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Some econometric results for the Blanchard-Watson bubble model*

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Abstract

The purpose of the present paper is to analyse a simple bubble model suggested by Blanchard and Watson. The model is defined by $y_t = s_t \rho y_{t-1} + \varepsilon_t$, $t = 1, \ldots, n$, where s_t is an i.i.d. binary variable with $p = P(s_t = 1)$, independent of ε_t i.i.d. with mean zero and finite variance. We take $\rho > 1$ so the process is explosive for a period and collapses when $s_t = 0$. We apply the drift criterion for non-linear time series to show that the process is geometrically ergodic when $p < 1$, because of the recurrent collapse. It has a finite mean if $p\rho < 1$, and a finite variance if $p\rho^2 < 1$. The question we discuss is whether a bubble model with infinite variance can create the long swings, or persistence, which are observed in many macro variables. We say that a variable is persistent if its autoregressive coefficient $\hat{\rho}_n$ of y_t on y_{t-1}, is close to one. We show that $\hat{\rho}_n \xrightarrow{P} \rho p$ if the variance is finite, but if the variance of y_t is infinite, we prove the curious result that $\hat{\rho}_n \xrightarrow{P} \rho^{-1}$. The proof applies the notion of a tail index of sums of positive random variables with infinite variance to find the order of magnitude of $\sum_{t=1}^n y_{t-1}^2$ and $\sum_{t=1}^n y_t y_{t-1}$ and hence the limit of $\hat{\rho}_n$.

Keywords: Time series, explosive processes, bubble models

JEL Classification: C32.

1 Introduction

The paper by Blanchard and Watson (1982) investigates the nature and the presence of bubbles in financial markets and whether the presence of bubbles in a particular market can be detected statistically.

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They suggested the model defined by ε_t i.i.d. $(0,\sigma^2)$ and independent binary variables s_t for which

$$P(s=1)=p; P(s=0)=1-p=q.$$

The process is generated by

$$y_t = s_t\rho y_{t-1}+\varepsilon_t,\ t=1,\dots,n,$$

where we assume for notational reasons that $s_0=0$ and $y_0=\varepsilon_0$. Throughout we consider the distribution of the data conditional on y_0 and denote the probability measure P, the expectation E, and the variance Var. The process y_t is explosive in the periods where $s_t=1$ and creates a bubble which busts when $s_t=0$, an event that has probability $q=1-p$.

We define persistence of y_t as a value close to one of the limit of the autoregressive estimator

$$\hat\rho_n=\frac{\sum_{t=1}^n y_t y_{t-1}}{\sum_{t=1}^n y_{t-1}^2}. \tag{1}$$

Frydman and Goldberg (2007) raised the question if such a bubble model with infinite variance model could create the long swings, or persistence, which is typical of macro variables.

In order to discuss this problem we first prove that if $p<1$ the process y_t is geometrically ergodic with an invariant distribution P_*. We denote mean and variance with repect to P_* by E_* and Var_*. Then, if $p\rho<1$, $E_*(y_t)=0$ and if $p\rho^2<1$, $Var_*(y_t)<\infty$.

We show that if $p\rho^2<1$ and $E\varepsilon_t^2<\infty$, then $\hat\rho_n\overset{P}{\to}p\rho$ and under the further condition $p\rho^{4+\delta}<1$ and $E(\varepsilon_t^{4+\delta})<\infty$, $\hat\rho_n$ is asymptotically Gaussian, whereas if $p\rho^2>1$, we prove the curious result that $\hat\rho_n\overset{P}{\to}\rho^{-1}$. Thus, in this sense the bubble model cannot create the long swings that are characteristic for unit root processes, because the only way $\hat\rho_n$ can converge to one is if $\rho=1$, and then the variance is finite because $p<1$.

2 A condition for stationarity and finite variance

The Blanchard Watson model is a special case of the autoregressive conditional root (ACR) model, see Bec, Rahbek, and Shepard (2008) and we use the methods developed there in this section.

Lemma 1 *Let $p<1<\rho$. Then y_t is a geometrically ergodic Markov chain and there exists an invariant distribution, P_*, so that y_t becomes stationary. If $p\rho^d<1$, for some $d\geq 1$ and $E|\varepsilon_t|^d<\infty$ then $E_*(|y_t|^d)<\infty$.*

Proof. We first note that y_t is a Markov chain with transition density

$$h(y_t|y_{t-1}=y)=\frac{1}{\sigma}\phi(\frac{y_t-\rho y}{\sigma})p+\frac{1}{\sigma}\phi(\frac{y_t}{\sigma})q>0,$$

where ϕ is the density function for the Gaussian distribution. Thus the transition kernel for the Markov chain y_t has a density with respect to the Lebesgue measure, which is strictly positive and bounded away from zero on compact sets. This establishes that the Markov chain is irreducible, aperiodic, and that compact sets are small, see Bec, Rahbek, and Shepard (2008) for similar results for general ACR models. Next we establish that the Markov chain satisfies a drift criterion for a drift function, $D(.)$ defined below, which by Theorem 15.0.1 (iii) of Meyn and Tweedie (1993) implies that the chain is geometrically ergodic with invariant distribution P_* and $E_*D(y_t) < \infty$. The condition we have to check for a continuous function $D(y) > 0$ is that

$$\frac{E(D(y_t)|y_{t-1} = y)}{D(y)} < 1 \text{ for } |y| \geq A, \tag{2}$$

for some constant $A > 0$ and $D(y)$ is bounded for $|y| \leq A$.

If $p\rho^d < 1$ we can take $D(y) = 1 + |y|^d$ and find from $y_t = s_t\rho y_{t-1} + \varepsilon_t$, that

$$\frac{E(D(y_t)|y_{t-1} = y)}{D(y)} = \frac{1 + E|s_t\rho y + \varepsilon_t|^d}{1 + |y|^d} \leq \frac{1 + [(p\rho^d|y|^d)^{1/d} + (E|\varepsilon_t|^d)^{1/d}]^d}{1 + |y|^d} \to p\rho^d < 1,$$

for $|y| \to \infty$ which shows that (2) is satisfied. We have used the Minkowsky inequality in the form $E|V_1 + V_1|^d \leq [E(|V_1|^d)^{1/d} + E(|V_2|^d)^{1/d}]^d$ for two random variables V_1 and V_2 with finite d'th moment. ■

Thus, although the process is explosive in the intervals where $s_t = 1$, it collapses to ε_t if $s_t = 0$, and the bubble busts. It is this repeated collapse that creates a stationary process, which starts each period in a new ε.

3 Asymptotic properties of the autoregressive estimator when the variance is finite

In this section we analyse the estimator $\hat{\rho}_n$ if $p\rho^2 < 1$ so the variance of y_t is finite and show that $\hat{\rho}_n \xrightarrow{P} p\rho$ and that $n^{1/2}(\hat{\rho}_n - p\rho)$ is asymptotically Gaussian.

Theorem 2 *Assume that $p < 1 < \rho$ and that $p\rho^2 < 1$ and $E\varepsilon_t^2 < \infty$, then*

$$\hat{\rho}_n \xrightarrow{P} p\rho. \tag{3}$$

If, furthermore, there exists a constant $\delta > 0$ such that $p\rho^{4+\delta} < 1$ and $E|\varepsilon_t|^{4+\delta} < \infty$, then

$$n^{1/2}(\hat{\rho}_n - p\rho) \xrightarrow{d} N(0, \frac{pq\rho^2 E_*(y_{t-1}^4) + \sigma^2 E_*(y_{t-1}^2)}{E_*(y_{t-1}^2)^2}). \tag{4}$$

Proof. If $p\rho^2 < 1$ and $E(\varepsilon_t^2) < \infty$, then by Lemma 1 for $d = 2$, $Var_*(y_t) < \infty$, and by the law of large numbers for ergodic processes

$$\hat{\rho}_n \xrightarrow{P} \frac{E_*(y_t y_{t-1})}{E_*(y_{t-1}^2)} = p\rho,$$

because $E_*(y_t y_{t-1}) = E_*[(s_t\rho y_{t-1} + \varepsilon_t)y_{t-1}] = p\rho E_*(y_t^2)$, which shows (3).

Next we define from

$$n^{1/2}(\hat{\rho} - p\rho) = \frac{n^{-1/2}\sum_{t=1}^n[(s_t - p)\rho y_{t-1} + \varepsilon_t]y_{t-1}}{n^{-1}\sum_{t=1}^n y_{t-1}^2}$$

the martingale difference sequence

$$X_{nt} = n^{-1/2}[(s_t - p)\rho y_{t-1} + \varepsilon_t]y_{t-1},$$

which satisfies

$$\sum_{t=1}^{n} Var_{t-1}(X_{nt}) = n^{-1}\sum_{t=1}^{n}[pq\rho^2 y_{t-1}^2 + \sigma^2]y_{t-1}^2 \xrightarrow{P} [pq\rho^2 E_*(y_{t-1}^4) + \sigma^2 E_*(y_{t-1}^2)] > 0,$$

by the law of large numbers because $E_* y_t^4 < \infty$, by Lemma 1 for $d = 4$. Finally we check the Lindeberg condition:

$$\begin{aligned}\sum_{t=1}^{n} X_{nt}^2 1_{\{|X_{nt}|\geq\eta\}} &\leq \eta^{-\delta/2}\sum_{t=1}^{n}|X_{nt}|^{2+\delta/2} \\ &= n^{-\delta/4}\eta^{-\delta/2}n^{-1}\sum_{t=1}^{n}|(s_t - p)\rho y_{t-1} + \varepsilon_t|^{2+\delta/2} y_{t-1}^{2+\delta/2} \xrightarrow{P} 0,\end{aligned}$$

as $E|\varepsilon_t|^{4+\delta} < \infty$, and $E|y_{t-1}|^{4+\delta}$ is finite by Lemma 1 and the assumption $p\rho^{4+\delta} < 1$. The central limit theorem for martingales, see Hall and Heyde (1980), now gives the result. ■

4 The probability limit of the autoregressive estimator when the variance is infinite

In this section we find an approximation to the autoregressive estimator and bound the remainder terms in the conditional distribution given the variables $\{s_t\}_{t=0}^{\infty}$. We then apply this result to prove the main result about the limit of the autoregressive coefficient.

Let $N_n = \sum_{t=1}^{n}(1 - s_t)$ be the number of busts before or at n, and let the times of bust, when $s_t = 0$, be $T_i^*, i = 0, \ldots, N_n + 1$. These satisfy

$$0 = T_0^* < T_1^* < T_2^* < \cdots < T_{N_n}^* \leq n < T_{N_n+1}^*,$$

and we let $T_i = T_i^* - T_{i-1}^*$ be the length of the periods, $i = 1, 2, \ldots$ The last period before n is of length $n - T_{N_n}^*$. The variables T_i are independent and have the same geometric distribution

$$P(T_i = m) = p^{m-1}q, m = 1, 2, \ldots$$

We now construct a double array of i.i.d. $(0, \sigma^2)$ random variables $\varepsilon_{it}, i = 1, 2, \ldots, t = 0, 1, \ldots$ and construct the process y_t as follows. In the first period we use $\varepsilon_{1t}, t = 0, 1, \ldots$ and find for $t = 1, \ldots, T_1 - 1$ that, starting at $y_0 = \varepsilon_{10}$, we get because $s_1 = \cdots = s_{T_1 - 1} = 1$, that

$$y_t = \sum_{v=0}^{t}\rho^{t-v}\varepsilon_{1v} = \rho^t\sum_{v=0}^{\infty}\rho^{-v}\varepsilon_{1v} - \rho^{-1}\sum_{v=0}^{\infty}\rho^{-v}\varepsilon_{1,v+t+1} = \rho^t Z_1 - \rho^{-1}Z_{1t}, \qquad (5)$$

where Z_{1t} has the same distribution as Z_1 with $E(Z_1) = 0$ and $Var(Z_1) = \sigma^2/(1 - \rho^{-2})$. The last observation of the first period, y_{T_1}, has $s_{T_1} = 0$, and we define

$$y_{T_1} = \varepsilon_{20},$$

which acts as initial value for the second bust, where $\varepsilon_{2t}, t = 0, 1, \ldots$ are used to construct the process.

Similar expressions can be found for the i'th period, $t = T^*_{i-1} + 1, \ldots, T^*_i - 1$

$$y_t = \sum_{v=0}^{t-T^*_{i-1}} \rho^{t-T^*_{i-1}-v}\varepsilon_{iv} = \rho^{t-T^*_{i-1}}\sum_{v=0}^{\infty}\rho^{-v}\varepsilon_{iv} - \rho^{-1}\sum_{v=0}^{\infty}\rho^{-v}\varepsilon_{i,v+t-T^*_{i-1}+1} = \rho^{t-T^*_{i-1}}Z_i - \rho^{-1}Z_{it}, \quad (6)$$

and define $y_{T^*_i} = \varepsilon_{i+1,0}$. Note that by using a double array $\{\varepsilon_{it}\}$ we have made sure that T_i and Z_i are independent and i.i.d. We next apply this representation to find an approximation to the autoregressive estimator.

Lemma 3 *The product moments have the representation*

$$\sum_{t=1}^{n} y^2_{t-1} = \frac{1}{\rho^2 - 1}[\sum_{i=1}^{N_n}\rho^{2T_i}Z_i^2 + \sum_{i=1}^{N_n+1}A_i],$$

$$\sum_{t=1}^{n} y_{t-1}y_t = \frac{\rho^{-1}}{\rho^2 - 1}[\sum_{i=1}^{N_n}\rho^{2T_i}Z_i^2 + \sum_{i=1}^{N_n+1}B_i],$$

where the remainder terms satisfy

$$E(|A_i|^{\xi} + |B_i|^{\xi}|T_i) \leq c\rho^{\xi T_i}, \; i = 1, \ldots, N_n + 1. \quad (7)$$

It follows that the estimator based on $y_t, t = 0, 1, \ldots, n$, has the representation

$$\hat{\rho}_n = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y^2_{t-1}} = \rho^{-1}\frac{\sum_{i=1}^{N_n}\rho^{2T_i}Z_i^2 + \sum_{i=1}^{N_n+1}B_i}{\sum_{i=1}^{N_n}\rho^{2T_i}Z_i^2 + \sum_{i=1}^{N_n+1}A_i}. \quad (8)$$

Proof. We find from (5) that

$$\sum_{t=1}^{T_1} y^2_{t-1} = \sum_{t=1}^{T_1}(\rho^{t-1}Z_1 - \rho^{-1}Z_{1t-1})^2 = \frac{\rho^{2T_1}}{\rho^2 - 1}Z_1^2 + A_1$$

$$A_1 = \frac{-1}{\rho^2 - 1}Z_1^2 + \rho^{-2}\sum_{t=1}^{T_1}Z^2_{1t-1} - 2Z_1\sum_{t=1}^{T_1}\rho^{t-2}Z_{1t-1}.$$

We need the inequality valid for $a \geq 0$ and $b \geq 0$

$$(a+b)^{\xi} = b^{\xi} + \xi\int_0^a (b+x)^{\xi-1}dx \leq b^{\xi} + \xi\int_0^a x^{\xi-1}dx = b^{\xi} + a^{\xi}, \; 0 < \xi \leq 1. \quad (9)$$

This implies that

$$E(|A_1|^{\xi}|T_1) \leq a_1 E(Z_1^{2\xi}) + a_1 T_1 E(Z_1^{2\xi}) + a_2\rho^{\xi T_1}E(Z_1^{2\xi}) \leq c\rho^{\xi T_1},$$

which shows (7) for A_1. The same proof can be used for $A_i, i = 2, \ldots, N_n$, and it is seen that the bound c does not depend on i. For $i = N_n + 1$, we have $A_{N_n+1} = \sum_{t=T^*_{N_n}+1}^{n} y^2_{t-1} \leq \sum_{t=T^*_{N_n}+1}^{T^*_{N_n+1}} y^2_{t-1}$ and the same proof works.

Next we find, noting that $y_{T_1} = \varepsilon_{20}$, that

$$\sum_{t=1}^{T_1} y_{t-1}y_t = \sum_{t=1}^{T_1-1}(\rho^{t-1}Z_1 - \rho^{-1}Z_{1t-1})(\rho^t Z_1 - \rho^{-1}Z_{1t}) + (\rho^{T_1-1}Z_1 - \rho^{-1}Z_{1T_1-1})\varepsilon_{20}$$

$$= \frac{\rho^{2(T_1-1)}}{\rho^2-1}\rho Z_1^2 + B_1$$

$$B_1 = \frac{-\rho Z_1^2}{\rho^2-1} + \rho^{-2}\sum_{t=1}^{T_1-1}[Z_{1t-1}Z_{1t} - \rho^{t+1}Z_{1t-1}Z_1 - \rho^t Z_1 Z_{1t}] + \rho^{-1}(\rho^{T_1}Z_1 - Z_{1T_1-1})\varepsilon_{20}.$$

The same proof shows that $E(|B_1|^\xi|T_1)$ satisfies (7). The terms $B_i, i = 2, \ldots, N_n + 1$ can be handled similarly. ■

In the following we assume that $p\rho^2 > 1$ so that the variance of y_t is infinite. We want to find the limit of the regression estimator given in (8) and for that we need the order of magnitude of the main term $\sum_{i=1}^{N_n}\rho^{2T_i}Z_i^2$ and bounds on the remainder terms.

We find the order of magnitude of these terms using the theory of sums of positive random variables with infinite variance, see for instance Feller (1971, Chapter IX, Section 8). It turns out that it is not possible to normalize the main term to convergence, because of the discrete nature of the geometric distribution, but instead we bound T_i using an exponentially distributed variable U_i, for which $\sum_{i=1}^{N_n}\rho^{2U_i}Z_i^2$ can be normalized to convergence.

Let U_i be i.i.d. exponentially distributed variables with parameter $\lambda = -\log p$, and represent the waiting times as one plus the integer part of U_i :

$$T_i = [U_i] + 1.$$

Then

$$P(T_i = m) = P(m-1 \leq U_i < m) = e^{-\lambda(m-1)} - e^{-\lambda m} = p^{m-1}q.$$

We have the evaluations

$$U_i \leq T_i \leq U_i + 1,$$

and hence the bounds for any finite m

$$\sum_{i=1}^{m}\rho^{2U_i}Z_i^2 \leq \sum_{i=1}^{m}\rho^{2T_i}Z_i^2 \leq \rho^2\sum_{i=1}^{m}\rho^{2U_i}Z_i^2. \tag{10}$$

This shows that it is enough to find the order of magnitude of $\sum_{i=1}^{m}\rho^{2U_i}Z_i^2$, and for this we need the so-called tail index of a positive random variable. We find from

$$P(\rho^{2U} > x) = P(U \geq \frac{\log x}{2\log\rho}) = e^{-\frac{\lambda \log x}{2\log\rho}} = x^{\frac{\log p}{2\log\rho}} = x^{-\alpha/2},\ \alpha = -\frac{\log p}{\log\rho},$$

that

$$P(\rho^{2U}Z^2 > x) = E[P(\rho^{2U}Z^2 > x)|Z)] = E(xZ^{-2})^{-\alpha/2} = x^{-\alpha/2}E(Z^\alpha).$$

Thus the tails of the distributions of ρ^{2U} and $\rho^{2U}Z^2$ decrease as $x^{-\alpha/2}$, and we say that the tail index of ρ^{2U} and $\rho^{2U}Z^2$ is $\alpha/2$. Note that $p < 1 < p\rho^2$ implies $0 < \alpha < 2$.

With these tools we can now prove the main result.

Theorem 4 *For $p < 1 < p\rho^2$, and $E(\varepsilon_t^2) < \infty$, it holds that*

$$m^{-2/\alpha} \sum_{i=1}^{m} \rho^{2U_i} Z_i^2 \xrightarrow{d} U_{\alpha/2}, m \to \infty, \tag{11}$$

where $Z_i = \sum_{v=0}^{\infty} \rho^{-v} \varepsilon_{iv}$ is given by (6) and $U_{\alpha/2}$ is a stable distribution of index $\alpha/2$. This implies that

$$\hat{\rho}_n = \frac{\sum_{t=1}^{n} y_t y_{t-1}}{\sum_{t=1}^{n} y_{t-1}^2} \xrightarrow{P} \rho^{-1}. \tag{12}$$

Proof. *Proof of* (11): By construction $\rho^{2U_i} Z_i^2, i = 1, \ldots, m$ are i.i.d. and Z_i is independent of U_i, so the tail index of $\rho^{2U_i} Z_i^2$ is $\alpha/2$. The result (11) now follows from Feller (1971, Theorem 2 p. 305, and (8.14)).

Proof of (12): We use the representation (8) for $\hat{\rho}_n$ but first replace the stochastic N_n by the nonstochastic m and show that

$$R_m = \frac{m^{-2/\alpha} \sum_{i=1}^{m} \rho^{2T_i} Z_i^2 + m^{-2/\alpha} \sum_{i=1}^{m+1} B_i}{m^{-2/\alpha} \sum_{i=1}^{m} \rho^{2T_i} Z_i^2 + m^{-2/\alpha} \sum_{i=1}^{m+1} A_i} \xrightarrow{P} 1, \ m \to \infty.$$

From the bounds (10) and (11) it follows that it is enough to show that

$$m^{-2/\alpha} \sum_{i=1}^{m+1} (|A_i| + |B_i|) \xrightarrow{P} 0, \ m \to \infty.$$

The expectation of this need not be finite when $\rho > 1$, but because $0 < \alpha < 2$ we can choose ξ so that $\alpha/2 < \xi < \min(1, \alpha)$. Then $p\rho^\xi < 1$ and $E(\rho^{\xi T_1}) = \sum_{m=0}^{\infty} (\rho^\xi p)^m p^{-1} q < \infty$. From (7) and (9) we find

$$E(m^{-2/\alpha} \sum_{i=1}^{m+1} (|A_i| + |B_i|))^\xi \le m^{-2\xi/\alpha}(m+1) E(|A_1|^\xi + |B_1|^\xi) \le c m^{1-2\xi/\alpha} E(\rho^{\xi T_1}) \to 0,$$

because $\alpha/2 < \xi$ and $E(\rho^{\xi T_i}) < \infty$ when $\xi < \alpha$.

Next we want to prove that we can replace m by $N_n = \sum_{t=1}^{n} (1 - s_t)$. By the law of large numbers we have $n^{-1} N_n \xrightarrow{P} q$ so that for given $\varepsilon > 0, \delta > 0$ we can choose an n_0 so that for $n \ge n_0$ we have with probability greater than $1 - \delta$

- $[n(q - \varepsilon)] \le N_n \le [n(q + \varepsilon)]$
- $(\frac{[n(q+\varepsilon)]}{[n(q-\varepsilon)]})^{2/\alpha} \le 1 + \varepsilon$
- $[n(q + \varepsilon)]^{-2/\alpha} \sum_{i=1}^{[n(q+\varepsilon)]+1} (|A_i| + |B_i|) \le \varepsilon/(1 + \varepsilon)$.

Then it follows that with probability greater than $1 - \delta$

$$N_n^{-2/\alpha} \sum_{i=1}^{N_n+1} (|A_i| + |B_i|) \le (\frac{[n(q + \varepsilon)]}{[n(q - \varepsilon)]})^{2/\alpha} [n(q + \varepsilon)]^{-2/\alpha} \sum_{i=1}^{[n(q+\varepsilon)]+1} (|A_i| + |B_i|) \le \varepsilon,$$

so that $N_n^{-2/\alpha} \sum_{i=1}^{N_n+1} (|A_i| + |B_i|) \xrightarrow{P} 0$. This proves (12). ∎

We conclude this section with a small simulation. Figure 1 shows the simulated values of the median (and 2.5% and 97.5% quantiles) of 1000 simulations of $\hat{\rho}_n$ for $n = 10.000$ and $\rho = 1.032$. It is seen that for $p < \rho^{-2} = 0.939$ (the finite variance case) the limit is almost proportional to p with slope ρ and for $p > \rho^{-2} = 0.939$ (the infinite variance case) the limit is almost constantly equal to $\rho^{-1} = 0.969$, which illustrates the result that $\hat{\rho}_n \xrightarrow{P} \min(p\rho, \rho^{-1})$.

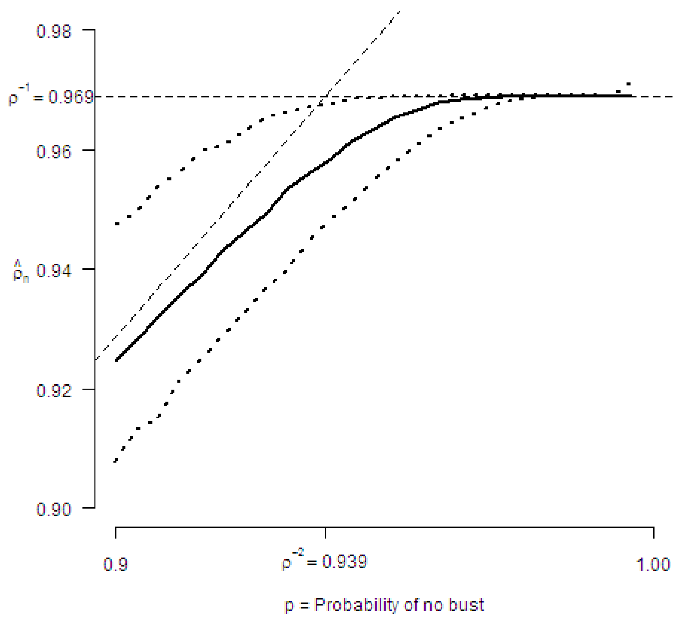

Figure 1: The figure shows the result of 1.000 simulations for $n = 10.000$ of $\hat{\rho}_n$ for $\rho = 1.032$ and $0.9 < p < 1$. We have plotted the median and the 2.5% and 97.5% quantiles of the simulations.

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