currents. These assumptions lead to several symmetry relations on the conductance matrix. Next, by considering a numerical model of a proximitized semiconductor wire with spin-orbit coupling and two normal contacts at its ends, we show how such symmetries can be used to identify the direction and relative strength of Rashba versus Dresselhaus spin-orbit coupling. Finally, we study how a voltage-bias-dependent electrostatic potential as well as quasiparticle leakage breaks the derived symmetry relations and investigate characteristic signatures of these two effects.

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I. INTRODUCTION

Tunneling spectroscopy is a powerful tool for studying superconductor-semiconductor hybrid devices as it provides a clear signature for Andreev bound states (ABS). Nonlocal conductance spectroscopy is the natural extension of local two-probe spectroscopy and overcomes some of its limitations. Initially used in the context of the search for signatures of Cooper-pair splitting [1–5], this type of measurement has been recently considered in the context of topological superconductivity (TS) [6–8] leading to its use in experiments [9,10], its inclusion in identification protocols for Majorana bound states (MBs) in nanowires [11] and unconventional superconductor vortex cores [12], as well as for the characterization of chiral Majorana edge states in two-dimensional (2D) TS [13] and the helical gap in two-dimensional electron gases (2DEGs) [14]. Moreover, the same concept appears in experiments involving quantum dots to probe the nonequilibrium dynamics of quasiparticles [4,15].

When an electron current flows across the device, aside from the electric charge current, energy and heat currents flow too. For this reason, the spectral features of the device, including peaks connected to the onset of the topological phase, can also be identified when analyzing the thermal conductance [16,17]. Nevertheless, measurements of thermal transport require a very complex and delicate experimental setup. Easier experiments are the ones that study thermoelectric transport, where the system is driven out of equilibrium by using leads thermalized at different temperatures while the measured output is still a charge current. Thermoelectric measurements have been proposed as an additional tool to investigate subgap features and identify MBs [18].

Motivated by recent experimental success in measuring multiterminal electric differential conductance [2,5,9,10,19–23] and the need to characterize hybrid superconductor-semiconductor devices, we here extend the theory of multiterminal tunneling spectroscopy to extract additional information on the electronic and Andreev transmission processes from linear combinations of local and nonlocal differential conductance measurements at different bias voltage or magnetic fields. These linear combinations are derived using conditions that follow from quasiparticle-number conservation, microreversibility, and particle-hole conjugation in the presence of superconductivity. Further relations can be derived in the presence of geometrical symmetries, such as mirror symmetry, or less general Hamiltonian symmetries like additional antiunitary symmetries.

For the specific case of a semiconductor nanowire proximity coupled to an s-wave superconductor, we show that the resulting symmetry relations of the conductance matrix can be used to identify the relative strength of Rashba versus Dresselhaus spin-orbit coupling (SOC). We furthermore show that these symmetry relations can be employed to identify signatures of deviations from the assumed symmetries, in particular, voltage bias-dependent electric potential landscapes and quasiparticle dissipation into environmental baths.

To achieve this objective, we discuss an extended version of the Landauer-Büttiker theory that accounts for bias-voltage-dependent electric potentials. These results are compared to the linear Landauer-Büttiker theory, where it is assumed that the potential landscape, in which the scattering occurs, does not depend on the bias voltages. We refer to this assumption as the constant landscape approximation (CLA).

Extensions of the CLA theory for both nonlinear electric conductance [24,25] and thermoelectric conductance [26–28] have been considered before. Several previous works focused on the quadratic correction (in voltage bias) to differential conductance obtained by the method of characteristic potential, e.g., [25,27,29–32], that in practical applications
relied on Thomas-Fermi approximation [32,33]. This method is more suited for mesoscopic metallic devices with a high density of states, in which the finite-size effects can be neglected.

In the case of superconductor-semiconductor nanoelectronic devices, instead, the electrostatic potential can be a complicated function of the gate voltage that in general requires the solution of the complete Schrödinger-Poisson problem [34–40]. An initial characterization of finite-bias effects in the fully nonlinear regime was obtained by a combination of approximate analytic and numerical methods in Ref. [41].

This paper is organized as follows. In Sec. II, we describe the general theory of nonlinear charge transport in multiterminal hybrid devices and the difference with CLA results. In Sec. III, we show how fundamental symmetries such as microrеversibility and particle-hole conjugation in the $S$ matrix generate conductance symmetries valid in CLA and how to use these to extract additional information on the transmission processes from the differential conductance matrix. As an example application, in Sec. IV we demonstrate how these symmetries can be exploited to identify the spin-orbit direction in a semiconductor nanowire proximitized by a superconductor. Finally, in Sec. V, with simple numerical simulations we show the effect of finite-bias deformation of the electrostatic potential and dissipation and discuss how violation of conductance symmetry can be used to distinguish between the two. We also illustrate an example of spin-orbit coupling characterization, and discuss how thermoelectric differential conductance can be used as a probe to avoid the finite-bias effect.

II. SCATTERING TRANSPORT THEORY

The Landauer-Büttiker formalism is a simple yet powerful technique to model transport phenomena. While usually employed in CLA, the nonlinear version can be constructed easily while paying attention to preserving the gauge invariance of the theory [25].

When the motion of quasiparticles is a coherent process and the interactions between quasiparticles beyond mean-field theory can be neglected, quasiparticle transport phenomena in a device are completely described by the single-particle scattering matrix $S$ that relates the probability amplitude of the incoming and outgoing quasiparticles in the leads. More specifically, the relation between the average currents and biases of a multiterminal device depends only on the transmission probability defined as

$$T_{\alpha\beta}^{\gamma\delta}(\varepsilon; P) \equiv \text{tr} |S_{\alpha\beta}^{\gamma\delta}(\varepsilon; P)|^2,$$

where $S_{\alpha\beta}^{\gamma\delta}$ is the subblock of the scattering matrix that connects the channel of the incoming particles of type $\delta$ from lead $\alpha$ to the channel of outgoing particles of type $\gamma$ in lead $\alpha$ for scattering events at energy $\varepsilon$. Since the device we are describing is a superconductor-semiconductor hybrid, the transmission matrix features a Nambu structure with particle and hole sectors, $\delta, \gamma \in \{ e, h \}$. The transmission probabilities depend on the set of electric potentials applied to all the electrodes in the system $\{ V_i \}$ and other external parameters like the applied magnetic field $\mathbf{B}$ and the pairing amplitude of superconductive leads $\{ \Delta_i \}$. We denote the set of external parameters by $P = \{ V_i \} \cup \{ \Delta_i \} \cup \{ \mathbf{B} \}$. In principle, also the temperature of the leads can enter as a parameter of the system. For example, temperature could affect the size of the superconductive gap or induce charge accumulation in the semiconductor. We will neglect these effects as we are assuming that the temperature differences involved are much smaller than the critical temperature of the superconductor and are too small to induce a relevant change in the electrostatic potential landscape. Indeed, while voltage bias enters directly into the Poisson equation as boundary conditions, the temperature can enter the electrostatic problem only through charge accumulation.

A generic multiterminal system comprises a number of normal and superconductive terminals, suggesting a division of the $S$ matrix into subblocks as follows:

$$S = \begin{pmatrix} S_{NN} & S_{NS} \\ S_{SN} & S_{SS} \end{pmatrix}. \quad (2)$$

In this paper, we will consider the simplest case in which all the superconductive terminals are grounded together and we will denote the voltage of the common superconductive lead as $V_S$.

If the system conserves the quasiparticle number, the $S$ matrix is unitary and it follows that

$$R_{\alpha}^{ee} + R_{\alpha}^{eh} + \sum_{\beta \neq \alpha} (T_{\beta\alpha}^{ee} + T_{\beta\alpha}^{he}) = N_{\alpha}^e(\varepsilon), \quad (3)$$

$$R_{\alpha}^{ee} + R_{\alpha}^{eh} + \sum_{\beta \neq \alpha} (T_{\alpha\beta}^{ee} + T_{\alpha\beta}^{he}) = N_{\alpha}^h(\varepsilon), \quad (4)$$

where we defined for clarity the reflection matrix $R_{\alpha}^{\gamma\delta} \equiv T_{\alpha\alpha}^{\gamma\delta}$ while $N_{\alpha}^\gamma(\varepsilon)$ is the number of eigenmodes for electrons in lead $\alpha$. Note that, unless the density in the lead is so low to break particle-hole symmetry, $N_{\alpha}^e(\varepsilon) \equiv N_{\alpha}^h(\varepsilon) = N_{\alpha}(\varepsilon)$. Equation (3) represents the conservation of incoming quasiparticles from lead $\alpha$ while Eq. (4) is the same for outgoing quasiparticles. The conservation law breaks down when dissipation effects are included as they do not conserve the particle number. When restricted to energies below the smallest parent superconductor gap $\varepsilon < \min |\Delta_i|$, the matrices $S_{SS}(\varepsilon)$, $S_{SN}(\varepsilon)$, $S_{NS}(\varepsilon)$ are null and therefore a stronger relation holds where the sum over $\beta$ is restricted to the nonsuperconductive leads.

As mentioned above, in the superconductive version of the Landauer-Büttiker theory quasiparticle conservation takes the place of electron conservation for conventional devices. This implies that the electric charge is not explicitly conserved. Indeed, we can distinguish between the charge-conserving normal processes $T_{\alpha\alpha}^{ee}$ and $T_{\alpha\alpha}^{eh}$ and the non-charge-conserving Andreev processes $T_{\alpha\beta}^{eh}$ and $T_{\alpha\beta}^{he}$. The last two imply, respectively, the creation and the destruction of one Cooper pair in the superconductive leads.

A correct theory of nonlinear conductance of a multiterminal device needs to be gauge invariant. This means that the transmission probabilities are not changed by the addition of a constant offset to all the voltages and particle energy:

$$T_{\alpha\beta}^{\gamma\delta}(\varepsilon + \Delta E; \{ V_i \} + \Delta E_E) = T_{\alpha\beta}^{\gamma\delta}(\varepsilon; \{ V_i \}). \quad (5)$$
The easiest way to guarantee gauge invariance is to define a reference voltage. We define the superconductor lead to be our reference voltage and set \( V_\beta = 0 \). The first argument of the transmission function is then the energy of the scattering particle with respect to the superconductive lead chemical potential \( \varepsilon = E_\beta + eV_\beta \). The parameters \( V_{\alpha\delta} \) and \( V_{\beta\delta} \) are the bias of the leads with respect to \( V_\delta \). These biases, together with the voltage biases applied to the capacitively coupled gates, determine the electrostatic potential landscape in which the scattering events take place. Therefore, they enter here as parameters of the \( S \) matrix as well as the transmission probability matrix.

### A. Currents

The nonlinear Landauer-Büttiker approach consists in first solving the electrostatic problem for the potential landscape given the applied biases. With the calculated potential landscape, the \( S \) matrix for the scattering processes can be evaluated. Finally, once the static scattering problem has been solved, from the \( S \) matrix all the transport properties of the system can be derived. In particular, we can use the transmission functions to determine the average currents in the system can be derived. In particular, we can use the transmission functions to determine the average currents in the nonequilibrium steady state. In this last step, the terminal biases \( V_{\alpha\delta} \) appear again also as parameters of the distribution functions of the leads. The application of this approach for superconductive systems has been derived on several occasions [42–46], and in the most general formulation the average electric current through a lead \( \alpha \) is then

\[
I^e_\alpha(P) = \frac{e}{\hbar} \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\alpha\delta}, \theta_\alpha) - f(\varepsilon, \theta_\delta) \right] \times \left[ N_\alpha \beta - \frac{1}{2} \frac{\partial f}{\partial \varepsilon}(\varepsilon - eV_\beta, \theta_\beta) \right] - \sum_{\beta \neq \alpha} \frac{e}{\hbar} \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\beta\delta}, \theta_\beta) - f(\varepsilon, \theta_\delta) \right] \times \left[ T_{\alpha\beta}^e(\varepsilon; P) - T_{\alpha\beta}^h(\varepsilon; P) \right],
\]

where \( f(\varepsilon, \theta) = (1 + e^{\varepsilon/k_B\theta})^{-1} \) is the Fermi-Dirac distribution, and the parameters \( \{\theta_\delta\} \) are the temperatures of the leads. We assumed for simplicity that the temperature of all the superconductive leads is equal to \( \theta_\delta \). A similar expression can be written for the energy current \( I^h_\alpha \) while the heat current \( I^h_\alpha = I^e_\alpha + eV_{\alpha\delta}I^e_\delta \) follows easily from the first law of thermodynamics [28].

We can write, for the \( I^e \) and \( I^h \) vectors of currents, the following differential relation with the voltages \( V \) and temperatures \( \theta \) vectors [27,30,47]:

\[
\frac{dI^h_\alpha}{dI^e_\alpha} = \left( \begin{array}{cc} G_{\alpha\beta} & L_{\alpha\beta} \\ N_{\alpha\beta} & M_{\alpha\beta} \end{array} \right) \frac{dV_\beta}{d\theta_\beta},
\]

where \( G \) is the differential electric conductance, \( N \) is the differential thermal conductance, while \( L \) is the thermoelectric and \( M \) is the electrothermal differential conductance. We do not consider thermal transport in this work.

An important observation about the charge current is that it does not satisfy Kirchhoff’s current law. This is because the expression derived describes only the quasiparticle current while the supercurrent is not captured in this formalism. By imposing charge conservation, the net supercurrent flowing into the device is

\[
P' = -\frac{2e}{\hbar} \sum_{\alpha \beta} \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\beta\delta}, \theta_\beta) - f(\varepsilon, \theta_\delta) \right] \times \left[ R_{\beta\delta}^h(\varepsilon; P) + T_{\alpha\beta}^h(\varepsilon; P) \right].
\]

In the case of a single superconductive lead, this is equal to the supercurrent flowing into the device. In the case of multiple superconductive leads, instead, the supercurrent divides between the different superconductive leads.

### B. Constant landscape approximation

In the CLA, the change in the potential landscape when a voltage bias is applied is neglected. We denote this by writing that \( P = P_0 \) where \( P_0 \) is the set of parameters at equilibrium. In this case, the CLA result for electrical conductance is

\[
G_{\alpha\alpha} = G_0 \sum_{-\infty}^{+\infty} d\varepsilon \left[ -\partial_\varepsilon f(\varepsilon - eV_{\alpha\delta}, \theta_\alpha) \right] \times \left[ N_\alpha - R_{\alpha\alpha}^e(\varepsilon; P = P_0) + R_{\alpha\alpha}^h(\varepsilon; P = P_0) \right]
\]

and for local differential conductance, while for the nonlocal differential conductance we have

\[
G_{\alpha\beta} = -G_0 \sum_{-\infty}^{+\infty} d\varepsilon \left[ -\partial_\varepsilon f(\varepsilon - eV_{\beta\delta}, \theta_\beta) \right] \times \left[ T_{\alpha\beta}^e(\varepsilon; P = P_0) - T_{\alpha\beta}^h(\varepsilon; P = P_0) \right].
\]

where \( G_0 = \frac{e^2}{\hbar} \) is the conductance quantum and

\[
\partial_\varepsilon f(\varepsilon, \theta) = \frac{1}{2k_B\theta} \frac{1}{1 + e^{\varepsilon/k_B\theta}}.
\]

A similar expression can be obtained for thermoelectric conductance. The local and nonlocal thermoelectric conductance reads as

\[
L_{\alpha\alpha} = +L_0 \sum_{-\infty}^{+\infty} d\varepsilon \left( k_\beta^h \partial_\varepsilon f(\varepsilon - eV_{\alpha\delta}, \theta_\alpha) \right) \times \left[ N_\alpha - R_{\alpha\alpha}^e(\varepsilon; P = P_0) + R_{\alpha\alpha}^h(\varepsilon; P = P_0) \right]
\]

and

\[
L_{\alpha\beta} = -L_0 \sum_{-\infty}^{+\infty} d\varepsilon \left( k_\beta^h \partial_\varepsilon f(\varepsilon - eV_{\beta\delta}, \theta_\beta) \right) \times \left[ T_{\alpha\beta}^e(\varepsilon; P = P_0) - T_{\alpha\beta}^h(\varepsilon; P = P_0) \right].
\]

where \( L_0 = \frac{e^2}{k_B} \) is the thermoelectric conductance quantum and we used the derivative of the distribution function with respect to temperature, that is \( k_\beta^h \partial_\varepsilon f(\varepsilon, \theta) = -\frac{e}{k_B^h} \partial_\varepsilon f(\varepsilon, \theta) \). Note that since \( \partial_\varepsilon f \) is an odd function of the energy, thermoelectric conductance is sensitive only to the antisymmetric component of the transmission spectrum of the device.

### C. Differential conductance in the nonlinear theory

In principle, the average current given by Eq. (6) is exact for a noninteracting system if the potential landscape is calculated self-consistently from the set of parameters \( P \). We do not attempt such a calculation here since the devices
treated often have a complicated three-dimensional geometry that cannot easily be reduced to a simple one-dimensional model when taking the electrostatic environment into account. Instead, in this section, we focus on general considerations, while in Sec. V we parametrize a potential landscape in a physically motivated way and look at differences with CLA results.

Evaluating the full derivatives of the charge current $I^q$ with respect to a terminal voltage bias we find that the electric differential conductance can be split into two parts

$$G_{\alpha\beta}([V_q]) = \frac{dI_q}{dV_{\beta S}} = G_{\alpha\beta}^{(m)} + G_{\alpha\beta}^{(def)},$$

where the first term is the marginal contribution that reads as, i.e., for the local conductance,

$$G_{\alpha\beta}^{(m)} = G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ -\partial_\varepsilon f(\varepsilon - eV_{\alpha S}, \theta_\alpha) \right] \times \left[ N_\alpha - R_{\alpha\beta}^{(e)}(\varepsilon; P) + R_{\alpha\beta}^{he}(\varepsilon; P) \right].$$

This term can be interpreted as the fact that when evaluating the additional current carried by higher-energy states, the $S$ matrix has to be calculated using the potential landscape that takes into account the modified voltage bias. The second term accounts for the deformation of the $S$ matrix for the already filled channels due to the effect of the biasing itself. The deformation contribution reads as

$$G_{\alpha\beta}^{(def)} = G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\alpha S}, \theta_\alpha) - f(\varepsilon, \theta_\beta) \right] \left[ -\frac{\partial R_{\alpha\beta}^{(e)}(\varepsilon; P)}{\partial V_{\beta S}} + \frac{\partial R_{\alpha\beta}^{he}(\varepsilon; P)}{\partial V_{\beta S}} \right] + G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\beta S}, \theta_\beta) - f(\varepsilon, \theta_\beta) \right] \left[ -\frac{\partial T_{\alpha\beta}^{(e)}(\varepsilon; P)}{\partial V_{\beta S}} + \frac{\partial T_{\alpha\beta}^{he}(\varepsilon; P)}{\partial V_{\beta S}} \right].$$

This correction is often neglected in previous works, e.g., Ref. [25]. The reason is that in the case of symmetric biasing $V_\alpha = V_\beta = V$ and fixed temperature $\theta_\alpha = \theta_\beta = \theta_\beta$, we can rewrite $G_{\alpha\beta}^{(def)}$ as

$$G_{\alpha\beta}^{(def)}(V) = -G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_\alpha, \theta_\alpha) - f(\varepsilon, \theta_\beta) \right] \frac{\partial}{\partial V_\beta} \times \left[ N_\alpha(\varepsilon) - R_{\alpha\beta}^{he}(\varepsilon; P) - R_{\alpha\beta}^{he}(\varepsilon; P) - \sum_\rho \left| T_{\alpha\beta}^{he}(\varepsilon; P) + T_{\alpha\beta}^{he}(\varepsilon; P) \right| \right].$$

Since $N_\alpha$ does not depend on the bias (it is a property of the leads) and all the other terms in the brackets are probabilities of Andreev's processes, it is clear that this quantity vanishes for nonsuperconductive devices.

However, this contribution shows an interesting interplay between electrostatic behavior and superconductivity. For example, let us consider a simple NS junction. The differential conductance can be written as

$$G_{\alpha\beta} = 2G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ -\partial_\varepsilon f(\varepsilon - eV_{\alpha S}, \theta_\alpha) \right] R_{\alpha\beta}^{he}(\varepsilon; P) + 2G_0 \int_{-\infty}^{+\infty} d\varepsilon \left[ f(\varepsilon - eV_{\alpha S}, \theta_\alpha) - f(\varepsilon, \theta_\beta) \right] \frac{\partial R_{\alpha\beta}^{he}(\varepsilon; P)}{\partial V_{\alpha S}}. $$

It is evident that in case $\frac{\partial R_{\alpha\beta}^{he}(\varepsilon; P)}{\partial V_{\alpha S}} > 0$ the second term in the sum can overcome the first one, which is always positive, resulting in negative local differential conductance. A simple case of this can be when the coupling of the scattering region with the superconductor decreases with the bias [41,48].

### III. Conductance Symmetries

In this section, we consider how the symmetries of the system manifest themselves first as symmetries of the $S$ matrix and, consequently, in the differential conductance matrix. We consider the ideal CLA case and, for this reason, we drop the biases as arguments in the $S$ matrix.

We choose the Nambu basis of time-reversed holes $\Psi^T = (\psi, \bar{T}\psi)^T = (\psi_1, \psi_2, -\psi_1^\dagger, \psi_2^\dagger)$ where we have chosen for the time-reversal symmetry $T = -i\sigma_y K$, where $K$ is the complex-conjugation operator, and therefore for the particle-hole symmetry $P = i\tau_y T$.

In the following analysis, we consider symmetric and antisymmetric linear combinations of the conductance matrix elements at opposite voltage bias and magnetic field. We have investigated all linear combinations. However, in the following, we only present the interesting cases in which the linear combination leads to a reduction in the number of terms.

#### A. Particle-hole symmetry

If the system features particle-hole symmetry, the energy-resolved scattering matrix satisfies the following relation:

$$S(\varepsilon) = PS(-\varepsilon)P^\dagger = \sigma_\gamma\tau_y S^*(\varepsilon)\sigma_\gamma\tau_y$$

This poses an additional constraint on Andreev transmission probabilities that read as

$$T_{\alpha\beta}^{\gamma\delta}(\varepsilon) = T_{\alpha\beta}^{\gamma\delta}(-\varepsilon),$$

where the overbar indicates that the index should be flipped $\varepsilon \leftrightarrow h$.

This property of the transmission probabilities has a number of consequences on differential conductance. For example, in a system with a single normal terminal (e.g., an NS junction), or for a terminal that is completely isolated from...
others such that there are no propagating channels (normal or Andreev) connecting it to other leads, the reflection coefficients have to be energy symmetric below the gap. As a consequence, the conductance has to be a symmetric function of the voltage for ideal devices. For the same reason, in these cases the thermoelectric conductance is always exponentially suppressed for temperatures much smaller than the superconducting gap.

Dissipation, inelastic scattering, and coupling to other leads are known effects that break this symmetry of the transmission matrix [49–51] while finite-bias effects can lead to the breakdown of the symmetry at conductance-matrix level [41,52].

A generalization of this conductance-matrix symmetry for the multiterminal case can be obtained by considering the quantity

$$G_{\alpha}^{\text{sum}}(V) = G_{\alpha\alpha}(V) + \sum_{\beta \neq \alpha} G_{\alpha\beta}(V),$$

that is the sum of the local conductance at terminal \(\alpha\) and the nonlocal conductances obtained measuring the current at \(\alpha\) while applying a voltage bias to all the other normal leads. It follows that, as a consequence of Eq. (19),

$$G_{\alpha}^{\text{sum}}(V) = G_0 \int_{-\infty}^{+\infty} d \epsilon [-\partial_\epsilon f(\epsilon - eV)] H_\alpha(\epsilon),$$

where we defined the quantity

$$H_\alpha(\epsilon) = R_{\alpha}^{he}(+\epsilon) + R_{\alpha}^{he}(-\epsilon) + \sum_{\beta \neq \alpha} \left[ T_{\alpha\beta}^{he}(+\epsilon) + T_{\alpha\beta}^{he}(-\epsilon) \right] + \sum_{\nu} \left[ T_{\alpha\nu}^{ee}(+\epsilon) + T_{\alpha\nu}^{ee}(+\epsilon) \right].$$

The first two terms in \(H_\alpha(\epsilon)\) are explicitly symmetric in \(\epsilon\), while the last sum is null for \(\epsilon < \min |\Delta_\nu|\). Therefore, as a consequence,

$$G_{\alpha}^{\text{sum}}(V) = G_{\alpha}^{\text{sum}}(V) - G_{\alpha}^{\text{sum}}(-V) = 0.$$

This relation is a generalization of the three-terminal case derived in Ref. [8]. This result has been derived for noninteracting systems in CLA, therefore, any deviation from zero in \(G^{\text{sum}}\) can be used as a tool to inspect deviations from the CLA and the contributions of quasiparticle dissipation or Coulomb repulsion between quasiparticles. We will discuss these effects in Sec. V.

We can split the local and nonlocal differential conductance into symmetric and antisymmetric components

$$G_{\alpha\beta}^{\text{sym}}(V) = \frac{G_{\alpha\beta}(V) + G_{\alpha\beta}(-V)}{2},$$

$$G_{\alpha\beta}^{\text{anti}}(V) = \frac{G_{\alpha\beta}(V) - G_{\alpha\beta}(-V)}{2}.$$

It has been shown that one can extract the BCS charge, i.e., \(\langle \tau_c \rangle\), of each ABS from the antisymmetric combination \(G_{\alpha\beta}^{\text{anti}}(V)\), given the ABSs are sufficiently separated in the spectrum [8].

A similar relation can be derived for thermoelectric differential conductance. Indeed, under the same conditions, we can define

$$L_{\alpha}^{\text{sum}}(\theta) = L_{\alpha\alpha} + \sum_{\beta \neq \alpha} L_{\alpha\beta},$$

$$= L_0 \int_{-\infty}^{+\infty} d \epsilon \, k_B \langle \partial_\epsilon f(\epsilon, \theta) \rangle H_\alpha(\epsilon).$$

Since \(H_\alpha(\epsilon)\) is an even function of \(\epsilon\) while \(\partial_\epsilon f(\epsilon)\) is an odd function, we have that \(L_{\alpha}^{\text{sum}} \approx 0\) for \(k_\beta \theta \ll \min \Delta_\nu\). This holds, again, for noninteracting systems but only for temperatures low enough to exclude excitations of states above the parent gap. Since the thermoelectric conductance is connected to the antisymmetric part of the transmission spectrum, its sign at low temperature can be linked to the BCS charge \(\langle \tau_c \rangle\) of ABSs, following a similar argument as for \(G_{\alpha\beta}^{\text{anti}}\) as presented in Ref. [8].

In the presence of dissipation, the previous transport symmetries do not hold. Here by dissipation, we mean the presence of a reservoir at the Fermi level that induces quasiparticle leakage. This can be due to various reasons, such as the presence of subgap states in the superconductor causing a softening of the gap or some other leakage mechanism that connects the scattering region to the common ground. A simple way to model quasiparticle leakage is by considering a fictitious lead \(\beta^*\) that is excluded when taking the calculation of \(G_{\alpha}^{\text{sum}}\). Focusing on energies below the gap, one finds that the antisymmetric part does not vanish but equals to

$$G_{\alpha}^{\text{sum}}(V) = G_0 \int_{-\infty}^{+\infty} d \epsilon [-\partial_\epsilon f(\epsilon)] \left[ T_{\alpha\beta}^{he}(\epsilon - eV) + T_{\alpha\beta}^{he}(\epsilon - eV) - T_{\alpha\beta}^{ee}(\epsilon + eV) - T_{\alpha\beta}^{ee}(\epsilon + eV) \right].$$

A similar relation can be obtained for thermoelectric conductance. To assess in a more quantitative way the effect of dissipation later we will switch to numerical simulations (see Sec. V).

B. Microreversibility

The microreversibility of the scattering process is a consequence of global time-reversal symmetry and implies that, upon inversion of the time-reversal breaking fields and spin direction, the motion can be reversed.\(^1\) As a consequence, the scattering matrix is equal to its transpose

$$S(\mathbf{B}, \Delta_\nu) = T S(-\mathbf{B}, \Delta_\nu^*) T^\dagger = \sigma_y S^T(-\mathbf{B}, \Delta_\nu^*) \sigma_y,$$

which, expanding in particle and lead labels, becomes

$$S^{y\gamma}_{\alpha\beta}(\mathbf{B}, \Delta_\nu^*) = \sigma_y \left[ S^{y\gamma}_{\beta\alpha}(-\mathbf{B}, \Delta_\nu^*) \right]^T \sigma_y.$$

If we consider only non-spin-polarized leads, we can take the trace over the internal spin indices and get the following symmetry relation:

$$T^{y\gamma}_{\alpha\beta}(\mathbf{B}, \Delta_\nu^*) = T^{y\gamma}_{\beta\alpha}(\mathbf{B}, \Delta_\nu^*).$$

\(^1\)Note that we are considering systems that do not break time-reversal symmetry internally.
Combining this relation with the one derived from particle-hole symmetry [Eq. (19)], we find that, for Andreev reflections and transmission coefficients,

$$T_{\alpha\beta}^{\text{ch}}(\epsilon, B) = T_{\beta\alpha}^{\text{ch}}(-\epsilon, -B, \Delta^*_\nu).$$

(31)

This has implications both in the electric and thermoelectric nonlocal conductance.

The connection between microreversibility and thermoelectric quantities has been explored on general grounds both in theory [53,54] and experiments [55]. In particular, microreversibility is the microscopic explanation of the Onsager-Casimir relations that lead to other transport symmetries in the charge, heat, and spin channels valid in linear response [56,57].

Microreversibility can be exploited to study separately normal and Andreev processes. To do so, we introduce two new quantities, $G^{a\beta}$ and $G^{e\beta}$, both in the local and nonlocal versions, that we call reciprocal conductances and that can be extracted from the electric differential conductance matrix:

$$G_{\alpha\beta}^{a\beta}(V, B) \equiv G_{\alpha\beta}(V, B) - G_{\rho\alpha}(+V, -B).$$

(32)

$$G_{\alpha\beta}^{e\beta}(V, B) \equiv G_{\alpha\beta}(V, B) - G_{\rho\alpha}(-V, -B).$$

(33)

By using microreversibility and particle-hole symmetry it is possible to show that

$$G_{\alpha\beta}^{a\alpha}(V, B) = G_0 \int_{-\infty}^{+\infty} d\epsilon [-\partial_\epsilon f(\epsilon)] \left[ T_{\alpha\beta}^{be}(\epsilon - eV, B) - T_{\alpha\beta}^{be}(\epsilon + eV, B) \right],$$

(34)

$$G_{\alpha\beta}^{e\alpha}(V, B) = G_0 \int_{-\infty}^{+\infty} d\epsilon [-\partial_\epsilon f(\epsilon)] \left[ T_{\alpha\beta}^{re}(\epsilon + eV, B) - T_{\alpha\beta}^{re}(\epsilon - eV, B) \right].$$

(35)

where $G^{a\beta}(V)$ is proportional to the antisymmetric part of the Andreev transmission probability while $G^{e\beta}(V)$ is proportional to the antisymmetric part of the normal electron transmission probability. For this reason, these two quantities can be used to analyze separately the two types of transport processes. Moreover, it can be verified from their definitions that these two quantities are the decomposition of the antisymmetric part of the local differential conductance:

$$G_{\alpha\beta}^{\text{anti}} = G_{\rho\alpha}^{a\alpha} + G_{\rho\alpha}^{e\alpha}. \quad (36)$$

The local versions $G_{\alpha\alpha}^{e\alpha}$ and $G_{\alpha\alpha}^{a\alpha}$ are proportional only to the antisymmetric part of the reflection probabilities. As mentioned before, if a lead is sufficiently isolated from the others such that there are no propagating channels connecting it to other leads, the reflection probabilities are bound to be energy symmetric, making the defined quantities null in absence of inelastic scattering.

The quantities (32) and (33) are the only symmetric or antisymmetric combinations of conductance matrix elements $G_{\alpha\beta}(V, B)$ that simplify to a difference of two transmission or reflection probabilities under the constraints imposed by unitarity [Eqs. (3) and (4)], particle-hole symmetry [Eq. (19)], and time-reversal symmetry [Eq. (30)]. Contrarily to particle-hole symmetry (PHS) derived conductance symmetries, the results in Eqs. (32) and (33) are not affected by dissipation since the derivation does not make use of the unitarity of the $S$ matrix.

Note that $\lim_{\nu \to -\nu} G^{a\beta}(V) = 0$ in agreement with Onsager-Casimir relation. The vanishing of $G^{a\beta}$ for normal (nonsuperconductive) devices can be explained as an extension of Onsager-Casimir relations beyond the linear-response regime.

C. Additional antiunitary symmetry

Several widely used models in the context of proximitized devices, e.g., the Lutchyn-Oreg Hamiltonian describing a topological phase transition in a proximitized semiconductor nanowire [58,59] satisfy an additional antiunitary symmetry $\mathcal{A} = U_A K$ aside from microreversibility that persists even in the presence of a Zeeman field. This symmetry implies additional constraints on the conductance matrix. In case the antiunitary symmetry is inherited from the normal state (i.e., it holds separately for electron and hole parts of the wave function), then the matrix $U_A$ does not mix the particle-hole and lead indices. In this case, the symmetry condition for the scattering matrix can be written as

$$S(B, \Delta_\nu) = U_A^* S(B, \Delta_\nu) U_A^*.$$

(37)

As a consequence, the transmission probabilities satisfy the symmetry relations

$$T_{\alpha\beta}^{gy}(B, \Delta_\nu) = T_{\rho\alpha}^{gy}(B, \Delta_\nu).$$

(38)

The validity of this symmetry on the transmission probabilities is due to the block-diagonal structure of the unitary $U_A$ combined with the definition of $T_{\alpha\beta}^{gy}(B, \Delta_\nu)$ in Eq. (1) that contains a trace over all single-lead indices that are present in the normal state.

In combination with PHS [Eq. (19)], we find

$$T_{\alpha\beta}^{gy}(+\epsilon, +B, \Delta_\nu) = T_{\rho\alpha}^{gy}(-\epsilon, +B, \Delta_\nu) \quad (39)$$

and, in particular for the Andreev transmission,

$$T_{\rho\alpha}^{ch}(+\epsilon, +B, \Delta_\nu) = T_{\rho\alpha}^{ch}(-\epsilon, +B, \Delta_\nu). \quad (40)$$

The combination with microreversibility in Eq. (30) instead gives

$$T_{\alpha\beta}^{gy}(+\epsilon, +B, \Delta_\nu) = T_{\alpha\beta}^{gy}(+\epsilon, -B, \Delta_\nu^*). \quad (41)$$

As a result, the conductance magnetic asymmetry, that we define as

$$G_{\alpha\beta}^{\text{m}}(V, B, \Delta_\nu) \equiv G_{\alpha\beta}(V, B, \Delta_\nu) - G_{\alpha\beta}(V, -B, \Delta_\nu^*),$$

(42)

vanishes.\footnote{Note that for local quantities $G_{uu}^{a\beta} = G_{uu}^{a\beta}$.} Violations of this symmetry relation can be attributed to perturbations that break the antiunitary symmetry...
All the superconductive leads are grounded.

superconductor (gray). The metallic contacts are depicted in yellow.

semiconductor nanowire (blue) proximitized by an s-wave superconductor. Such that it may feature approximate geometrical symmetries.

of the system can be controlled to some degree of accuracy by the reciprocal differential conductances, but with the advantage that is evaluated at only one lead and reversing only one component of the magnetic field

\[ G_{LR}^{\text{th}}(V, B) \equiv G_{LR}(V, B_x) - G_{LR}(V, -B_x), \]

\[ G_{LR}^{\text{ve}}(V, B) \equiv G_{LR}(V, B_x) - G_{LR}(-V, -B_x). \]

By using Eqs. (44) and (45) it is possible to show that

\[ G_{LR}^{\text{th}}(V, B) = G_0 \int_{-\infty}^{+\infty} d\epsilon [-\partial_x f(\epsilon)] \left[ T_{LR}^{\text{th}}(\epsilon - eV, B) - T_{LR}^{\text{th}}(\epsilon + eV, B) \right], \]

\[ G_{LR}^{\text{ve}}(V, B) = G_0 \int_{-\infty}^{+\infty} d\epsilon [-\partial_x f(\epsilon)] \left[ T_{LR}^{\text{ve}}(\epsilon + eV, B) - T_{LR}^{\text{ve}}(\epsilon - eV, B) \right]. \]

Similarly, for the thermoelectric conductance, we have

\[ L_{LR}(\theta, B_x) \equiv L_{LR}(\theta, B_x) - L_{LR}(\theta, -B_x) \cong 0. \]

Again, this quantity is exactly zero if we neglect the energies above the parent gap and thus deviation at low temperatures can be directly linked to dissipation effects.
If a system featuring mirror symmetry satisfies an additional antiunitary symmetry as discussed in Sec. III C, then it follows that $G_{ab}^\alpha = 0$. In absence of the additional antiunitary symmetry, the conductance symmetry $G_{ab}^\alpha = 0$ is present in case the magnetic field lies in the plane orthogonal to the mirror-symmetry axis. These properties can be used as an indication of whether the system satisfies mirror symmetry.

IV. ADDITIONAL ANTIUNITARY SYMMETRY IN A PROXIMITIZED SEMICONDUCTOR NANOWIRE

To introduce a concrete example application of the additional antiunitary symmetry discussed in Sec. III C, we now consider a quasi-1D semiconductor nanowire proximitized by a superconductor. An important question for these devices is the characterization of spin-orbit coupling. This can be achieved by leveraging the symmetry relations previously introduced.

We consider a system represented by the following low-energy effective Hamiltonian

$$\mathcal{H} = \frac{\hbar^2 k^2}{2m^*} + V(\mathbf{r}) + \mathcal{H}_{\text{SOC}} \tau_z + \mathbf{b} \cdot \mathbf{\sigma} \tau_0 + \Delta \tau_z,$$  \hspace{1cm} (51)

where $\mathbf{k} = -i\nabla$ is the wave vector, $m^*$ is the effective mass, $V(x) = -\mu(x)$ is the potential landscape (which can include disorder), $\Delta$ is the proximity-induced pairing potential in the weak coupling limit, and $\mathbf{b}$ is the Zeeman spin splitting in the semiconductor. Finally, the spin-orbit coupling $\mathcal{H}_{\text{SOC}} = \mathcal{H}_R + \mathcal{H}_D$ is the sum of the Rashba and the Dresselhaus terms.

The Rashba interaction reads as $\mathcal{H}_R = \mathbf{k} \times \mathbf{\alpha} \cdot \mathbf{\sigma}$ where the Rashba field $\mathbf{\alpha}$ is proportional to the electric field in the device. The Dresselhaus spin-orbit coupling arises from the lack of inversion symmetry of the material and can be written as $\mathcal{H}_D = \gamma_D \mathbf{1} \cdot \mathbf{\sigma}$. In zinc-blende crystals, the $\mathbf{l}$ vector components are $l_\alpha = k_\alpha (k^2_\alpha - k^2_\alpha)$ where $(a, b, c)$ are cyclic permutations of the coordinates $(x, y, z)$ [60].

Here we consider a quasi-1D system in the $\mathbf{e}_z$ direction as shown in Fig. 2. For sufficiently thin wires with sufficiently energy-separated eigenmodes with different radial momenta, we can replace the radial momentum operators by their expectation values evaluated on the transverse eigenmode wave function $\mathbf{k} \simeq (k_x, k_y, k_z)$. Under this assumption, we can rewrite the Dresselhaus Hamiltonian as $\mathcal{H}_D = \gamma_D \mathbf{k} \cdot \mathbf{\sigma} = \beta \mathbf{k} \cdot \mathbf{\sigma}$, while for the Rashba SOC, $\mathcal{H}_R = (\alpha_x \sigma_y - \alpha_y \sigma_x) k_z$, where a term $(\mathbf{k}_+ \alpha_y \sigma_x)$ vanishes due to $(\mathbf{k}_+ \alpha_y \sigma_x)$ for confined eigenmodes.

First, consider the case of pure Rashba spin-orbit coupling, i.e., $\gamma_D = 0$. The Hamiltonian satisfies an antiunitary symmetry when the magnetic field points within the plane spanned by the Rashba field $\mathbf{\alpha}$ and the direction of the wire. For $\mathbf{\alpha} = \alpha_x \mathbf{e}_z$, the antiunitary symmetry is complex conjugation and the real-space Hamiltonian in Eq. (51) is real.

If the system features both Rashba and Dresselhaus spin-orbit coupling, the plane spanned by the magnetic fields that preserve an antiunitary symmetry is tilted. Without loss of generality we choose a Rashba field perpendicular to the wire pointing along $z$, i.e., $\mathbf{\alpha}_z = \alpha_z \mathbf{e}_z$. We introduce a coordinate system for the magnetic field defined as $\mathbf{b} = b(\cos \theta \mathbf{\alpha} \cos \phi, \cos \theta \sin \phi, \sin \theta)$ where $\theta$ is the elevation and $\phi$ the azimuth with respect to the wire direction. In this case, the spin-orbit coupling term reads as $k_x (\alpha_x \sigma_y + \beta \sigma_x)$. A rotation $e^{\phi \sigma_z/2}$ in spin space by the angle $\tan \phi = \frac{\beta}{\alpha_x}$ transforms $e^{-i \phi \sigma_z/2} k_x (\alpha_x \sigma_y + \beta \sigma_x) e^{i \phi \sigma_z/2} = k_x \sqrt{\alpha_x^2 + \beta^2} \sigma_y$.

In this basis, spin-orbit coupling is real and the Hamiltonian satisfies $\mathcal{A} = \mathcal{K}$.

This antiunitary symmetry is preserved by a Zeeman field $\mathbf{b} = b_\perp \mathbf{\sigma} + b_\| \mathbf{e}_z$. The Hamiltonian satisfies an antiunitary symmetry as discussed in Sec. III C, then it follows that $G_{ab}^\alpha = 0$. In absence of the additional antiunitary symmetry, the conductance symmetry $G_{ab}^\alpha = 0$ is present in case the magnetic field lies in the plane orthogonal to the mirror-symmetry axis. These properties can be used as an indication of whether the system satisfies mirror symmetry.

V. NUMERICAL MODELS

In this section, we introduce two numerical models to show examples of how microscopic symmetries of the systems manifest themselves in the transport properties. In both cases, we model the grounded leads in the system with the method of self-energies. This can be useful when the lead is a metal with high density compared to the scattering region as the self-energy takes the simple form of a local complex-valued potential. This is added to the Hamiltonian to generate an effective non-Hermitian energy-dependent Hamiltonian that can be studied with the scattering approach.

We consider the most general case of a grounded soft-gap superconductor that can be described by the Dynes superconductor model [61–63]. In the case of a superconductive lead with a high density of states, the intermediate coupling regime can be adequately described by the following local self-energy:

$$\Sigma_\nu(\epsilon, \mathbf{r}) = \gamma_\nu(\mathbf{r}) \frac{-(\epsilon + i\Gamma_\nu) \tau_0 \sigma_0 + (\Delta_\nu \tau_z + \Delta_\nu^* \tau_-) \sigma_0}{\sqrt{\Delta_\nu^2 |\tau_0 \sigma_0 - [(\epsilon + i\Gamma_\nu) \tau_0 \sigma_0]^2}},$$  \hspace{1cm} (52)

where $\Delta_\nu = |\Delta_\nu| e^{i \phi}$ is the pairing amplitude with phase $\phi$. (that we assume constant in space), $\Gamma_\nu$ is the Dynes parameter that models pair-breaking scattering processes, the local coupling strength is $\gamma_\nu = \pi \mathcal{D}_\nu \tau_0^2$ with $\mathcal{D}_\nu$ being the density of states in the lead $\nu$, and $\tau_0$ is the interface hopping amplitude between the scattering region and the superconductive lead $\nu$. For a normal lead, this reduces to

$$\Sigma_0(\epsilon, \mathbf{r}) = -i \gamma_0(\mathbf{r}) \tau_0 \sigma_0,$$  \hspace{1cm} (53)

that is an imaginary potential that causes the decay of the quasiparticle wave function.

A. Double-dot Josephson junction

To study the effect of the additional antiunitary symmetry discussed in Sec. III C, we first consider a double-dot Josephson junction illustrated in Fig. 1. The effective Hamiltonian of
the system is
\[
\mathcal{H} = \left(\begin{array}{cc}
-\mu_1 & -t \\
-1 & -\mu_2
\end{array}\right) + \left(\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}\right) \Sigma_1(\epsilon) + \left(\begin{array}{cc}
0 & 0 \\
0 & 1
\end{array}\right) \Sigma_2(\epsilon),
\]
where $\mu_i$ are the local chemical potentials in the dots, $t$ is the hopping amplitude between the dots, and $\Sigma_i$ are the local self-energies induced by the superconductive leads $i = 1, 2$. The scattering matrix can be obtained by using the Weidenmüller formula (see, e.g., [64])
\[
S(\epsilon) = 1 - 2\pi i W^\dagger \frac{1}{\epsilon - \mathcal{H} + i\Gamma W^\dagger} W,
\]
where $W_i(\epsilon) = \sqrt{\rho_i(\epsilon)} \Pi_i(\epsilon)$ with $t_i$ the tunneling amplitude from the device to lead $i$, $\rho_i(\epsilon)$ the density of states in lead $i$, and $\Pi_i(\epsilon)$ the projector onto the eigenstates of lead $i$ at energy $\epsilon$.

In our model, we approximate the tunnel coupling between lead $L$ and dot 1 (lead $R$ and dot 2) by two energy-independent parameters such that $W = (w_L, w_R)\tau_z$.

When the phase difference between the two superconductive terminals $\phi_{12} = \phi_1 - \phi_2$ is zero or $\pi$, the system features the antiunitary symmetry $A = K$. As a consequence, the Andreev process probabilities are symmetric in the energy axis, i.e., $T_{\epsilon^\dagger}(\epsilon) = T_{\epsilon^\dagger}(-\epsilon)$, and $R_{\epsilon^\dagger}(\epsilon) = R_{\epsilon^\dagger}(-\epsilon)$. This can be verified by the zero in the conductance magnetic asymmetry $G^T$, as shown in the second column of Fig. 3. The reciprocal conductance can also be used to verify the presence of a mirror symmetry of the device. A mirror symmetry $M_x$ exchanges the two dots and reverses the sign of the phase difference $\phi_i \rightarrow -\phi_i$. The latter can be seen by noticing that the phase difference can be created by a magnetic field $B$, piercing a superconducting loop within the x-y plane connecting to the two dots. This mirror symmetry implies a zero in the nonlocal reciprocal conductances $G^\sigma_{LR}$.

B. Proximitized semiconductor nanowire

As an example of a three-terminal device, we consider the case of a semiconductor nanowire proximitized by an s-wave superconductor as shown in Fig. 4. We demonstrate how an antiunitary symmetry persisting at a finite magnetic field for specific directions can be employed to extract the ratio between Dresselhaus and Rashba spin-orbit coupling, as introduced in Sec. IV. We further study the effects of dissipation and voltage-bias-dependent potentials on the symmetry relations derived under CLA.

The Hamiltonian is similar to the one in Eq. (51), but here we treat the superconductive lead using the self-energy model
\[
\mathcal{H}(\epsilon) = \left[\frac{\hbar^2 k_z^2}{2m^*} + V(x)\right] \tau_z + [(-\alpha, \sigma_z - \alpha, \sigma_y)k_x + \beta k_z \sigma_y] \tau_z + b \cdot \sigma \tau_0 + \Sigma(\epsilon),
\]
where we take $m^* = 0.026m_e$, consistent with an InAs nanowire, and $\Sigma(\varepsilon)$ is the superconductive lead self-energy as given in Eq. (52).

To include the effect of a finite bias, we model the deformation of the potential landscape in a simplified effective manner. Following an approach proposed by Ref. [41], we assume that the density of states in the superconductor shell is high enough to guarantee perfect screening of the electric field. This means that the potential drop falls entirely in the (depleted) barrier region. The case of imperfect screening is analyzed in Ref. [11].

To provide a gauge-invariant description of the potential landscape we select the voltage applied to the superconductor as the reference voltage $V_S = 0$ such that $\varepsilon = E_p - \mu_S = E_p + eV_S$ is the energy of the scattering particle. We define the left and right biases as $V_{LS}$ and $V_{RS}$. The other parameters that enter in the effective potential landscape are the chemical potential in the lead and the wire, that we define as $\mu_l = -e(V_l - V_S)$ and $\mu_w = -e(V_w - V_S)$, and the zero-bias barrier height $\Delta V_b = e(V_b - V_S)$. We assume the absence of built-in biases in the junction by considering a flat potential barrier at zero bias. The effect of the zero-bias barrier $\Delta V_b$ can be connected to a reduction of the coupling with the leads that causes a reduction of the height of the peaks and increased sharpness in the differential conductance. All these parameters are shown in the sketch of the landscape shown in Fig. 4. We modeled the effect of the finite bias as a linear voltage drop (i.e., a constant electric field) within the barrier, and we smoothed the potential using a sigmoid function instead of Heaviside steps to avoid sharp transitions between the different parts of the system.

We used a nanowire length of $L_w = 500$ nm with barriers of length $L_b = 50$ nm. The local chemical potential in the nanowire is set to $\mu_w = 0.5$ meV while the zero-bias barrier height is set to $\Delta V_b = 0.3$ meV. The lead has $\mu_l = 25$ meV. For the superconductive lead, we set $\Delta = 0.35$ meV and $\gamma_{Sc} = 0.2$ meV. To simplify the evaluation of reciprocal conductance, we restrict the elevation angle $\theta \in [-\pi/2, \pi/2]$ while allowing the magnitude $b$ to take negative values. We discretized the Hamiltonian using the finite-differences method with step lengths $\alpha_s = 1$ nm, then evaluated the scattering matrix $S[\varepsilon, H(\varepsilon; P)]$ using the KWANT package for quantum transport [65]. After evaluation of the $S$ matrix, the conductance is calculated for the CLA case following Eqs. (9)–(12) while in the nonlinear case, the electric charge current is calculated by numerical integration of Eq. (6).

### 1. Identification of the spin-orbit coupling direction

We first focus on the newly introduced quantities, reciprocal differential conductances and conductance magnetic asymmetry, and their use for the determination of the spin-orbit coupling direction. To emphasize the effect and maximize $G^\alpha$, we consider a strongly asymmetric case in which the left barrier is set to $\Delta V_{b,L} = 0.3$ meV while the right barrier is in the open regime $\Delta V_{b,R} = 0$. We also choose to align the Rashba field in the out-of-plane direction $\alpha_R = (0, 0, -10)$ meVnm while we set the Dresselhaus energy to $\beta = 5$ meVnm.

A sweep in Zeeman energy $b$ is shown in Fig. 5. We find that $G^\alpha$ is much larger than $G^\beta$. By Eqs. (32) and (33), this
indicates that the antisymmetric part of the electron-electron transmission probability dominates over the antisymmetric part of the crossed Andreev reflection probability. Note that in the presence of both mirror symmetry \( \mathcal{M}_x \) (inverting the wire direction) and an antiunitary symmetry, the Andreev transmission probabilities are symmetric in energy such that \( G_{LR}^{\text{a}} \) vanishes. For our device, in the presence of only Rashba SOC and a magnetic field oriented in the wire direction, a mirror-symmetric device satisfies an antiunitary symmetry \( \mathcal{A} = \mathcal{K} \) and a mirror symmetry \( \mathcal{M}_x = \sigma_z \tau_0 \). Therefore, a signal in \( G_{LR}^{\text{a}} \) is correlated to the mirror-symmetry-breaking terms in the device geometry, such as the asymmetric barrier configuration used here, and Dresselhaus spin-orbit coupling \( \alpha e\kappa \), breaking both the antiunitary symmetry and the mirror symmetry.

The reciprocal conductances \( G^a \) and \( G^e \) can be used to characterize the spin-orbit coupling of the nanowire. Indeed, both are symmetric under reversal of magnetic field only if the system satisfies an antiunitary symmetry that persists at a finite magnetic field. An alternative and easier measurement is the conductance magnetic asymmetry \( G^m \), Eq. (42), since it requires the combination of only two differential conductances at the same terminal. This quantity vanishes if there is an antiunitary symmetry persisting at a finite magnetic field. Measuring this quantity while rotating the azimuth of the magnetic field allows for the identification of the direction of the generalized spin-orbit coupling vector \( e_x \), as shown in Fig. 6.

The zero of the quantity \( G^m(\phi = \phi_0) = 0 \) is achieved only when the orthogonality condition \( b \cdot \kappa = 0 \) is satisfied. Therefore, with the measured set of directions for which \( G^m = 0 \) it is possible to determine \( e_x \) and its relative angle with the wire direction \( \phi_x = \phi_0 - \pi/2 \). With this information, it is possible to determine both the direction of the Rashba field and the ratio of the orthogonal Rashba and Dresselhaus SOC. The orthogonal Rashba field \( \alpha_L \) is oriented in the direction \( e_x \times e_y \) while the ratio of the two fields is connected to the angle by \( \beta/\alpha_L = \tan(\phi_b) \).

Note that \( G^m(\phi) \) shows a linear behavior in \( \phi \) near \( \phi_0 \) (marked by a change of sign in the neighborhood). In the simulations we noticed an additional zero in the direction \( (\phi_b, \theta_b) \) that is \( \kappa \times b = 0 \). In this case, \( G^m(\phi, \theta) \) has a quadratic behavior in both \( \phi \) and \( \theta \) in the neighborhood of \( (\phi_b, \theta_b) \). Note that \( \theta_b = 0 \) in the chosen coordinate system.

### 2. Finite-bias effect and dissipation

Transport symmetries can be also exploited to assess the presence of nonideal effects and possibly distinguish between them. To illustrate the idea in this example system, we consider the dissipation and finite-bias effect. Indeed, in an ideal system the antisymmetric components of the local and nonlocal electrical conductance as a function of bias voltage are opposite to each other, such that \( G_{LR}^{\text{m}}(V) \equiv G_{LL}^{\text{m}}(V) + G_{LR}^{\text{m}}(V) = 0 \) [cf. Eq. (23)]. This is illustrated in Fig. 7.

This symmetry relation is broken by finite-bias effects, dissipation, and Coulomb scattering between quasiparticles. We verify the possibility of distinguishing between finite-bias and dissipation effects by calculating the quantity \( G_{LR}^{\text{m}} \) in presence
of these effects. We consider the same system as before in Sec. VB1 with the only difference of considering symmetric barrier of $\Delta V_b = 30 \,\text{meV}$ and we set the Dresselhaus SOC $\gamma = 0$.

First, we introduce the finite-bias effect by manually introducing the voltage drop in the barrier regions as shown in Fig. 4. The nonlinear differential conductance is then obtained by numerical differentiation of the total current calculated with nonlinear theory (left panels) and correction to the CLA can be seen in Fig. 8. It is possible to distinguish two corrections. One general background correction in the local conductance that can be attributed to an increase in the average barrier height as the potential is raised. On top of this, we can identify a shift in the position of the peaks. They finite-bias effect gets stronger and more evident as the barrier length is increased. In this simulation, $V_{L,0} = 30 \,\text{meV}$, $V_{b,0} = 30 \,\text{meV}$, $\beta = 0$, while we set the Zeeman field to $b = (1, 0, 0) 40 \,\text{meV}$.

FIG. 8. Electric differential conductance in the symmetric setup calculated with nonlinear theory (left panels) and correction to differential conductance in CLA. It is possible to distinguish two corrections. One general background correction in the local conductance that can be attributed to an increase in the average barrier height as the potential is raised. On top of this, we can identify a correction that moves and changes the position of the peaks. The finite-bias effect gets stronger and more evident as the barrier length is increased. In this simulation, $V_{L,0} = 30 \,\text{meV}$, $V_{b,0} = 30 \,\text{meV}$, $\beta = 0$, while we set the Zeeman field to $b = (1, 0, 0) 40 \,\text{meV}$.

The effects of the two dissipation terms are very similar and consistent with the result in Eq. (27). Therefore, it is not possible to distinguish between the two effects with this measurement. In the lower plots, nonlinear theory within perfect metallic screening approximation is considered. The effect of the dissipation on the symmetry relation appears qualitatively different from dissipation also in this case. It can be described by a background contribution that depends on the sign of the applied voltage together with a localized correction in the position of the peaks. More strikingly, after the topological transition, there is no evident oscillation in the sign connected to the local BCS charge in the dissipation case. Therefore, measurements of electrical conductance offer the possibility of distinguishing between the effect of finite bias and dissipation. These results are consistent with previous analyses [41,51].

3. Thermoelectric conductance

The clear advantage of thermoelectric conductance is that temperature-induced charge accumulation, which leads to potential change modification, can be safely ignored in the regime of interest. Therefore, it represents an alternative measurement free of problems related to the finite-bias effect. We stress here that by thermoelectric measurements we mean the measurement of the current as a change in the temperature of the leads. We note that for

![Fig. 9: Electric differential conductance in the symmetric setup calculated with nonlinear theory (left panels) and correction to differential conductance in CLA.](image)
CONDUCTANCE MATRIX SYMMETRIES OF … PHYSICAL REVIEW B 106, 104516 (2022)

negligible inelastic scattering we expect no local thermalization, such that the device parameters should remain unchanged by the temperatures in the leads.

As in the case of electric differential conductance, we can define local and nonlocal thermoelectric conductance. These satisfy the symmetry relation

\[ L_{\text{sum}}^\text{am} (\theta) = L_{LL}(\theta_L = \theta) + L_{LR}(\theta_R = \theta) \approx 0 \]

if we restrict the integral over the energy to values below the parent gap region. The latter just introduces a nonexactly balanced background contribution. As can be seen in Fig. 10, the interesting features are the lobes with an oscillating sign at low temperatures. These features can be linked to the BCS charge ($\tau_z$) of the Andreev bound states at the end of the wire by a straightforward extension of the derivation using nonlocal electric conductance presented in Ref. [8].

Finite-bias effects can also affect the procedure for the determination of the spin-orbit coupling outlined in Sec. V B 1. The same information can be obtained by thermoelectric measurements by evaluating the thermoelectric conductance magnetic asymmetry $L^m$ while rotating the magnetic field as shown in Fig. 11. As expected, when the magnetic field lies in the plane orthogonal to the generalized spin-orbit coupling vector $\mathbf{k}$, identified by the angle $\phi_0$, we observe a zero in $L^m$. In contrast to the electric conductance combination $G_{\text{uf}}^m$, the thermoelectric conductance combination $L_{LR}^m$ displays a quadratic behavior in $\phi$ around $\phi_0$ at the magnetic field angle $\theta = \pi/4$. Also for $L^m$ we observe an additional quadratic zero at ($\phi_0$, $\theta_0$), i.e., when $\mathbf{k} \times \mathbf{b} = 0$.

FIG. 10. Local and nonlocal thermoelectric conductance in a proximitized semiconductor nanowire. The low-temperature lobes with alternating signs can be associated with the BCS charge ($\tau_z$) similarly to the interpretation of $G^{am}$.

VI. CONCLUSIONS

In this work, we have explored the limits of local and nonlocal tunneling spectroscopy of hybrid devices within the extended Landauer-Büttiker formalism. We have derived symmetry constraints on the multiterminal conductance matrix that follow from the fundamental microreversibility and particle-hole conjugation in the presence of superconductivity. Our first result shows that the reciprocal conductances $G^a$ and $G^{re}$, defined in Eqs. (32) and (33), can be employed to extract the antisymmetric-in-voltage parts of the individual electron and Andreev transmission and reflection probabilities.

In the presence of an additional antiunitary symmetry that persists at a finite Zeeman field, a further relation can be derived for the conductance magnetic asymmetry $G^m$ in Eq. (42). This relation is particularly useful in the study of spin-orbit coupled semiconductor nanowires proximitized by an $s$-wave superconductor since it allows extracting the ratio between the Rashba and Dresselhaus spin-orbit coupling strength. We have demonstrated this result in an explicit numerical model. This result may be useful for material and device characterization because the characterization of the spin-orbit coupling in proximitized semiconductor devices is an open research question. Future work can study these quantities in a more realistic scenario modeling the cross section of the superconductor-semiconductor heterostructure to include multiple transverse modes and the orbital coupling of the magnetic field.

Furthermore, we have studied the effects of dissipation and the dependence of the electric potential on the bias voltage on the symmetry relations at an explicit model of a proximitized semiconductor nanowire. Generally, these symmetry relations are broken by these nonidealities. However, the two effects yield distinct signatures in the conductance matrix elements and their linear combinations.
In conclusion, nonlocal tunneling spectroscopy is a powerful tool employed in the study of Andreev bound states [8–10,12,21–23]. We hope that our work contributes to the interpretation of the experimental measurements and expands the scope of the method by allowing access to more detailed information on the system under study.

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