Dimension and degeneracy of solutions of parametric polynomial systems arising from reaction networks

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DIMENSION AND DEGENERACY OF SOLUTIONS OF PARAMETRIC POLYNOMIAL SYSTEMS ARISING FROM REACTION NETWORKS

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Abstract. We study the generic dimension of the solution set over $\mathbb{C}^*$, $\mathbb{R}^*$ and $\mathbb{R}_{>0}$ of parametric polynomial systems that consist of linear combinations of monomials scaled by free parameters. We establish a relation between this dimension, Zariski denseness of the set of parameters for which the system has solutions, and the existence of nondegenerate solutions, which enables fast dimension computations. Systems of this form are used to describe the steady states of reaction networks modeled with mass-action kinetics, and as a corollary of our results, we prove that weakly reversible networks have finitely many steady states for generic reaction rate constants and total concentrations.

1. Introduction

In this work we study parametric (Laurent) polynomial systems of the form
$$g(\alpha, x) = 0, \quad x \in \mathcal{X} \subseteq (\mathbb{C}^*)^n, \quad \alpha \in \mathcal{A} \subseteq \mathbb{C}^\ell,$$
where $g(\alpha, x) \in \mathbb{C}[\alpha, x^\pm]$ is a tuple of $s$ linearly independent polynomials in parameters $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ restricted to some set $\mathcal{A} \subseteq \mathbb{C}^\ell$, and variables $x = (x_1, \ldots, x_n)$ restricted to some set $\mathcal{X} \subseteq (\mathbb{C}^*)^n$.

Such systems arise naturally in many applications. Important examples include reaction network theory, where $\mathcal{X} = \mathbb{R}_{>0}^n$ and $x$ represents concentrations or abundances at steady states [Dic16, Fei19], algebraic statistics, where one often takes $\mathcal{X}$ to be the interior of the probability simplex and $x$ to be parameters of discrete probability distributions [Sul18], or robotics, where $\mathcal{X} = (\mathbb{R}^*)^n$ and $x$ represents configurations of various mechanisms [CR04].

A fundamental problem is to determine the generic dimension of the algebraic variety defined by the solutions to the system of interest, and, in particular, whether it agrees with the lower bound $n - s$ derived from the number of linearly independent equations.

For sparse polynomial systems with fixed support and freely varying parametric coefficients, this lower bound is always the generic dimension. As an illustration of this, consider the simplest example, namely a system of linear equations with free coefficients:
$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 = 0, \quad \alpha_4 x_1 + \alpha_5 x_2 + \alpha_6 = 0, \quad x \in (\mathbb{C}^*)^2, \quad \alpha \in \mathbb{C}^4.$$

The generic dimension of the solution set is 0, as there will be a unique solution in $(\mathbb{C}^*)^2$ whenever $(\alpha_1 \alpha_5 - \alpha_2 \alpha_4)(\alpha_1 \alpha_6 - \alpha_3 \alpha_4)(\alpha_2 \alpha_6 - \alpha_3 \alpha_5) \neq 0$. On the other hand, when there are algebraic dependencies between the coefficients, the situation can be more complicated. Consider for example the system
$$\alpha_1 x_1 - \alpha_2 x_2 = 0, \quad \alpha_1^2 x_1^2 - \alpha_2^2 x_2^2 = 0, \quad x \in (\mathbb{C}^*)^2, \quad \alpha \in \mathbb{C}^2.$$

As the first polynomial divides the second, the generic complex dimension is 1, despite the system having two (linearly independent) equations, and hence a lower dimension bound of 0.
The motivation of this work comes from the theory of reaction networks, where the parametric systems of interest are of the form

\[ f_\kappa(x) = N \text{diag}(\kappa)x^B, \quad x \in \mathbb{R}^n_>, \quad \kappa \in \mathbb{R}^r_>, \quad (1.4) \]

with \( N \in \mathbb{Z}^{s \times r}, \) \( \text{rk}(N) = s, \) and \( B \in \mathbb{Z}^{n \times r}. \) This system describes the steady states of a reaction network under the assumption of mass-action kinetics (more details are given in Section 4). Even though the study of reaction networks in the current mathematical formalism goes back at least to Feinberg, Horn and Jackson in the 70’s [Fei72, HJ72], many fundamental questions about the system (1.4) remain unclear, including the question about the generic dimension of the solution sets.

Since each parameter \( \kappa_i \) might appear as a coefficient in several equations, it is at first glance not obvious whether the generic dimension can deviate from \( n - s, \) similarly to (1.3). There are well-known examples of networks where the solution set always has a higher dimension than \( n - s \) whenever it is nonempty (see e.g. [Fei87, Appendix IV]), but for all such networks, the solution set has the additional pathology of being empty for almost all parameter values.

Analogous observations have been made for a second parametric system arising in reaction network theory, which describes the steady states constrained by the linear first integrals of the underlying system of differential equations:

\[ N \text{diag}(\kappa)x^B = 0, \quad Wx = c, \quad x \in \mathbb{R}^n_>, \quad (\kappa, c) \in \mathbb{R}^r_> \times \mathbb{R}^{n-s}, \quad (1.5) \]

with \( W \in \mathbb{Z}^{(n-s) \times n} \) of full rank. Here, the empirical observation is that for all realistic networks, the solution sets are finite (i.e., 0-dimensional) for generic parameters \( \kappa \) and \( c, \) and if not, then the solution sets are generically empty.

In this work, we settle these questions and confirm that the pathologies of higher dimension and generic emptiness always go hand in hand. If the complex solutions to (1.4) do not form a variety of dimension \( n - s \) for no \( \kappa \in \mathbb{R}^r_>, \) then at the same time, it holds that the variety is empty for all \( \kappa \) outside a proper Zariski closed subset. Likewise, if (1.5) has either infinitely many or no solutions for all \( (\kappa, c) \), then it also has no solution for any \( (\kappa, c) \) outside of a proper Zariski closed subset.

Our second main contribution is that deciding whether the set of parameters for which the system has solutions is Zariski dense reduces to checking whether the system has a nondegenerate solution (in the sense that the rank of the Jacobian agrees with the number of equations) for at least one choice of parameters. This can be easily checked computationally, by inspecting the generic rank of a family of matrices. In practice, this turns out to be a significantly cheaper computation than the standard methods for computing the generic dimension based on Gröbner bases (see e.g. [CLO15, Section 9.3] for a discussion of such methods).

To obtain these results, we study first general families of parametric systems of the form (1.1). Background theory establishes that the generic dimension of the complex variety of solutions to \( g(\alpha, x) = 0 \) for a fixed \( \alpha \) can be determined by employing the Theorem on the Dimension of Fibers, after checking that the incidence variety of the parametric system is irreducible. Then, the generic dimension depends on the dimension of the incidence variety as well as on whether the set of parameters \( D \) for which the variety is nonempty is Zariski dense in parameter space. This theory applies to the examples in (1.2) and (1.3) above. The underlying arguments are reviewed in Section 2 where we also compare the condition of \( D \) being Zariski dense to having nonempty Euclidean interior in \( A. \) The latter can be studied by means of the concept of nondegeneracy.
With these tools in place, we proceed in Section 3 to study the systems of the form \( (1.4) \) and \( (1.5) \), where we allow the coefficient matrix to take complex values. We show that for these systems, the existence or lack of nondegenerate solutions have very strong consequences in terms of the generic dimension of the associated complex varieties. Our results are gathered in the main theorem of this work, Theorem 3.7 which gives several equivalent conditions guaranteeing that the generic dimension is \( n - s \). The theorem is stated for sets \( X \) and \( A \) satisfying certain mild algebraic and topological conditions (which are satisfied for the positive orthant, and the real and complex torus). Furthermore, the equivalent conditions of Theorem 3.7 hold for a pair of subsets \( A \) and \( X \) satisfying the conditions if and only if they hold for any other such pair, in particular for complex parameters and solutions in the complex torus.

In Section 4 we connect the results of Section 3 to the theory of reaction networks. This section is presented with less technicalities, to make it accessible to the broader reaction network community. Furthermore, we specialize our conclusions to show that for the well-known and well-studied class of so-called weakly reversible networks, the generic dimension of the complex varieties of the two systems \( (1.4) \) and \( (1.5) \) is always the lower bound, namely \( n - s \) and \( 0 \), respectively. This settles a question posed by Boros, Craciun and Yu in [BCY20]. We illustrate further the simplicity of our conditions by exploring the database ODEbase of biologically relevant reaction networks [LSR22], and are able to verify within minutes that for the vast majority of them, the systems of interest attain the lower bound on the generic dimension, whereas for a handful of exceptions, the steady state variety is generically empty. Given the size of the networks in the database, these computations are nontrivial with standard Gröbner basis techniques.

**Notation and conventions.** We let \( \circ: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}^n \) denote the Hadamard product, given by \((x \circ y)_i = x_i y_i\). For a field \( F \), we let \( F^* = F \setminus \{0\} \) be the group of units of \( F \). For a matrix \( A = (a_{ij}) \in \mathbb{Z}^{n \times m} \) and a vector \( x \in (F^n)^n \), we let \( x^A \in F^m \) be defined by \((x^A)_j = x_1^{a_{1j}} \cdots x_n^{a_{nj}} \) for \( j \in \{1, \ldots, m\} \).

In this work we will consider both the Euclidean and Zariski topologies on \( \mathbb{C}^n \), and their restrictions to subsets. By default, we use the Zariski topology, and for a set \( S \subseteq \mathbb{C}^n \), we let \( \overline{S} \) denote the Zariski closure of \( S \) in \( \mathbb{C}^n \). When we say that a set \( U \subseteq X \) is open or has nonempty interior in \( X \), for some \( X \subseteq \mathbb{C}^n \), we mean with respect to the subspace topology topology on \( X \). For example, if we say that \( U \) has nonempty Euclidean interior in \( X \), we mean there exists an open ball \( B \subseteq \mathbb{C}^n \) such that \( B \cap X \subseteq U \).

When saying that a property holds generically in a family indexed by some parameters \( \alpha \in \mathcal{A} \), we mean that it holds outside a proper Zariski closed subset of \( \mathcal{A} \).

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**2. The dimension of solution sets of parametric polynomial systems**

In this section we study the generic dimension of the solution set of parametric systems for which the incidence variety is irreducible of known dimension. This sets the background theory for the application to the systems of polynomials indicated in the introduction. We
We will identify the algebraic variety \( \mathcal{V}(g, x) \). A proof of the result can be found in [SR13, Thm. 1.25], [Vak17, Section 11.4].

We will use the following version of the Theorem on the Dimension of Fibers.

2.1. Framework. We consider a parametric family of systems of (Laurent) polynomials of the form

\[
g(\alpha, x) = 0, \quad \alpha = (\alpha_1, \ldots, \alpha_\ell) \in \mathbb{C}^\ell, \quad x = (x_1, \ldots, x_n) \in (\mathbb{C}^*)^n, \quad (2.1)
\]

where \( g \in \mathbb{C}[\alpha, x^\pm] \) for some \( s \in \mathbb{Z}_{\geq 0} \), and where we view \( \alpha = (\alpha_1, \ldots, \alpha_\ell) \) as parameters, and \( x = (x_1, \ldots, x_n) \) as variables. We consider the incidence variety

\[
\mathcal{E}_g := \{ (\alpha, x) \in \mathbb{C}^\ell \times (\mathbb{C}^*)^n : g(\alpha, x) = 0 \}, \quad (2.2)
\]

and the projection map to parameter space

\[
\pi : \mathcal{E}_g \to \mathbb{C}^\ell, \quad (\alpha, x) \mapsto \alpha. \quad (2.3)
\]

For each choice of parameters \( \alpha \in \mathbb{C}^\ell \), we get a polynomial \( g_\alpha := g(\alpha, \cdot) \in \mathbb{C}[x^\pm]^s \), which leads to the specialized system

\[
g_\alpha(x) = 0, \quad x \in (\mathbb{C}^*)^n. \quad (2.4)
\]

We will identify the algebraic variety \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) consisting of solutions to (2.4) with the fiber \( \pi^{-1}(\alpha) \) of the projection map. We will sometimes restrict the parameter space to a subset \( \mathcal{A} \subseteq \mathbb{C}^\ell \). (The main example will be \( \mathcal{A} = \mathbb{R}_{>0}^\ell \).) As there is no guarantee that system (2.4) has solutions, we introduce the following set:

\[
\mathcal{D}_{g, \mathcal{A}} = \pi(\mathcal{E}_g \cap (\mathcal{A} \times (\mathbb{C}^*)^n)) = \text{im}(\pi) \cap \mathcal{A} = \{ \alpha \in \mathcal{A} : \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \neq \emptyset \}. \quad (2.5)
\]

In what follows we will distinguish between the coefficient matrix \( N \) with entries in \( \mathbb{C} \) obtained by viewing \( g \) as a polynomial in \( \mathbb{C}[\alpha, x^\pm] \), and the coefficient matrix \( \Sigma_\alpha \) with entries in \( \mathbb{C}[\alpha] \), obtained by viewing \( g \) as a polynomial in \( \mathbb{C}[\alpha][x^\pm] \).

2.2. Dimension over \( \mathbb{C}^* \). The first concept of interest is the dimension of the family of complex varieties \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) for \( \alpha \in \mathcal{A} \). An immediate first observation is that for each fixed \( \alpha \in \mathcal{A} \), the principal ideal theorem [Eis95, Thm. 10.2] gives that

\[
\dim \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \geq n - s, \quad (2.6)
\]

since \( g_\alpha \) is a tuple of \( s \) polynomials. Moreover, all irreducible components of \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) have dimension at least \( n - s \). Hence, if \( \dim(\mathcal{V}_\mathbb{C}^\ell(g_\alpha)) = n - s \) for a given \( \alpha \in \mathcal{A} \), then all irreducible components have dimension \( n - s \), and we say that \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) is equidimensional of dimension \( n - s \). With this in mind, it makes sense to remove redundancies in the entries of \( g(\alpha, x) \) such that the coefficient matrix \( N \) of \( g \) as a polynomial in \( \mathbb{C}[\alpha, x^\pm] \) has full rank.

For a fixed \( \alpha \), the codimension of \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) is additionally bounded above by the rank of the coefficient matrix \( \Sigma_\alpha \). Therefore, if the generic rank of \( \Sigma_\alpha \) is strictly smaller than \( s \), the dimension of \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) is strictly larger than \( n - s \).

When the incidence variety \( \mathcal{E}_g \) is irreducible, the generic dimension of \( \mathcal{V}_\mathbb{C}^\ell(g_\alpha) \) is well understood. We will use the following version of the Theorem on the Dimension of Fibers. A proof of the result can be found in [SR13, Thm. 1.25], [Vak17, Section 11.4].
**Theorem 2.1** (Dimension of fibers). Let $\varphi : X \to Y$ be a dominant morphism of irreducible varieties. Then, for any $y \in \varphi(X)$, it holds that

$$\dim(\varphi^{-1}(y)) \geq \dim(X) - \dim(Y).$$

(2.7)

Moreover, there exists a nonempty Zariski open subset $U \subseteq Y$ such that (2.7) holds with equality for all $y \in U \cap \varphi(X)$.

With this in place, we state the first theorem on the generic dimension of $\mathcal{V}_{C^\ast}(g_\alpha)$.

**Theorem 2.2** (Generic dimension of $\mathcal{V}_{C^\ast}(g_\alpha)$). Let $g \in \mathbb{C}[\alpha, x^+]^s$ and consider the specialized polynomials $g_\alpha$ for parameters $\alpha \in A$, with $A \subseteq \mathbb{C}^\ell$ being Zariski dense. Assume that the incidence variety $E_g$ is irreducible, and let $d = \dim(E_g) - \ell$. Then the following holds:

(i) If $D_{g,A}$ is Zariski dense in $\mathbb{C}^\ell$, then there exists a nonempty Zariski open subset $U$ of $D_{g,A}$ such that for all $\alpha \in U$, it holds that

$$\dim \mathcal{V}_{C^\ast}(g_\alpha) = d.$$

Furthermore, if $d = n - s$, then $\mathcal{V}_{C^\ast}(g_\alpha)$ is equidimensional.

(ii) If $D_{g,A}$ is not Zariski dense in $\mathbb{C}^\ell$, then it holds for all $\alpha \in D_{g,A}$ that

$$\dim \mathcal{V}_{C^\ast}(g_\alpha) > d.$$

**Proof.** The projection map $\pi$ over parameter space defined in (2.3) is a regular map of irreducible varieties. Let $\operatorname{im}(\pi)$ denote the Zariski closure of the image of $\pi$. We can now apply the Theorem on the Dimension of Fibers (Theorem 2.1), to conclude that nonempty fibers $\pi^{-1}(\alpha)$ have dimension at least

$$e = \dim(E_g) - \dim(\overline{\operatorname{im}(\pi)}) = \ell + d - \dim(\overline{\operatorname{im}(\pi)}),$$

with generic dimension equal to $e$. That is, there exists a nonempty Zariski open subset $U \subseteq \operatorname{im}(\pi)$ such that, for any $\alpha \in U$, either $\dim(\pi^{-1}(\alpha)) = e$ or $\pi^{-1}(\alpha) = \emptyset$.

Let us consider scenario (i). Note that $D_{g,A} \subseteq \operatorname{im}(\pi)$. Hence, taking Zariski closures in $\mathbb{C}^\ell$ gives $\mathbb{C}^\ell = \overline{D_{g,A}} \subseteq \overline{\operatorname{im}(\pi)}$, and hence $\overline{\operatorname{im}(\pi)} = \mathbb{C}^\ell$. Therefore, the generic dimension of the fibers of $\pi$ is $e = \ell + d - \ell = d$. It follows that

$$\dim(\mathcal{V}_{C^\ast}(g_\alpha)) = d$$

for all $\alpha \in U \cap D_{g,A}$, since in this case $\pi^{-1}(\alpha) = \mathcal{V}_{C^\ast}(g_\alpha) \neq \emptyset$. Noting that $U \cap D_{g,A} \neq \emptyset$ as $D_{g,A} \subseteq \mathbb{C}^\ell$ is Zariski dense, we obtain statement (i) by taking $U = U \cap D_{g,A}$.

Consider now scenario (ii). By (2.8), the result follows if $\overline{\operatorname{im}(\pi)} \subseteq \mathbb{C}^\ell$ is proper, and hence of lower dimension than $\ell$. By Chevalley’s theorem [Sta23, 00F5], the image of a constructible set under a projection map is constructible. Hence $\operatorname{im}(\pi)$ is a constructible set, and it can be written as $\overline{\operatorname{im}(\pi)} = \bigcup_{i=1}^{m} Z_i \cap U_i$, for some $m \in \mathbb{Z}_{>0}$, irreducible Zariski closed subsets $Z_i \subseteq \mathbb{C}^\ell$ and nonempty Zariski open subsets $U_i \subseteq \mathbb{C}^\ell$. It follows that

$$\overline{\operatorname{im}(\pi)} = \bigcup_{i=1}^{m} Z_i.$$

Assume for a contradiction that $\overline{\operatorname{im}(\pi)} = \mathbb{C}^\ell$. Then $Z_i = \mathbb{C}^\ell$ for some $i \in \{1, \ldots, m\}$, and we have $U_i \subseteq \operatorname{im}(\pi)$. Hence,

$$U_i \cap A \subseteq \operatorname{im}(\pi) \cap A = D_{g,A},$$

which contradicts the assumption that $D_{g,A}$ is Zariski dense.
where \( U_i \cap A \neq \emptyset \), since \( A \) is Zariski dense. Taking Zariski closures, as \( \overline{U_i \cap A} = \mathbb{C}^d \), we have \( \overline{D}_{g,A} = \mathbb{C}^d \), which is a contradiction. This shows that \( \text{im}(\pi) \) is proper and concludes the proof of (ii).

Combining cases (i) and (ii) of Theorem 2.2, we obtain that the dimension of \( V_{C^*}(g_\alpha) \) is 
\( d \) for some \( \alpha \) if and only if \( D_{g,A} \) is Zariski dense in \( \mathbb{C}^d \). Furthermore, if the dimension is \( d \) for one value of \( \alpha \), then it is generically \( d \) in \( D_{g,A} \).

**Example 2.3.** Consider \( g(\alpha, x) \) with \( n = \ell = s = 2 \) giving rise to the system
\[
\alpha_1 x_1 - \alpha_2 x_2 = 0, \quad \alpha_1^2 x_1^2 - \alpha_2^2 x_2^2 = 0,
\]
which was briefly discussed in the Introduction in (1.3). Let \( A = \mathbb{C}^2 \). In this case, \( D_{g,A} = \mathbb{C}^2 \). Furthermore, the incidence variety is defined by the equation \( \alpha_1 x_1 - \alpha_2 x_2 = 0 \) as \( \alpha_1^2 x_1^2 - \alpha_2^2 x_2^2 = (\alpha_1 x_1 - \alpha_2 x_2)(\alpha_1 x_1 + \alpha_2 x_2) \). Hence, \( \mathcal{E}_g \) is irreducible of dimension 3 and by Theorem 2.2(i), the generic dimension of \( V_{C^*}(g_\alpha) \) is \( d = 3 - 2 = 1 \).

In the previous example, the generic dimension is not the minimal one from (2.6), which would be \( n - s = 0 \). However, in the systems of interest in the coming sections, the goal will be to determine when the generic dimension is exactly \( n - s \), in which case the variety also is equidimensional.

**Remark 2.4.** Recall that \( N \) denotes the coefficient matrix of \( g \), and \( \Sigma_\alpha \) the coefficient matrix of \( g_\alpha \). Let us assume for simplicity that \( N \) has rank \( s \). Then the dimension of \( \mathcal{E}_g \) cannot be smaller than \( n + \ell - s \). Under the setting of Theorem 2.2 if \( D_{g,A} \) is Zariski dense and additionally \( \dim(\mathcal{E}_g) = n + \ell - s \), then the generic dimension of \( V_{C^*}(g_\alpha) \) is exactly \( n - s \). In this case, the generic rank of \( \Sigma_\alpha \) is necessarily \( s \), as the corank of \( \Sigma_\alpha \) is a lower bound for the dimension of \( V_{C^*}(g_\alpha) \).

2.3. **Restricting the ambient space.** Let \( F \subseteq \mathbb{C} \) be a subfield, and consider \( A \subseteq \mathbb{F}^\ell \). Then, for \( \alpha \in A \), we let \( V_{F^*}(g_\alpha) \) denote the set of solutions to system (2.4) in \( (\mathbb{F}^*)^n \). Additionally, for a set \( X \subseteq (\mathbb{F}^*)^n \) we let
\[
V^X_{F^*}(g_\alpha)
\]
denote the variety defined as the union of the irreducible components of \( V_{F^*}(g_\alpha) \) that intersect \( X \). We will also consider the set of solutions in \( X \), that is, \( V_{C^*}(g_\alpha) \cap X \). The main example we have in mind, which will become relevant in the next section, is the case \( F = \mathbb{R} \) and \( X = \mathbb{R}_{>0}^n \), hence of real varieties. We generalize (2.5), and introduce the set
\[
D_{g,A}(X) = \pi(\mathcal{E}_g \cap (A \times X)) = \{ \alpha \in A : V_{C^*}(g_\alpha) \cap X \neq \emptyset \}.
\]
With this notation, \( D_{g,A}((\mathbb{C}^*)^n) = D_{g,A} \) and we have the following inclusions:
\[
D_{g,A}(X) \subseteq D_{g,A}, \quad D_{g,A}(X) \subseteq D_{g,C^*}(X).
\]

**Remark 2.5.** The fact that \( V^X_{C^*}(g_\alpha) \) is a Zariski closed set gives that
\[
\overline{V_{C^*}(g_\alpha) \cap X} \subseteq V^X_{C^*}(g_\alpha)
\]
but the reverse inclusion is not necessarily true. The inclusion (2.10) holds with equality if \( X \) is an Euclidean open subset of \( (\mathbb{C}^*)^n \), since, in that case, \( \overline{V_{C^*}(g_\alpha) \cap X} \) is, whenever nonempty, Zariski dense in \( V_{C^*}(g_\alpha) \). When \( X \) is an Euclidean open subset of \( (\mathbb{R}^*)^n \), a sufficient condition for equality in (2.10) is that each irreducible component of \( V^X_{C^*}(g_\alpha) \) has a nonsingular point (see [BCR08], [PEF22] Thm. 6.5, [HHS21] and [Mar08] Thm. 12.6.1]).
On the same line of thought, the lower bound for the complex dimension in (2.6) does not hold over $\mathbb{R}$, as the next example illustrates.

**Example 2.6.** A simple example that illustrates the previous remark is the polynomial

$$g(\alpha, x) = \alpha_1 x_1^2 - \alpha_2 x_1 + \alpha_3 x_2^2 - \alpha_4 x_2 + \alpha_5$$

with $X = \mathbb{R}^2$ and $\alpha \in \mathbb{R}^5$. For $\alpha = (1, 2, 1, 2, 2)$, we get

$$\mathbb{V}^X_\alpha(g_\alpha) = \mathbb{V}^\big((x_1 - 1)^2 + (x_2 - 1)^2\big) \supseteq \{(1, 1)\} = \mathbb{V}^\big((g_\alpha) \cap X\big).$$

Hence the real variety $\mathbb{V}^\big((g_\alpha) \cap X\big)$ is zero-dimensional, while the lower bound for the dimension of $\dim(\mathbb{V}^\big((g_\alpha) \cap X\big))$ is $n - s = 1$. Note that the point $(1, 1)$ is singular in this case.

From Theorem 2.2 we obtain the following result on the generic dimension of $\mathbb{V}^X_\alpha(g_\alpha)$, which is in general bounded above by the generic dimension of $\mathbb{V}^\big((g_\alpha) \cap X\big)$.

**Corollary 2.7** (Generic dimension of $\mathbb{V}^X_\alpha(g_\alpha)$). Let $g \in \mathbb{C}[\alpha, x]^{\mathbb{N}}$ and $g_\alpha$ as in (2.4), $x \in (\mathbb{C}^*)^n$, $\alpha \in A \subseteq C$ with $A$ being Zariski dense. Assume that the incidence variety $\mathcal{E}_g$ is irreducible and has dimension $d$ for some $d \geq 0$.

For $X \subseteq (\mathbb{C}^*)^n$, if $D_{g,A}(X) \subseteq A$ is Zariski dense in $\mathbb{C}^\ell$, then there exists a nonempty Zariski open set $U \subseteq \mathbb{C}$ such that $U \cap D_{g,A}(X) \neq \emptyset$ and

$$\dim \mathbb{V}^X_\alpha(g_\alpha) = \dim \mathbb{V}^\big((g_\alpha) \cap X\big) = d, \quad \text{for all} \quad \alpha \in U \cap D_{g,A}(X).$$

If $d = n - s$, then $\mathbb{V}^X_\alpha(g_\alpha)$ is equidimensional.

**Proof.** This is an immediate consequence of Theorem 2.2(i) as $D_{g,A}(X) \subseteq D_{g,A}$, with the set $U$ in the statement of the theorem, after noting that $U \cap D_{g,A}(X) \neq \emptyset$ as $D_{g,A}(X)$ is Zariski dense.

Observe that Corollary 2.7 extends only Theorem 2.2(i) to $X$. As discussed in Remark 2.5 some special care needs to be taken for certain instances of $X$.

**2.4. Zariski denseness and nonempty Euclidean interior.** One of the hypotheses of Theorem 2.2 and Corollary 2.7 relies on checking that $D_{g,A}$ or $D_{g,A}(X)$ are Zariski dense. In this subsection, we relate this hypothesis to the concept of nondegeneracy, under the assumption that $A$ has the following property.

**Definition 2.8.** A subset $X \subseteq \mathbb{C}^n$ is said to be **locally Zariski dense** (in $\mathbb{C}^n$) if for any Euclidean open subset $U \subseteq \mathbb{C}^n$ with $U \cap X \neq \emptyset$, it holds that $U \cap X = \mathbb{C}^n$ with respect to the Zariski topology on $\mathbb{C}^n$.

Simple examples of locally Zariski dense sets in $\mathbb{C}^n$ include $\mathbb{R}^n_{>0}$ and $\mathbb{R}^n$. Note that any locally Zariski dense set is in particular Zariski dense, but the converse is not true: for instance, $\mathbb{Z}^n$ is Zariski dense in $\mathbb{C}^n$, but not locally Zariski dense.

**Lemma 2.9.** Let $X \subseteq (\mathbb{C}^*)^n$ and suppose that $A \subseteq \mathbb{C}^\ell$ is locally Zariski dense. If $D_{g,A}(X)$ has nonempty Euclidean interior in $A$, then $D_{g,A}(X)$ is Zariski dense in $\mathbb{C}^{\ell}$.

**Proof.** By hypothesis, there exists an open (Euclidean) ball $B \subseteq \mathbb{C}^\ell$ such that $\emptyset \neq B \cap A \subseteq D_{g,A}(X)$. In particular, by local Zariski denseness of $A$, the Zariski closures satisfy $\mathbb{C}^\ell = \overline{B \cap A} \subseteq \overline{D_{g,A}(X)}$. Hence $\mathbb{C}^\ell = \overline{D_{g,A}(X)}$. \qed
Example 2.10. The set $D_{g,A}(X)$ might be nonempty but not (Euclidean) dense in $A$. Consider the single parametric polynomial

$$g(\alpha, x) = \alpha_1 x - \alpha_2 x + \alpha_3$$

with $n = 1$, $\ell = 3$, $X = \mathbb{R}_{>0}$ and $A = \mathbb{R}^3_{>0}$. Then

$$D_{g,A}(X) = \{ \alpha \in \mathbb{R}^3_{>0} : \alpha_1 < \alpha_2 \}.$$

We have that $D_{g,A}(X)$ is full dimensional in $\mathbb{R}^3_{>0}$ and so is its complement. As $\mathbb{R}^3_{>0}$ is locally Zariski dense in $\mathbb{C}^3$ and $D_{g,A}(X)$ has nonempty Euclidean interior in $\mathbb{R}^3_{>0}$, by Lemma 2.9 we obtain that $D_{g,A}(X)$ is Zariski dense and hence we are in the scenario (i) of Corollary 2.7.

The next results show that for most interesting sets, the converse of Lemma 2.9 also holds, that is, being Zariski dense or having nonempty Euclidean interior are equivalent.

Lemma 2.11.

(i) A constructible set $S \subseteq \mathbb{C}^n$ is Zariski dense in $\mathbb{C}^n$ if and only if it has nonempty Euclidean interior in $\mathbb{C}^n$.

(ii) A semialgebraic set $S \subseteq \mathbb{R}^n$ is Zariski dense in $\mathbb{C}^n$ if and only if it has nonempty Euclidean interior in $\mathbb{R}^n$.

Proof. For part (i), note that $S$ being constructible means that it can be written in the form $S = \bigcup_{i=1}^{m} Z_i \cap U_i$, for irreducible Zariski closed sets $Z_i \subseteq \mathbb{C}^n$ and Zariski open subsets $U_i \subseteq \mathbb{C}^n$, with $Z_i \cap U_i \neq \emptyset$ for each $i = 1, \ldots, m$. The statement now follows from the fact that the Zariski closure satisfies $\overline{S} = \bigcup_{i=1}^{m} Z_i$. In case (ii), note that [BCR98, Prop. 2.8.2] gives that $S$ is Zariski dense in $\mathbb{C}^n$ if and only if its semialgebraic dimension is $n$, which in turn is equivalent to $S$ having nonempty Euclidean interior in $\mathbb{R}^n$. \hfill $\square$

We now apply Lemma 2.11 to $D_{g,A}(X)$.

Proposition 2.12. Assume that $A$ and $X$ satisfy any of the following cases:

(i) $A \subseteq \mathbb{R}^\ell$ and $X \subseteq (\mathbb{R}^*)^n$ are both semialgebraic.

(ii) $A \subseteq \mathbb{C}^\ell$ and $X \subseteq (\mathbb{C}^*)^n$ are both constructible.

(iii) $A \subseteq \mathbb{R}^\ell$ is semialgebraic and $X \subseteq (\mathbb{C}^*)^n$ is constructible.

Assume additionally that $A$ is locally Zariski dense in $\mathbb{C}^\ell$. Then $D_{g,A}(X)$ is Zariski dense in $\mathbb{C}^\ell$ if and only if it has nonempty Euclidean interior in $A$.

Proof. The reverse implication is Lemma 2.9. The forward implication is a consequence of Lemma 2.11. Note that if $D_{g,A}(X)$ has nonempty Euclidean interior in $\mathbb{R}^\ell$ in cases (i) and (iii) and in $\mathbb{C}^\ell$ in case (ii), then it has nonempty Euclidean interior in $A$. In case (i), the Tarski–Seidenberg theorem [BCR98, Thm. 2.2.1] gives that $D_{g,A}(X) \subseteq \mathbb{R}^\ell$ is semialgebraic. In case (ii) the set $D_{g,A}(X) \subseteq \mathbb{C}^\ell$ is constructible by Chevalley’s theorem. Finally, in case (iii), Chevalley’s theorem gives that $D_{g,C}(X) \subseteq \mathbb{C}^\ell$ is constructible. This in turn implies that

$$D_{g,A}(X) = D_{g,C}(X) \cap A = (D_{g,C}(X) \cap \mathbb{R}^\ell) \cap A$$

is an intersection of semialgebraic sets and therefore semialgebraic (note that the real points of a constructible set in $\mathbb{C}^\ell$ form a semialgebraic set in $\mathbb{R}^\ell$). \hfill $\square$
2.5. **Nondegeneracy and real dimension.** An approach to decide whether \( D_{g,A}(X') \) has nonempty Euclidean interior is to use the concept of nondegeneracy. In turn, nondegenerate solutions will allow us to compare the real and complex dimension of the varieties in an easier way.

**Definition 2.13.** Given a polynomial system \( g(x) = 0 \) with \( g = (g_1, \ldots, g_s) \in \mathbb{C}[x^\pm]^s \), and \( x = (x_1, \ldots, x_n) \), a solution \( x^* \) is called **nondegenerate** if \( \text{rk}(J_g(x^*)) = s \). Otherwise, it is called **degenerate**.

Note that the concept of nondegeneracy refers to the system and not to the variety defined by the system. Note also that if the coefficient matrix of \( g \) does not have full rank \( s \), then all solutions to the system will be degenerate. Hence, a natural preprocessing step of any system \( g(x) = 0 \) is to remove redundant equations by keeping a maximal set of linearly independent equations. In particular, we could assume that the rank of the coefficient matrix of \( g \) is \( s \).

The following proposition gathers well-known results about nondegenerate solutions, which will become relevant later on.

**Proposition 2.14.** Let \( g = (g_1, \ldots, g_s) \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]^s \).

(i) If \( x^* \in (\mathbb{C}^*)^n \) is a nondegenerate solution to \( g(x) = 0 \), then \( x^* \) is a nonsingular point of the variety \( V_{\mathbb{C}^*}(g) \), and belongs to a unique irreducible component of dimension \( n - s \).

(ii) The set of nondegenerate solutions of \( g(x) = 0 \) is either empty or a Zariski open subset of \( V_{\mathbb{C}^*}(g) \).

(iii) If \( g \in \mathbb{R}[x_1^\pm, \ldots, x_n^\pm]^s \) and \( x^* \) is a nondegenerate real solution to \( g(x) = 0 \), then the irreducible component of \( V_{\mathbb{R}^*}(g) \) containing \( x^* \) has dimension \( n - s \).

**Proof.** Statement (i) is Theorem 9 in [CLO15, Section 9.6]. For statement (ii), note that the degenerate locus is given by the vanishing of all \( s \times s \) minors of \( J_g(x) \) for \( x \in V_{\mathbb{C}^*}(g) \), and therefore is Zariski closed. For (iii), consider (i) and Remark 2.5. \( \square \)

We note that nonsingular points of a variety are not necessarily nondegenerate solutions of a defining polynomial set.

**Lemma 2.15.** Let \( g \in \mathbb{C}[\alpha, x^\pm]^s \), and consider the specialized polynomials \( g_\alpha \) for \( \alpha \in A \) for some subset \( A \subseteq \mathbb{C}^l \). Suppose that there is some parameter value \( \alpha^* \in A \) such that \( g_{\alpha^*}(x) = 0 \) has a nondegenerate solution in \( (\mathbb{C}^*)^n \). Then the following holds:

(i) \( D_{g,A} \) has nonempty Euclidean interior in \( A \).

(ii) If in addition \( A \) is locally Zariski dense in \( \mathbb{C}^l \), then \( D_{g,A} \) is Zariski dense in \( \mathbb{C}^l \).

**Proof.** For part (i), suppose that \( (\alpha^*, x^*) \in A \times (\mathbb{C}^*)^n \) is such that \( \text{rk}(J_{g_{\alpha^*}}(x^*)) = s \). Let \( A \in \mathbb{C}^{(n-s) \times n} \) be a matrix whose rows extend the rows of \( J_{g_{\alpha^*}}(x^*) \) to a basis for \( \mathbb{C}^n \). Then \( x^* \) is a nondegenerate solution of the square system \( g_{\alpha^*}(x) = 0 \), where

\[
\tilde{g}(\alpha^*, x) = \begin{bmatrix} g(\alpha^*, x) \\ Ax - Ax^* \end{bmatrix} \in \mathbb{C}[\alpha, x^\pm]^n.
\]

The complex implicit function theorem [Huy05, Prop. 1.1.11] now gives that there exists an open Euclidean neighborhood of \( \alpha^* \) contained in \( D_{g,\mathbb{C}^l} \). Intersecting this open set with \( A \), we obtain the statement. Part (ii) is immediate from (i) and Lemma 2.9. \( \square \)
Corollary 2.16. Let \( g \in \mathbb{C}[\alpha, x^\pm] \) and \( g_\alpha \) be as in (2.4). Let \( \mathcal{A} \subseteq \mathbb{C}^\ell \) be locally Zariski dense. Assume that the incidence variety \( \mathcal{E}_g \) is irreducible of dimension \( n + \ell - s \), and that \( g_\alpha(x) = 0 \) has a nondegenerate solution in \((\mathbb{C}^*)^n\) for some \( \alpha^* \in \mathcal{A} \).

Then \( \mathbb{V}_{\mathbb{C}^*}(g_\alpha) \) is generically equidimensional of dimension \( n - s \), that is, this holds for all \( \alpha \) in a nonempty Zariski open subset of \( \mathcal{D}_{g,\mathcal{A}} \).

If in addition \( \mathcal{A} \subseteq \mathbb{R}^\ell \), then \( \mathbb{V}_{\mathbb{R}^*}(g_\alpha) \) is generically of dimension \( n - s \).

Proof. Lemma 2.15 gives that \( \mathcal{D}_{g,\mathcal{A}} \) has nonempty Euclidean interior in \( \mathcal{A} \) and by Lemma 2.9, it is Zariski dense in \( \mathbb{C}^\ell \). As \( \mathcal{E}_g \) is irreducible, Theorem 2.2 gives that \( \mathbb{V}_{\mathbb{C}^*}(g_\alpha) \) has dimension \( \dim(\mathcal{E}_g) - \ell = n - s \) for generic \( \alpha \in \mathcal{D}_{g,\mathcal{A}} \). For the last statement, we use Proposition 2.14(iii).

Note that in the setting of Corollary 2.16, the real dimension being \( n - s \) does not directly imply equidimensionality.

Corollary 2.16 gives a condition for the generic dimension of \( \mathbb{V}_{\mathbb{C}^*}(g_\alpha) \) to be the smallest possible, namely \( n - s \). This condition is not satisfied in Example 2.3, where there are no nondegenerate solutions. Consistently, we saw that the generic dimension was strictly higher than \( n - s \) in that case.

In the next section we introduce three families of polynomial systems where \( \mathcal{D}_{g,\mathcal{A}}(X) \) has nonempty Euclidean interior if and only if there exists a nondegenerate solution, and if and only if the generic dimension is the minimal, namely \( n - s \). This phenomenon does not happen for general families, as Example 2.3 shows.

We conclude this subsection by noting that in certain scenarios within the setting of Theorem 2.2, the system \( g_\alpha(x) = 0 \) will only have nondegenerate solutions for all \( \alpha \) in a nonempty Zariski open subset of \( \mathcal{A} \).

Theorem 2.17. Let \( g \in \mathbb{C}[\alpha, x^\pm] \) and \( g_\alpha \) as in (2.4), for parameters \( \alpha \in \mathcal{A} \), with \( \mathcal{A} \subseteq \mathbb{C}^\ell \) being Zariski dense. Assume that the incidence variety \( \mathcal{E}_g \) is irreducible of dimension \( \ell \) and that there exists a nondegenerate solution to \( g_\alpha(x) = 0 \) for some \( \alpha \in \mathcal{A} \). Then there exists a nonempty Zariski open subset \( U \subseteq \mathcal{A} \) such that for all \( \alpha \in U \), any solution to \( g(x) = 0 \) is nondegenerate.

Proof. The points \((\alpha, x) \in \mathcal{E}_g \) for which \( x \) is a degenerate solution of \( g_\alpha \) form a proper Zariski closed subset \( Z \) of \( \mathcal{E}_g \), as they are the zeroes of a collection of polynomial equations and there exists a nondegenerate solution. As \( \dim(\mathcal{E}_g) = \ell \), \( \dim(Z) < \ell \), and hence \( \overline{\pi(Z)} \subseteq \mathbb{C}^\ell \) has dimension strictly smaller than \( \ell \). For any \( \alpha \) in the nonempty Zariski open set \( U := \mathbb{C}^\ell \setminus \overline{\pi(Z)} \), all solutions to \( g_\alpha(x) = 0 \) are nondegenerate. As \( \mathcal{A} \) is Zariski dense, the Zariski open set \( U \) in the statement is \( U \cap \mathcal{A} \neq \emptyset \).

3. Generic dimension for the three parametric systems

We consider now three polynomial systems of a specific form, motivated by the application to the study of steady states of reaction networks. These systems are built from parametric systems that are linear in the parameters, and, in addition, each parameter accompanies always the same monomial in \( x \).

3.1. The parametric systems. The first parametric system we consider takes the form

\[
f(\kappa, x) = N(\kappa \circ x^B), \quad x \in \mathcal{X} \subseteq (\mathbb{C}^*)^n, \quad \kappa \in \mathcal{K} \subseteq \mathbb{C}^r, \tag{3.1}
\]
with $N \in \mathbb{C}^{s \times r}$ of full rank $\text{rk}(N) = s$, and $B \in \mathbb{Z}^{n \times r}$. We let $f_{\kappa}$ denote the specialization of $f$ to a chosen value of $\kappa$. In the notation of the previous section, $A = \mathbb{K}$ and $\ell = r$.

For a vector subspace $S \subseteq \mathbb{C}^n$ of dimension $s$, we are also interested in the intersection of $\mathcal{V}_{\mathbb{C}^r}(f_{\kappa}) \cap \mathcal{X}$ with parallel translates of $S$. To this end, consider any (full rank) matrix $W \in \mathbb{C}^{d \times n}$ such that $S = \ker(W)$ and $d = n - s$. Then we consider the family of linear varieties parametrized by $c = (c_1, \ldots, c_d) \in C \subseteq \mathbb{C}^d$ as the solutions to the polynomial family

$$Wx - c = 0.$$ 

The intersections of the parallel translates of $S$ with $\mathcal{V}_{\mathbb{C}^r}(f_{\kappa})$ are the solutions to the extended polynomial system $F_{\kappa,c}(x) = 0$ with

$$F(\kappa, c, x) = \begin{bmatrix} f(\kappa, x) \\ Wx - c \end{bmatrix}, \quad x \in \mathcal{X} \subseteq (\mathbb{C}^s)^n, \quad \kappa \in \mathcal{K} \subseteq \mathbb{C}^r, \quad c \in C \subseteq \mathbb{C}^d.$$ 

For this second parametrized system, the parameter vector is $\alpha = (\kappa, c)$ and the parameter space becomes $A = \mathbb{K} \times C \subseteq \mathbb{C}^r \times \mathbb{C}^d$ and $\ell = r + d$.

Note that if $C \cap W(\mathcal{X}) = \emptyset$, then the system $F_{\kappa,c}(x) = 0$ does not have a solution in $\mathcal{X} \subseteq \mathbb{C}^n$. Since $W$ has full rank $d$, it defines a continuous and surjective map. If $\mathcal{X} \subseteq \mathbb{C}^n$ is locally Zariski dense, then so is $W(\mathcal{X})$, and in particular,

$$W(\mathcal{X}) = \mathbb{C}^d.$$

Finally, the third parametric system we consider is given by the restriction of $F$ to a value of $c$:

$$F_{\kappa,c}(\kappa, x) = F(\kappa, c, x).$$ 

### 3.2. The incidence varieties.

We now consider the incidence varieties (2.2) for the polynomial functions $f$, $F$ and $F_{\kappa,c}$, and derive some basic facts about their geometry.

**Proposition 3.1.** Let $N \in \mathbb{C}^{s \times r}$ of rank $s$, $W \in \mathbb{C}^{(n-s) \times n}$ of rank $d = n - s$, and $B \in \mathbb{Z}^{n \times r}$. Construct $f$, $F$ and $F_{\kappa,c}$ as in (3.1), (3.2) and (3.3) for $\kappa \in \mathbb{C}^r$ and $c \in \mathbb{C}^d$.

(i) The incidence varieties $\mathcal{E}_f \subseteq \mathbb{C}^r \times (\mathbb{C}^s)^n$ and $\mathcal{E}_F \subseteq \mathbb{C}^{r+d} \times (\mathbb{C}^s)^n$ admit injective rational parametrizations

$$\mathbb{C}^{r-s} \times (\mathbb{C}^s)^n \rightarrow \mathcal{E}_f, \quad \mathbb{C}^{r-s} \times (\mathbb{C}^s)^n \rightarrow \mathcal{E}_F.$$

Hence, $\mathcal{E}_f$ and $\mathcal{E}_F$ are irreducible varieties of dimension $r + d$.

(ii) For any $c \in \mathbb{C}^d$, the incidence variety $\mathcal{E}_{F_{\kappa,c}} \subseteq \mathbb{C}^r \times (\mathbb{C}^s)^n$ admits an injective rational parametrization

$$\mathbb{C}^{r-n} \times (\mathbb{C}^s)^n \rightarrow \mathcal{E}_{F_{\kappa,c}}.$$

Hence, $\mathcal{E}_{F_{\kappa,c}}$ is an irreducible variety of dimension $r$.

(iii) The incidence varieties $\mathcal{E}_f$, $\mathcal{E}_F$ and $\mathcal{E}_{F_{\kappa,c}}$ (for any $c \in \mathbb{C}^d$) have no singular points.

**Proof.** Viewing $f$ as a linear function in $\kappa_1, \ldots, \kappa_r$, the coefficient matrix is $N \text{diag}(x^B)$. As $x \in (\mathbb{C}^s)^n$, the rank of the coefficient matrix is $\text{rk}(N) = s$. Hence, the equation $f(\kappa, x) = 0$ can be solved for $s$ of the $\kappa_i$’s, giving a parametrization $\varphi(\kappa', x)$ of the solution set in terms of the $r - s$ remaining $\kappa_i$’s (forming the vector $\kappa' \in \mathbb{C}^{r-s}$) and the entries of $x$.

Similarly, as the first $s$ components of $F(\kappa, c, x)$ agree with $f$, we consider the parametrization $\varphi(\kappa', x)$ above. The last $d$ components are linear in $x$, with coefficient matrix $W$ of rank $d$. Hence, $d$ of the $x_i$’s can be solved in terms of the rest of the $x_i$’s and $c$, yielding a function
\[ \psi(x', c). \] The desired parametrization is \( \varphi(\kappa', \psi(x', c)) \), which has \( r - s + n - d + d = r + d \) free parameters.

Finally, the parametrization of \( E_{F,c} \) arises by specializing \( \varphi(\kappa', \psi(x', c)) \) to the chosen \( c \), and hence has \( r + d - d = r \) free parameters.

To prove (iii) it suffices to show that all points in the variety are nondegenerate solutions of the equations defining the variety as in (2.2). To this end, we find a minor of each of the Jacobian matrices \( J_f(\kappa, x), J_F(\kappa, c, x), J_{F,c}(\kappa, x) \) with maximal rank. For \( f \), the submatrix arising from the partial derivatives of \( f \) with respect to \( \kappa \) is \( \text{N diag}(x^B) \), which has maximal rank \( s \). For \( F \) and \( F_c \), the Jacobian matrix has block form

\[
\begin{bmatrix}
\text{N diag}(x^B) & * \\
0 & W
\end{bmatrix},
\]

where 0 denotes the zero matrix of size \( d \times r \). As \( \text{N diag}(x^B) \) and \( W \) have maximal rank, so has the Jacobian matrix.

Proposition 3.1 tells us that the hypothesis of Theorem 2.2 on the incidence variety being irreducible holds for the three parametrized systems considered here. Therefore, for each of the families, the minimal dimension is attained if scenario (i) of Theorem 2.2 holds.

**Corollary 3.2.** Let \( g = F \) as in (3.2), \( g = F_c \) as in (3.3), or \( g = f \) as in (3.1) with additionally \( s = n \). If there exists a nondegenerate solution to \( g_{\alpha^*}(x) = 0 \) for some parameter value \( \alpha^* \), then, for \( \alpha \) in a nonempty Zariski open subset of the respective parameter spaces, all solutions to \( g_\alpha(x) = 0 \) are nondegenerate.

**Proof.** This is a consequence of Theorem 2.17 and Proposition 3.1.

### 3.3. Nondegenerate solutions and nonempty Euclidean interior

By Corollary 2.16, if any of the polynomial functions \( g \) in play have a nondegenerate solution, then \( D_{g,A} \) has nonempty Euclidean interior and the complex algebraic varieties have the minimal dimension given by the number of equations. We will see next that the converse also holds for the three families \( g \) under consideration, that is, if \( D_{g,A} \) has nonempty Euclidean interior for \( A \) the largest possible parameter space (in each case), then necessarily the system has a nondegenerate solution.

We start with a description of the sets \( D_{f,K}(X) \), \( D_{F,c,K}(X) \) and of \( D_{F,A}(X) \) with \( A \subseteq \mathbb{C}^r \times \mathbb{C}^d \). We proceed then to study nondegenerate solutions of the systems given by these families and conclude with the main theorem relating nondegenerate solutions to the Euclidean interior of \( D_{g,A} \) for \( A \) begin the maximal complex subspace of parameters used.

**Proposition 3.3.** With \( f, F, F_c \) as in (3.1), (3.2) and (3.3), let \( \mathbb{M} \) be a multiplicative subgroup of \( \mathbb{C}^* \), \( K \in \{\mathbb{M}', (\mathbb{M} \cup \{0\})^*\} \), and \( X \subseteq \mathbb{M}^n \) a multiplicative subgroup. Let \( A \subseteq K \times \mathbb{C}^d \). Then it holds that

\[
D_{f,K}(X) = \{w \circ h^B : w \in \ker(N) \cap K, h \in X\},
\]

\[
D_{F,c,K}(X) = \{w \circ h^B : w \in \ker(N) \cap K, h \in X, c = Wh^{-1}\},
\]

\[
D_{F,A}(X) = \{(w \circ h^B, Wh^{-1}) : w \in \ker(N) \cap K, h \in X\}.
\]

**Proof.** We note that \( \kappa \in D_{f,K}(X) \) if and only if \( \kappa \circ x^B \in \ker(N) \cap K \) for some \( x \in X \). This holds if and only if \( \kappa = w \circ (x^{-1})^B \) for some \( w \in \ker(N) \cap K \) and \( x \in X \). By letting \( h = x^{-1} \)
we obtain the desired equality. Similarly, for \(F\) and \(F,c\), all we need is to include the extra equation \(c = Wh^{-1}\), which translates into \(c = Wh^{-1}\) as \(h = x^{-1}\).

**Proposition 3.4.** With \(f, F, F,c\) as in (3.1), (3.2) and (3.3), let \(M\) be a multiplicative subgroup of \(C^*, K \in \{\mathbb{M}^r, (\mathbb{M} \cup \{0\})^r\}\), and \(A \subseteq \mathbb{M}^n\) a multiplicative subgroup. Let \(A \subseteq K \times C^r\). Then the following holds:

(i) There exists a nondegenerate solution of \(f_\kappa(x) = 0\) in \(X\) for some \(\kappa \in K\) if and only if \(N \text{diag}(w)B^\top \in \mathbb{C}^{n \times n}\) has rank \(s\) for some \(w \in \ker(N) \cap K\).

(ii) There exists a nondegenerate solution of \(F_{\kappa,c}(x) = 0\) in \(X\) for some \((\kappa, c) \in A\) (resp. for some \(\kappa \in K, c\) fixed) if and only if the matrix

\[
\begin{bmatrix}
N \text{diag}(w)B^\top \text{diag}(h) \\
W
\end{bmatrix} \in \mathbb{C}^{n \times n}
\]

has rank \(n\) for some \(w \in \ker(N) \cap K\) and some \(h \in X\) (resp. some \(h \in X\) such that \(c = Wh^{-1}\)).

**Proof.** An easy computation shows that

\[J_{f_\kappa}(x) = N \text{diag}(\kappa \circ x^B)B^\top \text{diag}(x^{-1}).\]

Statement (i) follows directly from this, by noting that the set of vectors \(\kappa \circ x^B\) for which \(f(\kappa, x) = 0\), \(\kappa \in K\) and \(x \in X\) is exactly \(\ker(N) \cap K\) as \((1, \ldots, 1) \in X\). Statement (ii) follows similarly by noting that

\[J_{F_{\kappa,c}}(x) = \begin{bmatrix}
N \text{diag}(\kappa \circ x^B)B^\top \text{diag}(x^{-1}) \\
W
\end{bmatrix}
\]

and letting \(h = x^{-1}\). \(\square\)

**Remark 3.5.** Suppose \(M \subseteq C^*\) is a multiplicative subgroup that is Zariski dense in \(C\) (for instance \(\mathbb{R}_{>0}\) or \(\mathbb{R}^*\)). If in addition \(\ker(N) \cap M^r \neq \emptyset\), then \(\ker(N) \cap M^r\) is Zariski dense in \(\ker(N)\). As a consequence, (i) in Proposition 3.4 is equivalent to the existence of some \(w \in \ker(N)\) such that \(\text{rk}(N \text{diag}(w)B^\top) = s\). Similarly, (ii) is equivalent to the existence of some \(w \in \ker(N)\) and \(h \in X\) such that

\[\text{rk} \begin{bmatrix}
N \text{diag}(w)B^\top \text{diag}(h) \\
W
\end{bmatrix} = s.
\]

This makes it significantly easier to verify the conditions in (i) and (ii) computationally in concrete examples, as we will see in Subsection 3.5.

We now use these propositions to show the main result of this subsection. The statement is given in the full setting where \(X = (C^*)^n\) and the parameter space is the maximal complex space considered in each situation. We will see later on that the statement also holds for common sets \(A, X\) (Theorem 3.7).

**Theorem 3.6.** Consider \(f\) as in (3.1) and \(F\) as in (3.2). Let \((g, A)\) be either

(i) \((f, C^n)\),

(ii) \((F, C^r \times C^d)\), or

(iii) \((F,c, C^*)\) for some fixed \(c\).

For each of these cases, it holds that, under the assumption that \(D_{g,A} \neq \emptyset\), all solutions \(x^* \in (C^*)^n\) of \(g_\alpha(x) = 0\) for all \(\alpha \in D_{g,A}\) are degenerate if and only if \(D_{g,A}\) has empty Euclidean interior in \(A\).
Proof. For the three polynomial systems, the reverse implication follows from Lemma 2.15.

For the forward implication, we will use Proposition 3.3 and 3.4 with \( M = \mathbb{C}^* \) and \( \mathcal{X} = (\mathbb{C}^*)^n \). Let us start with (i). We assume all solutions \( x^* \in (\mathbb{C}^*)^n \) of \( f_N(x) = 0 \) are degenerate. Proposition 3.3 gives that \( \mathcal{D}_{f,k}(\mathcal{X}) \) is contained in the Zariski closure in \( \mathbb{C}^r \) of the image of the map

\[
\varphi: \mathbb{C}^{r-s} \times (\mathbb{C}^*)^n \to \mathbb{C}^r, \quad (u,h) \mapsto Gu \circ h^B,
\]

where \( G \in \mathbb{C}^{r \times (r-s)} \) is any matrix whose columns form a basis for \( \ker(N) \) (recall we are assuming \( N \) has full rank). We show now that \( \text{im}(\varphi) \) has dimension strictly less than \( r \), by proving that the rank of the Jacobian of \( \varphi \),

\[
J_\varphi(u,h) = \begin{bmatrix} \text{diag}(h^B)G \mid \text{diag}(Gu \circ h^B)B^\top \text{diag}(h^{-1}) \end{bmatrix} \in \mathbb{C}^{r \times (r-s+n)},
\]

(3.4)
is strictly smaller than \( r \) for all \( (u,h) \). Specifically, we will do this by exhibiting more than \( n-s \) linearly independent vectors in \( \ker(\text{diag}(Gu \circ h^B)) \). The rank of \( J_\varphi(u,h) \) agrees with the rank of

\[
C(u,h) = \begin{bmatrix} \text{diag}(h^B)G \\ \text{diag}(Gu \circ h^B)B^\top \end{bmatrix} \in \mathbb{C}^{r \times (r+n-s)}
\]

so we focus on this matrix instead. The first block of \( C(u,h) \) has \( r-s \) columns and the second block has \( n \) columns.

We can immediately find \( n - \text{rk}(B) \) linearly independent vectors \( \delta_j \in \mathbb{C}^n \) for \( j = 1, \ldots, n - \text{rk}(B) \) from \( \ker(B^\top) \). Then for each \( j \), \( (0, \delta_j) \in \mathbb{C}^{r-s+n} \) belongs to \( \ker(C(u,h)) \). If \( \text{rk}(B) < s \), we are done. If not, then our assumption that all solutions are degenerate gives us that \( \text{rk}(N \text{diag}(u)B^\top) < s \) for all \( w \in \ker(N) \cap (\mathbb{C}^s)^r \) by Proposition 3.4(i). This in turn implies that \( \text{rk}(N \text{diag}(u)B^\top) < s \) for all \( w \in \ker(N) \) as \((\mathbb{C}^s)^r \) is Zariski dense in \( \mathbb{C}^r \). Hence,

\[
n - s < \dim(\ker(N \text{diag}(Gu \circ h^B))) = \dim(\ker(B^\top)) + \dim(\ker(N) \cap \text{im}(\text{diag}(Gu \circ h^B))) = n - \text{rk}(B) + \dim(\ker(G) \cap \text{im}(\text{diag}(Gu \circ h^B))),
\]

(3.5)

for any \( u \in \mathbb{C}^{r-s} \). From this we conclude that there are \( p > \text{rk}(B) - s \geq 0 \) linearly independent vectors

\[
\gamma_i \in \text{im}(\ker(B^\top)) \cap \text{im}(\text{diag}(Gu \circ h^B)), \quad \text{for } i = 1, \ldots, p.
\]

For each \( i \), \( \gamma_i = G\alpha_i = \text{diag}(Gu \circ h^B)\beta_i \) for some \( \alpha_i \in \mathbb{C}^{r-s}, \beta_i \in \mathbb{C}^n \). Now, for \( i = 1, \ldots, p \), \((-\alpha_i, \beta_i) \in \ker(C(u,h)) \) by construction.

All that is left is to see that the collection of vectors \((0, \delta_j)\) and \((-\alpha_i, \beta_i)\) is linearly independent. Assume that we have a linear combination

\[
\sum_{i=1}^p a_i(-\alpha_i, \beta_i) + \sum_{j=1}^{n-\text{rk}(B)} b_j(0, \delta_j) = 0.
\]

Then \( \sum a_i \alpha_i = 0 \), which after multiplication by \( G \) gives \( \sum a_i \gamma_i = 0 \), and we conclude that \( a_i = 0 \) for all \( i \). Then from \( 0 = \sum b_j \delta_j \) we conclude that \( b_j = 0 \) for all \( j \). All in all, we have now found \( p + n - \text{rk}(B) > \text{rk}(B) - s + n - \text{rk}(B) = n - s \) linearly independent vectors in the kernel of \( C(u,h) \). This concludes the proof for case (i).

To show the forward implication in (ii), Proposition 3.3 gives that \( \mathcal{D}_{f,k}(\mathcal{X}) \) is contained in the Zariski closure in \( \mathbb{C}^r \times \mathbb{C}^d \) of the image of the map

\[
\psi: \mathbb{C}^{r-s} \times (\mathbb{C}^*)^n \to \mathbb{C}^r \times \mathbb{C}^d, \quad (u,h) \mapsto (Gu \circ h^B, Wh^{-1}),
\]

(3.6)
where \( G \in \mathbb{C}^{r \times (r-s)} \) is any matrix whose columns form a basis for \( \ker(N) \). We proceed as in case (i). The Jacobian matrix of \( \psi \) is the square matrix
\[
J_{\psi}(u, h) = \begin{bmatrix}
\text{diag}(h^T)G & \text{diag}(Gu \circ h^T)B^T \text{diag}(h^{-1}) \\
0 & -W \text{diag}(h^{-2})
\end{bmatrix} \in \mathbb{C}^{(r+d) \times (r-s+n)}.
\]
By multiplying this matrix from the right by \( \text{diag}((1, \ldots, 1, h^{-2})) \in \mathbb{C}^{r-s} \times \mathbb{C}^n \), the rank of this matrix agrees with the rank of
\[
C(u, h) := \begin{bmatrix}
\text{diag}(h^T)G & \text{diag}(Gu \circ h^T)B^T \text{diag}(h) \\
0 & W
\end{bmatrix} \in \mathbb{C}^{(r+d) \times (r+d)}.
\]
We now show that \( \overline{\text{im}(\psi)} \) has dimension strictly less than \( r + d \), by finding a nonzero vector in the kernel of \( C(u, h) \) for all \((u, h) \in \mathbb{C}^{r-s} \times (\mathbb{C}^*)^n \). Proceeding as above, by Proposition 3.4 ii), as all solutions of \( F_{x,c}(x) = 0 \) are degenerate, the kernel of the matrix
\[
\begin{bmatrix}
N \text{diag}(Gu)B^T \text{diag}(h) \\
W
\end{bmatrix}
\]
is nonzero for all \((u, h) \in \mathbb{C}^{r-s} \times (\mathbb{C}^*)^n \). Let \( \beta \in \mathbb{C}^n \) be a nonzero vector in the kernel of this matrix. Hence \( W\beta = 0 \) and \( N \text{diag}(Gu)B^T \text{diag}(h)(\beta) = 0 \) and it follows that \( \text{diag}(Gu)B^T(h \circ \beta) = Go \) for some \( \alpha \in \mathbb{C}^{r-s} \). Then, the vector \((\alpha, \beta)\), which is nonzero, belongs to \( \ker(C(u, h)) \). This shows case (ii).

To show the forward implication for (iii), we note that the set \( \{h \in (\mathbb{C}^*)^n : Wh^{-1} = c\} \) admits an injective rational parametrization
\[
\Phi: U \to (\mathbb{C}^*)^n
\]
where \( U \) is a nonempty Zariski open subset of \( \mathbb{C}^s \). This parametrization is obtained as the composition of a parametrization of the linear variety \( \{x \in (\mathbb{C}^*)^n : Wx = c\} \) with the inverse map from \((\mathbb{C}^*)^n \) to \((\mathbb{C}^*)^n\) sending \( x \) to \( x^{-1} \). It follows that \( J_\Phi(z) \) has full rank \( s \) for all \( z \in U \).

Using this and Proposition 3.3, we obtain that \( \mathcal{D}_{F_{x,c},A}(\mathcal{X}) \) is contained in the Zariski closure in \( \mathbb{C}^r \) of the image of the composition
\[
\tilde{\varphi} := \varphi \circ (\text{Id}_{r-s} \times \Phi) : \mathbb{C}^{r-s} \times U \to \mathbb{C}^r,
\]
with \( \varphi \) as given in (3.4) and \( \text{Id}_{r-s} \) the identify map on \( \mathbb{C}^{r-s} \), corresponding to the \( u \) variables. We show that the Jacobian of \( \tilde{\varphi} \) has rank strictly smaller than \( r \), and from this the statement follows as in the previous cases.

By the multivariate chain rule, we have that
\[
J_{\tilde{\varphi}}(u, z) = J_{\tilde{\varphi}}(u, \Phi(z)) \begin{bmatrix}
\text{Id}_{(r-s) \times (r-s)} & 0_{(r-s) \times s} \\
0_{n \times (r-s)} & J_\Phi(z)
\end{bmatrix}.
\]
As \( W\Phi(z) = 0 \) identically, implicit differentiation gives \( WJ_\Phi(z) = 0 \). Furthermore, as by construction \( J_\Phi(z) \in \mathbb{C}^{n \times s} \) has full rank \( s \), its columns form a basis of \( \ker(W) \).

We proceed as in case (ii) and use that the kernel of the matrix
\[
\begin{bmatrix}
N \text{diag}(Gu)B^T \text{diag}(\Phi(z)) \\
W
\end{bmatrix}
\]
is nonzero for all \( u \) and all \( \Phi(z) \) by assumption, to construct a nonzero vector \((\alpha, \beta)\) in the kernel of \( J_{\tilde{\varphi}}(u, \Phi(z)) \), see (3.5), which additionally satisfies \( W\beta = 0 \). This last condition implies that \( \beta \in \text{im}(J_\Phi(z)) \). Hence,
\[
(\alpha, \beta)^T = \begin{bmatrix}
\text{Id}_{(r-s) \times (r-s)} & 0_{(r-s) \times s} \\
0_{n \times (r-s)} & J_\Phi(z)
\end{bmatrix} \gamma.
\]
for some $\gamma \in \mathbb{C}^r$. By (3.7), we found a vector $\gamma$ in the kernel of $J_{\overline{\varphi}}(u, z)$. As this holds for all $(u, z) \in \mathbb{C}^{r-s} \times \mathcal{U}$, the closure of the image of $\overline{\varphi}$ is a proper Zariski closed set of $\mathbb{C}^r$. □

3.4. The main theorem on generic dimension. Putting all this together, we have obtained the following theorem.

**Theorem 3.7.** Consider $f$ and $\mathcal{K}$ as in (3.1) and $F$ and $\mathcal{C}$ as in (3.2). Let the objects $g, \mathcal{A} \subseteq \mathbb{C}^\ell, \ell, \delta$ be either

(a) $g = f, \mathcal{A} = \mathcal{K}, \ell = r, \delta = d$,
(b) $g = F, \mathcal{A} = \mathcal{K} \times \mathcal{C}, \ell = r + d, \delta = 0$, or
(c) $g = F_c, \mathcal{A} = \mathcal{K}, \ell = r, \delta = 0$ for some fixed $c \in \mathcal{C}$.

We consider any such pair $(g, \mathcal{A})$, and $\mathcal{X} \in (\mathbb{C}^*)^n$ satisfying:

- $\mathcal{A}$ is locally Zariski dense in $\mathbb{C}^\ell$,
- $(\mathcal{A} \times \mathcal{X}) \cap \mathcal{E}_g$ is Zariski dense in $\mathcal{E}_g$,
- $\mathcal{A}, \mathcal{X}$ are such that Proposition 2.12 applies.

Then the following are equivalent:

(i) There is a nondegenerate solution of $g_{\alpha}(x) = 0$ in $(\mathbb{C}^*)^n$ for some $\alpha \in \mathbb{C}^\ell$.

(ii) There is a nondegenerate solution of $g_{\alpha}(x) = 0$ in $\mathcal{X}$ for some $\alpha \in \mathcal{A}$.

(iii) There exists a nonempty Zariski open subset $\mathcal{U} \subseteq \mathcal{E}_g$ such that for any $(\alpha^*, x^*) \in \mathcal{U} \cap (\mathcal{A} \times \mathcal{X}), x^*$ is a nondegenerate solution of $g_{\alpha^*}(x^*) = 0$.

(iv) $D_{g, \mathcal{A}}(\mathcal{X})$ is Zariski dense in $\mathbb{C}^\ell$.

(v) $D_{g, \mathcal{A}}(\mathcal{X})$ has nonempty Euclidean interior in $\mathcal{A}$.

(vi) $\forall \mathbb{C}^r(g_{\alpha})$ is equidimensional of dimension $\delta$ for generic $\alpha \in D_{g, \mathcal{A}}(\mathcal{X})$, that is, for all $\alpha$ in a nonempty Zariski open subset $\mathcal{U} \cap D_{g, \mathcal{A}}(\mathcal{X})$ of $D_{g, \mathcal{A}}(\mathcal{X})$.

(vii) $\forall \mathbb{C}^r(g_{\alpha})$ is equidimensional of dimension $\delta$ for at least one $\alpha \in D_{g, \mathcal{A}}(\mathcal{X})$.

Furthermore, any of these imply:

(1) $\forall \mathbb{C}^r(g_{\alpha})$ is equidimensional of dimension $\delta$ for generic $\alpha \in D_{g, \mathcal{A}}(\mathcal{X})$.

(2) If $\mathcal{A} \subseteq \mathbb{R}^\ell$ and $\mathcal{X} \subseteq (\mathbb{R}^*)^n$, then $\forall \mathbb{R}^r(g_{\alpha})$ and $\forall \mathbb{R}^r(g_{\alpha})$ have dimension $\delta$ for generic $\alpha \in D_{g, \mathcal{A}}(\mathcal{X})$.

**Proof.** First, we show that (i)⇔(ii)⇔(iii). We have that (iii)⇒(ii), as $\mathcal{U}$ is a nonempty Zariski open set and $(\mathcal{A} \times \mathcal{X}) \cap \mathcal{E}_g$ is Zariski dense in $\mathcal{E}_g$, guaranteeing that the intersection $\mathcal{U} \cap (\mathcal{A} \times \mathcal{X})$ is nonempty. (ii)⇒(i) holds trivially. To prove (i)⇒(iii), note that the subset of points $(\alpha, x)$ of $\mathcal{E}_g$ for which $x$ is a nondegenerate solution of $g_{\alpha}(x) = 0$ is Zariski open if nonempty.

The equivalence (iv)⇔(v) is Proposition 2.12 which applies by the assumptions on $\mathcal{A}$ and $\mathcal{X}$.

We now show (iv)⇒(vi)⇒(vii)⇒(i). The implication (iv)⇒(vi) is Corollary 2.7. The implication (vi)⇒(vii) is clear. For the implication (vii)⇒(i), we note that (vii) implies that $D_{g, \mathcal{A}}$ is Zariski dense in $\mathbb{C}^\ell$ by Theorem 2.2 but, as $D_{g, \mathcal{A}} \subseteq D_{g, \mathcal{C}^l}$, so is $D_{g, \mathcal{C}^l}$. Hence $D_{g, \mathcal{C}^l}$ has nonempty Euclidean interior in $\mathbb{C}^\ell$ by Proposition 2.12 which leads to (i) by Theorem 3.6.

All that is left is to show that (ii)⇒(iv). By Proposition 2.12 and Theorem 3.6, (ii) gives that $D_{g, \mathcal{C}^l}$ is Zariski dense in $\mathbb{C}^\ell$. The assumption that $(\mathcal{A} \times \mathcal{X}) \cap \mathcal{E}_g$ is Zariski dense
in $\mathcal{E}_g$, gives that
\[
\overline{D}_{g,A}(X) = \pi((A \times X) \cap \mathcal{E}_g) = \pi((A \times X) \cap \mathcal{E}_g) = \pi(\mathcal{E}_g) = \overline{D}_{g,C} = \mathbb{C}^\ell.
\]

Hence, $D_{g,A}(X)$ is Zariski dense in $\mathbb{C}^\ell$, which gives (iv). This concludes the first part of the proof.

Finally, (iv) $\Rightarrow$ (1) follows by Corollary 2.7 and (ii) $\Rightarrow$ (2) follows by Corollary 2.16.

**Remark 3.8.** For any $(g, A)$ and $X$ under the assumptions of Theorem 3.7, any statement (ii)-(vii) in Theorem 3.7 is equivalent to (i), and hence to any of the statements for $A = \mathbb{C}^\ell$ and $X = (\mathbb{C}^*)^n$ instead.

**Remark 3.9.** We note that the equivalences (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) in Theorem 3.7 work in the more general setting of Proposition 3.3 and 3.4, but for the equivalences (iv) $\Leftrightarrow$ (v) and (vii) $\Rightarrow$ (v), Proposition 2.12 is required, which relies on Lemma 2.11.

**Remark 3.10.** The following scenarios that are common in applications satisfy the hypotheses in Theorem 3.7:

(i) $K \in \{\mathbb{R}^\ell_0, (\mathbb{R})^d, \mathbb{R}^\ell\}, C \in \{\mathbb{R}^\ell_0, (\mathbb{R})^d, \mathbb{R}^\ell\}$ and $X \in \{\mathbb{R}^n_0, (\mathbb{R})^n, (\mathbb{C})^n\}$.

(ii) $K \in \{(\mathbb{C})^*, \mathbb{C}^*\}, C \in \{(\mathbb{C})^d, \mathbb{C}^d\}$ and $X = (\mathbb{C}^*)^n$.

In general, we note that if $A$ and $X$ are Euclidean open subsets of $\mathbb{R}^\ell$ and $\mathbb{R}^n$ respectively, or of $\mathbb{C}^\ell$ and $\mathbb{C}^n$ respectively, then $(A \times X) \cap \mathcal{E}_g$ is Zariski dense in $\mathcal{E}_g$ if this intersection is nonempty. In the real case, this follows from the fact that $\mathcal{E}_g$ is nonsingular by Proposition 3.1(iii), together with the fact that for a complex irreducible variety defined by polynomials with real coefficients that has at least one nonsingular point, the real points form a Zariski dense subset [Set19, Thm. 5.1].

### 3.5. Computational considerations.

In the case when the matrix $N$ has rational entries, the criterion (i) in Theorem 3.7 can be checked computationally through Proposition 3.3.

The idea is the following: Let $G \in \mathbb{Q}^{r \times (n-s)}$ be a matrix whose columns form a basis for $\ker(N)$. For $f$, we want to check whether there exists some $u \in \mathbb{Q}^{n-s}$ such that
\[
\text{rk}(N \text{ diag}(Gu)B^\top) = s.
\]

Note that the set of $u$ for which the rank is $s$ forms a Zariski open subset of $\mathbb{Q}^{n-s}$, so if it is nonempty, a randomly chosen $u \in \mathbb{Q}^{n-s}$ will give $\text{rk}(N \text{ diag}(Gu)B^\top) = s$ with probability 1. Hence, we pick a random $u \in \mathbb{Q}^{n-s}$, and compute $N \text{ diag}(Gu)B^\top$ with exact arithmetic. If the rank is $s$, we conclude that criterion (i) in Theorem 3.7 holds. If not, we can suspect that it does not hold, and to conclusively prove this, we view $N \text{ diag}(Gu)B^\top$ as a symbolic matrix with indeterminate $u$ and computationally verify that all $s \times s$ minors are the zero polynomial. For $F$, we instead want to determine whether there is some $u$ such that
\[
\text{rk} \left[ N \text{ diag}(Gu)B^\top \right] = n,
\]
which is done analogously.

In the setting of Proposition 3.4 for $f$, where $K \in \{M^\ell, (M \cup \{0\})^\ell\}$ and $M$ is a multiplicative subgroup of $\mathbb{C}^*$, we know by Remark 3.10 that to check denseness of $(K \times X) \cap \mathcal{E}_g$ in $\mathcal{E}_g$ it is enough to check that $(K \times X) \cap \mathcal{E}_g \neq \emptyset$. This, in turn, is equivalent to $\ker(N) \cap K \neq \emptyset$. Indeed, if $\ker(N) \cap K = \emptyset$, then $\forall_{C^*}(f) = \emptyset$ for all $k \in K$. On the other hand, if $\ker(N) \cap K \neq \emptyset$, then $(1, \ldots, 1) \in \forall_{C^*}(f)$ for any $k \in \ker(N) \cap K$. In the special case $M = \mathbb{R}^\ell_0$, checking $\ker(N) \cap \mathbb{R}^\ell_0 \neq \emptyset$ corresponds to showing the existence of an
interior point of the polyhedral cone \( \ker(N) \cap \mathbb{R}^r_{\geq 0} \), which is a straightforward computation using either linear programming, or satisfiability modulo theories solvers [BFC17].

### 4. Application to Reaction Networks

As explained in the introduction, the main motivation behind the development of the results in Section 3 comes from the study of reaction networks. This connection will be explained in this section. To make this section accessible to readers familiar with reaction networks but not necessarily acquainted with the background and language of algebraic geometry, we will use a less technical approach. In particular, when saying that a property holds for almost all parameters in a set \( A \subseteq \mathbb{C}^\ell \), we mean that the property holds generically in the set, that is, in a nonempty Zariski open subset of \( A \). In the reaction network scenario where \( A = \mathbb{R}^\ell \) or \( \mathbb{R}^\ell_{>0} \), this implies that the property holds outside a subset of \( A \) of Lebesgue measure zero (so for almost all parameters, measure theoretically speaking, it holds too).

#### 4.1. Reaction networks

In what follows we present some generalities, necessary to establish the connection of our results with some questions arising in this field, but we refer to [Fei19] for further details.

A reaction network is normally pictured by means of a graph, whose vertices correspond to linear combinations of the species involved (complexes), connected by a directed edge whenever an interaction is assumed to lead from any of these complexes (source) to another one (product). A simple reaction network modeling the enzymatic transfer of calcium ions is

\[
\begin{align*}
0 & \xrightleftharpoons[\kappa_1/\kappa_2]{\kappa_3} X_1 \\
X_1 + X_2 & \xrightarrow[\kappa_3]{\kappa_4} 2X_1 \\
X_1 + X_3 & \xrightarrow[\kappa_4/\kappa_5]{\kappa_6} X_4 \xrightarrow[\kappa_6]{\kappa_5} X_2 + X_3.
\end{align*}
\]

(4.1)

Here, \( X_1 \) stands for cytosolic calcium, \( X_2 \) for calcium in the endoplasmic reticulum, and \( X_3 \) is an enzyme catalyzing the transfer via the formation of an intermediate protein complex \( X_4 \) [GES05]. The labels of the reactions are positive real numbers called reaction rate constants. Given any reaction network with species \( X_1, \ldots, X_n \) and reactions

\[
b_{i1}X_1 + \cdots + b_{ni}X_n \xrightarrow[\kappa_i]{\kappa_i} a_{1i}X_1 + \cdots + a_{ni}X_n, \quad i = 1, \ldots, r,
\]

under the assumption of mass-action kinetics, the concentration \( x = (x_1, \ldots, x_n) \) of the species in time is modeled by means of a system of ordinary differential equations (ODEs) of the form

\[
\frac{dx}{dt} = \Gamma(\kappa \circ x^B), \quad x \in \mathbb{R}^n_{\geq 0},
\]

(4.2)

for reaction rate constants \( \kappa = (\kappa_1, \ldots, \kappa_r) \in \mathbb{R}^r_{>0} \), stoichiometric matrix \( \Gamma \in \mathbb{Z}^{n \times r} \) with entries \( (a_{ji} - b_{ji}) \), and reactant matrix \( B \in \mathbb{Z}^{n \times r} \) with entries \( b_{ji} \). The trajectories of the ODE system are confined into parallel translates of the image of \( \Gamma \), which can be described by linear equations \( Wx - c = 0 \) with \( c \) depending on the initial condition and \( W \in \mathbb{R}^{d \times n} \) with \( d = n - \text{rk}(\Gamma) \). The intersection of each such linear subspace with \( \mathbb{R}^n_{\geq 0} \) is called a stoichiometric compatibility class. We note that for common networks in biochemistry, the rank of \( \Gamma \) is not maximal.

The reader might already notice the similarities with the parametric systems (3.1) and (3.2). These arise naturally in this context as follows:
• The positive steady state variety $V_κ$ for a given $κ ∈ R^r_≥0$ is the set of steady states of (4.2) in $R^n_>0$. By letting $N ∈ Z^{s×n}$ be a matrix of full rank $s = \text{rk}(Γ)$ and $\text{im}(N^\top) = \text{im}(Γ^\top)$, the positive steady state variety is the solution set to

\[ N(κ ⊙ x^B) = 0, \quad x ∈ R^n_>0. \tag{4.3} \]

This corresponds to the parametric system $f(κ, x)$ from (3.1) with $K = R^n_≥0$ and $X = R^n_>0$. The complex (resp. real) steady state variety is the set of solutions to (4.3) in $(C^n)^n$ (resp. $(R^n)^n$).

• The set of positive steady states within stoichiometric compatibility classes $P_{κ,c}$ for a given $κ ∈ R^r_>0$ and $c ∈ R^d$ is the set of steady states of (4.2) in $R^n_>0$ that additionally belong to the stoichiometric compatibility class defined by $c$, that is, solutions to

\[ N(κ ⊙ x^B) = 0, \quad Wx - c = 0. \tag{4.4} \]

This corresponds to the parametric system $F(κ, c, x)$ from (3.2) with $K = R^n_≥0$ and $C = R^d$ with $d = n - s$. By $P_{κ,c}^C$ we denote the set of solutions to (4.3) in $(C^n)^n$.

Since the reaction rate constants are normally unknown, studying the dynamics of a reaction network often implies understanding the behavior of the system for all parameters, maybe generically, or being able to describe the parameter regions leading to different behaviors.

In the reaction network literature, a steady state $x^*$ is said to be degenerate if it is degenerate as a solution to the system $F_{κ,Wx^*}(x) = 0$ (equivalently, if $\ker(J_{F_{κ,Wx^*}}(x^*)) \cap \text{im}(Γ) \neq \{0\}$.)

The existence of nondegenerate steady states is assumed to be common, although not guaranteed for all networks as we will see below. The existence of nondegenerate steady states is often a requirement to “lift” steady states from one network to another with arguments relying on the implicit function theorem or homotopy continuation [JS13, FW13, CF06, BP18]. The relation between nondegenerate steady states and nonsingularity pointed out in Proposition 2.14 allows also to employ machinery from algebraic geometry to the study of steady states [PEF22].

4.2. Theorems on dimension and finiteness of steady states. In the language of reaction network theory, Theorem 3.7 gives rise to the following results.

**Theorem 4.1** (Expected dimension of the complex steady state variety). Consider a reaction network with $n$ species, $r$ reactions, and stoichiometric matrix $Γ$. Consider the polynomial function $f(κ, x)$ defining $V_κ$ as in (4.3) with $B$ the reactant matrix, and $N ∈ Z^{s×n}$ of full rank $s = \text{rk}(Γ)$ and such that $\text{im}(N^\top) = \text{im}(Γ^\top)$. The following are equivalent:

(i) The system $f_κ(x) = 0$ has a nondegenerate solution in $R^n_>0$ for some $κ ∈ R^r_≥0$.

(ii) The set of parameters $κ$ for which $V_κ ≠ ∅$ has nonempty Euclidean interior in $R^r$.

(iii) The dimension of the complex steady state variety is $n - s$ for at least one $κ ∈ R^r_>0$ for which $V_κ ≠ ∅$.

(iv) The matrix $N \text{diag}(w)B^\top$ has rank $s$ for some $w ∈ \ker(N) ∈ R^n_≥0$.

If any of these hold, then the dimension of the complex and real steady state varieties is $n - s$ for almost all $κ$ for which $V_κ ≠ ∅$.

If none of these holds, then the complex steady state variety is either empty or has dimension strictly larger than $n - s$. 


Proof. The statement follows from Proposition 3.4(i) and from Theorem 3.7 case (a) together with Remark 3.10 as $K = \mathbb{R}_r^r > 0$ and $X = \mathbb{R}_s^s > 0$. □

**Corollary 4.2.** A reaction network such that for all $\kappa \in \mathbb{R}_r^r > 0$ the positive steady state variety is either empty or has dimension higher than $n - s$, has no positive steady states for almost all $\kappa \in \mathbb{R}_r^r > 0$.

**Example 4.3.** For the calcium network in (4.1), the stoichiometric matrix and the reactant matrix are given by

$$
\Gamma = \begin{bmatrix}
1 & -1 & 1 & -1 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 \\
0 & 0 & 0 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & -1
\end{bmatrix}
$$

and

$$
B = \begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}.
$$

Here, $n = 4$, $r = 6$, $s = 3$, and we can choose

$$W = \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}.$$

We note that $w = (1, 1, 1, 2, 1, 1) \in \ker(\Gamma) \cap \mathbb{R}_r^r > 0$, and that

$$\text{rk} \left[ \Gamma \text{diag}(w) B^T \right] = 4.$$

Hence Theorem 4.1(iv) holds and from this we conclude that there exist positive steady states for $\kappa$ in a set with nonempty Euclidean interior, and that the complex and real steady state varieties are 1-dimensional for almost all such reaction rate constants.

**Example 4.4.** Consider the network

$$
\begin{array}{c}
X \xrightarrow{\kappa_1} Y \\
\downarrow \kappa_2 \\
Z
\end{array}
\hspace{2cm}
\begin{array}{c}
Y + Z \xrightarrow{\kappa_3} X + Y + Z \\
\downarrow \kappa_4 \\
\emptyset
\end{array}
$$

We have

$$f_\kappa(x) = \begin{bmatrix}
-k_1 x - k_2 x + k_3 y z \\
k_1 x - k_4 y z \\
k_2 x - k_4 y z
\end{bmatrix},$$

and $V_\kappa \neq \emptyset$ precisely when $k_1 = k_2$ and $k_3 = 2k_4$. So Theorem 4.1(ii) does not hold and hence, the complex steady state variety has dimension strictly larger than $n - s = 1$ in these cases. In fact, for these particular parameter values, the complex steady state variety is defined by the equation $k_1 x - k_4 y z = 0$, and hence has complex dimension 2.

**Theorem 4.5** (Finiteness of the number of steady states in stoichiometric compatibility classes). Consider a reaction network with $n$ species, $r$ reactions, and stoichiometric matrix $\Gamma$. Consider the polynomial function $F(\kappa, c, x)$ defining $P_{\kappa, c}$ as in (4.4) with $B$ the reactant matrix, $N \in \mathbb{Z}^{s \times n}$ of full rank $s = \text{rk}(\Gamma)$ such that $\text{im}(N^T) = \text{im}(\Gamma^T)$, and $W$ a full matrix whose rows form a basis of $\text{ker}(\Gamma^T)$.

The following are equivalent:

(i) The network has a nondegenerate steady state in $\mathbb{R}_s^n_0$ for some $\kappa \in \mathbb{R}_r^r > 0$.

(ii) The set of parameters $(\kappa, c)$ for which $P_{\kappa, c} \neq \emptyset$, has nonempty Euclidean interior in $\mathbb{R}_r^r + d$.

(iii) $P_{\kappa, c}^C \neq \emptyset$ is finite for at least one $\kappa \in \mathbb{R}_r^r > 0$ and $c \in \mathbb{R}^d$ for which $P_{\kappa, c} \neq \emptyset$. 
(iv) The matrix
\[
\begin{bmatrix}
N \operatorname{diag}(w) B^\top \operatorname{diag}(h)
\end{bmatrix}
\]
has full rank \( n \) for some \( w \in \ker(N) \in \mathbb{R}^r_{>0} \) and \( h \in \mathbb{R}^n_{>0} \).

If any of these hold, then \( P_{\kappa,c} \) is finite for almost all \((\kappa,c) \in \mathbb{R}^r_{>0} \times \mathbb{R}^d\). Additionally, for almost all values of \((\kappa,c)\), the elements of \( P_{\kappa,c} \) are nondegenerate steady states.

\[\text{Proof.}\] This follows from Proposition 3.4(ii) and from Theorem 3.7 case (b) together with Remark 3.10 as \( K = \mathbb{R}^r_{>0} \), \( C = \mathbb{R}^d \) and \( X = \mathbb{R}^n_{>0} \). The last part follows from Corollary 3.2. \(\square\)

We note that condition (iv) in Theorem 4.1 and 4.5 holds if \( \ker(N) \setminus \mathbb{R}^r_{>0} \neq \emptyset \) and the respective conditions hold over \( \mathbb{C}^* \) (see Subsection 3.5). We also note that Theorem 4.5(i) implies Theorem 4.1(i), and we obtain the following corollary.

\textbf{Corollary 4.6.} If \( P_{\kappa,c}^C \neq \emptyset \) is finite for at least one \( \kappa \in \mathbb{R}^r_{>0} \) and \( c \in \mathbb{R}^d \) for which \( P_{\kappa,c} \neq \emptyset \), then the complex (and the real) steady state variety has dimension \( n - s \) for almost all \( \kappa \) for which it is not empty, and moreover it is nonempty for a set of parameters with nonempty Euclidean interior.

The converse is not necessarily true, as the following example shows.

\textbf{Example 4.7.} Consider the simple reaction network
\[ X_1 + X_2 \xrightarrow{\kappa_1} X_1 \quad X_2 \xrightarrow{\kappa_2} 2X_2. \]
The first row of the stoichiometric matrix \( \Gamma \) is zero, and hence the steady states are described by the single parametric polynomial
\[ f_\kappa(x_1, x_2) = -\kappa_1 x_1 x_2 + \kappa_2 = x_2(-\kappa_1 x_1 + \kappa_2). \]
We obtain \( V_\kappa = \{(x_1, x_2) \in \mathbb{R}^2_{>0} : x_1 = \frac{\kappa_2}{\kappa_1}\} \), which is nonempty for all \( \kappa \).

On the other hand, the stoichiometric compatibility classes are defined by the equation \( x_1 = c \) for \( \kappa \in \mathbb{R} \). The set \( P_{\kappa,c} \) is described by \( x_1 = \frac{\kappa c}{\kappa_1} = c \), and hence is empty unless \( \frac{\kappa c}{\kappa_1} = c \). This shows that Theorem 4.1(i) does not imply Theorem 4.5(i).

In the next corollaries, we rewrite some of the implications of Theorem 4.5 to emphasize the consequences that our results have on the existence and number of positive (nondegenerate) steady states.

\textbf{Corollary 4.8.} If a reaction network has infinitely many steady states within stoichiometric compatibility classes for all \((\kappa,c)\) for which \( P_{\kappa,c} \neq \emptyset \), then \( P_{\kappa,c} \neq \emptyset \) for almost all \((\kappa,c) \in \mathbb{R}^r_{>0} \times \mathbb{R}^d\).

\textbf{Corollary 4.9.} A reaction network that has a nondegenerate steady state for some \( \kappa \in \mathbb{R}^r_{>0} \) cannot have infinitely many steady states in all (not even in most) stoichiometric compatibility classes.

We note that even when the conditions of Theorem 4.5 hold, there might be values of \( \kappa \) for which all steady states are degenerate. Moreover, all steady states can be degenerate even if the dimension of the steady state variety has dimension \( n - s \). This is illustrated in the following example.
Example 4.10. Consider the reaction network with \( s = 1 \)

\[
\begin{align*}
3X_1 + X_2 \xrightarrow{\kappa_1} 4X_1 & \quad 2X_1 + X_2 \xrightarrow{\kappa_2} 3X_2 & \quad X_1 + X_2 \xrightarrow{\kappa_3} 2X_1,
\end{align*}
\]
giving rise to the polynomial function

\[
f(\kappa, x) = \kappa_1 x_1^2 - 2\kappa_2 x_1 + \kappa_3,
\]

after suitably choosing \( N \). For any choice of \( \kappa \) such that \( \kappa_3^2 = \kappa_1\kappa_3 \), all positive steady states are degenerate. For any other \( \kappa \), the steady states are all nondegenerate and therefore, the conditions in Theorem 4.11 and 4.5 apply: for almost all \( \kappa \), the dimension of the complex steady state variety is \( n - s = 1 \) when nonempty, and for almost all \( (\kappa, c) \), there is a finite number of positive steady states in the corresponding stoichiometric compatibility class.

Example 4.11 (Real networks and steady states). As an illustration of the applicability of our methods to realistic networks, we applied the computational steps described in Subection 3.5 to the networks from the database ODEbase [LSR22], under the assumption of mass-action kinetics. Out of a total of 610 networks appearing in the database, we found that precisely 368 have positive steady states. Out of these, 6 networks only have steady states that are degenerate solutions to \( f(\kappa, x) = 0 \), whereas 362 networks have a nondegenerate steady state, and therefore satisfy the equivalent properties in Theorems 4.1 and 4.5.

The largest network in the database which admits a positive nondegenerate steady state is BIOMD0000000014, which has 86 species. The computation verifying this took \(<2\) seconds in Maple. For comparison, an attempt of computing the dimension for a fixed random choice of the rate constants via \( \text{PolynomialIdeals}[\text{HilbertDimension}] \) in Maple failed due to memory problems.

4.3. Weakly reversible networks and other particularities. Weakly reversible networks are those for which all connected components of the underlying digraph are strongly connected. These networks are known to admit positive steady states for all choices of \( \kappa \) [Bor19] and are conjectured to display some strong dynamical behavior such as persistence [Fei87] or bounded trajectories [And11].

In [BCY20], the authors considered the following reaction network for which \( n = s = 2 \)

\[
\begin{align*}
2Y & \xrightarrow{\kappa_1} Y \xrightarrow{\kappa_2} X + Y & X + 3Y & \xrightarrow{\kappa_5} X + 2Y \xrightarrow{\kappa_6} 2X + 2Y & 2X + Y & \xrightarrow{\kappa_7} 2X \xrightarrow{\kappa_8} 3X,
\end{align*}
\]

and fine tuned the reaction rate constants in such a way that the two equations defining the steady states had a common factor and the other factors did not admit positive solutions. Hence the complex steady state variety has dimension 1, while \( d = n - s = 0 \). So the dimension is higher than expected. Specifically, all parameters were set to one except for \( \kappa_3 = \kappa_6 = \kappa_{12} = 4 \). With this trick, they illustrated that even for weakly reaction networks, the dimension of the steady state variety could be larger than expected for some choice of parameter values. Motivated by this, the authors posed the following question [BCY20, Section 5]]:

\[\text{Is it possible for a weakly reversible network to have infinitely many positive steady states [within a stoichiometric class] for each choice of reaction rate constants in a [Euclidean] open set of the parameter space } \mathbb{R}^r_{>0} ?\]

*All computations were run in Maple 2023.0 on a 2.70 GHz Intel Core i5 with 8GB RAM.
Theorem 4.5 gives an answer in the negative to this question: at least for almost all stoichiometric classes, this cannot happen if the network is weakly reversible. We note that weakly reversible networks admit so called complex balanced steady states for some choice of reaction rate constants (Hor72), and these are always nondegenerate (see Fei19 Section 15.2.2). It follows that weakly reversible networks admit nondegenerate steady states, and hence both Theorem 4.1 and Theorem 4.5 as well as Corollary 4.9 apply. This leads to the following result.

Corollary 4.12. For a weakly reversible network, and with notation as in Theorem 4.5, it holds that $P_{\kappa,c}$ is finite for almost all $(\kappa,c) \in \mathbb{R}_{>0}^r \times \mathbb{R}^d$

Other classes of networks that are known to have nondegenerate steady states are those that satisfy the criteria in the Deficiency One Theorem of Feinberg Fei19 Section 17.1, and injective networks MFR15 Thm. 1.4. 

To conclude this subsection, we make a connection to some classical objects in the study of reaction networks: the kinetic and the stoichiometric subspaces. 

The stoichiometric subspace $S$ is simply $\text{im}(\Gamma)$, while the kinetic subspace $S_\kappa$ for a given $\kappa$ is given by $\text{im}(\Sigma_\kappa)$, with $\Sigma_\kappa$ the coefficient matrix of $f_\kappa$ from (4.3). In FH77, criteria for when the two subspaces agree are given, but one can construct networks for which they do not agree for any $\kappa$ (e.g. Example 4.4), or for some values of $\kappa$. By Remark 2.4, if the network admits a nondegenerate solution to (4.3), then the generic rank of $\Sigma_\kappa$ agrees with $\text{rk}(\Gamma)$ and hence the two subspaces agree generically. It follows that if the two subspaces do not agree generically, then there is no nondegenerate solution and consequently, by Corollary 4.2, for almost all values of $\kappa$ there are no positive steady states.

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