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Post-Minkowskian expansion from scattering amplitudes

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The SAGEX review on scattering amplitudes
Chapter 13: Post-Minkowskian expansion from scattering amplitudes

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Abstract
The post-Minkowskian expansion of Einstein’s general theory of relativity has received much attention in recent years due to the possibility of harnessing the computational power of modern amplitude calculations in such a classical context. In this brief review, we focus on the post-Minkowskian expansion as applied to the two-body problem in general relativity without spin, and we describe how relativistic quantum field theory can be used to greatly simplify analytical calculations based on the Einstein–Hilbert action. Subtleties related to the extraction of classical physics from such quantum mechanical calculations highlight the care which must be taken when both positive and negative powers of Planck’s constant are at play. In the process of obtaining classical results in both Einstein gravity and supergravity, one learns new aspects of quantum field theory that are obscured when using units in which Planck’s constant is set to unity. The scattering amplitude approach provides a self-contained framework for deriving the two-body scattering valid in all regimes of energy. There is hope that the full impact of amplitude computations in this field may significantly alter the way in which gravitational wave predictions will advance in the coming years.

Keywords: scattering amplitude, classical general relativity, perturbative expansion in general relativity

(Some figures may appear in colour only in the online journal)
1. Introduction

The observation of gravitational waves radiated by binary systems of massive astrophysical objects has opened a new and exciting astrophysical avenue for testing Einstein’s theory of gravity. To unlock the full discovery potential of gravitational-wave astrophysics, and to keep abreast with modern observational advancements, the development of new analytical methods is now urgently needed. This prompts for both refinements and complements to the existing theoretical framework as well as a general call for new and more efficient methods of computation. Based on the evident advantages of relativistic quantum field theory, it has been suggested to focus on the latter by means of an application of modern amplitude techniques to the post-Minkowskian expansion of general relativity [1–5]. During the inspiral phase the gravitational field is weak and perturbation theory can reliably be applied up to a few cycles before the merging phase. Progress has been swift. For the case of non-spinning black holes the relativistic amplitude analysis has run from the second post-Minkowskian order in the above references to third post-Minkowskian order [6–18]. Most recently, the relativistic amplitude approach to general relativity has even reached fourth post-Minkowskian order [19, 20] and, in the probe limit, all the way up to fifth post-Minkowskian order [21]. In this brief review we will attempt to describe how this rapid sequence of events unfolded. The corresponding case of spinning black holes is evidently of great phenomenological importance but the corresponding description in terms of the amplitude approach to general relativity is far more complex compared to the non-spinning case. We shall not be able to cover that fascinating story of spin but refer to some relevant papers here [22–41]. There is also a parallel development to the post-Minkowskian expansion based on the world-line approach [41–47] that we cannot cover here, but we will make some brief comments on the relation with the velocity cut formalism in section 5.2.

Key to progress in this interdisciplinary field of scattering amplitudes and classical gravity is that long-distance particle exchanges of gravitons between matter lines can be uniquely tied to observables in general relativity. This allows for on-shell amplitude methods to yield new ways of efficient computation in classical general relativity, thus prompting the radical viewpoint of classical general relativity being suitably defined perturbatively from a path-integral loop expansion. What at first may have looked like a technicality and perhaps a promising new research direction has turned out to lead to an advancement of our understanding of quantum field theory itself. Particularly surprising is the manner in which competing factors of Planck’s constant $\hbar$ conspire to combine into purely classical observables. The discovery of this fundamental property of the quantum field theoretic loop expansion has been prompted by early derivations of the classical two-body gravitational interactions from amplitudes [48] and only recently systematized to all loop order [3, 49, 50]. Interestingly, although we shall seek only classical observables, quantum mechanical unitarity will be seen to play a crucial role. Combined with integrand localization on what has been dubbed velocity cuts [15] and an exponential representation of amplitudes in the semi-classical limit it provides a most efficient procedure for the extraction of the classical long-distance parts of scattering amplitudes. An alternative approach, with certain similarities, is based on an heavy-mass effective field theory expansion [18]. An alternative track follows from the evaluation of expectation values [50] rather than a computation of a scattering amplitude per se. Finally, the eikonal formalism, based on the exponentiation of the gravitational scattering amplitude in impact parameter space [10, 12, 13, 51–54], provides an independent procedure for separating classical physics from quantum mechanical scattering amplitudes. The existence of these different field theoretic avenues highlights the richness of the problem at hand. In fact, it is remarkable that Einstein’s classical theory of general relativity turns out to be more easily solved (at least for the purpose of the post-Minkowskian expansion) by all these different
quantum field theoretic methods than by solving the classical equations of motion directly. Even more surprisingly, several of the mathematical structures known from the quantum mechanical loop amplitudes do survive upon taking the classical limit, thus implying a deep connection between solutions to the differential equations from Einstein’s non-linear field equations and these mathematical structures. This would presumably only with great difficulty be understandable without the link to quantum mechanical scattering amplitudes and their associated loop expansions.

The amplitude approach to gravitational scattering of two black holes ignores what can be called the internal structure of black holes. We do not seek a quantum mechanical description of two black holes interacting gravitationally. Because only large-distance scattering (and a weak coupling expansion in Newton’s constant $G_N$) will be considered, black holes are treated as point-like massive objects without a horizon. This is a simple observation and there is no need to invoke more complicated argumentation for this obvious separation of scales. We shall always imagine the two black holes as being separated by enormous distances while scattering off each other. Nevertheless, once this scattering regime is under control [55, 56] one can attempt to approach also the bound state regime through the effective potential [57] by analytic continuation. The scattering angle, which has been derived in many independent methods in different regimes [8, 9, 12–14, 42, 47, 55, 57], is the main link for connecting the scattering amplitudes to the dynamics of the two-body system. The effective one-body (EOB) formalism [58, 59] has historically proven remarkably efficient in extending the validity range of the post-Newtonian expansion. It is reassuring that this EOB formalism is robust under an extension to the post-Minkowskian regime [1, 2]. One will feed in this formalism the exact expressions from the scattering amplitudes valid in all regimes of energy. Being clearly coordinate-dependent there is much freedom in choosing such an EOB metric. From our viewpoint, it is interesting that an EOB formalism exists where an energy-dependent one-body metric exactly reproduces the scattering angle up to third post-Minkowskian order while at the same time being determined immediately by the effective potential of the two-loop scattering amplitude [60]. As the accuracy of the post-Minkowskian increases, it becomes phenomenologically relevant to explore the consequences of post-Minkowskian EOB formalisms in terms of gravitational wave predictions. The amplitude formalism is indeed a natural framework for investigating the most fundamental principles of Einstein’s theory of gravity while at the same time establishing new predictions that can be tested by observations.

2. Classical gravitational scattering from quantum field theory

2.1. Einstein’s theory of gravity coupled to scalars

Our starting point is the Einstein–Hilbert action of gravity coupled to matter through the energy–momentum tensor $T_{\mu\nu}$,

$$S = \int d^4 x \sqrt{-g} \left[ \frac{R}{16\pi G_N} + g^{\mu\nu} T_{\mu\nu} \right].$$

Here, Newton’s constant is denoted by $G_N$, the Ricci scalar is $R$, and we define a weak field expansion around the flat Minkowski space–time metric $\eta_{\mu\nu}$, by

$$g_{\mu\nu}(x) \equiv \eta_{\mu\nu} + \sqrt{32\pi G_N} h_{\mu\nu}(x).$$
For scalar fields $\phi(x)$, we have the minimal stress–energy tensor

$$T_{\mu\nu} \equiv \frac{\partial_\mu \phi \partial_\nu \phi}{2} - \frac{\eta_{\mu\nu}}{2} \left( \partial_\rho \phi \partial_\rho \phi - m^2 \phi^2 \right).$$

We shall only be concerned with the scattering of two massive scalar fields coupled to gravity labelled by

$$\varphi_1(p_1, m_1), \varphi_2(p_2, m_2) \rightarrow \varphi_1(p'_1, m_1), \varphi_2(p'_2, m_2),$$

where incoming momenta have been denoted by $p_i$ and outgoing momenta by $p'_i$ with $i = 1, 2$. The on-shell conditions are $p_i^2 = p'_i^2 = m_i^2$.

Now comes an interesting observation. We know that general relativity is a non-linear theory where all physical quantities (such as scattering angles, periastron shifts, time delays, etc) do not truncate at linear order in $G_N$. Let us here focus on the scattering angle. For two-body gravitational scattering in the post-Minkowskian expansion we must thus compute the two-to-two scattering amplitude in an expansion in $G_N$. To leading order this is given by a one-graviton exchange as shown in the first diagram below

In the non-relativistic limit this of course just gives rise to the Newtonian potential. Although the theory appears linear at this level, the scattering angle is a non-trivial function of $G_N$ (it is the famous arctan-formula of Newtonian scattering) which will have an infinite-order expansion in $G_N$. But we do know from elementary considerations of Einstein’s equations of motion that there must be corrections to that formula. How can such corrections appear if we compute the scattering in terms of Feynman diagrams? There is only one possibility: without gravitational radiation it is immediately evident that this must entail the computation of scalar four-point $L$-loop scattering processes (with $L$ arbitrarily large). There are simply no other ways to increase the power of $G_N$ while keeping fixed the number of external legs at four for the two-body problem. So there must be classical contributions to the scattering process residing in the loops! This simple argument explains why loop amplitudes for gravity necessarily must contain classical pieces despite the folk-theorem that claims loops to be of quantum origin only (see [61]). We shall return to this important point in some detail below where we shall identify precisely those parts of the loop integrations that give rise to eventually classical contributions to the scattering.

For the scattering matrix $\mathcal{M}(p_1, p_2, p'_1, p'_2)$ we use the following conventions. We denote incoming momenta by $p_1$ and $p_2$ and outgoing moment by $p'_1$ and $p'_2$. The $\gamma$-factor of the relative velocity is related to the momenta through

$$\gamma \equiv \frac{p_1 \cdot p_2}{m_1 m_2}.$$  

The invariant momentum transfer is defined as usual by

$$q^2 \equiv (p_1 - p'_1)^2 \equiv (p_2 - p'_2)^2.$$  

$^3$ For reasons that are obscure to us it has become common in the gravity-amplitude community to denote the $\gamma$-factor of the relative velocity by $\sigma$. Here we use the normal notation $\gamma$. 


and we also introduce the center-of-mass energy
\[ E_{\text{CM}}^2 \equiv (p_1 + p_2)^2 \equiv (p_1' + p_2')^2 = m_1^2 + m_2^2 + 2m_1m_2\gamma. \] (8)

With these conventions we have
\[ q \cdot p_1 = -q \cdot p_1' = \frac{q^2}{2}. \] (9)

We schematically define the four-point amplitudes \( \mathcal{M}(\gamma, q^2) \) expanded into loop amplitudes \( \mathcal{M}_L(\gamma, q^2) \) of order \( G_N^{L+1} \) by
\[ = \mathcal{M}(\gamma, q^2) = \sum_{L=0}^{\infty} \mathcal{M}_L(\gamma, q^2). \] (10)

### 2.2. Graviton unitarity cuts

In principle, loop amplitudes for scalars interacting gravitationally can be straightforwardly enumerated in terms of a standard Feynman diagram expansion. In practice, such a brute-force approach was abandoned long ago, and instead we today rely almost exclusively on generalized unitarity methods (see [62, 63] for some reviews). Although quantum mechanical in nature, unitarity more generally can be used to relate products of tree amplitudes to amplitudes of loops. This can be done in stages, from an ordinary Cutkosky cut to generalized unitarity cuts where more and more internal propagators are taken on-shell. Such generalized unitarity methods are able to produce, step-by-step, all those contributions from loop amplitude which are ‘cut constructible’ [64, 65]. All non-analytic elements of loop amplitudes can be obtained in this manner (and the remaining rational terms can be extracted with a bit more effort). Generalised unitarity thus provides an immediate simplification of amplitude computations. In fact, most precision calculation of scattering cross sections in the Standard Model of particle physics today rely on this. Most importantly for the present purposes is the fact that in order to obtain the long-distance classical gravity contributions to the scattering of two heavy objects we should discard all analytical pieces. Those terms correspond to ultra-local pieces after a Fourier transform. This means that generalized unitarity methods are almost ideally suited for the calculation of classical gravitational scattering. Physically, we can understand this from the fact that long-distance gravitational effects literally can be viewed as coming from those parts of the loop diagrams where the exchanged (virtual) gravitons are almost on mass shell and thus propagating over large distances. For massive scattering, these parts of the amplitude are functions of dimensionless ratios such as \( m/\sqrt{-q^2} \), where \( m \) is a heavy mass scale.

Let us illustrate our way of using unitarity to reconstruct the non-analytical parts of the four-point scattering amplitude. An \((L + 1)\)-graviton cut is defined by\(^4\)}
\[ i \mathcal{M}_{L+1}^{\text{cut}}(\gamma, q^2) \equiv \mathcal{M}_{L+1}(q, L+1) = \int (2\pi)^d \delta(q + L) \prod_{i=2}^{L+2} \int \frac{d^d \ell_i}{(2\pi)^d} \]

\[
\frac{1}{(L+1)!} \sum_{h_i = \pm 2} \mathcal{M}_{\text{left}}^{\text{tree}}(p_1, \ell_2, \ldots, \ell_{L+2}, -p'_1) \mathcal{M}_{\text{right}}^{\text{tree}}(p_2, -\ell_2, \ldots, -\ell_{L+2}, -p'_2) \]

which can be represented by

\[
\mathcal{M}_{L+1}^{\text{cut}}(\gamma, q^2) =
\]

where the middle dots represent \( L - 1 \) additional cut graviton lines. The amplitudes \( \mathcal{M}_{\text{left}}^{\text{tree}}(p_1, \ell_2, \ldots, \ell_{L+2}, -p'_1) \) and \( \mathcal{M}_{\text{right}}^{\text{tree}}(p_2, -\ell_2, \ldots, -\ell_{L+2}, -p'_2) \) denote tree-level multi-graviton emission from a massive scalar line. The two pieces are glued together with the inserted propagator factors so that indeed the full expression provides the cut of the shown part of the amplitude stemming from only graviton exchanges between the two massive legs.

Note that we have universal conventions with all graviton lines incoming in the left tree factor and out-going in the right tree factor, i.e.

\[
q = p_1 - p'_1 = -p_2 + p'_2 = -\sum_{i=2}^{L+2} \ell_i.
\]

These shown cut diagrams clearly describe only a subset of all diagrams for the full amplitude. We can understand intuitively from the above arguments that these should contain the bulk of the long-distance non-analytical parts. Indeed, it was shown in references [15, 16] that the classical contributions from the full amplitude at one and two loop level can be calculated from these \((L = 1)\) and \((L = 2)\) cuts except for a small set of diagrams that are not amenable to three-graviton cuts. These diagrams, which happen to vanish in maximal supergravity, must be included if we want the complete classical part of the amplitude in Einstein gravity. They correspond to self-energy and vertex corrections (some, leaping over two vertices are denoted as ‘mushroom diagrams’) and share the common property of always involving a graviton exchange from one massive line back to itself. Physically, this will clearly correspond to what can be considered radiation-reaction terms: it is the gravitational field reacting back on the same scalar line. However, not all radiation-reaction terms come from this class of diagrams, a class which, as mentioned above, vanishes in maximal supergravity. The other pieces come from the three-graviton cut, once all classical contributions from the integrations are properly included. This was first realized by a beautiful argument based on analyticity and crossing symmetry in reference [10] and later verified by the first explicit computation of the
full classical part of the two-loop amplitude in reference [15]. We shall discuss this in much greater detail below.

3. The classical potential from a Lippmann–Schwinger equation

It is a classical problem in scattering theory to relate the scattering amplitude $\mathcal{M}$ to an interaction potential $\mathcal{V}$. This is typically phrased in terms of non-relativistic quantum mechanics, but it is readily generalized to the relativistic case. Crucial in this respect is the fact that we shall consider particle solutions to the relativistic equations only. There will thus be, in the language of old-fashioned (time-ordered) perturbation theory, no back-tracking diagrams corresponding to multi-particle intermediate states. This is trivially so since we neither wish to treat the macroscopic classical objects such as heavy neutron stars as indistinguishable particles with their corresponding antiparticles nor do we wish to probe the scattering process in any potential annihilation channel. The classical objects that scatter will always be restricted to classical distance scales. The proper language for the Hamiltonian $H$ and hence potential $\mathcal{V}$ is therefore the so-called Salpeter equation for the two massive scalars. In the center-of-mass frame with three-momenta $p = |\vec{p}_1| = |\vec{p}_2|$, $H = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2} + \mathcal{V}(r, p)$, (14)

which indeed, as seen, excludes antiparticles of the massive states.

It is convenient to introduce the amplitude $\tilde{\mathcal{M}}$ in the non-relativistic normalization convention

$$\tilde{\mathcal{M}}(p, p') \equiv \frac{\mathcal{M}(p, p')}{4E_1E_2},$$

with $p_1 = (E_1, \vec{p})$, $p'_1 = (E_1, \vec{p}')$, $p_2 = (E_2, -\vec{p})$ and $p'_2 = (E_2, -\vec{p}')$. As a first observation one notices that to leading order, and in the non-relativistic limit, the classical potential is simply equal to the amplitude after a Fourier transform:

$$\mathcal{V}(r, p) = \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q} \cdot \vec{r}} \tilde{\mathcal{M}}(p, q).$$

(15)

The extension of this relationship beyond tree-level was established long ago in terms of the Born series. It is most succinctly phrased in terms the Lippman–Schwinger equation [5] that relates the scattering amplitude $\mathcal{M}$ and the potential $\mathcal{V}$ in an exact closed form:

$$\tilde{\mathcal{M}}(p, p') = \mathcal{V}(p, p') + \int \frac{d^3k}{(2\pi)^3} \mathcal{V}(p, k) \mathcal{M}(k, p'),$$

(17)

which is given here in momentum space. This framework is a relativistic extension, through a one-particle Hamiltonian and the associated Salpeter equation, of the conventional approach to determining the interaction potential in perturbation gravity by means of Born subtractions. The Lippmann–Schwinger equation summarizes this procedure in a very clear form and it furnishes in a transparent and systematic manner the needed Born subtractions at arbitrary loop order.

The Lippmann–Schwinger approach is equivalent to the effective field theory approach introduced in reference [4] and pursued in the two-loop calculation of references [6, 8].
This is quite easily shown [54]. First, we can solve the Lippmann–Schwinger equation (17) perturbatively in the potential \( V \) to get
\[
\tilde{M}(\vec{p}, \vec{p}') = \mathcal{V}(\vec{p}, \vec{p}') \\
+ \sum_{n=1}^{\infty} \int \frac{d^{D-1}\vec{k}_1}{(2\pi \hbar)^{D-1}} \frac{d^{D-1}\vec{k}_2}{(2\pi \hbar)^{D-1}} \ldots \frac{d^{D-1}\vec{k}_n}{(2\pi \hbar)^{D-1}} \frac{\mathcal{V}(\vec{p}, \vec{k}_1) \ldots \mathcal{V}(\vec{k}_n, \vec{p}')}{(E_p - E_{k_1} + i\epsilon) \ldots (E_{k_{n-1}} - E_{k_n} + i\epsilon).}
\] (18)
or, by expanding both \( \tilde{M} \) and \( \mathcal{V} \) in powers of \( G_N \) and, for illustration, truncating at one-loop order,
\[
\mathcal{V}_{1\text{PM}}(\vec{p}, \vec{p}') + \mathcal{V}_{2\text{PM}}(\vec{p}, \vec{p}') = \tilde{\mathcal{M}}_{\text{tree}}(\vec{p}, \vec{p}') + \tilde{\mathcal{M}}_{\text{one-loop}}(\vec{p}, \vec{p}') + \tilde{\mathcal{M}}_{\text{B}}(\vec{p}, \vec{p}'),
\] (19)
where the first Born subtraction is given by
\[
\tilde{\mathcal{M}}_{\text{B}}(\vec{p}, \vec{p}') \equiv - \int \frac{d^{D-1}\vec{k}}{(2\pi \hbar)^{D-1}} \frac{\tilde{\mathcal{M}}_{\text{tree}}(\vec{p}, \vec{k}) \tilde{\mathcal{M}}_{\text{tree}}(\vec{k}, \vec{p}')} {E_p - E_k + i\epsilon}.
\] (20)
The solution at one-loop level is thus found recursively as illustrated above and equation (19) provides the one-loop potential \( \mathcal{V}_{2\text{PM}} \).

The effective field theory approach for classical gravity [4] is based on a different principle but the resulting equation for the potential is the same. The starting point of this effective field theory formalism is the parametrization of the potential \( \mathcal{V} \) in an operator basis,
\[
\mathcal{V}(\vec{p}, \vec{p}') = G_N c_1 \left( \frac{p^2 + p'^2}{2} \right) \times \left( \frac{q^2}{\hbar^2} \right)^{D-5} + G_N^2 c_2 \left( \frac{p^2 + p'^2}{2} \right) \left( \frac{q^2}{\hbar^2} \right)^{\frac{D-4}{2}} + \cdots,
\] (21)
including terms here up to one-loop order. Here \( c_i \)-coefficients (chosen here to depend symmetrically on the momenta, as shown) are to be fixed by a matching condition between the effective field theory and the fundamental underlying gravity theory. At \( L \)-loop order it reads
\[
\tilde{\mathcal{M}}_{L-\text{loop}}(\vec{p}, \vec{p}') = \tilde{\mathcal{M}}_{\text{EFT}}^{L_{\text{tree}}+\text{PM}}(\vec{p}, \vec{p}'),
\] (22)
which at tree level reduces to
\[
\tilde{\mathcal{M}}_{\text{tree}}(\vec{p}, \vec{p}') = \tilde{\mathcal{M}}_{\text{PM}}^{\text{EFT}}(\vec{p}, \vec{p}').
\] (23)
Since in the effective theory the first interaction term is the contact term given in \( D = 4 \) dimensions by
\[
\mathcal{V}(\vec{p}, \vec{p}') = G_N c_1 \left( \frac{p^2 + p'^2}{2} \right) \times \left( \frac{q^2}{\hbar^2} \right)^{-1},
\] (24)
this fixes \( c_1 \) to be what we already knew: the tree-level potential is given entirely by the tree-level amplitude whose precise form is recalled below. So at this stage the first coefficient \( c_1 \) has been fixed. At one-loop order, the matching condition remains unchanged but when expanded in \( G_N \) there will be two contributions to the effective field theory amplitude: one from the
new vertex determined by the so far unknown $c_2$-coefficient, another from the loop (bubble) contribution from the leading-order interaction. The matching condition

$$\tilde{M}_{\text{one-loop}}(\vec{p}, \vec{p}') = M_{\text{2PM}}^{\text{EFF}}(\vec{p}, \vec{p}'),$$

is easily shown to give

$$\tilde{M}_{\text{one-loop}}(\vec{p}, \vec{p}') = V_{\text{2PM}}(\vec{p}, \vec{p}') - \tilde{M}_B(\vec{p}, \vec{p}').$$

We learn that the first Born subtraction term $\tilde{M}_B(\vec{p}, \vec{p}')$ is simply the bubble graph of the one-loop effective field theory. Moreover, the new coefficient $c_2$ has now been fixed by the remaining parts of the one-loop amplitude. This equivalence between the Lippmann–Schwinger and effective field theory approaches is easily shown to hold to all orders [54].

Neither the Lippmann–Schwinger approach nor the effective field theory approach outlined here have been pursued beyond two-loop order. Instead, alternative strategies have been pursued that are closer in spirit to the eikonal formalism. Nevertheless, the subtraction schemes that have been learned from the Lippmann–Schwinger and effective field theory methods do have analogues in the alternative methods—in interesting ways. This will be discussed below.

### 4. The $\hbar$ expansion from the exchange of gravitons between matter lines

At a given order in perturbation theory there are exchanges of gravitons (curly lines) between massive external matters (solid lines)

as well as diagrams with gravitons beginning and terminating on the same matter lines. Graviton self-energy diagrams can be discarded since they will never contribute to the classical result: even if included, their contributions will always be cancelled when computing the scattering angle.

A standard textbook argument states that the $L$-loop contribution should be of order $M_L(\gamma, q^2) = O(h^{-1})$ [61]. However, a different scaling emerges when keeping the wave-number $q = q/\hbar$ fixed and taking both the $\hbar \to 0$ and the small momentum transfer $q \to 0$ limits [3, 49, 50]. The $L$-loop two-body scattering amplitude has a Laurent expansion around four dimensions [15]

$$M_L(\gamma, q^2, \hbar) = \frac{M_L^{(-L-1)}(\gamma, q^2)}{h^{L+1}|q|^{2L+2}} + \cdots + \frac{M_L^{(-1)}(\gamma, q^2)}{h^{L}|q|^{2L+2-L}} + O(\hbar^0).$$

The part of the amplitude that will contribute to the classical interactions depicted in the above figure is the contribution of order $1/\hbar$. The full quantum amplitude contains three types of contributions: (1) a term of order $1/\hbar^r$ with $2 \leq r \leq L + 2$ that are more singular than the
classical piece, (2) a classical piece of order $1/\hbar$ and (3) quantum corrections of order $\hbar^r$ with $r \geq 0$. All these contributions are constrained (and in fact dictated) by unitarity of the $S$-matrix, as we will explain in section 6.

4.1. Tree level and one-loop amplitudes in Einstein gravity

The tree-level amplitude is given by

$$M_0(\gamma, q^2, \hbar) = \frac{2\pi m_1^2 m_2^2 G_N (2\gamma^2 - 1)}{|q|^2} + O(h^0).$$

(28)

In references [15, 66] the one-loop two-body scattering amplitude was computed by means of two-particle cuts (in the following we employ the notation in those papers)

$$M_1(\gamma, q^2, \hbar) = M_1(\gamma, q^2, \hbar) + M_1(\gamma, q^2, \hbar) + M_1(\gamma, q^2, \hbar) + O(h^0).$$

(29)

The one-loop amplitude can be decomposed terms of master integrals as follows

$$M_1(\gamma, q^2, \hbar) = M_1^{(0)}(\gamma, q^2) + M_1^{(1)}(\gamma, q^2) + M_1^{(2)}(\gamma, q^2),$$

(30)

with coefficients provided by the two-graviton unitarity cut. The amplitude has a Laurent expansion in $\hbar$ as explained above. Including the first quantum correction it reads

$$M_1(\gamma, q^2, \hbar) = \frac{1}{|q|^{d-2}} \left( \frac{M_1^{(2)}(\gamma, q^2)}{h^2} + \frac{M_1^{(1)}(\gamma, q^2)}{h} + M_1^{(0)}(\gamma, q^2) + O(h) \right),$$

(31)

with

$$M_1^{(0)}(\gamma, q^2) = M_1^{(0)}(\gamma, q^2),$$

$$M_1^{(1)}(\gamma, q^2) = M_1^{(1)}(\gamma, q^2) + M_1^{(1)}(\gamma, q^2) + M_1^{(1)}(\gamma, q^2),$$

$$M_1^{(2)}(\gamma, q^2) = M_1^{(2)}(\gamma, q^2) + M_1^{(2)}(\gamma, q^2) + M_1^{(2)}(\gamma, q^2) + M_1^{(2)}(\gamma, q^2).$$

(32)

Defining amplitudes in $b$-space by a suitably normalized Fourier transform,

$$\tilde{M}(\gamma, b) = \frac{1}{4m_1 m_2 \sqrt{\gamma^2 - 1}} \int_{R^{d-2}} \frac{d^{d-2}q}{(2\pi)^{d-2}} M(\gamma, q^2, \hbar) e^{i\vec{q} \cdot \vec{b}}.$$

(33)
we find the classical and leading quantum pieces from $\mathcal{M}_1^{(-1)}$ and $\mathcal{M}_1^{(0)}$, respectively. The classical part is

$$\tilde{\mathcal{M}}_1^{\text{Cl}} (\gamma, b, \hbar) = \frac{3\pi G^2 (m_1 + m_2) m_1 m_2 (5\gamma^2 - 1)}{4b \sqrt{\gamma^2 - 1}} \langle \pi b^2 e^{-\pi \sqrt{\gamma^2 - 1}} \rangle^{4-D} + O(4-D),$$

while the leading quantum correction reads

$$\tilde{\mathcal{M}}_1^{\text{Qt}} (\gamma, b) = \frac{G^2 (\pi b^2 e^{-\pi \sqrt{\gamma^2 - 1}})^{4-D}}{b^2} \left( \frac{4 - D (2\gamma^2 - 1)^2 e_{\text{CM}}^2}{\pi (\gamma^2 - 1)^2} - \frac{m_1 m_2}{(\gamma^2 - 1)^2} \frac{1 - 49\gamma^2 + 18\gamma^4}{15} \right) + O((4-D)^2).$$

The dimension-dependent pieces of these amplitudes will feed into certain contributions at the next loop order. They are needed for a proper identification of the classical part of the two-loop amplitude in (44).

4.2. The two-loop amplitude for Einstein gravity

The two-loop amplitude

$$\mathcal{M}_2 (\gamma, q^2) = \mathcal{M}_2^{\text{three-cut}} (\gamma, q^2) + \mathcal{M}_2^{\text{SE}} (\gamma, q^2),$$

receives a contribution from the three-particle cut, $\mathcal{M}_2^{\text{three-cut}} (\gamma, q^2)$, with only gravitons propagating across the cut, plus a set of diagrams with a graviton beginning and terminating on the same matter lines, $\mathcal{M}_2^{\text{SE}} (\gamma, q^2)$. This includes massive self-energy diagrams, vertex corrections, and so-called mushroom diagrams. Their contributions to the two-loop amplitude have collectively been denoted by $\mathcal{M}_2^{\text{SE}} (\gamma, q^2)$. The three particle-cut is evaluated in dimension $D$, with $D > 4$ for regulating the infrared divergences of the classical part.

$$\mathcal{M}_2^{\text{cut}} (\gamma, q^2, \hbar) = \text{tree} \quad \text{tree}$$

$$= \int \frac{d^D l_1 d^D l_2 d^D l_3}{(2\pi)^D 3!} (2\pi)^D \delta(D) (l_1 + l_2 + l_3 + q)^{\frac{1}{2}} \frac{1}{l_1 l_2 l_3}$$

$$\times \frac{1}{3!} \sum_{\text{perm}(l_1, l_2, l_3)} \mathcal{M}_0 (p_1, p_1', l_1^\lambda, l_2^\lambda, l_3^\lambda) \mathcal{M}_0 (p_2, p_2', -l_1^\lambda, -l_2^\lambda, -l_3^\lambda)$$

which involves two five-point tree-level amplitudes. The sum is over the physical states across the cut and can conveniently be done employing spinor-helicity variables $\lambda_i$ or using expressions for covariant tree amplitudes as outlined in [21] which is based on the representation of tree amplitudes discussed in references [67, 68]. As explained in the introduction, we need only
the cut-constructible part of the amplitude in order to extract the long-range classical contributions. The $h$-counting established in equation (3.6) of [15], makes it clear that at least two massive propagators are needed in order to obtain a classical contribution from the amplitude. Employing a partial-fraction decomposition of the tree-level amplitudes with references to the linear propagators $p_i \cdot l_i$ and $p_{2i} \cdot l_i$ with $i = 1, 2, 3$, the three-particle cut can be reorganized into five specific topologies that contribute to the classical result

$$\mathcal{M}_2^{\text{cut}}(\gamma, \frac{q^2}{2}, h) = \mathcal{M}_2^{\text{cut}} + \mathcal{M}_2^{\text{cut}} + \mathcal{M}_2^{\text{cut}} + \mathcal{M}_2^{\text{cut}} + \mathcal{M}_2^{\text{cut}}. \quad (38)$$

The two-loop amplitude for Einstein gravity involves integrals with non-trivial numerators and corresponding to many graph topologies. It is a remarkable fact that the reduction to master integrals (for instance using the automatic integral reduction programme \textbf{LiteRed} [69]) involves only nine such basis integrals.

As mentioned above, the three-particle graviton cut does not provide the full two-loop contribution to classical gravitational scattering in Einstein gravity. Contributions from diagrams with one graviton beginning and terminating on the same matter lines are needed as well. We write them as

$$\mathcal{M}_2^{\text{self-energy}}(\gamma, \frac{q^2}{2}) = -4(16\pi G_N)^3 \sum_{i=1}^{IV} (J_{SE}^{\gamma} + J_{SE}^{\alpha}) + (m_1 \leftrightarrow m_2), \quad (39)$$

where the integrals $J_{SE}^{\gamma}$ are given in equations (40)–(43) below. The contributions $J_{SE}^{\alpha}$ are obtained by the exchange of the legs $p_2$ and $p_2'$ and there are of course also all the symmetric contributions with a graviton line beginning and terminating on the massive line of mass $m_2$.

$$J_{SE}^{\gamma} = \int \frac{d^D l_1 d^D l_2}{(2\pi)^D} \frac{(16\pi G_N)^3 h^7}{(p_1 - l_1 - l_2)^2 - m_1^2 + i\epsilon} \times \frac{m_1^4 m_2^4 (2\sigma^2 - 1)^2 + 2m_1^4 m_2^4 (2\sigma^2 - 1)^2 |h| g}{(p_1 - l_1 - l_2 - q)^2 - m_1^2 + i\epsilon} \frac{1}{(p_1 - l_1 - l_2 - q)^2 - m_1^2 + i\epsilon} \frac{1}{(p_2 - l_1 - l_2 - q)^2 - m_2^2 + i\epsilon} \frac{1}{(l_1 + q)^2 (l_1 + l_2)^2}, \quad (40)$$

$$J_{SE}^{\alpha} = \int \frac{d^D l_1 d^D l_2}{(2\pi)^D} \frac{(16\pi G_N)^3 h^7 (m_1^4 m_2^4 (2\sigma^2 - 1)^2)}{(p_1 - l_1 - l_2)^2 - m_1^2 + i\epsilon} \frac{1}{(p_1 - l_1 - l_2)^2 - m_1^2 + i\epsilon} \frac{1}{(p_2 - l_1 - l_2)^2 - m_2^2 + i\epsilon} \frac{1}{(l_1 + q)^2 (l_1 + l_2)^2}, \quad (41)$$
The complete classical part of the two-loop amplitude (36) is then finally given by

\[
M_2^{(\gamma, q^2)} |_{\text{classical}} = 4(4\pi e^{-\gamma E})^4 \frac{G_N^3 m_1^2 m_2^2}{2(2\pi)^{2D}} \left( \frac{16\pi G_N^3 h^7 (m_1^2 m_2^2 (2\sigma^2 - 1)^2)}{((p_1 + l_1)^2 - m_1^2 + i\varepsilon)((p_2 - l_2)^2 - m_2^2 + i\varepsilon)} \right) 
\]

\[
\times \frac{1}{((p_1 - l_1 - l_2 - q)^2 - m_1^2 + i\varepsilon)(m_1^2 + i\varepsilon)(l_1 + q)^2(l_1 + l_2)^2},
\]

(42)

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\]

\[
\times \frac{1}{((p_1 - l_1 - l_2 - q)^2 - m_1^2 + i\varepsilon)(m_1^2 + i\varepsilon)(l_1 + q)^2(l_1 + l_2)^2},
\]

(43)

In the last line we have used a curiously simplified way of expressing part of the result as a \(\gamma\)-derivative. Writing it this way is an \textit{a posteriori} observation that does not fall out of our way of computing this part. It should be noted that there are, perhaps not surprisingly, both real and imaginary parts of the classical part of the amplitude. In particular, the real and imaginary parts of the last line of (44),

\[
M_2^{(\gamma, q^2)} |_{\text{classical}} = \frac{4(4\pi e^{-\gamma E})^4 \pi G_N^3 m_1^2 m_2^2}{3(4 - D) q^2} \left( \frac{3(2\gamma - 1)^3 \xi^2_{\text{CM}}}{(\gamma^2 - 1)^2} \right) 
\]

\[
+ \frac{2m_1 m_2 (2\gamma^2 - 1)}{\pi(4 - D)(\gamma^2 - 1)^2} \left( 1 - 49\gamma^2 + 18\gamma^4 \right) \sqrt{\gamma^2 - 1} 
\]

\[
- \frac{9(2\gamma^2 - 1)(1 - 5\gamma^2) \xi^2_{\text{CM}}}{2(\gamma^2 - 1)} + \frac{3}{2}(m_1^2 + m_2^2)(18\gamma^2 - 1) - m_1 m_2 \gamma(103 + 2\gamma^2) 
\]

\[
+ \frac{12m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4 \arccosh(\gamma))}{\sqrt{\gamma^2 - 1}} - \frac{12m_1 m_2 (2\gamma^2 - 1)^2}{\pi(4 - D) \sqrt{\gamma^2 - 1}} \left( 1 + i\frac{4-D}{4(\gamma^2 - 1)} \right) 
\]

\[
\times \left( \frac{-11}{3} + \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1) \arccosh(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right),
\]

(44)

In the last line we have used a curiously simplified way of expressing part of the result as a \(\gamma\)-derivative. Writing it this way is an \textit{a posteriori} observation that does not fall out of our way of computing this part. It should be noted that there are, perhaps not surprisingly, both real and imaginary parts of the classical part of the amplitude. In particular, the real and imaginary parts of the last line of (44),

\[
M_2^{(\gamma, q^2)} |_{\text{classical}} = \frac{4(4\pi e^{-\gamma E})^4 \pi G_N^3 m_1^2 m_2^2}{3(4 - D) q^2} \left( \frac{3(2\gamma - 1)^3 \xi^2_{\text{CM}}}{(\gamma^2 - 1)^2} \right) 
\]

\[
\times \left[ \frac{12m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4 \arccosh(\gamma))}{\sqrt{\gamma^2 - 1}} \right],
\]

(45)
are seen to satisfy the relation
\[
\lim_{D \to 4} 2(4 - D) R (45) = - \lim_{D \to 4} (4 - D)^2 \pi I (45),
\]
which was first argued on the basis of analyticity and crossing symmetry in [12, 13]. The additional real part from (45) plays a crucial role in the story as will be briefly reviewed below.

5. The scattering angle from the eikonal formalism

In quantum field theory the scattering amplitude provides us with the scattering cross section, computed order by order in perturbation theory by means of the Born expansion. Retaining only the classical information from the quantum mechanical scattering amplitude we might think that we should be able to compute the actual classical trajectories and not just the (classical) cross section. But in general relativity the trajectories of two massive objects scattering off each other will penetrate into regions where the metric will be non-trivial and hence coordinate dependent. Instead, if we compute the classical scattering angle from Minkowski space at far infinity to Minkowski space at far infinity this is an unambiguous and coordinate-independent quantity. The whole quantum mechanical thinking in terms of perturbation theory is in fact ideally set up to describe this situation, and we can immediately proceed even when we retain only the classical information. This is the backbone of the post-Minkowskian expansion which uses Minkowski space to define observables in a manner completely analogous to the quantum field theoretic expansion based on the interaction-picture Hamiltonian and matrix elements evaluated on a basis of free fields.

While the Hamiltonian framework can be used to compute the scattering angle, it is easier at two-loop order to employ the eikonal formalism. For this, one again converts the amplitude to \(b\)-space\(^5\) by performing a Fourier transform with respect to the momentum transfer,

\[
M_L(\gamma, b) = \frac{1}{4m_1 m_2 \sqrt{\gamma^2 - 1}} \int_{\mathbb{R}^{D-2}} \frac{d^{D-2} \vec{q}}{(2\pi)^{D-2}} M_L(\gamma, q^2, \bar{h}) e^{i \vec{q} \cdot \vec{b}}.
\]

The classical eikonal phase \(\delta(\gamma, b)\) is defined by an exponentiation of the S-matrix in \(b\)-space,

\[
1 + i T = (1 + i 2 \Delta(\gamma, b, \hbar)) e^{\frac{2i \delta}{\hbar}},
\]

where all other terms, order by order in perturbation theory, are kept at linear level and lumped into \(\Delta(\gamma, b, \hbar)\) as shown. Contrary to what one might think naively, this does not imply that this quantity \(\Delta(\gamma, b, \hbar)\) simply contains all quantum mechanical bits of the amplitude. Indeed, by expanding the exponent it is immediately evident that the resulting Laurent expansion in \(\hbar\) will combine non-trivially with the terms left at linear level through \(\Delta(\gamma, b, \hbar)\). The fact that the amplitude in \(b\)-space displays such a neat exponentiation is in fact a manifestation

\(^5\)This is not the impact parameter \(b_J\) orthogonal to the asymptotic momentum in the center-of-mass frame. The relation between the two quantities is \(b_J = b \cos(\chi/2)\) [9, 13].
of unitarity of the \( S \)-matrix \[54\]. In fact, one can easily understand the exponentiation of the \( b \)-space amplitude as the eikonal analog of introducing Born subtractions.

By construction, the eikonal phase \( \delta(\gamma, b) \) is independent of \( \bar{\hbar} \) and has a perturbative expansion in \( G_N \) through the sum over loops,

\[
\delta(\gamma, b) = \sum_{L \geq 0} \delta_L(\gamma, b),
\]

which is connected to the Laurent expansion in \( \bar{\hbar} \) of the scattering amplitude in \( (27) \) in \( b \)-space,

\[
1 + iT = 1 + i \sum_{L \geq 0} M_L(\gamma, b, \bar{\hbar}).
\]

Having carefully extracted the classical eikonal contribution at a given loop order we can then in principle evaluate the scattering angle at this order in perturbation theory by the saddle-point condition

\[
\sin(\chi_L^{\text{PM}}) = - \frac{\bar{\xi}_{\text{CM}}}{m_1 m_2 \sqrt{\gamma^2 - 1}} \frac{\partial \delta_L(\gamma, b)}{\partial b}.
\]

In practice, also this procedure becomes quite involved once accuracy is increased but it works fine up to two-loop order. From the results for the amplitude we quoted above we immediately get

\[
\delta_0(\gamma, b) = G_N m_1 m_2 \frac{2\gamma^2 - 1}{\sqrt{\gamma^2 - 1}} \frac{\pi b^2 e^\gamma}{D - 4} + O((D - 4)^0),
\]

at tree level. At second post-Minkowskian order we likewise get, straightforwardly from the amplitude,

\[
\delta_1(\gamma, b) = G_N^2 (m_1 + m_2) m_1 m_2 \frac{3\pi(5\gamma^2 - 1)}{8b \sqrt{\gamma^2 - 1}} \pi b^2 e^\gamma (\pi b^2 e^\gamma)^{4-D} + O(4 - D),
\]

but at third post-Minkowskian order we need to carefully extract the exponent by taking into account the iterations from lower orders as implied by the eikonal exponentiation formula \( (48) \).

After some algebra, one finds

\[
\delta_2(\gamma, b) = \frac{G_N^3 m_1 m_2 (\pi b^2 e^\gamma)^{\frac{14}{D}}}{2b^2 \sqrt{\gamma^2 - 1}} \left( \frac{2(12\gamma^2 - 10\gamma^2 + 1) C_M^2}{\gamma^2 - 1} \right) \\
- \frac{4m_1 m_2^2}{3} (25 + 14\gamma^2) + \frac{4m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4) \arccosh(\gamma)}{\sqrt{\gamma^2 - 1}} \\
+ \frac{4m_1 m_2 (2\gamma^2 - 1)^2}{\sqrt{\gamma^2 - 1}} \frac{1}{(4\gamma^2 - 1)^{\frac{D}{2}}} \\
\times \left( \frac{11}{3} + \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1) \arccosh(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right) + O(4 - D).
\]

This result was first found in reference \[10\] by means of the relation between real and imaginary parts discussed in the previous section. Here we see how it actually follows directly from the
complete amplitude calculation at third post-Minkowskian order, once all classical parts have been correctly computed.

Expressing the results in terms of angular momentum $J$ through the relation

$$J = \frac{m_1 m_2 \sqrt{\gamma^2 - 1}}{\epsilon_{C.M.}} b \cos \left( \frac{\chi}{2} \right),$$

(55)

we can derive the scattering angle at the first and second post-Minkowskian order

$$\chi_{1PM} = \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{G_N m_1 m_2}{J},$$

$$\chi_{2PM} = \frac{3\pi}{4} \left( \frac{m_1 + m_2}{\epsilon_{C.M.}} \right) (5\gamma^2 - 1) \left( \frac{G_N m_1 m_2}{J} \right)^2.$$ (56)

To this order they can in fact be related to the scattering of a test particle of mass $m_1 m_2/(m_1 + m_2)$ in a static Schwarzschild background of mass $m_1 + m_2$.

At third post-Minkowskian order the result gets more interesting. Collecting the contributions from the three-graviton cuts and the self-energy diagrams the scattering angle reads

$$\chi_{3PM} = \frac{2 \left( 64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5 \right)}{3(\gamma^2 - 1)^2} \left( \frac{G_N m_1 m_2}{J} \right)^3 + \frac{8m_1 m_2 \sqrt{\gamma^2 - 1}}{3\epsilon_{C.M.}} (-\gamma(25 + 14\gamma^2))$$

$$+ \frac{3(3 + 12\gamma^2 - 4\gamma^4) \text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \left( \frac{G_N m_1 m_2}{J} \right)^3$$

$$+ \frac{1}{(4(\gamma^2 - 1))^{\frac{1}{2}}} \left( \frac{11}{3} \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1) \text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right)$$

$$\times \frac{4m_1 m_2 (2\gamma^2 - 1)^2}{\epsilon_{C.M.}^2} \left( \frac{G_N m_1 m_2}{J} \right)^3 + O(4 - D).$$ (57)

At this order the scattering angle deviates from the geodesic scattering of a test particle by the contribution in the second line. The third line of (54) is what can be viewed as radiation-reaction terms in [11, 12, 14, 16]. The classical piece of the two-loop amplitude without gravitational radiation is the sum of all these contributions. Because of infrared singularities the scattering amplitude has been dimensionally regularized at intermediate stages but the final classical result is infrared finite in the limit $D \to 4$. If we had computed the amplitude in a truncated region of integration such as that of the so-called potential region we would have missed these additional classical terms. In fact, it is not natural at this third post-Minkowskian order to separate the pieces that a posteriori can be understood as radiation-reaction terms from the other pieces because all we do is to compute the complete classical contribution from the two-to-two scattering amplitude. At fourth post-Minkowskian order the amplitude even contains a divergent piece in four dimensions if the integrations are restricted to the so-called potential region [19]. It is only cancelled after the inclusion of all diagrams and an inclusion of all integration regions that contribute to the classical result [20]. The radiation-reactions contributions have also been derived using different amplitude based methods: (1) high-energy scattering
[10, 12], (2) linear response to the angular momentum [11, 70, 71], (3) reverse unitarity and the KMOC\textsuperscript{6} formalism [50, 73–75].

From the scattering angle, one can reconstruct a classical potential \( V = \frac{2 \gamma^2}{\sqrt{\gamma^2 - 1}} \) that produces it by matching the expression for the angle from the Hamiltonian formalism [1, 56]. We will return to this below.

5.1. Maximal supergravity

Although not of physical interest it is nevertheless illuminating to compare the results of Einstein gravity with the case of the two-body scattering in maximal supergravity [9, 10, 15, 76]. At first post-Minkowskian order the eikonal phase of \( N = 8 \) supergravity is given by

\[
\delta_{N=8}^1(\gamma, b) = m_1 m_2 G_N \frac{2 \gamma^2}{\sqrt{\gamma^2 - 1}} \frac{(b \sqrt{\pi})^{4-D}}{D-4} + \mathcal{O}(D-4)^0). \tag{58}
\]

This expression is quite similar to the one for Einstein gravity in (52) with the replacement of \( 2 \gamma^2 - 1 \) by \( 2 \gamma^2 \) in the numerator because of cancellations. The external states are massive half-BPS states constructed by Kaluza–Klein reduction [76], and the scattering angle is independent of the relative orientation of the momenta in the extra dimensions [9, 10, 15]. Such supersymmetric cancellations lead also to the well-known simplification of the one-loop amplitude which is given only by the box integral [77]. Consequently, a vanishing contribution to the one-loop potential ensues and the second-order post-Minkowskian scattering angle equals that of tree level [76] in \( D = 4 \) dimensions. In terms of the eikonal phase one has the one-loop expression

\[
\delta_{N=8}^2(\gamma, b) = G_N^2(\gamma + 1)m_1 m_2 \frac{2 \pi \gamma^4}{b(\gamma^2 - 1)^2}(D-4) + \mathcal{O}((D-4)^2), \tag{59}
\]

which indeed vanishes in four dimensions.

At two-loop order the supersymmetric cancellations make the tensorial reductions much simpler than those of Einstein gravity, but the basis of master integrals is identical to that of Einstein gravity. One finds [15]:

\[
\delta_{N=8}^3(\gamma, b) = \frac{8G_N^3 m_1^2 m_2^2 \gamma^4}{b^3} \left[ \frac{\text{arccosh} (\gamma)}{\gamma^2 - 1} \right. \\
+ \left. \frac{1}{4(\gamma^2 - 1)} \right] \frac{\frac{1}{2(\gamma^2 + 1)}}{d \gamma} \left( \frac{2 \gamma^2 \text{arccosh} (\gamma)}{\sqrt{\gamma^2 - 1}} \right) + \mathcal{O}(D-4) \right]. \tag{60}
\]

The scattering angle at the third post-Minkowskian order in maximal supergravity is thus given by

\[
\chi_{3\text{PM}}^{N=8} = -16 \gamma^4 \left( \frac{\gamma^2}{3(\gamma^2 - 1)^2} + \frac{m_1 m_2}{\epsilon_{\text{C.M.}}^2} \text{arccosh} (\gamma) \right) \left( \frac{G_N m_1 m_2}{J} \right)^3 \\
+ \frac{16 m_1 m_2}{\epsilon_{\text{C.M.}}^2} \frac{\gamma^4}{(4(\gamma^2 - 1))^{\frac{D-2}{2}}} \frac{d}{d \gamma} \left( \frac{2 \gamma^2 \text{arccosh} (\gamma)}{\sqrt{\gamma^2 - 1}} \right) \left( \frac{G_N m_1 m_2}{J} \right)^3 \\
+ \mathcal{O}(D-4). \tag{61}
\]

\textsuperscript{6} See, chapter 14 of this review [72].
The leading ultra-relativistic limit, $\gamma \gg 1$ of the third post-Minkowskian scattering angle in Einstein gravity and maximal supergravity (recall that the center-of-mass depend on $\gamma$ as well) is the same

$$\lim_{\gamma \to \infty} (57) = \lim_{\gamma \to \infty} (61) = \frac{32}{3} \left( \frac{G N m_1 m_2 \gamma}{J} \right)^3 + \mathcal{O}(\gamma^2),$$

as expected by universality of the ultra-relativistic gravitational scattering [10, 13, 53].

5.2. Velocity cuts

Velocity cuts can be viewed as unitarity cuts adapted to the post-Minkowskian expansion. They allow for an efficient extraction of the classical piece of order $1/\hbar$ in the Laurent expansion in (27) of the two-body scattering amplitude. The organisation of the $\hbar$ expansion of the integrand is closely related to the heavy-mass effective field theory adapted to the binary system used in [18].

The idea of velocity cuts relies on the observation that the combination of linear propagators

$$\frac{1}{(p_A \cdot \ell_A + i\epsilon)(p_A \cdot \ell_B - i\epsilon)} \times \frac{1}{(p_B \cdot \ell_B + i\epsilon)(p_A \cdot \ell_A - i\epsilon)} \times \frac{1}{(p_B \cdot \ell_C - i\epsilon)(p_B \cdot \ell_A + i\epsilon)},$$

can be expressed in terms of delta functions

$$\left( \frac{\delta(p_A \cdot \ell_A)}{p_A \cdot \ell_B + i\epsilon} - \frac{\delta(p_A \cdot \ell_B)}{p_B \cdot \ell_A + i\epsilon} \right) \times \left( \frac{\delta(p_B \cdot \ell_C)}{p_B \cdot \ell_A + i\epsilon} - \frac{\delta(p_B \cdot \ell_A)}{p_B \cdot \ell_C + i\epsilon} \right),$$

thanks to the identity

$$\frac{1}{x + i\epsilon} - \frac{1}{x - i\epsilon} = -2i\pi \delta(x).$$

This provides considerable simplifications to reorganise some variety of propagators in the integrand. We can illustrate this by the case of the sum of the one-loop box and crossed box diagrams

$$I_\square = \frac{p_1'}{p_1} \frac{p_2'}{p_2} + \frac{p_1'}{p_1} \frac{p_2'}{p_2},$$

$$= \int \frac{d^D \ell}{(2\pi \hbar)^D} \frac{1}{(\ell^2 + q^2)^2} \left( \frac{1}{(-p_1 + \ell)^2 - m_1^2 + i\epsilon} + \frac{1}{(p_1' + \ell)^2 - m_1^2 + i\epsilon} \right) \times \left( \frac{1}{(-p_2 + \ell)^2 - m_2^2 + i\epsilon} + \frac{1}{(p_2' + \ell)^2 - m_2^2 + i\epsilon} \right).$$
We scale the loop variable $\ell = h|q|l$ and set $q = |q|u_q$, and shift the external momenta $p_1 = p_1 + \frac{h}{2}q$, $p'_1 = p'_1 - \frac{h}{2}q$, $p_2 = p_2 + \frac{h}{2}q$, $p'_2 = p'_2 + \frac{h}{2}q$ to get

$$I_\Box = -\frac{|q|^{D-6}}{8h^2} \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2(k + u_q)^2} \times \left( \frac{1}{p_1 \cdot k + \frac{h|q|u_q k}{2} + i\varepsilon} - \frac{1}{p_1 \cdot k - \frac{h|q|u_q k}{2} - i\varepsilon} \right) \times \left( \frac{1}{p_2 \cdot k + \frac{h|q|u_q k}{2} + i\varepsilon} - \frac{1}{p_2 \cdot k - \frac{h|q|u_q k}{2} - i\varepsilon} \right),$$

(67)

where we have kept the dependence on $|q|$ in the denominators. We then do a small $\frac{h}{2}q$ expansion and neglect tadpoles (they are readily shown to not contribute), to get

$$I_\Box = I_{\text{one-cut}}^{\text{cut}} + \frac{|q|^{D-5}}{16h} \int \frac{d^D l}{(2\pi)^D} \frac{1}{l^2(l + u_q)^2} \left( \frac{\delta(p_2 \cdot l)}{(p_1 \cdot l)^2} + \frac{\delta(p_1 \cdot l)}{(p_2 \cdot l)^2} \right) + \mathcal{O}(|q|^{D-4}).$$

(68)

We have made a repeated use of the identity (65) on the linear propagators from the massive legs, e.g., $(-p_1 + \ell)^2 - m_1^2 + i\varepsilon = -2h(p_1 \cdot l + i\varepsilon) + \mathcal{O}(h^2)$, and defined

$$I_{\text{one-cut}}^{\text{cut}} = \frac{|q|^{D-6}}{4h^2} \left( 1 + \frac{h^2|q|^2 \epsilon^2_{C,M}}{4m_1^2m_2^2(\gamma^2 - 1 - \frac{h^2|q|^2 \epsilon^2_{C,M}}{4m_1^2m_2^2})} \right)^{\frac{D}{2}} \int \frac{d^D k}{(2\pi)^D} \frac{\delta(p_1 \cdot k)\delta(p_2 \cdot k)}{k^2(k + u_q)^2},$$

(69)

which evaluates to

$$I_{\text{one-cut}}^{\text{cut}} = \frac{|q|^{D-6}}{4h^2m_1m_2\sqrt{\gamma^2 - 1}} \left( 1 - \frac{h^2|q|^2 \epsilon^2_{C,M}}{4m_1^2m_2^2(\gamma^2 - 1)} \right)^{\frac{D}{2}} \frac{\Gamma(D-4)^2 \Gamma(\frac{D}{2}) \Gamma(\frac{6-D}{2})}{(4\pi)^{\frac{D}{2}} \Gamma(D) \Gamma(D-4)}. $$

(70)

Because of our conventions, this is the real part of the box integral which is related, by unitarity, to the phase-space integral over two trees. Remarkably, it can be evaluated in the soft region by an explicit resummation of the $q$-expansion. The final result has the highly compact form shown. The second term in (68) corresponds to the next-to-leading contribution of equation (2.27) of [15] and leads to a classical contribution from the $D$-dimensional one-loop amplitude.

The velocity cuts impose delta functions on the massive scalar legs. With these the delta-function insertions the Feynman integrals that arise are identical to the ones from the world-line formalism in [41–44, 46, 47]. The correspondence, illustrated in (71), between the two formalism arises thanks identity (65) that rewrites linear propagators from the Feynman prescription as the retarded Feynman propagators plus a delta-function.

This observation can be used to systematize the $h \to 0$ expansion of the integrand in terms of a multiple-soft graviton limit of tree amplitudes. As will be shown below, it permits the re-grouping of the integrand according to unitarity relations in an exponential representation of the $S$-matrix. This in turn will identify the precise subtractions from amplitude that leaves
The tree-level graviton emission amplitude has the universal behaviour

\[
M \quad \text{can be rewritten as (setting } M \text{)}
\]

The classical limit

5.3. A multi-soft graviton expansion

The classical limit \( h \to 0 \) of the scattering amplitude in (27) can be organized using a multi-soft limit of the tree-level amplitudes for the emissions of gravitons from a massive scalar line in the cut integral in (11).

Let us illustrate by considering the tree-level amplitudes \( M_{\ell+1}^{\text{tree}}(p, \ell_2, \ldots, \ell_{L+2}, -p') \) and make the substitution \( \ell_i \to \vec{h}|\vec{q}i \ell_i \) as \( h \to 0 \), so that

\[
q = p - p' = -h|\vec{q}| \sum_{i=2}^{L+2} \vec{\ell}_i.
\]

The tree-level graviton emission amplitude has the universal behaviour

\[
\lim_{|\vec{q}| \to 0} M_{\ell+1}^{\text{tree}}(p, h|\vec{q}|, \ell_2, \ldots, h|\vec{q}|, \ell_{L+2}, -p') \propto (|\vec{q}|)^{-L}.
\]

By repeated use of the identity

\[
\frac{1}{(p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q)^2 - m^2 + i\epsilon} = -2\pi i \delta((p_1 - \ell_{i_2} - \cdots - \ell_{i_j} - q)^2 - m^2)
\]

we can rewrite the four-point tree-level amplitudes as (setting \( \tilde{\delta}(x) \equiv -2\pi i \delta(x) \))

\[
M_{\ell}^{\text{tree}}(p_1, \ell_2, \ell_3, -p_1') = M_{\ell}^{\text{tree}}(p_1, \ell_2, \ell_3, -p_1')
\]

\[
+ \tilde{\delta}((p_1 + \ell_2)^2 - m_1^2)M_{\ell}^{\text{tree}}(p_1 + \ell_2, -p_1 - \ell_2, \ell_1, \ell_4, -p_1'),
\]

and in the five-point case

\[
M_{\ell}^{\text{tree}}(p_1, \ell_2, \ell_3, \ell_4, -p_1') = \tilde{\delta}((p_1 + \ell_2)^2 - m_1^2)\tilde{\delta}((p_1 + \ell_2 + \ell_4)^2 - m_1^2)
\]

\[
\times M_1^{\text{tree}}(p_1, \ell_4, -p_1 - \ell_4)M_1^{\text{tree}}(p_1 + \ell_4, \ell_2, -p_1 - \ell_4 - \ell_2)M_1^{\text{tree}}(p_1 + \ell_4, \ell_2, \ell_3, -p_1')
\]

\[
+ \tilde{\delta}((p_1 + \ell_2)^2 - m_1^2)\tilde{\delta}((p_1 + \ell_2 + \ell_4)^2 - m_1^2)
\]

\[
\times M_1^{\text{tree}}(p_1, \ell_4, -p_1 - \ell_4)M_1^{\text{tree}}(p_1 + \ell_4, \ell_2, -p_1 - \ell_4 - \ell_2)M_1^{\text{tree}}(p_1 + \ell_4, \ell_2, \ell_3, -p_1')
\]

\[
+ \tilde{\delta}((p_1 + \ell_2 + \ell_4)^2 - m_1^2)M_1^{\text{tree}}(p_1, \ell_2, \ell_4, -p_1 - \ell_2 - \ell_4)M_1^{\text{tree}}(p_1 + \ell_2, \ell_2, \ell_3, -p_1')
\]

\[
+ \tilde{\delta}((p_1 + \ell_3 + \ell_4)^2 - m_1^2)M_1^{\text{tree}}(p_1, \ell_3, \ell_4, -p_1 - \ell_3 - \ell_4)M_1^{\text{tree}}(p_1 + \ell_4, \ell_1, \ell_2, -p_1').
\]
We can have three types of contributions:

- The terms with \( k = L \) delta-functions behave as

\[
\frac{1}{\hbar |q|^{2+D-2L}}.
\]  

In all generality we have an expansion organised by powers of delta-function insertions on the massive lines as given in equation (4.8) of [18] and equation (4.36) of [21]

\[
\mathcal{M}_{L+1}^{\text{tree}} \sim (\mathcal{M}_1^{\text{tree}(+)})^{L+1} \prod_i \delta(\cdots) + (\mathcal{M}_2^{\text{tree}(+)})^{L-1} (\mathcal{M}_2^{\text{tree}(+)})^{L-1} \prod_i \delta(\cdots)
\]

\[
+ \cdots + \mathcal{M}_L^{\text{tree}(+)},
\]

The tree-level amplitudes \( \mathcal{M}_{L+1}^{\text{tree}(+)\pm} (p_1, \ell_2, \ldots, \ell_{L+2}, -p') \) are defined with the opposite prescription for the propagators involving a marked graviton leg \( \hat{\ell}_i \) by using (74). The point of this rewriting is that the amplitudes \( \mathcal{M}_{L+1}^{\text{tree}(\pm)} \) have the multi-soft behaviour

\[
\lim_{|q| \to 0} \mathcal{M}_{L+1}^{\text{tree}(\pm)} (p, h|q|\hat{\ell}_2, \ldots, h|q|\hat{\ell}_{L+2}, -p') \propto (h|q|^0).
\]  

By combining (78) with the scaling of the delta function

\[
\delta\left( \left( p_1 + \sum \ell_i \right)^2 - m_1^2 \right) = \delta \left( 2h|q|p_1 \cdot \sum \hat{\ell}_i + \mathcal{O}(|q|^2) \right) = \frac{1}{h|q|} \delta \left( 2p_1 \cdot \sum \hat{\ell}_i \right) + \mathcal{O}(|q|^0),
\]

we deduce that the multi-soft expansion of the tree-level amplitude \( \mathcal{M}_{L+1}^{\text{tree}} \) is organised by the different powers of delta-function, implementing the velocity cut insertions. When plugged into the expression for the integrand of the cut integral in (11), it becomes a sum of contributions organized as follows

\[
\mathcal{M}_{L+1}^{\text{cut}} \sim \sum_{k=0}^{2L} h^{2L+1} \int \frac{(d^D \ell')^k \left( \delta \left( \left( p_1 + \sum \ell_{\alpha} \right)^2 - m_1^2 \right) \right)^k}{\ell'^{L+1}} \times (\prod \mathcal{M}_{L+1}^{\text{tree}(+)}) \times (\prod \mathcal{M}_{L+1}^{\text{tree}(-)})
\]

\[
= \sum_{k=0}^{2L} h^{2L+1-k} \frac{\mathcal{M}_L^{\text{cut}}}{|q|^{2+D-2L}}.
\]

Setting \( \ell = h|q|\tilde{\ell} \) we see that the generic integrals in the multi-graviton cut behave as

\[
\mathcal{M}_L^{\text{cut}} \sim \sum_{k=0}^{2L} \frac{\mathcal{M}_L^{\text{cut}}}{|q|^{2+D-2L}}.
\]
which is of classical order and given by terms with \( r = L - 2 \) in (27). Thus in this case,

\[
\mathcal{M}_L(\gamma, q^2)_{\text{classical}} = \frac{1}{\hbar} \mathcal{M}_L^{L-2}(\gamma, D)_{\text{classical}}.
\]

(83)

which implies that for computing the classical part of the amplitude, we can approximate the unitarity delta-function constraint as a velocity cut \( \delta((p + \ell)^2 - m^2) \sim \delta(2p \cdot \ell) \), which hugely simplifies the integral computation.

- The terms with \( k < L \) delta-functions are of order \( \mathcal{O}(\hbar^0) \) and correspond to quantum contributions.
- The terms with \( k > L \) delta-functions correspond to contributions with \( -2 \leq r \leq L - 3 \) in the Laurent expansion (27). These contributions are precisely the ones arising from the expansion of the exponential representation of the \( S \)-matrix described in section 6.

5.4. Mass expansion of the amplitude

The classical \( L \)-loop amplitude has the generic mass expansion

\[
\mathcal{M}_L(\gamma, q^2)_{\text{classical}} = \frac{G_{\gamma}^{L+1} m_1^2 m_2^2}{\hbar |q|^2 + \frac{1}{m_1^2 m_2^2}} \sum_{i=0}^{L} c_{L-i+2,i+2}(\gamma, D) m_{1i}^{L-i} m_{2i}^{L},
\]

(84)

Using the reorganisation of the \( \hbar \) expansion in the cut multi-loop amplitude from the soft expansion of the tree-level amplitudes, the coefficients \( c_{i,j}(\gamma, D) \) are easily identified. This identification of the coefficient is valid for generic values of the masses \( m_1 \) and \( m_2 \).

For instance, at two-loop order the coefficient \( c_{4,2}(\gamma, D) \) is obtained by evaluating the following graph

\[
m_1^4 m_2^2 c_{4,1}(\gamma, D) \sim
\]

where we have imposed two velocity cuts (depicted as red dashed lines) on the left tree-level factor in the unitarity cut of the amplitude in (11). The blobs represent the multi-graviton tree-level emission amplitudes constructed in [21] based on the tree construction outlined in references [67, 68]. Similar graphs arise in the heavy-mass effective theory used in [18, 78, 79]. The coefficient \( m_1^2 m_2^2 c_{2,i}(\gamma, q^2) \) is obtained by putting the velocity cuts on the right tree-level factor in the cut of the amplitude in (11). With the obvious generalisation for higher-loop order coefficients \( c_{2+i,2}(\gamma, D) \) and \( c_{2,2+i}(\gamma, D) \) [18, 21]. The coefficient \( c_{3,3}(\gamma, D) \) is obtained by the
And so, on, the other monomials in the masses are obtained by distributing the velocity cuts on each side of the tree factor in the multiple graviton cut in (11).

6. An exponential representation of the $S$-matrix

The approaches presented above do lead to fairly complicated computations. In the first place, the eikonal exponentiation in (48) is obtained after a careful separation, order by order, of the various terms that go into the exponent and those terms that must remain as prefactor at the linear level. A second complication is that after exponentiation in impact-parameter space one must apply the inverse transformation and seek from it two crucial ingredients: (1) the correct identification of the transverse momentum transfer $\vec{q}$ in the center-of-mass frame and (2) the correct identification of the scattering angle from the saddle point. At low orders in the eikonal expansion, this procedure works well but it hinges on the impact-parameter transformation being able to undo the convolution product of the momentum–space representation. When $q^2$-corrections are taken into account (and they do need to be included at higher orders) it is well-known that this procedure requires amendments. While doable in principle, it becomes increasingly difficult with each new order of accuracy. This motivates why we must seek alternative pathways. One particularly promising method is rooted in the semi-classical WKB approximation rather than the eikonal formalism.

The eikonal formalism is built on the transformation of the scattering amplitude to $b$-space by the Fourier transform discussed above. With care, this can exponentiated, leaving the rest of the amplitude at linear level. But only what eventually exponentiates contributes to the eikonal saddle-point condition. It would be interesting if one could reverse the order in which one takes the $b$-space transform so that one instead considers matrix elements of an exponent from the outset. Such seems to be the prescription suggested in references [19, 20], although it is not expressed in those terms there. If we could pursue such a strategy of working with the $S$-matrix in an exponential representation and consider the radical idea of computing matrix elements of that operator in the exponent rather than the $S$-matrix itself we might effectively be computing matrix elements of a phase shift operator. By extension, this should be the WKB-limit of the
S-matrix, and the phase would then be the radial action

\[ S(r, \varphi; E, J) = J \varphi + \int p_t(r, E, J) \, dr, \]

as known from the match to the Hamilton–Jacobi equation in that semi-classical limit. If possible, this should provide a most efficient way to compute the classical scattering angle from amplitudes.

To this end, let us consider the exponential representation of the \( S \)-matrix at the operator level that has recently been introduced in [17].

\[ \tilde{S} = I + \frac{i}{\hbar} \tilde{T} = \exp \left( \frac{iN}{\hbar} \right), \]

with the completeness relation

\[ I = \sum_{n=0}^{\infty} \frac{1}{n!} \int \prod_{i=1}^{2} \frac{d^{D-1}k_i}{(2\pi \hbar)^{D-1}} \frac{1}{2E_{k_i}} \prod_{j=1}^{n} \frac{d^{D-1}s_j}{(2\pi \hbar)^{D-1}} \frac{1}{2E_{s_j}} |k_1, k_2, \ell_1, \ldots, \ell_n; k_1, k_2, \ell_1, \ldots, \ell_n\rangle \langle k_1, k_2, \ell_1, \ldots, \ell_n|, \]

which includes all the exchange of gravitons for \( n \geq 1 \) entering the radiation-reaction contributions \( \hat{N}^{\text{rad}} \). With this exponential representation of the \( S \)-matrix, we systematically relate matrix elements of the operator in the exponential \( \hat{N} \) to ordinary Born amplitudes minus pieces provided by unitarity cuts [17]. This is seen by the perturbation expansion

\[ \hat{N}_0 = T_0, \quad \hat{N}_0^{\text{rad}} = \hat{T}_0^{\text{rad}}, \]

\[ \hat{N}_1 = \tilde{T}_1 - \frac{i}{2\hbar} \tilde{T}_0^2, \quad \hat{N}_1^{\text{rad}} = \tilde{T}_1^{\text{rad}} - \frac{i}{2\hbar} (\tilde{T}_0^{\text{rad}} \tilde{T}_0 + \tilde{T}_0 \tilde{T}_0^{\text{rad}}), \]

\[ \hat{N}_2 = \tilde{T}_2 - \frac{i}{2\hbar} (\tilde{T}_0^{\text{rad}})^2 - \frac{i}{2\hbar} (\tilde{T}_0 \tilde{T}_1 + \tilde{T}_1 \tilde{T}_0) - \frac{1}{3\hbar^2} \tilde{T}_0^3, \]

and similarly for higher orders. The simplicity of this method seems very appealing and suggests that it may be used to streamline post-Minkowskian amplitudes in gravity by means of a diagrammatic technique that systematically avoids the evaluation of the cut diagrams that must be subtracted, but simply discards them at the integrand level. This decomposition is in correspondence with the \( 1/\hbar^2 q^2 \) expansion of the scattering amplitude in (27). The scattering matrix operator \( \tilde{T} \) is related to the scattering amplitude \( \mathcal{M}_q \propto \langle p_1, p_2 | \hat{T}_0 | p'_1, p'_2 \rangle \). The tree-level matrix element for the two-body scattering \( \mathcal{M}_0 \propto \langle p_1, p_2 | \hat{T}_0 | p'_1, p'_2 \rangle \) is of order \( O(1/\hbar) \). At one-loop order amplitude decomposes into two pieces

\[ \mathcal{M}_1 \propto \frac{1}{\hbar} \langle p_1, p_2 | \hat{T}_1 | p'_1, p'_2 \rangle \propto \frac{1}{\hbar} \langle p_1, p_2 | \hat{N}_1 | p'_1, p'_2 \rangle + \frac{i}{2\hbar^2} \langle p_1, p_2 | \hat{T}_0^2 | p'_1, p'_2 \rangle. \]

By unitarity the coefficient of the \( O(1/\hbar^2) \) contribution in the scattering amplitude is \( \langle p_1, p_2 | \hat{T}_0^2 | p'_1, p'_2 \rangle \), and the matrix element \( \langle p_1, p_2 | \hat{N}_1 | p'_1, p'_2 \rangle \) is given by the classical piece is of order \( O(1/\hbar) \). Therefore, for the classical two-body scattering only the matrix elements of \( \hat{N} \) are needed.
The explicit expression for the full one-loop two-body scattering amplitude in general relativity becomes [17]

\[
M_1(\vec{q}, \gamma, h) = \frac{i\hbar}{2} (16\pi G_N m_1^2 m_2^2(2\gamma^2 - 1))^2 \mathcal{P}^{\text{one-cut}} - \mathcal{N}_1(\vec{q}, \gamma) + \mathcal{O}(h), \tag{92}
\]

and it hence follows that the one-loop contribution to \(N\) is

\[
\mathcal{N}_1(\vec{q}, \gamma) = \frac{3\pi^2 G_N^2 m_1^3 m_2^3 (m_1 + m_2)(5\gamma^2 - 1)(4\pi e^{-\gamma})^{4-D}}{|\vec{q}|^{3-D}} - \frac{8G_N^2 m_1^3 m_2^3 (4\pi e^{-\gamma})^{4-D} h}{(4 - D)|\vec{q}|^{4-D}}
\times \left( \frac{2(2\gamma^2 - 1)(7 - 6\gamma^2)\text{arccosh}(\gamma)}{(\gamma^2 - 1)^2} + \frac{1 - 49\gamma^2 + 18\gamma^4}{15(\gamma^2 - 1)} \right) + \mathcal{O}(h) \tag{93}
\]

We now quote from [17] the two-loop result in Einstein gravity

\[
M_2(\vec{q}, \gamma) = \frac{\hbar}{6} (16\pi G_N m_1^2 m_2^3(2\gamma^2 - 1))^3 \mathcal{P}^{\text{two-cut}}
+ \frac{12\pi^2 G_N^2 (m_1 + m_2)m_1^3 m_2^3 (2\gamma^2 - 1)(1 - 5\gamma^2)(4\pi e^{-\gamma})^{4-D}}{(4 - D)|\vec{q}|^{9-2D}}
+ \frac{4\pi G_N^2 (4\pi e^{-\gamma})^{4-D} m_1^3 m_2^3 h}{(4 - D)|\vec{q}|^{8-2D}} \left( \frac{2m_1 m_2 (2\gamma^2 - 1)}{\pi(4 - D)(\gamma^2 - 1)^2} \right)
\times \left( 1 - 49\gamma^2 + 18\gamma^4 \frac{2\gamma(7 - 20\gamma^2 + 12\gamma^4)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right)
+ \frac{\mathcal{E}_M^2 (64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3(\gamma^2 - 1)^2} - \frac{4m_1 m_2 (14\gamma^2 + 25)}{3}
+ \frac{4m_1 m_2 (3 + 12\gamma^2 - 4\gamma^4)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}}
- \frac{4im_1 m_2 (2\gamma^2 - 1)^2}{\pi(4 - D)\sqrt{\gamma^2 - 1} (4(\gamma^2 - 1))^{1/2}}
\times \left( - \frac{11}{3} + \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right) + \mathcal{O}(h), \tag{94}
\]

where we have introduced the two-cut integral

\[
\mathcal{P}_{\text{two-cut}}^{\text{two-cut}} = \left( 1 - \frac{(11 - 2D)|\vec{q}|^2}{12m_1^2 m_2^2 (\gamma^2 - 1)} \right) \frac{1}{16\pi^{2D}} \times \int \frac{d^D l_1 d^D l_2 \delta(p_1 \cdot l_1) \delta(p_1 \cdot l_2) \delta(p_2 \cdot l_1) \delta(p_2 \cdot l_2)}{(2\pi)^{2D-4}} \frac{l_1^2 l_2^2 (l_1 + l_2 - q)^2}{l_1^2 + l_2 - q} + \mathcal{O}(|\vec{q}|^{1-4r}) \tag{95}
\]
which is evaluated to

\[
\mathcal{I}_{\text{two-cut}} = - \left( 1 - \frac{(11 - 2D)\hbar^2|\tilde{q}|^2E_{\text{CM}}^2}{12m_1^2m_2^2(\gamma^2 - 1)} \right) \frac{1}{16\hbar^4|\tilde{q}|^{10-2D}} \\
\times \frac{1}{m_1^2m_2^2(\gamma^2 - 1)} \frac{\Gamma(\frac{D-4}{2})^3\Gamma(5-D)}{(4\pi)^{D-2}\Gamma(\frac{D-2}{2})} + O(|\tilde{q}|^{D-7}). \quad (96)
\]

Subtractions of tree and one-loop terms as dictated by equation (90) lead to the following two-loop contribution to \( N \)

\[
N_2(\tilde{q}, \gamma) = \frac{4\pi G_N^2(4\pi e^{-\gamma}e^4 - 1)m_1^2m_2^2}{(4 - D)|\tilde{q}|^{8-2D}} \left( \frac{E_{\text{CM}}^2(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3(\gamma^2 - 1)^2} \right) \\
- \frac{4}{3}m_1m_2\gamma(14\gamma^2 + 25) + \frac{4m_1m_2(3 + 12\gamma^2 - 4\gamma^4)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \\
+ \frac{2m_1m_2(2\gamma^2 - 1)^2}{\sqrt{\gamma^2 - 1}} \left( -\frac{11}{3} + \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right) \right) + O(h). \quad (97)
\]

We can now collect all contributions to \( N \) up to two-loop order, keeping only the leading terms in \( D - 4 \), and performing the Fourier transform to \( b \)-space. Expressing everything in terms of angular momentum \( J \) in (55), we get

\[
\tilde{N}(J, \gamma) = \frac{G_Nm_1m_2(2\gamma^2 - 1)\Gamma(\frac{D-4}{2})}{\sqrt{\gamma^2 - 1}} J^{D-3} \\
+ \frac{3\pi G_N^2m_1^2m_2^2(m_1 + m_2)(5\gamma^2 - 1)^2}{4E_{\text{CM}}} \\
+ \frac{G_N^2m_1^2m_2^2}{E_{\text{CM}}^2} \frac{\sqrt{\gamma^2 - 1}}{3(\gamma^2 - 1)^2} \left( \frac{E_{\text{CM}}^2(64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5)}{3(\gamma^2 - 1)^2} \right) \\
- \frac{4}{3}m_1m_2\gamma(14\gamma^2 + 25) + \frac{4m_1m_2(3 + 12\gamma^2 - 4\gamma^4)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \\
+ \frac{2m_1m_2(2\gamma^2 - 1)^2}{\sqrt{\gamma^2 - 1}} \left( -\frac{11}{3} + \frac{d}{d\gamma} \left( \frac{(2\gamma^2 - 1)\text{arccosh}(\gamma)}{\sqrt{\gamma^2 - 1}} \right) \right) \right) \right) \frac{1}{J^2} + O(h). \quad (98)
\]

Taking this to be the interacting part of the radial action to third post-Minkowskian order, we obtain the scattering angle

\[
\chi = -\lim_{\hbar \to 0} \frac{\partial}{\partial J} \tilde{N}(J, \gamma) = \frac{2(2\gamma^2 - 1)}{\sqrt{\gamma^2 - 1}} \frac{G_Nm_1m_2}{J} \\
+ \frac{3\pi(m_1 + m_2)(5\gamma^2 - 1)}{4E_{\text{CM}}} \left( \frac{G_Nm_1m_2}{J} \right)^2 \\
+ 2\sqrt{\gamma^2 - 1} \left( \frac{64\gamma^6 - 120\gamma^4 + 60\gamma^2 - 5}{3(\gamma^2 - 1)^2} \right) \frac{1}{J^2}. \quad (99)
\]
One route to connect the scattering regime to the bound-state regime is based on the EOB formalism [58, 59], suitably adapted from post-Newtonian to post-Minkowskian formulations [1, 2, 60, 80]. One important lesson from the scattering amplitude approach to gravitational scattering in general relativity is that at least up to and including third post-Minkowskian order, there exists, in isotropic coordinates, a very simple relationship between center-of-mass three-momentum $p$ and the effective classical potential $V_{\text{eff}}(r, p)$ of the form

$$p^2 = p_{\infty}^2 - V_{\text{eff}}(r, E), \quad V_{\text{eff}}(r, E) = -\sum_{n\geq 1} f_n \left( \frac{G_{\text{N}}(m_1 + m_2)}{r} \right)^n,$$

where the coefficients $f_n$ are directly extracted from the scattering angle [56]

$$\frac{\chi}{2} = \sum_{k=1} b \int_0^\infty d\alpha \left( \frac{d}{d\alpha} \right)^k \left[ \frac{1}{u^2 + b^2} \left( \frac{V_{\text{eff}}(\sqrt{u^2 + b^2})}{\gamma^2 - 1} \right) \right].$$

Since $p^2 = p_{\infty}^2 + J^2/r^2$ where $J$ is the angular momentum we have

$$\frac{\chi}{2} = -\int_{r_m}^{\infty} dr \frac{\partial p_t}{\partial J} = \frac{\pi}{2} = b \int_{r_m}^{\infty} \frac{dr}{r^2} \sqrt{1 - \frac{b^2}{r^2} - \frac{V_{\text{eff}}(r, E)}{E_m}} = \frac{\pi}{2},$$

where $r_m$ is solution to $1 - \frac{b^2}{r^2} - \frac{V_{\text{eff}}(r, E)}{E_m} = 0$. Considering a general parametrization of the effective metric $g_{\mu\nu}^{\text{eff}}$ in isotropic coordinates

$$ds_{\text{eff}}^2 = A(r)dr^2 - B(r)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

and the principal function $S = E_{\text{eff}}t + J_{\text{eff},\varphi} + W(r)$ of the associated Hamilton–Jacobi equation $g_{\alpha\beta}^{\text{eff}} \partial_{\alpha} S \partial_{\beta} S = \mu^2$, the scattering angle is given by

$$\frac{\chi}{2} = J_{\text{eff}} \int_{r_m}^{\infty} \frac{dr}{r^2} \sqrt{\frac{B(r)J_{\text{eff}}^2}{A(r)} - \frac{E_{\text{eff}}^2}{\mu^2} - B(r)\mu^2} = \frac{\pi}{2}.$$

which agrees with the literature [6, 7] after inclusion of the crucial radiation reaction terms [10, 11, 16].

7. A post-Minkowskian effective one-body formalism

One route to connect the scattering regime to the bound-state regime is based on the EOB formalism [58, 59], suitably adapted from post-Newtonian to post-Minkowskian formulations [1, 2, 60, 80]. One important lesson from the scattering amplitude approach to gravitational scattering in general relativity is that at least up to and including third post-Minkowskian order, there exists, in isotropic coordinates, a very simple relationship between center-of-mass three-momentum $p$ and the effective classical potential $V_{\text{eff}}(r, p)$ of the form

$$p^2 = p_{\infty}^2 - V_{\text{eff}}(r, E), \quad V_{\text{eff}}(r, E) = -\sum_{n\geq 1} f_n \left( \frac{G_{\text{N}}(m_1 + m_2)}{r} \right)^n,$$

where the coefficients $f_n$ are directly extracted from the scattering angle [56]

$$\frac{\chi}{2} = \sum_{k=1} b \int_0^\infty d\alpha \left( \frac{d}{d\alpha} \right)^k \left[ \frac{1}{u^2 + b^2} \left( \frac{V_{\text{eff}}(\sqrt{u^2 + b^2})}{\gamma^2 - 1} \right) \right].$$

Since $p^2 = p_{\infty}^2 + J^2/r^2$ where $J$ is the angular momentum we have

$$\frac{\chi}{2} = -\int_{r_m}^{\infty} dr \frac{\partial p_t}{\partial J} = \frac{\pi}{2} = b \int_{r_m}^{\infty} \frac{dr}{r^2} \sqrt{1 - \frac{b^2}{r^2} - \frac{V_{\text{eff}}(r, E)}{E_m}} = \frac{\pi}{2},$$

where $r_m$ is solution to $1 - \frac{b^2}{r^2} - \frac{V_{\text{eff}}(r, E)}{E_m} = 0$. Considering a general parametrization of the effective metric $g_{\mu\nu}^{\text{eff}}$ in isotropic coordinates

$$ds_{\text{eff}}^2 = A(r)dr^2 - B(r)(d\theta^2 + \sin^2 \theta d\varphi^2),$$

and the principal function $S = E_{\text{eff}}t + J_{\text{eff},\varphi} + W(r)$ of the associated Hamilton–Jacobi equation $g_{\alpha\beta}^{\text{eff}} \partial_{\alpha} S \partial_{\beta} S = \mu^2$, the scattering angle is given by

$$\frac{\chi}{2} = J_{\text{eff}} \int_{r_m}^{\infty} \frac{dr}{r^2} \sqrt{\frac{B(r)J_{\text{eff}}^2}{A(r)} - \frac{E_{\text{eff}}^2}{\mu^2} - B(r)\mu^2} = \frac{\pi}{2}.$$
In order to relate the scattering amplitude data in $V_{\text{eff}}(r, E)$ to the effective metric, energy and angular momentum, we reconsider the maps on which the EOB formalism is based on [58, 59, 81]. We have

- The energy map

$$E = (m_1 + m_2) \sqrt{1 + \frac{2m_1m_2}{(m_1 + m_2)^2}(\frac{m_1 + m_2}{m_1m_2}E_{\text{eff}} - 1)}$$

(105)

- The momentum map

$$p^\infty_\nu = \frac{(E^2 - (m_1 + m_2)^2)(E^2 - (m_1 - m_2)^2)}{4E^2}, \quad \frac{p_{\text{eff}}}{\mu} = \frac{p^\infty E}{m_1m_2}$$

(106)

- An angular momentum map

$$b = \frac{J}{p^\infty} = \frac{J_{\text{eff}}}{p_{\text{eff}}} \implies J_{\text{eff}} = J\frac{p_{\text{eff}}}{p^\infty} = J \frac{E}{M^2}$$

(107)

This map differs from the one used in [58, 59, 81]. With this relation we choose to keep fixed the impact parameter $b$, whereas in [58] the angular momentum is kept fixed. The possibility of fixing $b$ instead of $J$ has been mentioned in reference [23] but not pursued there.

One can now identify the two expressions in (102) and (104) if one equates the expressions under the square root. This leads to a relation between the metric coefficients and the effective potential

$$1 - \frac{V_{\text{eff}}(r, E)}{p^\infty_\nu} = \frac{B(r)}{\gamma^2 - 1} \left(\frac{\gamma^2}{A(r)} - 1\right).$$

(108)

In order to fix the parametrisation ambiguity we parameterise the metric coefficient using the ansatz

$$A(r) = \left(\frac{1 - h(r)}{1 + h(r)}\right)^2; \quad B(r) = (1 + h(r))^4,$$

(109)

to get

$$\left(\frac{h(r) + \gamma - 1}{\gamma - 1}\right)\left(h(r) + \frac{\gamma - 1}{\gamma + 1}\right)(1 + h(r))^4$$

$$= (1 - h(r))^2\left(1 + \frac{E^2}{(\gamma^2 - 1)M^2}\right)\frac{V_{\text{eff}}(r, E)}{\nu^2M^2}.$$  

(110)

Using the known perturbative expansion of the effective potential $V_{\text{eff}} = - \sum_{n \geq 1} f_n(GM/r)^n$ one can solve for the metric coefficients $h_n$ at each order in $(GM/r)^n$. The resulting metric deviates from the Schwarzschild solution and there is no need to introduce the non-metric terms as in the traditional EOB approach, as they are reabsorbed in the metric coefficients.

By computing the two-body scattering in perturbation one derives a Lorentz invariant expression valid in all regimes of relative velocity between the two interacting massive bodies. Importantly, the relation between the scattering amplitude and the effective-one-body effective potential in (101) is valid in any space–time dimension and applies to gravity in higher dimensions [52, 54].
8. Summary

The amplitude approach to the binary problem of general relativity is a highly promising new avenue for computations in the post-Minkowskian approximation. From the scattering regime, one can infer the dynamics at arbitrarily high energies. From this also the bound pseudo-elliptic regime can be reached and one can hence, eventually, generate waveforms that go beyond the post-Newtonian approximation. While the latter is motivated by the virial theorem, having access to an effective Hamiltonian that is unrestricted in terms of kinetic energy promises to have definite phenomenological value. Interestingly, the scattering amplitude viewpoint on gravity has also led to a renewed understanding of how classical physics can be extracted from quantum mechanical $S$-matrix elements. This interplay is bound to continue in the coming years.

The general framework presented in this review shows how the post-Minkowskian expansion from scattering amplitudes can be used to gain complete control of the two-body scattering in all regimes of energy. As explained, the gravitational radiation field leads to a back-reaction on the participating massive bodies which plays a crucial role in obtaining correct scattering dynamics already at two-loop order. This is all automatically contained in the ordinary $S$-matrix element of the gravitational two-body scattering to that order. The radiation reaction to this order thus need not be added separately but is an essential ingredient of the amplitude calculation, correctly taking into account all contributing diagrams and all loop-integration regimes that lead to classical terms in the amplitude. From this viewpoint the separation of radiation-reaction terms at two-loop order is not natural since all terms generating classical contributions are already correctly included in the standard Feynman diagram expansion.

One major technical issue is the extraction of the classical part of the amplitude (all terms of order $1/\bar{h}$ in the Laurent expansion (27)). By combining unitarity and the concept of velocity cuts introduced in [16], we can identify exactly those elements of integrands that lead to classical physics upon integration. Our approach uses an organisation of the integrand of the multi-loop amplitude with unitarity cuts on the massive scalar propagator lines together with detailed knowledge of the correspondence between the multi-soft graviton expansion and $\bar{h} \to 0$ limit classical integrand matching the exponential representation of the $S$-matrix of [17]. In the classical limit, this approach systematically relates the classical part of the scattering amplitude to the matrix elements of the operator $\hat{N}$ of the exponential representation of the $S$-matrix, without having to actually having to perform the subtractions that are determined by unitarity in a simple manner. Considering the rapid pace at which progress has taken place we can expect much clarification of the optimal path for amplitude computations for gravity in the coming years.

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Data availability statement

No new data were created or analysed in this study.

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