Vertical Configuration Spaces and Their Homology

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Vertical configuration spaces 
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We introduce ordered and unordered configuration spaces of ‘clusters’ of points in an Euclidean space \( \mathbb{R}^d \), where points in each cluster satisfy a ‘verticality’ condition, depending on a decomposition \( d = p + q \). We compute the homology in the ordered case and prove homological stability in the unordered case.

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1 Introduction

We fix integers \( p \geq 0 \) and \( q \geq 1 \) and let \( d := p + q \) throughout the article. For \( k \geq 1 \), a cluster of size \( k \) in \( \mathbb{R}^d \) is a tuple of \( k \) distinct points of \( \mathbb{R}^d \), i.e. a point in the ordered configuration space \( \tilde{C}_k(\mathbb{R}^d) \) of \( k \) points in \( \mathbb{R}^d \). For \( r \geq 0 \) and a tuple \( K = (k_1, \ldots, k_r) \) of integers \( k_i \geq 1 \), we consider the subspace

\[
\tilde{C}_K(\mathbb{R}^d) \subseteq \prod_{i=1}^r (\mathbb{R}^d)^{k_i}
\]

consisting of all configurations of \( r \) ordered, pairwise disjoint clusters of sizes \( k_1, \ldots, k_r \) in \( \mathbb{R}^d \). Note that if we put \( |K| := k_1 + \cdots + k_r \), the space \( \tilde{C}_K(\mathbb{R}^d) \) is, up to reindexing, homeomorphic to the more familiar space \( \tilde{C}_{|K|}(\mathbb{R}^d) \).

Now decompose \( \mathbb{R}^d = \mathbb{R}^p \times \mathbb{R}^q \), and denote by \( \text{pr}_1: \mathbb{R}^d \to \mathbb{R}^p \) the projection on the first \( p \) coordinates. A cluster \( z = (z^1, \ldots, z^k) \) of \( k \) points in \( \mathbb{R}^d \) is vertical if \( \text{pr}_1(z^1) = \cdots = \text{pr}_1(z^k) \), i.e. the \( k \) points in the cluster share their first \( p \) coordinates. For \( p = 1 \) and \( q = 1 \), we are requiring the \( k \) points of the cluster to lie on the same vertical line of \( \mathbb{R}^2 \), whence the terminology: see Figure 1.

Definition 1.1. For \( r \) and \( K = (k_1, \ldots, k_r) \) as above, we introduce a subspace

\[
\tilde{V}_K(\mathbb{R}^{p,q}) \subseteq \tilde{C}_K(\mathbb{R}^{p+q}).
\]

A sequence of clusters \( (z_1, \ldots, z_r) \) belongs to \( \tilde{V}_K(\mathbb{R}^{p,q}) \) if and only if each cluster \( z_i = (z_i^1, \ldots, z_i^{k_i}) \) is vertical.
These spaces have already been studied in [Lat17]. Our interest for the spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ has the following reasons.

1. There is a coloured operad $\mathcal{V}_{p,q}$, in spirit similar to the operad of little cubes and, even more closely, to the extended Swiss cheese operad [Wil17]: the second author has introduced this operad in his PhD thesis, and has studied the problem of delooping $\mathcal{V}_{p,q}$-algebras in [Kra21]. The spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ occur, up to a mild homotopy equivalent replacement, in the description of the operad $\mathcal{V}_{p,q}$. The first occurrence of operations related to the operad $\mathcal{V}_{p,q}$ is in [Böd90, §5].

2. For $n \geq 0$, the cohomology of the ordered configuration space $\tilde{C}_n(\mathbb{R}^d)$ is known to be free abelian; more precisely, for every choice of
   - $r \geq 1$, and a sequence $K = (k_1, \ldots, k_r)$ with $|K| = n$;
   - a partition of the set $\{1, \ldots, n\}$ into pieces of sizes $k_1, \ldots, k_r$,

   we have a proper embedding $\tilde{V}_K(\mathbb{R}^{d-1,1}) \hookrightarrow \tilde{C}_n(\mathbb{R}^d)$, and a basis for $H^*(\tilde{C}_n(\mathbb{R}^d); \mathbb{Z})$ can be chosen to consist of Poincaré duals of components of the submanifolds $\tilde{V}_K(\mathbb{R}^{d-1,1})$ obtained in this way.

3. The spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ give an example of ordered configuration spaces for which the Fadell–Neuwirth maps fail, in general, to be fibrations.

There is an unordered counterpart of the construction above: consider the partition of $\{1, \ldots, |K|\}$ into $r$ consecutive segments of lengths $k_1, \ldots, k_r$, and denote by $\mathcal{G}_K \subseteq \mathcal{G}_{|K|}$ the subgroup of the symmetric group containing all permutations $\sigma$ which preserve this partition, i.e. $\sigma$ maps each partition component to a (possibly different) partition component. The group $\mathcal{G}_K$ can be described as follows: for all $k \geq 1$ we denote by $r(k) \geq 0$ the number of occurrences of $k$ in $K$; then $\mathcal{G}_K$ is isomorphic to the product

$$\mathcal{G}_K \cong \prod_{k=1}^\infty \mathcal{G}_k \wr \mathcal{G}_{r(k)} = \prod_{k=1}^\infty (\mathcal{G}_k)^{r(k)} \rtimes \mathcal{G}_{r(k)}.$$

**Definition 1.2.** The group $\mathcal{G}_K$ acts freely on $\tilde{V}_K(\mathbb{R}^{p,q})$ by permuting the labels $1 \leq i \leq r$ of clusters of the same size, and permuting the labels $1 \leq j \leq k_i$ of the points of each cluster; we denote the quotient space by

$$V_K(\mathbb{R}^{p,q}) := \tilde{V}_K(\mathbb{R}^{p,q}) / \mathcal{G}_K.$$
Roughly speaking, and using the notation above, a point in $V_k(\mathbb{R}^{p,q})$ consists of a collection of $r$ clusters, of which $r(k)$ have size $k$; clusters of the same size are unordered, and points inside a cluster are also unordered, see Figure 2. One can thus regard $V_k(\mathbb{R}^{p,q})$ as a subspace of

$$\prod_{k=1}^{\infty} \left( C_k(\mathbb{R}^d)^{r(k)} / S_{r(k)} \right),$$

where $C_k(\mathbb{R}^d)$ denotes the unordered configuration space of $k$ points in $\mathbb{R}^d$. Our interest for the spaces $V_k(\mathbb{R}^{p,q})$ has the following reasons.

1. These spaces occur naturally in the description of free algebras over the operad $\mathcal{V}_{p,q}$ mentioned earlier.

2. We note that for $p = 0$ and $q = d$, the space $\tilde{V}_k(\mathbb{R}^{0,d})$ is homeomorphic to the ordered configuration space $\tilde{C}_{|K|}(\mathbb{R}^d)$; however the unordered version $V_k(\mathbb{R}^{0,d})$ is in general not homeomorphic to $C_{|K|}(\mathbb{R}^d)$, rather it is a covering of the latter space; the space $\tilde{V}_k(\mathbb{R}^{0,d})$ is an unordered configuration space of clusters of points in $\mathbb{R}^d$, without any ‘verticality’ condition. For $d = 2$, spaces of unordered configurations of clusters have been considered in their own sake in [TP14] and in relation to Hurwitz spaces in [Tie16].

3. For general $p,q$, the spaces $V_k(\mathbb{R}^{p,q})$ are related to the spaces of parallel submanifolds in an ambient manifold, see [Pal21] and [Lat17].

We will occasionally restrict to situations where all clusters have the same size, i.e. $K = (k, \ldots, k)$ for some $k \geq 1$ and let $r$ be the length of this tuple. In these situations, we will we will simplify our notation and write

$$\tilde{V}_r^k(\mathbb{R}^{p,q}) := \tilde{V}_{(k, \ldots, k)}(\mathbb{R}^{p,q}) \quad \text{and} \quad V_r^k(\mathbb{R}^{p,q}) := V_{(k, \ldots, k)}(\mathbb{R}^{p,q}).$$

**Results** The first aim of this article is to compute the integral homology of the spaces $\tilde{V}_r^k(\mathbb{R}^{p,q})$, for all choices of $p$, $q$, and $K$. We will see that $H_*(\tilde{V}_r^k(\mathbb{R}^{p,q}); \mathbb{Z})$ is free abelian, and for $q = 1$ it is supported in degrees multiple of $p$.

The second aim is to prove a homological stability result: for all $k \geq 1$, the stabilisation map $V_r^k(\mathbb{R}^{p,q}) \to V_{r+1}^k(\mathbb{R}^{p,q})$, which adjoins a new cluster of size $k$, induces isomorphisms in integral homology in degrees $* \leq \frac{r}{2}$. This extends the results of [TP14], [Pal21], and [Lat17] who covered all cases with $p + q \geq 3$. 

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**Figure 2.** A configuration in $V_{(3,4,2,2)}(\mathbb{R}^{1,1})$
Outline Our article is organised as follows: in Section 2, we introduce some notation and make some first observations about the basic properties of these configuration spaces. In Section 3, we calculate the integral homology of the ordered vertical configuration spaces \( \tilde{V}_K(\mathbb{R}^p) \). Then we turn in Section 4 to the question of homological stability for the unordered vertical configuration spaces \( V^k_\mathbb{R}(\mathbb{R}^p) \) with fixed cluster size \( k \).

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2 Preliminaries

In this section we introduce notation for the spaces \( \tilde{V}_K(\mathbb{R}^p) \) and \( V^k_\mathbb{R}(\mathbb{R}^p) \), and make some basic observations about the topology of these spaces.

Notation 2.1. Recall that \( \text{pr}_1 : \mathbb{R}^d \to \mathbb{R}^p \) denotes the projection to the first \( p \) coordinates. However, in several situations, we will make use of the decomposition \( \mathbb{R}^d = \mathbb{R}^{p+q-1} \times \mathbb{R} \) and write \((\zeta, t)\) for a generic point in \( \mathbb{R}^d \). Hence, we have two other projections, namely \( \text{pr}_\zeta : \mathbb{R}^d \to \mathbb{R}^{p+q-1} \) and \( \text{pr}_t : \mathbb{R}^d \to \mathbb{R} \). Clearly, if \( q = 1 \), then \( \text{pr}_\zeta \) and \( \text{pr}_1 \) coincide.

Notation 2.2. We denote elements in \( \tilde{V}_K(\mathbb{R}^p) \) resp. \( V^k_\mathbb{R}(\mathbb{R}^p) \) as follows:

- An element in \( \tilde{V}_K(\mathbb{R}^p) \) is an ordered collection \( Z := (z_1, \ldots, z_r) \) of (vertical) clusters \( z_i := (z_i^1, \ldots, z_i^k) \). We will also often write \( Z = (z_1^1, \ldots, z_r^k) \).
• For the unordered version, we use the suggestive sum notation: A generic element in $V_k(\mathbb{R}^p,q)$ is an unordered collection $[Z] := \sum_{i=1}^{r} [z_i]$ of unordered (vertical) clusters $[z_i] := [z_{i1}, \ldots, z_{ik_i}] = \{z_{i1}, \ldots, z_{ik_i}\}$.

**Remark 2.3** (Path components of $\tilde{V}_K(\mathbb{R}^p,q)$ and $V_K(\mathbb{R}^p,q)$). For the ordered vertical configuration spaces, the following hold:

• For $q \geq 2$ the space $\tilde{V}_K(\mathbb{R}^p,q)$ is connected.

• For $q = 1$ and $p \geq 1$, the space $\tilde{V}_K(\mathbb{R}^p,q)$ has one component $\tilde{V}_K(\mathbb{R}^p,q)_{\Sigma}$ for each tuple $\Sigma = (\sigma_1, \ldots, \sigma_r) \in \prod_i \mathfrak{S}_{k_i}$ of permutations. This component contains all configurations $(z_{i1}^{\sigma_1(1)}, \ldots, z_{ik_i}^{\sigma_r(i)}, z')$ with $\text{pr}_i(z_{i1}^{\sigma_1(1)}) < \text{pr}_i(z_{i2}^{\sigma_1(2)})$ for all $1 \leq i \leq r$ and $1 \leq j < k_i$.

• For $q = 1$ and $p = 0$, we note that $\tilde{V}_K(\mathbb{R}^{0,1}) = \tilde{C}_{|K|}(\mathbb{R})$, so each permutation $\sigma \in \mathfrak{S}_{|K|}$ corresponds to a connected component which contains all configurations $(z_{i1}^{\sigma_1(1)}, \ldots, z_{ik_i}^{\sigma_r(i)}, z')$ with $z' < z^{\sigma(i)}$.

We have inclusions $\prod_i \mathfrak{S}_{k_i} \subseteq \mathfrak{S}_{|K|}$ and the group $\mathfrak{S}_K = \prod_k \mathfrak{S}_{k_i} \acts \mathfrak{S}_{|K|}$ acts on $\pi_0 \tilde{V}_K(\mathbb{R}^p,q)$ with quotient equal to $\pi_0 V_K(\mathbb{R}^p,q)$. Since the action is transitive in the first two cases listed above, the space $V_K(\mathbb{R}^p,q)$ is connected for $(p,q) \neq (0,1)$, whereas for $(p,q) = (0,1)$ we can identify

$$\pi_0 V_K(\mathbb{R}^{0,1}) \cong \mathfrak{S}_{|K|}/\mathfrak{S}_K.$$

The latter set can also be identified with the set of unordered partitions of $\{1, \ldots, |K|\}$ into subsets of sizes $k_1, \ldots, k_r$; such that for all $k \geq 1$ there are $r(k)$ partition components of size $k$.  

**Remark 2.4** ($\bar{V}$ and $V$ are manifolds). The space $\tilde{V}_K(\mathbb{R}^p,q)$ is an open subspace of $(\mathbb{R}^p)^r \times (\mathbb{R})^{\left|K\right|}$ and hence an orientable smooth manifold of dimension $p \cdot r + q \cdot |K|$. The action of $\mathfrak{S}_K$ is free, so $V_K(\mathbb{R}^p,q)$ is again a manifold of the same dimension. The manifold $V_K(\mathbb{R}^p,q)$ is non-orientable if and only if at least one of the following holds:

• $q \geq 3$ is odd and there is at least one cluster of some size $k \geq 2$; then a path in $V_K(\mathbb{R}^p,q)$ interchanging two points of this cluster, while fixing all other points, reverses the local orientation;

• $p + q \geq 2$ and there is some $k \geq 1$ such that $p + q \cdot k$ is odd and $r(k) \geq 2$; then, interchanging two clusters of size $k$ while preserving their internal ordering and fixing all other points reverses the local orientation.

**Remark 2.5** (Poincaré–Lefschetz duality). For a topological space $X$, we denote by $X^\infty$ its one-point compactification, and denote the point at infinity by $\infty$. Since $\tilde{V}_K(\mathbb{R}^p,q)$ is an open and orientable manifold of dimension $p \cdot r + q \cdot |K|$, we can apply Poincaré–Lefschetz duality and obtain

$$H^*(\tilde{V}_K(\mathbb{R}^p,q)) \cong H_{p\cdot r + q \cdot |K| - *}(\tilde{V}_K(\mathbb{R}^p,q)^\infty, \infty).$$
3 The cohomology of $\tilde{V}_K(\mathbb{R}^{p,q})$

In this section we calculate the integral cohomology of the spaces $\tilde{V}_K(\mathbb{R}^{p,q})$ for all dimensions $p \geq 0$ and $q \geq 1$. In the case $p = 0$ we recover the calculations of [Arn69] and [CLM76, §III.6] for the classical configuration spaces $\tilde{C}_{|K|}(\mathbb{R}^d)$. Let us exclude the case $(p,q) = (0,1)$, where all components are contractible.

3.1 Ray partitions

We fix a partition $K = (k_1, \ldots, k_r)$ for the entire section. Before we state our main result about the cohomology of $\tilde{V}_K(\mathbb{R}^{p,q})$, we need to introduce a few combinatorial notions.

**Definition 3.1.** The *table* associated with the partition $K$ is the set

$$T_K := \{(i, j); \ 1 \leq i \leq r \text{ and } 1 \leq j \leq k_i\}.$$

We order $T_K$ lexicographically, which means we write $(i, j) < (i', j')$ if either $i < i'$, or $i = i'$ and $j < j'$ holds.

**Notation 3.2.** For each partition $Q$ of $T_K$ into non-empty subsets $Q_1, \ldots, Q_l$, we consider two positive integers:

- The number $l(Q) := l$ is called the *length* of the partition, and in general we have $1 \leq l(Q) \leq |K|$.

- Consider on $\{1, \ldots, l\}$ the equivalence relation spanned by $\beta \sim \beta'$ if there are $1 \leq i \leq r$ and $1 \leq j, j' \leq k_i$ with $(i, j) \in Q_\beta$ and $(i, j') \in Q_{\beta'}$ (i.e. the $i$th cluster intersects both $Q_\beta$ and $Q_{\beta'}$). The number of equivalence classes $1 \leq a(Q) \leq \min(l(Q), r)$ will be called the *agility* of the partition.

**Definition 3.3.** A *ray partition* $Q$ of type $K$ is a partition $Q_1, \ldots, Q_l$ of $T_K$, with a total order $<_\beta$ on each piece $Q_\beta$ (called *ray*), such that the following hold:

**R1.** the components are labelled from 1 to $l$ according to their minimum with respect to the global order $<$, i.e.

$$\min(Q_1, <) < \cdots < \min(Q_l, <);$$

**R2.** for each $1 \leq \beta \leq l$, the minima with respect to $<$ and $<_\beta$ coincide

$$\min(Q_\beta, <_\beta) = \min(Q_\beta, <).$$

**Definition 3.4.** Let $Z = (z_1^1, \ldots, z_r^k) \in \tilde{V}_K(\mathbb{R}^{p,q})$. We say that a ray partition $Q$ is *witnessed by $Z$* if the following conditions hold:

**W1.** all $z_i^j$ with $(i, j) \in Q_\beta$ project along $pr_\xi$ to the same point in $\mathbb{R}^{d-1}$.

**W2.** if $(i, j) \prec_\beta (i', j')$ in $Q_\beta$, then $pr_1(z_i^j) < pr_1(z_{i'}^{j'})$ in $\mathbb{R}$. 

6
Theorem 3.3

The rest of the section is devoted to the proof of Theorem 3.3 parallel to the proof of Theorem 3.1. We assign to each ray partition $\Sigma_q$ in order to do so, we assign to each ray partition $\Sigma_q$ the property that it restricts on this line according to the order $\preceq$. In particular, the points $z_{ij}$ with $(i,j) \in Q_\beta$ lie on a ray, namely the half-line starting at $z_{\min}(Q_\beta, \prec)$ and running in the positive direction. See Figure 3 for an example.

Remark 3.5. If $q = 1$, recall that we consider only $p \geq 1$. Then $\hat{V}_K(\mathbb{R}^{p,1})$ is disconnected, with path components indexed by tuples $\Sigma \in \prod_{i=1}^r \mathcal{S}_{k_i}$ (see Remark 2.3), and we would like to calculate the homology of a single path component. In order to do so, we assign to each ray partition $Q$ of type $K$ such a tuple $\Sigma$ as follows.

Definition 3.6. Given a ray partition $Q$, the ‘stacked’ total order $\prec$ on

$$T_K = (Q_1, \prec_1) \sqcup \cdots \sqcup (Q_r, \prec_1),$$

is determined by the property that it restricts on $Q_\beta$ to $\prec_\beta$ and that all elements from $Q_{\beta+1}$ are $\prec$-smaller than all elements from $Q_\beta$.

For each $1 \leq i \leq r$, there is a unique $\sigma_i \in \mathcal{S}_{k_i}$ with $(i, \sigma_i(j)) \prec (i, \sigma_i(j+1))$ for all $1 \leq j < k_i$; we define $\Sigma(Q) := (\sigma_1, \ldots, \sigma_r) \in \prod_{i=1}^r \mathcal{S}_{k_i}$.

The rationale for the previous definition is the following: a configuration $z \in \hat{V}_K(\mathbb{R}^{p,1})_\Sigma$ can only witness ray partitions $Q$ with $\Sigma(Q) = \Sigma$.

The following is the main theorem of the section.

Theorem 3.7. Let $p \geq 0$, $q \geq 1$ and $K = (k_1, \ldots, k_r)$ with $k_i \geq 1$.

1. The integral cohomology $H^*(\hat{V}_K(\mathbb{R}^{p,d}))$ is freely generated by classes $u_Q$ for each ray partition, and the cohomological degree of $u_Q$ is

$$|u_Q| = p \cdot (r - a(Q)) + (q - 1) \cdot (|K| - l(Q)).$$

2. For $q = 1$, the cohomology class $u_Q$ is supported on the component $\hat{V}_K(\mathbb{R}^{p,1})_{\Sigma(Q)}$.

The rest of the section is devoted to the proof of Theorem 3.7.

Figure 3. A configuration in the space $\hat{V}_{(3,4,3)}(\mathbb{R}^{1,2})$ which witnesses the ray partition $(Q_1, \ldots, Q_7)$, where e.g. $Q_2 = \{(1,2) \prec (3,3) \prec (1,3)\}$. The components $Q_\beta$ are numbered according to their smallest label (R4), and the point carrying the minimal label lies at the bottom of each ray (R2). Recall that the verticality condition demands that all points belonging to the same cluster have to lie in the same affine plane orthogonal to the first axis. Note that this ray partition seems not to be the most ‘efficient’ one: we may merge $Q_4$ and $Q_5$. We will introduce a measure for ‘efficiency’ soon.
3.2 The weight filtration and the proof of Theorem 3.7

Throughout this section we fix \( K, p \) and \( q \) as before. We treat simultaneously the cases \( q \geq 2 \) and \( q = 1 \), putting in parentheses the differences needed in the case \( q = 1 \). For \( q \geq 2 \) we abbreviate \( \hat{V} := \hat{V}_k(\mathbb{R}^{p,q}) \); for \( q = 1 \) we fix \( \Sigma \in \prod_i \mathfrak{S}_k \) throughout the section and abbreviate \( \hat{V} := \hat{V}_k(\mathbb{R}^{p,q})_\Sigma \).

**Notation 3.8.** For a positive integer \( \Lambda \geq 0 \) we denote by \( \mathcal{P}(\Lambda) \) the set of all sequences \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of integers \( \lambda_i \geq 1 \), for some \( 1 \leq l \leq \Lambda \), satisfying \( \lambda_1 + \cdots + \lambda_l = \Lambda \). The number \( l \) is called the length of the sequence.

We have a natural injection \( \mathcal{P}(\Lambda) \hookrightarrow \{0, \ldots, \Lambda\}^\Lambda \), by adding a suitable number of zeroes at the end of each sequence; we consider on \( \mathcal{P}(\Lambda) \) the inherited lexicographic order.

We denote by \( \mathcal{P}(K) \) the set \( \mathcal{P}(|K|) \), and by \( N \) its cardinality.

**Definition 3.9.** The weight of a ray partition \( Q \) is defined as

\[ \omega(Q) := (|Q_1|, \ldots, |Q_l|) \in \mathcal{P}(K). \]

In the following we state three lemmata and postpone their proofs to the next subsection.

**Lemma 3.10.** Let \( Z \in \hat{V} \). There is a unique ray partition, called \( Q^Z \), which is witnessed by \( Z \) and has maximal weight among all ray partitions witnessed by \( Z \). (If \( q = 1 \), we have moreover that \( \Sigma(Q^Z) = \Sigma \).)

**Definition 3.11.** Given a ray partition \( Q \), we denote by \( W_Q \subset \hat{V} \) the subspace containing all points \( Z \) with \( Q^Z = Q \) (see Lemma 3.10). We define a filtration \( F_* \) on \( \hat{V}^\infty \) (see Remark 2.5) indexed by the linearly ordered set \( \mathcal{P}(K) \); for all \( \lambda \in \mathcal{P}(K) \) define the \( \lambda \)th filtration level \( F_\lambda = F_\lambda \hat{V}^\infty \) as the subspace containing \( \infty \) and all \( Z \in \hat{V} \) with \( \omega(Q^Z) \geq \lambda \). Note that for \( \lambda < \lambda' \) in \( \mathcal{P}(K) \) we have an inclusion \( F_{\lambda'} \subset F_\lambda \).

**Lemma 3.12.** Let \( \lambda \in \mathcal{P}(K) \). Then the inclusion \( F_\lambda \subseteq \hat{V}^\infty \) is closed.

**Notation 3.13.** We can switch our indexing set of the filtration \( F_* \) from Definition 3.11 from \( \mathcal{P}(K) \) to the natural numbers \( 1 \leq \nu \leq N \) in the following way: let \( \chi : \{1, \ldots, N\} \to \mathcal{P}(K) \) be the unique order-reversing bijection; then for \( 1 \leq \nu \leq N \) we define \( F_\nu = F_{\chi(\nu)} \). Moreover we set \( F_0 := \{\infty\} \subset \hat{V}^\infty \). We obtain an ascending filtration of \( \hat{V}^\infty \) with closed levels (see Lemma 3.12):

\[ \{\infty\} = F_0 \subseteq F_1 \subseteq \cdots \subseteq F_N = \hat{V}^\infty. \]

We also denote \( F_{-1} := \emptyset \), and for \( 0 \leq \nu \leq N \) we denote by \( \mathfrak{F}_\nu \) the \( \nu \)th filtration stratum of the filtration \( F_* \), i.e. the difference \( \mathfrak{F}_\nu = F_\nu \setminus F_{\nu-1} \).
Lemma 3.14. The strata satisfy the following properties:

1. For each ray partition \( Q \) (with \( \Sigma(Q) = \Sigma \)), the subspace \( W_Q \) is a contractible open manifold of dimension \( |K| + p \cdot a(Q) + (q - 1) \cdot l(Q) \) and a path component of the stratum \( \mathfrak{S}_v \), where \( 1 \leq v \leq N \) satisfies \( \chi(v) = \omega(Q) \).

2. All connected components of a stratum \( \mathfrak{S}_v \) with \( v \geq 1 \) arise in this way.

3. The closure \( \overline{W}_Q \) of \( W_Q \) inside \( \overline{V} \) is also a smooth, orientable submanifold of dimension \( |K| + p \cdot a(Q) + (q - 1) \cdot l(Q) \).

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. We consider the Leray spectral sequence associated with the filtered space \( \overline{V}^\infty \) and compute its reduced homology. The \( E^1 \)-page reads

\[
E^1_{v,\mu} = H_{v+\mu}(F_v/F_{v-1}) = H_{v+\mu}(F_v/F_{v-1}).
\]

By Lemma 3.14, for all \( 1 \leq v \leq N \), \( \mathfrak{S}_v \) is the disjoint union of the open manifolds \( W_Q \) for \( Q \) varying in the finite set of ray partitions with \( \omega(Q) = \chi(v) \). By Lemma 3.12 we have homeomorphisms

\[
F_v/F_{v-1} \cong \mathfrak{S}_v^\infty \cong \bigvee_{\omega(Q) = \chi(v)} W_Q^\infty.
\]

Even for \( v = 0 \) we have that \( F_0 = F_0/F_{-1} = \{\infty\} \) is formally homeomorphic to the empty wedge. By Lemma 3.14, \( W_Q \) is an open manifold of dimension \( d(Q) := |K| + p \cdot a(Q) + (q - 1) \cdot l(Q) \) for all ray partitions; hence we can apply Poincaré–Lefschetz duality and obtain for all \( v, \mu \geq 0 \) an isomorphism

\[
E^1_{v,\mu} \cong \bigoplus_{\omega(Q) = \chi(v)} H_{v+\mu}(W_Q^\infty, \infty) \cong \bigoplus_{\omega(Q) = \chi(v)} H^{d(Q)-v-\mu}(W_Q).
\]

Again by Lemma 3.14, \( W_Q \) is contractible for all ray partitions \( Q \); hence \( H^{d(Q)-v-\mu}(W_Q) \) contributes to the first page of the spectral sequence only in the case \( \mu + v = d(Q) \). We can rewrite, for all \( v \geq 0 \) and considering all degrees \( \mu \) at the same time

\[
E^1_{v,*} \cong \bigoplus_{\omega(Q) = \chi(v)} H_{d(Q)}(W_Q^\infty, \infty).
\]

Since by Lemma 3.12, \( F_* \) is a closed filtration and \( W_Q \) is a path component of \( \mathfrak{S}_v \), we can now, for all ray partitions \( Q \), replace the relative homology of the pair \( (W_Q^\infty, \infty) \) with the relative homology of the pair \( (F_v, F_v \setminus W_Q) \) or, by excision, the relative homology of the pair \( (\overline{W}_Q^\infty, \overline{W}_Q^\infty \setminus W_Q) \). Here, as in Lemma 3.14, we denote by \( \overline{W}_Q \) the closure in \( \overline{V} \) of \( W_Q \), and by \( \overline{W}_Q^\infty \) the one-point compactification of \( \overline{W}_Q \) (it coincides, for \( v \geq 1 \), with the closure of \( W_Q \) in \( \overline{V}^\infty \)). We obtain

\[
E^1_{v,*} \cong \bigoplus_{\omega(Q) = \chi(v)} H_{d(Q)}(\overline{W}_Q^\infty, \overline{W}_Q^\infty \setminus W_Q).
\]
Each direct summand in the previous decomposition is isomorphic to $\mathbb{Z}$, generated by the fundamental class of the relative manifold $(\overline{W}_Q^\infty, \overline{W}_Q^\infty \setminus W_Q)$.

By Lemma 3.14, also $(\overline{W}_Q^\infty, \infty)$ is a relative manifold, and its fundamental class projects to that of $(\overline{W}_Q^\infty, \overline{W}_Q^\infty \setminus W_Q)$ under the natural map

$$H_d(Q)(\overline{W}_Q^\infty, \infty) \to H_d(Q)(\overline{W}_Q^\infty, \overline{W}_Q^\infty \setminus W_Q).$$

The previous analysis shows in particular that for all $\nu \geq 0$ the natural map $H_\nu(F, \infty) \to H_\nu(F, F_{\nu-1})$ is surjective. This suffices to prove that the spectral sequence collapses on its first page: any element in the first page is represented by a genuine relative cycle of a pair $(F, \infty)$, so it must survive to the limit. This shows that $H_\nu(\overline{W}_Q^\infty, \infty)$ is freely generated by the fundamental classes of the relative submanifolds $(\overline{W}_Q^\infty, \infty)$, so by Poincaré–Lefschetz duality, $H^*(\overline{V})$ is generated by their duals, which we call $u_Q$; for each ray partition $Q$ we finally see

$$|u_Q| = p \cdot r + q \cdot |K| - d(Q)$$

$$= p \cdot (r - a(Q)) + (q - 1) \cdot (|K| - l(Q)).$$

For $q = 1$ and a fixed component $\Sigma \in \prod_k \mathcal{G}_k$, the entire argument takes place inside $\overline{V} = \overline{V}_k(\mathbb{R}^{p_d})_\Sigma$; more precisely, for each ray partition $Q$ with $\Sigma(Q) = \Sigma$, we have $\overline{W}_Q \subset \overline{V}_k(\mathbb{R}^{p_d})_\Sigma$. Thus, the second claim of the theorem follows.

3.3 Proofs of the three lemmata

Proof of Lemma 3.10. We construct $Q^\Sigma$ by recursively constructing a sequence $(Q_1^\Sigma, \prec_1^\Sigma), \ldots, (Q_\gamma^\Sigma, \prec_\gamma^\Sigma)$ with non-empty and disjoint subsets $Q_1^\Sigma, \ldots, Q_\gamma^\Sigma$ of $T_K$ satisfying the axioms $R1$ and $R2$, see Figure 4.

- For ‘$\gamma = 1$’, we write $z_1^\Sigma = (z_1^\Sigma_1, t_1^\Sigma_1) \in \mathbb{R}^d$, and let $Q_1^\Sigma$ contain all $(i, j) \in T_K$ such that $z_i^\Sigma_1 \in \mathbb{R}^d$ has the form $(z_i^\Sigma_1, t)$ for some $t \geq t_1^\Sigma_1$; in other words, $Q_1^\Sigma$ contains all $(i, j) \in T_K$ such that $z_i^\Sigma_1$ lies on the ray starting at $z_1^\Sigma_1$ and running in the positive $t$-direction. The order $\prec_1^\Sigma$ on $Q_1^\Sigma$ is defined in such a way that condition $W2$ holds.

- For ‘$\gamma = 1 \to \gamma$’, if $Q_1^\Sigma \sqcup \cdots \sqcup Q_{\gamma-1}^\Sigma \neq T_K$ have been constructed, let $(i_{\gamma}, j_{\gamma})$ be the minimal pair in $T_K \setminus (Q_1^\Sigma \sqcup \cdots \sqcup Q_{\gamma-1}^\Sigma)$. Write $z_{i_{\gamma}}^\Sigma = (z_{i_{\gamma}}^\Sigma_1, t_{i_{\gamma}}^\Sigma_1)$ and let $Q_\gamma^\Sigma$ contain all $(i, j) \in T_K \setminus (Q_1^\Sigma \sqcup \cdots \sqcup Q_{\gamma-1}^\Sigma)$ such that $z_{i,j}^\Sigma \in \mathbb{R}^d$ has the form $(z_{i,j}^\Sigma_1, t)$ for some $t \geq t_{i_{\gamma}j_{\gamma}}^\Sigma$, and define the order $\prec_{\gamma}^\Sigma$ on $Q_\gamma^\Sigma$ in such a way that condition $W2$ holds.

Since $T_K$ is finite, this algorithm terminates, and the resulting sequence $Q^\Sigma := (Q_1^\Sigma, \prec_1^\Sigma), \ldots, (Q_{\gamma}^\Sigma, \prec_{\gamma}^\Sigma)$ is a ray partition, which is witnessed by $Z$.

In the case $q = 1$, we have $\Sigma(Q^\Sigma) = \Sigma$: let $z \in \mathbb{R}^{d-1}$ and let $1 \leq \beta, \beta' \leq l(Q^\Sigma)$ be two indices such that for all $(i, j) \in Q_\beta$ and all $(i', j') \in Q_{\beta'}$ we have $\text{pr}_{z^\Sigma}(z_i^\Sigma) = \text{pr}_{z^\Sigma}(z_i'') = z_i^\Sigma$; by construction we have $\text{pr}_{z^\Sigma}(z_i^\Sigma) < \text{pr}_{z^\Sigma}(z_i'')$ if and only
We want to show that \( \omega \) we additionally require \( t(i', j') \prec_\beta (i, j) \). In particular, if \( i = i' \), we have \( \sigma_i(j) < \sigma_i(j') \) if and only if \( \text{pr}_i(z_{i'}) < \text{pr}_i(z'_i) \), and this holds if and only if \( (i, j) \prec (i', j') \); this shows that \( \Sigma(Q^Z) = \Sigma \).

Suppose that \( Q \) is another ray partition witnessed by \( Z \), and let \( l := l(Q) \).

We want to show that \( \omega(Q) < \omega(Q^Z) \) and we do this by showing for each \( 1 \leq \gamma \leq \min(l, l^Z) \) that if \( Q^Z = Q^Z_{\beta} \) for \( 1 \leq \beta < \gamma \), then \( Q^Z \subseteq Q^Z_\gamma \), from which we can deduce inductively that since \( Q \neq Q^Z \) by assumption, there is a \( \gamma \) such that \( Q^Z = Q^Z_\gamma \) for \( 1 \leq \beta < \gamma \) and \( Q^Z_\gamma \subseteq Q^Z_\gamma \), so by definition of the lexicographic ordering, we get \( \omega(Q) < \omega(Q^Z) \).

To do so, assume \( Q^Z_\beta = Q^Z_\gamma \) for \( 1 \leq \beta < \gamma \) and let \((i_\gamma, j_\gamma)\) be the minimum of \( T_K \setminus (Q^Z_1 \sqcup \cdots \sqcup Q^Z_{\gamma-1}) = T_K \setminus (Q^Z_1 \sqcup \cdots \sqcup Q^Z_{\gamma-1}) \) as before. By \( R_\gamma \), the pair \((i_\gamma, j_\gamma)\) has to lie inside \( Q^Z_\gamma \), and by \( R_2 \) it is the minimum with respect to \( \prec_\gamma \).

By \( W_1 \), all \((i, j) \in Q^Z_\gamma \) have to satisfy \( z_i = (\zeta_i^\gamma, t) \) for some \( t \in \mathbb{R} \), and by \( W_2 \), we additionally require \( t \geq t_\gamma^i \). Hence \( Q^Z_\gamma \subseteq Q^Z_\gamma \) as desired.

**Proof of Lemma 3.12.** We show that \( \mathbb{V}^\infty \setminus F_\lambda \) is open. Let \( \hat{Z} \in \mathbb{V}^\infty \setminus F_\lambda \), then we have \( \omega(Q^Z) < \lambda \). Let \( \varepsilon > 0 \) be defined as follows: we consider all (Euclidean) distances in \( \mathbb{R}^{d-1} \) between any two distinct projections \( \text{pr}_\varepsilon(z_i^\varepsilon) \) and \( \text{pr}_\varepsilon(z'_i) \), and also all distances in \( \mathbb{R} \) between any two distinct projections \( \text{pr}_\varepsilon(z_i^\varepsilon) \) and \( \text{pr}_\varepsilon(z'_i) \), for \((i, j) \neq (i', j') \) in \( T_K \). We obtain a finite set of strictly positive real numbers, and \( \varepsilon \) is defined as the minimum of all these numbers.

Let \( Z \) be any configuration in \( \mathbb{V} \) such that, for all \((i, j) \in T_K \), the distance in \( \mathbb{R}^d \) between \( \hat{z}_i \) and \( z_i \) is less than \( \varepsilon \). We claim that \( \omega(Q^Z) \leq \omega(Q^Z_{\varepsilon}) \): the claim implies that for \( Z \) in a neighbourhood of \( \hat{Z} \) in \( \mathbb{V} \), also \( Z \notin F_\lambda \). This would conclude the proof, as \( \mathbb{V} \) is open in \( \mathbb{V}^\infty \).

To prove the claim, we use a method similar to the proof of Lemma 3.10, namely we show for each \( 1 \leq \gamma \leq \min(l^Z, l^Z_{\varepsilon}) \) that if \( Q^Z_\beta = Q^Z_{\varepsilon} \) for \( 1 \leq \beta < \gamma \), then \( Q^Z_\gamma \subseteq Q^Z_{\varepsilon} \), which immediately implies \( \omega(Q^Z) \leq \omega(Q^Z_{\varepsilon}) \).

The minimum \((i_\gamma, j_\gamma)\) of \( T_K \setminus (Q^Z_1 \sqcup \cdots \sqcup Q^Z_{\gamma-1}) = T_K \setminus (Q^Z_1 \sqcup \cdots \sqcup Q^Z_{\gamma-1}) \) has to lie in both \( Q^Z_{\beta} \) and \( Q^Z_{\varepsilon} \). Now for \((i, j) \in T_K \setminus (Q^Z_1 \sqcup \cdots \sqcup Q^Z_{\gamma-1}) \), the following holds: if \( \hat{z}_i \) does not lie on the ray starting at \( \hat{z}_i^\gamma \) and running in the positive \( t \)-direction, then either \( \text{pr}_\varepsilon(\hat{z}_i^\gamma) \neq \text{pr}_\varepsilon(\hat{z}_j^\gamma) \), or the two projections are equal but \( \text{pr}_\varepsilon(\hat{z}_i^\gamma) > \text{pr}_\varepsilon(\hat{z}_j^\gamma) \). In both cases, by the choice of \( \varepsilon \), we would also have that \( z_i^\gamma \) does not lie on the ray starting at \( z_i^\gamma \) and running in positive \( t \)-direction. This shows in particular that \( Q^Z_\gamma \subseteq Q^Z_{\varepsilon} \) as desired. \( \square \)
Proof of Lemma 3.14. Define $H_Q \subseteq \tilde{V}$ as the subspace of configurations of the form $Z = (z_1^1, \ldots, z_k^k)$ such that the following condition holds: for each $1 \leq \beta \leq l$ and $(i, j) \prec_{\beta} (i', j')$, we have $pr_{\zeta}(z_i^j) = pr_{\zeta}(z_i^{j'})$ and $pr_{i}(z_i^j) \leq pr_{i}(z_i^{j'})$. Then $H_Q$ is a closed subspace of $\tilde{V}$, as it is defined by imposing some equalities and some weak inequalities (using \textquoteleft\textquoteleft \leq \textquoteright\textquoteleft) between the coordinates.

Note, however, that the same space $H_Q$ can be defined, as a subspace of $\tilde{V}$, by replacing the second condition \textquoteleft$pr_{i}(z_i^j) \leq pr_{i}(z_i^{j'})$\textquoteright with \textquoteleft$pr_{i}(z_i^j) < pr_{i}(z_i^{j'})$\textquoteright.

As subspaces of $(\mathbb{R}^d)^{|Q|}$, we then have the following:

- There is a linear subspace of $(\mathbb{R}^d)^{|Q|}$ determined by $pr_{t}(z_i^j) = pr_{t}(z_i^{j'})$ for all $1 \leq i \leq r$ and $1 \leq j, j' \leq k_i$, and inside this linear subspace, $\tilde{V}$ is open, defined by the strict inequalities $z_i^j \neq z_i^{j'}$ for each $(i, j) \neq (i', j') \in T_k$;

- There is a linear subspace of $(\mathbb{R}^d)^{|Q|}$ determined by the linear equations
  - $pr_{t}(z_i^j) = pr_{t}(z_i^{j'})$ for all $1 \leq i \leq r$ and $1 \leq j, j' \leq k_i$;
  - $pr_{\zeta}(z_i^j) = pr_{\zeta}(z_i^{j'})$ for all $1 \leq \beta \leq l(Q)$ and $(i, j), (i', j') \in Q_{\beta}$;

inside this linear subspace, $H_Q$ is open, defined by the strict inequalities
  - $z_i^j \neq z_i^{j'}$ for $(i, j) \neq (i', j') \in T_k$;
  - $pr_{t}(z_i^j) < pr_{t}(z_i^{j'})$ for all $1 \leq \beta \leq l(Q)$ and $(i, j) \prec_{\beta} (i', j') \in Q_{\beta}$.

It follows that $H_Q$ is an orientable submanifold without boundary of $\tilde{V}$, which in turn is an orientable submanifold without boundary of $(\mathbb{R}^d)^{|Q|}$: here we are using the simple observation that the intersection inside a real vector space of an open subset and a linear subspace is an orientable submanifold without boundary. The dimension of $H_Q$ is computed by noting that, locally, we have the following parameters describing a configuration $Z \in H_Q$:

- For all $1 \leq \beta \leq l$, we have a parameter $\zeta_{\beta} = (\zeta_{\beta}^1, \zeta_{\beta}^2) \in \mathbb{R}^p \times \mathbb{R}^{q-1}$ which corresponds to the (unique) value attained by $pr_{\zeta}(z_i^j)$ for all $(i, j) \in Q_{\beta}$.

However, if two rays $Q_{\beta}$ and $Q_{\beta'}$ share a cluster, their further projections $\zeta_{1}^\beta$ and $\zeta_{2}^\beta$ have to coincide inside $\mathbb{R}^p$. Hence, we get for each equivalence class of rays a choice in $\mathbb{R}^p$, and for each ray a choice in $\mathbb{R}^{q-1}$. This yields $p \cdot a(Q) + (q - 1) \cdot l(Q)$ parameters in $\mathbb{R}$.

- For each $(i, j) \in T_k$ we have a parameter $t_i^j = pr_{t}(z_i^j)$ in $\mathbb{R}$.

We clearly have $W_Q \subseteq H_Q$, and $W_Q$ can be characterised as the subspace of $H_Q$ containing configurations $Z$ for which the following condition holds: for all $1 \leq \beta < \beta' \leq l$ and for all $(i, j) \in Q_{\beta}$ and $(i', j') \in Q_{\beta'}$, either $pr_{\zeta}(z_i^j) \neq pr_{\zeta}(z_i^{j'})$ or $pr_{i}(z_i^j) \neq pr_{i}(z_i^{j'})$. Thus, $W_Q \subseteq H_Q$ is an open subspace (it is a finite intersection of open subspaces). To see that $W_Q$ is dense in $H_Q$, note that one can slightly perturb all parameters of any configuration $Z \in H_Q$ of the forms

- $\zeta_{\beta}^2 \in \mathbb{R}^{q-1}$, for $1 \leq \beta \leq l(Q)$;

- $\zeta_{\beta}^1 \in \mathbb{R}^p$, for $\beta$ ranging in a set of representatives of the $a(Q)$ equivalence classes of rays,
to ensure that they attain all different values, so that the new perturbed configuration $Z'$ lies in $W_Q$. Hence, $H_Q$ is the closure $\overline{W_Q}$ inside $V$.

To prove that $W_Q$ is contractible, we choose distinct numbers $t_1, \ldots, t_k \in \mathbb{R}$ such that for $1 \leq \beta, \beta' \leq l$ and $(i, j) \in Q_\beta$ and $(i', j') \in Q_{\beta'}$, we have $t_{\beta'} < t_{\beta}$ if and only if $\beta' > \beta$ or $\beta' = \beta$ and $(i', j') \prec_\beta (i, j)$, i.e. the $t_i$ are ordered exactly as the stacked order on $Q_1 \sqcup \cdots \sqcup Q_1$ from Definition 3.6 prescribes.

We define $\bar{z}_i := (0, \bar{t}_i) \in \mathbb{R}^d$ for all $(i, j) \in T_K$; note that the configuration $\tilde{Z} = (\bar{z}_1, \ldots, \bar{z}_k)$ lies in $W_Q$. We can connect any configuration $Z \in W_Q$ to $\tilde{Z}$ by *linear interpolation* inside $(\mathbb{R}^d)^{|K|}$, as shown in Figure 5: for all $0 \leq s \leq 1$ we consider the configuration $s \cdot Z + (1 - s) \cdot \tilde{Z}$, where we set

$$(s \cdot Z + (1 - s) \cdot \tilde{Z})_i := s \cdot z_i + (1 - s) \cdot \bar{z}_i \in \mathbb{R}^d.$$ 

Since $pr_t$ is a linear map, for all $0 \leq s \leq 1$ we have that, for fixed $1 \leq \beta \leq l$, the map $pr_t$ attains the same value on all points of the form $s \cdot z_i + (1 - s) \cdot \bar{z}_i$, for $(i, j)$ ranging in $Q_\beta$; similarly, for all $(i, j) \prec_\beta (i', j') \in Q_{\beta'}$ we have an inequality $pr_t((s \cdot Z + (1 - s) \cdot \tilde{Z})_i) < pr_t((s \cdot Z + (1 - s) \cdot \tilde{Z})_{i'})$. This means that the linear interpolation takes place inside the subspace $\overline{W_Q}$.

With some more effort one can show that the linear interpolation takes place inside $W_Q$ by using the characterisation of the points of $W_Q$ inside $\overline{W_Q}$: for all $0 < s \leq 1$, $1 \leq \beta < \beta' \leq l$, $(i, j) \in Q_\beta$, and $(i', j') \in Q_{\beta'}$, if $pr_t(z_i) \neq pr_t(z_{i'})$ then also we have the inequality

$$pr_t\left((s \cdot Z + (1 - s) \cdot \tilde{Z})_i\right) \neq pr_t\left((s \cdot Z + (1 - s) \cdot \tilde{Z})_{i'}\right),$$

and if $pr_t(z_{i'}) < pr_t(z_i)$ then also

$$pr_t\left((s \cdot Z + (1 - s) \cdot \tilde{Z})_i\right) < pr_t\left((s \cdot Z + (1 - s) \cdot \tilde{Z})_{i'}\right),$$

by the same argument used above. At time $s = 1$, we already know that $z$ lies in $W_Q$. Thus, we have exhibited a contraction of $W_Q$ onto its point $\tilde{Z}$.

In particular we have shown that $W_Q$ is a connected component of $\mathfrak{F}_v$. All connected components arise in this way, since every point $Z$ in $\mathfrak{F}_v$ belongs to the subspace $W_Qz$, with $\chi(v) = \omega(Q^z)$, see Definition 3.11.
3.4 Growth of Betti numbers and cup product indecomposables

We will not attempt to give a precise description of $H^*(\tilde V_K(\mathbb{R}^{p,q}))$ as a ring. The aim of this short subsection is to disprove a natural, yet na"{i}ve conjecture on multiplicative generators of $H^*(\tilde V_K(\mathbb{R}^{p,q}))$.

**Notation 3.15.** For all $1 \leq i < j \leq r$ there is a Fadell–Neuwirth map of the form $\text{pr}_{i,j}: \tilde V_K(\mathbb{R}^{p,q}) \to \tilde V_{(K,k)}(\mathbb{R}^{p,q})$ which forgets all clusters but the $i^{th}$ and $j^{th}$ ones. The map $\text{pr}_{i,j}$ is in general not a fibration, though it is a fibration in the quite special case in which $k_l = 1$ for all $l \neq i,j$.

In the case $k_1,\ldots,k_r = 1$, the space $\tilde V_K(\mathbb{R}^{p,q})$ is homeomorphic to the classical ordered configuration space $\tilde C_r(\mathbb{R}^d)$, and the maps $\text{pr}_{i,j}$ reduce to a version of the classical Fadell–Neuwirth fibrations $\text{pr}_{i,j}: \tilde C_r(\mathbb{R}^d) \to \tilde C_2(\mathbb{R}^d)$. Denote by $\vartheta \in H^{d-1}(\tilde C_2(\mathbb{R}^d)) \cong \mathbb{Z}$ a generator, and let $\vartheta_{i,j} := \text{pr}_{i,j}^*\vartheta \in H^{d-1}(\tilde C_r(\mathbb{R}^d))$ be the pulled back cohomology class. It is then a classical result by Arnol’d [Arn69] that the classes $\vartheta_{i,j}$ generate $H^*(\tilde C_r(\mathbb{R}^d))$ as a ring. A natural conjecture would then be the following:

**Conjecture 3.16** (Na"{i}ve conjecture). The ring $H^*(\tilde V_K(\mathbb{R}^{p,q}))$ is generated in arity 2, i.e. by all cohomology classes that can be obtained as a pullback along $\text{pr}_{i,j}$, for some $1 \leq i < j \leq r$, from a cohomology class in $H^*(\tilde V_{(k,k)}(\mathbb{R}^{p,q}))$.

The following example shows that Conjecture 3.16 is wrong in general.

**Example 3.17.** Consider the case $r = 3, k \geq 2$, and $p \geq 1$ and select the component of $\tilde V_{3}^{K}(\mathbb{R}^{p,1}) = \tilde V_{(k,k,k)}(\mathbb{R}^{p,1})$ corresponding to $\text{id} := (\text{id}, \text{id}, \text{id}) \in (\mathbb{S}_k)^3$.

Via the stacked total order from Definition 3.6, a ray partition $Q$ of type $K$ with $\Sigma(Q) = \text{id}$ is the same as a shuffle of the columns of $T_K$, i.e. a total order $\prec$ on $T_K$ which preserves the ordering of each column: for the inverse construction, given such a shuffle $\prec$, we let $Q_1$ be the subset of $T_K$ containing $(1,1)$ and all $\prec$-larger elements, and $Q_\beta$ be the subset of $T_K \smallsetminus (Q_1 \sqcup \cdots \sqcup Q_{\beta-1})$ containing the $\prec$-minimal element and all $\prec$-larger ones.

By Theorem 3.7, $H^*(\tilde V_{3}^{K}(\mathbb{R}^{p,1})_{\text{id}})$ is concentrated in degrees 0, $p$ and $2p$, with Betti numbers equal, respectively, to the following:

- 1 in degree 0. There is indeed a unique ray partition of agility 3, having three rays containing each one cluster.
- $3 \cdot \binom{2k}{k} - 1$ in degree $p$. To count ray partitions of agility 2, we first choose which of the three clusters forms on its own an equivalence class according to Notation 3.2. Without loss of generality, we assume to have selected the third cluster to stay on its own. The other two clusters can either form a single ray with minimum $(1,1)$, for which there are $\binom{2k-1}{k-1}$ possibilities, or they are divided into two rays which are equivalent: this second case corresponds to a shuffle of the columns
\{ (1,1), \ldots, (1,k) \} and \{ (2,1), \ldots, (2,k) \}, beginning with (2,1) and different from the shuffle (2,1) \prec \cdots \prec (2,k) \prec (1,1) \prec \cdots \prec (1,k); there are \( (2^k - 1) - 1 \) possibilities for such a shuffle.

- \( \left( \frac{3^k}{k} \cdot \left( \frac{2^k}{k} \right) - 3 \cdot \left( \frac{2^k}{k} \right) \right) + 2 \) in degree 2p. There are precisely \( \left( \frac{3^k}{k} \cdot \left( \frac{2^k}{k} \right) \right) \) shuffles of the three columns, and \( 3 \cdot \left( \frac{2^k}{k} \right) - 2 \) of these shuffles correspond to ray partitions already considered before.

Similarly, \( H^*(\tilde{V}_2^k(\mathbb{R}^{p,q})_{id}) \) is concentrated in degrees 0 and \( p \), with Betti numbers equal, respectively, to 1 and \( \left( \frac{2^k}{k} \right) - 1 \). The projections \( pr_{1,2}, pr_{1,3}, pr_{2,3} \) exhibit an isomorphism between \( H^p(\tilde{V}_3^k(\mathbb{R}^{p,q})_{id}) \) and \( H^p(\tilde{V}_2^k(\mathbb{R}^{p,q})_{id})^{\otimes 3} \).

If Conjecture 3.16 were true, the set of all cup products of pairs of classes in \( H^p(\tilde{V}_3^k(\mathbb{R}^{p,q})_{id}) \) would suffice to generate the entire cohomology group \( H^{2p}(\tilde{V}_2^k(\mathbb{R}^{p,q})_{id}) \); in particular we would have

\[ \left( \frac{3^k}{k} \cdot \left( \frac{2^k}{k} \right) - 3 \cdot \left( \frac{2^k}{k} \right) \right) + 2 \leq \left( 3 \cdot \left( \frac{2^k}{k} \right) - 1 \right)^2. \]

However, using Stirling’s approximation, the left hand side grows as fast as \( 27^k \cdot \frac{\sqrt{3}}{2\pi k} \) for \( k \to \infty \), whereas the right hand side grows as fast as \( 9 \cdot 16^k \cdot \frac{1}{\sqrt{k}} \). Hence for large \( k \) the inequality does not hold; in particular the graded ring \( H^*(\tilde{V}_3^k(\mathbb{R}^{p,q})_{id}) \) has non-trivial indecomposable elements in degree 2p.

The example generalises for fixed \( r \geq 4 \) to show that the rank of the indecomposables of \( H^*(\tilde{V}_r^k(\mathbb{R}^{p,q})_{id}) \) in degree \( (r-1) \cdot p \) grows as fast as \( r^{c \cdot k} \).

We do not expect that the situation becomes better when considering more than three clusters, or taking a value of \( q \) higher than 1.

### 4 Homological stability

In this section we will prove homological stability for the unordered configuration spaces \( V^k_r(\mathbb{R}^{p,q}) \) of vertical clusters of size \( k \) for all values of \( p \) and \( q \) except for the one pair where it obviously does not hold. This extends results by [Lat17], [TP14] and [Pal21].

#### 4.1 Setting and results

We fix throughout the section a cluster size \( k \geq 1 \) and we will abbreviate \( \tilde{V}_r(\mathbb{R}^{p,q}) := \tilde{V}_r^k(\mathbb{R}^{p,q}) \) and \( V_r(\mathbb{R}^{p,q}) := V_r^k(\mathbb{R}^{p,q}) \). If \( p \) and \( q \) are fixed and clear from the context, we may also just write \( \tilde{V}_r \) resp. \( V_r \).

**Construction 4.1.** For each \( r \geq 0, p \geq 0, \) and \( q \geq 1 \), we have **stabilisation maps**

\[
\text{stab}: V_r^k(\mathbb{R}^{p,q}) \to V_{r+1}^k(\mathbb{R}^{p,q})
\]

by adding an extra cluster on the ‘far right’ with respect to the first coordinate of \( \mathbb{R}^{p+q} \), as depicted in Figure 6.
Figure 6. The stabilisation map \( \text{stab} : V_2(R^{1,1}) \to V_6(R^{1,1}) \), which adds a new cluster on the far right.

**Proposition 4.2.** The induced maps in homology

\[ \text{stab}_*: H_m(V_r(R^{p,q})) \to H_m(V_{r+1}(R^{p,q})). \]

are split monic for each \( m, r, p \geq 0 \) and \( q \geq 1 \).

**Proof.** This proof generalises the one from [Pal21, Lem. 5.1] which uses the classical idea of a ‘power-set map’: from [Dol62], we want to use the following

**Lemma 2.** Suppose we are given a sequence \( (0 = A_0 \xrightarrow{\tau_0} A_1 \xrightarrow{\tau_1} \cdots) \) of abelian groups, and assume that there are \( \tau_{i,r} : A_r \to A_j \) for \( 1 \leq j \leq r \) such that \( \tau_{r,r} = \text{id} \) and \( \tau_{i,r} \circ s_r : A_r \to A_r \) lies in the image of \( s_j \). Then every \( s_r \) is split monic.

In order to do so, we first note that \( V_0 = \ast \), so all spaces \( V_r = V_r(R^{p,q}) \) are canonically based by \( \text{stab}_r(\ast) \in V_r \), and the stabilisation maps are basepoint-preserving by definition. For a fixed \( m \geq 0 \), let \( A_r := \tilde{H}_m(V_r) \), so we have maps \( s_r : A_r \to A_{r+1} \) induced by the stabilisation.

Now recall for \( \ell \geq 0 \) the \( \ell \)-fold symmetric product \( \text{SP}^\ell V_r := (V_r)^\ell / \Sigma_\ell \) where \( \Sigma_\ell \) acts by coordinate permutation. We will denote elements of \( \text{SP}^\ell V_r \) as formal sums of elements of \( V_r \), but we will use a sign \( + \) resp. \( \sum \) in order to distinguish the notation for the symmetric product from Notation 2.2. For the binomial coefficient \( \ell := \binom{r}{j} \), consider the maps

\[ \gamma_{j,r} : V_r \to \text{SP}^\ell V_j, \quad \sum_{i=1}^r [z_i] \mapsto \sum_{S \subseteq \{1, \ldots, r\}} \sum_{\# S = j} [z_i]. \]

A priori, \( \gamma_{j,r} \) is not based, but it can be homotoped to a based map since \( V_r \) is well-based. Then \( \gamma_{r,r} = \text{id} \) and we have a homotopy

\[ \gamma_{j,r+1} \circ \text{stab}_r \simeq \gamma_{j,r} + \text{SP}^{\binom{r}{j-1}}(\text{stab}_{j-1}) \circ \gamma_{j-1,r}. \]

of maps \( V_r \to \text{SP}^{\binom{r+1}{j}} V_j \). Applying the functor \( \pi_m \circ \text{SP}^\infty \simeq \tilde{H}_m \) and using the ‘flattening’ map \( \varphi_\ell : \text{SP}^\infty \text{SP}^\ell V_r \to \text{SP}^\infty V_r \), we obtain the desired system \( (\tau_{j,r}) \) of
morphisms for Dold’s lemma by

\[ A_r \xrightarrow{\tau_j} A_i \]

\[ \pi_m(SP^\infty V_r) \xrightarrow{\pi_m(SP^\infty \gamma_r)} \pi_m(SP^\infty V_j) \xrightarrow{\pi_m(\varphi_i)} \pi_m(SP^\infty V_i). \]

In contrast, surjectivity of \( \text{stab}^*: H_m(V^k_r(\mathbb{R}^{p,q})) \rightarrow H_m(V^k_{r+1}(\mathbb{R}^{p,q})) \) holds only in a certain range. The rest of this section is devoted to the proof of the following stability theorem:

**Theorem 4.3.** For all \( p \geq 0 \) and \( q \geq 1 \) with \( (p,q) \neq (0,1) \), the induced maps

\[ H_m(V^k_r(\mathbb{R}^{p,q})) \rightarrow H_m(V^k_{r+1}(\mathbb{R}^{p,q})). \]

are isomorphisms for \( m \leq \frac{r}{2} \).

Many cases of Theorem 4.3 have already been solved:

- We know that \( \pi_0 V_r(\mathbb{R}^{0,1}) \cong \mathbb{G}_k \cap \mathbb{G}_r \), so there is no stability result to be expected in the case \( p = 0 \) and \( q = 1 \).

- For \( p = 0 \), we are in the case without any vertical coupling condition. This can alternatively be described by embeddings of (disconnected) 0-dimensional manifolds into \( \mathbb{R}^q \). For these cases, the theorem was proven for \( q \geq 3 \) in [Pal21] and for \( q = 2 \) in [TP14].

- In [Lat17], the case \( p + q \geq 3 \) was considered and proven. Actually, Latifi writes down the proof only for \( p = 2 \) and \( q = 1 \), but her strategy works whenever \( p + q \geq 3 \).

Hence we only have to prove the single remaining case \( (p,q) = (1,1) \). However, since the method is the same, we will provide a proof for arbitrary \( (p,1) \) with \( p \geq 1 \). Our proof uses different methods than Latifi’s proof.

### 4.2 The dexterity filtration

**Notation 4.4.** In the remainder of the section we assume \( q = 1 \). In this case, a vertical cluster \( [z] = \{z^1,\ldots,z^k\} \subseteq \mathbb{R}^{p+1} \) is canonically ordered by the last coordinate \( t^\ell := \text{pr}_{\xi}(z^\ell) \in \mathbb{R} \), and \( [z] \) is determined by their common projection \( \zeta := \text{pr}_{\xi}(z) \in \mathbb{R}^k \) and the real numbers \( t^1,\ldots,t^k \). Hence, we can write

\[ \{z^1,\ldots,z^k\} = (\zeta; t^1 < \cdots < t^k). \]

**Definition 4.5.** Let \( Z := (z_1,\ldots,z_r) \in \check{V}_r \) be an ordered configuration, where \( z_i = (z^1_i,\ldots,z^k_i) \). We define an equivalence relation \( \sim_Z \) on the set \( \{1,\ldots,r\} \).

First, set \( i \sim_Z i' \) whenever the two following conditions hold:
• $z_i$ and $z_i'$ are aligned, i.e. they are contained in the same $t$-line, or equivalently $\text{pr}_t(z_i) = \text{pr}_t(z_i')$ in $\mathbb{R}^p$, since $q = 1$;

• $z_i$ and $z_i'$ are entangled, i.e. their convex hulls (contained in the vertical line) intersect each other, see Figure 7.

Let $\sim_Z$ be the equivalence relation generated by the above basic relations $\sim_Z$. We define the dexterity of $Z$, denoted $\delta(Z)$, to be the number $s$ of equivalence classes of $\sim_Z$. Since the notion of dexterity is invariant under the permutation action of the group $\mathfrak{S}_k \wr \mathfrak{S}_r$, we obtain a notion of dexterity also for unordered configurations in $V_r$.

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Figure 7. The leftmost two upper clusters are entangled and hence form an equivalence class. Therefore, the dexterity is 5, while the number of clusters is 6.

\textbf{Definition 4.6 (Dexterity filtration).} For $s \geq -1$ we let $F_s V_r$ be the subspace of all $[Z] \in V_r$ satisfying $\delta[Z] \geq r - s$. We have inclusions

$$\emptyset = F_{-1} V_r \subseteq F_0 V_r \subseteq \cdots \subseteq F_{r-1} V_r = V_r.$$ We denote by $\mathcal{F}_s V_r$ the $s^{th}$ stratum of the filtration

$$\mathcal{F}_s V_r := F_s V_r \setminus F_{s-1} V_r \subset V_r.$$

Note that each filtration level $F_s V_r$ is an open subspace of $V_r$, in particular it is a manifold of the same dimension $p \cdot r + r \cdot k$; the stratum $\mathcal{F}_s V_r$ is a closed subset of $F_s V_r$.

Additionally, the stabilisation map $\text{stab}: V_r \to V_{r+1}$ is filtration-preserving, i.e. it restricts to maps $F_s V_r \to F_s V_{r+1}$ and even to maps of strata $\mathcal{F}_s V_r \to \mathcal{F}_s V_{r+1}$.

\textbf{Lemma 4.7.} The stratum $\mathcal{F}_s V_r \subset F_s V_r$ is a closed submanifold of codimension $s \cdot p$, i.e. of dimension $p \cdot (r - s) + r \cdot k$.

\textbf{Proof.} Let $[\hat{Z}] = \sum_i [\hat{z}_i] \in \mathcal{F}_s V_r$ with $[\hat{z}_i] = (\hat{\xi}_i; \hat{r}_1^i, \ldots, \hat{r}_k^i)$, where $\hat{\xi}_i \in \mathbb{R}^p$ and $\hat{r}_1^i < \cdots < \hat{r}_k^i \in \mathbb{R}$. A small neighbourhood of $[\hat{Z}]$ in $V_r$ is described by the following local parameters constituting $[Z] = \sum_i (\xi_i; r_1^i, \ldots, r_k^i)$:

• $\xi_i$, ranging in a neighbourhood of $\hat{\xi}_i \in \mathbb{R}^p$, for all $1 \leq i \leq r$;

• $r_j^i$, ranging in a neighbourhood of $\hat{r}_j^i \in \mathbb{R}$, for all $1 \leq i \leq r$ and $1 \leq j \leq k$. 

\text{18}
For \([Z]\) in a neighbourhood of \([\tilde{Z}]\), the condition \([Z] \in \mathcal{G}_s V_r\) is, up to a permutation of clusters of \(Z\), equivalent to the equality \(\tilde{\zeta}_i = \zeta_i\) whenever \(i \sim 2 i'\).

Let \(1 \leq i_1 < \cdots < i_{r-s} \leq r\) be the unique representatives of the \(r-s\) equivalence classes of the relation \(\sim\), satisfying for each \(1 \leq c \leq r-s\) and each \(i_1 \sim 2 i_c\) the inequality \(t_i^c \geq \tilde{t}_i^c\). Thus, a neighbourhood of \([\tilde{Z}]\) in \(\mathcal{G}\) is described by the following local parameters:

- \(\zeta_{i_1}\) for all \(1 \leq c \leq r-s\);
- \(t_i^c\) for all \(1 \leq i \leq r\) and \(1 \leq j \leq k\).

\(\square\)

**Remark 4.8.** The argument in the previous proof can be pushed a bit further to describe the normal cotangent bundle \(N^*(\mathcal{G}_s V_r, F_s V_r)\) of \(\mathcal{G}_s V_r\) in \(F_s V_r\). Recall that, for fixed \([\tilde{Z}] \in \mathcal{G}_s V_r\), the normal cotangent space \(N^*[\tilde{Z}](\mathcal{G}_s V_r, F_s V_r)\) is the subspace of the cotangent space \(T^*[\tilde{Z}] \mathcal{G}_s V_r\) of all linear functionals that vanish on the vector subspace \(T^*[\tilde{Z}] \mathcal{G}_s V_r\). The previous proof shows that \(N^*[\tilde{Z}](\mathcal{G}_s V_r, F_s V_r)\) is 'spanned' by the parameters \(\zeta_i\) for \(i \in \{1, \ldots, r\} \setminus \{i_1, \ldots, i_{r-s}\}\). By this we formally mean the following:

- for each index \(i \in \{1, \ldots, r\} \setminus \{i_1, \ldots, i_{r-s}\}\) we consider the list of \(p\) linear functionals \(d\zeta_{i_1}^1, \ldots, d\zeta_{i_1}^p\), where \(\zeta_{i_1}^1, \ldots, \zeta_{i_1}^p \in \mathbb{R}\) are the \(p\) coordinates of the parameter \(\zeta_{i_1}\) which takes values in \(\mathbb{R}^p\);
- a basis for \(N^*[\tilde{Z}](\mathcal{G}_s V_r, F_s V_r)\) is given by the linear functionals \(d\zeta_{i_1}^j\) where \(i\) ranges in \(\{1, \ldots, r\} \setminus \{i_1, \ldots, i_{r-s}\}\) and \(1 \leq j \leq p\).

### 4.3 Coloured configuration spaces

**Notation 4.9 (Distributions).** Let \(E\) be an index set. A distribution is a map \(\alpha : E \to \mathbb{N}\) with finite support. We write \(\alpha(e) := \alpha(e)\) and \(\alpha = \sum e \alpha(e)\).

In particular, for a fixed \(e_0 \in E\), we denote by \(\alpha + e_0\) the distribution which coincides with \(\alpha\), except for the fact that it increases \(\alpha(e)\) by \(1\).

**Definition 4.10 (Coloured labelled configuration spaces).** Let \(E\) be a set and \(\alpha : E \to \mathbb{N}\) a distribution. Define \(|\alpha| := \sum e \alpha(e)\) and \(\mathcal{G}(\alpha) := \prod \alpha(e) \subseteq \mathcal{G}_{|\alpha|}\). Moreover, let \(X := (X_e)_{e \in E}\) be a family of spaces. Then we define

\[
C_\alpha(\mathbb{R}^{p+1}; X) := C_{|\alpha|}(\mathbb{R}^{p+1}) \times \mathcal{G}(\alpha) \prod e \in E X_e^{\alpha(e)}.
\]

In case \(X_e = *\) for all \(e\), we just write \(C_\alpha(\mathbb{R}^{p+1}) = C_{|\alpha|}(\mathbb{R}^{p+1}) / \mathcal{G}(\alpha)\).

Informally, we consider \(C_\alpha(\mathbb{R}^{p+1}; X)\) as the space of unordered configurations of \(|\alpha|\) unordered points, each equipped with a label in \(\bigcup X_e\), such that for all \(e \in E\), there are precisely \(\alpha(e)\) points carrying a label in \(X_e\).

**Notation 4.11.** For unordered labelled configurations as before, we will use again the suggestive 'sum notation': For distinct points \(y_1, \ldots, y_{|\alpha|} \in \mathbb{R}^{p+1}\) and
labels \( x_1, \ldots, x_{|a|} \in X \), we will denote the unordered labelled configuration \( \{y_1, \ldots, y_n\} \subseteq \mathbb{R}^{p+1} \) in which the point \( y_i \) carries the label \( x_i \) by

\[
\Theta := \sum_{i=1}^{|a|} y_i \otimes x_i \in C_a(\mathbb{R}^{p+1}; X).
\]

**Definition 4.12.** For \( w \geq 1 \), an unordered partition of \( \{1, \ldots, w \cdot k\} \) into subsets \( S_1, \ldots, S_w \) of size \( k \) is irreducible if there is no \( 1 \leq i \leq w - 1 \) for which the subset \( \{1, \ldots, i \cdot k\} \) is a union of some pieces \( S_b \) of the partition, see Figure 8. We denote by \( E_w \) the set of all irreducible, unordered partitions of \( \{1, \ldots, w \cdot k\} \).

**Notation 4.13.** We denote \( E := \bigsqcup_{w \geq 1} E_w \) the union of all \( E_w \). For \( e \in E_w \subset E \), we will write \( w(e) := w \geq 1 \) for the weight of \( e \). Note that there is precisely one partition \( e_0 \in E \) with \( w(e_0) = 1 \). For all \( e \in \{S_1, \ldots, S_w\} \in E_w \), we use the convention that \( \min(S_b) < \min(S_{b'}) \) for \( 1 \leq b < b' \leq w(e) \).

**Figure 8.** We can picture partitions of \( \{1, \ldots, w \cdot k\} \) as in this figure. Here we see a reducible partition (left) and an irreducible one (right) with \( k = 2 \) and \( w = 3 \).

**Remark 4.14.** The notion of irreducibility is related to the dexterity filtration: given an irreducible partition \( e = (S_1, \ldots, S_w) \) with \( S_b = \{h_b^1 < \cdots < h_b^i\} \) and \( \zeta_1, \ldots, \zeta_w \in \mathbb{R}^p \), consider the configuration \( [Z] := \sum_{b=1}^w [z_b^1, \ldots, z_b^i] \in V_w \) with \( z_b^i = (\zeta_b, h_b^i) \). Then \( [Z] \) has dexterity 1, see Definition 4.5, if and only if \( \zeta_1 = \cdots = \zeta_w \) holds. This is used in the following, for the special case \( \zeta_1 = 0 \).

**Construction 4.15.** For each \( e \in E \) we denote by \( D^e \) the product \((D^p)^{w(e)-1}\), where \( D^p \subset \mathbb{R}^p \) denotes the standard, Euclidean unit disc; note that \( D^e \) is, topologically, also a disc. We thus obtain a family of discs \( D := (D^e)_{e \in E} \), and regard each \( D^e \) as a subset of \( \mathbb{R}^{p \cdot (w(e)-1)} \). For reasons that will become clearer later, we denote by \( z_2, \ldots, z_{w(e)} \) the \( w(e) - 1 \) parameters of \( D^e \), each taking values in \( D^p \); each parameter \( z_b \) consists of \( p \) coordinates \( z_{b1}, \ldots, z_{bp} \in \mathbb{R} \).

For a distribution \( \alpha : \mathbb{R} \rightarrow \mathbb{N} \), we define the degree \( \deg(\alpha) \) as the pair of non-negative numbers

\[
\deg(\alpha) = (r(\alpha), s(\alpha)) := \left( \sum_e a_e \cdot w(e), \sum_e a_e \cdot (w(e) - 1) \right).
\]

We let \( C_{r,s} := C_{r,s}(\mathbb{R}^{p+1}; D) \) be the union of all spaces \( C_a(\mathbb{R}^{p+1}; D) \), where \( a \) ranges among distributions \( \alpha : \mathbb{R} \rightarrow \mathbb{N} \) with \( \deg(\alpha) = (r, s) \).

Each disc \( D^e \) contains a centre \( 0^e \in D^e \), and we denote by \( 0 := (0^e)_{e \in E} \) the family of centres; we obtain an inclusion \( C_a(\mathbb{R}^{p+1}; 0) \subseteq C_a(\mathbb{R}^{p+1}; D) \) which
is a closed embedding of a submanifold of codimension \( s(\alpha) \cdot p \) for each distribution \( \alpha : \mathbb{E} \to \mathbb{N} \). We write \( C^*_s(\mathbb{R}^{p+1}; D) := C_\alpha(\mathbb{R}^{p+1}; D) \setminus C_\alpha(\mathbb{R}^{p+1}; 0) \).

Moreover, we define \( C^0_{r,s} \subseteq C_{r,s} \) to be the union of all \( C_\alpha(\mathbb{R}^{p+1}; 0) \) with \( \deg(\alpha) = (r,s) \) and define its complement \( C^*_r := C_{r,s} \setminus C^0_{r,s} \).

**Remark 4.16.** A generic point in \( C_{r,s} \) is of the form

\[
\Theta = \sum_{i=1}^{r-s} y_i \otimes (e_i, \xi_i),
\]

where the points \( y_1, \ldots, y_{r-s} \in \mathbb{R}^{p+1} \) are distinct, \( e_i \in \mathbb{E} \), and \( \xi_i \in D_{e_i} \) is expanded as \( \xi_i = (\xi_{i,2}, \ldots, \xi_{i,n(e_i)}) \) with \( \xi_{i,b} \in D^p \). Consider a distribution \( \alpha \) of degree \( \deg(\alpha) = (r,s) \), and let \( \Theta := \sum_{i=1}^{r-s} y_i \otimes (e_i, 0) \in C_\alpha(\mathbb{R}^{p+1}; 0) \subseteq C^0_{r,s} \).

In order to describe a small neighbourhood of \( \Theta \) in \( C^0_{r,s} \), we can use the local parameters \( y_i \in \mathbb{R}^{p+1} \), each ranging in a small neighbourhood of \( \hat{y}_i \); to describe a small neighbourhood of \( \Theta \) in \( C_{r,s} \) we can additionally use the parameters \( \xi_i = (\xi_{i,2}, \ldots, \xi_{i,n(e_i)}) \in D_{e_i} \), each ranging in a small neighbourhood of \( 0_{\xi_i} \). It follows that the \( N^*_\Theta(C^0_{r,s}, C_{r,s}) \) is ‘spanned by the parameters \( \xi_i \) for \( 1 \leq l \leq r-s \).

By this we mean the following:

- for each \( 1 \leq l \leq r-s \) we consider the list of \( p \cdot (w(e_l) - 1) \) linear functionals \( d_{\xi_{l,b}}^\mathbb{T} \), for \( 2 \leq b \leq w(e_l) \) and \( 1 \leq \tau \leq p \). Here \( \xi_{l,b}^\mathbb{T} \), for \( 2 \leq b \leq w(e_l) \) and \( 1 \leq \tau \leq p \), are the \( p \cdot (w(e_l) - 1) \) coordinates of the parameter \( \xi_{l,b} \), which takes values in \( D_{e_l} \subseteq \mathbb{R}^{p-\{w(e_l)\}-1} \);

- a basis for \( N^*_\Theta(C^0_{r,s}, C_{r,s}) \) is given by all linear functionals \( d_{\xi_{l,b}}^\mathbb{T} \), indexed by \( 1 \leq l \leq r-s, 2 \leq b \leq w(e_l), \) and \( 1 \leq \tau \leq p \).

There are stabilisation maps \( C_{r,s} \to C_{r+1,s} \) given by placing a new point with label in \( D_{e_0} = * \) on the ‘right’ with respect to the first coordinate of \( \mathbb{R}^{p+1} \). The stabilisation increases the parameter \( r \) by 1, but leaves \( s \) constant. Moreover, \( C^0_{r,s} \) is sent to \( C^0_{r+1,s} \) under the stabilisation map, and \( C^*_{r,s} \) is sent to \( C^*_{r+1,s} \).

**4.4 The insertion map**

We will now connect the filtration pairs \( (F_sV_r, F_{s-1}V_r) \) to the pairs \( (C_{r,s}, C^*_{r,s}) \) of labelled configurations via an ‘insertion map’.

**Construction 4.17.** For each \( 1 \leq s \leq r \), we have a map of pairs

\[
\varphi_{r,s} : (C_{r,s}, C^*_{r,s}) \to (F_sV_r, F_{s-1}V_r),
\]

which pictorially does the following, see Figure 9: given a labelled configuration in \( C_{r,s} \), we draw pairwise disjoint cylinders around each point in \( \mathbb{R}^{p+1} \), and place inside each of them a small ‘standard configuration’ which corresponds to the given indecomposable partition, and is ‘perturbed’ by the \( w(e) - 1 \) disc parameters from the label, where in each cylinder, one cluster stays in the centre.
By Remark 4.14, the dexterity of the resulting vertical configuration is \( r - s \) if and only if all clusters inside each cylinder stay in the centre, i.e. if all disc parameters are 0. Thus, if the labelled configuration to start with lies in \( C_{r,s} \), then the dexterity is at least \( r - (s - 1) \), whence we land in the filtration component \( F_{s-1} V_r \). Formally, the map \( \varphi_{r,s} \) is constructed as follows:

- For each \( w \geq 1 \), each subset \( S \subseteq \{1, \ldots, w \cdot k\} \) of cardinality \( k \), and each \( \xi \in D^p \), we define the unordered ‘standard cluster’
  \[
  T_S(\xi) := (\xi, (-1 + \frac{2}{\kappa w+1} \cdot h)_{h \in S}).
  \]
  Pictorially, \( T_S(\xi) \) is the unordered vertical cluster of \( k \) points which projects to \( \xi \in D^p \) and whose \( t \)-coordinate takes the values corresponding to \( S \), among all values arising from a uniform distribution of \( w \cdot k \) points in the interior of the interval \([-1;1]\).

- For a partition \( e \in \mathcal{E} \) write \( S_1, \ldots, S_{w(e)} \subseteq \{1, \ldots, w \cdot k\} \) for the partition components; since \( e \) is an unordered partition, we assume without loss of generality that \( \min(S_1) < \min(S_{i+1}) \). For all \( \xi_2, \ldots, \xi_{w(e)} \in D^p \), we set \( \xi_1 := 0 \in D^p \) and define
  \[
  T_e(\xi_2, \ldots, \xi_{w(e)}) := \sum_{b=1}^{w(e)} T_{S_b}(\xi_b) \in V_{w(e)}.
  \]
  Note that the ‘lowest cluster’ (the one attaining the lowest value of the \( t \)-coordinate) is ‘in the middle’ (projects to the centre of \( D^p \)). Note also that \( S_b \) and \( S_{b'} \) are disjoint for \( b \neq b' \), hence the clusters \( T_{S_b}(\xi_b) \) and \( T_{S_{b'}}(\xi_{b'}) \) are also disjoint, and the sum defining \( T_e(\xi_2, \ldots, \xi_{w(e)}) \) is well-defined.

- Consider on \( \mathbb{R}^p \times \mathbb{R} \) the ‘product distance’
  \[
  d((\xi, t), (\zeta, t')) := \max(d(\xi, \zeta), d(t, t'))
  \]
  with respect to the Euclidean distances on \( \mathbb{R}^p \) resp. \( \mathbb{R} \). This means that for a radius \( \rho > 0 \), the closed \( \rho \)-ball around \((\xi, t)\) is given by the cylinder
  \[
  B_\rho(\xi, t) = (\xi + \rho \cdot D^p) \times [t - \rho ; t + \rho].
  \]

Now we have everything together to define the desired insertion map: given an element \( \Theta = \sum_i y_i \otimes (e_i, \xi_i) \) in \( C_{r,s} \), we define
\[
\rho = \rho(\Theta) := \begin{cases} 
  \frac{1}{5} \cdot \min_{l \neq l'} d(y_l, y_{l'}) & \text{for } r - s \geq 2, \\
  1 & \text{for } r - s = 1.
\end{cases}
\]
and accordingly, the map
\[
\varphi_{r,s} : C_{r,s} \to F_s V_r, \quad \Theta = \sum_{l=1}^{r-s} y_l \otimes (e_l, \xi_l) \mapsto \sum_{l=1}^{r-s} (y_l + \rho(\Theta) \cdot T_{e_l}(\xi_l)).
\]
First of all, note that the signs ‘+’ and ‘·’ in the expression \( y_l + \rho(\Theta) \cdot T_l(\xi_l) \) denote a translation and a dilation in \( \mathbb{R}^{p+1} \); the sum sign always describes an unordered collection (of points, or of vertical clusters). Note that the second sum is well-defined as the configurations \( y_l + \rho(\Theta) \cdot T_l(\xi_l) \) lie inside the cylinders \( B_\rho(y_l) \) and are hence disjoint. Finally, we have \( \delta(\varphi_{r,s}(\Theta)) \geq r - s \), so the image of \( \varphi_{r,s} \) is actually contained in the filtration level \( F_s V_r \subset V_r \).

As indicated before, Remark 4.14 ensures that \( \varphi_{r,s}(\Theta) \) lies in the stratum \( \mathfrak{g}_s V_r \) if and only if \( \xi_l = 0_s \) for all \( 1 \leq l \leq r - s \). In particular \( \varphi_{r,s} \) restricts to maps \( C^*_{r,s} \rightarrow F_{s-1} V_r \) and \( C^0_{r,s} \rightarrow \mathfrak{g}_s V_r \).

Remark 4.18. 1. In fancier language, and up to homotopy, we constructed the following: Let \( D := \bigsqcup_e D_e \) and define a map \( \varphi: D \rightarrow V := \bigsqcup_{r \geq 0} V_r \), restricting for all \( e \in E \) to a map \( D_e \rightarrow V_{\omega(e)} \) similar to the map \( T_r \) above. Then we use that \( V \) is an \( \mathcal{C}_{p+1} \)-algebra and consider the adjoint map of \( \mathcal{C}_{p+1} \)-algebras, with source the free \( \mathcal{C}_{p+1} \)-algebra on \( D \):

\[
\varphi: C(\mathbb{R}^{p+1}; D) \simeq F^{\mathcal{C}_{p+1}}(D) \rightarrow V.
\]

The left hand side decomposes as a disjoint union of the spaces \( C_{r,s} \) as before, while the right hand side decomposes into the spaces \( V_r \), which are filtered by the spaces \( F_s V_r \), and the map \( \varphi \) is compatible with this decomposition and filtration. We spelled out the insertion map directly for three reasons: firstly, we need the explicit choice of \( \rho \) later in the proof; secondly, this adjoint description gives a definition of \( \varphi \) which is sensible only up to homotopy, and in particular we could not immediately make sense of a statement like: \( \varphi \) is compatible with the filtrations and their strata, and thirdly, we need each \( \varphi_{r,s} \) to be smooth in the proof of the upcoming Proposition 4.19.

2. The insertion maps respect the stabilisation maps on both sides up to homotopy: the stabilisation maps in the following diagram of maps of
pairs can be chosen so that the diagram commutes on the nose

\[(C_{r,s}, C^*_{r,s}) \xrightarrow{\varphi_{r,s}} (F_s V_r, F_{s-1} V_r) \]
\[\text{stab} \downarrow \quad \text{stab} \]
\[(C_{r+1,s}, C^*_{r+1,s}) \xrightarrow{\varphi_{r+1,s}} (F_s V_{r+1}, F_{s-1} V_{r+1}) \]

3. In order to ensure that \(\varphi_{r,s}\) is well-defined, it would have been enough to choose \(\rho\) slightly smaller than \(\frac{1}{2} \cdot \min_{y',y''} d(y, y')\). However, we wanted to ensure that \(\varphi_{r,s} : C_{r,s} \to F_s V_r\) is even an embedding, so we need to ensure that \(\varphi_{r,s}(\Theta)\) still 'knows' which clusters come from the same label.

**Proposition 4.19.** For each \(1 \leq s \leq r\), the map \(\varphi_{r,s} : (C_{r,s}, C^*_{r,s}) \to (F_s V_r, F_{s-1} V_r)\) induces an isomorphism in relative homology.

The proof of Proposition 4.19 relies on the following lemma.

**Lemma 4.20.** For \(1 \leq s \leq r\), the map \(\varphi_{r,s} : C_{r,s}^0 \to \mathfrak{F}_s V_r\) is a homotopy equivalence.

We first finish the proof of Proposition 4.19 assuming Lemma 4.20, and then prove Lemma 4.20.

**Proof of Proposition 4.19.** Since \(\mathfrak{F}_s V_r \subseteq F_s V_r\) is an embedded, closed submanifold, the relative homology groups of \((F_s V_r, F_{s-1} V_r)\) can be computed using excision and the Thom isomorphism for the normal bundle of \(\mathfrak{F}_s V_r\) in \(F_s V_r\),

\[H_s(F_s V_r, F_{s-1} V_r) = H_s(F_s V_r, F_s V_r \setminus \mathfrak{F}_s V_r) \cong H_{s-s_p}(\mathfrak{F}_s V_r; \mathcal{O}_{r,s}),\]

for a suitable choice of local coefficients \(\mathcal{O}_{r,s}\) on \(\mathfrak{F}_s V_r\). Similarly, the relative homology groups of \((C_{r,s}, C^*_{r,s})\) are given by

\[H_s(C_{r,s}, C^*_{r,s}) = H_s(C_{r,s}, C_{r,s} \setminus C^*_{r,s}) \cong H_{s-s_p}(C^0_{r,s}; \mathcal{O}'_{r,s}),\]

for a suitable choice of local coefficients \(\mathcal{O}'_{r,s}\) on \(C^0_{r,s}\).

Recall Remark 4.8 and Remark 4.16: the key observation is that the insertion map \(\varphi_{r,s}\) induces a map of normal bundles \(N(C^0_{r,s}, C_{r,s}) \to N(\mathfrak{F}_s V_r, F_s V_r)\), which is an isomorphism on fibres. To see this, fix \(\hat{\Theta} \in C^0_{r,s}\) as in Remark 4.16, denote \([\hat{Z}] = \varphi_{r,s}(\hat{\Theta})\), and use the notation from Remark 4.8. By construction \(\varphi_{r,s}\) is a smooth map, and our aim is to check that

\[\varphi^*_{r,s} : N^*_{[\hat{Z}]}(\mathfrak{F}_s V_r, F_s V_r) \to N^*_{\hat{\Theta}}(C_{r,s}, C^*_{r,s})\]

is an isomorphism of vector spaces. This follows directly from the observation that \(\varphi^*_{r,s}\) sends the linear functional \(d_{\mathfrak{F}_s V_r}^{\hat{Z}}\) to the linear functional \(\frac{1}{\rho(\hat{\Theta})} \cdot d_{C^0_{r,s}}^{\hat{Z}}\), whenever the cluster \([\vec{z}_i]\) of the collection \([\hat{Z}]\) is obtained from the labelled
point \( \tilde{y}_l \otimes \xi_l \), using the \( b \)th component \( S_b \) of the partition \( c_l \). Hence the local coefficient system \( O^0_{r,s} \) coincides with \( \varphi^*_{r,s} O_{r,s} \), and so,

\[
H_* (C_{r,s}^0, C_{r,s}^s) \xrightarrow{\cong} H_{* - p} (C_{r,s}^0, O^0_{r,s})
\]

\[
H_* (F_r V_r, F_{r-1} V_r) \xrightarrow{\cong} H_{* - p} (\mathcal{O}, V_{r}; O_{r,s}).
\]

commutes. By Lemma 4.20, the right vertical map is an isomorphism; hence also the left vertical map is an isomorphism. 

**Proof of Lemma 4.20.** The insertion map \( \varphi_{r,s} : C_{r,s}^0 \to \mathcal{O} V_r \) is a closed injection, and we define a deformation retraction of \( \mathcal{O} V_r \) onto \( C_{r,s}^0 \): for each \( [Z] \in \mathcal{O} V_r \) there is, up to permutation, a unique sequence \( e_1, \ldots, e_{r-s} \in \mathbb{E} \) such that

\[
[Z] = \sum_{l=1}^{r-s} \sum_{s \in \xi_l} (\xi_l, (u_l^h)_{h \in S}),
\]

where we assume \( u_l^h < u_l^{h+1} \) for all \( 1 \leq h < w(e_l) \cdot k \). We define

\[
u_l := \frac{1}{k \cdot w(e_l)} \cdot (u_l^1 + \cdots + u_l^{w(e_l)})
\]

Moreover, let \( y_l := (\xi_l, u_l) \in \mathbb{R}^{p+1} \) and \( \rho := \rho (y_1, \ldots, y_{r-s}) \). The homotopy \( H: \mathcal{O} V_r \times [0;1] \to \mathcal{O} V_r \) is given by linear interpolation of the local parameters: we move \( u_l^h \) to \( u_l^1 + \rho \cdot (1 + \frac{2}{k \cdot w(e_l) + 1} \cdot h) \), and keep the values \( \xi_l \) fixed. For each \( 1 \leq l \leq r - s \), the interpolation takes place in the convex hull of the points \( (\xi_l, u_l^h) \in \mathbb{R}^{p+1} \); these are \( r - s \) disjoint vertical segments in \( \mathbb{R}^{p+1} \), and at each time the original vertical order of the points lying on each of these segments is preserved. Therefore no collision between distinct points of the vertical configuration occurs.

We note that \( H ([Z], 1) \) is in the subspace \( \varphi_{r,s} (C_{r,s}^0) \), and by construction, \( H \) fixes pointwise at all times the subspace \( \varphi_{r,s} (C_{r,s}^0) \). 

**4.5 The stability proof**

**Lemma 4.21.** The stabilisation map \( C_{r,s} \to C_{r+1,s} \) induces isomorphisms

\[
H_m (C_{r,s}, C_{r,s}^s) \to H_m (C_{r+1,s}, C_{r+1,s}^s)
\]

in homology for \( m \leq \frac{r}{2} \) and all \( 1 \leq s \leq r \).

**Proof.** For each distribution \( \alpha \) of degree \( (r, s) \), we have a (not always orientable) disc bundle \( C_{\alpha} (\mathbb{R}^{p+1}; D) \to C_{\alpha} (\mathbb{R}^{p+1}) \) of dimension \( p \cdot s \) and structure group \( \mathbb{S} (\alpha) \), which gives us a Thom isomorphism

\[
H_m \left( C_{\alpha} (\mathbb{R}^{p+1}; D), C_{\alpha} (\mathbb{R}^{p+1}) \right) \cong H_{m-p-s} \left( C_{\alpha} (\mathbb{R}^{p+1}); \text{pr}_\alpha^* O_{\alpha} \right) =: M_{m,a},
\]

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where \( \text{pr}_\alpha : \pi_1(C_\alpha(\mathbb{R}^{p+1})) \to \mathcal{O}_\alpha \) is the projection and \( \mathcal{O}_\alpha \) is of the form

\[
\mathcal{O}_\alpha : \mathcal{S}(\alpha) \to \{ \pm 1 \}, \quad (\sigma_\epsilon)_{\epsilon \in E} \mapsto \prod_\epsilon \text{sg}(\sigma_\epsilon)^{p \cdot (w(e)-1)}.
\]

In particular, we have a natural isomorphism \( \text{pr}_\alpha^* \mathcal{O}_\alpha \cong \text{stab}^* \text{pr}_\alpha^* \mathcal{O}_{\alpha+\epsilon_0} \) for the canonical stabilisation \( C_\alpha(\mathbb{R}^{p+1}) \to C_{\alpha+\epsilon_0}(\mathbb{R}^{p+1}) \), which gives us induced stabilisation morphisms \( M_{m,\alpha} \to M_{m,\alpha+\epsilon_0} \). Now we have a commutative square

\[
\begin{array}{ccc}
H_m(C_{r,s}, C^*_{r,s}) & \xrightarrow{\cong} & \bigoplus_{\deg(\alpha)=(r,s)} M_{m,\alpha} \\
\downarrow & & \downarrow \\
H_m(C_{r+1,s}, C^*_{r+1,s}) & \xrightarrow{\cong} & \bigoplus_{\deg(\alpha)=(r,s)} M_{m,\alpha+\epsilon_0} \oplus \bigoplus_{\deg(\alpha)=(r+1,s), \alpha_0=0} M_{m,\alpha}
\end{array}
\]

where the left vertical arrow is the desired stabilising map and the right side is a sum of maps \( M_{m,\alpha} \to M_{m,\alpha+\epsilon_0} \). Therefore, we can prove the statement by showing that for \( m \leq \frac{r}{2} \), we have \( M_{m,\alpha} = 0 \) for each distribution \( \alpha \) of degree \((r+1,s)\) with \( \alpha_0 = 0 \), and secondly, that the stabilising map \( M_{m,\alpha} \to M_{m,\alpha+\epsilon_0} \) is an isomorphism for each distribution \( \alpha \) of degree \((r,s)\). For the first part, we use \( w(e) \geq 2 \) for all \( e \in E \) with \( \alpha_e \neq 0 \) to obtain

\[
p \cdot s \geq p \cdot \sum_e \alpha_e \cdot (w(e) - \frac{1}{2} \cdot w(e)) = p \cdot \frac{r+1}{2} \geq \frac{r+1}{2},
\]

so \( m - p \cdot s < 0 \), whence \( M_{m,\alpha} = 0 \). For the second part, have to check that \( H_{m-p \cdot s}(C_\alpha(\mathbb{R}^{p+1}); \text{pr}_\alpha^* \mathcal{O}_\alpha) \to H_{m-p \cdot s}(C_{\alpha+\epsilon_0}(\mathbb{R}^{p+1}); \text{pr}_\alpha^* \mathcal{O}_{\alpha+\epsilon_0}) \) is an isomorphism for \( m \leq \frac{r}{2} \). To do so, we first observe that \( \frac{r}{2} \geq \frac{1}{2} \cdot \alpha_0 + \sum_{e \neq \epsilon_0} \alpha_e \) since \( w(e) \geq 2 \) for \( e \neq \epsilon_0 \); and as \( p \geq 1 \), we obtain

\[
m - p \cdot s \leq \frac{r}{2} - p \cdot \sum_e \alpha_e \cdot (w(e) - 1) \leq -\frac{r}{2} + \sum_e \alpha_e \leq \frac{1}{2} \cdot \alpha_0.
\]

Now we want to use a technique from [Pal18], so we adapt his notation by writing \( \lambda := (\alpha_e)_{e \neq \epsilon_0} \), so \( |\lambda| = \sum_{e \neq \epsilon_0} \alpha_e \) as well as \( \lambda[n] \) for the distribution with \( \lambda[n](\epsilon_0) = n - |\lambda| \) and \( \lambda[n](e) = \alpha_e \) for \( e \neq \epsilon_0 \), so we have \( \lambda[r-s] = \alpha \) and \( \lambda[r-s+1] = \alpha + \epsilon_0 \). This notation has the advantage that \( n = \sum_e \lambda[n](e) \). We have a stabilisation map \( C_{\lambda[n]}(\mathbb{R}^{p+1}) \to C_{\lambda[n+1]}(\mathbb{R}^{p+1}) \) by placing an additional point with label \( \epsilon_0 \), which for \( n = r-s \) is our map from before.

Now we construct a signed version of [Pal18, Ex. 4.6]: let \( \text{PINj} \) be the category whose objects are non-negative integers and whose morphisms \( n \to n' \) are partially defined injections \( \eta : \{ 1, \ldots, n \} \to \{ 1, \ldots, n' \} \). We define a functor \( \mathcal{P}_\lambda : \text{PINj} \to \text{Ab} \) to the category of abelian groups as follows: we set

\[
\mathcal{P}_\lambda(n) := \mathbb{Z}(\langle P_e \rangle_{e \neq \epsilon_0}; \ P_e \subseteq \{ 1, \ldots, n \}, P_e \cap P_{e'} = \emptyset, \text{ and } \#P_e = \lambda_e),
\]
and for each partially-defined injection \( \eta: n \to n' \) and \( P := (P_c)_{c \neq 0} \), we define

\[
\eta_* (P) := \begin{cases}
\prod_{c \neq 0} 
\sg (\eta|_P)_c (w_c) \cdot (\eta(P_c))_c 
& \text{if } \eta \text{ is defined on } \bigcup_c P_c,
0 
& \text{else},
\end{cases}
\]

where in the first case, the restriction \( \eta|_P: P_c \to \eta(P_c) \) can canonically be identified with a permutation in \( \mathfrak{S}_n \), since \( P_c \) and \( \eta(P_c) \) are totally ordered as subsets of \( \{1 < \cdots < n\} \) resp. \( \{1 < \cdots < n'\} \). By the same inductive argument as in [Pal18, Lem. 4.7], \( \mathcal{P} \) is a polynomial coefficient system with \( \deg(\mathcal{P}) = |\lambda| = r - s + \alpha_0 \) and since \( \mathbb{Z} \mathfrak{S}_n \otimes \mathbb{Z}[\mathfrak{S}_n] \mathcal{O}_{\lambda[n]} \cong \mathcal{P}(n) \) as \( \mathfrak{S}_n \)-representations, we have natural isomorphisms

\[
H^{m'}(C_n(\mathbb{R}^{p+1}); \text{pr}_n^* P(n)) \xrightarrow{\cong} H^{m'}(C_{n+1}(\mathbb{R}^{p+1}); \text{pr}_n^* \mathcal{P}(n + 1)),
\]

where \( \text{pr}_n: \pi_1(C_n(\mathbb{R}^{p+1})) \to \mathfrak{S}_n \) is the projection. Since \( p + 1 \geq 2 \), the bottom map is an isomorphism for \( m' \leq \frac{1}{2} \cdot (n - r + s + \alpha_0) \) by [Pal18, Thm. A]. For us, \( n = r - s \), we get the isomorphism for \( m' \leq \frac{1}{2} \cdot \alpha_0 \) as desired. \( \square \)

Now we have all tools to prove Theorem 4.3.

**Proof of Theorem 4.3.** This is now a standard argument: let \( E(r) \) denote the Leray homology spectral sequence associated with the filtered space \( V_r(\mathbb{R}^{p+1}) \), the filtration-preserving stabilisation \( V_r(\mathbb{R}^{p+1}) \to V_{r+1}(\mathbb{R}^{p+1}) \) induces a morphism \( f: E(r) \to E(r+1) \) of spectral sequences, and on the first page we have exactly the morphisms

\[
f^1_{s,t}: E(r)_{s,t}^1 = H_{s+t}(F_s V_r, F_{s-1} V_r) \to E(r+1)_{s,t}^1 = H_{s+t}(F_s V_{r+1}, F_{s-1} V_{r+1}).
\]

By Proposition 4.19 and Lemma 4.21, we know that \( f^1_{s,t} \) is an isomorphism for \( s + t \leq \frac{r}{2} \), so by a standard comparison argument between spectral sequences [Zee57] we obtain that \( H_m(V_r) \to H_m(V_{r+1}) \) is an isomorphism for \( m \leq \frac{r}{2} \). \( \square \)

**Outlook 4.22.** 1. As already remarked, the space \( V^k(\mathbb{R}^{p+q}) = \coprod_{j \geq 0} V^j(\mathbb{R}^{p+q}) \) is a \( \mathfrak{C}_{p+q} \)-algebra with \( \mathbb{N} \) as monoid of path components, hence the stable homology \( H_* (V^k(\mathbb{R}^{p+q})) := \text{colim}_{r \to \infty} H_*(V^r(\mathbb{R}^{p+q})) \) agrees with the homology of (a component of) some \( \Omega^{p+q} \)-space. The second author [Kra21] provides a geometric model for the \( p \)-fold delooping of \( V^k(\mathbb{R}^{p+q}) \), and in the case \( q = 1 \) even for the \( (p+1) \)-fold delooping. We still lack a geometric description of the \( (p+q) \)-fold delooping of \( V^k(\mathbb{R}^{p+q}) \) for \( q \geq 2 \), even in the seemingly innocent case \( p = 0 \).
2. We believe that the Leray spectral sequence for the filtration $F_rV_r$ collapses on its first page and that the extension problem is trivial. This would then imply that, using the notation from the proof of Lemma 4.21,

$$H_m(V_r(\mathbb{R}^{p,1})) \cong \bigoplus_{s=0}^{r-1} H_m(C_{r,s}, C_{r,s}^+) \cong \bigoplus_{s=0}^{r-1} \bigoplus_{\deg(\alpha) = (r,s)} M_{m,\alpha}.$$  

Our motivation is again the description of the stable homology $H_m(V_\infty(\mathbb{R}^{p,1})) \cong \bigoplus_{s=0}^{\infty} \bigoplus_{\alpha} M_{m,\alpha}$, given in [Kra21], where the last direct sum is extended over all distributions $\alpha : \mathbb{E} \to \mathbb{N}$ with $\alpha_{s_0} = 0$ and $s(\alpha) = s$.

3. The strategy of proof of Theorem 4.3 generalises to the following case: Let $K = (k_1, \ldots, k_r)$, and for $k \geq 1$ let $r(k) \geq 0$ be the number of indices $1 \leq i \leq r$ with $k_i = k$; define a stabilisation map $V_K(\mathbb{R}^{p,1}) \to V_{(K,k)}(\mathbb{R}^{p,1})$ by inserting a new vertical cluster; then the induced map in homology

$$\text{stab}_{r} : H_m(V_K(\mathbb{R}^{p,1})) \to H_m(V_{(K,k)}(\mathbb{R}^{p,1}))$$

is an isomorphism for $m \leq r(k)^2$. We leave to the reader the details of the generalisation of the proof.

References


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