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Polariton dynamics in one-dimensional arrays of atoms coupled to waveguides

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Abstract

Photons strongly coupled to material systems constitute a novel system for realizing non-linear optics at the level of individual photons and studying the dynamics of non-equilibrium quantum many-body system. We give a simple physical polariton-picture of the dynamics of photons coupled to a one-dimensional array of two-level atoms. This picture allows a fully analytical description of the dynamics in terms of polariton scattering inside the medium and reflections of the polaritons from the edge of the array. We show that inelastic collisions, previously identified in small systems, also occur in infinite systems and are related to the existence of multiple bands in the dispersion relation. The developed theory constitutes an effective field theory for the dynamics, which can be used for studies of non-linear optics and many-body dynamics. As a specific example we map the system to the Lieb–Liniger model and show that a so-called Tonks–Girardeau gas of photons is a stable eigenstate of the system in the limit of many emitters.

1. Introduction

The interaction between photons in free space is completely negligible. As a consequence investigations of photon-interaction rely on indirect interaction via non-linear coupling to matter [1–3]. The absorption and re-emission of light in such non-linear media gives rise to a plethora of phenomena [4] like super- [5–7] and sub-radiance [7–12], electromagnetically induced transparency [13], light-matter quantum interfaces [14], single photon transistors [15], bound states of light [16–25], the emergence of dimer states [26, 27], as well as elastic and inelastic scattering [28–30]. Understanding such phenomena is of immense technological importance as a crucial ingredient for extreme non-linear optics at the single-photon level as well as for quantum computation and communication hardware based on photonic systems [1]. At the same time, such systems with multiple photons constitute an intriguing many body quantum system [14, 31–33]. The exploration of strongly coupled light-matter quantum systems has recently lead to the observation of several effects arising from effective photon-photon interactions [34–39], including photon bound states [40]. Even few emitter setups, however, show complex behavior and their description typically requires a large theoretical overhead based on e.g. scattering matrix theory [41–43], or numerical integration of a wave function Ansatz [44–46] and effective non-Hermitian Hamiltonians [8–10, 47, 48]. On the other hand, the cleanest realizations of the aforementioned phenomena (and many-body dynamics in general) are achieved in the limit of many emitters [8–10, 23, 47, 49]. These systems are also technologically promising, e.g. for generating photon-photon gates [50–52]. To understand such systems with multiple excitations it is essential to first establish a solid understanding of the few-photon dynamics and basic optical properties of the systems, which can then be used to understand and control the complex many-body dynamics.

Here, we develop a concise fully analytical description of the dynamics of photons coupled to a regular array of multiple two-level emitters, as illustrated in figure 1(a). This provides an easy to understand picture of the essential dynamics of these systems. Our theory provides a full description of the input and output of the system (figures 1(b) and (d)), mapping any photon states to coupled light-matter polariton states and...
vice versa, as well as the propagation and two-body interaction of the excitations (figure 1(c)). Furthermore
we identify an inelastic scattering mechanism corresponding to four-wave mixing, where the polaritons (and
in extension the photons) exchange energy, as illustrated in figure 1(d). Such inelastic collisions have
previously been identified for few-emitter systems [30]. Here we show that such effect also exists in bulk
systems, fully solve the resulting dynamics, and elucidate the physics behind them: they occur due to
structure of the dispersion relation, which contain multiple polariton branches. From the dispersion relation
one can thus control the collision dynamics and e.g. the energy of the outgoing states. These results set the
stage for explorations of more advanced non-linear optical effects and many-body phenomena. We map the
dynamics to the Lieb–Liniger model of many-body physics and show that the so-called Tonks–Girardeau gas
[53, 54] becomes a solution in the limit of many emitters. In this state the photons effectively behave as
fermions rather than bosons. A related heuristic mapping has recently been used to understand the
dissipative part of evolution [10], but we provide a complete mapping for the entire dynamics.

2. System

The system of interest consists of left- and right-propagating waveguide modes with field operators \( \mathcal{E}_L(z) \) and
\( \mathcal{E}_R(z) \), respectively, with \( c(z_1, z_2) = c \delta(z_1 - z_2) \) and \( c \) is the group velocity of light in the
waveguide. The two level emitters are described by Pauli operators, e.g. \( \sigma_\mu^+ = |\mu\rangle \langle \mu | \) exciting the
\( \mu \)th emitter, all with the same distance \( d \) to the neighbors. We consider the typical one-dimensional
photon-emitter-interaction Hamiltonian under the rotating wave approximation, \( H_{\text{int}} = H_p + H_{\text{int}} \) with [42]

\[
H_p = -i\hbar \int dz \left[ \mathcal{E}_R^\dagger(z) \partial_z \mathcal{E}_R(z) - \mathcal{E}_L^\dagger(z) \partial_z \mathcal{E}_L(z) \right]
\]

and

\[
H_{\text{int}} = \sqrt{\frac{1}{2\pi}} \sum_\mu \left[ \mathcal{E}_R^\dagger(z_\mu) \sqrt{\Gamma_R} + \mathcal{E}_L^\dagger(z_\mu) \sqrt{\Gamma_L} \right] \sigma_\mu^+ + \text{h.c.}
\]

Here, \( H_{\text{int}} \) has been transformed to the interaction picture absorbing the excitation energy of the emitters.
All photon frequencies \( \omega \) are thus considered relative to the resonance frequency. By considering potentially
deviating couplings \( \Gamma_L \neq \Gamma_R \) we incorporate an arbitrary level of chirality in the atom chain [55].

A polariton is a quasiparticle emerging from the coupling of a photon to a two level system. These
polaritons can most easily be described after integrating out the photon degree of freedom. Then the
dynamics of re-emission and absorption from emitter to emitter is contained in the effective Hamiltonian
[21, 23, 32]

\[
H_{\text{sys}}/\hbar = -i \sum_{\mu > \nu} e^{ik_0 \rho_\mu - z_\mu} \left[ \Gamma_R \sigma_\mu^+ \sigma_\nu^- + \Gamma_L \sigma_\nu^+ \sigma_\mu^- \right] - \frac{i}{2} \sum_\mu \Gamma_R + \Gamma_L + \Gamma_S \sigma_\mu^+ \sigma_\mu^- ,
\]

with \( k_0 \) being the resonance wave number. The rate \( \Gamma_S \) of photons ejected out of the waveguide leads to an
overall depletion of the polariton number. While this constant depletion rate can be a considerable limitation
for experimental realizations, it does not qualitatively influence the calculations to come apart from
introducing a constant loss rate. It will thus be omitted from this point onward.

We note that the effective Hamiltonian is non-Hermitian reflecting that we consider an open system
where polaritons can enter and leave the system at the edges. The Hamiltonian describes the dynamics within

![Figure 1.](image-url)
a manifold of a fixed polariton number including their decay probabilities from the edges. To also describe the state after a decay, the current description could be supplemented with quantum jumps from the Monte–Carlo wavefunction approach [56]. As shown below, however, the Hamiltonian captures the essential dynamics. For an infinite chain losses at the edges play no role and the non-Hermitian Hamiltonian conserves the number of excitations. The eigenstates of the system can then be found using Bloch’s theorem and the effective Hamiltonian (3) leads to the dispersion relation

\[ \omega_k = -\frac{\Gamma_R \cos[(k - k_0)d/2]}{2 \sin[(k - k_0)d/2]} + \frac{\Gamma_L \cos[(k + k_0)d/2]}{2 \sin[(k + k_0)d/2]}, \]  

(4)

plotted in figure 2.

3. Input and output

We first study the coupling of photons into and out of the atom array as illustrated in figures 1 (b) and (d). Using the well-known input-output formalism [42] we derive in appendix A the input relation for an incoming field \( \sigma_k^+ = t(k) E_{\text{in}}(z_1, \omega_k) \) with the transmission amplitude

\[ t(k) = -i \sqrt{\Gamma_R d} \left[ f(k - k_0) + r(k) f(k' - k_0) \right], \]  

(5)

and the Pauli operators in momentum space \( \sigma_k^- = \sqrt{d} \sum_{\mu} e^{ik'\mu} \sigma_{\mu}^- \). Here, we have a factor

\[ f(k - k_0) = 1/\left[ 1 - e^{i(k - k_0)d} \right] \]  

and the reflection coefficient is \( r(k) = -\left( e^{-i(k' - k_0)d} - 1 \right)/\left( e^{-i(k + k_0)d} - 1 \right) \), where \( k' \) is the degenerate wavenumber for which \( \omega_{k'} = \omega_k \).

To obtain a probability for being coupled into the atom array as a polariton, we have to correct for the group velocity \( v_k = |\partial_k \omega_k| \) yielding

\[ |\sigma_k^+|^2 dk = \frac{|t(k)|^2}{v_k} |E_{\text{in}}(z_1, \omega_k)|^2 d\omega_k \]  

(6)

From here we can extract the transmission probability \( |t(k)|^2/v_k \) (we note that since the transmission coefficient \( t(k) \) relates quantities with different dimensions, the coefficient \( t(k) \) has physical dimensions, whereas the probability \( |t(k)|^2/v_k \) is dimensionless). This probability becomes smallest for \( k \to n\pi, n \in \mathbb{Z} \) (i.e. the points closest to resonance), where the polariton is entirely reflected if \( \Gamma_L = \Gamma_R \) (so that \( v_k/v_k = 1 \)) extending the well-known mirror-like behavior for \( k_0d = \pi \) [47, 57, 58] to general \( k_0 \), as shown in figure 2.

On the other hand, for a completely chiral setup with \( \Gamma_L = 0 \) we get \( |t(k)|^2/v_k = 1 \) and the photonic wavefunction enters the medium in its entirety.

For the output scenario we find in appendix A

\[ E_{\text{out}}(z_2, \omega) = \frac{1}{v_k(\omega)} e^{iL(k_k - k_0)} t(k(\omega)) \sigma_k^-(\omega) \]  

(7)
analogous to the transport scenario, only rescaled by the group velocity. Also, we have an additional phase factor depending on the length \( L = d(N - 1) \) of the atom array. From here we can calculate the probability for a polariton to leave the array as a photon

\[
|\mathcal{E}_{\text{out}}(z_N, \omega)|^2 \, \text{d}\omega = \frac{|r(k)|^2}{v_k} |\sigma_{\mu_1}^{\mu_2}|^2 \, \text{d}k(\omega) = \left( 1 - \frac{v_q}{v_k} |r(k)|^2 \right) |\sigma_{\mu_1}^{\mu_2}|^2 \, \text{d}k(\omega),
\]

which matches the input probability in accordance with Helmholtz reciprocity. We furthermore note that the transmission \(|t(k)|^2/v_k\) and reflection \(v_q/r(k)^2/v_k\) probabilities add up to unity.

As a direct application, this result allows us to reproduce the previously reported \(N^{-3}\) scaling of the subradiant states for \(\Gamma_K = \Gamma_L\) [8, 10]: The lowest energy eigenstates for an array with \(N\) atoms are found at \(k = \pi \xi/N d\) with \(\xi \in \mathbb{N}^+\). Wavepackets localized around a certain \(k\) will reach the boundaries at a rate \(v_q/N d\) altogether, this leads to a total ejection rate of \(\gamma_{\text{em}} = v_k (1 - |r(k)|^2)/N d\) and expanding the transmission \(t(k)\) to lowest order in \(k\) reproduces the results from [10] with a simplified physical interpretation: The subradiance is caused by the near perfect reflection of slow moving polaritons at the edges.

### 4. Polariton scattering

The only ingredient missing for a complete picture of the two-photon dynamics is the scattering event shown in figure 1(c). To describe the scattering of two polaritons we change to center of mass and relative coordinates with absolute momentum \(K = (k_1 + k_2)/2\) and relative momentum coordinate \(q = (k_1 - k_2)/2\). The two-excitation momentum basis states can then be expressed as

\[
|q, K) = \sum_{\mu_1, \mu_2} e^{i\mu_1 z_{\mu_1} + i\mu_2 z_{\mu_2}} \sigma_{\mu_1}^{\mu_2} \sigma_{\mu_3}^{\mu_4} |0),
\]

where \(|0)\) is the polariton vacuum state where all atoms are in the ground states \(|g)\). We show in appendix B that

\[
|\Psi_{q,K}) = |q, K) + t_1 | - q, K) + t_2 | - q', K)
\]

is an eigenstate of the effective Hamiltonian (3) with eigenenergy \(\omega_{q,K} = \omega_{q,K} + \omega_{q,K}\) and \(q'\) the degenerate momentum number fulfilling \(\omega_{q,K} = \omega_{q,K}\) \(|q') \neq |q)\). Together with (5) and (7) this provides a completely analytical description of two photon dynamics in the limits of many emitters.

In contrast to scattering in free space, the scattering amplitudes depend not only on the relative momentum \(q\) but also on their mean momentum \(K\). Due to the presence of multiple bands in the dispersion relation, the nature of the scattering depends heavily on the degeneracy of the total energy \(\omega_{q,K}\) shown in figures 3(a) and (d). By symmetry the relative opposite momenta \(q \rightarrow -q\) always have the same energy \(\omega_{q,K} = \omega_{-q,K}\), but in some regions of phase space (marked with blue in figures 3(a) and (d)) there is a four-fold degeneracy with real \(q\), \(-q\), \(q'\), and \(-q'\) fulfilling \(\omega_{q,K} = \omega_{q', K}\). In this case the scattering has an elastic component with a probability \(|t_1|^2\) to scatter into \(|-q)\) and an inelastic component scattering into \(|-q')\) with a probability \(|t_2|^2 v_{q', K}/v_{q, K}\); we show in appendix B that the continuity equation \(|t_1|^2 + |t_2|^2 v_{q, K}/v_{q, K} = 1\) holds. Here \(v_{q,K} = \partial \omega_{q,K}/\partial q\) is the relative group velocity and the probabilities are symmetric under \(q \rightarrow -q\) or \(q \leftrightarrow q'\).

Physically an inelastic collision means that the photons will be going out with different individual momenta and thus also different energies. This is illustrated in figure 2 where symbols indicate examples of such inelastic collisions. For instance, two polaritons with momenta indicated by the open triangles can scatter and produce polaritons at the momenta indicated by the filled triangles and vice-versa.

Experimentally such events could be measured by the emergence of new components of the outgoing field and represent a peculiar form of four wave mixing, which could, e.g. be used for frequency conversion of two single photons by picking \(t_1 \rightarrow 0\), i.e. the dark blue areas in figure 3(f).

In most cases this change of energy results in a sign flip of the energy (relative to resonance \(\omega = 0)\) such that one of the photons switches to a different branch of the dispersion relation, see figure 2. In case of a non-chiral setup this is always the case as the two branches are either purely convex or concave, and we cannot have momentum and energy conservation without branch-hopping. For strong chirality, however, this convexity can be broken and inelastic scattering can take place within the branches, see the red dots in figure 2 which are part of the region of inelastic scattering inside the triangle of figure 3(e).

As opposed to the situation above, for some combinations of \(q, K\) (marked with yellow in figures 3(a) and (d)) the two-polariton dispersion relation only shows a two fold degeneracy \(\omega_{q,K} = \omega_{-q,K}\). This means that we cannot directly solve the eigenequation of the effective Hamiltonian with real relative momenta, and thus
Figure 3. The scattering of two polaritons for different relative and absolute momentum $q$ and $K$ with (a)–(c) $kd = \pi/2$ and $\Gamma_R/\Gamma_L = 1$, (d)–(f) $kd = 0.9 \pi$ and $\Gamma_R/\Gamma_L = 4$. (a) and (d): the two-polariton dispersion relation $\omega_{q,K}$ (solid lines) for a fixed $Kd = -1$ marked by dashed lines in (b)–(f). For energy regions with four-fold degeneracy (blue region on axis) we have inelastic scattering with $|t_1|^2 \leq 1$. In (d) instead of a minimum at $qd = 0$ we have a tiny local maximum in energy which secures a broad region of inelastic scattering. In the yellow regions we require complex $q'$ to solve the eigenequation of the effective Hamiltonian (3) and the imaginary part is plotted as dashed-dotted lines. (b) and (e): Imaginary part of the degenerate $q'$. Whenever this imaginary part is non-zero we have scattering into meta-stable resonance states with finite extension, while for $\text{Im}\{q'd\} = 0$ the scattering only goes to localized resonance states, so that $|t_1|^2 = 1$ is required to conserve probability [23], whereas $|t_1|^2 < 1$ for $\text{Im}\{q'd\} = 0$.

5. Many particle limit

An interesting application of the above results is to use them for studying more complex many-body dynamics in a long chain of atoms. A particular example of such systems is the Lieb-Lininger model describing massive particles with short range interactions in one dimension [59]. Here the dynamics is governed by a scattering length $a$. In the limit $a \rightarrow 0$ the ground state of this model is a Tonks–Girardeau gas describing impenetrable particles and fermionization of bosons [53, 54]. Below we show that a similar behavior can also be produced in our setup for a non-chiral system $\Gamma_R = \Gamma_L = \Gamma_0$. To resemble the Lieb–Lininger model, we are interested in a situation where we only have elastic scattering. The most promising approach is thus to stay as close to $k \rightarrow 0$ (corresponding to $K, q \rightarrow 0$) as possible, where
we are dominated by elastic scattering, see figures 3(c) and (f). By expanding \( \omega_n \) quadratically at the origin we find that the polaritons in this region can be described as massive particles with an effective mass 
\[
m_{\text{eff}} = \hbar (1 - \cos k_0 d)^2 / \Gamma_0 d^2 \sin k_0 d.
\]
To fully match the dynamics of the Lieb–Liniger model we also need to ensure that the scattering dynamics match the model. For \( K = 0 \) the solution is especially simple and only involves scattering into \(-q\) with an amplitude
\[
t_1 = \frac{e^{i q d} - \cos k_0 d}{e^{-i q d} - \cos k_0 d},
\]
and \( t_2 = 0 \) (independent of the degree of chirality). By expanding the amplitude (11) for small \( q \) we find the scattering length
\[
a = \frac{1}{2} \frac{d}{1 - \cos k_0 d}
\]
allowing a complete mapping to the Lieb–Lininger model by reducing the interaction to a single quantity \( a \) corresponding to an effective potential \( V = \lambda \delta(z_n - z_{n'}) \) with \( \lambda = \hbar^2 / m_{\text{eff}} a \). At low energies \( qa \ll 1 \) the scattering from this potential leads to a relative phase of \( t_1 = -1 \) (see figure 4) corresponding to a phase change when interchanging two photons. For sufficiently dilute gases this leads to the Tonks–Girardeau ground state where the photons can be exactly described by a fermionic wavefunction. For the high energy limit \( qa \gg 1 \), on the other hand, the scattering yields \( t_1 \approx 1 \) corresponding to non-interacting particles. As opposed to the Lieb–Liniger model, however, the model we use here is fundamentally a lattice model. This difference is explored in figure 4 where we compare the scattering amplitude with the Lieb–Lininger model. As seen in the figure the resemblance is best for small \( k_0 d \) where it matches the continuum result in both the strongly interacting regime \((t_1 \approx -1 \text{ for } qa \ll 1)\) as well as the weakly interacting regime \((t_1 \approx 1 \text{ for } qa \gg 1)\) until the resemblance finally breaks down when the momentum becomes comparable to the inverse lattice spacing \((q d \sim 1)\). On the other hand, to maximize the region of phase space with \( TG \sim 1 \), i.e. low energies and (c) a low loss rate. We will below consider the limit \( k_0 d \sim 1 \) although the high energy behavior is different in this limit.

Above we have rigorously verified the mapping of the full scattering theory to the Lieb–Lininger model for two polaritons in the low energy limit. To extend the model to the many-body limit and the Tonks–Girardeau gas we have to ensure that the approximation of only two-body scatterings is fulfilled and that the Tonks–Girardeau ground state of the Lieb–Liniger model is stable. For this we require (a) low polariton densities, (b) small \( k_0 \), i.e. low energies and (c) a low loss rate. We will first show how these requirements lead to a Tonks–Girardeau gas in the continuum limit and then quantitatively assess the loss rates of the Tonks–Girardeau ground state for finite chain lengths.

We consider a system with \( N \) emitters and \( n \) polaritons and assume \( n \ll N \) to ensure that we are in the low density limit such that the dynamics is accurately described by two-body interaction. We examine the quantum states with low \( k_0 \), where we can expand the dispersion relation quadratically corresponding to an effective mass and take the continuum limit \( N \to \infty \). In this limit the Tonks–Girardeau gas becomes the exact ground state for a dilute gas [54]. The condition for this reads \( \lambda / d \gg \hbar^2 k_0^2 / 2m_e \) where typical momenta are \( k_{TG} \sim \pi n / Nd \), and is equivalent to \( t_1 \approx -1 \) for typical momenta \( a \sim k_{TG} \) in figure 4. For \( k_0 d \) of order unity this translates into \( n \ll N \), i.e. the low density assumption already made to neglect three-body interactions. If this condition is fulfilled the lowest energy state of the many-body system will thus be the Tonks–Girardeau ground state.

Figure 4. Exact real and imaginary part of \( t_1 \), respectively, in our setup at \( K = 0 \) with \( k_0 d = 0.05 \pi \) (red), \( k_0 d = 0.5 \pi \) (purple), and \( k_0 d = \pi \) (blue). The dotted lines are the result of the short range interaction assumed in the Lieb–Liniger model. While this provides the best description for small \( k_0 \) the broadest region of \( t_1 = -1 \) is found for \( k_0 d \to \pi \) (although still being restricted to very small \( q \)).
To ensure that the Tonks–Girardeau gas is also stable we need to consider the effect of losses (for simplicity we assume $k_0d$ of order unity). As above, the emission probability at the edges scales as $P_{\text{em}} \sim k_{\text{TCG}}d \sim n/N$. With a density $g = n/Nd$ and typical group velocity $v_g \sim \Gamma_0 nd/N$ the loss rate per polariton is

$$\gamma_{\text{em}} = g v_g P_{\text{em}}/n \sim \Gamma_0 (n/N)^3/n,$$

consistent with the scaling $N^{-3}$ obtained for fixed $n$ in [60]. This loss rate vanishes for $n \to \infty$ as the edges become irrelevant. Additionally, polaritons may be lost due to inelastic scattering. The probability for this scales as $P_{\text{in}} \sim k_{\text{TCG}}d^2$ and if we assume all inelastically scattered polaritons to be lost, the respective loss rate reads

$$\gamma_{\text{in}} \sim g v_g P_{\text{in}} \sim \Gamma_0 (n/N)^5,$$

only depending on the polariton density. Despite the inherent losses in the model, the Tonks–Girardeau gas is thus long lived for a sufficiently dilute gas $n \ll N$.

In case of a (partly) chiral coupling with $\Gamma_R \neq \Gamma_L$ it is still possible to realize a Tonks–Girardeau gas, although the conditions are not as favorable as for the non-chiral case. We can maximally suppress the inelastic scattering by having $k \approx 0$ corresponding to a distribution around $q$, $K \approx 0$. Expanding the dispersion relation around this point $\omega(q) \approx v_g q + \hbar q^2/2m_{\text{eff}}$ we find that the system has the same effective mass as for the non-chiral case, $m_{\text{eff}} = 2\hbar (1 - \cos k_0d)/(\Gamma_R + \Gamma_L)d^2 \sin k_0d$, but additionally has an overall drift with a group velocity $v_g = d(\Gamma_R - \Gamma_L)/4 \sin^2 k_0d/2$. The dynamics is thus the same but in a moving frame. For a finite system this will lead to complicated dynamics as the system reach the edge, where chirality furthermore leads to losses, see figure 2. For a finite chiral system the Tonks–Girardeau gas is thus only achievable near $k = 0$ as a transient phenomenon and not as a steady state. These issues can be alleviated by centering the individual momenta at a finite $k$ around the minima of the dispersion relation (thus around zero group velocity). Here we still find $t_1 \approx -1$ and $t_2 = 0$ as well as perfect reflection at the edges allowing a stable Tonks–Girardeau gas. The non-zero $k$ leads to finite $K$, which we find shrinks the region with $t_1(q) \approx -1$ compared to the non-chiral case, and thus in turn leads to greater losses due to inelastic scattering. Consequently, the non-chiral setup is better suited for creating a Tonks–Girardeau gas of photons.

6. Conclusion

We have described the dynamics of polaritons in atom arrays in the limit of many two-level emitters. Our concise analytical model allows for a precise grasp of the rich physics in the system, featuring an interplay of elastic and inelastic scattering as well as scattering into scattering resonances. The results we obtain allow a better understanding of previously obtained results for reflection on resonance [47] and sub-radiance [9, 10]. At the same time the analytical description of the scattering process paves the way for understanding extreme non-linear optics effects at the level of individual photons as well as complex many-body effects. As a first step in this direction we have used our result to map the system to the Lieb–Liniger model and shown that the Tonks–Girardeau gas of photons becomes an eigenstate of the system in the limit of many emitters. This is, however, only one out of many possibilities. In particular the analytical approach brought forward in this paper may be extended to more involved setups including three-level emitters with deviating chiral coupling [24] and atom arrays in two or three dimensions. Such extensions may widen the range of non-linear phenomena and many body dynamics, which may be realized with these systems. Furthermore, the theory developed here may also be applied to enhance photonic quantum technologies [50, 51].

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

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Appendix A. Coupling of photons in and out of the atom array

A.1. Output
We start with the easier scenario of a polariton coupling out of the atom array. The generalized input-output-formalism for a chain of emitters (see e.g. [42]) results in the output field

$$\mathcal{E}_{\text{out}}(z,t) = \mathcal{E}_{\text{in}}(z,t) - i \sqrt{\Gamma} \sum_{\mu} \sigma_{\mu}^+(t) e^{i k_{\mu} z - i \omega_{\mu} t}$$

(A1)

with $a_{\mu}(t) = \langle \sigma_{\mu}^+ \rangle$. For now we are only interested in what happens at the boundaries. We therefore for simplicity consider a single excitation in a semi-infinite chain with $z_0 \in (-\infty, z_N]$. Without the homogeneous solution $\mathcal{E}_{\text{in}}(z,t)$ (we assume a vacuum state here) the output field right after the last emitter at $z = \epsilon \to 0^+$ reads

$$\mathcal{E}_{\text{out}}(z_N, t) = -i \sqrt{\Gamma} \sum_{\mu=-\infty}^{N} \sigma_{\mu}^-(t) e^{-i k_{\mu} z_0}$$

$$= \frac{1}{2 \pi i} \sqrt{\Gamma} \int_{-\pi/d}^{\pi/d} \frac{\sigma_{\mu}^-(t) \sum_{\mu=-\infty}^{N} e^{i(k_{\mu} - k_{\nu}) z_0}}{\pi/d} dk \sigma_{\nu}^-(t) = \frac{1}{2 \pi i} \sqrt{\Gamma} \int_{-\pi/d}^{\pi/d} dk \sigma_{\mu}^-(t) f(k - k_0),$$

(A2)

after inserting the inverse Z-transform from first to second line to transform the $\sigma_{\mu}^+ = \sqrt{\pi} \int_{-\pi/d}^{\pi/d} dk \sigma_{\mu}^- / 2 \pi$ into momentum space. Here we have introduced the abbreviation $f(k - k_0) = (1 - \exp[i(k_0 - k)d])^{-1}$ independent from the chirality properties of the waveguide. $L = d(N - 1)$ is the length of the waveguide and gives an overall $k$-dependent phase that has to be considered for the whole process of photons coupling in and out of the atom array.

For a setup of arbitrary chirality $|k\rangle = \sigma_k^+ |\text{vac}\rangle$ is not an eigenstate of the effective (non-hermitian) Hamiltonian

$$H_{\text{eff}} = -i \Gamma \sum_{\mu > \nu} e^{ik_{\mu} z_{\nu} - i \omega_{\mu} t} \sigma_{\mu}^+ \sigma_{\nu}^- - i \Gamma_L \sum_{\mu < \nu} e^{ik_{\mu} z_{\nu} - i \omega_{\mu} t} \sigma_{\mu}^+ \sigma_{\nu}^- - \frac{\Gamma_R + \Gamma_L}{2} \sum_{\mu} \sigma_{\mu}^+ \sigma_{\mu}^-$$

(A3)

in the Hilbert space of only the emitters with a single excitation. Applying this effective Hamiltonian on $|k\rangle$ with an half infinite chain of emitters ($\mu \in (-\infty, z_N]$) leads to

$$H_{\text{eff}} |k\rangle = \left[ -\Gamma_R \cos[\frac{(k - k_0) d}{2}] - \frac{\Gamma_L \cos[\frac{(k + k_0) d}{2}]}{2 \sin[\frac{(k + k_0) d}{2}]} \right] |k\rangle + i e^{i(k - k_0)d} \frac{\Gamma_L}{e^{-i(k + k_0)d} - 1} |k_0\rangle,$$

(A4)

with an eigenenergy $\omega_k$. From the dispersion relation we can always find a degenerate $k'$ with $\omega_{k'} = \omega_k$ but this will be different from $-k$ if a (partially) chiral setup is considered. From this we can construct eigenstates $|\psi_{k'}\rangle = |k\rangle + r(k) e^{i(k - k')} |k'\rangle$ of the effective Hamiltonian with

$$r(k) = - \frac{e^{-i(k' + k_0)d} - 1}{e^{-i(k + k_0)d} - 1} = \begin{cases} 1 & \text{for } k \to 0, \pi \\ 0 & \text{for } k \to k_0. \end{cases}$$

(A5)

with the latter equality if the system is perfectly non-chiral ($\Gamma_R = \Gamma_L$) and thus $k' = -k$. $r(k)$ is the reflection amplitude with the respective probability $|r(k)|^2 v_{k'} / v_k$ of polaritons to be reflected at the boundaries, $v_k = \partial_\omega k$ being the group velocity.

In general, plane waves with positive $k$ get reflected as plane wave with negative wave numbers $k'$ which also contribute to (A2). The electric field can then be written with help of the eigenstates as

$$\mathcal{E}_{\text{out}}(z_N, t) = \frac{1}{2 \pi i} \sqrt{\Gamma} \int_{0}^{\pi/d} dk \left[ f(k - k_0) e^{i k_{\mu} z} + f(k' + k_0) e^{i k_{\nu} z} \right]$$

$$\Rightarrow \mathcal{E}_{\text{out}}(z_N, \omega) = \lim_{\epsilon \to 0^+} \frac{1}{2 \pi i} \sqrt{\Gamma} \int_{0}^{\pi/d} dk \left[ f(k - k_0) e^{i k_{\mu} z - i \omega t} + f(k' + k_0) e^{i k_{\nu} z - i \omega t} \right]$$

$$= - i e^{i(k - k_0)d} \sqrt{\Gamma} \int_{-\infty}^{\omega_k} \frac{f(k - k_0) e^{i k_{\mu} z}}{v_k} + r(k') f(k' - k_0) e^{i k_{\nu} z}$$

(A6)
where we used in the second line Laplace transform into frequency space

\[ \sigma_k^-(\omega) = \lim_{\epsilon \to 0} \int_0^\infty dt' e^{i\omega t' - \epsilon t'} \sigma_k^-(t') = \frac{i}{\omega - \omega_k} \sigma_k^-(t = 0), \] (A7)

together with a substitution \( \partial_k \omega_k = v_k \), and the third line is achieved via the residue theorem. In case of
\( \Gamma_R = \Gamma_L = \Gamma_0 \) we have \( k'(\omega) = -k(\omega) \) and

\[ v_{-k(\omega)} = -v_k(\omega) = -\Gamma_0 d \sin (k_0 d) \sin (kd) / (\cos (kd) - \cos (k_0 d))^2 \] (A8)

which simplifies the infinitesimal energy range of emitted photons to

\[ |E_{out}(\omega)|^2 = \frac{2 \sin (kd) \sin (k_0 d)}{1 - \cos (k (k + k_0)d)} \frac{2 \sin ( kd ) \sin ( k_0 d )}{(1 - \cos ( k (k + k_0)d))} = \frac{2 \Gamma_0 d^2 E_{in}^2}{N^3 (1 - \cos (k_0 d))^3} + \mathcal{O}(k^3 d^3), \] (A10)

which is in agreement with results found by diagonalization [10].

A.2. Input

For the input scenario we can start with a semi-infinite emitter chain with \( z_{\mu} \in [0, \infty) \)

\[ \partial_t \sigma_{-\mu}^-(t) = -i \sqrt{\Gamma_R} \sigma_{\mu}^+ e^{i k z_{\mu}} + H_{\mu} \sigma_{-\mu}^-(t) \] (A11)

and directly change to momentum space, \( \sigma_{-k}^-(t) = \sqrt{\eta} \sum_{\mu} e^{-ikz_{\mu}} \sigma_{-\mu}^-(t) \), for which we need to know the time evolution of the annihilation operators \( \sigma_{-k}^- \). We use the full Lindblad master equation of the reduced system [21]

\[ \mathcal{L} \sigma_{-k}^- = i \hbar \left[ |H|^2 \sigma_{-k}^- - \sigma_{-k}^- H_{\epsilon \sigma}^+ \right] + (\Gamma_R + \Gamma_L) \sum_{\mu} \sigma_{\mu}^+ \sigma_{-\mu}^- \sigma_{-\mu}^+ \sigma_{\mu}^- 
\]

\[ + \sum_{\nu > \mu} \left[ \left( \Gamma_R e^{ikz_{\mu} - z_{\nu}} + \Gamma_L e^{-ikz_{\mu} - z_{\nu}} \right) \sigma_{-\mu}^+ \sigma_{-\nu}^- + \left( \Gamma_R e^{-ikz_{\mu} - z_{\nu}} + \Gamma_L e^{ikz_{\mu} - z_{\nu}} \right) \sigma_{\mu}^+ \sigma_{\nu}^- \right]. \] (A12)

Since \( \sigma_{-k}^- = \sqrt{\eta} \sum_{\lambda = 0}^{\infty} e^{-ikz_{\lambda}} \bar{\sigma}_{-k}^- \) this results in

\[ \mathcal{L} \sigma_{-k}^- = -\frac{\Gamma_R + \Gamma_L}{2} \sigma_{-k}^- - \sqrt{\eta} \sum_{\nu > \mu} \left[ \sigma_{\nu}^- \Gamma_L e^{ikz_{\mu} - z_{\nu}} + \sigma_{\mu}^- \Gamma_R e^{ikz_{\mu} - z_{\nu}} \right] \] (A13)

Thus, for \( c_k = \sigma_{-k}^- + r(k) \sigma_{-k}^+ \) we have \( \partial_t c_k = -i \omega_k c_k \). For this operator the input equation becomes

\[ \partial_t c_k(t) = -i \sqrt{\Gamma_R} e^{ik z_{\mu}} e^{-ik z_{\nu}} = i \omega_k c_k(t) \]

\[ = -i \sqrt{\Gamma_R} e^{ik z_{\mu}} e^{-ik z_{\nu}} f(k - k_0) f(k' - k_0) - i \omega_k c_k(t), \] (A14)

and isolation of \( c_k(t) \) leads after back and forth transformation between time and frequency (analogously to the output calculation) to

\[ c_k(t) = -i \sqrt{\Gamma_R} e^{ik z_{\mu}} e^{-ik z_{\nu}} f(k - k_0) f(k' - k_0) e^{-i \omega_k t}, \] (A15)
The entrance probability for a photon into the chain can be calculated just like in the output case as

\[|c_k(t=0)|^2dk = \Gamma_R|f(k-k_0) + r(k)f'(k-k_0)|^2|\psi_{\text{in}}(z_1,\omega_k)|^2 \frac{\partial k}{\partial \omega} d\omega = \frac{1}{d} F(k, k_0)|\psi_{\text{in}}(z_1, \omega_k)|^2d\omega, \quad (A16)\]

with \(F(k, k_0) = (1 - |r(k)|^2)\) if the setup is completely non-chiral. After sufficiently long time (after which the wave package in question is localized far enough away from the boundaries) the \(\sigma_k^+; in\) \(c_k(t)\) do no longer meaningfully contribute and we can map \(c_k \to \sigma_k\).

**Appendix B. Two polariton scattering**

To solve the scattering problem we apply the effective Hamiltonian (3) on \(|q, K\rangle\) to obtain [23]

\[H_{\text{sys}}(q, K) = \hbar\omega_{q, K}|q, K\rangle + \hbar\Gamma_R[|i - a(q, k_0 - K)|k_0 - K, K] + \hbar\Gamma_L[|i - a(q, k_0 + K)|k_0 + K, K], \quad (B1)\]

where the total energy \(\omega_{q, K} = \omega_{k_1} + \omega_{k_2}\) and \((q, p) = \sin q d/|\cos q d - \cos pd|\). To get rid of the second and third line in (B1) and turn it into an eigen-equation we make a scattering ansatz

\[|\Psi_{q, K}\rangle = |q, K\rangle + t_1|q - K, K\rangle + t_2|q, K\rangle\]

where \(q\) is a degenerate momentum number fulfilling \(\omega_{q', K} = \omega_{q, K}\) with \(|q'| \neq |q|\) (see below). Requiring \(|k_0 - K, K\rangle\) and \(|k_0 + K, K\rangle\) to vanish from \(H_{\text{sys}}|\Psi_{q, K}\rangle\), then give two equations determining \(t_1\) and \(t_2\) as a function of \(q, K, k_0\) and \(\Gamma_R, \Gamma_L\).

To solve (B1) we have to find the degenerate \(q'\) which fulfill \(\omega_{q', K} = \omega_{q, K}\), i.e.

\[\Gamma_L \frac{\sin([k_0 + K]d)}{\cos[qd] - \cos([k_0 + K]d)} + \Gamma_L \frac{\sin([k_0 - K]d)}{\cos[qd] - \cos([k_0 - K]d)} = \Gamma_R \frac{\sin([k_0 + K]d)}{\cos[q'd] + \cos([k_0 + K]d)} + \Gamma_R \frac{\sin([k_0 - K]d)}{\cos[q'd] - \cos([k_0 - K]d)}. \quad (B2)\]

This equation leads to a quadratic function

\[c_1 \cos^2 q' + c_2 \cos q' + c_3 = 0\]

with \(r = \Gamma_R/\Gamma_L\). The solution of this quadratic equation can be simplified to

\[r \cos^2([k_0 + K]d) \sin((k_0 - K)d) + \cos(qd)(r - 1) \sin(2Kd) - (r + 1) \sin(2kd) + 2 \cos^2([k_0 - K]d) \sin((k_0 + K)d)\]

\[2(r - 1) \sin Kd \cos qd \cos k_0 d - \cos Kd) + 2(1 + r) \sin k_0 d \cos k_0 d - \cos qd \cos Kd\]

\[= \cos q', \quad (B4)\]

or \(q' = \pm q\). Inserting these \(q'\) into (B1) to get rid of the second and third line results in the two conditions

\[1 + ia(q, k_0 + K) + t_1[1 - ia(q, k_0 + K)] + t_2[1 - ia(q', k_0 + K)] = 0\]

\[1 + ia(q, k_0 - K) + t_1[1 - ia(q, k_0 - K)] + t_2[1 - ia(q', k_0 - K)] = 0, \quad (B5)\]

which is solved by

\[t_1 = \frac{a(q, k_0 + K)[a(q', k_0 - K) - i] + i[a(q, k_0 - K) - a(q', k_0 - K)] - a(q', k_0 + K)[a(q, k_0 - K) - i]}{a(q, k_0 + K)[a(q', k_0 - K) - i] + i[a(q, k_0 - K) + a(q', k_0 - K)] - a(q', k_0 + K)[a(q, k_0 - K) + i]}\]

\[t_2 = \frac{2i[a(q, k_0 + K) - a(q, k_0 - K)] - i[a(q, k_0 - K) + a(q', k_0 - K)] - a(q', k_0 + K)[a(q, k_0 - K) - i]}{a(q, k_0 + K)[a(q', k_0 - K) - i] + i[a(q, k_0 - K) + a(q', k_0 - K)] - a(q', k_0 + K)[a(q, k_0 - K) + i]} \quad (B6)\]

Inserting the degenerate momentum (B1) into the transmission coefficient (B6) leads to a cumbersome analytical expression. The scattering fulfills the continuity equation with the relative group velocities \(v_g(q) = \partial \omega(q)/\partial q\). To show this we note that

\[|t_1|^2 + |t_2|^2v_g(q')v_g(q) - 1 = \frac{(a(q', k_0 + K) - a(q', k_0 - K)v_g(q')/v_g(q) - a(q, k_0 + K) + a(q, k_0 - K)}{f(q, q', k_0, K)} \quad (B7)\]
for every $q$ with degenerate, real $q'$. The complicated function $f(q,q',k_0,K)$ is of no further interest since the numerator is proportional to

$$
\sin[(k_0 + K)d] \cos[|q'|d] - \cos[(k_0 - K)d] \cos[(k_0 + K)d] + r \sin[(k_0 - K)d] \cos[|q'|d] - \cos[(k_0 - K)d] 
$$

which is very close to the degeneracy condition (B2), leads to a quadratic equation with coefficients proportional to (B3) (the factor being $\cos[(k_0 + K)d] - \cos[(k_0 - K)d]$) and vanishes when (B4) holds. Thus for degenerate $k$ the term (B7) always vanishes and the continuity equation is fulfilled at all degrees of chirality $r = \Gamma_R/\Gamma_L$. Whenever $k$ is complex, the scattering involves meta-stable states with finite extension and $|t_k| = 1$.

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