Solution group representations as quantum symmetries of graphs

Roberson, David E.; Schmidt, Simon

Published in:
Journal of the London Mathematical Society

DOI:
10.1112/jlms.12664

Publication date:
2022

Document version
Publisher's PDF, also known as Version of record

Document license:
CC BY

Citation for published version (APA):
Solution group representations as quantum symmetries of graphs

David E. Roberson1 | Simon Schmidt2

1Department of Applied Mathematics and Computer Science, Technical University of Denmark, Lyngby, Denmark
2QMATH, Department of Mathematical Sciences, University of Copenhagen, Copenhagen, Denmark

Correspondence
Simon Schmidt, QMATH, Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark.
Email: sisc@math.ku.dk

Funding information
VILLUM Centre of Excellence for the Mathematics of Quantum Theory (QMATH), Grant/Award Number: 35544; Horizon 2020, Grant/Award Number: 101030346; EPSRC, Grant/Award Number: EP/T03064X/1

Abstract
In 2019, Aterias et al. constructed pairs of quantum isomorphic, non-isomorphic graphs from linear constraint systems. This article deals with quantum automorphisms and quantum isomorphisms of colored versions of those graphs. We show that the quantum automorphism group of such a colored graph is the dual of the homogeneous solution group of the underlying linear constraint system. Given a vertex- and edge-colored graph with certain properties, we construct an uncolored graph that has the same quantum automorphism group as the colored graph we started with. Using those results, we obtain the first-known example of a graph that has quantum symmetry and finite quantum automorphism group. Furthermore, we construct a pair of quantum isomorphic, non-isomorphic graphs that both have no quantum symmetry.

MSC 2020
46LXX (primary), 20B25, 05CXX (secondary)

1 | INTRODUCTION

Generalizing Mermin’s magic square [13], linear system games were introduced by Cleve and Mittal in [7]. It was then shown by Cleve, Liu and Slofstra [6] that perfect quantum strategies for those games are related to the representations of the associated solution group; a finitely presented group with generators and relations reflected by the underlying linear system.
Together with his collaborators, the first author [1] constructed quantum isomorphic, but non-isomorphic graphs from linear system games having perfect quantum strategies but no such classical strategies. In this case, a quantum isomorphism corresponds to a perfect quantum strategy of the isomorphism game which was also introduced in [1].

Quantum automorphism groups of graphs were first introduced by Banica [2] and Bichon [5] to obtain examples of quantum permutation groups. They are generalizations of the automorphism group of a graph in the framework of Woronowicz’ compact matrix quantum groups. An interesting question to ask is which quantum permutation groups can be realized as the quantum automorphism group of some graph. This question has, for example, been considered in [4].

In this work, we give a correspondence between representations of solution groups and quantum isomorphisms/quantum automorphisms of colored versions of the graphs appearing in [1]. More specifically, we will see that the quantum automorphism groups of the colored versions of those graphs are given by the dual of the homogeneous solution group of the underlying linear constraint system. Furthermore, representations of the non-homogeneous solution group provide quantum isomorphisms between the colored versions of the graphs.

Additionally, we will discuss a decoloring procedure for vertex- and edge-colored graphs which does not change the quantum automorphism group for certain graphs. This procedure decolors the vertices by adding paths of different lengths to vertices of different colors. In a second step, we get rid of the edge-colors by subdividing the colored edges and then adding paths of different lengths to the newly added vertices.

The previous results allow us to construct two explicit examples of graphs that were not known before: First, we construct a graph that has quantum symmetry and finite quantum automorphism group. Second, we obtain a pair of graphs that are quantum isomorphic and non-isomorphic, where both graphs additionally do not have quantum symmetry.

The article is structured as follows. In Section 2, we briefly discuss compact quantum groups and give the definition of the quantum automorphism group of a vertex- and edge-colored graph. In Section 3, we define colored versions of the graphs appearing in [1]. Here we prove that the dual of the solution group is the quantum automorphism group of such a graph. Section 4 deals with decoloring the graphs from the previous section without changing the quantum automorphism group. In Section 5, we provide (uncolored) graphs whose quantum automorphism group is the dual of a solution group. We in particular obtain a graph with quantum symmetry and finite quantum automorphism group, see Corollary 5.6. Finally, we discuss quantum isomorphisms of the colored graphs in Section 6. Here, we present our example of a pair of graphs that are quantum isomorphic but non-isomorphic, where both graphs do not have quantum symmetry, see Corollary 6.11.

## 2 | PRELIMINARIES

We start with the definition of a compact quantum group, see [21]. Throughout this article, we write $A \otimes B$ for the minimal tensor product of the $C^*$-algebras $A$ and $B$.

**Definition 2.1.** A compact quantum group $\mathbb{G}$ is a pair $(C(\mathbb{G}), \Delta)$, where $C(\mathbb{G})$ is a unital $C^*$-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism with the following properties.

- $(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ (coassociativity).
- $\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))$ and $\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)$ are dense in $C(\mathbb{G}) \otimes C(\mathbb{G})$. 
Definition 2.2. Let $G = (C(G), \Delta_G)$ and $H = (C(H), \Delta_H)$ be compact quantum groups. Then $G$ and $H$ are isomorphic as compact quantum groups if there exists an $\ast$-isomorphism $\varphi : C(G) \to C(H)$ such that $\Delta_H \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G$. We then write $G \cong H$.

Definition 2.3. Let $G = (C(G), \Delta_G)$ be a compact quantum group. We say that $G$ is finite if the $C^\ast$-algebra $C(G)$ is finite dimensional.

The following example can, for example, be found in [12, Example 4.5].

Example 2.4. Let $\Gamma$ be a discrete group. Note that the group $C^\ast$-algebra $C^\ast(\Gamma)$ can be written as $C^\ast(\Gamma) = C\ast(u_g, g \in \Gamma | u_g \text{ unitary, } u_{gh} = u_g u_h, u_{g^{-1}} = u_g^\ast)$, see, for example, [19, A.6]. Let $\Delta$ be the $\ast$-homomorphism $\Delta : C^\ast(\Gamma) \to C^\ast(\Gamma) \otimes C^\ast(\Gamma)$, $\Delta(u_g) = u_g \otimes u_g$. Then the pair $(C^\ast(\Gamma), \Delta)$ is a compact quantum group known as the dual of $\Gamma$ which we denote by $\hat{\Gamma}$.

We will now define compact matrix quantum groups, a subclass of compact quantum groups.

Definition 2.5 [20]. A compact matrix quantum group $G$ is a pair $(C(G), u)$, where $C(G)$ is a unital $C^\ast$-algebra and $u = (u_{ij}) \in M_n(C(G))$ is a matrix such that:

- the elements $u_{ij}$, $1 \leq i, j \leq n$, generate $C(G)$;
- the $\ast$-homomorphism $\Delta : C(G) \to C(G) \otimes C(G)$, $u_{ij} \mapsto \sum_{k=1}^n u_{ik} \otimes u_{kj}$ exists;
- the matrix $u$ and its transpose $u^T$ are invertible.

The matrix $u$ is usually called the fundamental representation of $G$.

A very important example is the quantum symmetric group, the quantum analogue of the symmetric group. It was defined by Wang in [18].

Definition 2.6. The quantum symmetric group $S^+_n = (C(S^+_n), u)$ is the compact matrix quantum group, where

$$C(S^+_n) := C^\ast(u_{ij}, 1 \leq i, j \leq n | u_{ij} = u_{ij}^\ast = u_{ij}^2, \sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1).$$

Note that a matrix $u = (u_{ij})_{i,j \in [n]}$ with entries from a non-trivial unital $C^\ast$-algebra satisfying $u_{ij} = u_{ij}^\ast = u_{ij}^2$ and $\sum_{k=1}^n u_{ik} = \sum_{k=1}^n u_{kj} = 1$, as in the definition above, is known as a magic unitary.

Quantum automorphism groups of finite graphs were defined in [2, 5]. We give a more general definition in the following, for vertex- and edge-colored graphs. Note that edge-colorings are similar to distances in finite metric spaces, as considered in [3]. For us, graphs are undirected and do not have loops or multiple edges. A colored graph is a graph $G$ with vertex set $V$ and edge set $E$ along with a coloring function $c : V \cup E \to S$ for some set $S$. We refer to $c(x)$ as the color of the vertex/edge $x$. To be specific, we will sometimes refer to colored graphs as vertex- and edge-colored graphs. Furthermore, we will also consider edge-colored graphs where the coloring function is
defined only on the edge set $E$, that is, edges but not vertices are colored. In either case, we will use $E_c$ to refer to the set of edges of color $c$. Also, $A_{G_c}$ will denote the adjacency matrix of the edge color $c$, that is, $(A_{G_c})_{ij} = 1$ if $(i, j) \in E_c$ and $(A_{G_c})_{ij} = 0$ otherwise.

**Definition 2.7.** Let $G$ be a vertex- and edge-colored graph. The quantum automorphism group $\text{Qut}(G)$ is the compact matrix quantum group $(C(\text{Qut}(G)), u)$, where $C(\text{Qut}(G))$ is the universal $C^*$-algebra with generators $u_{ij}, i, j \in V(G)$ and relations

$$u_{ij} = u_{ij}^* = u_{ij}^2, \quad i, j \in V(G),$$

$$\sum_l u_{il} = \sum_l u_{li}, \quad i \in V(G),$$

$$u_{ij} = 0, \quad \text{for all } i, j \in V(G) \text{ with } c(i) \neq c(j),$$

$$A_{G_c} u = u A_{G_c}, \quad \text{for all edge colors } c,$$  
(4)

where (4) is nothing but $\sum_k u_{ik}(A_{G_c})_{kj} = \sum_k (A_{G_c})_{ik}u_{kj}$ for all $i, j \in V(G)$ and all edge-colors $c$.

It is not immediately obvious that this defines a compact matrix quantum group. By Definition 2.5, we have to show that $u$ and $u^T$ are invertible as well as that the comultiplication $\Delta : C(\text{Qut}(G)) \to C(\text{Qut}(G)) \otimes C(\text{Qut}(G))$, $u_{ij} \mapsto \sum_k u_{ik} \otimes u_{kj}$ is a $*$-homomorphism. It is enough to show $\Delta(\sum_k u_{ik}(A_{G_c})_{kj}) = \Delta(\sum_k (A_{G_c})_{ik}u_{kj})$ for $i, j \in V(G)$, $c$ edge-color and $\Delta(u_{ij}) = 0$ for $c(i) \neq c(j)$, since we know that $S^+_n$ is a compact matrix quantum group. The first equation follows, for example, from [16, Lemma 2.1.2]. For the second equation, we take vertices $i, j$ with $c(i) \neq c(j)$. We have $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$. Note that there is no $k \in V(G)$ with $c(k) = c(i)$ and $c(k) = c(j)$, since we assumed $c(i) \neq c(j)$. We deduce $u_{ik} \otimes u_{kj} = 0$ for all $k$ and thus $\Delta(u_{ij}) = 0$. Summarizing, we see that $\text{Qut}(G)$ is indeed a compact matrix quantum group.

The following lemma gives relations that are equivalent to Relation (4), see [16, Proposition 2.1.3].

**Lemma 2.8.** Let $u_{ij}, 1 \leq i, j \leq n$, be the generators of $C(S^+_n)$. Then Relation (4) is equivalent to the relations

$$u_{ij}u_{kl} = 0 \text{ if } (i, k) \in E_c, (j, l) \notin E_c \text{ for some edge-color } c,$$

$$u_{ij}u_{kl} = 0 \text{ if } (i, k) \notin E_c, (j, l) \in E_c \text{ for some edge-color } c.$$

### 3 | COLORED GRAPHS WHOSE QUANTUM AUTOMORPHISM GROUP IS THE DUAL OF A SOLUTION GROUP

In the following definition, we use $1$ to denote the identity element of a group.

**Definition 3.1.** Let $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$ with $b \neq 0$. The solution group $\Gamma(M, b)$ of the linear system $Mx = b$ is the group generated by elements $x_i$ for $i \in [n]$ and an element $\gamma$ satisfying the following relations.
1. $x_i^2 = 1$ for all $i \in [n]$.
2. $x_i x_j = x_j x_i$ if there exists $k \in [m]$ s.t. $M_{ki} = M_{kj} = 1$.
3. $\prod_{k \in S} x_i^a = y^{b_k}$ for all $k \in [m]$.
4. $\gamma^2 = 1$.
5. $x_i \gamma = \gamma x_i$ for all $i \in [n]$.

If $b = 0$, then we refer to $\Gamma(M, b)$ as the homogeneous solution group of the system $Mx = 0$, and define this the same as above except that we add the relation $\gamma = 1$. This is equivalent to removing $\gamma$ from the list of generators, changing the right-hand side of the equation in (3) to 1, and removing items (4) and (5). We will sometimes also use $\Gamma_0(M)$ to denote this group.

We will denote by $S_k(M)$ the set $\{i \in [n] : M_{ki} = 1\}$, often writing simply $S_k$ when $M$ is clear from context. We use $\pm 1^S$ to denote the set of functions $\alpha : S \to \{1, -1\}$, and will typically write $\alpha_i$ instead of $\alpha(i)$. We will also use $\pm 1^S_0$ to denote the subset of such functions satisfying $\prod_{i \in S} \alpha_i = 1$, and similarly use $\pm 1^S_1$ for the set of such functions satisfying $\prod_{i \in S} \alpha_i = -1$.

**Definition 3.2.** Let $M \in \mathbb{F}^{m \times n}$ and $b \in \mathbb{F}^m$. Define the colored graph $G := G(M, b)$ as follows.

The vertex set of $G$ is $\{(k, \alpha) : k \in [m], \alpha \in \pm 1^S_{b_k}\}$. The color of a vertex $v = (k, \alpha)$, denoted $c(v)$, is $k$. We also define the subsets $V_k = \{v \in V(G) : c(v) = k\}$ for all $k \in [m]$, and these partition the vertex set. For any $l, k \in [m]$ such that $S_l \cap S_k = \emptyset$, the graph $G$ contains all (non-loop) edges between $V_l$ and $V_k$ (thus each $V_k$ induces a complete subgraph). Given an edge $e$ between adjacent vertices $(l, \alpha)$ and $(k, \beta)$, the color of $e$, denoted $c(e)$, is equal to the function $\alpha \triangle \beta \in \pm 1^{S_l \cap S_k}$ defined as $(\alpha \triangle \beta)_i = \alpha_i \beta_i$.

Note that $\alpha \triangle \beta = \beta \triangle \alpha$ and so the edge colors defined above actually are edge colors and not arc (directed edge) colors.

**Remark 3.3.** According to the above definition, it is possible for two edges $e$ and $e'$ between pairs of vertices $(l, \alpha), (k, \beta)$ and $(l', \alpha'), (k', \beta')$ to be colored the same color even if $l$ is not equal to either $l'$ nor $k'$, that is, the edges are between different pairs of subsets $V_l, V_k$ and $V_{l'}, V_{k'}$. This can happen since it is possible that $S_l \cap S_k = S_{l'} \cap S_{k'} = \emptyset$ and $\alpha \triangle \beta = \alpha' \triangle \beta'$. However, we do wish such edges to be distinguished by the (quantum) automorphism group of $G(M, b)$, that is, $u(l, \alpha, (l', \alpha')) U(k, \beta, (k', \beta')) = 0$ where $u$ is the fundamental representation of $\text{Out}(G(M, b))$. This could be done explicitly by defining the color of the edge $e$ (for instance) to be $c(e) = (l, k, \alpha \triangle \beta)$. However, this is redundant for our purposes since such edges $e$ and $e'$ as described above are already distinguished by the (quantum) automorphism group due to the vertex colors of the endpoints of the edges. In other words, the edges between $V_l$ and $V_k$ can be thought of as being implicitly colored distinctly from those between $V_{l'}$ and $V_{k'}$ whenever $\{l, k\} \neq \{l', k'\}$.

**Remark 3.4.** We also note that in order to reduce the total number of colors used, for each pair $l, k \in [m]$, we can choose one color $\alpha \in \pm 1^{S_l \cap S_k}$ and replace these edges with non-edges. Moreover, instead of coloring the edges between $V_l$ and $V_k$ with functions from $\pm 1^{S_l \cap S_k}$, we can simply use $\{1, \ldots, 2^{\mid S_l \cap S_k\mid} - 1\}$, and this will not change $\text{Out}(G)$ as explained in the previous remark. In fact, we will need to do this for some of our results later on.

\[\text{This is the quantum analog of there being no automorphism mapping the edge } (l, \alpha)(k, \beta) \text{ to } (l', \alpha')(k', \beta').\]
FIGURE 1  The graph $G(M, b)$ for $M$ and $b$ as above. The vertices on the left-hand side are the solutions of $x_1 x_2 x_3 = 1$, the vertices on the right-hand side are the solutions of $x_1 x_4 x_5 = -1$. We used Remarks 3.3 and 3.4 to reduce the number of colors needed in the graph

Example 3.5. Consider the linear system $Mx = b$, where
\[
M = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.
\]
Then the graph $G(M, b)$ is the one given in Figure 1.

Lemma 3.6. Let $M \in \mathbb{F}^{m \times n}_2$ and $b \in \mathbb{F}^m_2$. Set $G = G(M, b)$. If $u$ is the fundamental representation of $\text{Out}(G)$, then $u$ has the form
\[
\begin{pmatrix}
V_1 & V_2 & \cdots & \cdots & V_m \\
V_1 & 0 & 0 & \cdots & 0 \\
V_2 & 0 & u^{(2)} & 0 & \cdots & 0 \\
\vdots & 0 & 0 & u^{(3)} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
V_m & 0 & 0 & 0 & 0 & u^{(m)}
\end{pmatrix}
\]
where each $u^{(k)}$ is a magic unitary. Furthermore, $u^{(k)}_{\alpha, \beta} := u_{(k, \alpha), (k, \beta)}$ depends only on $k$ and the value of $\alpha \bigtriangleup \beta$ and therefore the entries of $u^{(k)}$ pairwise commute for each $k \in [m]$.  

Proof. If $l \neq k$, then $c((l, \alpha)) = l \neq k = c((k, \beta))$ and thus $u_{(l, \alpha), (k, \beta)} = 0$. This shows that $u$ has the block form given in the lemma statement. Moreover, each diagonal block must be a magic unitary since $u$ is a magic unitary.

Now fix $k \in [m]$. Pick $\tilde{\alpha}, \tilde{\beta} \in \pm 1_{S_k}$ arbitrarily. Suppose that $\alpha, \beta \in \pm 1_{S_k}$ are such that $\alpha \bigtriangleup \beta = \tilde{\alpha} \bigtriangleup \tilde{\beta}$. Note that since all these functions are elements of $\pm 1_{S_k}$, the operation $\bigtriangleup$ is simply the pointwise product, and therefore we have that $\tilde{\alpha} \bigtriangleup \alpha = \tilde{\beta} \bigtriangleup \beta$. In particular, this implies that the color of the edge between $(k, \tilde{\alpha})$ and $(k, \alpha)$ is equal to the color of the edge between $(k, \tilde{\beta})$ and $(k, \beta)$ if and only if $\beta' = \tilde{\beta}$. Therefore, $u^{(k)}_{\tilde{\alpha}, \tilde{\beta}} u^{(k)}_{\alpha, \beta} = 0$ unless $\beta' = \tilde{\beta}$. We similarly have that $u^{(k)}_{\tilde{\alpha}, \tilde{\beta}} u^{(k)}_{\alpha, \beta'} = 0$ unless $\tilde{\beta}' = \tilde{\beta}$. It follows that
\[
u^{(k)}_{\tilde{\alpha}, \tilde{\beta}} = \sum_{\beta' \in \pm 1_{S_k}} u^{(k)}_{\tilde{\alpha}, \beta'} = \sum_{\tilde{\beta}' \in \pm 1_{S_k}} u^{(k)}_{\tilde{\alpha}, \tilde{\beta}} = u^{(k)}_{\alpha, \beta} = u^{(k)}_{\alpha, \beta'}.
\]
Thus the operator $u^{(k)}_{\alpha,\beta}$ depends only on the value of $\alpha \triangle \beta$ (and $k$) as claimed. Let us therefore use $u^{(k)}_\delta$ to denote $u^{(k)}_{\alpha,\beta}$ where $\delta = \alpha \triangle \beta$. For any fixed $\alpha \in \pm 1^{S_k}_{b_k}$, the set $\{\alpha \triangle \beta : \beta \in \pm 1^{S_k}_{b_k}\}$ is equal to $\pm 1^0_{S_k}$. Thus the set of operators appearing in any row (and similarly any column) of $u^{(k)}$ is $\{u^{(k)}_\delta : \delta \in \pm 1^0_{S_k}\}$. Since the entries of any given row of a magic unitary commute, and every row of $u^{(k)}$ contains the same set of entries, we have that all entries of $u^{(k)}$ commute. □

Remark 3.7. Let $\Gamma = \Gamma(M,0)$ be the homogeneous solution group of the system $Mx = 0$. By Example 2.4, the group $C^*$-algebra $C^*(\Gamma)$ can be written as

$$C^*(\Gamma) = C^*(x_i, i \in [n] | x_i = x_i^*, x_i^2 = 1, x_i x_j = x_j x_i \text{ if there exists } k \in [m] \text{ s.t. } M_{ki} = M_{kj} = 1,$$

$$\prod_{i: M_{ki} = 1} x_i = 1 \text{ for all } k \in [m]).$$

Note that we slightly abuse the notation by writing $x_i$ instead of $u x_i$.

Theorem 3.8. Let $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$. Set $G = G(M,b)$ and $\Gamma = \Gamma(M,0) = \Gamma_0(M)$. Let $\Delta_\Gamma : C^*(\Gamma) \to C^*(\Gamma) \otimes C^*(\Gamma)$ denote the coproduct given by $\Delta_\Gamma(g) = g \otimes g$ for $g \in \Gamma$, and let $\Delta_G : C(\text{Qut}(G)) \to C(\text{Qut}(G)) \otimes C(\text{Qut}(G))$ denote the coproduct given by $\Delta_G(u_{a,c}) = \sum_{b \in V(G)} u_{a,b} \otimes u_{b,c}$ where $u = (u_{a,c})$ is the fundamental representation of $\text{Qut}(G)$. Then there exists an isomorphism $\varphi : C^*(\Gamma) \to C(\text{Qut}(G))$ such that $\Delta_G \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_\Gamma$.

Proof. The proof is structured as follows: we use the universal properties of $C^*(\Gamma)$ and $C(\text{Qut}(G))$, respectively, to first obtain a $*$-homomorphism $\varphi_1 : C^*(\Gamma) \to C(\text{Qut}(G))$, and then obtain a $*$-homomorphism $\varphi_2 : C(\text{Qut}(G)) \to C^*(\Gamma)$. We will then show that $\varphi_1$ and $\varphi_2$ are inverses of each other thus proving that they are in fact isomorphisms. Lastly, we will prove that $\varphi_1$ intertwines the coproducts as described in the theorem statement.

Step 1: Construction of a $*$-homomorphism $\varphi_1 : C^*(\Gamma) \to C(\text{Qut}(G))$.

For this step, we will construct elements $y_i$ of $C(\text{Qut}(G))$ that satisfy the relations of the generators $x_i$ of $C^*(\Gamma)$ given in Remark 3.7. This proves that there is a $*$-homomorphism $\varphi_1$ from $C^*(\Gamma)$ to $C(\text{Qut}(G))$ such that $\varphi_1(x_i) = y_i$. Later we will see that $\varphi_1$ is in fact an isomorphism.

Let $u$ be the fundamental representation of $\text{Qut}(G)$. By Lemma 3.6, $u = \bigoplus_{k \in [m]} u^{(k)}$ where each $u^{(k)}$ is a magic unitary indexed by $V_k = \{(k,\alpha) : \alpha \in \pm 1^{S_k}_{b_k}\}$. Moreover, the operator $u^{(k)}_{\alpha,\beta} = u((k,\alpha),(k,\beta))$ depends only on $k$ and the value of $\alpha \triangle \beta \in \pm 1^0_{S_k}$. Thus, as in the proof of Lemma 3.6, for each $\delta \in \pm 1^0_{S_k}$ we let $u^{(k)}_\delta$ denote $u^{(k)}_{\alpha,\beta}$ such that $\alpha \triangle \beta = \delta$. Note that for any $\alpha \in \pm 1^0_{S_k}$, we have that the set $\{\alpha \triangle \beta : \beta \in \pm 1^0_{S_k}\}$ is equal to $\pm 1^0_{S_k}$ and thus every row/column of $u^{(k)}$ contains the same set of operators and

$$\sum_{\delta \in \pm 1^0_{S_k}} u^{(k)}_\delta = \sum_{\alpha,\beta \in \pm 1^0_{S_k}} u^{(k)}_{\alpha,\beta} = \sum_{\alpha \in \pm 1^0_{S_k}} u^{(k)}_{\alpha,\beta} = 1. \quad (6)$$

Now we define $y_i^{(k)} = \sum_{\alpha \in \pm 1^0_{S_k}} \alpha_i u^{(k)}_{\alpha}$ for all $k \in [m]$ and $i \in S_k$. We first aim to show that $y_i^{(k)}$ does not depend on $k$.

Fix $i \in [n]$ and $k, l \in [m]$ such that $i \in S_l \cap S_k$. Consider the edges between the subsets $V_l$ and $V_k$. For each $\delta \in \pm 1^{S_l \cap S_k}$, let $A_\delta$ be the adjacency matrix of the graph consisting of the edges of $G$.
colored $\delta$. Further, let $B^\delta$ be the submatrix of $A^\delta$ consisting of the rows indexed by $V_l$ and columns indexed by $V_k$. In other words,

$$B^\delta_{(l,\alpha),(k,\beta)} = \begin{cases} 1 & \text{if } \alpha \bigtriangledown \beta = \delta \\ 0 & \text{o.w.} \end{cases}$$

Also define $A^+ = \sum_{\delta \in \pm 1} S_l \cap S_k, \delta = +1 A^\delta$ and $A^- = \sum_{\delta \in \pm 1} S_l \cap S_k, \delta = -1 A^\delta$, and similarly define $B^+$ and $B^-$. Then

$$B^+_{(l,\alpha),(k,\beta)} = \begin{cases} 1 & \text{if } \alpha_i = \beta_i \\ 0 & \text{o.w.} \end{cases} \quad & B^-_{(l,\alpha),(k,\beta)} = \begin{cases} 1 & \text{if } \alpha_i = -\beta_i \\ 0 & \text{o.w.} \end{cases}$$

Since $u$ must commute with each $A^\delta$ by definition of $\text{Qut}(G)$, it must also commute with both $A^+$ and $A^-$. This implies that $u^{(l)} B^+ = B^+ u^{(k)}$ and $u^{(l)} B^- = B^- u^{(k)}$. Considering the $(l, \alpha), (k, \beta)$ entry of both sides of the former equation, we see that

$$\sum_{\alpha' \in \pm 1}^{S_l} b^{(l)}_{\alpha'} \text{ s.t. } \alpha' = \beta_i \sum_{\alpha \in \pm 1}^{S_l} u^{(l)}_{\alpha} = \sum_{\beta' \in \pm 1}^{S_k} b^{(k)}_{\beta'} \text{ s.t. } \beta' = \alpha_i$$

Note that for every term in the above sums, we have that $\alpha, \alpha' = \alpha_i \beta_i = \beta_i' \beta_i$. Thus, reexpressing $u^{(l)}_{\alpha, \alpha'}$ as $u^{(l)}_{\tilde{\alpha}}$ where $\tilde{\alpha} = \alpha \bigtriangledown \alpha'$, and similarly for $u^{(k)}_{\beta', \beta'}$ we have that

$$\sum_{\tilde{\alpha} \in \pm 1}^{S_l} u^{(l)}_{\tilde{\alpha}} = \sum_{\tilde{\beta} \in \pm 1}^{S_k} u^{(k)}_{\tilde{\beta}}$$

Doing the same for $u^{(l)} B^- = B^- u^{(k)}$ yields the same equation but with $\alpha_i \beta_i$ replaced with $-\alpha_i \beta_i$ and combining these proves that

$$\sum_{\tilde{\alpha} \in \pm 1}^{S_l} u^{(l)}_{\tilde{\alpha}} = \sum_{\tilde{\beta} \in \pm 1}^{S_k} u^{(k)}_{\tilde{\beta}}$$

for any $x \in \{+1, -1\}$. Therefore,

$$y^{(l)}_i = \left( \sum_{\alpha \in \pm 1}^{S_l} u^{(l)}_{\alpha} \right) - \left( \sum_{\alpha \in \pm 1}^{S_l} u^{(l)}_{\alpha} \right) = \left( \sum_{\alpha \in \pm 1}^{S_k} u^{(k)}_{\alpha} \right) - \left( \sum_{\alpha \in \pm 1}^{S_k} u^{(k)}_{\alpha} \right) = y^{(k)}_i.$$ 

So we have shown that the value of $y^{(k)}_i$ does not depend on $k$, and thus we will simply denote this operator by $y_i$.

Now note that since $y_i$ is a linear combination (with real coefficients) of the operators $u^{(k)}_{\alpha}$ for $k \in [m]$ such that $i \in S_k$, and these operators are entries of the magic unitary $u$, we have that
Also, by equation (6) we have that $u^{(k)}_{\alpha} u^{(k)}_{\beta} = 0$ for $\alpha \neq \beta$ and thus

$$y^2_i = \left( \sum_{\alpha \in \pm 1^{S_k}} \alpha_i u^{(k)}_\alpha \right)^2 = \sum_{\alpha \in \pm 1^{S_k}} \alpha_i^2 (u^{(k)}_\alpha)^2 = \sum_{\alpha \in \pm 1^{S_k}} u^{(k)}_\alpha = 1.$$ 

Thus the $y_i$ satisfy relation (1) from Definition 3.1.

Next we will show that relation (2) of Definition 3.1 holds, that is, that $y_i y_j = y_j y_i$ if there exists $k \in [m]$ such that $i, j \in S_k$. Suppose that $i, j, k$ are as described. Then $y_i = y^{(k)}_i$ and $y_j = y^{(k)}_j$ are both linear combinations of the entries of $u^{(k)}$ which pairwise commute by Lemma 3.6. Therefore, $y_i$ and $y_j$ commute as desired.

Lastly, we must show that relation (3) of Definition 3.1 holds. Recall that we are trying to show that $C^*(\Gamma_0(M)) \cong C(\text{Out}(G))$, that is, we are in the homogeneous solution group case. Thus we must show that $\prod_{i \in S_k} y_i = 1$ for all $k \in [m]$. We have that

$$\prod_{i \in S_k} y_i = \prod_{i \in S_k} y^{(k)}_i = \prod_{i \in S_k} \left( \sum_{\alpha \in \pm 1^{S_k}} \alpha_i u^{(k)}_\alpha \right).$$

Since the elements $u^{(k)}_\alpha$ are pairwise orthogonal, when we expand the above product all cross terms disappear and we obtain

$$\prod_{i \in S_k} y_i = \sum_{\alpha \in \pm 1^{S_k}} \left( \prod_{i \in S_k} \alpha_i \right) u^{(k)}_\alpha = \sum_{\alpha \in \pm 1^{S_k}} u^{(k)}_\alpha = 1,$$

as desired. Therefore, the elements $y_i \in C(\text{Out}(G))$ for $i \in [n]$ satisfy the relations of the generators of $C^*(\Gamma)$ and thus there exists an $*$-homomorphism $\varphi_1$ from $C^*(\Gamma)$ to $C(\text{Out}(G))$. Such that $\varphi_1(x_i) = y_i$ for all $i \in [n]$.

**Step 2: Construction of a $*$-homomorphism $\varphi_2 : C(\text{Out}(G)) \to C^*(\Gamma)$**

For this step, we will construct elements $u^{(l,\alpha),(k,\beta)}_\delta$ of $C^*(\Gamma)$ that satisfy all the relations of the generators $u^{(l,\alpha),(k,\beta)}_\delta$ of $C(\text{Out}(G))$. This proves that there is a $*$-homomorphism $\varphi_2$ from $C(\text{Out}(G))$ to $C^*(\Gamma)$ such that $\varphi_2(u^{(l,\alpha),(k,\beta)}_\delta) = u^{(l,\alpha),(k,\beta)}_\delta$. Later we will see that $\varphi_2$ is an isomorphism.

For each $i \in [n]$, define $p^{\pm}_i = \frac{1}{2}(1 \pm x_i)$. It is straightforward to check that $p^{\pm}_i$ is a projection, that is, $(p^{\pm}_i)^* = p^{\pm}_i = (p^{\pm}_i)^2$, and that $x_i = p^{+}_i - p^{-}_i$ and $1 = p^{+}_i + p^{-}_i$. Note that the latter implies that $p^{+}_i$ and $p^{-}_i$ are orthogonal, that is, $p^{+}_i p^{-}_i = 0$. We will abuse notation somewhat and write $p^{\alpha}_{i,j}$ to denote $p^{+}_i$ whenever $\alpha = +1$ and similarly for $p^{-}_i$ when $\alpha = -1$. Next, for all $k \in [m]$ and $\alpha \in \pm 1^{S_k}$ define $v^{(k)}_\alpha = \prod_{i \in S_k} p^{\alpha}_{i,j}$. Note that the $v^{(k)}_{\alpha}$ are indeed well defined since the commutativity of the elements $p^{\pm}_i$ for $i \in S_k$ follows from the commutativity of the $x_i$ for $i \in S_k$ (that is, relation (2) from Definition 3.1). Since it is the product of pairwise commuting projections, we have that $v^{(k)}_{\alpha}$ is a projection. Furthermore,

$$\sum_{\alpha \in \pm 1^{S_k}} v^{(k)}_{\alpha} = \sum_{\alpha \in \pm 1^{S_k}} \prod_{i \in S_k} p^{\alpha}_{i,j} = \prod_{i \in S_k} (p^{+}_i + p^{-}_i) = \prod_{i \in S_k} 1 = 1.$$ 

(7)

This also implies that for a fixed $k \in [m]$ the projections $v^{(k)}_{\alpha}$ are pairwise orthogonal.
Now suppose that $\alpha \in \pm 1_{S_k}$, that is, that $\prod_{i \in S_k} \alpha_i = -1$. Then, using the easily checked fact that $x_i p_i^{\alpha_i} = \alpha_i p_i^{\alpha_i}$, we have that
\[
v^{(k)}_{\alpha} = \left( \prod_{i \in S_k} x_i \right) v^{(k)}_{\alpha} = \prod_{i \in S_k} \alpha_i p_i^{\alpha_i} = \left( \prod_{i \in S_k} \alpha_i \right) v^{(k)}_{\alpha} = -v^{(k)}_{\alpha}.
\]
(8)
Thus $v^{(k)}_{\alpha} = 0$ for all $\alpha \in \pm 1_{S_k}$. Combining this with equation (7), we have that
\[
\sum_{\alpha \in \pm 1_{S_k}} v^{(k)}_{\alpha} = 1.
\]
(9)

Next, for all $k \in [m]$ and $\alpha, \beta \in \pm 1_{b_k}$ we define $v^{(k)}_{\alpha, \beta} := v^{(k)}_{\delta}$ where $\delta = \alpha \triangle \beta$. Lastly, define $v$ to be the matrix indexed by $V(G)$ such that
\[
v(l, \alpha, \beta) = \begin{cases} v^{(l)}_{\alpha, \beta} & \text{if } l = k \\ 0 & \text{o.w.} \end{cases}
\]
We aim to show that $v$ satisfies all of the relations satisfied by the fundamental representation $u$ of Qut(G). First, $v$ is a magic unitary since all of its entries are projections and the sum of its $(l, \alpha)$-row is simply $\sum_{\beta \in \pm 1_{b_l}} v^{(l)}_{\alpha, \beta} = \sum_{\delta \in \pm 1_{b_l}} v^{(l)}_{\delta} = 1$. Next we must show that for $a, b \in V(G)$, we have $v_{a,b} = 0$ whenever $c(a) \neq c(b)$, that is, the colors of $a$ and $b$ are different. Recall from the definition of $G(M, b)$ that the color of the vertex $(l, \alpha)$ is $l$. Thus $v$ satisfies this relation by definition. Lastly, we must show that for $a, b, a', b' \in V(G)$, we have $v_{a,b} v_{a', b'} = 0$ whenever the edges $\{a, a'\}$ and $\{b, b'\}$ have different colors (or when one is an edge and the other is not). The previously shown relation already implies this one unless $c(a) = c(b)$ and $c(a') = c(b')$. So we may assume that $a = (l, \alpha), b = (l, \beta), a' = (k, \alpha'),$ and $b' = (k, \beta')$. Thus, letting $\delta = \alpha \triangle \beta$ and $\delta' = \alpha' \triangle \beta'$, we have $v_{a,b} = v^{(l)}_{\alpha, \beta} = v^{(l)}_{\delta}$ and $v_{a',b'} = v^{(k)}_{\alpha', \beta'} = v^{(k)}_{\delta'}$. Suppose the edges $\{a, a'\}$ and $\{b, b'\}$ do have different colors. By definition of $G(M, b)$, this is equivalent to $\alpha \triangle \alpha' \neq \beta \triangle \beta'$. This means there exists $j \in S_l \cap S_k$ such that $\alpha_j \alpha'_j \neq \beta_j \beta'_j$. Since all these terms are in $\{+1, -1\}$, we have that $\delta_j = \alpha_j \beta_j \neq \alpha'_j \beta'_j = \delta'_j$. This implies $p^j_{\delta_j} p^j_{\delta'_j} = 0$. Therefore,
\[
v_{a,b} v_{a', b'} = v^{(l)}_{\delta} v^{(k)}_{\delta'} = \left( \prod_{i \in S_l} p^j_i \right) \left( \prod_{i \in S_k} p^j_i \right) = p^j_{\delta_j} p^j_{\delta'_j} \left( \prod_{i \in S_l \setminus \{j\}} p^j_i \right) \left( \prod_{i \in S_k \setminus \{j\}} p^j_i \right) = 0.
\]
So we have shown that $v$ is a magic unitary satisfying all of the relations satisfied by $u$ and thus by the universal property of $C(Qut(G))$, there exists an *-homomorphism $\varphi_2$ from $C(Qut(G))$ to $C^*(\Gamma)$ such that $\varphi_2(u(l, \alpha, (k, \beta)) = v(l, \alpha, (k, \beta))$, which is equivalent to $\varphi_2(u^{(k)}_{\delta}) = v^{(k)}_{\delta}$ for all $k \in [m]$ and $\delta \in \pm 1_{S_k}$.

**Step 3: Showing that $\varphi_1$ and $\varphi_2$ are inverses of each other.**

We first show that $\varphi_2 \circ \varphi_1 : C^*(\Gamma) \to C^*(\Gamma)$ is the identity. Since both $\varphi_1$ and $\varphi_2$ are *-homomorphisms, it suffices to show that $\varphi_2 \circ \varphi_1$ acts as the identity on the generators $x_i$ of $C^*(\Gamma)$. From Step 1, we have that $\varphi_1(x_i) = y_i = \sum_{\alpha \in \pm 1_{S_k}} \alpha_i u^{(k)}_{\alpha}$ for any $k \in [m]$ such that $i \in S_k$. From
Step 2, we have that \( \varphi_2(u^{(k)}_\alpha) = u^{(k)}_\alpha = \prod_{i \in S_k} p^{\alpha_i}_i \). Therefore,
\[
\varphi_2 \circ \varphi_1(x_i) = \sum_{\alpha \in \pm^{S_k}_0} \alpha_i \varphi_2(u^{(k)}_\alpha) = \sum_{\alpha \in \pm^{S_k}_0} \alpha_i u^{(k)}_\alpha.
\]

To see that the final expression above is equal to \( x_i \), we will show that multiplying it by \( x_i \) yields 1. First, note that since \( x_i p^{\alpha_i}_i = \alpha_i p^{\alpha_i}_i \), we have that
\[
x_i u^{(k)}_\alpha = x_i \prod_{j \in S_k} p^{\alpha_j}_j = x_i p^{\alpha_i}_i \prod_{j \in S_k \setminus \{i\}} p^{\alpha_j}_j = \alpha_i \prod_{j \in S_k \setminus \{i\}} p^{\alpha_j}_j = \alpha_i v^{(k)}_\alpha.
\]

Thus, making use of equation (9) in the last equality, we have that
\[
x_i(\varphi_2 \circ \varphi_1(x_i)) = x_i \left( \sum_{\alpha \in \pm^{S_k}_0} \alpha_i u^{(k)}_\alpha \right) = \sum_{\alpha \in \pm^{S_k}_0} \alpha_i x_i u^{(k)}_\alpha = \sum_{\alpha \in \pm^{S_k}_0} \alpha_i^2 u^{(k)}_\alpha = \sum_{\alpha \in \pm^{S_k}_0} u^{(k)}_\alpha = 1,
\]
as desired. It follows that \( \varphi_2 \circ \varphi_1(x_i) \) is the inverse of \( x_i \), which is its own inverse by definition. Therefore, \( \varphi_2 \circ \varphi_1(x_i) = x_i \) and thus \( \varphi_2 \circ \varphi_1 \) is the identity map on \( C^*(\Gamma) \).

Now we will show that \( \varphi_1 \circ \varphi_2 : C(\text{Out}(G)) \to C(\text{Out}(G)) \) is the identity map. As above, it suffices to show that \( \varphi_1 \circ \varphi_2 \) acts as the identity on all entries of the fundamental representation \( u \).

Other than 0, the entries of \( u \) are precisely the elements \( u^{(k)}_\alpha \) for \( k \in [m] \) and \( \alpha \in \pm^{S_k}_0 \). From Step 2, we have that \( \varphi_2(u^{(k)}_\alpha) = v^{(k)}_\alpha = \prod_{i \in S_k} p^{\alpha_i}_i = \prod_{i \in S_k} \frac{1}{2}(1 + \alpha_i x_i) \), and from Step 1 we have that \( \varphi_1(x_i) = y_i = \sum_{\alpha \in \pm^{S_k}_0} \alpha_i u^{(k)}_\alpha \). Therefore,
\[
\varphi_1 \circ \varphi_2(u^{(k)}_\alpha) = \varphi_1 \left( \prod_{i \in S_k} \frac{1}{2}(1 + \alpha_i x_i) \right) = \prod_{i \in S_k} \frac{1}{2}(1 + \alpha_i y_i),
\]
where we have used the fact that \( \varphi_1 \) maps \( 1 \in C^*(\Gamma) \) to \( 1 \in C(\text{Out}(G)) \) which follows from \( \varphi_1(1) = \varphi_1(x_i^2) = \varphi_1(x_i)^2 = y_i^2 = 1 \). Let us use \( \tilde{u}^{(k)}_\alpha \) to denote the final expression of equation (10).

We wish to show that \( \tilde{u}^{(k)}_\alpha = u^{(k)}_\alpha \). Since the \( y_i \) satisfy all of the relations that the \( x_i \) satisfy, the same argument as for the \( u^{(k)}_\alpha \) can be used to show that, for fixed \( k \in [m] \) the \( \tilde{u}^{(k)}_\alpha \) are pairwise orthogonal projections satisfying \( \sum_{\alpha \in \pm^{S_k}_0} \tilde{u}^{(k)}_\alpha = 1 \). For \( i \in S_k \) and \( \beta \in \pm^{S_k}_0 \), we have that
\[
\tilde{u}^{(k)}_\alpha y_i = \tilde{u}^{(k)}_\alpha \sum_{\alpha \in \pm^{S_k}_0} \alpha_i \tilde{u}^{(k)}_\alpha = \beta \tilde{u}^{(k)}_\alpha.
\]

Suppose \( \alpha \neq \beta \). Then there exists \( j \in S_k \) such that \( \alpha_j \neq \beta_j \), and therefore
\[
\tilde{u}^{(k)}_\alpha \tilde{u}^{(k)}_\beta = \tilde{u}^{(k)}_\beta \prod_{i \in S_k} \frac{1}{2}(1 + \alpha_i y_i) = \tilde{u}^{(k)}_\beta \prod_{i \in S_k \setminus \{j\}} \frac{1}{2}(1 + \alpha_i y_i).
\]
\[
\begin{align*}
&= \frac{1}{2}(u_\beta^{(k)} + \alpha_j u_\beta^{(k)} y_j) \prod_{i \in S_k \setminus \{j\}} \frac{1}{2}(1 + \alpha_i y_i) \\
&= \frac{1}{2}(u_\beta^{(k)} + \alpha_j \beta_j u_\beta^{(k)}) \prod_{i \in S_k \setminus \{j\}} \frac{1}{2}(1 + \alpha_i y_i) \\
&= \frac{1}{2}(u_\beta^{(k)} - u_\beta^{(k)} \prod_{i \in S_k \setminus \{j\}} \frac{1}{2}(1 + \alpha_i y_i) \\
&= 0,
\end{align*}
\]

where the penultimate equality comes from the fact that \(\alpha_j \neq \beta_j\) implies \(\alpha_j \beta_j = -1\). Using the above we have that

\[
u_\alpha^{(k)} = \nu_\alpha^{(k)} \sum_{\beta \in \pm^1 S_k} \nu_\beta^{(k)} = \sum_{\beta \in \pm^1 S_k} u_\beta^{(k)} \nu_\alpha^{(k)} = \nu_\alpha^{(k)}.
\]

This completes the proof that \(\varphi_1 \circ \varphi_2(u_\alpha^{(k)}) = u_\alpha^{(k)}\), and thus that \(\varphi_1 \circ \varphi_2\) is the identity on \(C(\text{Qut}(G))\). Combining with the above, this proves that \(\varphi_1 : C^*(\Gamma) \to C(\text{Qut}(G))\) and \(\varphi_2 : C(\text{Qut}(G)) \to C^*(\Gamma)\) are isomorphisms of these \(C^*\)-algebras which are inverse to each other.

**Step 4: Showing that** \(\Delta_G \circ \varphi_1 = (\varphi_1 \otimes \varphi_1) \circ \Delta_\Gamma\).

Since \(\varphi_1\) and the coproducts are \(*\)-homomorphisms, it suffices to prove the identity on the generators, that is, that \(\Delta_G(\varphi_1(x_i)) = (\varphi_1 \otimes \varphi_1)(\Delta_\Gamma(x_i))\) for all \(i \in [n]\). From the above, we have that \(\varphi_1(x_i) = y_i = \sum_{\delta \in \pm^1 S_k} \delta_i u_\delta^{(k)}\) for any \(k \in [m]\) such that \(i \in S_k\). Thus \((\varphi_1 \otimes \varphi_1)(\Delta_\Gamma(x_i)) = (\varphi_1 \otimes \varphi_1)(y_i) = y_i \otimes y_i\). We aim to show that \(\Delta_G(\varphi_1(x_i)) = \Delta_G(y_i)\) is equal to \(y_i \otimes y_i\).

First, let us determine \(\Delta_G(u_\delta^{(k)})\) for \(\delta \in \pm^1 S_k\). Picking \(\alpha \in \pm^1 S_k\) arbitrarily, we have that \(u_\delta^{(k)} = u_\delta^{(k)}\alpha,\delta \Delta \alpha\) and thus

\[
\Delta_G\left(u_\delta^{(k)}\right) = \Delta_G\left(u_\delta^{(k)}\alpha,\delta \Delta \alpha\right)
\]

\[
= \sum_{\beta \in \pm^1 S_k} u_{(k,\alpha),(k,\beta)} \otimes u_{(k,\beta),(k,\delta \Delta \alpha)}
\]

\[
= \sum_{\beta \in \pm^1 S_k} u_{(k,\alpha),(k,\beta)} \otimes u_{(k,\beta),(k,\delta \Delta \alpha)}
\]

\[
= \sum_{\delta' \in \pm^1 S_k} u_{\delta'}^{(k)} \otimes u_{\delta \delta'}^{(k)}.
\]

Now we will show that \(\Delta_G(y_i) = y_i \otimes y_i\). We have that

\[
\Delta_G(y_i) = \Delta_G\left(\sum_{\delta \in \pm^1 S_k} \delta_i u_\delta^{(k)}\right)
\]
\[
\sum_{\delta \in \pm 1} \delta \Delta_G (u_{(k)}^{(\delta)})
\]
\[
= \sum_{\delta, \delta' \in \pm 1} \delta_i (u_{\delta}^{(k)} \otimes u_{\delta \Delta \delta'}^{(k)})
\]
\[
= \sum_{\delta, \delta' \in \pm 1} \delta_i \delta_i' (u_{\delta}^{(k)} \otimes u_{\delta \Delta \delta'}^{(k)})
\]
\[
= \sum_{\delta, \delta' \in \pm 1} \left( \delta_i' u_{\delta}^{(k)} \otimes (\delta_i \delta_i' u_{\delta \Delta \delta'}^{(k)}) \right)
\]
\[
= \sum_{\delta' \in \pm 1} \left( \delta_i' u_{\delta}^{(k)} \otimes \left( \sum_{\delta \in \pm 1} (\delta \Delta \delta') u_{\delta \Delta \delta'}^{(k)} \right) \right)
\]
\[
= \sum_{\delta' \in \pm 1} \left( \delta_i' u_{\delta}^{(k)} \otimes \left( \sum_{\delta' \in \pm 1} \delta_i'' u_{\delta' \Delta \delta''}^{(k)} \right) \right)
\]
\[
= \sum_{\delta' \in \pm 1} \delta_i' u_{\delta}^{(k)} \otimes y_i
\]
\[
= y_i \otimes y_i.
\]

Since we have already shown that \((\varphi_1 \otimes \varphi_1) \circ \Delta_G (x_i) = y_i \otimes y_i\), this completes the proof. \(\square\)

**Remark 3.9.** Note that after using Lemma 3.6, the \(*\)-isomorphism \(\varphi : C^*(\Gamma) \to C(\text{Qut}(G))\) is the same as the one between the solution group and the game algebra appearing in [9, Theorem 4.1]. We only became aware of this result after the initial version of this paper.

## 4 Constructing Uncolored Graphs That Have the Same Quantum Automorphism Group as a Given Colored Graph

In the following, we will describe a procedure that given certain vertex- and edge-colored graphs produces a decolored graph with isomorphic quantum automorphism group. This procedure is divided into two steps: We first decolor the vertices and then the edges. We start with a lemma known from [8, Lemma 3.2.3]. The degree \(\text{deg}(v)\) of a vertex \(v\) denotes the number of neighbors of \(v\) in the graph \(G\).

**Lemma 4.1.** Let \(G\) be a finite graph, \(A_G\) be its adjacency matrix and \(u_{vw}, 1 \leq v, w \leq n\) be the generators of \(C(\text{Qut}(G))\). If \((A_G^l)_{vw} \neq (A_G^l)_{uw}\) for some \(l \in \mathbb{N}\), then \(u_{vw} = 0\). Particularly, if \(\text{deg}(v) \neq \text{deg}(w)\), then \(u_{vw} = 0\).
The following lemma can be found in [15, Lemma 3.2]. The distance $d(v, w)$ between two vertices $v, w \in V(G)$ is the length of a shortest path connecting $v$ and $w$.

**Lemma 4.2.** Let $G$ be a finite graph and $u_{vw}, 1 \leq v, w \leq n$ be the generators of $C(\text{Qu}(G))$. If we have $d(v, p) \neq d(w, q)$, then $u_{vw}u_{pq} = 0$.

**Lemma 4.3.** Let $G$ be a finite graph and $u$ the fundamental representation of $\text{Qu}(G)$. Let $v, w \in V(G)$. If there exists $p \in V(G)$ with $d(v, p) = k$ such that $\deg(p) \neq \deg(q)$ for all $q \in V(G)$ with $d(w, q) = k$, then $u_{vw} = 0$.

**Proof.** Let $v, w \in V(G)$ and $p \in V(G), d(v, p) = k$ as above. Using Relation (2) and Lemma 4.2, we get

$$u_{vw} = u_{vw}\left(\sum_{q} u_{pq}\right) = u_{vw}\left(\sum_{q : d(w, q) = k} u_{pq}\right).$$

We conclude

$$u_{vw} = u_{vw}\left(\sum_{q : d(w, q) = k} u_{pq}\right) = 0,$$

since we have $\deg(p) \neq \deg(q)$ and thus $u_{pq} = 0$ by Lemma 4.1. \hfill \square

**Remark 4.4.** Lemmas 4.1–4.3 also work for vertex- and edge-colored graphs (take the decolored adjacency matrix in Lemma 4.1), since colors just add more relations on the generators of $C(\text{Qu}(G))$.

We will now define an edge-colored graph $G'$ from the vertex- and edge-colored graph $G$.

**Definition 4.5.** Let $G$ be a vertex- and edge-colored graph. Depending on the color $c$, attach a path of length $n_c \in \mathbb{N}_0$ to every vertex colored $c$, where $n_{c_1} \neq n_{c_2}$ for colors $c_1 \neq c_2$ and then decolor the vertices of the graph. We choose one of the edge-colors of $G$ and let the edges in the paths all have this edge-color. We denote this new edge-colored (but not vertex-colored) graph by $G'$ (for an example, see Figure 2).

The next lemma gives a description of the fundamental representation of $\text{Qu}(G')$. We will see in the following theorem that the quantum automorphism groups of $G$ and $G'$ are isomorphic.
Lemma 4.6. Let \( G \) be a vertex- and edge-colored graph with \( \deg(v) \geq 3 \) for all \( v \in V(G) \) and let \( G' \) be the graph as in Definition 4.5. Denote by \( v_i \), \( 0 \leq i \leq n_{c(v)} \), the vertices in the added path with \( d(v, v_i) = i \) (thus \( v = v_0 \)) and let \( V_i = \{ x \in V(G') ; x = v_i \text{ for some } v \in V(G) \} \). Then the fundamental representation \( u \) of \( \text{Out}(G') \) is of the form

\[
\begin{pmatrix}
V_0 & V_1 & \ldots & \ldots & V_m \\
V_0 & u_0 & 0 & \cdots & 0 \\
V_1 & 0 & u_1 & \cdots & 0 \\
\vdots & 0 & 0 & \ddots & \vdots \\
\vdots & \vdots & \vdots & \ddots & 0 \\
V_m & 0 & 0 & 0 & u_m
\end{pmatrix}
\]

where additionally \( u_{v_iw_j} = u_{v_0w_0} \) for all \( i \) and \( u_{v_iw_j} = 0 \) for \( c(v) \neq c(w) \), \( c(.) \) being the vertex-colors in the original graph \( G \).

Proof. Step 1: We have \( u_{v_iw_0} = u_{v_0w_0} \) for all \( i \neq 0 \).

We know that it holds \( \deg(v_i) \in \{1, 2\} \) for vertices \( v_i \) with \( i > 0 \). Since we have \( \deg(w_0) \geq 3 \) by assumption, we get \( u_{v_iw_0} = 0 \) by Lemma 4.1. We similarly obtain \( u_{v_0w_0} = 0 \).

Step 2: We have \( u_{v_iw_j} = 0 \) for \( i \neq j \), \( i, j > 0 \).

First assume \( i < j \). Then \( v_0 \) is a vertex with \( d(v_i, v_0) = i \) and \( \deg(v_0) \geq 3 \). Since \( i < j \), we know that the vertices \( q \) with \( d(w_j, q) = i \) are in the path added to \( w \) and thus \( \deg(q) \in \{1, 2\} \). We deduce \( u_{v_iw_j} = 0 \) by Lemma 4.3. The case \( i > j \) follows similarly.

Step 3: We have \( u_{v_1w_0} = u_{v_0w_0} \) for all \( i \).

We first show \( u_{v_1w_0} = u_{v_0w_0} \). It holds

\[
u_{v_0w_0} = u_{v_0w_0} \left( \sum_{k; (k,w_0) \in E} u_{v_1k} \right) = u_{v_0w_0} \left( u_{v_1w_1} + \sum_{p_0; (p_0, w_0) \in E \cap V_0} u_{v_1p_0} \right) = u_{v_0w_0} u_{v_1w_1} = (u_{v_0w_0} + u_{v_0w_2}) u_{v_1w_1} = \left( \sum_{k; (k,v_1) \in E} u_{v_0k} \right) u_{v_1w_1} = u_{v_1w_1},
\]

since \( u_{v_1p_0} = 0 \) and \( u_{v_0w_2} = 0 \) by Step 1. Furthermore, for \( i \geq 1 \), we have

\[
u_{v_iw_1} = u_{v_iw_i} \left( \sum_{k; (k,w_i) \in E} u_{v_{i+1}k} \right) = u_{v_iw_i} (u_{v_iw_{i+1}} + u_{v_iw_{i-1}})
\]
\[ = u_{v_{i+1}w_{i+1}} u_{v_{i+1}w_{i+2}} \]

\[ = \begin{pmatrix} \sum_{k: (k,w_{i+1}) \in E} u_{v_{i}k} \end{pmatrix} u_{v_{i+1}w_{i+1}} \]

\[ = u_{v_{i+1}w_{i+1}} \]

since it holds that \( u_{v_{i+1}w_{i+2}} = 0 \) and \( u_{v_{i+1}w_{i+1}} = 0 \) by Step 1 or Step 2. Note that if there is no \( w_{i+2} \), we still get \( u_{v_{i}w_{i}} = u_{v_{i+1}w_{i+1}} \) by a similar calculation, since then \( w_i \) is the only neighbor of \( w_{i+1} \).

**Step 4:** We have \( u_{v_{i}w_{i}} = 0 \) for \( c(v) \neq c(w) \).

Since \( c(v) \neq c(w) \), we know \( n_{c(v)} \neq n_{c(w)} \). Assume \( n := n_{c(v)} < n_{c(w)} \). Then we have \( u_{v_{n}w_{n}} = 0 \) by Lemma 4.1, since \( \text{deg}(v_{n}) = 1 \neq 2 = \text{deg}(w_{n}) \). We deduce \( u_{v_{i}w_{i}} = u_{v_{0}w_{0}} = u_{v_{n}w_{n}} = 0 \) for all \( i \) by Step 3. The case \( n_{c(v)} > n_{c(w)} \) follows similarly. \( \square \)

**Theorem 4.7.** Let \( G \) be a vertex- and edge-colored graph with \( \text{deg}(v) \geq 3 \) for all \( v \in V(G) \) and let \( G' \) be as in Definition 4.5. Then there exists a \(*\)-isomorphism \( \varphi : \text{C}(\text{Qut}(G)) \rightarrow \text{C}(\text{Qut}(G')) \) such that \( \Delta_{G'} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{G} \).

**Proof.** Let \( u \) be the fundamental representation of \( \text{Qut}(G') \). As in Lemma 4.6, we denote by \( v_i \), \( 0 \leq i \leq n_{c(v)} \) the vertices in the added path with \( d(v, v_i) = i \) and let \( V_i = \{ x \in V(G'); x = v_i \text{ for some } v \in V(G) \} \). Let \( c_e \) be the color of the edges in the attached paths in \( G' \). For colors \( c \neq c_e \), we get that \( A_{G'_c} \) is equal to

\[
\begin{pmatrix}
V_0 & V_1 & V_2 & \cdots & V_m \\
A_{G'_c} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
B_1 & 0 & B_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\] (12)

By Lemma 4.6, we directly see that \( uA_{G'_c} = A_{G'_c} u \) implies \( u_{v}A_{G'_c} = A_{G'_c} u_{0} \) for \( c \neq c_e \), \( u_0 \) being the \((V_0 \times V_0)\)-block in \( u \) (see (11)). The adjacency matrix \( A_{G'_c} \) is of the form

\[
\begin{pmatrix}
V_0 & V_1 & V_2 & \cdots & V_m \\
A_{G'_c} & B_1 & 0 & \cdots & 0 \\
B_1 & 0 & B_2 & \cdots & 0 \\
0 & B_2 & 0 & \ddots & 0 \\
\vdots & \vdots & \vdots & \ddots & B_m \\
0 & 0 & 0 & 0 & B'_m
\end{pmatrix}
\] (13)

where \( B_i \) is the \((V_{i-1} \times V_i)\)-matrix with \((B_i)_{a_{i-1}b_i} = \delta_{ab} \). Also by Lemma 4.6, we see that \( uA_{G'_c} = A_{G'_c} u \) implies \( u_{0}A_{G'_c} = A_{G'_c} u_{0} \). Furthermore, since \((u_{v})_{v_{i}w_{i}} = (u_{0})_{v_{0}w_{0}} = 0 \) for all \( i \), we see that \( \text{C}(\text{Qut}(G')) \) is generated by the entries of \( u_0 \). We conclude that \( \text{C}(\text{Qut}(G')) \) is generated by a magic unitary \( u_0 \) that fulfills \( u_{0}A_{G'_c} = A_{G'_c} u_{0} \) for all edge-colors \( c \) and \((u_{0})_{w_{i}w_{i+1}} = 0 \) for \( c(v) \neq c(w) \).
(see Lemma 4.6). This yields a surjective $*$-homomorphism $\varphi : C(\text{Qut}(G)) \to C(\text{Qut}(G'))$, $w_{ab} \mapsto (u_0)_{ab}$, where $w$ is the fundamental representation of Qut($G$).

For the other direction, take the fundamental representation $w$ of Qut($G$) and build the matrix $w'$ as follows

$$
\begin{pmatrix}
V_0 & V_1 & \ldots & \ldots & V_m \\
V_0 & 0 & 0 & \ldots & 0 \\
V_1 & 0 & w_1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\
V_m & 0 & 0 & 0 & 0 & \ldots & w_m
\end{pmatrix}
$$

(14)

where we put $(w_i)_{a,b} = w_{a_0b_0}$. Note that $w'$ is a magic unitary if and only if the matrices $w_i$ are magic unitaries. We compute

$$
\sum_{b : n_c(b) \geq i} (w_i)_{a,b} = \sum_{b : n_c(b) \geq i} w_{ab} = \sum_b w_{ab} = 1,
$$

where we used $w_{ab} = 0$ for $c(a) \neq c(b)$ and the fact that $w$ is a magic unitary. Similarly, we get $\sum_{a : n_c(a) \geq i} (w_i)_{a,b} = 1$ and thus $w'$ is a magic unitary. It remains to show that $w'$ commutes with $A_{G'_c}$ for all edge colors $c$. We deduce from (12) and (14) that $wA_{G'_c} = A_{G'_c} w$ implies $w'A_{G'_c} = A_{G'_c} w'$ for $c \neq c_e$. From (13) and (14), we see that $w'A_{G'_c} = A_{G'_c} w'$ is equivalent to $wA_{G'_c} = A_{G'_c} w$, $w_i B_{i+1} = B_{i+1} w_i$, $w_i B_{i+1} = B_{i+1} w_i$, $w_i B_{i+1} = B_{i+1} w_i$. Note that we have

$$
(w_i B_{i+1})_{a,b_{i+1}} = \sum_k (w_i)_{a,k} (B_{i+1})_{k,b_{i+1}}
= (w_i)_{a,b_i}
= (w)_{a_0b_0}
= (w_{i+1})_{a+1,b_{i+1}}
= \sum_k (B_{i+1})_{a,k} (w_{i+1})_{k,b_{i+1}}
= (B_{i+1} w_{i+1})_{a,b_{i+1}}
$$

by definition of $w_i$ and $B_i$. We similarly get that $w_i B_{i+1} = B_{i+1} w_i$ is automatically fulfilled by definition of $w_i$ and $B_i$. Since we have $wA_{G'_c} = A_{G'_c} w$ by assumption, we obtain $w'A_{G'_c} = A_{G'_c} w'$. Summarizing, we have $w'A_{G'_c} = A_{G'_c} w'$ for all edge-colors $c$. This yields a surjective $*$-homomorphism $\tilde{\varphi} : C(\text{Qut}(G')) \to C(\text{Qut}(G))$, $u_{xy} \mapsto w'_{xy}$ which is inverse to $\varphi$. Thus, $\varphi$ is a $*$-isomorphism.

It remains to show that $\Delta_{G'} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G$. But this is true because

$$
\Delta_{G'} \circ \varphi(w_{ij}) = \Delta_{G'}((u_0)_{ij})
= \sum_k (u_0)_{ik} \otimes (u_0)_{kj}
$$
FIGURE 3 Example for the construction of the graph $G''$ from $G'$. We chose black to be the edge-color $c_0$

$$=(\varphi \otimes \varphi)\left(\sum_k w_{ik} \otimes w_{kj}\right)$$

$$=(\varphi \otimes \varphi) \circ \Delta(w_{ij}).$$

We will now deal with the edge-colors of $G'$. First, we need the following easy lemma.

**Lemma 4.8.** Let $w_{ij}, 1 \leq i, j \leq n$ be elements in a $C^*$-algebra such that the matrix $w = (w_{ij})_{1 \leq i, j \leq n}$ is a magic unitary. Then $w_{ij}w_{kl} + w_{il}w_{kj}$ is a projection if and only if $w_{ij}w_{kl} = w_{kl}w_{ij}$ and $w_{il}w_{kj} = w_{kj}w_{il}$.

**Proof.** Let $w_{ij}w_{kl} + w_{il}w_{kj}$ be a projection. Since it is self-adjoint, we have

$$w_{ij}w_{kl} + w_{il}w_{kj} = w_{kl}w_{ij} + w_{kj}w_{il}.$$  

Multiplying by $w_{ij}$ and $w_{il}$, respectively, yields $w_{ij}w_{kl} = w_{ij}w_{kl}w_{ij}$ and $w_{il}w_{kj} = w_{il}w_{kj}w_{il}$. But, by taking adjoints, this implies $w_{ij}w_{kl} = w_{kl}w_{ij}$ and $w_{il}w_{kj} = w_{kj}w_{il}$. The other direction is clear, since $w_{ij}w_{kl} + w_{il}w_{kj}$ is the sum of two orthogonal projections if $w_{ij}w_{kl} = w_{kl}w_{ij}$ and $w_{il}w_{kj} = w_{kj}w_{il}$.

We will define a (uncolored) graph $G''$ from the edge-colored graph $G'$. In the next theorem, we will then see that for certain graphs, the quantum automorphism groups of $G'$ and $G''$ are isomorphic. Let $G$ be a graph and $e = (u, v) \in E(G)$. We say that we subdivide $e$ if we delete the edge $e = (u, v)$ from $G$ and add a vertex $w$ as well as edges $(u, w)$ and $(w, v)$ to the graph.

**Definition 4.9.** Let $G$ be a vertex- and edge-colored graph. First decolor the vertices by applying the construction as in Definition 4.5 to obtain the edge-colored graph $G'$. We denote the color of the newly added edges in the graph $G'$ by $c_0$. We subdivide each colored edge with $c(e) \neq c_0$ and add a path of length $m_c$ to the subdivision, where $m_{c_1} \neq m_{c_2}$ for colors $c_1 \neq c_2$ and then decolor the edges in the graph $G'$. We call this graph $G''$ (for an example, see Figure 3).

**Lemma 4.10.** Let $G$ be a vertex- and edge-colored graph such that $\deg(v) \geq 3$ for all $v \in V(G)$ and let $G''$ be the graph as in Definition 4.9. We denote the vertex that subdivided the edge $e$ in $G$ by


e_0 and the vertices in the added path with \( d(e_0, e_i) = i \) by \( e_i \), \( 1 \leq i \leq m_c(e) \). Furthermore, \( E'_i = \{ x \in V(G''); x = e_i \text{ for some } e \in E(G) \}. \) Then, the fundamental representation \( u \) of \( Qut(G'') \) is of the form

\[
\begin{pmatrix}
V(G') & E'_0 & \cdots & E'_m \\
E'_0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E'_m & 0 & 0 & 0 & u_m \\
\end{pmatrix}
\]

(15)

where \( u_{e_if_i} = u_{wx} u_{wy} + u_{xy} u_{wx} \) for \( c(e) = c(f), e = (v, w), f = (x, y) \) and \( u_{e_if_i} = 0 \) for \( c(e) \neq c(f) \).

**Proof. Step 1:** We have \( u_{e_i v_j} = u_{v_j e_i} = 0 \) for \( v_j \in V(G'), e_i \in E'_i \).

First assume \( i > 0, j = 0 \). Then \( \deg(e_i) \in \{1, 2\} \) and \( \deg(v_0) \geq 3 \), therefore \( u_{e_0 v_j} = 0 \) by Lemma 4.1.

Now, let \( i = 0, j > 0 \). Then \( n_c(e_0) > 0 \), \( m_c(e_0) = 0 \). If \( n_c(v_0) = 1 \), then \( \deg(v_0) = 1 \neq \deg(e_0) \) and thus \( u_{e_0 v_j} = 0 \) by Lemma 4.1. If \( n_c(v_0) \geq 2 \), then \( v_j \) has at least one neighbor of degree 1 or 2. Since \( e_0 \) only has neighbors of degree \( \geq 3 \), we get \( u_{e_0 v_j} = 0 \) by Lemma 4.3.

Assume \( i \neq j \) and \( i, j \neq 0 \). Let furthermore \( i < j \). Then \( e_0 \) is a vertex with \( d(e_i, e_0) = i \) and \( \deg(e_0) = 3 \). Since \( i < j \), we know that the vertices \( q \) with \( d(v_j, q) = i \) are in the path and thus \( \deg(q) \in \{1, 2\} \). We deduce \( u_{e_i v_j} = 0 \) by Lemma 4.3. The case \( i > j \) follows similarly.

It remains to show \( u_{e_i v_j} = 0 \). If \( n_c(v_i) > 0 \), then \( \deg(v_i) \geq 4 \) and \( d(v_0, v_i) = i \). We know \( \deg(q) \leq 3 \) for vertices \( q \) with \( d(q, e_i) = i \), since either \( q = e_0 \) or \( q \) is a vertex in the added path. We deduce \( u_{e_i v_j} = 0 \) by Lemma 4.3. Now assume \( n_c(v_i) = 0 \). If \( m_c(e_0) = 0 \), then we know \( \deg(e_0) = 2 \) and thus \( u_{e_0 v_j} = 0 \) by Lemma 4.1. If \( m_c(e_0) > 0 \), then \( e_0 \) has a neighbor of degree 1 or 2. If \( v_0 \) has no neighbor of this degree, then \( u_{e_0 v_0} = 0 \). The vertex \( v_0 \) only has a neighbor of degree 2 if there exists a subdivision \( f_0 \) of some edge \( f = (v_0, w_0) \) with \( m_c(f) = 0 \). Then, it holds

\[
u_{e_0 v_0} = u_{e_0 v_0} \left( \sum_{k; (k, e_0) \in E(G'')} u_{k f_0} \right) = u_{e_0 v_0} u_{e_1 f_0}.
\]

If \( m_c(e_0) = 1 \), then \( \deg(e_1) = 1 \neq 2 = \deg(f_0) \) which yields \( u_{e_1 f_0} = 0 \) by Lemma 4.1. If \( m_c(e_0) \geq 2 \), then \( e_1 \) has a neighbor \( e_2 \) with \( \deg(e_2) \in \{1, 2\} \). Since the neighbors \( v_0, w_0 \) of \( f_0 \) have degree \( \geq 3 \), we get \( u_{e_1 f_0} = 0 \) by Lemma 4.3. We deduce \( u_{e_0 v_0} = 0 \) in all those cases.

**Step 2:** We have \( u_{e_i f_0} = u_{f_0 e_i} = 0 \) for \( i \neq 0 \).

We know that it holds \( \deg(e_i) \in \{1, 2\} \) for vertices \( e_i \) with \( i > 0 \). If \( m_s(f_0) > 0 \), then we have \( \deg(f_0) = 3 \) and get \( u_{e_i f_0} = 0 \) by Lemma 4.1. Let \( m_s(f_0) = 0 \). If \( m_c(e_i) = 1 \), then \( \deg(e_1) = 1 \) and thus \( u_{e_1 f_0} = 0 \) by Lemma 4.1. If \( m_c(e_i) \geq 2 \), then \( e_i \) has a neighbor with degree 1 or 2. Since the neighbors of \( f_0 \) have degree \( \geq 3 \) by assumption, we get \( u_{e_i f_0} = 0 \) by Lemma 4.3. We similarly obtain \( u_{e_0 f_0} = 0 \).

**Step 3:** We have \( u_{e_i f_j} = 0 \) for \( i \neq j, i, j > 0 \).

First assume \( i < j \). Then \( e_0 \) is a vertex with \( d(e_i, e_0) = i \) and \( \deg(e_0) = 3 \). Since \( i < j \), we know that the vertices \( q \) with \( d(f_j, q) = i \) are in the path and thus \( \deg(q) \in \{1, 2\} \). We deduce \( u_{e_i f_j} = 0 \) by Lemma 4.3. The case \( i > j \) follows similarly.
Step 4: We have $u_{e_i f_i} = u_{e_0 f_0}$ for all $i$.
We first show $u_{e_0 f_0} = u_{e_1 f_1}$. It holds
\[
u_{e_0 f_0} = u_{e_0 f_0} \left( \sum_{k: (k, f_0) \in E(G'')} u_{e_1 k} \right) = u_{e_0 f_0} u_{e_1 f_1} = (u_{e_0 f_0} + u_{e_0 f_2}) u_{e_1 f_1} = \left( \sum_{k: (k, f_1) \in E(G'')} u_{e_0 k} \right) u_{e_1 f_1} = u_{e_1 f_1},
\]
since $u_{e_1 p_0} = 0$ by Step 1 and $u_{e_0 f_2} = 0$ by Step 2. Furthermore, for $i \geq 1$, we have
\[
u_{e_i f_i} = u_{e_i f_i} \left( \sum_{k: (k, f_i) \in E(G'')} u_{e_{i+1} k} \right) = u_{e_i f_i} (u_{e_{i+1} f_{i+1}} + u_{e_{i+1} f_{i-1}}) = u_{e_i f_i} u_{e_{i+1} f_{i+1}} = (u_{e_i f_i} + u_{e_i f_{i+2}}) u_{e_{i+1} f_{i+1}} = \left( \sum_{k: (k, f_{i+1}) \in E(G'')} u_{e_i k} \right) u_{e_{i+1} f_{i+1}} = u_{e_{i+1} f_{i+1}}
\]
since we have $u_{e_{i+1} f_{i-1}} = 0$ and $u_{e_i f_{i+2}} = 0$ by Step 2 or Step 3. Note that if there is no $f_{i+2}$, we still get $u_{e_i f_i} = u_{e_{i+1} f_{i+1}}$ by a similar calculation, since then $f_i$ is the only neighbor of $f_{i+1}$.

Step 5: We have $u_{e_i f_i} = 0$ for $c(e) \neq c(f)$. Since $c(e) \neq c(f)$, we know $m_{c(e)} \neq m_{c(f)}$. First assume $m := m_{c(e)} < m_{c(f)}$. Then we have $u_{e_m f_m} = 0$ by Lemma 4.1, since $\deg(e_m) = 1 \neq 2 = \deg(f_m)$. We deduce $u_{e_i f_i} = u_{e_0 f_0} = u_{e_m f_m} = 0$ for all $i$ by Step 3. The case $m_{c(e)} > m_{c(f)}$ is similar.

Step 6: We have $u_{e_i f_i} = u_{vx} u_{wy} + u_{vy} u_{wx}$ for $e = (v, w), f = (x, y)$. We have
\[
u_{e_0 f_0} = \left( \sum_{k: (k, f_0) \in E(G'')} u_{ek} \right) u_{e_0 f_0} \left( \sum_{l: (l, f_0) \in E(G'')} u_{wl} \right) = (u_{vx} + u_{vy} + u_{e_1 f_1}) u_{e_0 f_0} (u_{wx} + u_{wy} + u_{w f_1}) = (u_{vx} + u_{vy}) u_{e_0 f_0} (u_{wx} + u_{wy}),
\]
where we used \( u_{e, f_1} = 0, u_{w, f_1} = 0 \) by Step 1. Note that \( e_0 \) is the only vertex in \( E'_0 \) that is a common neighbor of \( v \) and \( w \). Thus, we have 
\[
(u_{ux} + u_{uy})u_{k, f_0}(u_{ux} + u_{wy}) = 0 \quad \text{for all } k \neq e_0 \text{ by Lemma 4.2 and Step 1.}
\]
Therefore, we get
\[
u_{e_0, f_0} = (u_{ux} + u_{uy})u_{e_0, f_0}(u_{ux} + u_{wy})
\]
\[
= (u_{ux} + u_{uy})\left(\sum_{k \in \{k, w\} \in E(G''),(k, w) \in E(G'')} u_{k, f_0}\right)(u_{ux} + u_{wy})
\]
\[
= (u_{ux} + u_{uy})(u_{ux} + u_{wy})
\]
\[
= u_{ux}u_{wy} + u_{uy}u_{ux}.
\]
By Step 4, we obtain
\[
u_{e, f_1} = u_{ux}u_{wy} + u_{uy}u_{ux}.
\]
\[\square\]

**Remark 4.11.** Note that the operator \( u_{(v, w)(x, y)} = u_{ux}u_{wy} + u_{uy}u_{wx} \) does not depend on the order of \((v, w)\) or \((x, y)\), since
\[
u_{(v, w)(x, y)} = u_{ux}u_{wy} + u_{uy}u_{wx} = u_{vy}u_{wx} + u_{ux}u_{wy} = u_{(v, w)(y, x)},
\]
\[
u_{(v, w)(y, x)} = u_{vy}u_{wx} + u_{ux}u_{wy} = u_{wx}u_{vy} + u_{wy}u_{ux} = u_{(w, v)(x, y)}.
\]

Before stating the theorem, we first need to define a quantum subgroup of the quantum automorphism group of the graph \( G' \). It is straightforward to check that the comultiplication is an \( * \)-homomorphism.

**Definition 4.12.** Let \( G \) be a vertex- and edge-colored graph and \( G' \) as in Definition 4.5. We define \( \text{Qut}^*(G') \) to be the compact matrix quantum group whose corresponding \( C^* \)-algebra is generated by a magic unitary \( x \) with \( xA_G c = A_G cx \) for every edge color \( c \) and \( x_{ij}x_{jl} = x_{jl}x_{ij} \) for \( c((i, j)) = c((k, l)), c \neq c_0 \), where \( c_0 \) is the edge-color we choose for the newly added edges in \( G' \).

**Theorem 4.13.** Let \( G \) be a vertex- and edge-colored graph such that \( \text{deg}(v) \geq 3 \) for all \( v \in V(G) \). Let \( G'' \) be the graph as in Definition 4.9 and let \( \text{Qut}^*_0(G') \) be the compact matrix quantum group as in Definition 4.12. Then there exists an \( * \)-isomorphism \( \varphi : C(\text{Qut}^*_0(G')) \to C(\text{Qut}(G'')) \) such that \( \Delta_{G''} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_G \).

**Proof.** Let \( u \) be the fundamental representation of \( \text{Qut}(G'') \). As in Lemma 4.10, we denote the vertex that subdivided the edge \( e \) in \( G \) by \( e_0 \) and the vertices in the added path with \( d(e_0, e_i) = i \) by \( e_i, 1 \leq i \leq m_{c(e)} \). Moreover, we denote by \( v_i, 0 \leq i \leq n_{c(v)} \) the vertices in the added path with \( d(v, v_i) = i \). The adjacency matrix of \( G'' \) is of the form
\[
\begin{pmatrix}
V(G') & E'_0 & E'_1 & \ldots & E'_m \\
A_{G'} & B & 0 & \ldots & 0 \\
E'_0 & 0 & 0 & \ldots & 0 \\
E'_1 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
E'_m & 0 & 0 & 0 & C_m
\end{pmatrix}
\]
where $A_{G_0}$ is the adjacency matrix of the edge-color that is not subdivided together with the paths from the construction of $G'$, $B_{G''}$ is the matrix with

$$(B_{G''})_{ve} = \begin{cases} 
1 & \text{for } v \text{ incident to } e, \\
0 & \text{otherwise}, 
\end{cases}$$

and $C_i$ is the $(E'_1 \times E'_1)$-matrix with $(C_i)_{a_{i-1}b_i} = \delta_{ab}$. Using Lemma 4.10, we see that $uA_{G''} = A_{G''}u$ implies $wA_{G'_0} = A_{G'_0}w$, $wB_{G''} = B_{G''}w$, $wB_{G'_0} = B_{G'_0}w$, where $w$ and $u_0$ are blocks in $u$ (see (15)). Furthermore, since $(u_i)_{e,f} = w_{ai}w_{bl} + w_{al}w_{bk}$ for $e = (a, b), f = (k, l)$, we also have that $C(\text{Qut}(G''))$ is generated by the entries of the matrix $w$. We will now show that $w$ fulfills the relations of the generators of $C(\text{Qut}_0^c(G'))$. We already know $wA_{G'_0} = A_{G'_0}w$. We will now show $wA_{G_c} = A_{G_c}w$ for every edge color $c \neq c_0$. By Lemma 2.8, this is equivalent to $w_{ik}w_{jl} = 0$ for $c((i, j)) \neq c((k, l))$ if $(i, j) \in E'_0$ or $(k, l) \in E'_0$. We will show that the elements of $w$ fulfill those relations. Let $v \in V(G')$, $(a, b) = e \in E'_0$. It holds

$$w_{va} + w_{vb} = \sum_{k: k \text{ inc. to } e} w_{vk}$$

$$= \sum_{k} w_{vk}(B_{G''})_{ke}$$

$$= (wB_{G''})_{ve}$$

$$= (B_{G''}u_0)_{ve}$$

$$= \sum_{k} (B_{G''})_{vk}(u_0)_{ke}$$

$$= \sum_{k: k \text{ inc. to } v} (u_0)_{ke}$$

$$= \sum_{s: (v, s) \in E_0} (w_{va}w_{sb} + w_{vb}w_{sa}).$$

(17)

Multiplying $w_{va}$ from left and right yields

$$w_{va} = \sum_{s: (v, s) \in E_0} w_{va}w_{sb}w_{sa}. $$

We deduce $\sum_{s: (v, s) \in E_0} w_{va}w_{sb}w_{va} = 0$ and since all elements in the sum are positive, we get $w_{va}w_{sb}w_{va} = 0$ for all $v, s \in V(G')$ with $(v, s) \notin E'_0$. We deduce $w_{va}w_{sb} = 0$ for all $v, s \in V$ with $(v, s) \notin E'_0$. Similarly, by using $u_0B_{G''} = B_{G''}w$, we get $w_{va}w_{bs} = 0$ for all $v, s \in V(G')$ with $(v, s) \notin E'_0$. It remains to show $w_{ik}w_{jl} = 0$ for $c(e) \neq c(f)$, $e = (i, j) \in E'_0$, $f = (k, l) \in E'_0$. We know $\sum (u_0)_{ef} = 0$ for $c(e) \neq c(f)$ from Lemma 4.10. Then

$$w_{ik}w_{jl} + w_{il}w_{jk} = (u_0)_{ef} = 0$$

and by multiplying $w_{ik}$ and $w_{il}$ from the left, respectively, we get $w_{ik}w_{jl} = 0$ and $w_{il}w_{jk} = 0$. Thus, we have shown $w_{ik}w_{jl} = 0$ for $c((i, j)) \neq c((k, l))$ if $(i, j) \in E'_0$ or $(k, l) \in E'_0$ which is equivalent to $wA_{G_c} = A_{G_c}w$ for every edge color $c \neq c_0$. 
Summarizing, we get \( w A_G = A_G w \) for every edge color \( c \). By Lemma 4.10, we furthermore know that
\[
(w_{ab})_{ab} = w_{ai} w_{ji} + w_{aj} w_{jk}
\]
for \( a = (i, j) \) and \( b = (k, l) \) and \( c(a) = c(b) \), \( c(a) \neq c_0 \) and thus \( w_{ai} w_{ij} + w_{aj} w_{jk} \) is a projection. Then Lemma 4.8 yields \( w_{ij} w_{kl} = w_{kl} w_{ij} \) and \( w_{ij} w_{kl} = w_{kj} w_{il} \). Therefore, we get a surjective \( * \)-homomorphism \( \varphi : C(Qut^*_c(G')) \rightarrow C(Qut(G'')) \), \( w_j' \rightarrow w_{ij} \), where \( w' \) is the fundamental representation of \( Qut^*_c(G') \).

For the other direction, take the fundamental representation \( w' \) of \( Qut^*_c(G') \) and build the matrix \( w'' \) as follows

\[
\begin{bmatrix}
V(G') & E_0' & \ldots & E_m' \\
E_0' & u_{ij}' & 0 & \ldots & 0 \\
\vdots & 0 & u_{ij}' & \ldots & 0 \\
E_m' & 0 & 0 & \ldots & 0
\end{bmatrix}
\]

where we put \( (u_{ij}')_{ij} = w_{ab} w_{bl} + w_{ai} w_{lk} \) for \( e = (a, b) \), \( f = (k, l) \). Those elements are projections by Lemma 4.8, because we have \( w_{ab} w_{bl} = w_{bl} w_{ab} \) for \( (a, b), (k, l) \in E_0' \) (for the same color by assumption, for different colors because the product is 0). Note that \( w'' \) is a magic unitary if and only if the matrices \( u_{ij}' \) are magic unitaries. Let \( e = (a, b) \in E_0' \). We compute

\[
\sum_{f: n_c(f) \geq i} (u_{ij}')_{ij} = \sum_{k, l, n_c((k, l)) \geq i} (w_{ab} w_{bl} + w_{ai} w_{lk}) = \sum_{k, l, (k, l) \in E_0'} (w_{ab} w_{bl} + w_{ai} w_{lk}) = 1,
\]

where we used \( w_{ab} w_{bl} = 0 \) for \( c((a, b)) \neq c((k, l)) \) and the fact that \( w' \) is a magic unitary. We get

\[
\sum_{f, n_c(f) \geq i} (u_{ij}')_{ij} = \sum_{x; (i, x) \in E_0'} (w_{ia} w_{xb} + w_{ib} w_{xa}) = (w'B_{G''})_{i(a, b)} + w'_{ia} + w'_{ib}
\]

and therefore \( w'B_{G''} = B_{G''} w' \). We obtain \( u_{ij}' B_{G''} = B_{G''} u_{ij}' \) similarly. From (16) and the form of \( w'' \), we see that \( w''A_{G''} = A_{G''} w'' \) is equivalent to \( w'A_G = A_G w' \), \( w'B_{G''} = B_{G''} w' \), \( u_{ij}' C_{i+1} = C_{i+1} u_{ij}' \), \( u'_{i+1} C_i = C_i u'_{i+1} \). Note that \( u'_{i+1} C_i = C_i u'_{i+1} \) are automatically fulfilled by definition of \( u'_{ij} \) and \( C_i \). Therefore, we get \( w'' A_{G''} = A_{G''} w'' \). This yields a surjective \( * \)-isomorphism \( \varphi : C(Qut(G'')) \rightarrow C(Qut^*(G'')) \), \( u_{ij} \mapsto w''_{ij} \) which is inverse to \( \varphi \). Thus, \( \varphi \) is an \( * \)-isomorphism.

It remains to show that \( \Delta_{G''} \circ \varphi = (\varphi \otimes \varphi) \circ \Delta_{G'} \), which is true because of the following:

\[
\Delta_{G''} \circ \varphi(w'_{ij}) = \Delta_{G'}(w_{ij}) = \sum_k w_{ik} \otimes w_{kj}
\]
\begin{align*}
&= (\varphi \otimes \varphi) \left( \sum_k w'_{ik} \otimes w'_{kj} \right) \\
&= (\varphi \otimes \varphi) \circ \Delta_{G'}(w'_{ij}).
\end{align*}

We get the following corollary, which we will use in the next section to construct a graph with quantum symmetry and finite quantum automorphism group.

**Corollary 4.14.** Let \( G \) be a vertex- and edge-colored graph such that \( \deg(v) \geq 3 \) for all \( v \in V(G) \). Denote by \( G' \) and \( G'' \) the graphs constructed in Definitions 4.5 and 4.9. If \( \text{Qut}^*_c(G') \cong \text{Qut}(G') \), then \( \text{Qut}(G'') \cong \text{Qut}(G) \).

## 5 (UNCOLORED) GRAPHS WHOSE QUANTUM AUTOMORPHISM GROUP IS THE DUAL OF A SOLUTION GROUP

In this section, we will look at certain graphs from Definition 3.2, where we replace one of the edge colors by non-edges. Furthermore, we restrict to graphs coming from linear constraint systems as in the following definition. By using Theorem 3.8 and the decoloring procedure in Section 4, we will obtain (decolored) graphs whose quantum automorphism group is the dual of the corresponding solution group.

\begin{definition}
Let \( H \) be a connected graph with vertex set \([m]\) and label the edges \(1, ..., n := |E(H)|\). Let \( M_H \in \mathbb{F}_2^{m \times n} \) be the matrix, where

\[
(M_H)_{ki} = \begin{cases} 
1 & \text{if } k \in V(H) \text{ is incident to } i \in E(H); \\
0 & \text{otherwise.}
\end{cases}
\]

\end{definition}

\begin{definition}
Let \( H \) be a connected graph, let \( M_H \) be as in Definition 5.1 and let \( G(M_H, b) \) be the colored graph as in Definition 3.2. Recall \( S_k(M_H) = \{i \in [n] : (M_H)_{ki} = 1\} \). Note that it holds \( |S_l(M_H) \cap S_k(M_H)| \leq 1 \) for all \( l \neq k \), since otherwise there would be multiple edges between \( l \) and \( k \) in the graph \( H \) by definition of \( M_H \). By Remarks 3.3 and 3.4, we know that we can color the edges between any \( V_l \) and \( V_k \), \( k \neq l \), by the same \((2^{|S_k|} - 1)\) numbers. In our case, this means that for every \( |S_l \cap S_k| \neq 0 \), we can use the same two numbers for any \( V_l, V_k \). We color the edge between \((l, \alpha), (k, \beta)\) by \( \alpha_i \beta_i \in \{-1, 1\} \), where \( i \in [n] \) is the number for which \( (M_H)_{ki} = (M_H)_{li} = 1 \). We then define the graph \( G_*(M_H, b) \) to be the graph obtained from the previous graph by replacing the edges of color 1 by non-edges.

\begin{remark}
As already mentioned in Remark 3.4, Theorem 3.8 is true if we replace \( G(M_H, 0) \) by \( G_*(M_H, 0) \).

For the next lemma, recall the definition of \( \text{Qut}^*_c(G') \) from Definition 4.12. Note that a matching in a graph is a set of non-adjacent edges, that is, no two edges share a common vertex.

\begin{lemma}
Let \( H \) be a graph and \( G_*(M_H, 0) \) as in Definition 5.2. We choose the color of the added paths in \( G'(M_H, 0) \) to be \(-1\), that is, the same color as any edge between \( V_l \) and \( V_k \) for \( k \neq l \). Choosing \( c_0 = -1 \), we get \( \text{Qut}^*_{-1}(G'_*(M_H, 0)) \cong \text{Qut}(G'_*(M_H, 0)) \).
\end{lemma}
Proof. We start by showing that every edge-color class except $c_0$ in $G'_l(M_H, 0)$ is a matching. For this, first note that every edge between $V_k$ and $V_1, k \neq l$ as well as every edge in an added path has color $c_0$. Thus, the remaining edges are between vertices $(k, \alpha), (k, \beta)$ in $V_k$ for some $k$. Thus, the color of the edge is the function $\alpha \triangle \beta : S_k \to \{\pm 1\}^{S_k}, (\alpha \triangle \beta)_i = \alpha_i \beta_i$. Fix a vertex $(k, \alpha)$. For every $\beta \neq \beta'$, it holds $\alpha \triangle \beta \neq \alpha \triangle \beta'$, as otherwise $\beta_i = \beta'_i$ for all $i$. Thus, no two edges of the same color share a common vertex, which means that the color classes are matchings.

Let $c((i, k)) = c((j, l)) \neq c_0$. Then

$$u_{ij}u_{kl} = u_{ij}\left(\sum_{a : c((j, a)) = c((i, k))} u_{ka}\right) = u_{ij}\left(\sum_a u_{ka}\right) = u_{ij},$$

because $l$ is the only neighbor of $j$ such that $c((i, k)) = c((j, l))$, since the color classes are matchings. Using the adjoint then yields $u_{ij}u_{kl} = u_{kl}u_{ij}$. We deduce $\text{Qut}_{-1}(G'_l(M_H, 0)) \cong \text{Qut}(G'_s(M_H, 0))$.

We get the following theorem.

Theorem 5.5. Let $H$ be a graph with $\deg(v) \geq 2$ for all $v \in V(H)$ and $G_s(M_H, 0)$ be as in Definition 5.2. Let $G''_s(M_H, 0)$ be the graph constructed from $G'_s(M_H, 0)$ as in Lemma 5.4. Then $\text{Qut}(G''_s(M_H, 0)) \cong \hat{\Gamma}_H$.

Proof. First note that we have $\deg(v) \geq 3$ for all $v \in V(G_s(M_H, 0))$ (since $\deg(w) \geq 2$ for $w \in V(H)$, we see that every $v \in V(G_s(M_H, 0))$ has at least one neighbor that is associated to the same equation and at least two neighbors that come from different equations but share one of the at least two variables in the equation). We also know $\text{Qut}_{-1}^*(G'_s(M_H, 0)) \cong \text{Qut}(G'_s(M_H, 0))$ by Lemma 5.4. Thus, Corollary 4.14 yields $\text{Qut}(G''_s(M_H, 0)) \cong \text{Qut}(G_s(M_H, 0))$. Using Theorem 3.8, we deduce $\text{Qut}(G''_s(M_H, 0)) \cong \hat{\Gamma}_H$.

Using the theorem above, we are able to obtain a graph with quantum symmetry and finite quantum automorphism group. This graph is constructed from the constraint system associated to $K_{3,4}$.

Corollary 5.6. The graph $G''_{s}(M_{K_{3,4}}, 0)$ has quantum symmetry and finite quantum automorphism group.

Proof. It is easy to see that $\deg(v) \geq 2$ for all $v \in V(K_{3,4})$. Therefore, $C(\text{Qut}(G''_{s}(M_{K_{3,4}}, 0))) \cong C^*(\Gamma_{K_{3,4}})$ by the previous theorem. Furthermore, it was shown in [14] (by using Sage [17]) that $\Gamma_{K_{3,4}}$ is finite and non-abelian, which yields that $C^*(\Gamma_{K_{3,4}})$ is finite-dimensional and non-commutative. By the isomorphism above, we get that $C(\text{Qut}(G''_{s}(M_{K_{3,4}}, 0)))$ is finite-dimensional and non-commutative which yields the assertion.

6 | QUANTUM ISOMORPHISMS

In this section, we consider quantum isomorphisms between the colored graphs $G(M, b)$ and $G(M, b')$ for $b \neq b'$. In particular, we give analogs of Lemma 3.6 and Theorem 3.8 for this case. From this, we are able to obtain non-isomorphic colored graphs $G(M, b)$ and $G(M, b')$ that are
quantum isomorphic but neither \(G(M, b)\) nor \(G(M, b')\) has quantum symmetry. Further, applying decoloring techniques as in Section 4 allows us to produce non-isomorphic uncolored graphs \(G\) and \(G'\) without quantum symmetry that are nevertheless quantum isomorphic. This appears to be the first known such example.

To define quantum isomorphism of colored graphs, we must provide a suitable generalization of the isomorphism game. The way to do this is quite natural.

**Definition 6.1** (Isomorphism game for colored graphs). Given colored graphs \(G\) and \(G'\), with respective color functions \(c\) and \(c'\), the \((G, G')\)-isomorphism game has as inputs and outputs for both Alice and Bob the set \(V(G) \cup V(G')\). To win, upon receiving \(x \in V(G)\) Alice (respectively, Bob) must respond with \(y \in V(G')\) and vice versa. If this condition is met, then there is a vertex \(g_a \in V(G)\) that is either Alice’s input or output, and there is similarly \(g_b \in V(G)\) and \(g_a', g_b' \in V(G')\). Alice and Bob then win if the following conditions are met.

1. \(c(g_a) = c'(g_a')\) and \(c(g_b) = c'(g_b')\).
2. \(g_a = g_b\) if and only if \(g_a' = g_b'\).
3. \((g_a, g_b)\) is an edge of color \(c\) if and only if \((g_a', g_b')\) is an edge of color \(c'\).

We then say that two colored graphs \(G\) and \(G'\) are **quantum isomorphic** if there is a quantum strategy\(^1\) that wins the \((G, G')\)-isomorphism game with probability 1. In [11], it was shown that (uncolored) graphs \(G\) and \(G'\) are quantum isomorphic if and only if there exists a magic unitary \(u\) such that \(A_G u = u A_{G'}\). Precisely, the same proof applied to colored graphs gives the following:

**Proposition 6.2.** Colored graphs \(G\) and \(G'\) with coloring functions \(c\) and \(c'\) are quantum isomorphic if and only if there exists a \(V(G) \times V(G')\) magic unitary \(u\) such that \(c(g) \neq c'(g')\) implies \(u_{gg'} = 0\) for \(g \in V(G)\), \(g' \in V(G')\), and \(A_G u = u A_{G'}\) for all edge colors \(c\).

Based on the above, we define the following\(^2\):

**Definition 6.3** (Isomorphism algebra). Given colored graphs \(G\) and \(G'\) with coloring functions \(c\) and \(c'\), we define the **isomorphism algebra**, denoted \(\text{Iso}(G, G')\) as the universal \(C^*\)-algebra generated by elements \(u_{gg'}\) for \(g \in V(G)\), \(g' \in V(G')\) such that

\[
\begin{align*}
    u_{gg'}^2 & = u_{gg'}^* = u_{gg'} \quad \text{for all } g \in V(G), g' \in V(G'); \\
    \sum_{h \in V(G)} u_{hg} & = 1 = \sum_{h' \in V(G')} u_{g'h'} \quad \text{for all } g \in V(G), g' \in V(G'); \\
    u_{gg'} & = 0 \text{ if } c(g) \neq c'(g'); \\
    A_G u & = u A_{G'} \quad \text{for all edge colors } c.
\end{align*}
\]

\(^1\) We are implicitly assuming here that the vertex sets of \(G\) and \(G'\) are disjoint. Alternatively, we can assume that as part of their input they are told which graph their input should be associated to, and similarly for their output.

\(^2\) There are a few different models of quantum strategies that can be considered. In this work, we are always referring to the commuting operator framework when we mention quantum strategies.

\(^3\) This \(C^*\)-algebra is analogous to the \(\ast\)-algebra of a non-local game defined in [10], applied to the isomorphism game.
It is clear from Proposition 6.2 and the above definition that the following holds:

**Proposition 6.4.** Let $G$ and $G'$ be colored graphs. Then $G \cong_q G'$ if and only if $\text{Iso}(G, G')$ is non-trivial.

We will also need to define the following algebra for our results in this section:

**Definition 6.5.** For $M \in \mathbb{F}_2^{m \times n}$ and $b \in \mathbb{F}_2^m$, we define $\mathcal{A}(M, b)$ to be the universal $C^*$ algebra generated by $x_1, \ldots, x_n$ such that:

1. $x_i^2 = 1$ for all $i \in [n]$;
2. $x_i x_j = x_j x_i$ if there exists $k \in [m]$ such that $M_{ki} = M_{kj} = 1$;
3. $\prod_{i : M_{ki} = 1} x_i = (-1)^{b_k}$ for all $k \in [m]$.

Note that $\mathcal{A}(M, b)$ is not $C^*(\Gamma)$ for $\Gamma = \Gamma(M, b)$, unless $b = 0$. Rather, for $b \neq 0$, the algebra $\mathcal{A}(M, b)$ is isomorphic to the algebra $pC^*(\Gamma(M, b)) := \{ px : x \in C^*(\Gamma(M, b)) \}$ where $p = \frac{1}{2}(1 - \gamma)$, which may be trivial for some $M$ and $b$. We will show that $\text{Iso}(G(M, b), G(M, b'))$ is isomorphic to $\mathcal{A}(M, b + b')$, but first we need an analog of Lemma 3.6. We omit the proof because it is almost identical to the proof of Lemma 3.6. The only difference is that one index of the magic unitary $u$ is a vertex of $G(M, b)$ and the other is a vertex of $G(M, b')$ instead of both being from the same graph.

**Lemma 6.6.** Let $M \in \mathbb{F}_2^{m \times n}$ and $b, b' \in \mathbb{F}_2^m$. Set $G = G(M, b)$ and $G' = G(M, b')$. If $u$ is a $V(G) \times V(G')$ magic unitary that gives a quantum isomorphism from $G$ to $G'$, then $u$ has the form

$$
\begin{pmatrix}
V_1(G) & V_2(G) & \cdots & \cdots & V_m(G')
\end{pmatrix}
\begin{pmatrix}
0 & 0 & \cdots & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & u^{(2)} & 0 & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & 0 & u^{(3)} & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\begin{pmatrix}
u^{(m)}
\end{pmatrix}
$$

where each $u^{(k)}$ is a magic unitary. Furthermore, $u^{(k)}_{\alpha, \beta} := u_{\alpha, \beta}(k, \beta)$ depends only on $k$ and the value of $\alpha \triangle \beta$ and therefore the entries of $u^{(k)}$ pairwise commute for each $k \in [m]$.

We are now able to prove the following partial analog of Theorem 3.8:

**Theorem 6.7.** Let $M \in \mathbb{F}_2^{m \times n}$ and $b \neq b' \in \mathbb{F}_2^m$. Set $G = G(M, b)$, $G' = G(M, b')$, and $\mathcal{A} = \mathcal{A}(M, b + b')$. Then there exists an isomorphism $\varphi : \mathcal{A} \to \text{Iso}(G, G')$.

**Proof.** The proof is very similar to that of Theorem 3.8. We will first give a homomorphism $\varphi_1$ from $\mathcal{A}$ to $\text{Iso}(G, G')$ and then a homomorphism $\varphi_2$ in the other direction, and finally show that these are inverses of each other and therefore isomorphisms.

**Step 1:** Construction of an $*$-homomorphism $\varphi_1 : \mathcal{A} \to \text{Iso}(G, G')$.

For this step, we will construct elements $y_i$ of $\text{Iso}(G, G')$ that satisfy the relations of the generators $x_i$ of $\mathcal{A}$. This proves that there is an $*$-homomorphism $\varphi_1$ from $\mathcal{A}$ to $\text{Iso}(G, G')$ such that $\varphi_1(x_i) = y_i$. Later we will see that $\varphi_1$ is in fact an isomorphism.
Let \( u \) be magic unitary of generators for \( \text{Iso}(G, G') \). By Lemma 6.6, \( u = \bigoplus_{k \in [m]} u^{(k)} \) where each \( u^{(k)} \) is a \( V_k(G) \times V_k(G') \) magic unitary. Moreover, the operator \( u^{(k)}_{\alpha, \beta} := u^{(k)}(\alpha, \beta) \) depends only on \( k \) and the value of \( \alpha \triangle \beta \in \pm 1^{S_k}_{b_k+b'} \). Thus, as in the proof of Lemma 3.6, for each \( \delta \in \pm 1^{S_k}_{b_k+b'} \), we let \( u^{(k)}_{\delta} \) denote \( u^{(k)}_{\alpha, \beta} \) such that \( \alpha \triangle \beta = \delta \). Note that for any \( \alpha \in \pm 1^{S_k}_{b_k} \), we have that

\[
\{u^{(k)}_{\delta} : \delta \in \pm 1^{S_k}_{b_k+b'}\} = \{u^{(k)}_{\alpha, \beta} : \beta \in \pm 1^{S_k}_{b'}\},
\]

and thus every row (and similarly every column) of \( u^{(k)} \) contains the same set of operators.

Now we define \( y^{(k)}_i = \sum_{\alpha \in \pm 1^{S_k}_{b_k+b'} \atop \alpha_i=\beta_i} \alpha_i u^{(k)}_{\alpha, \beta} \) for all \( k \in [m] \) and \( i \in S_k \). We first aim to show that \( y^{(k)}_i \) does not depend on \( k \).

Fix \( i \in [n] \) and \( l, k \in [m] \) such that \( i \in S_l \cap S_k \). For each \( \delta \in \pm 1^{S_l \cap S_k} \), let \( A_\delta \) (respectively, \( A'_\delta \)) be the adjacency matrices of the graph consisting of the edges of \( G \) (respectively, \( G' \)) colored \( \delta \). Further, let \( B_\delta \) (respectively, \( B'_\delta \)) be the submatrix of \( A_\delta \) (respectively, \( A'_\delta \)) consisting of the rows indexed by \( V_l(G) \) and columns indexed by \( V_k(G) \) (respectively, \( V_l(G') \) and \( V_k(G') \)). In other words,

\[
(B_\delta)_{(l, \alpha), (k, \beta)} = \begin{cases} 1 & \text{if } \alpha \triangle \beta = \delta \\ 0 & \text{o.w.} \end{cases},
\]

and similarly for \( B'_\delta \). Also define \( A_+ = \sum_{\delta \in \pm 1^{S_l \cap S_k} \atop \delta_i=+1} A_\delta \) and \( A_- = \sum_{\delta \in \pm 1^{S_l \cap S_k} \atop \delta_i=-1} A_\delta \), and similarly define \( A'_+ \), \( A'_- \), \( B_+ \), \( B_- \), \( B'_+ \), and \( B'_- \). Then

\[
(B_+)_{(l, \alpha), (k, \beta)} = \begin{cases} 1 & \text{if } \alpha_i = \beta_i \\ 0 & \text{o.w.} \end{cases} \quad \& \quad (B_-)_{(l, \alpha), (k, \beta)} = \begin{cases} 1 & \text{if } \alpha_i = -\beta_i \\ 0 & \text{o.w.} \end{cases},
\]

and similarly for \( B'_+ \) and \( B'_- \). Since \( u \) must satisfy \( A_\delta u = u A'_\delta \) by definition, it must also satisfy \( A_\pm u = u A'_\pm \). This implies that \( B_+ u^{(k)} = u^{(l)} B'_+ \) and \( B_- u^{(k)} = u^{(l)} B'_- \). Considering the \( (l, \alpha), (k, \beta) \) entry of both sides of the former equation, we see that

\[
\sum_{\beta' \in \pm 1^{S_k}_{b_k+b'} \atop \beta'_i=\alpha_i} u^{(k)}_{\beta', \beta} = \sum_{\alpha' \in \pm 1^{S_l}_{b_l+b'_l} \atop \alpha'_i=\beta_i} u^{(l)}_{\alpha, \alpha'}. \quad \text{for all } k \in [m].
\]

Note that for every term in the above sums, we have that \( \alpha, \alpha' = \alpha_i \triangle \alpha'_i = \beta'_i \beta_i \). Thus, reexpressing \( u^{(l)}_{\alpha, \alpha'} \) as \( u^{(l)}_{\bar{\alpha}} \) where \( \bar{\alpha} = \alpha \triangle \alpha' \), and similarly for \( u^{(k)}_{\beta', \beta} \) we have that

\[
\sum_{\beta \in \pm 1^{S_k}_{b_k+b'} \atop \beta_i=\alpha_i \beta_i} u^{(k)}_{\beta} = \sum_{\bar{\alpha} \in \pm 1^{S_l}_{b_l+b'_l} \atop \bar{\alpha}\beta_i=\alpha_i \beta_i} u^{(l)}_{\bar{\alpha}}. \quad \text{for all } k \in [m].
\]
Doing the same for $B_- u^{(k)} = u^{(l)} B'_-$ yields the same equation but with $\alpha_i \beta_i$ replaced with $-\alpha_i \beta_i$ and combining these proves that

$$
\sum_{\hat{\beta} \in \pm 1_{\hat{b}_k + b'_k}} u^{(k)}_{\hat{\beta}} = \sum_{\hat{\alpha} \in \pm 1_{\hat{b}_l + b'_l}} u^{(l)}_{\hat{\alpha}}
$$

for any $x \in \{+1, -1\}$. Therefore,

$$
y_i^{(l)} = \left( \sum_{\alpha \in \pm 1_{\hat{b}_k}} u_{\alpha}^{(l)} \right) - \left( \sum_{\alpha \in \pm 1_{\hat{b}_l}} u_{\alpha}^{(l)} \right) = \left( \sum_{\alpha \in \pm 1_{\hat{b}_k}} u_{\alpha}^{(k)} \right) - \left( \sum_{\alpha \in \pm 1_{\hat{b}_l}} u_{\alpha}^{(k)} \right) = y_i^{(k)}.
$$

So we have shown that the value of $y_i^{(k)}$ does not depend on $k$, and thus we will simply denote this operator by $y_i$.

Now note that since $y_i$ is a linear combination (with real coefficients) of the operators $u_{\alpha}^{(k)}$ for $k \in [m]$ such that $i \in S_k$, and these operators are entries of the magic unitary $u$, we have that $y_i^* = y_i$. Also, by (23) we have that $u_{\alpha}^{(k)} u_{\beta}^{(k)} = 0$ for $\alpha \neq \beta$ and thus

$$
y_i^2 = \left( \sum_{\alpha \in \pm 1_{\hat{b}_k}} \alpha_i u_{\alpha}^{(k)} \right)^2 = \sum_{\alpha \in \pm 1_{\hat{b}_k}} \alpha_i^2 \left( u_{\alpha}^{(k)} \right)^2 = \sum_{\alpha \in \pm 1_{\hat{b}_k}} u_{\alpha}^{(k)} = 1.
$$

Thus the $y_i$ satisfy relation (1) from Definition 6.5.

Next we will show that relation (2) of Definition 6.5 holds, that is, that $y_i y_j = y_j y_i$ if there exists $k \in [m]$ such that $i, j \in S_k$. Suppose that $i, j, k$ are as described. Then $y_i = y_i^{(k)}$ and $y_j = y_j^{(k)}$ are both linear combinations of the entries of $u^{(k)}$ which pairwise commute by Lemma 6.6. Therefore $y_i$ and $y_j$ commute as desired.

Lastly, we must show that relation (3) of Definition 6.5 holds, that is, that $\prod_{i \in S_k} y_i = (-1)^{b_k + b'_k}$ for all $k \in [m]$. We have that

$$
\prod_{i \in S_k} y_i = \prod_{i \in S_k} y_i^{(k)} = \prod_{i \in S_k} \left( \sum_{\alpha \in \pm 1_{\hat{b}_k + b'_k}} \alpha_i u_{\alpha}^{(k)} \right)
$$

Since the elements $u_{\alpha}^{(k)}$ are pairwise orthogonal, when we expand the above product all cross terms disappear and we obtain

$$
\prod_{i \in S_k} y_i = \sum_{\alpha \in \pm 1_{\hat{b}_k + b'_k}} \left( \prod_{i \in S_k} \alpha_i \right) u_{\alpha}^{(k)} = \sum_{\alpha \in \pm 1_{\hat{b}_k + b'_k}} (-1)^{b_k + b'_k} u_{\alpha}^{(k)} = (-1)^{b_k + b'_k},
$$

since \( \prod_{i \in S_k} \alpha_i = (-1)^{b_k + b'_k} \) for all \( \alpha \in \pm 1_{S_k}^{b_k + b'_k} \), by definition. Therefore, the elements \( y_i \in Iso(G,G') \) for \( i \in [n] \) satisfy the relations of the generators of \( A \) and thus there exists an \( * \)-homomorphism \( \varphi_1 \) from \( A \) to \( Iso(G,G') \) such that \( \varphi_1(x_i) = y_i \) for all \( i \in [n] \).

**Step 2: Construction of an \( * \)-homomorphism \( \varphi_2 : Iso(G,G') \rightarrow A \).**

This step is almost identical to Step 2 of the proof of Theorem 3.8, and so we omit it. We only remark that the biggest change is that when showing that \( v^{(k)}(\alpha) = 0 \) for \( \alpha \notin \pm 1_{S_k}^{b_k + b'_k} \), the analog of equation (8) gets an additional factor of \( (-1)^{b_k + b'_k} \) in the first and last expressions.

**Step 3: Showing that \( \varphi_1 \) and \( \varphi_2 \) are inverses of each other.**

This step is nearly identical to Step 3 of the proof of Theorem 3.8 and so we omit it.

\[ \square \]

**Remark 6.8.** For the uncolored versions of \( G(M,b) \) and \( G(M,b') \), it was shown in [9, Theorem 4.1 and Theorem 5.3] that \( A(M,b + b') \) and \( Iso(G(M,b),G(M,b')) \) are \( * \)-equivalent, meaning that there are \( * \)-homomorphisms in both directions. Note that the \( * \)-homomorphisms in Theorem 6.7 are the same ones as those in [9], which we only became aware of after writing the initial version of this paper. We get a \( * \)-isomorphism in our case by coloring the graphs in the right way.

**Corollary 6.9.** Let \( M \in F_2^{m \times n} \) and \( b \neq b' \in F_2^m \). Then the following are equivalent.

1. There is a quantum isomorphism from \( G(M,b) \) to \( G(M,b') \).
2. The algebra \( A(M,b + b') \) is non-trivial.
3. The element \( \gamma \) of the solution group \( \Gamma(M,b + b') \) is not equal to the identity.

**Proof.** Proposition 6.2 states that (1) is equivalent to \( Iso(G(M,b),G(M,b')) \) being non-trivial, which is equivalent to (2) by Theorem 6.7. To see that (2) and (3) are equivalent, recall that \( A(M,b + b') \) is isomorphic to \( pC^*(\Gamma(M,b + b')) = \{ px : x \in nC^*(\Gamma(M,b + b')) \} \) where \( p = \frac{1}{2}(1 - \gamma) \), and so these algebras are non-trivial if and only if \( \gamma \neq 1 \) in \( C^*(\Gamma(M,b + b')) \) which is equivalent to \( \gamma \) not being equal to the identity in \( \Gamma(M,b + b') \).

Note that the equivalence of (1) and (3) in the corollary above differs from earlier results (for example, from [1]) because the graphs \( G(M,b) \) and \( G(M,b') \) here are colored graphs, rather than uncolored ones.

We are now able to give two non-isomorphic colored graphs without quantum symmetry that are quantum isomorphic. Recall the definition of \( G_s(M,0) \) from Definition 5.2.

**Corollary 6.10.** Let \( M \) be the incidence matrix of \( K_{3,3} \), and let \( b \in F_2^V(K_{3,3}) \) be any vector with exactly one entry equal to 1. Then \( G_s(M,0) \) and \( G_s(M,b) \) are non-isomorphic colored graphs with no quantum symmetry which are nevertheless quantum isomorphic.

**Proof.** In this case, it is well known that the element \( \gamma \) of the solution group \( \Gamma(M,b + 0) = \Gamma(M,b) \) is not equal to the identity [7], since this is the solution group of the Mermin–Peres magic square game. It then follows from Corollary 6.9 that \( G_s(M,0) \) and \( G_s(M,b) \) are quantum isomorphic. On the other hand, it is known that even the uncolored versions of these graphs are non-isomorphic [1], and thus the colored versions are as well. Lastly, it was shown in [14] that \( \Gamma(M,0) \) is abelian, and therefore by Theorem 3.8 the algebras \( C(\text{Quot}(G_s(M,0))) \) and \( C(\text{Quot}(G_s(M,b))) \) are commutative, that is, the graphs \( G_s(M,0) \) and \( G_s(M,b) \) have no quantum symmetry.

\[ \square \]
Corollary 6.11. Let $M$ be the incidence matrix of $K_{3,3}$, and let $b \in F_2^{V(K_{3,3})}$ be any vector with exactly one entry equal to 1. Let $G'_s(M,0)$ and $G'_s(M,b)$ the graphs constructed from $G_s(M,0)$ and $G_s(M,b)$ as in Lemma 5.4, where we choose the added paths in both graphs to have the same length for vertices of the same color and edges of the same color. Then $G'_s(M,0)$ and $G'_s(M,b)$ are non-isomorphic graphs with no quantum symmetry which are nevertheless quantum isomorphic.

Proof. We start by showing that $G'_s(M,0)$ and $G'_s(M,b)$ have no quantum symmetry. By the previous corollary, we know that $G_s(M,0)$ and $G_s(M,b)$ have no quantum symmetry. We have $C'(\text{Out}^{-1}_s(G'_s(M,0))) \cong C'(\text{Out}(G'_s(M,0)))$ and $C'(\text{Out}^{-1}_s(G'_s(M,b))) \cong C'(\text{Out}(G'_s(M,b)))$ by Lemma 5.4. Therefore, Corollary 4.14 yields that $G'_s(M,0)$ and $G'_s(M,b)$ have no quantum symmetry.

It remains to show that $G''_s(M,0)$ and $G''_s(M,b)$ are quantum isomorphic. By Corollary 6.10, we know that $G_s(M,0)$ and $G_s(M,b)$ are quantum isomorphic. Thus, there exists a $C^*$-algebra $A$ and a matrix $U \in M_n(A)$ such that Relations (18)–(21) hold. In $G''_s(M,0)$, we denote the vertex that subdivided the edge $e$ in $G_s(M,0)$ by $e_0$ and the vertices in the added path with $d(e_0,e_i) = i$ by $e_i$, $1 \leq i \leq m_c(e)$. Moreover, we denote by $v_i$, $0 \leq i \leq n_c(v)$ the vertices in the added path with $d(v,v_i) = i$. We similarly label the vertices of $G''_s(M,b)$.

Note that by a similar reasoning as in Lemma 5.4, we get $u_{ij}u_{kl} = u_{kl}u_{ij}$ for all $e = (i,k), f = (j,l)$ with $c(e) \neq c_0 \neq c'(f)$. Define the magic unitary $W$ as follows:

$$w_{ij} = \delta_{ji}$$
$$w_{ikj} = \delta_{ki}u_{ij} \text{ for } i \in V(G(M,0)), j \in V(G(M,b)),$$
$$w_{ikj} = \delta_{ki}(u_{ac}u_{bd} + u_{bd}u_{ac}) \text{ for } e = (a,b) \in E(G(M,0)), f = (c,d) \in E(G(M,b)),$$
$$w_{ikj} = 0 \text{ for } i \in V(G(M,0)), f \in E(G(M,b)),$$
$$w_{ikj} = 0 \text{ for } e \in E(G(M,0)), j \in V(G(M,b)),$$

for all occurring $k,l$. Similar to the proofs of Theorem 4.7 and Theorem 4.13, we see that $A_{G''_s(M,0)}W = WA_{G''_s(M,b)}$. We conclude that $G''_s(M,0)$ and $G''_s(M,b)$ are quantum isomorphic.

Acknowledgements

DR was partially supported by the VILLUM Centre of Excellence for the Mathematics of Quantum Theory (QMATH) project no. 35544. SSch would like to thank Matthew Daws and Christian Voigt for helpful discussions. This project has received funding from the European Union’s Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie grant agreement No. 101030346. It was also supported by the EPSRC project ‘Quantum groups in action’, EP/T03064X/1.

Journal Information

The Journal of the London Mathematical Society is wholly owned and managed by the London Mathematical Society, a not-for-profit Charity registered with the UK Charity Commission. All surplus income from its publishing programme is used to support mathematicians and mathematics research in the form of research grants, conference grants, prizes, initiatives for early career researchers and the promotion of mathematics.
REFERENCES