Lee model and its resolvent analysis

Jagvaral, Yesukhei; Turgut, O. Teoman; Ünel, Meltem

Published in:
International Journal of Geometric Methods in Modern Physics

DOI:
10.1142/S021988782350055X

Publication date:
2023

Document version
Early version, also known as pre-print

Document license:
CC BY

Citation for published version (APA):
Relativistic Lee model and its Resolvent Analysis

Yesukhei Jagvaral¹, O. Teoman Turgut *¹,², and Meltem Unel³

¹Department of Physics, Carnegie Mellon University, Pittsburgh, PA 15213, USA
²Department of Physics, Bogazici University, Bebek, 34342, Istanbul, Turkey
³Department of Mathematical Sciences, University of Copenhagen, DK-2100, Copenhagen, Denmark

Abstract

We reexamine the relativistic 2+1 dimensional Lee model in light-front coordinates on flat space and on a space-time with a spatial section given by a compact manifold in the usual canonical formalism. The simpler 2+1 dimension is chosen because renormalization is needed only for the mass difference but not required for the coupling constant and the wavefunction. The model is constructed non-perturbatively based on the resolvent formulation [1]. The bound state spectrum is studied through its “principal operator” and bounds for the ground state energy are obtained. We show that the formal expression found indeed defines the resolvent of a self-adjoint operator—the Hamiltonian of the interacting system. Moreover, we prove an essential result that the principal operator corresponds to a self-adjoint holomorphic family of type-A in the sense of Kato.

Introduction

Lee model is a nontrivial toy model originally proposed to understand renormalization in a nonperturbative way. There are two fermion species, called \(V\) and \(N\) particles, considered so heavy that their energies are assumed to be given only by their masses but they are allowed to carry nontrivial momenta [2, 3]. They interact with a relativistic real scalar field, called \(\theta\) particle, in a very specific way; only reaction we allow is \(N + \theta \leftrightarrow V\) (so no crossing symmetry). In a relativistic theory this is possible by truncating the field operators of \(\theta\) to positive and negative frequencies respectively (which then is not truly relativistic except the energy dispersion relation). We refrain from discussing in detail various aspects of this model, instead we refer to the existing literature [4–12]. In some of these works the heavy particles are assumed to have no recoil, hence there is no momentum transfer between the heavy and light particles (which is a further simplification of the model). Since the total fermion number is conserved, we can restrict to the subspace corresponding to a single \(V\) particle or a single \(N\) particle but no restriction on the boson number. Moreover, there is another conservation law, due to the interaction term, we now either have \(n + 1\) bosons and the \(N\) particle or \(n\) bosons and the \(V\) particle. If we apply this reduction and assume \(V\) and \(N\) particles as two distinct states of some system, as well as no recoil, we have a fixed two level system interacting with bosons. This is the reduced model we consider (a similar reduction of a Lee type model with crossing symmetry was performed by Wilson when he studied coupling constant renormalization in a non-perturbative manner in [13]).

Since in the second part of the paper we study the model on a compact manifold, we choose to study this reduced version in the light-front coordinates first, to be discussed completely in the following sections. We aim to establish some of the technical tools in a familiar situation, before the manifold case is introduced, albeit in a slightly different coordinate system (light-cone coordinates are well-known to be better suited to bound state problems, as to be seen, it is also somewhat more advantageous here. A Poincare invariant version of the Lee model is constructed in light-front coordinates in [14] and many subtleties about the relativistic invariance are discussed in that work.)

If the bosons are treated by a non-relativistic dispersion relation, the model becomes somewhat simpler (this is the version proposed by Thirring and Henley in their book [15]). In an unpublished inspiring work Rajeev applied an algebraic approach to obtain the resolvent of this simpler version [16]. His construction

*corresponding author: turgute@boun.edu.tr
naturally leads to an operator— the principal operator which contains all the information about the bound states. Rajeev uses operator methods to find a lower bound for the ground state for any number of bosons (we essentially follow his approach in our estimates). Moreover, he applies mean field theory techniques to find an estimate for the ground state for large number of bosons. The more conventional version we study here must also be amenable to a mean field estimate, yet we look at a more mathematical problem. We aim to justify the formal manipulations that is used to obtain most of the aforementioned results and indeed show that there is a well-defined Hamiltonian underlying the obtained resolvent.

To establish this we use some ideas about pseudo-resolvents from semi-group theory. One could object the use of pseudo-resolvents and limit conditions instead of the more widely used resolvent convergence. We think our approach has some advantages over the resolvent convergence, for example, when we introduce a model via dimensional regularization, it is not clear that there is a sensible Hamiltonian underlying the dimensionally regularized system. However, the resulting expressions can be taken as such and it can be tested whether they make sense. This is our approach here: we define the theory via the resolvent and it does not matter what limit process is used in its construction. Having found the resolvent, we need to check the self-adjointness and as a result compute the ground state wave function via spectral projection. This requires a notion invented by Kato, the principal operator as a function of the complex parameter $E$ must be a self-adjoint holomorphic family of type-A. It is one of the main results of this work to verify this claim.

Due to the effective nature of quantum field theory, it is expected that some parameters are to be redefined, if we do not have a natural cut-off in the theory we need to push the unknown cut-off to infinity, more often than not, this leads to fine tuning of the "bare parameters". It is essential to verify that the resulting theory indeed makes sense after the removal of all divergences. Up to the present time this has not been possible in realistic theories, despite heroic efforts by many people (the existing literature is huge, one possible starting point could be the classic work by Glimm and Jaffe [17]). Our work is a modest attempt in this direction for a very simple model in which there is some hope to disentangle all mathematical complications.

From the perspective of a pure mathematician our presentation may not be deemed rigorous, we are not trying to justify all the manipulations, we justify essential ideas as much as we can. On the other hand, a typical theoretical physicist may find the presentation too technical. Nevertheless, we think it is essential to introduce these ideas in the recent period of quantum field theory. At present quantum field theory is full of exciting ideas and techniques derived from special models. We do have the main ideas and basic constructions in this vast subject, especially from a perturbation point of view. However, it is hard to give a rigorous basis in a perturbative formalism since most of the expansions are meant to be asymptotic. Therefore, it is important to find special models for which a nonperturbative method works.

**Part I: Lee model in the Light-front Coordinates**

The Lee model is one of the simplest field theory models where the renormalization can be done non-perturbatively. Despite the simplicity of the model, the exact spectrum of the model is difficult to obtain. The resolvent formulation (which can be written down exactly for this model) first introduced by Rajeev, enables us to study the spectrum of the model without the need of an explicit formula for the renormalized quantum Hamiltonian. To understand this model in more depth, we recall some basic ideas which were originally worked out in [1]. To avoid the repetition we deliberately work with the light-front formalism (some aspects of which were also covered in an appendix of [1]). Moreover, it is believed that the light-front coordinates provide a better approximation when truncated field operators are used (note that in the interaction term we suppress positive frequency modes for up state and negative ones for the down state in the field operators). Let us briefly discuss the oblique coordinate system we use, which was recently used and reviewed in [18]. In the oblique light-front coordinate system $u = t + x$ was chosen to be the evolution parameter, so $u$ is the light-front "time". Otherwise we keep the coordinates $x, y$ as before, although we call $y$ as $x^\perp$ (the transverse coordinate). As a result, the line element, the metric tensor and its inverse are given by

$$ds^2 = du^2 - 2dudx - (dx^\perp)^2,$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad g^{\mu\nu} = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$
The scalar product between the coordinate and the conjugate momenta is

\[ p_\mu x^\mu = p_u u + p x + p_\perp x^\perp \]  

where \( x \) and \( x^\perp \) are the longitudinal and the transverse coordinates; \( p_u, p \) and \( p_\perp \) are the light-front energy, the longitudinal and the transverse momenta, respectively. Note that we have down indices for momentum variables, as it is more convenient to do so in the oblique formalism. Quantization of a scalar field in this coordinate system is reviewed in detail in our recent work [18], since the present work is fairly technical (and formally needed) and the limit can be taken.

After introducing a proper counter term (since we work with the resolvent directly, the limiting process is only formally needed) and the limit can be taken.

We note that it is essential to use our oblique coordinate system in this problem, since the two level system is at a fixed location in space, this can be meaningfully defined in our light front system, the usual choice of variables, as it is more convenient to do so in the oblique formalism. Quantization of a scalar field in this coordinate system is reviewed in detail in our recent work [18], since the present work is fairly technical (and formally needed) and the limit can be taken.

The positive and the negative frequencies at zero light-front time evaluated at the origin are

\[ \phi^{(+)}(0) = \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{a(p, p_\perp)}{\sqrt{2p}} \]  

\[ \phi^{(-)}(0) = \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{a^\dagger(p, p_\perp)}{\sqrt{2p}} \]  

Bosonic \( n \)-particle wave function in the coordinate space are given by

\[ |\psi\rangle = \frac{1}{\sqrt{n!}} \int dx_1 dx_1^\perp ... dx_n dx_n^\perp \psi(x_1, x_1^\perp, ..., x_n, x_n^\perp) \phi^{(-)}(x_1, x_1^\perp) ... \phi^{(-)}(x_n, x_n^\perp) |0\rangle \]

with \( \psi \) being symmetric on all particle coordinates. As a result of this, we use a relativistically invariant norm; the Hilbert space norm for bosons in the momentum space decomposition for \( n \) particles is given by:

\[ \langle \psi|\psi\rangle = \frac{1}{4\pi^2} \int \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} |\psi(p_1, p_1^\perp, ..., p_n, p_n^\perp)|^2 \]

The last expression is manifestly positive definite and the wave functions are chosen to make the norm finite.
I.1 Principal operator

Since the details of this model were discussed in \[1\] we keep our presentation brief. The idea of the method is to calculate the resolvent by means of a formal identity in an algebraic way and isolate the divergence of the problem in an additive manner. We compute a formal inverse of $H - E$ algebraically in a special way (this idea is due to Rajeev for the nonrelativistic version of this model). If we write the Hamiltonian in $2 \times 2$ decomposition according to up and down states \[10\]:

$$H - E = \begin{bmatrix} H_0 - E & \lambda \phi^{(-)}(0) \\ \lambda \phi^{(+)}(0) & H_0 - E + \mu \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$$

(11)

Note that this formal Hamiltonian acts on a direct sum $F^{(n+1)}_B(\mathcal{H}) \otimes \chi^{\uparrow} \otimes F^{(n)}_B(\mathcal{H}) \otimes \chi^{\downarrow}$, without mixing different sectors due to a conserved quantity mentioned. Thus each sector can be studied independently, consequently, the Hamiltonian is restricted to such sectors. Then, the resolvent is

$$R(E) = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$$

(12)

where:

$$\begin{align*}
\alpha &= a^{-1} + a^{-1}b^\dagger \Phi^{-1}(E) b a^{-1} \\
\beta &= -\Phi^{-1}(E) b a^{-1} \\
\gamma &= -a^{-1}b^\dagger \Phi^{-1}(E) \\
\delta &= \Phi^{-1}(E) \\
\Phi &= d - b a^{-1}b^\dagger
\end{align*}$$

(13)

and $\Phi$ is defined as the principal operator. Note that here, we have a formal expression for the inverse which involves the inverse of the free bosonic Hamiltonian as well as the $\Phi$ operator. The free resolvent is well defined as long as $\text{Im}(E) \neq 0$ or if $\text{Im}(E) = 0$, then for $E$ less than $(n + 1)m$. When searching for the bound states, we must look for the poles of the resolvent below the free bosonic spectrum. In \[12\] these poles can only appear as zeros of the operator $\Phi(E)$. Thus we have a great simplification; solutions of the eigenvalue equation $\Phi(E)|\omega(E)\rangle = 0$ determines the corresponding $E$ values: if they are below the free spectrum we have bound states, otherwise a resonance. The price we pay for this simplification is that a linear eigenvalue problem is turned into a nonlinear one, since in general $\Phi(E)$ is a complicated function of $E$. We remark that in all blocks of the resolvent, $\Phi^{-1}(E)$ operates always on the bosonic Fock space of $n$ particles $F^{(n)}_B(\mathcal{H})$.

After normal-ordering the creation and annihilation operators in the operator $\Phi(E)$, we face a divergent term which can be cancelled by a counter term coming from the mass difference of up and down states. The divergent expression is actually an operator, whereas the counter term is a scalar; thus after renormalization and the physical mass condition for the down state (a single boson with the up state and no boson with the down state belonging to the same Hilbert space we require $\Phi_R(E = \mu_p)|0\rangle = 0$, that is we impose the condition that the physical binding energy in this sector be $E = \mu_p$), the principal operator takes the form:

$$\Phi_R(E) = (H_0 - E + \mu_p) + \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p \omega(p, p_\perp) - \mu_p} - \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p \omega(p, p_\perp) - \mu_p}$$

$$- \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \int \frac{dq}{2\pi} \frac{1}{2\sqrt{p^2 q^2 - (q, q_\perp)^2}} a^\dagger(q, q_\perp) \frac{1}{H_0 - E + \omega(p, p_\perp) + \omega(q, q_\perp) a(p, p_\perp)}$$

(14)

The difference of the two integrals in the first line above is now finite, thus we find a well-defined expression for the principal operator. Indeed it should be written as,

$$\Phi_R(E) = (H_0 - E + \mu_p) \left[ 1 + \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p \omega(p, p_\perp) - \mu_p} \frac{1}{H_0 - E + \omega(p, p_\perp)} \right]$$

$$- \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \int \frac{dq}{2\pi} \frac{1}{2\sqrt{p^2 q^2 - (q, q_\perp)^2}} a^\dagger(q, q_\perp) \frac{1}{H_0 - E + \omega(p, p_\perp) + \omega(q, q_\perp) a(p, p_\perp)}$$

(15)
For simplicity, we will drop $R$-subscript and often will write $\Phi(E)$. Above form makes $\Phi(E = \mu_p)|0\rangle = 0$ condition manifest. In other sectors, that is for $\mathcal{F}_B^{(n)}(H)$ which represents the $n$-particle bosonic Fock space, the zeros of the eigenvalues are much more complicated. The resolvent formula (12), with $\Phi(E)$ operator (15) as found after normal ordering, defines $\frac{1}{\pi - E}$ for our interacting system (yet we do not have a formula for $H$ itself). We show that it is possible to associate a well-defined quantum Hamiltonian to this operator family and all the information about the system is contained in this formula.

To accomplish these tasks, we next compute the flow of eigenvalues on the real axis of $E$, below $nm + \mu$. Since $\langle \frac{d\Phi(E)}{dE}\rangle < 0$ the eigenvalues flow monotonically with $E$ and by the Feynman-Hellman formula [19] we get:

$$\frac{\partial \omega_k}{\partial E} = \langle \omega_k | \frac{\partial \Phi(E)}{\partial E} | \omega_k \rangle \implies \frac{\partial \omega_k}{\partial E} < 0,$$

where $\Phi(E) | \omega_k \rangle = \omega_k(E) | \omega_k \rangle$, i.e. $k$th isolated eigenvalue of $\Phi(E)$ in a fixed particle sector. This means $\omega_k(E) = 0$ has a unique solution and these solutions correspond to the possible bound states or resonances.

Here we assume formally that the operator $\Phi(E)$ has a discrete set of eigenvalues, we need to further comment on this assumption below. Because of the flow of eigenvalues, in any sector, $\omega_0(E) = 0$ i.e. the zero of the lowest eigenvalue of $\Phi(E)$, then gives us the ground state and this observation can be used to find a lower bound for the ground state energy for a fixed number of bosons.

Note that the above observations rely on the assumption that the eigenvalues are differentiable functions of the parameter $E$. Indeed we aim for much more: $\omega(E)$'s are actually holomorphic functions of the complex parameter $E$. This is a bit tricky in the noncompact case since due to infinite size there may be arbitrarily small excitations with almost no change of energy. Nevertheless a fixed two level system used here breaks translational invariance and subsequently the ground state energy of this system is not expected to be connected to a continuum of states. In general, it is possible to have a continuum of states attached to the lowest eigenvalue, the zeros of which then typically coalesce to form a branch cut in the complex plane.

If we are around an isolated zero, we can use the Riesz projection to find the precise form of the bound state wave function thanks to holomorphicity. In the following discussion we plan to establish the holomorphicity, proving that the infimum of the spectrum corresponds to an isolated eigenvalue is a harder problem which we postpone for the time being.

### I.2 Lower bound on the ground state

In this section we review the lower bound for the ground state energy. Note that, if the real part of $E$ is below a certain value which defines a half plane, the operator $\Phi$ becomes invertible with a bounded inverse, then there cannot be a pole of the resolvent in this region. For simplicity, assuming that the operator $\Phi$ becomes self-adjoint on the real axis (something to be justified later), it is enough to show that the operator is invertible along the real axis below a certain value, since we are looking for zeros along the real axis.

Incidentally most interesting part of the complex $E$ plane for us is $\text{Re}(E) < nm + \mu_p$ for $n$ boson sector (recall that in the no boson and only the down state we choose the energy as $\mu_p$) since we expect the bound states to appear below this level. In the second part we provide a variational proof of this claim on a compact manifold. Before we review this bound, it is insightful to consider the term coming from renormalization, which we can think of as a kind of a "kinetic term", so we separate the free part and call the remaining term as $K_1(E)$:

$$K_1(E) = \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p} \left( \frac{1}{\omega(p, p_\perp) - \mu_p} - \frac{1}{H_0 - E + \omega(p, p_\perp)} \right).$$

We first establish a lower bound on this term, for real values of $E$, then it is clear that the result is a positive operator (it can be defined via the spectral decomposition of $H_0$). After doing the $p_\perp$ integral we end up with

$$K_1(E) = \int_0^\infty \frac{dp}{2\pi} \left( \frac{-1}{\sqrt{2p(H_0 - E) + m^2 + p^2}} + \frac{1}{\sqrt{p^2 + m^2 - 2p\mu_p}} \right).$$

Let us collect the terms under a common denominator and multiply the top and the bottom by $\sqrt{2p(H_0 - E) + m^2 + p^2} + \sqrt{p^2 + m^2 - 2p\mu_p}$ to find
With the above lower bound on the kinetic part, we can ignore the rest and use the free part only, this actually gives an upper bound on
\[ \int_0^\infty \frac{dp}{2\pi} \left( \frac{1}{\sqrt{2p(H_0 - E) + m^2 + p^2}} + \frac{1}{\sqrt{p^2 + m^2 - 2p\mu_P}} \right) \times \frac{\sqrt{2p(H_0 - E) + m^2 + p^2} + \sqrt{p^2 + m^2 - 2p\mu_P}}{2p(H_0 - E) + m^2 + p^2 + \sqrt{p^2 + m^2 - 2p\mu_P}} \geq \int_0^\infty \frac{dp}{2\pi} \frac{p}{(H_0 - E + \mu)} \frac{p}{(2p(H_0 - E) + m^2 + p^2)\sqrt{p^2 + m^2 - 2p\mu_P}} \]  
(19)

where in the last inequality we replaced the smaller term \( p^2 + m^2 - 2p\mu_P \) by the bigger one with \( H_0 - E \) to get a lower bound. Let us separate the multiplicative factor \( (H_0 - E + \mu_P) \) for the time being. For the remaining part we use Feynman parametrization (removing the numerical factor) to get:
\[ \int_0^\infty \frac{dp}{2\pi} \int_0^1 \frac{2up(H_0 - E) + p^2 + m^2 - 2p\mu_P(1-u)^3/2}{(2up(H_0 - E + \mu_P) + p^2 + m^2)3/2} \, du \geq \int_0^\infty \frac{dp}{2\pi} \int_0^1 \frac{(1 - u)^{-1/2}}{(2up(H_0 - E + \mu_P) + p^2 + m^2)^{3/2}} \, du \]
\[ = \int_0^\infty \frac{dp}{2\pi} \int_0^1 \frac{(1 - u)^{-1/2}}{\bigl((p + (H_0 - E + \mu_P)u)^2 - (H_0 - E + \mu_P)^2u^2 + m^2\bigr)^{3/2}} \, du \]  
(20)

where we drop \( 2p\mu_P \) term in the last inequality to get a lower bound again. As a result we write the above expression as,
\[ \int_0^\infty \frac{dp}{2\pi} \int_0^1 \frac{(1 - u)^{-1/2}}{(p + (H_0 - E + \mu_P)u)^2 - (H_0 - E + \mu_P)^2u^2 + m^2}\]  
(21)

performing the \( p \) and \( u \) integrations and multiplying the result with the left out \( (H_0 - E + \mu_P) \) term again, we arrive at (collecting all the numerical/constant factors as \( C_0 \)):
\[ K_1(E) \geq C_0 \ln \left[ \frac{H_0 - E + \mu_P + m}{m} \right] \]  
(22)

Consequently the full kinetic part, \( K(E) = (H_0 - E + \mu_P) + K_1(E) \), satisfies the lower bound
\[ K(E) \geq (H_0 - E + \mu_P) + C_0 \ln \left[ \frac{H_0 - E + \mu_P + m}{m} \right] \]  
(23)

For the principal operator to be invertible, it is sufficient to satisfy the condition
\[ ||\hat{U}(E)|| = ||K(E)^{-1/2}U(E)K(E)^{-1/2}|| < 1. \]

With the above lower bound on the kinetic part, we can ignore the rest and use the free part \( H_0 - E + \mu_P \) only, this actually gives an upper bound on \( ||\hat{U}(E)|| \) (when we find an upper bound the free term is seen to dominate over this additional positive term). To this purpose we recall the following operator inequality:
\[ \left| \left| \int dPdQF(P,Q)a^\dagger(P)a(Q) \right| \right| \leq n \left[ \int dPdQ|F(P,Q)|^2 \right]^{1/2} \]  
(24)

Inside the estimate we replace all \( H_0 \) by their lower bounds \( (n - 1)m \) and then apply the above inequality. Then by defining \( \Delta = (n - 1)m + \mu_P - E \), the relative potential term above satisfies:
\[ ||\hat{U}(E)|| \leq \lambda^2 n \left[ \int_0^\infty \frac{dPdQ}{4\pi^2} \int \frac{dp_{\perp}dq_{\perp}}{4\pi^2} \frac{1}{4Pq_{\perp}} \left( \Delta + \omega(p,p_{\perp}) + \omega(q,q_{\perp}) \right)^2 (\Delta + \omega(p,p_{\perp})) (\Delta + \omega(q,q_{\perp})) \right]^{1/2} \]  
(25)

\[ \leq \lambda^2 n \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{2p} \left( \Delta + \omega(p,p_{\perp}) \right)^2 \]  
(26)

\[ = \lambda^2 n \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{2p}{2\Delta p + p^2 + m^2 + p_{\perp}^2} \]  
(27)

\[ \leq \frac{n^2}{2} m(n - 1) + \mu_P - E \]  
(28)
In the first line of the integral above, we eliminate \( \omega(p, p_\perp) \) in one of the terms, and \( \omega(q, q_\perp) \) in the other term and then recognize that we have the same integral repeated. The resulting integral can be evaluated exactly. We then impose the condition,

\[
\frac{\lambda^2}{2} \frac{\pi n}{m(n-1) + \mu_p - E} < 1,
\]

which gives the lower bound on ground state energy for \( n \) particle sector as,

\[
E_{gr} \geq m(n-1) + \mu_p - \frac{\lambda^2 \pi n}{2}.
\]

### 1.3 Spectral Projections

Let us now digress briefly on the use of spectral projections to calculate the bound state wave functions. If the resulting renormalized resolvent indeed corresponds to the resolvent of a well-defined self-adjoint quantum Hamiltonian, we can calculate the ground state wave function via the contour integral,

\[
P_{\Psi_0} = \frac{1}{2\pi i} \int_{E_{gr}} dE R(E),
\]

where we assume that a small contour is picked around the isolated ground state \( E_{gr} \). Although we do not know in our model at light-cone that the ground state is unique and corresponds to an isolated eigenvalue, since the two level system is fixed, this seems to be a reasonable assumption. This assumption is indeed correct when we deal with a non-relativistic version of this model on a compact manifold as shown in [20] (otherwise the jump along the spectral cut should be considered, but this possibility will not be considered here). If \( \Phi(E) \) has a unique lowest eigenvector, the point at which it vanishes, gives us the desired ground state energy but the above formula reveals that the ground state of the original model is then also unique. To find an explicit formula it is essential to know that the family \( \Phi(E) \) is self-adjoint holomorphic of type-A in the sense of Kato, that guarantees the holomorphicity of the corresponding eigenvalues and eigenvectors as functions of \( E \). If we write the wave functions in the two component form,

\[
|\Psi_0 \rangle = \begin{pmatrix} |\Psi_0^{(n+1)} \rangle \\ |\Psi_0^n \rangle \end{pmatrix}
\]

we have

\[
|\Psi_0^{(n+1)} \rangle = \left[ -\frac{\partial \omega_0(E)}{\partial E} \right]_{E_{gr}}^{-1/2} (H_0 - E_{gr})^{-1} \phi^{(-)}(0) |\omega_0(E_{gr}) \rangle,
\]

\[
|\Psi_0^n \rangle = \left[ -\frac{\partial \omega_0(E)}{\partial E} \right]_{E_{gr}}^{-1/2} |\omega_0(E_{gr}) \rangle,
\]

justification of which requires holomorphicity of the eigenvalues \( \omega_0(E) \) and its associated eigenvector. Here, in the coordinate basis we express the wave function for lowest eigenvector,

\[
|\omega_0(E_{gr}) \rangle = \int dx_1...dx_n \psi_0(x_1,...,x_n) \phi^{(-)}(x_1)...\phi^{(-)}(x_n) |0 \rangle.
\]

We use \( x_1,...,x_n \) as general coordinates, in fact the above expression is valid if we interpret these coordinates as our light-front variables as well as assuming them coordinates on a compact manifold as we do in a later section. We remark that we should interpret the operator expression in the wave function, in the coordinate representation, as

\[
(H_0 - E_{gr})^{-1} \phi^{(-)}(0) = \int_0^{+\infty} ds \int dx \phi^{(-)}(x) k_s(x,0) e^{-s(H_0 - E_{gr})},
\]

where \( k_s(x, \bar{x}) = \int [dp dp_\perp] e^{-s\omega(p, p_\perp)} e^{-ip(x-\bar{x}) - ip_\perp(x^+ - \bar{x}^+)} \) corresponds to a relativistic version of the heat kernel for the light-front model (an analogous expression in the manifold case is linked to the heat kernel by the subordination identity as to be seen). The normalization is preserved by these formulae (can be checked by a tedious calculation), that is we can see that the expressions above lead to

\[
\langle \Psi_0^n | \Psi_0^n \rangle + \langle \Psi_0^{(n+1)} | \Psi_0^{(n+1)} \rangle = \left[ -\frac{\partial \omega_0(E)}{\partial E} \right]_{E_{gr}}^{-1} \langle \omega_0(E_{gr}) \rangle \left[ -\frac{\partial \Phi(E)}{\partial E} \right]_{E_{gr}} \omega(E_{gr}) = 1.
\]
I.4 Resolvent Defining a Hamiltonian

In order to establish the fact that the formal expression we find actually defines a Hamiltonian, we borrow some ideas from the theory of semi-groups. Let us recall the definition of a pseudo-resolvent family,

**Definition 1.** Let Δ be a subset of the complex plane. A family \( J(\lambda), \lambda \in \Delta, \) of bounded linear operators on \( X \) (\( X \) being the Banach Space) satisfying:

\[
J(\lambda) - J(\mu) = (\lambda - \mu)J(\lambda)J(\mu)
\]

is called a pseudo-resolvent on \( \Delta \).

**Theorem 1.** Let Δ be an unbounded subset of \( \mathbb{C} \) and let \( R(E) \) be a pseudo-resolvent on \( \Delta \). If there is a sequence \( E_k \in \Delta \) such that \( |E_k| \to \infty \) as \( k \to \infty \) and

\[
\lim_{k \to \infty} E_k R(E_k)x = -x \quad \text{for all } x \in X
\]

then \( R(E) \) is the resolvent of a unique densely defined closed operator \( H \).

The resolvent we introduce does indeed satisfy the resolvent identity (as can be explicitly checked, we outline the main steps in the next part when we discuss the model on a manifold, the verification essentially is algebraic). We show that there exists an operator \( H \), such that the resolvent \( R(E) \) is the resolvent family of \( H \), where \( R(E) = \frac{1}{E - P} \) by means of the above decay conditions. Note that our initial (ill-defined) Hamiltonian is operating on \( \mathcal{F}_{B}^{(n+1)}(\mathcal{H}) \otimes \chi_{\uparrow} \oplus \mathcal{F}_{B}^{(n)}(\mathcal{H}) \otimes \chi_{\downarrow} \). Therefore the resolvent is defined over this Hilbert space that we call \( \mathcal{H}_Q \). Below we verify the decay conditions of the above theorem.

I.4.1 Verifying the decay conditions

In order to show that \( R(E) \) satisfies Theorem 1, we pick a sequence \( \lambda_k \) on the negative real axis for every \( k \), \( \lambda_k < 0 < E_{gr} \). Since \( \lambda_k = -|\lambda_k| \), the condition (36) becomes:

\[
\lim_{k \to \infty} \left\| |\lambda_k| R(-|\lambda_k|) - 1 \right\|_{\mathcal{H}_Q} = 0
\]

And using the triangular inequality repeatedly we get:

\[
\left\| \left[ |\lambda_k| R(-|\lambda_k|) - 1 \right] \left( \frac{f^{n+1}}{f^n} \right) \right\|_{\mathcal{H}_Q} \leq \left\| |\lambda_k| \alpha(-|\lambda_k|) - 1 \right\| \left( \frac{f^{n+1}}{f^n} \right) \right\| + \left\| |\lambda_k| \gamma(-|\lambda_k|) \right\| \left( \frac{f^{n+1}}{f^n} \right) \right\| + \left\| |\lambda_k| \delta(-|\lambda_k|) \right\| \left( \frac{f^{n+1}}{f^n} \right) \right\| \left( \frac{f^{n+1}}{f^n} \right) \right\|
\]

Thus it is sufficient to show that as \( k \to \infty \) all the terms on the right hand side of the inequality go to zero. To establish this we need a more detailed analysis of the behaviour of the operator \( \Phi(E) \) for large negative values of \( E \). We know that \( (H_0 - E)^{-1} \) is the resolvent family of the free bosonic part, thus it satisfies,

\[
\left[ |\lambda_k| (H_0 + |\lambda_k|)^{-1} - 1 \right] f^n \to 0 \quad \text{as} \quad k \to \infty
\]

Let us concentrate on the decay of \( \Phi \) operator. We examine how the principle operator behaves in order to check the validity of the pseudo-resolvent condition mentioned above.

\[
\Phi(E) = (H_0 - E + \mu_F) \left[ 1 + \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p} \left( \omega(p, p_\perp) - \mu_F \right) (H_0 - E + \omega(p, p_\perp)) \right]^{N_k} \right. \\
- \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \int_0^\infty \frac{dq}{2\pi} \int \frac{dq_\perp}{2\pi} \frac{1}{2\sqrt{pq}} H_0 - E + \mu_F \left( \frac{1}{H_0 - E + \omega(p, p_\perp) + \omega(q, q_\perp)} \right)^{N_k} \right. \\
- \frac{1}{\omega(p, p_\perp)} \right. \\
\left. \frac{1}{\omega(q, q_\perp)} \right. \\
\left. a(p, p_\perp) \right)
\]

(39)
The idea is this: for large negative values of $E$, we can make $||\tilde{K}(E)|| < 1/4$ as well as $||\tilde{U}(E)|| < 1/4$ thus we have the estimate,

$$
|\lambda_k||\Phi^{-1}(-|\lambda_k|)|| \leq |\lambda_k| ||(H_0 + |\lambda_k| + \mu_P)^{-1}|| [1 + \tilde{K}(-|\lambda_k|) - \tilde{U}(-|\lambda_k|)]^{-1}|| \\
\leq |\lambda_k| ||(H_0 + |\lambda_k| + \mu_P)^{-1}|| \frac{1}{1 - ||\tilde{K}(-|\lambda_k|)|| - ||\tilde{U}(-|\lambda_k|)||} \leq 2|\lambda_k| ||(H_0 + |\lambda_k| + \mu_P)^{-1}||.
$$

This implies that as $k \to \infty$, $|\lambda_k||\Phi^{-1}(-|\lambda_k|)||$ remains bounded.

To establish the above claims, let us look at the behavior of $\tilde{K}$ for real values of $E$. Later we actually estimate these norms for complex values, there is a certain degree of repetition here but it is nice to see the differences, a more proper thing to do is to use norm inequalities (since we do not have a formal proof that the operators are self-adjoint).

$$
||\tilde{K}(E)|| = \lambda^2 \left| \int_0^\infty \frac{dp}{2\pi} \int \frac{dp}{2\pi} \frac{2p}{(-2p\mu_P + p^2 + m^2 + p^2)}[(H_0 - E)2p + p^2 + m^2 + p^2] \right| \left| \int_0^\infty \frac{dp}{2\pi} \frac{2p}{(-2p\mu_P + p^2 + m^2)(H_0 - E)2p + p^2 + m^2 + p^2} \right|
\leq \frac{\pi \lambda^2}{2} \int_0^\infty \frac{dp}{2\pi} \frac{2p}{(p^2 + m^2 - 2p\mu_P)^{1/2}}[(nm - E)2p + p^2 + m^2]^{1/2}[(H_0 - E)2p + p^2 + m^2]^{1/2} + [(nm - E)2p + p^2 + m^2]^{1/2} + [(nm - E)2p + p^2 + m^2]^{1/2} (40)
$$

Where in the third line we replaced $H_0$ with its lower bound $nm$, for the last inequality we use the fact that $-\mu_P \leq nm - E$ we replace the bigger term $(nm - E)2p + p^2 + m^2$ with the smaller term $-2p\mu_P + p^2 + m^2$ to get an upper bound.

Now using Feynman parametrization we get:

$$
||\tilde{K}(E)|| \leq C_1 \frac{\pi \lambda^2}{2} \int_0^\infty \frac{dp}{2\pi} \int_0^1 \frac{p(1-u)^{-1/2}du}{(nm - E - (nm - E + \mu_P)u)2p + p^2 + m^2} \left(41\right)
\leq C_1 \frac{\pi \lambda^2}{2} \int_0^1 (1-u)^{-1/2}du \left[ \frac{1}{(n+1)m - E - u(nm - E + \mu_P)} \right] \left(42\right)
\leq C_2 \lambda^2 \int_0^1 v^{-1/2}dv \left[ \frac{1}{(m - \mu_P) + (nm + \mu_P - E)v} \right] \left(43\right)
= C_2 \lambda^2 \int_0^1 \frac{1}{(m - \mu) + (nm + \mu_P - E)\eta^2} \quad \eta \to 0
$$

This establishes that $||\tilde{K}(E)||$ goes to zero as $E \to -\infty$. Moreover, it shows that the operator remains bounded as long as $E < nm + \mu_P$, since the expression in (40) is well defined within this region, which is important for our later purposes. In fact, we can assume that the variable $E$ is complex, and the region of interest becomes $\text{Re}(E) < nm + \mu_P$. Indeed, in the above inequalities we may replace $E$ with its real part as long as we use it as a norm inequality, that gives an upper bound. If we further keep the imaginary part, we see that there is never a problem but we do not need this fact for our purpose (indeed we can extend the region of validity of the formulae by keeping a nonzero imaginary part).

So far we have shown that both $||\tilde{U}||$ (from eq. (28) it is obvious) and $||\tilde{K}||$ go to zero as $E$ goes to infinity. This indicates that $|\lambda_k| ||\Phi^{-1}(-|\lambda_k|)||$ remains finite (this estimate is required for the $\delta$ term as well).

1.4.1.1 $\beta$ term

$$
||\lambda_k|\Phi^{-1}(-|\lambda_k|)\phi^{(+)} \frac{1}{H_0 + |\lambda_k|} |f^{n+1}\rangle|| \leq |\lambda_k| ||\Phi^{-1}|| \int_0^{\text{finite}} \frac{1}{H_0 + |\lambda_k|} |\phi^{(+)}\rangle ||f^{n+1}\rangle||
$$

(44)
Therefore we need to estimate the second norm,
\[
\left\| \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{a(p, p_{\perp})}{\sqrt{2p}} \frac{1}{H_0 + |\lambda_k|} |f^{n+1}\rangle \right\| = \left\| \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} H_0 + |\lambda_k| + \omega(p, p_{\perp}) a(p, p_{\perp}) |f^{n+1}\rangle \right\|
\]
(45)
\[
\leq \left\| \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} nm + |\lambda_k| + \omega(p, p_{\perp}) a(p, p_{\perp}) |f^{n+1}\rangle \right\|
\]
(46)

We now recall the inequality
\[
||\phi^{(+)}(g)||f^n|| \leq n||g||||f^n||
\]
(48)
where all the norms are in the Hilbert space, which can be proven similar to the integral operator version. We have
\[
||g|| = \frac{1}{2\pi} \left[ \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{nm + |\lambda_k| + p^2 + p_{\perp}^2 + m^2} \right]^{1/2}
\]
(49)
Evaluating the integral we get the final result for \(\beta\):
\[
\left[ \frac{nm + |\lambda_k|}{(nm)^2 + |\lambda_k|^2 - 4m^2} \right] \left[ \frac{4m}{(nm + |\lambda_k|^2 - 4m^2) - \lambda_k \cdot 0} \right] \rightarrow 0
\]
(50)
Here \(\alpha\) term requires a bit more work but \(\alpha, \gamma, \delta\) terms all go to zero as \(|\lambda_k|\) goes to infinity in a similar way.

I.4.1.2 \(\alpha\) term

We start with the inequality:
\[
\left\| \left[ |\lambda_k| \alpha(-|\lambda_k|) - 1 \right] |f^{n+1}\rangle \right\| \leq \left\| \left[ \frac{|\lambda_k|}{H_0 + |\lambda_k|} - 1 \right] |f^{n+1}\rangle \right\|
\]
(51)
\[+ |\lambda_k| ||\phi^{-1}|| \left\| \frac{1}{H_0 + |\lambda_k|} \phi^{-1} \right\| ||\phi^{(+)}|| \left\| \frac{1}{H_0 + |\lambda_k|} |f^{n+1}\rangle \right\|
\]
Notice that the first term involving only the resolvent \((H_0 + |\lambda_k|)^{-1}\) actually goes to zero. Let us therefore concentrate on the next piece,
\[
\frac{|\lambda_k||\Phi^{-1}|}{\text{finite}} \left\| \frac{1}{H_0 + |\lambda_k|} \phi^{-1} \right\| ||\phi^{(+)}|| \left\| \frac{1}{H_0 + |\lambda_k|} |f^{n+1}\rangle \right\|
\]

It is enough to look at,
\[
\lim_{|\lambda_k| \rightarrow \infty} \left\| \frac{1}{H_0 + |\lambda_k|} \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} a'(p, p_{\perp}) |f^n\rangle \right\| \leq \lim_{|\lambda_k| \rightarrow \infty} \left\| \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} a'(p, p_{\perp}) \frac{1}{nm + |\lambda_k| + \omega(p, p_{\perp})} |f^{n+1}\rangle \right\|
\]
This factor behaves much like the preceding factor, by means of a similar inequality for the creation part. Thus, we get
\[
\lim_{|\lambda_k| \rightarrow \infty} \left\| \left[ |\lambda_k| \alpha(-|\lambda_k|) - 1 \right] |f^{n+1}\rangle \right\| = 0
\]

I.4.1.3 \(\gamma\) term

The \(\gamma\) term is identical to the previous expression it is the formal adjoint, written explicitly
\[
\left\| |\lambda_k| \frac{1}{H_0 + |\lambda_k|} \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} \Phi^{-1}(-|\lambda_k|) |f^n\rangle \right\| \leq |\lambda_k| ||\Phi^{-1}|| \left\| \frac{1}{H_0 + |\lambda_k|} \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{\perp}}{2\pi} \frac{1}{\sqrt{2p}} a'(p, p_{\perp}) |f^n\rangle \right\|
\]
and by the previous arguments,
\[
\lim_{|\lambda_k| \rightarrow \infty} \left\| |\lambda_k| \gamma(-|\lambda_k|) |f^n\rangle \right\| = 0
\]
(52)
I.4.1.4 δ term

Let us briefly mention the δ term. The difference can be written as

$$\left\| \left[ |\lambda_k| \Phi^{-1}(-|\lambda_k|) - 1 \right] f^m \right\| \leq \left\| \left[ |\lambda_k| (H_0 + |\lambda_k| + \mu)^{-1} - 1 \right] f^m \right\|$$

$$+ \left\| \lambda_k (H_0 + |\lambda_k| + \mu)^{-1} \right\| \frac{1}{1 - \|K\| - \|U\|}$$

The finiteness of term as indicated above is due to the principle of uniform boundedness. Thus the whole expression goes to zero again using the above estimates.

Thus, we conclude that $R(E)$ is the resolvent of a quantum Hamiltonian (which we cannot write down explicitly). Let us emphasize that this already implies that $R(E)$ is the resolvent family of a self-adjoint operator if we can justify that $R^\dagger(E) = R(E)$ for complex values of $E$. This is formally true, but we must justify the formal operation of $\Phi(E) = \Phi(E)$ carefully, since unlike the resolvent itself this is an unbounded operator. In fact its inverse shows up in the resolvent formula, and we do not have an explicit expression for this inverse, in any case, we must show that $\Phi(E)$ is a self-adjoint holomorphic family of type-A in the sense of Kato (for $\text{Re}(E) < nm + \mu \rho$) to justify many formal manipulations that we perform with $\Phi(E)$ (one consequence of which is to verify the formal equality above). Thus we turn to this issue now. It is in some sense more abstract and technical but essential to justify the spectral projection formula alluded above as well.

I.5 Holomorphic Structure and Self-Adjointness of the Principal Operator

Here we introduce the concept of holomorphic family of type-A and show that indeed $\Phi(E)$ defines such a family. Moreover, there is a concept of self-adjointness for operator families defined over a complex domain as well, we show that this is actually true for our family.

Definition 2. A family $T(E) \in \mathcal{C}(X, Y)$ (closed linear operators from Banach spaces $X$ to $Y$) defined for $E$ in a domain $\Omega$ of the complex plane is said to be holomorphic of type-A if:

- $D(T(E)) = D$ is independent of $E$,
- $T(E)u$ is holomorphic for $E \in \Omega$ for every $u \in D$.

I.5.1 Finding the Common Domain

To start we first find a common domain for the family $\Phi(E)$, most reasonable choice seems to be $D(H_0)$. To justify $D(H_0)$ to be the common domain of $\Phi(E)$, we want to show that $\tilde{K}(E)$ and $\tilde{U}(E)$ are bounded, $E$ being complex.

$$\Phi(E) = \left[ 1 + \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p} \left( \omega(p, p_\perp) - \mu \rho \right) \left( H_0 - E + \omega(p, p_\perp) \right) \right] \frac{1}{\tilde{K}}$$

$$- \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \int_0^\infty dq \int \frac{dq_\perp}{2\pi} \frac{a^\dagger(q, q_\perp)}{\sqrt{2q}} \frac{1}{\tilde{U}} \left( H_0 - E + \omega(q, q_\perp) + \omega(p, p_\perp) \right) \left( H_0 - E + \mu \rho + \omega(p, p_\perp) \right) \frac{1}{2p} \right]$$

$$\times (H_0 - E + \mu \rho)$$

Let us now work out a norm estimate on the $\tilde{U}$ term since we already commented on the $\tilde{K}$ term above. We use a slightly different approach since our aim is to show that this term is bounded as long as $\text{Re}(E) < nm + \mu$. We remind again the inequality

$$\left\| \int dp dq a^\dagger(p, q) a(q) \right\| < n \left[ \int dp dq |F(p, q)|^2 \right]^{1/2}$$
where the integral refers to multi-parameters and norm is taken in a Fock space of $n$ particles. This implies immediately that if we drop $\text{Im}(E)$ we get an upper bound, so we replace $E$ with $\text{Re}(E)$. Moreover inside the norm we replace the positive operator $H_0$ by its lower bound $(n-1)m$,

$$
\| \tilde{U} \| = \lambda^2 \int_0^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dp_\perp}{2\pi} \int_0^{\infty} \frac{dq}{2\pi} \int_0^{\infty} \frac{dq_\perp}{2\pi} \frac{1}{\sqrt{2p}} \frac{a(p,p_\perp)}{\sqrt{2q}} \frac{1}{|H_0 - E + \omega(p,p_\perp) + \omega(q,q_\perp)|} |H_0 + \mu_p - E + \omega(p,p_\perp)| \frac{1}{\sqrt{2q}} \frac{1}{2q}^{1/2}
$$

Let us now note that $(n-1)m - \text{Re}(E) = nm + \mu_p - \text{Re}(E) + 2\frac{1}{2}(m + \mu_p)$ and let us suppose we always keep $\text{Re}(E) < nm + \mu_p$, which is the region of interest for possible bound states. We now use a generalized arithmetic-geometric mean inequality for positive numbers,

$$
\omega(p,p_\perp) - \frac{1}{2}m + \mu_p + \omega(q,q_\perp) - \frac{1}{2}(m + \mu_p) > \left[ \omega(p,p_\perp) - \frac{1}{2}m + \mu_p \right]^{1/4} \left[ \omega(q,q_\perp) - \frac{1}{2}(m + \mu_p) \right]^{3/4}.
$$

This splits the integration. We drop the difference $\Delta = nm + \mu_p - \text{Re}(E)$ for simplicity in $(q,q_\perp)$-integrals. Furthermore we write $\omega(p,p_\perp) + (n-1)m + \mu_p - \text{Re}(E)$ term as $\omega(p,p_\perp) - (m - \Delta)$, and combine the two $(p,p_\perp)$-terms with the largest of $\frac{m + \mu_p}{2}$ and $m - \Delta$, calling the largest one $m_*$, which is strictly less than $m$. After cancelling the $p$ and $q$ products, we have the product of two integrals below,

$$
\| \tilde{U} \| \leq C_1 \lambda^2 \left[ \int_0^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dp_\perp}{2\pi} \int_0^{\infty} \frac{dq}{2\pi} \int_0^{\infty} \frac{dq_\perp}{2\pi} \cos^{3/2}(\theta) \frac{\cos^{1/2}(\beta)}{(\rho^2 + m^2 - 2m_*\rho \cos(\theta))^5/2} \frac{1}{\sqrt{2pq}} \frac{1}{\sqrt{2pq}} \frac{1}{(\rho^2 + m^2 - (m + \mu_p)\rho \cos(\beta))^3/2} \right]^{1/2},
$$

both of which are finite integrals. Consequently, we have the desired result for complex $E$ for which $\text{Re}(E) < nm + \mu_p$. The holomorphicity requirement will make use of this bound as well. Although the above argument can be generalized to discuss the decay of $\tilde{U}$ for large negative values of $\text{Re}(E)$, it is instructive to get another bound by means of Feynman parametrization and exponentiation. Therefore we give an alternative estimate for the norm,

$$
\| \tilde{U} \| = \lambda^2 \int_0^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dp_\perp}{2\pi} \int_0^{\infty} \frac{dq}{2\pi} \int_0^{\infty} \frac{dq_\perp}{2\pi} \frac{1}{\sqrt{2pq}} \frac{a(p,p_\perp)}{\sqrt{2pq}} \frac{1}{\sqrt{2pq}} \frac{1}{(\rho^2 + m^2 - (m + \mu_p)\rho \cos(\beta))^3/2} \right]^{1/2},
$$

For ease of calculation let us define:

$$
\phi^{(+)}(h(s)) = \int_0^{\infty} \frac{dp}{2\pi} \int_0^{\infty} \frac{dp_\perp}{2\pi} \frac{1}{\sqrt{p}} a(p,p_\perp) e^{-s \omega(p,p_\perp)}
$$

$$
\phi^{(-)}(g(su)) = \int_0^{\infty} \frac{dq}{2\pi} \int_0^{\infty} \frac{dq_\perp}{2\pi} \frac{1}{\sqrt{q}} a(q,q_\perp) e^{-s \omega(q,q_\perp)}
$$

Then the norm inside $\tilde{U}$ can be estimated as follows, after pulling out $s$ and $u$ integrals, $e^{sE}$ is replaced with $e^{s \text{Re}(E)}$, moreover we replace $H_0$ by its lower bound $nm$. We also use the inequality,

$$
\| \phi^{(-)}(g(su)) e^{-sH_0} \phi^{(+)}(h(s)) \| \leq e^{-s(n-1)m \| g \| \| h \|}
$$

(55)
Then the norm becomes,
\[
\|\phi^{-1}(\phi)\| \leq C_1 n \frac{e^{-s(1+u)m/2}}{u^{1/2} s}
\]
(59)

Here we use the integral
\[
\int_0^\infty \frac{dp}{2\pi} \frac{1}{\sqrt{p^2}} e^{-s\left(p + \frac{m^2}{p}\right)} = \frac{1}{2\sqrt{\pi} s} e^{-ms}
\]
(61)

Putting back this into the inequality for \(\tilde{U}\):
\[
\|\tilde{U}\| \leq C_1 n \lambda^2 \int_0^1 du \int_0^\infty s ds e^{-(1-u)\mu p - s(n-1)m - s\text{Re}(E)} \frac{e^{-s(1+u)m/2}}{s}
\]
\[
\leq C_2 \lambda^2 n \int_0^\infty ds e^{s[(n-1)m + \mu p - \text{Re}(E)]} \int_0^1 du \frac{1}{u^{1/2}} e^{-s(n-1)m - s\mu p - s(1-u)m/2}
\]
\[
\leq C_3 \frac{\lambda^2 n}{(n-1)m + \mu p - \text{Re}(E)}
\]
(64)

This estimate shows that if we choose \(\text{Re}(E)\) sufficiently small we can make the norm of these parts less than 1 and the operator becomes invertible. This will be important for our discussion below.

### 1.5.2 Operator Family \(\Phi(E)\) is Closed on its Common Domain

We remind the reader the definition of a closed operator.

**Definition 3.** An operator \(T\) is said to be closed if, for any sequence \(x_k\) in its domain \(D(T)\), \(x_k \to x\) and \(Tx_k \to y\) implies that \(Tx = y\).

We want to show that \(\Phi(E)\) is closed in its domain \(D(\Phi(E)) = D = D(H_0)\). Let us suppose that we have a sequence \(x_k \in D(H_0)\) that converges to \(x\) as well as,
\[
\Phi(E) x_k \to y
\]
(65)

When \(\text{Re}(E) \leq \text{Re}(E_*)\) where \(\text{Re}(E_*)\) is sufficiently small, such that \(\Phi(E)\) becomes invertible (notice that due to the bound found above, there is a value for \(E\) in the complex plane, below which the operator becomes invertible):
\[
[1 + \tilde{K}(E) - \tilde{U}(E)] (H_0 + \mu p - E) x_k \to y ,\ x_k \to x
\]
\[
\implies (H_0 + \mu p - E) x_k \to [1 + \tilde{K}(E) - \tilde{U}(E)]^{-1} y ,\ x_k \to x
\]
(66)

Since \(H_0\) is closed on its domain:
\[
(H_0 + \mu p - E) x_k \to (H_0 + \mu p - E) x = [1 + \tilde{K}(E) - \tilde{U}(E)]^{-1} y
\]
\[
\implies y = [1 + \tilde{K}(E) - \tilde{U}(E)] (H_0 + \mu p - E) x
\]
(67)

Subsequently, for \(\text{Re}(E) \leq \text{Re}(E_*)\), \(\Phi(E)\) is closed. For \(\text{Re}(E) > \text{Re}(E_*)\), we rearrange according to \(124\):
\[
\Phi(E) - \Phi(E_*) = T(E, E_*)(E_* - E) =
\]
(68)
\[
= (E_* - E) \left[1 + \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \frac{1}{2p^2(p_\perp)(H_0 - E_\perp + \omega(p, p_\perp))(H_0 - E - \omega(p, p_\perp))} \right]
\]
(69)
\[
- \lambda^2 \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_\perp}{2\pi} \int_0^\infty \frac{dp_\perp}{2\pi} \frac{1}{2\sqrt{pq}} q \left[ (H_0 - E_\perp + \omega(q, q_\perp))(H_0 - E + \omega(q, q_\perp) + \omega(q, q_\perp)) \right]
\]
(70)
We can now choose $E$ are actually bounded. The remaining matrix elements thus become, and an invertible part.

Theorem 2. Let $\Phi(E) \subseteq \mathbb{C}$ be an open subset of $\mathbb{C}$ and $L^1(V)$ the Lebesgue integration space of complex valued functions on $V$. Define $\Theta(E) : \Omega \rightarrow \mathbb{C}$ by:

$$\Theta(E) := \int_V \phi(t, E)dt \quad , \quad E \in \Omega$$

where $\phi(t, E) : V \times \Omega \rightarrow \mathbb{C}$ satisfies:

- $\phi(t, E) \in L^1(V) \quad , \quad E \in \Omega$
- $\phi(t, \cdot) \in H(\Omega) \quad , \quad t \in V$

where $H(\Omega)$ denotes all functions that are holomorphic on $\Omega$ and $t$ stands for all the parameters related to our product measure (since there could be more than just one). If the mapping:

$$E \rightarrow \int_V |\phi(t, E)|dt$$

is bounded on every compact subset of $\Omega$, then $\Theta(E)$ is holomorphic on $\Omega$.

Since $\tilde{K}(E)$ can be defined by the spectral measure of $H_0$ and we have shown its boundedness for any choice of $E$ in the symmetric domain $\Omega$, its holomorphicity is easier to check (moreover a similar calculation for a compact manifold is to be presented in the next part). Therefore, we concentrate on the potential like part.

Let us write down explicitly the matrix elements of the operator $\tilde{U}(E)$ to see that it consists of integrable (over the parameter space) functions for all $E$ and holomorphic functions (for fixed parameters) in the variable $E$. Note that once we isolate the piece $H_0 - E + \mu_p$, which is clearly holomorphic in $E$, the proceeding operators are actually bounded. The remaining matrix elements thus become,

$$\langle \psi_1 | \tilde{U}(E) | \psi_2 \rangle = \lambda^2 \int_0^1 du \int_0^\infty ds \int \left( \phi^{(+)}(g(s)u) e^{-s(H_0 - E)} \phi^{(+)}(h(s)) \right) \psi_1 \psi_2.$$  

(74)
Note that these are well defined expressions and for any fixed complex $E$, real part of which is below $nm + \mu_p$, they are finite. To clarify our claims, we write the measure more explicitly in this case (to exhibit the full parameter space),

$$\langle \phi^{(+)}(g(su))\psi_1|e^{-sH_0}\phi^{(+)}(h(s))\psi_2\rangle = \int \frac{dp_1 dp_2 \ldots dp_n dp_{n\perp}}{4\pi^2} \int \frac{dp dp_{n\perp}}{4\pi^2} \frac{\psi_2(p, p_{n\perp}, p_{n\perp}, \ldots, p_n, p_{n\perp}) e^{-s\omega(p, p_{n\perp})}}{2^{n/2} \sqrt{p p_{n\perp} \cdots p_n}} \frac{\psi_1(q, q_{n\perp}, q_{n\perp}, \ldots, q_n, q_{n\perp}) e^{-s\omega(q, q_{n\perp})}}{2^{n/2} \sqrt{q p_{n\perp} \cdots q_n}} \times e^{-s \sum_{i=1}^{n\perp} \omega(p_i, p_{i\perp})}.$$ 

If we choose any two measurable functions $\psi_1, \psi_2$, the above expression defines a measurable function over all the variables, since the full expression is absolutely integrable as shown above, as a result we obtain a product (Lebesgue) measure over $s, u$ and all the momentum variables we have. As a result of the above theorem, we establish that $\tilde{U}(E)$ defines a holomorphic family of type-A in the sense of Kato for $\Omega = \{ E \in \mathbb{C} | \text{Re}(E) < nm + \mu_p \}$. This in turn, combined with $\tilde{K}(E)$ claim, proves that $\Phi(E)$ is a holomorphic family of type-A. We need to establish that this family is self-adjoint (in the appropriate sense).

### 1.5.4 Self-Adjointness of the Family $\Phi(E)$

Note that, formally, $\Phi^\dagger(E) = \Phi(\tilde{E})$, then at least, $D(\Phi(E)) \subset D(\Phi^\dagger(E))$. But to conclude self-adjointness, we need to show that they admit the same domain.

Our strategy will be the following: we make use of the well-known Kato-Rellich Theorem [22] to show that $\Phi(E)$ is self-adjoint on some region on the real axis for $E$ chosen to be sufficiently small and then employ Wüst’s theorem [23] (quoted below) to generalize it to the whole symmetric open domain of interest.

**Theorem 3.** Let $A : D(A) \to \mathcal{H}$ be a self-adjoint operator and $B : D(B) \to \mathcal{H}$ be symmetric. For $D(A) \subset D(B)$, if the following is satisfied:

$$||Bx|| \leq a||Ax|| + b||x||, \quad \forall x \in \mathcal{H}$$

(75)

with $a < 1, b < \infty$; then $A + B : D(A) \to \mathcal{H}$ is self-adjoint.

A proof of this well-known theorem is in the classical reference by Reed and Simon [22]. Recall the form of the Principal Operator for $E$:

$$\Phi(E) = (1 + \tilde{K}(E) - \tilde{U}(E))(H_0 - E + \mu_p)$$

$$= \begin{pmatrix} H_0 - E + \mu_p + \tilde{K}(E)(H_0 - E + \mu_p) & -\tilde{U}(E)(H_0 - E + \mu_p) \\ A & B \end{pmatrix}$$

(76)

If $A$ is invertible, we can write $x = A^{-1}y$ for some $y \in \mathcal{H}$, so the inequality above becomes:

$$||BA^{-1}y|| \leq a||y|| + b||A^{-1}y||$$

(77)

We will work on the real axis where $E < E_*$, $E_*$ chosen to be sufficiently small such that $\tilde{K}(E)$ is strictly positive. By the spectral theorem it is self-adjoint (being a continuous function of $H_0$) and well defined on the original domain $D(H_0)$. Hence $A$ is a self-adjoint operator on $D(H_0)$. It takes some work to justify that $BE$ is symmetric, since for continuous variables creation-annihilation operators are distributional valued. It is more natural to think of this expression via the following representation,

$$B(E) = \int_0^\infty ds \phi^{(-)}(f(s)) e^{-s(H_0 - E)}\phi^{(+)}(f(s))$$

(78)

where

$$\phi^{(-)}(f(s)) = \int_0^\infty \frac{dp}{2\pi} \int \frac{dp_{n\perp}}{2\pi} e^{-s\omega(p, p_{n\perp})},$$

similarly for the adjoint operator. The adjoint operation commutes with the integration since the integral is absolutely convergent. One can then show that this expression is indeed a symmetric operator for real values.
(It is still a delicate matter to prove the equivalence of all these representation, which we leave to the reader). Note that for \( b = 0 \), if the following is true in some region:

\[
\|BA^{-1}\| < 1
\]

then the conditions stated in Theorem [4] are satisfied and \( A + B \) is self-adjoint in that region. Rearranging:

\[
BA^{-1} = -\tilde{U}(E)(H_0 - E + \mu_p)[(1 + \tilde{K}(E))(H_0 - E + \mu_p)^{-1} - \tilde{U}(E)][1 + \tilde{K}(E)]^{-1}
\]

(80)

since \( \tilde{K}(E) \) is positive,

\[
\|\tilde{U}(E)(1 + \tilde{K}(E))^{-1}\| \leq \|\tilde{U}(E)\|
\]

(81)

Recall that while searching for the bound on the ground state, we show that \( \|\tilde{U}(E)\| < 1 \) if we choose \( E < (n - 1)m + \mu_p - Cn\lambda^2 \). In the same spirit, we can see that, from the estimate in the complex case, we can choose a sufficiently low value of \( E \) (say less than or equal to \( E_* \)) on the real axis to make the above norm less than 1. Then, \( \|BA^{-1}\| \leq a \) where \( a < 1 \) and by the theorem statement, \( A + B = \Phi(E) \) is self-adjoint at least in some region where \( E \leq E_* \).

**Theorem 4.** (Wüst) Let \( \Omega \) be a domain in the complex plane which is symmetric around the real axis and \( \{\Phi(E), E \in \Omega\} \) be a holomorphic family of type-A in \( \mathcal{H} \) with dense domain \( D_0 \) such that \( \Phi(\bar{E}) \subset \Phi^\dagger(E) \). Define \( M \) by:

\[
M := \{E \mid E \in U, \; \Phi^\dagger(E) = \Phi(\bar{E})\}
\]

(82)

If \( M \) is not empty, it extends to all of \( \Omega \); i.e. \( M \neq \emptyset \) \( \Rightarrow \) \( M = \Omega \).

The formal relation \( \Phi^\dagger(E) = \Phi(\bar{E}) \) implies that domain inclusion alluded in the theorem holds. As we have shown previously that at least in some region on the real line below a sufficiently small \( E_* \), \( \Phi(E) \) is self-adjoint. Thanks to the Wüst’s theorem, the equality (not only formally but in the real sense; meaning that domains are also equal) \( \Phi^\dagger(E) = \Phi(\bar{E}) \) extends to all \( \{E \in \mathbb{C}|\text{Re}(E) < nm + \mu_p\} \). Hence we conclude that \( \Phi(E) \) is a self-adjoint holomorphic family of type-A on the domain of interest.

**Part II: Lee model on 2D compact Riemannian manifolds**

In this second part of the paper, we analyze a version of the Lee model where the two level system is fixed on a Riemannian manifold interacting with an arbitrary number of bosons. Again, we employ the nonperturbative renormalization method proposed by Rajeev [16], where the resolvent is expressed in terms of the “Principal Operator” \( \Phi(E) \). Once a finite expression for \( \Phi(E) \) is found, the spectral information can be obtained from it. The zeros of the eigenvalues of \( \Phi(E) \), as discussed in the previous part, correspond to the bound states of the quantum Hamiltonian, if they are below the free spectrum. The compact version is technically much simpler, thanks to spectral results known in the case of compact manifolds. We plan to investigate uniqueness of the ground state only for this compact version in a forthcoming work. To apply this technique to our model, we make use of an essential mathematical tool; the heat kernel. We give a brief overview of this tool on Riemannian manifolds here not to interrupt the flow of the main work. Here we follow [24].

The heat equation on a Riemannian manifold \( M \) is given as:

\[
\frac{\partial u}{\partial t} = -\nabla_G^2 u,
\]

(83)

where \( -\nabla_G^2 \) is the Laplace-Beltrami operator on \( M \). The heat kernel \( K_t(x, y) \), defined on \( (0, \infty) \times M \times M \), is a fundamental solution of the heat equation which is \( C^2 \) with respect to \( x \) and \( C^1 \) with respect to \( t \) satisfying:

\[
\frac{\partial}{\partial t} K_t(x, y) = -\nabla_G^2 x K_t(x, y) \quad ; \quad \lim_{t \to 0^+} K_t(\cdot, y) = \delta_{\cdot, y},
\]

(84)

as well as being positive and symmetric and it satisfies the semi-group property respectively:

\[
K_t(x, y) = K_t(y, x)
\]

(85)

\[
K_t(x, y) > 0 \quad ; \quad x, y \in M \quad , \quad t \geq 0
\]

(86)

\[
\int_M d_2z K_t(x, z) K_{t_2}(z, y) = K_{t_1 + t_2}(x, y)
\]

(87)
For $\mathcal{M}$ compact, there exists a *complete orthonormal basis* consisting of eigenfunctions $f_\sigma$ of the Laplace-Beltrami operator $-\nabla^2_g$ and the Sturm-Liouville decomposition of the heat kernel reads:

$$K_t(x, y) = \sum_{\sigma} \infty e^{-\sigma t} f_\sigma(x) f_\sigma(y)$$

where $\sigma$’s are the corresponding positive eigenvalues (counting multiplicities as well). It is essential that this *set of eigenvalues is countable with no accumulation point other than infinity*. The short time asymptotics for the diagonal heat kernel is given by:

$$K_t(x, x) \sim \frac{1}{(4\pi t)^{d/2}} \sum_{k=0}^{\infty} a_k(x)t^k \quad t \to 0^+$$

where $d$ is the dimension of $\mathcal{M}$ and the smooth functions $a_k(x)$ (restricted to the diagonal) are given by explicit formulas in terms of local geometric invariants [25] with $a_0 = 1$. We can directly recognize the singular nature of the heat kernel near $0^+$. This is an important point to keep in mind throughout the work when giving estimates to some integrals and searching for the sources of possible divergences.

When estimating some expressions that we face, the following upper bound for the heat kernel on compact manifolds will be of great importance [26]:

$$K_t(x, x) \leq \frac{1}{V(\mathcal{M})} + C t^{-d/2}$$

for all $t > 0$ and $x \in \mathcal{M}$ where $d = \dim(\mathcal{M})$, $V(\mathcal{M})$ is the volume of the manifold and $C$ is a positive constant which can be computed explicitly in terms of geometric invariants. We introduce the resolvent in terms of the Principle Operator $\Phi(E)$ as before. We explicitly construct $\Phi(E)$ and observe that the bound state solutions come from the poles of $\Phi(E)^{-1}$. Identifying the divergence, we first put a cut-off to the allowed eigenvalues of the Laplacian and let the mass difference $\mu$ depend on $\Lambda$. Imposing the physical mass condition and solving for $\mu(\Lambda)$, we remove the divergence then take the limit $\Lambda \to \infty$ (as mentioned in the first part this process is not essential, we actually only need the resulting expression).

Once we establish a finite principal operator, we start searching for upper and lower bounds to the ground state energy. In this part thanks to the compactness, the variational method is employed to show that the ground state energy is indeed below the trivial guess $nm + \mu_p$ where $n$ is the number of bosons and $\mu_p$ is the physical binding energy. $E_0$, below which the Principal Operator is observed to be invertible serves as a lower bound to the ground state energy. As shown in the previous part, we establish that $R(E) = \frac{1}{H - E}$ is indeed the resolvent of a densely defined closed operator. Subsequently, we study the holomorphicity of the Principal Operator. To show that $\Phi(E)$ is a self-adjoint holomorphic family of type-A in the sense of Kato, we fix the common domain $D(H_0)$ as is done in the previous part and show that $\Phi(E)$ is closed on it. We conclude that $\Phi(E)$ is a self-adjoint holomorphic family of type-A following the same method as before.

### II.1 Hamiltonian and the Renormalized Resolvent

For the relativistic Lee model on a 2+1 dimensional *compact* Riemannian manifold $(\mathcal{M}, g)$, the formal Hamiltonian is (for more details see [1]):

$$H = H_0 + \mu \frac{1 - \sigma_3}{2} + H_I$$

Here, we have the free Hamiltonian,

$$H_0 = \sum_\sigma \omega_\sigma a_\sigma^\dagger a_\sigma$$

in a similar fashion, we have the interaction part,

$$H_I = \lambda [\sigma_+ \phi^{(-)}(\vec{x}) + \sigma_- \phi^{(+)}(\vec{x})]$$
where \( \sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i\sigma_2) \), \( \omega_\sigma = \sqrt{\sigma + m^2} \), \( m \) the mass of the boson, \( \bar{x} \) the location of the two level system. Compactness is not an essential restriction for the formalism presented below, but it simplifies the rigorous analysis we attempt in our work.

Since the manifold we are working on is compact, the Laplacian has a discrete spectrum and there is a family of orthonormal complete eigenfunctions \( f_\sigma(x) \in L^2(M) \) which satisfy \([27]\):

\[
\int_M d_\sigma x \ f_\sigma^*(x) f_\sigma(x) = \delta_{\sigma \sigma'},
\]

\[
\sum_{\sigma} f_\sigma^*(x) f_\sigma(y) = \delta_\sigma(x, y)
\]

(94)

where \( d_\sigma x = \sqrt{\det[g_{ij}]} dx \) is the volume element and we introduce:

\[
\phi^{(-)}(x) = \sum_{\sigma} \frac{1}{\sqrt{2\omega_\sigma}} f_\sigma^*(x) a_\sigma^{-}
\]

\[
\phi^{(+)}(x) = \sum_{\sigma} \frac{1}{\sqrt{2\omega_\sigma}} f_\sigma(x) a_\sigma
\]

(95)

Since \( f_\sigma(x) \)'s can be chosen to be real, the complex conjugate will not be important in the following calculations. We use \( n \)-particle Hilbert space with an invariant norm as before,

\[
|\psi\rangle = \frac{1}{\sqrt{n!}} \int_M d_\sigma x_1...d_\sigma x_n \psi(x_1, ..., x_n) \phi^{(-)}(x_1)...\phi^{(-)}(x_n)|0\rangle,
\]

(96)

where \( \psi(x_1, ..., x_n) \) is symmetric in all its entries. The inner product can be written in a nicer form in the eigenfunction decomposition,

\[
\langle \psi | \psi \rangle = \sum_{\sigma_1, ..., \sigma_n} \frac{1}{2^n} \frac{\langle \psi(\sigma_1, ..., \sigma_n) \rangle^2}{\omega_{\sigma_1} ... \omega_{\sigma_n}}
\]

(97)

or in coordinate space as

\[
\langle \psi | \psi \rangle = \frac{1}{2^n} \int_M dx_1...dx_n \psi(x_1, ..., x_n)[-\nabla^2 + m^2]^{-1/2}...[-\nabla^2 + m^2]^{-1/2} \psi(x_1, ..., x_n).
\]

(98)

The coordinate version will not be used in this work, although for some other purposes, such as uniqueness of the ground state coordinate space measure becomes important (this is because the ground state wave function is strictly positive in coordinate space representation).

### II.2 Principal Operator and Spectral Flow

To construct a finite model, we calculate the resolvent by an alternative method as is done for the light-front version, and arrive at the principal operator:

\[
\Phi(E) = [H_0 - E + \mu] - \sum_{\sigma, \tau} \frac{\lambda^2}{2\omega_\sigma} f_\sigma a_\sigma \frac{1}{H_0 - E} \frac{1}{\sqrt{2\omega_\tau}} f_\tau a_\tau^+.
\]

(99)

This is a formal expression and we now normal order this operator using the commutation relation,

\[
a_\sigma \frac{1}{H_0 - E} \frac{1}{H_0 - E + \omega_\sigma} a_\sigma
\]

(100)

Changing the order of \( a_\sigma \) and \( a_\sigma^+ \), then repeating the calculation above for \( \frac{1}{H_0 - E + \omega_\sigma} a_\sigma^+ \), we arrive at:

\[
\Phi_\Lambda(E) = \left[ H_0 - E + \mu(\Lambda) \right] - \sum_{\sigma < \Lambda} \frac{\lambda^2}{2\omega_\sigma} |f_\sigma|^2 \frac{1}{H_0 - E + \omega_\sigma} - \sum_{\sigma, \tau < \Lambda} \frac{\lambda^2}{\sqrt{2\omega_\tau}} f_\tau a_\tau^+ \frac{1}{H_0 - E + \omega_\sigma} \frac{1}{\sqrt{2\omega_\tau}} f_\sigma
\]

(101)
Note that we introduce a cut-off anticipating a divergence; the second term in \([101]\) diverges as \(\Lambda \to \infty\). In order to make sense of all formal operations, we should put a cut-off to the allowed eigenvalues of \(-\nabla^2\). We choose \(\mu(\Lambda)\) such that we remove the divergence in \([101]\). This still leaves out some ambiguity in the finite parts. But if we impose the condition that we get a zero when \(E =\mu_p\), where \(\mu_p\) is the physical binding energy for the boson and the down state composite, which is the vacuum sector for the principal operator, i.e.

\[
\Phi_R(E = \mu_p)|0\rangle = 0,
\]

this fixes the finite part to be \(\mu_p\). This is a renormalization condition typical of such problems. As a result for \(\mu(\Lambda)\) we get:

\[
\mu(\Lambda) = \sum_{\sigma < \Lambda} \frac{\lambda^2}{2\omega_\sigma} |f_\sigma|^2 \frac{1}{(\omega_\sigma - \mu_p)} + \mu_p
\]

(103)

the principal operator becomes after the physical mass condition imposed,

\[
\Phi(E) = (H_0 - E + \mu_p)\left[1 + \sum_{\sigma} \frac{\lambda^2}{2\omega_\sigma} \frac{1}{(H_0 - \omega_\sigma)(\omega_\sigma - \mu_p)} f^2_\sigma(\bar{x})\right] - \sum_{\sigma, \tau} \lambda^2 f_\sigma(\bar{x}) a_\sigma^\dagger \frac{a_\tau}{\sqrt{2\omega_\sigma}} H_0 - E + \omega_\sigma + \omega_\tau \sqrt{2\omega_\sigma} f_\tau(\bar{x})
\]

Recall that the eigenvalues of \(\Phi(E)\) carry essential information about the spectrum,

\[
\Phi(E)|\omega_k(E)\rangle = \omega_k(E)|\omega_k(E)\rangle,
\]

(104)

here we assume again that the eigenvalues are differentiable functions of \(E\). As to be anticipated, we prove later that they behave better than that, they are actually holomorphic functions of \(E\). We now compute the flow of eigenvalues as we change \(E\) along the real axis while staying below \(nm + \mu_p\). This can be accomplished by means of Feynman-Hellman formula \([19]\) (equation 3.18 page 391):

\[
\frac{\partial \omega_k(E)}{\partial E} = \langle \omega_k(E)|\frac{\partial \Phi(E)}{\partial E}|\omega_k(E)\rangle = -1 + \frac{\lambda^2}{2} \sum_{\sigma} \omega_k(E) |\frac{\sum_{\sigma} f^2_\sigma(\bar{x})}{\omega_\sigma(H_0 - E + \omega_\sigma)^2}| \omega_k(E)\rangle
\]

\[
- \frac{\lambda^2}{2} \sum_{\sigma, \tau} \omega_k(E) |\frac{a_{\sigma}^\dagger f_\tau(\bar{x}) f_\sigma(\bar{x})}{\sqrt{\omega_\sigma^2(H_0 - E + \omega_\sigma + \omega_\tau)^2}}| \frac{a_\tau}{\sqrt{2\omega_\sigma}} |\omega_k(E)\rangle
\]

\[
\int_0^\infty ds \left| \sum_{\sigma} e^{-s(H_0 + \omega_\sigma - \frac{i}{2} E)} f_\sigma(\bar{x}) a_\sigma \right|^2 |\omega_k(E)\rangle\rangle
\]

\[
\implies \frac{\partial \omega_k(E)}{\partial E} < 0
\]

(105)

II.3 Upper and Lower Bounds on the Ground State

We want to show that there is an upper bound to the ground state energy by means of the variational principle. We choose a trial function:

\[
|\Omega_*\rangle = \frac{1}{\sqrt{n!}} a^\dagger_0 \cdots a^\dagger_n |0\rangle
\]

(106)

where we have \(n\) creation operators with \(\sigma = 0\). This is possible on a compact manifold since \((-\nabla^2)\frac{1}{\sqrt{V(M)}} = 0\) is a constant solution \([27]\), where \(\frac{1}{\sqrt{V(M)}}\) is chosen for the sake of normalization \(\int |f|^2 dv = 1\),

The zero’s of the principal operator give us bound state energies since they are the poles of the resolvent. Accordingly, if we show that:

\[
\omega_0(E_\ast) \leq \langle \Omega_*|\Phi_R(E_\ast)|\Omega_*\rangle < 0,
\]

(107)

where by the variational principle \(\omega_0(E)\) refers to the smallest eigenvalue, we can deduce, using \(\frac{\partial \omega_0(E)}{\partial E} < 0\), that

\[
E_{\sigma\tau} < E_\ast
\]
Making a trivial guess, we set $E_\ast = nm + \mu_p$, corresponding to the sector $Q = n + 1$.

\[
\langle \Omega_\ast | \Phi_R(E_\ast) | \Omega_\ast \rangle = \langle \Omega_\ast | (H_0 - nm) | \Omega_\ast \rangle + \langle \Omega_\ast | (H_0 - nm) \sum_{\sigma} \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - nm - \mu_p + \omega_\sigma)(\omega_\sigma - \mu_p)} | \Omega_\ast \rangle
\]

\[
- \langle \Omega_\ast | \sum_{\sigma, \tau} \lambda^2 f_\sigma a_\tau^\dagger \sqrt{2\omega_\sigma} (H_0 - nm - \mu_p + \omega_\sigma + \omega_\tau) \frac{1}{\sqrt{2\omega_\tau}} a_\tau | \Omega_\ast \rangle (108)
\]

The first part becomes zero as can be easily checked, the last “potential” part gives

\[
- \lambda^2 \langle \Omega_\ast | \sum_{\sigma, \tau} f_\sigma \frac{a_\tau^\dagger}{\sqrt{2\omega_\sigma}} (H_0 - nm - \mu_p + \omega_\sigma + \omega_\tau) \frac{1}{\sqrt{2\omega_\tau}} a_\tau | \Omega_\ast \rangle
\]

\[
= \lambda^2 \langle 0 | (a_0^\dagger)^n \sum_{\sigma, \tau} f_\sigma f_\tau \frac{a_\tau^\dagger}{\sqrt{2\omega_\sigma}} H_0 - nm + \mu_p + \omega_\sigma + \omega_\tau \frac{1}{\sqrt{2\omega_\tau}} (a_0^\dagger)^n | 0 \rangle
\]

\[
= -n \lambda^2 \langle 0 | a_0 \ldots a_0 \sum_{n-1} \frac{|f_0|^2}{m} \frac{1}{H_0 - nm + \mu_p + m + m} | a_0^\dagger \ldots a_0^\dagger | 0 \rangle
\]

\[
= - \frac{n \lambda^2}{m(m + \mu_p)} |f_0|^2,
\]

consequently the desired inequality is established. Note that we have $\omega_0(nm + \mu_p) < 0$ and we know, $\frac{\partial \omega_0(E)}{\partial E} < 0$ (Equation 105). Thus, we need to reduce $E$ to get $\omega_0(E_{gr}) = 0$ which is the sought after result for the bound state energy. This implies the following inequality for the actual ground state energy:

\[
E_{gr} < nm + \mu_p
\]

Let us now think about a lower bound for the ground state energy. We write $\Phi(E)$ in a symmetrical form assuming real values of $E$ and the term,

\[
K(E) = \sum_{\sigma} \frac{\lambda^2}{2\omega_\sigma} \frac{1}{(H_0 - E + \omega_\sigma)(\omega_\sigma - \mu_p)} f_\sigma^2(\bar{x}) (111)
\]

is strictly positive, therefore it can be dropped, which leads us to the following operator inequality provided that $E$ is real:

\[
\Phi_R \geq (H_0 - E + \mu_p)^{1/2} \left[ 1 - \frac{\lambda^2}{2} \sum_{\sigma, \tau} f_\sigma a_\tau^\dagger \frac{1}{\sqrt{\omega_\sigma}} (H_0 - E + \mu_p + \omega_\sigma)^{1/2} (H_0 - E + \omega_\sigma + \omega_\tau)^{1/2} (H_0 - E + \mu_p + \omega_\tau)^{1/2} \right]
\]

\[
\times (H_0 - E + \mu_p)^{1/2}
\]

(112)

Call the second term in the square brackets as $U$. Then,

\[
\Phi_R \geq (H_0 - E + \mu_p)^{1/2} \left[ 1 - U(E) \right] (H_0 - E + \mu_p)^{1/2}
\]

(113)

If $||U(E)|| < 1$, the right hand side is invertible and so is the Principal Operator. Therefore, if we can find $E_\ast$ below which $||U(E)|| < 1$, we can deduce directly:

\[
E_{gr} \geq E_\ast
\]

(114)

Let us define $\chi = (n - 1)m - E$. Noting that $H_0 \geq (n - 1)m$, we can replace $H_0 - E$’s by $\chi$ and bring $U$ in to the operator inequality form we used in the previous part:

\[
||U(E)|| \leq \frac{n \lambda^2}{2} \left[ \sum_{\sigma, \tau} \omega_\sigma \omega_\tau (\chi + \omega_\sigma)(\chi + \omega_\sigma + \omega_\tau)^2(\chi + \omega_\tau)^2 \right]^{1/2}
\]

(115)
where we have also omitted \( \mu_p \)'s for convenience. Using the crude inequality:

\[
(\chi + \omega_\sigma + \omega_\tau) > (\chi + \omega_\sigma)(\chi + \omega_\tau)
\]  

(116)

we decouple \( \sigma \) and \( \tau \) to get:

\[
||\mathcal{U}(E)|| \leq \frac{n\lambda^2}{2} \sum_\sigma \frac{|f_\sigma|^2}{\omega_\sigma(\chi + \omega_\sigma)^2}
\]  

(117)

Using Feynman parametrization, exponentiation and subordination identity consecutively we get:

\[
||\mathcal{U}(E)|| \leq \frac{n\lambda^2}{4\sqrt{\pi}} \int_0^1 \xi d\xi \int_0^\infty s^3 ds e^{-s^2/2} \int_0^\infty u^{-3/2} e^{-s^2/4u} \sum_\sigma |f_\sigma|^2 e^{-u\omega_\sigma} \int K_u(\bar{x}, x)e^{-um^2}
\]

(118)

Using the heat kernel estimate (90) we have:

\[
||\mathcal{U}(E)|| \leq \frac{n\lambda^2}{4\sqrt{\pi}} \int_0^1 \xi d\xi \int_0^\infty s^3 ds e^{-s\xi} \int_0^\infty u^{-3/2} e^{-s^2/4u} \left( \frac{1}{V(\mathcal{M})} + \frac{C}{u} \right) e^{-mu^2}
\]

\[
\leq n\lambda^2 \left\{ \int_0^1 \xi d\xi \frac{1}{V(\mathcal{M})} \left( \chi + m + \chi^2 \right) + \frac{C}{m + \chi} \right\}
\]

\[
\leq n\lambda^2 \left\{ \frac{1}{2mV(\mathcal{M})\chi^2} + \frac{C}{\chi} \right\}
\]

(119)

For \( \chi > m \), we can replace one of the \( \chi \)'s by \( m \):

\[
||\mathcal{U}(E)|| \leq \frac{n\lambda^2}{2m^2V(\mathcal{M})} + C \frac{1}{\chi}
\]

(120)

If we impose the condition:

\[
\frac{n\lambda^2}{\chi} \left\{ \frac{1}{2m^2V(\mathcal{M})} + C \right\} < 1
\]

\[
||\mathcal{U}(E)|| < 1
\]

is guaranteed. Substituting \( \chi = (n-1)m - E \) we get the lower bound for the ground state energy:

\[
(n-1)m - n\lambda^2 \left( \frac{1}{2m^2V(\mathcal{M})} + C \right) < E_{gr}
\]

(121)

which was first presented in [1].

II.4 Resolvent Defining a Hamiltonian

As discussed in the previous part we need to check that \( R(E) \) is a pseudo-resolvent, since we have the resolvent defined as a summation over the eigenmodes we will explicitly check it. To show that \( R(E) \) is a pseudo-resolvent, we need to check:

\[
R(E_1) - R(E_2) \equiv (E_1 - E_2) \left( R(E_1) - R(E_2) \right)
\]

(122)

which is equivalent, according to two by two matrix form:

\[
\begin{bmatrix}
\alpha(E_1) - \alpha(E_2) & \gamma(E_1) - \gamma(E_2) \\
\beta(E_1) - \beta(E_2) & \delta(E_1) - \delta(E_2)
\end{bmatrix} = \begin{bmatrix}
\alpha(E_1)\alpha(E_2) + \gamma(E_1)\beta(E_2) & \alpha(E_1)\gamma(E_2) + \gamma(E_1)\delta(E_2) \\
\beta(E_1)\alpha(E_2) + \delta(E_1)\beta(E_2) & \beta(E_1)\gamma(E_2) + \delta(E_1)\delta(E_2)
\end{bmatrix}
\]

(refer to [14] for the definitions of the terms). We remark that all operators are bounded here hence there are no issues about domains. Noting that the free resolvent \( R_0 = \frac{1}{\mathcal{H}_{0-E}} \) satisfies (35), it is straightforward to show that (122) reduces to:

\[
R_0(E_1) b_1^\dagger \Phi^{-1}(E_2) \left[ \Phi(E_1) - \Phi(E_2) + b \left( R_0(E_1) - R_0(E_2) \right) b_1^\dagger + E_1 - E_2 \right] \Phi^{-1}(E_2)bR_0(E_2) \equiv 0
\]

(123)
We can check the equality in (123) by direct substitution. Calculating the term in square brackets term by term:

\[
\mathcal{A} = \Phi(E_1) - \Phi(E_2) = (H_0 - E_1 + \mu_p) - (H_0 - E_2 + \mu_p) \\
+ \lambda^2 \sum_{\sigma} \frac{|f_\sigma|^2}{2\omega_\sigma} \frac{E_2 - E_1}{(H_0 - E_1 + \omega_\sigma)(H_0 - E_2 + \omega_\sigma)} \\
+ \lambda^2 \sum_{\sigma,\tau} f_{\sigma,\tau}(\bar{x}) \frac{a_\tau^\dagger}{\sqrt{2\omega_\tau}} \frac{E_2 - E_1}{(H_0 - E_1 + \omega_\sigma + \omega_\tau)(H_0 - E_2 + \omega_\sigma + \omega_\tau)} \frac{1}{\sqrt{2\omega_\tau}}
\]

(124)

\[
\mathcal{B} = b \left( R_0(E_1) - R_0(E_2) \right) b^\dagger = (\lambda \sum_{\sigma} \frac{f_{\sigma}(\bar{x})}{\sqrt{2\omega_\sigma}} a_\sigma) \left[ \frac{1}{H_0 - E_1} - \frac{1}{H_0 - E_2} \right] (\lambda \sum_{\tau} \frac{f_\tau(\bar{x}) a_\tau^\dagger}{\sqrt{2\omega_\tau}})
\]

\[
= \frac{\lambda^2}{2} \sum_{\sigma,\tau} \frac{f_{\sigma} f_\tau}{\sqrt{\omega_\sigma \omega_\tau}} (E_1 - E_2) \frac{1}{H_0 - E_1 + \omega_\sigma} \frac{a_\sigma a_\tau^\dagger}{H_0 - E_2 + \omega_\tau} \\
= \frac{\lambda^2}{2} \sum_{\sigma,\tau} \frac{f_{\sigma} a_\tau^\dagger}{\sqrt{\omega_\tau}(H_0 - E_1 + \omega_\sigma + \omega_\tau)(H_0 - E_2 + \omega_\sigma + \omega_\tau)} \frac{E_1 - E_2}{\sqrt{\omega_\tau}} \\
+ \frac{\lambda^2}{2} \sum_{\sigma} \frac{|f_\sigma|^2}{\omega_\sigma} \frac{E_1 - E_2}{(H_0 - E_1 + \omega_\sigma)(H_0 - E_2 + \omega_\sigma)}
\]

(125)

As we can see now, we have,

\[
\mathcal{A} + \mathcal{B} + \mathcal{C} = 0
\]

(126)

Thus, \( R(E) \) is indeed a pseudo-resolvent family depending on a complex parameter \( E \).

### II.4.1 The Decay Condition

To show that the resolvent \( R \) satisfies Theorem 1 as before, we choose a series \( \lambda_k \) on the negative real axis such that for every \( k, \lambda_k < 0 < E_{\gamma r} \). Since \( \lambda_k = -|\lambda_k| \), we can write down the condition (10) as:

\[
\lim_{k \to \infty} \left\| \left| |\lambda_k| R(-|\lambda_k|) - 1 \right| x \right\|_H = 0
\]

(127)

Substituting the resolvent expression in the previous equation, and applying the triangular inequality twice, we again concludes that,

\[
\left\| \left| |\lambda_k| R(-|\lambda_k|) - 1 \right| \left| f_{n+1}^\gamma \right| \right\| \\
\left\| \left| (|\lambda_k| \alpha(-|\lambda_k|) - 1) f_{n+1}^\gamma \right| \right\| + \left\| \left| |\lambda_k| \gamma(-|\lambda_k|) f_{n}^\gamma \right| \right\| + \left\| \left| |\lambda_k| \beta(-|\lambda_k|) \right| f_{n+1}^\gamma \right\| + \left\| \left| |\lambda_k| \delta(-|\lambda_k|) - 1 \right| f_{n}^\gamma \right\|
\]

So, if we can show that as \( k \to \infty \), each term in (125) goes to zero separately, (125) can be immediately deduced.

#### II.4.1.1 Behaviour of the \( \Phi \) Operator

Since we need to reconsider the behaviour of the operator for complex values of \( E \), the estimates below use \( E \) as a complex variable. Let us write the operator \( \Phi \) below, as before, removing the operator \( H_0 - E + \mu_p \) to
Now applying the subordination identity to $e$ here. Applying Feynman parametrization to $K$ is examined separately, we call them $||$. To understand the behaviour of $\Phi$ and make use of it in the upcoming calculations, we estimate $K$ and $U$ here.

Applying Feynman parametrization to $K$:

\[
K = C \sum_{\kappa} f_{\kappa}(\bar{x}) \int_{0}^{1} du_{1} du_{2} du_{3} \frac{\delta(1 - \sum_{i=1}^{3} u_{i})}{[u_{1}\omega_{\sigma} + u_{2}(\omega_{\sigma} - \mu_{p}) + u_{3}(\omega_{\sigma} + H_{0} - E)]^{3}}
\]

\[
= C \sum_{\kappa} f_{\kappa}(\bar{x}) \int_{0}^{1} du_{1} du_{2} du_{3} \delta(1 - \sum_{i=1}^{3} u_{i}) \int s^{2} dse^{-sA}
\]

(129)

where all constants are absorbed into $C$ and we define as $A = u_{1}\omega_{\sigma} + u_{2}(\omega_{\sigma} - \mu_{p}) + u_{3}(\omega_{\sigma} + H_{0} - E)$. Note that because of the Dirac delta function, $u_{1} + u_{2} + u_{3} = 1$ and we have:

\[
K = C \int_{0}^{1} du_{1} du_{2} du_{3} \delta(1 - \sum_{i=1}^{3} u_{i}) \int s^{2} dse^{-sA}
\]

(130)

Now applying the subordination identity to $e^{-s\omega_{\sigma}}$ and substituting the heat kernel:

\[
K = C \int_{0}^{1} du_{1} du_{2} du_{3} \delta(1 - \sum_{i=1}^{3} u_{i}) \int s^{3} ds \int d\xi e^{-\frac{\pi}{m^{2}2} \xi^{2}} K_{\xi}(\bar{x}, \bar{x}) e^{s\mu_{p} u_{2} e^{-su_{3}(H_{0} - E)}}
\]

(131)

If we substitute the estimate for the heat kernel (50) and take the norm:

\[
||K|| \leq C \int s^{3} ds du_{2} du_{3} e^{su_{2}\mu_{p} e^{-s(nm - \text{Re}(E))u_{3}}} \int d\xi \frac{me^{-\frac{\pi}{m^{2}2} \xi^{2}}}{\xi^{3/2}} e^{-\xi} \left[ \frac{C}{\xi/m^{2}} + \frac{1}{V(M)} \right]
\]

(132)

where we estimate $H_{0}$ as $(nm)$ and note that $E \to \text{Re}(E)$ when we use the norm. The two parts of the integral is examined separately, we call them $||K||_{(1)}$ and $||K||_{(2)}$. Looking at the first part ($\frac{C}{\xi/m^{2}}$ term), this is the integral representation of the modified Bessel function of the second kind:

\[
K_{\nu}(z) = \frac{1}{2} \left( \frac{1}{2} \right)^{\nu} \int_{0}^{\infty} \frac{1}{\xi^{\nu+1}} e^{-\xi - \frac{z^{2}}{2\xi}} d\xi
\]

(133)

where in our case $z = ms$ and $\nu = 3/2$. Then the integral can be computed to get:

\[
K_{3/2}(ms) = \sqrt{\frac{\pi}{2ms}} e^{-ms} \left[ 1 + \frac{1}{ms} \right]
\]

(134)

we arrive at:

\[
||K||_{(1)} \leq C \int ms ds du_{2} du_{3} e^{su_{2}\mu_{p} e^{-s(nm - \text{Re}(E))u_{3}}} e^{-ms} \left[ 1 + \frac{1}{ms} \right]
\]

(135)

where we absorb every constant into $C$. If we find an upper bound to the most singular part of the integral (i.e. the second term in square brackets), the bound would also be valid for the other part. As we take the
limit $E = \lambda_k \to -\infty$, the bound we are to find is enough for our calculations below as well. Denote the norm coming from the most singular part as $||K||_{(1)-\text{sing}}$. Introducing $u_1 + u_2 + u_3 = 1$:

$$
||K||_{(1)-\text{sing}} \leq C \int dsdu_1du_2du_3\delta(1 - \sum_{i=1}^{3} u_i)e^{su_2u_3}\rho\mu\rho^2 e^{-s(nm-\text{Re}(E))u_3} e^{-ms(u_1+u_2+u_3)}
$$

$$
\leq C \int dsdu_1du_2du_3\delta(1 - \sum_{i=1}^{3} u_i)e^{-su_2u_3}\rho\mu\rho^2 e^{-s(m-\mu_p)u_2} e^{-s((n+1)m-\text{Re}(E))u_3}
$$

$$
\leq C \int du_1du_2du_3\delta(1 - \sum_{i=1}^{3} u_i)\rho\mu\rho^2 e^{-su_2u_3}\rho\mu\rho^2 e^{-s((n+1)m-\text{Re}(E))u_3}
$$

If we ignore the term $(m - \mu_p)u_2$ and take the $u_2$ integral:

$$
||K||_{(1)-\text{sing}} \leq C \int_{0 \leq u_1+u_3 \leq 1} \frac{du_1du_3}{mu_1 + [(n+1)m - \text{Re}(E)]u_3}
$$

(136)

Note that the region $0 \leq u_1 + u_3 \leq 1$ is contained in $u_1^2 + u_3^2 \leq 1$, so we can integrate in the latter since the integrand is positive. We can go to polar coordinates where $u_1 = \rho \cos \theta$ and $u_3 = \rho \sin \theta$:

$$
||K||_{(1)-\text{sing}} \leq C \int_{0}^{\pi/2} \frac{\rho d\rho d\theta}{m \rho \cos \theta + [(n+1)m - \text{Re}(E)]\rho \sin \theta}
$$

(137)

Since in the first quadrant both sine and cosine are positive, we can write the inequalities $\cos \theta \geq \cos^2 \theta$ and $\sin \theta \geq \sin^2 \theta$. We then replace cosine and sine with the squares and turn the integral to a more familiar form:

$$
||K||_{(1)-\text{sing}} \leq C \int_{0}^{\pi/2} \frac{d\rho d\theta}{m \cos^2 \theta + [(n+1)m - \text{Re}(E)]\sin^2 \theta} \leq C \frac{\pi}{\sqrt{m} \sqrt{(n+1)m - \text{Re}(E)}}
$$

Thus in the limit

$$
E = \lambda_k \to -\infty \ , \ ||K||_{(1)} \to 0
$$

(138)

For $||K||_{(2)}$, we proceed similarly:

$$
||K||_{(2)} = C \int s^3 dsdu_1du_2du_3\delta(1 - \sum_{i=1}^{3} u_i)e^{su_2u_3}\rho\mu\rho^2 e^{-s(nm-\text{Re}(E))u_3} \int d\xi e^{-\frac{s^2\xi^2}{2}} \left[ \frac{1}{\text{V}(M)} \right] e^{-\xi} \int d\xi e^{-\frac{s^2\xi^2}{2}} \left[ \frac{1}{\text{V}(M)} \right] e^{-\xi}
$$

(139)

$$
||K||_{(2)} \leq \frac{C}{V(M)} \int dsdu_2du_3s^{5/2} e^{su_2u_3}\rho\mu\rho^2 e^{-s(nm-\text{Re}(E))u_3} e^{-ms} \sqrt{m} s^{5/2} e^{su_2u_3}\rho\mu\rho^2 e^{-s(nm-\text{Re}(E))u_3} e^{-ms}
$$

$$
\leq \frac{C}{V(M)} \int dsdu_2du_3s^{5/2} e^{su_2u_3}\rho\mu\rho^2 e^{-s(nm-\text{Re}(E))u_3} e^{-ms}
$$

$$
\leq \frac{C}{V(M)} \int \int_0^1 du_1du_2du_3\delta(1 - \sum_{i=1}^{3} u_i)\frac{1}{[mu_1 + (m - \mu_p)u_2 + ((n+1)m - \text{Re}(E))u_3]^3}
$$

$$
\leq \frac{C}{V(M)} \int_0^1 du_3\frac{1}{[2(nm + \mu - \text{Re}(E))(m - \mu_p)^2]}
$$

(140)
It is straightforward to see that:
\[ E = \lambda_k \to -\infty \quad \Longrightarrow \quad ||K||_2 \to 0 \quad (141) \]

it follows that:
\[ E = \lambda_k \to -\infty \quad \Longrightarrow \quad ||U|| \to 0 \quad (142) \]

We also have the bound for \( ||U|| \), see eq (119). Substituting back \( \chi = nm - \text{Re}(E) \), it is straightforward to take the limit and see that:
\[ E = \lambda_k \to -\infty \quad \Longrightarrow \quad ||U|| \to 0 \quad (143) \]

We add one more result to this section for future simplicity. Note that:
\[ ||\lambda_k|| \Phi^{-1}(-|\lambda_k|) = ||\lambda_k|| (1 + K(-|\lambda_k|) - U(-|\lambda_k|))^{-1} || (H_0 + \mu_p + |\lambda_k|)^{-1} || \quad (144) \]

When \( ||A|| < 1 \), we can write down the Neumann series:
\[ (1 - A)^{-1} = \sum_{l=0}^{\infty} A^l \quad (145) \]

Thus using (142) and (143), we can deduce that for proper choice of \( \lambda_k \) we can set:
\[ ||U|| < \frac{1}{4} \quad \text{and} \quad ||K|| < \frac{1}{4} \quad (146) \]

Hence the Neumann series expansion leads to an upper bound:
\[ ||(1 + K(-|\lambda_k|) - U(-|\lambda_k|))^{-1} || < 2 \quad (147) \]

Consequently we deduce that,
\[ ||\lambda_k|| \Phi^{-1}(-|\lambda_k|) \leq C||\lambda_k|| (nm + |\lambda_k|)^{-1} \quad (148) \]

It is now obvious that in the limit \( \lambda_k \to \infty \) the right hand side goes to a constant, which means \( ||\lambda_k|| \Phi^{-1}(-|\lambda_k|) \) is finite.

II.4.1.2 The \( \beta \) term

We can write the inequality:
\[ \left\| |\lambda_k| \Phi^{-1}(-|\lambda_k|) \phi^{(+)} \right\| \frac{1}{H_0 + |\lambda_k|} |f^{n+1}| \leq ||\lambda_k|| \Phi^{-1} || \phi^{(+)} \frac{1}{H_0 + |\lambda_k|} |f^{n+1}| \quad (149) \]

without any problems since each term on the right hand side is bounded. We know that \( |\lambda_k| \Phi^{-1}(-|\lambda_k|) \) has finite operator norm. We need to work on the following term:
\[ || \phi^{(+)} \frac{1}{H_0 + |\lambda_k|} |f^{n+1}| \| = \| \sum_{\sigma} \frac{f_{\sigma}}{\sqrt{2\omega_{\sigma}}} \frac{1}{H_0 + |\lambda_k| + \omega_{\sigma}} a_{\sigma} |f^{n+1}| \| \quad (150) \]

Using the estimates as is done in the previous part (substituting \( H_0 \) as \( nm \) to get an upper):
\[ \left\| \sum_{\sigma} \frac{f_{\sigma}}{H_0 + |\lambda_k| + \omega_{\sigma} \sqrt{2\omega_{\sigma}}} |f^{n+1}| \right\| \leq (n + 1) \left[ \sum_{\sigma} \frac{|f_{\sigma}|^2}{(nm + |\lambda_k| + \omega_{\sigma}^2)^{1/2}} \right]^{1/2} |f^{n+1}| \|
\]
\[ \leq (n + 1) \left[ \sum_{\sigma} \frac{|f_{\sigma}|^2}{(nm + |\lambda_k|)^2 + \omega_{\sigma}^2} \right]^{1/2} |f^{n+1}| \|
\]

---

25
If we apply Feynman parametrization to the term in square brackets:

\[
\sum_{\sigma} \frac{|f_{\sigma}|^2}{(nm + |\lambda_k|)^2 + \omega_\sigma^2 (\omega_\sigma^2)^{1/2}} = \frac{1}{2} \int_0^1 \frac{d\xi}{\sqrt{1 - \xi}} \int_0^\infty ds \sqrt{s} e^{-s(nm + |\lambda_k|)^2} \sum_{\sigma} |f_{\sigma}|^2 e^{-s\omega_\sigma^2} K_s(\bar{x}, \bar{x}) e^{-s\omega_\sigma^2}
\]

(151)

Substituting the estimate for the heat kernel (90) and computing the integrals we arrive at the following inequality:

\[
\sum_{\sigma} \frac{|f_{\sigma}(\bar{x})|^2}{(nm + |\lambda_k|)^2 + \omega_\sigma^2 (\omega_\sigma^2)^{1/2}} \leq \frac{1}{2} \int_0^1 \frac{d\xi}{\sqrt{\xi(1 - \xi)}} C \int_0^\infty ds e^{-s(nm + |\lambda_k|)^2} \left[ \frac{1}{2} \frac{d}{ds} \sqrt{s} e^{-s(nm + |\lambda_k|)^2} \right] \frac{1}{\pi} \frac{1}{nm + |\lambda_k|} \left( \frac{C}{2} + \frac{\tilde{C}}{V(M)} \right) \frac{\pi}{\pi} \frac{1}{nm + |\lambda_k|^{1/2}}
\]

(152)

where \(\tilde{C}\) is a finite constant introduced for notational simplicity (which can be computed). Now we can substitute everything into (149) to get:

\[
\left\| |\lambda_k| \Phi^{-1} \phi(+) \frac{1}{H_0 + |\lambda_k|} |f^{n+1}| \right\| \leq |\lambda_k||\Phi^{-1}| C \left[ \frac{1}{nm + |\lambda_k|} \right]^{1/4}
\]

(153)

where we collect all constants (including the volume of the manifold) in \(C\) for simplicity and make use of the fact that \(|f^{n+1}|\) is normalized. This implies directly that the left hand side of (153) goes to zero, exactly what we aimed to show:

\[
\lim_{|\lambda_k| \to \infty} || |\lambda_k| \beta(-|\lambda_k|) |f^{n+1}| || = 0
\]

(154)

**II.4.1.3 The \(\gamma\) Term**

We proceed as in the previous part.

\[
\left\| |\lambda_k| \frac{1}{H_0 + |\lambda_k|} \sum_{\sigma} f_{\sigma}(\bar{x}) a_\sigma^+ \Phi^{-1}(-|\lambda_k|) |f^{n}| \right\| \leq |\lambda_k| ||\Phi^{-1}(-|\lambda_k|)|| \left\| \sum_{\sigma} \frac{1}{H_0 + |\lambda_k|} f_{\sigma}(\bar{x}) a_\sigma^+ |f^{n}| \right\|
\]

We consider the last factor, similar to previous estimates we see that:

\[
\left\| \sum_{\sigma} \frac{1}{H_0 + |\lambda_k|} f_{\sigma}(\bar{x}) a_\sigma^+ |f^{n}| \right\| \leq \sqrt{n + 1} \left( \sum_{\sigma} \frac{|f_{\sigma}|^2}{(nm + |\lambda_k| + \omega_\sigma^2)^2} \right)^{1/2} \left\| |f^{n}| \right\|
\]

Note that on the right hand side, we have exactly what we have above up to some constant and we can use the result in (153) directly to establish the result we seek after:

\[
\lim_{|\lambda_k| \to \infty} || |\lambda_k| \gamma(-|\lambda_k|) |f^{n}| || = 0
\]

(155)

**II.4.1.4 The \(\alpha\) Term**

We again start with an inequality:

\[
\left\| \left[ |\lambda_k| \alpha(-|\lambda_k|) - 1 \right] |f^{n+1}| \right\| \leq \left\| \left[ \frac{|\lambda_k|}{H_0 + |\lambda_k|} - 1 \right] |f^{n+1}| \right\| + \frac{1}{H_0 + |\lambda_k|} \left\| \Phi^{-1}(-|\lambda_k|) \right\| \left\| \Phi^{(+)} \right\| \left\| \frac{1}{H_0 + |\lambda_k|} |f^{n+1}| \right\|
\]

(156)
Taking the limit, it is straightforward to see that the first term on the right hand side goes to zero. The second term should be worked out, in a similar way we see that

$$\lim_{|\lambda_k| \to \infty} \left| \frac{1}{H_0 + |\lambda_k|} \phi(-) |f^n| \right| = 0.$$  

Note that the last term is exactly the same as we have above for $\gamma$ and so,

$$\lim_{|\lambda_k| \to \infty} \left| |\Phi(-)| \right| \left| \frac{1}{H_0 + |\lambda_k|} \phi(-) |f^{n+1}| \right| \leq \lim_{|\lambda_k| \to \infty} C \left( \frac{1}{|n+1|m + |\lambda_k|} \right)^{1/4} = 0$$

(157)

where $C$ is constant. Thus we conclude that:

$$\lim_{|\lambda_k| \to \infty} \left[ |\lambda_k| |\Phi(-)| - 1 \right] |f^{n+1}| = 0$$

Hence follows the result:

$$\lim_{|\lambda_k| \to \infty} \left[ |\lambda_k| |\alpha(-)| - 1 \right] |f^{n+1}| = 0$$

(158)

II.4.1.5 The $\delta$ Term

Let us repeat the steps in the light-front model,

$$\left\langle |\lambda_k| \Phi(-) |f^n| \right\rangle = \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1}(1 + K - U)^{-1} - 1 \right\rangle |f^n|$$

(159)

Remember that in the limit $|\lambda_k| \to \infty$, $\left( 1 - (U - K) \right)^{-1}$ can be expanded as Neumann series:

$$\left( 1 - (U - K) \right)^{-1} = 1 + \sum_{l=1}^{\infty} (U - K)^l$$

(160)

and (159) reduces to:

$$\left\langle |\lambda_k| \Phi(-) |f^n| \right\rangle = \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} - 1 \right\rangle |f^n| + \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} \sum_{l=1}^{\infty} (U - K)^l \right\rangle |f^n|$$

Using the triangle inequality:

$$\left\langle |\lambda_k| |\Phi(-)| - 1 \right\rangle |f^n| \leq \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} - 1 \right\rangle |f^n|$$

(161)

$$+ \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} \sum_{l=1}^{\infty} (U - K)^l \right\rangle$$

finite $\to 0$

where we take the limit $|\lambda_k| \to \infty$ in the second line and we deduce:

$$\lim_{|\lambda_k| \to \infty} \left\langle |\lambda_k| |\delta(-)| - 1 \right\rangle = \lim_{|\lambda_k| \to \infty} \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} + 1 \right\rangle |f^n|$$

(162)

Note that the resulting equation is (30) in Theorem-1, $R(E)$ being the free resolvent. Since $\frac{1}{H_0 - E}$ is indeed a resolvent, it must satisfy (30) and we arrive at:

$$\lim_{|\lambda_k| \to \infty} \left\langle |\lambda_k|(H_0 + |\lambda_k| + \mu_p)^{-1} - 1 \right\rangle |f^n| = 0$$

(163)

Having shown that each term on the right hand side of the equations (128) goes to zero as $|\lambda_k| \to \infty$, we conclude that:

$$\lim_{|\lambda_k| \to \infty} \left\langle |\lambda_k|R(-|\lambda_k|) - 1 \right\rangle \left\langle \frac{|f^{n+1}|}{|f^n|} \right\rangle = 0$$

(164)

Therefore, we showed that $R(E) = \frac{1}{H_0 - E}$ indeed defines a resolvent.
II.5 Holomorphic structure of the Principal Operator

It is well-known that to obtain a spectral decomposition of a family of operators in which eigenvalues and the corresponding projections are holomorphic functions of the parameter, we need the notion of a self-adjoint holomorphic family of type-A in the sense of Kato. This in turn justifies the fact that our resolvent formula defines a self-adjoint quantum Hamiltonian as well as putting our estimates on a firmer ground.

First, we aim to establish the following claim: The family $\Phi(E)$, defined for $\text{Re}(E) < nm + \mu_p$, on a symmetric domain of the complex plane is holomorphic of type-A, that is

- $D(\Phi(E)) = D(H_0)$, independent of $E$,
- $\Phi(E)$ is closed on this common domain,
- $\Phi(E)u$ is holomorphic for $E \in D(H_0)$ for every $E$ in the open symmetric domain.

We start by showing that the family can be given a common dense domain for $\text{Re}(E)$ with $\text{Re}(E) < nm + \mu_p$ on which it is closed. To establish self-adjointness of the family $\Phi(E)$, we rely on the Wüst’s theorem that it is enough to establish the self-adjointness condition even at a single point. This in turn is true due to Kato-Rellich theorem on self-adjointness when $E$ is sufficiently small on the real axis [28]. This part of the proof is essentially identical to the light-front case, since the proof given there is formal, it does not require the explicit forms of the operators.

II.5.1 Common Domain of the Family $\Phi(E)$

We start by organizing the Principal Operator $\Phi(E)$ in the following way:

$$\Phi(E) = \left[1 + \sum_{\sigma} \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - E + \omega_\sigma)(\omega_\sigma - \mu_p)} \right]$$

$$- \sum_{\omega,\tau} \frac{\lambda^2 f_\omega a_\tau}{\sqrt{2\omega_\tau}} \frac{1}{H_0 - E + \omega_\tau + \mu_\tau} \frac{1}{H_0 - E + \omega_\tau + \mu_\tau} \frac{a_\tau f_\omega}{\sqrt{2\omega_\tau}} (H_0 - E + \mu_p)$$

Recall that we are working on a sector of the full Fock space, $\mathcal{H} = \mathcal{F}^{(n+1)_1} \otimes \mathcal{F}^{(n)} \otimes \chi_+^\dagger$, which is a Hilbert space. Call the domain of $H_0$ as $D(H_0)$, which is dense in $\mathcal{F}^{(n)}$ for any $n$. Moreover, $H_0$ is closed on this domain being a self-adjoint operator. Renaming the terms in (165) we rewrite $\Phi(E)$ as:

$$\Phi(E) = [1 + K(E) + U(E)](H_0 - E + \mu_p)$$

To fix $D(H_0)$ to be the common domain of $\Phi(E)$, we want to show that $K(E)$ and $U(E)$ are bounded, $E$ being complex. Since $K(E)$ and $H_0$ commute, the new splitting of $\Phi(E)$ does not effect the bound we found for $K(E)$ previously in 1.3.2, which is:

$$||K(E)|| \leq C \frac{\pi}{\sqrt{m}} \frac{1}{\sqrt{(n+1)m - \text{Re}(E)}} + \frac{C}{\sqrt{V(M)}} \left[ \frac{1}{2(nm + \mu_p - \text{Re}(E))(m - \mu_p)^2} \right]$$

For $U(E)$, we had previously worked with $E$ chosen on the real axis and the result must be generalized to the complex case. We start by collecting the terms using Feynman parametrization:

$$U(E) = \sum_{\sigma,\tau} \lambda^2 f_\sigma \frac{a_\tau}{\sqrt{2\omega_\tau}} \int_0^1 \frac{du}{d u + (1 - u)\mu_\tau + u\omega_\tau + \mu_\tau} \frac{a_\tau f_\omega}{\sqrt{2\omega_\tau}}$$

$$= \sum_{\sigma,\tau} \lambda^2 f_\sigma \frac{a_\tau}{\sqrt{2\omega_\tau}} \int_0^1 du \int_0^\infty ds dse^{-s(H_0 - E) - s\mu_\tau (1 - u) e^{-s\omega_\tau} e^{-s\omega_\tau} \frac{a_\tau f_\omega}{\sqrt{2\omega_\tau}}}$$

Taking the norm:

$$||U(E)|| \leq \lambda^2 \int_0^\infty ds \int_0^1 du e^{-s\mu_\tau (1 - u)} \left[ \sum_{\sigma} \frac{a_\tau f_\sigma}{\sqrt{2\omega_\tau}} e^{-s\omega_\tau} e^{-s\omega_\tau} \frac{a_\tau f_\omega}{\sqrt{2\omega_\tau}} \right]$$

$$\leq \lambda^2 \int_0^\infty ds \int_0^1 du e^{-s\mu_\tau (1 - u)} e^{s\text{Re}(E)} \left[ \phi^-(f)e^{-sH_0\phi}(g) \right]$$

$$\leq \lambda^2 \int_0^\infty ds \int_0^1 du e^{-s\mu_\tau (1 - u)} e^{s\text{Re}(E)} \left[ \phi^-(f)e^{-sH_0\phi}(g) \right]$$

(168)
where we define:

\[
\phi^-(f(s)) = \sum_{\sigma} a_{\sigma} f_{\sigma}(\bar{s}) e^{-s_{\sigma}\sigma} \quad \text{and} \quad \phi^+(g(su)) = \sum_{\tau} b_{\tau} f_{\tau}(\bar{t}) e^{-t_{\tau}\tau}.
\]

Now we can estimate \(e^{-sH_0}\) as \(e^{-s(n-1)m}\), we end up with the following inequality:

\[
||U|| \leq \lambda_0 \int_0^\infty ds \int_0^1 du e^{s\mu_p u} e^{-s(n-1)m + \mu_p - \text{Re}(E)||\phi^- (g) \phi^+(f)||} (169)
\]

We need to show that the integral is finite since the generalized heat kernels appearing in \(\phi^-\) and \(\phi^+\) are singular around 0\(^+\). We recall the estimate \(\|\phi^-(f) \phi^+(g)\| \leq n\|f\|\|g\|\) as discussed in the previous part.

As a result we have,

\[
||f||^2 = \sum_{\sigma} |f_{\sigma}(\bar{s})|^2 e^{-2s_{\sigma}\sigma}, \quad ||g||^2 = \sum_{\tau} |f_{\tau}(\bar{t})|^2 e^{-2s_{\tau}\tau} (170)
\]

We first estimate \(||f||\) and \(||g||\) by employing an integral identity:

\[
e^{-2s_{\sigma}\sigma} = \frac{1}{\sqrt{s}} \int_0^\infty e^{-t(m^2 + \sigma^2) - 4s^2 / t} \frac{dt}{t^{1/2}}, (171)
\]

, then we have

\[
||f||^2 = C_1 \int_0^\infty \frac{dt}{t^{1/2}} e^{-4s^2 / t - m^2 t} \sum_{\sigma} e^{-s_{\sigma}\sigma} |f_{\sigma}(\bar{s})|^2 (172)
\]

We work with the most singular part:

\[
||f||_{\text{sing}}^2 \leq C_2 \int_0^\infty t^{-3/2} e^{-m^2 t - 4s^2 / t} dt \leq C_2 e^{-2ms} (173)
\]

and similarly,

\[
||g||_{\text{sing}} \leq C_2 e^{-2ms} (173)
\]

We need to substitute these estimates into (169) to show that \(||U||\) is bounded:

\[
||U(E)||_{\text{sing}} \leq C\lambda^2 n \int_0^\infty s ds \int_0^1 du e^{s\mu_p u} e^{-s[(n-1)m + \mu_p - \text{Re}(E)]} e^{-ms} e^{-mus} (174)
\]

where we have used:

\[
\int_0^1 du \frac{e^{-s(m-\mu_p)u}}{\sqrt{u}} \leq \int_0^1 \frac{du}{\sqrt{u}} = 2 (175)
\]

Since the most singular part is finite, the rest certainly is. This concludes that \(K(E)\) and \(U(E)\) are bounded, thus we can choose the domain of \(H_0\) as the common domain of the family \(\Phi(E)\) on the open domain \(\Omega = \{E \in C | \text{Re}(E) < nm + \mu_p\}\).
II.5.2 $\Phi(E)$ is Closed on its Domain

We want to show that $\Phi(E)$ is closed in its domain $D(\Phi(E)) = D = D(H_0)$, that is, having a sequence $x_k$ that converges to $x$ as well as,

$$\Phi(E) x_k \rightarrow y,$$

we establish that $\Phi(E)x = y$.

Let us recall the argument in the previous part. We essentially follow the same reasoning, only the precise estimates being different. When $\text{Re}(E) \leq \text{Re}(E_*)$ where $\text{Re}(E_*)$ is sufficiently small such that $\Phi(E)$ becomes invertible (see section 7.2):

$$[1 + K(E) - U(E)] \ (H_0 + \mu_p - E) x_k \rightarrow y, \quad x_k \rightarrow x$$

$$\Rightarrow \quad (H_0 + \mu_p - E)x_k \rightarrow [1 + K(E) - U(E)]^{-1} y, \quad x_k \rightarrow x$$

Since $H_0$ is closed on its domain:

$$(H_0 + \mu_p - E)x_k \rightarrow (H_0 + \mu_p - E) x = [1 + K(E) - U(E)]^{-1} y$$

$$\Rightarrow \quad y = [1 + K(E) - U(E)] \ (H_0 + \mu_p - E) x$$

consequently for $\text{Re}(E) \leq \text{Re}(E_*)$, $\Phi(E)$ is closed. For $\text{Re}(E) > \text{Re}(E_*)$, we rearrange according to (124):

$$\Phi(E) - \Phi(E_*) = T(E, E_*)(E_* - E)$$

$$= (E_* - E) \left[ 1 + \lambda^2 \sum_{\sigma} \frac{|f_{\sigma}|^2}{2\omega_{\sigma}} \frac{1}{(H_0 - E + \omega_{\sigma})(H_0 - E_* + \omega_{\sigma})} \right]$$

$$+ \lambda^2 \sum_{\sigma, \tau} f_{\sigma}(\bar{x}) \frac{a_1}{2\omega_{\sigma}} \frac{1}{(H_0 - E + \omega_{\sigma} + \omega_{\tau})(H_0 - E_* + \omega_{\sigma} + \omega_{\tau})} \frac{a_{\tau}}{\sqrt{2\omega_{\tau}}} f_{\tau}(\bar{x})$$

We want to show that $T(E, E_*)$ is bounded. Calling the second term in square brackets $A$, we proceed similar to previous calculations and show that it is bounded. Again we apply Feynman parametrization followed by a subordination and take the norm. Estimating $H_0 > nm$ as well as recognizing the heat kernel as before and substituting the bound given in (90), we find:

$$||A|| \leq \int_0^1 du_1 du_2 du_3 \int_0^\infty s^3 dse^{-s(nm - \text{Re}(E))u_2} e^{-s(nm - \text{Re}(E_*) u_3} \int d\xi \frac{me^{-s^2m^2/4\xi} e^{-\xi}}{\xi^{3/2}} \left( \frac{C}{\xi/m^2} + \frac{1}{V(M)} \right)$$

We compute the most singular term (the first part) of the integral:

$$||A||_{\text{sing}} \leq C \int dsdu_1 du_2 du_3 \delta(1 - \sum_i u_i) e^{-s(nm - \text{Re}(E))u_2} e^{-s(nm - \text{Re}(E_*)u_3} e^{-sm(u_1 + u_2 + u_3)}$$

$$\leq C \int du_1 du_2 du_3 \delta(1 - \sum_i u_i) \frac{1}{mu_1 + ((n + 1)m - \text{Re}(E))u_2 + ((n + 1)m - \text{Re}(E_*)u_3}$$

$$\leq C \int_{0 \leq u_2 + u_3 \leq 1} \frac{1}{[(n + 1)m - \text{Re}(E)]u_2 + [(n + 1)m - \text{Re}(E_*)u_3}$$

where in the last line we have ignored a positive term $m(1 - u_2 - u_3)$ in the denominator. Passing to polar coordinates:

$$||A||_{\text{sing}} \leq C \int_0^{\pi/2} \int_0^1 \frac{\rho d\rho d\theta}{[(n + 1)m - \text{Re}(E)]\rho \cos \theta + [(n + 1)m - \text{Re}(E_*)]\rho \sin \theta}$$

$$\leq C \frac{1}{\sqrt{[(n + 1)m - \text{Re}(E)]^2 + [(n + 1)m - \text{Re}(E_*)]^2}}$$

Note that we absord every constant we encounter into $C$. If the most singular part is bounded, the other part certainly is. Therefore, we have shown that $A$ is bounded.

We now show the boundedness of the third term in square brackets in (179), call it $B$ for simplicity.
\[
B = \lambda^2 \sum_{\sigma, \tau} f_\sigma(x) \frac{a^\dagger_\sigma}{\sqrt{2\omega_\sigma}} \left( H_0 - E + \omega_\sigma + \omega_\tau \right) \frac{1}{\left( H_0 - E + \omega_\sigma + \omega_\tau \right)} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau(x)
\]

Taking the norm and replacing \(\text{Re}(E_*)\) by \(\text{Re}(E)\) since \(\text{Re}(E_*) < \text{Re}(E)\):

\[
\|B\| \leq \lambda^2 \int_0^\infty \int_0^1 du e^{s \text{Re}(E)} \|\phi^{(-)}(f) e^{-sH_0} \phi^{(+)}(f)\| \tag{182}
\]

we are faced with this estimate above, using the same notation \(f\) as before and concentrating on the most singular part, we have an estimate

\[
\|B\|_{(\text{sing})} \leq n^2 \int_0^\infty \int_0^1 du e^{-s[(n-1)m-\text{Re}(E)]} \|f\|^2 \leq C n \int_0^\infty ds e^{-s[(n+1)m-\text{Re}(E)]} \tag{183}
\]

which is finite. As we have shown that \(T(E, E_*)\) is indeed bounded and since every bounded operator on a Hilbert space is closable, we conclude that for fixed \(E_*\):

\[
[\Phi(E) - \Phi(E_*)] x_k \rightarrow [\Phi(E) - \Phi(E_*)] x
\]

\[
\Phi(E_*) x_k = [1 + \mathcal{K}(E_*) - \mathcal{U}(E_*)] (H_0 - E_* + \mu_p) x_k
\]

\[
= [1 + \mathcal{K}(E_*) - \mathcal{U}(E_*)] (H_0 - E_* + \mu_p) x
\]

We can now add them up to see that:

\[
y = [(1 + \mathcal{K}(E_*)) - \mathcal{U}(E_*)] (H_0 - E_* + \mu_p) + \Phi(E) - \Phi(E_*)] x \Rightarrow \Phi(E) x = y \tag{184}
\]

Hence, we conclude that \(\Phi(E)\) is closed on its domain \(D(\Phi(E)) = D(H_0)\).

### II.5.3 Holomorphicity of the Matrix Elements

We now want to show that the family \(\Phi(E)\) satisfies the second criteria in the Definition \(2\) Note that operator family \(\Phi(E)\) is not given by an explicit formula it is an integral of a parameter dependent operator. To understand its holomorphic structure, it is essential to look into the matrix elements, by definition. For this we can employ the following Theorem \(2\) from \(21\) as it is done in the previous part.

To make contact with this theorem, we note that in our case, \(\Theta(E) = \langle \lambda | \Phi(E) | \Psi \rangle\) where \(|\lambda\rangle \in \mathcal{F}^{(n)}\) and \(|\Psi\rangle \in D(H_0)\). For a family of unbounded operators, operator acts on the domain then we can take an inner product of the resulting vector with any vector in the Hilbert space (here it is essential that the family has a common domain for any value of the complex parameter \(E\)). Recall that the Principal Operator reads as:

\[
\Phi(E) = \left[ 1 + \sum_{\sigma} \frac{\lambda^2}{2\omega_\sigma} \frac{|f_\sigma|^2}{(H_0 - E + \omega_\sigma) (\omega_\sigma - \mu_p)} \frac{1}{(H_0 - E + \omega_\sigma + \omega_\tau)} \frac{a_\tau}{\sqrt{2\omega_\tau}} f_\tau \right] (H_0 - E + \mu_p) \tag{185}
\]

\(H_0 - E + \mu_p\) is already obviously holomorphic on the entire complex plane. We will again call the second term in square brackets as \(\mathcal{K}(E)\) and the third as \(\mathcal{U}(E)\). The main thing is to prove that these bounded operators are actually holomorphic in the desired open domain of the complex plane. Note that the operator
$H_0 + \mu_p - E$ is invertible for our choice of $E$, so its range is the full Hilbert space. Therefore we show that for any choice of $|\lambda\rangle, |\psi\rangle$ the matrix elements $\langle \lambda | [1 + K(E) - \mathcal{U}(E)] | \psi \rangle$, apart from 1, when considered as an integral representation, becomes a sum of two pieces, each of which is defined over the same domain of the complex plane. They carry different measures, but they can be put into the form,

$$
\hat{\Theta}(E) = \int_V \phi(E, t) d\mu(t) \quad \text{with} \quad \phi(E, \cdot) \quad \text{holomorphic for almost all } t \in V \quad \text{(186)}
$$

where we use the heat kernel and subordination identity for the $\phi(\cdot, t)$, measurable and integrable for any $Re(E) < nm + \mu_p \quad \text{(187)}$

$E \mapsto \int_V |\phi(E, t)| d\mu(t)$ is a bounded map on any compact subset of $Re(E) < nm + \mu_p \quad \text{(188)}$

Theorem 2 then implies that the sum is a holomorphic function of $E$.

Let us verify these for $\mathcal{K}(E)$ first, using our previous estimates, we readily find that,

$$
\langle \lambda | \mathcal{K}(E) | \psi \rangle = C \int_{0 \leq u_2 + u_3 \leq 1} du_2 du_3 \int s^3 ds \int d\xi \int_{\xi^{3/2}} e^{-\frac{2m^2\xi}{\pi}} K_\xi(\bar{x}, \bar{e}) e^{s\mu_p u_2} \langle \lambda | e^{-su_3(H_0 - E)} | \psi \rangle 
$$

$$
= \int_{0 \leq u_2 + u_3 \leq 1} du_2 du_3 \int s^3 ds \int d\xi \phi(s, u_2, u_3, \xi, E) \quad \text{(190)}
$$

where we use the heat kernel and subordination identity for the $f_\sigma(\bar{x}) e^{-s\omega_\sigma}$ to get the heat kernel, after this we collect everything into $\phi$ except the integral measures. To show integrability, we employ the well-known result, let $|\phi| \leq g$, $\phi$ being a measurable function. If $g$ is integrable, so is $\phi$. Taking the absolute value of $\phi(s, u_2, u_3, \xi, E)$, we get:

$$
\int du_2 du_3 \int ds |\phi| = C \int du_2 du_3 \int s^3 ds \int d\xi \int_{\xi^{3/2}} e^{-\frac{2m^2\xi}{\pi}} K_\xi(\bar{x}, \bar{e}) |\langle \lambda | e^{-su_3(H_0 - E)} | \psi \rangle| 
$$

$$
\leq C |\langle \lambda | \psi \rangle| \int du_2 du_3 \int s^3 ds \int d\xi K_\xi(\bar{x}, \bar{e}) e^{s\mu_p u_2} e^{-\frac{2m^2\xi}{\pi}} e^{s\mu_p u_2} e^{-su_3(nm - Re(E))} 
$$

$$
= \int du_2 du_3 \int ds \int d\xi g(s, u_2, u_3, \xi, E) \quad \text{(191)}
$$

where we have defined $|\phi| < g$ by estimating $H_0$ as $nm$ as usual and $E$ is taken to have a fixed value below $nm + \mu_p$. $\phi$ consists of well-defined continuous functions hence measurable. Note that the integral (191) is the same as (131) up to some constants. Thus, we can estimate it following the same steps and show that it is bounded. For the integral (131) is already shown to be finite in (11.4.1), so is (191). Consequently, we can conclude that $\phi(\cdot, E)$ is indeed in $L^1$. Incidentally the true parameter space depends on the wave functions and the heat kernel coming from the exponential of $H_0$, therefore we have a multiple integral over the compact manifold weighted by heat kernels. In fact, it is easier to directly establish holomorphicity of $e^{-s(H_0 - E)}$ by this argument and not to think of these integrals as part of the measure. For simplicity this is what we assume.

Note that for the parameters $u_2, u_3, s, \xi$ fixed, $\phi$ is simply an entire function of $E$ where the only factor depending on $E$ is $e^{-s(H_0 - E)}$. Thus, the holomorphicity of $\phi(\cdot, E)$ is straightforward. The extra condition stated in the above Theorem does not require more work since we have already shown the integrability through $\langle \int du_2 du_3 dsd\xi |\phi| \rangle$ being bounded above by $(\int du_2 du_3 dsd\xi g)$, for $Re(E) < nm + \mu_p$. The explicit bound can be found in (167).

As the explicit construction done above for $\mathcal{K}(E)$, for $\mathcal{U}(E)$ we make use of some inequalities based on the bounds we found before. Now let us note that

$$
\langle \psi_1 | \mathcal{U}(E) | \psi_2 \rangle = \lambda^2 \int_0^\infty ds \int_0^1 du e^{su} \langle \psi_1 | \mathcal{U}(E) | \psi_2 \rangle \langle \phi^{(s)}(g(su)) | \psi_1 | e^{-sH_0} | \phi^{(s)}(f(s)) | \psi_2 \rangle \quad \text{(192)}
$$

The inner product is a well-defined measurable function of $s, u$ for fixed $\psi_1, \psi_2$, and moreover it is bounded thanks to the norm inequality that $|\langle \psi_1 | \mathcal{U}(E) | \psi_2 \rangle| \leq |\mathcal{U}(E)||\psi_1|| \psi_2||$. The explicit form of the functions
are cumbersome but it can be found (indeed we need the explicit expression for the compactness of these operators which we plan to discuss in a later publication). Overall boundedness can be repeated as before, for the sake of brevity we do not give these expressions. (It is most natural to use the coordinate representation of the inner product in $\mathcal{U}(E)$, then we express these in terms of product measures over $\mathcal{M}$). Holomorphicity in $E$ is again straightforward since the only function containing $E$ is an entire one, $e^{sE}$. Since the boundedness of $\mathcal{U}(E)$ is satisfied and integrability condition follows in a similar way as for $\mathcal{K}(E)$, we have shown that $\Phi(E)$ is indeed a holomorphic family of type-A on the domain of $\{E \in \mathbb{C} | \text{Re}(E) < nm + \mu_p\}$ (with the common operator domain $D(H_0)$ over this set). This is essential to establish the spectral projections via a contour integral as discussed previously.

II.5.4 Self-Adjointness of $\Phi(E)$

Note that, formally, $\Phi^\dagger(E) = \Phi(E)$, hence at least, $D(\Phi(E)) \subset D(\Phi^\dagger(E))$. But to conclude self-adjointness, we need to show that they admit the same domain. As it is done in the first part, we make use of the well-known Kato-Rellich Theorem [22] to show that $\Phi(E)$ is self-adjoint on some region along the real axis for which $E$ is chosen to be sufficiently negative and then employ Wüst’s theorem [23] to generalize it to the whole region of concern.

Note that the argument used in the preceding part about the light-front version is purely formal, therefore it carries over to the manifold case exactly. We identify $A$ and $B$ parts in the Principal Operator:

$$\Phi(E) = (1 + \mathcal{K}(E) - \mathcal{U}(E))(H_0 - E + \mu_p)$$

$$= \frac{(H_0 - E + \mu_p + \mathcal{K}(E)(H_0 - E + \mu_p) - \mathcal{U}(E)(H_0 - E + \mu_p)}{A \quad B}, (193)$$

since $A$ is invertible, we can write $x = A^{-1}y$ for some $y \in \mathcal{H}$ (with $b = 0$ is set),

$$||BA^{-1}y|| \leq a||y|| (194)$$

We work on the real axis if we show that (for some choices of $E$),

$$||BA^{-1}|| = ||\mathcal{U}(E)(1 + \mathcal{K}(E))^{-1}|| \leq ||\mathcal{U}(E)|| < 1 (195)$$

then the conditions stated in Theorem[3] are satisfied. Note that, by the spectral theorem, $A(E)$ is a self-adjoint operator for real values of the parameter $E$ belonging to the symmetric region $\Omega$, defined on a domain $D(H_0)$. Moreover, $B(E)$ is a symmetric operator for real values of $E$ (in the compact manifold case this is easier to see since creation and annihilation operators in energy representation are ordinary unbounded operators, many of the formal properties can be justified). Recall that we show previously $||\mathcal{U}(E)|| < 1$ if we choose $E$ sufficiently below along the real axis (say below $E_\ast$). Then $||BA^{-1}y|| \leq 1$ and accordingly, by the theorem stated, $A + B = \Phi(E)$ is self-adjoint at least in some region where $E < E_\ast$.

We now invoke the Theorem of Wiist, let $\Omega = \{E \in \mathbb{C} | \text{Re}(E) < nm + \mu_p\}$ (our domain in the complex plane which is symmetric around the real axis) and $\{\Phi(E), E \in \Omega, D(\Phi(E)) = D(H_0)\}$ a holomorphic family of type-A defined in $\mathcal{H}$ (with domain $D(H_0)$ is dense inside) such that $\Phi(E) \subset \Phi^\dagger(E)$ (which is clear since formally the equality holds, thus the domain is included). Let us define,

$$M := \{E \mid E \in \Omega, \quad \Phi^\dagger(E) = \Phi(E)\}. (196)$$

We show above that at least in some region, on the real line below a sufficiently negative $E_\ast$, $\Phi(E)$ is self-adjoint. Now, thanks to the Wüst’s theorem, $M = \Omega$, that is the equality $\Phi^\dagger(E) = \Phi(E)$ (not only formally but also the equality in the sense of domains) extends to all $\{E \in \mathbb{C} | \text{Re}(E) < nm + \mu_p\}$. Therefore, we conclude that $\Phi(E)$ is a self-adjoint holomorphic family of type-A on the domain of interest.

Conclusion

The relativistic Lee model is reanalyzed in the 2+1 dimensional oblique light-front coordinates in more detail. The resolvent formulation, developed by Rajeev, enables us to study the spectrum of this model, in
particular the ground state energy can be estimated from below and above by analyzing the principle operator. We show that the resolvent obtained by a formal process indeed corresponds to the resolvent of an operator. To establish self-adjointness and obtain spectral projections, we show that the principal operator as a family dependent on a complex parameter $E$ (in some symmetric domain) is a self-adjoint holomorphic family of type-A in the sense of Kato. In the second part of the paper, these results are extended to the model defined over a compact manifold by means of the heat kernel techniques. In the near future, we plan to prove uniqueness (or the non-degeneracy) of the ground state for the compact case.

Acknowledgements

We would like to note that the extension of the first part to the compact manifold case is based on M. Unel’s master thesis which was completed while she was a graduate student at Bogazici University, Department of Physics. M. Unel acknowledges support from the Villum Foundation via the QMATH Center of Excellence. The authors would like to thank B.T. Kaynak and F. Erman for discussions. The basic ideas of this project was conceived while O. T. Turgut was visiting J. Hoppe at KTH, Stockholm, he is grateful to J. Hoppe for these invitations and the scientific support he has received during all these years. We would like to thank M. Znojil, A. Michelangeli and G. Dell’Antonio for their kind interest to our work, moreover O.T. Turgut would like to thank them for their continual support over the years. Last but not least, O. T. Turgut would like to thank M. Deserno for arranging a long term visit to Carnegie-Mellon Physics Department which brought this work to a completion.

References


