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Jakobsen, Hans P.; Levichev, Alexander V.

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A REPRESENTATION OF $SU(2,2)$ WHICH CAN BE INTERPRETED AS DESCRIBING CHRONOMETRIC FERMIONS (PROTON, NEUTRINO, AND ELECTRON) IN TERMS OF A SINGLE COMPOSITION SERIES

HANS P. JAKOBSEN

University of Copenhagen, Department of Mathematical Sciences
Universitetsparken 5, DK-2100, Copenhagen, Denmark
(e-mail: jakobsen@math.ku.dk)

and

ALEXANDER V. LEVICHEV

Sobolev Institute of Mathematics, 4 Acad. Koptyug avenue, 630090 Novosibirsk, Russia
(e-mail: alevichev@gmail.com)

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In 1991 I. E. Segal stated (on the basis of a certain induced representation of the conformal group, but without proof) that there are four chronometric elementary particles of spin $\frac{1}{2}$. In terms of the same representation (but using a different parallelization), we discuss a model where there are just three of those particles. They are interpreted as the proton, the (electronic) neutrino, and the electron, respectively. Mathematically, the three particles correspond to factors of the composition series. Each of the three representation spaces ($F_p$, $F_\nu$, and $F_e$, respectively) is provided with an invariant unitary structure and, in each of the three cases, energy positivity holds. These findings may shed some light on why there are lepton generations at all. The conclusions are mathematically based on the specific formula for the action of our representation in the full representation space $F$. This formula is presented in terms of the so-called reproducing kernel functions which are related to coherent states.

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1. Motivation and introduction

Our current research can be viewed as a discussion of, and supplement to, Segal’s list of chronometric$^1$ elementary particles of spin $\frac{1}{2}$ [1]. That article is in

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$^1$The adjective chronometric indicates that we are within the Chronometric Theory. This is a theory based on the group $SU(2,2)$ and a deep analysis of the notion of time and causality. The particles suggested by this theory should be compared to ‘relativistic’ particles, where the (10-dimensional) Poincaré group is the main symmetry group, or to ‘Galilean’ particles, when (also 10-dimensional) Galilean group is the main one.
some sense a summary of Segal's findings, and this, 5 pages long, paper should be kept within reach of the reader of the present paper. In [1], there are few, if any, clues of how to obtain results outlined in it. One of our article's goals is to prove (some of) Segal's statements.

The most remarkable of these is that there are four elementary chronometric particles of spin 1/2. Namely, there is a massive neutral particle named the exon, the electron $e$, and two types of neutrino (interpreted as $\nu_\mu$ and $\nu_e$). In Segal's theory, a particle (e.g. each of the above) is mathematically associated with an irreducible unitary positive-energy representations of the symmetry group $G$ (in our case, the conformal group SU(2,2) or, which is essentially the same, the universal cover of it). Let us stress that when we, below, associate certain mathematical objects to specific particles, we try (as much as possible) to stay in line with what Segal has done before in this regard. Here is the only significant exception to the above.

Around 2010 A. Levichev suggested [2, Section 7] that rather than a hypothetical neutral particle, the exon, we have just the proton $p$. Later that suggestion gave birth to Levichev's multi-level model of quarks, MLM [3, 4] which claims that each quark can be viewed as if it is a state of the proton. Segal himself thought (see [1, p. 994]) that the exon could be the main ingredient of both the neutron and the proton.

Leaving aside the validity of Levichev's switch from exon to proton as well as the MLM validity, we display an algebraic model (in terms of the composition series, which is inescapable since we deal with an indecomposable representation) in which there is just one (rather than two, as in the Segal article) light (i.e. mass-less) particle. Trying to stay in line with Segal, we associate it with the electronic neutrino $\nu_e$. In distinguishing between a neutrino and the two 'heavier' leptons (his exon and an electron), Segal found it helpful to introduce the sophisticated notion of the Gelfand–Kirillov (G–K) dimension. We keep this notion in the hope of reaching a broader audience. Its precise definition can be found in [5, p. 524 and p. 538] but below (in our Remark 10) we provide a much simpler way to indicate how the different G–K dimensions (namely, being 4 for both the proton and the electron, while just 3 for the the neutrino) reveal themselves mathematically for the particles involved.

Remark 1. It is worth mentioning that an earlier conjecture by Segal (about the number of chronometric spin 1/2 particles) was in compliance with our current findings: see [6, Theorem 16.7.1] which worked with a 3-step composition series. The representation is a limiting case of representations already studied by Jakobsen in [7]. Here, purely algebraically, a 3-step series may be obtained after-the-fact. Later, Segal's original conclusion has been withdrawn: [1, Table 1] states (without proof) the existence of the 4-step composition series.

Overall, we follow the approach of [1] (see the Section Indecomposable elementary particle associations ibid.), and we now proceed to give an overview of the main mathematical tools involved in our paper. Parallelization (of a vector bundle over space-time, see [8, Section 4]) is an important part of the chronometric approach.
Before discussing this, we recall a few quantum–mechanical features. The word ‘world’ (below) is used as a synonym of space-time.

According to quantum mechanics, each object is assigned a state (or wave function but this latter notion we better reserve for a more specialized situation, namely, after a parallelization has been applied). An elementary particle (it ‘lives’ in a certain world $E$ of events) is described by the totality of its possible states. The latter set is a certain subspace of the section space (sections can later be specified as smooth, or square integrable, etc.) of a certain vector bundle over $E$. At this point, states are not, yet, number-valued (for a scalar particle) or vector-valued – with numerical components of those vectors (for particles of nonzero spin). Typically, we then need to convert to parallelized sections (to wave functions, in other words). This is a terminology from Segal, where an induction method which starts with a closed subgroup $H$ of $G$ and a representation $\rho$ of $H$ in a finite-dimensional space $V_\rho$ is used to construct a vector bundle over $G/H$ with fibers linearly isomorphic to $V_\rho$. This is sometimes called geometric induction and is used in e.g. [9]. The sections of this bundle then carry a representation of $G$. Using a suitable subgroup, possibly passing to a covering space, these sections can be “parallelized” so that one obtains a space of $V_\rho$-valued functions on $G/H$. The end result is then the same as Ind$_H^G(V_\rho)$. We stress that the bundles need not be trivial (parallelizable). We work modulo null sets.

Below, $H$ is the extended Poincaré group and $G/H = U(2)$ (or a covering thereof).

The respective Hilbert space can then be determined. It has become an acknowledged way of modern theoretical physics to describe elementary particles and their interactions in terms of induced representations of the (respective) symmetry group $G$. As it is put in [10], “the main philosophical point of these developments is perhaps the importance of induced representations, not purely as representations, but as actions on the homogeneous vector bundles that naturally emerge from the induction process. This additional structure provides a spatio-temporal labeling of the vectors in the group representation space that is absolutely essential for the formation of local nonlinear interactions, and relatedly, for causality considerations”.

Conventional quantum mechanics uses representations of the Poincaré group, which are induced from certain subgroups obtained from orbits of the Lorentz subgroup in connection with the translation subgroup as in Wigner’s seminal work [11]. These representations can be formulated in spaces of vector-valued functions on the Minkowski space-time $M$. However, there the inner product becomes complicated. In Fourier-transformed versions, the Hilbert spaces become sections over Lorentz orbits in energy-momentum space. Here the Hilbert space inner products are straightforward, but one has to work with sections of bundles over the orbits, see [9, 12].

In the general setting of space-time symmetry groups, the parallelization procedure is essentially defined by the choice of the parallelizing (four-dimensional but not necessarily commutative) subgroup $N$ of the symmetry group $G$. Typically, $N$ will be
a finite cover of the original space-time $E$. In Segal’s (with co-authors) publications the most used parallelizations are the $N = M$, and the $N = U(2)$ ones. The former parallelization is frequently called the flat parallelization, while the latter can be called the curved parallelization.

In our article we deal with the $M$-parallelization: see expression (6) below. Additionally to the already mentioned references [1, 8, 9], all necessary definitions and properties can be found in [7, 13–15]. In particular, [13, Table I] lists all fifteen generators (both as four by four matrices, as well as vector fields) of the conformal group. The publication [16] is recommended as having most of what is needed in a single reference (however, as regards elementary particles, no assignments are made in this work).

The use of conformal symmetry in particle physics has been discussed in a huge number of papers and monographs. We apologize in advance for citing only such publications that have fairly directly and consciously influenced the current article. So, we mention [1, 6–7, 14, 16–20] but point to references cited in these for further information.

2. The two representations: the inducing and the induced one

We will be dealing with the Minkowski space-time $M$ in its matrix form. That is, $M$ consists of all matrices

$$
\begin{bmatrix}
  x_0 + x_1 & x_2 + ix_3 \\
  x_2 - ix_3 & x_0 - x_1
\end{bmatrix}
$$

with real $x_0, x_1, x_2, x_3$. We view the symmetry group $SU(2,2)$ as the subgroup of $SL(4,\mathbb{C})$ consisting of those matrices $g$ from $SL(4,\mathbb{C})$ that satisfy

$$g S_1 g^* = S_1,$$

where

$$S_1 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$$

with $i$ being the $2 \times 2$ scalar matrix.

**Remark 2.** In [8, p. 82, 16, (3.2)], the choice of the matrix $S_1$ is different. For more details on that, see our Remark 6 in the next section.

Writing $g$ in $2 \times 2$ blocks, (1) equivalent to

$$ad^* - bc^* = 1, \quad ab^* = ba^*, \quad cd^* = dc^*, \quad (2)$$

and to

$$a^*d - c^*b = 1, \quad a^*c = c^*a, \quad b^*d = d^*b, \quad (3)$$

where (3) is (2) for $g^{-1}$.

Consider the following subgroup in $SU(2,2)$,

$$P_- = \begin{bmatrix} a & 0 \\ c & a^*-1 \end{bmatrix} \quad (4)$$
where determinant of \( a \) is real and \( ca^* = ac^* \). It is known that \( P_- \) is isomorphic to the extended (that is, to the 11-dimensional) Poincaré group with two connected components.

Introduce the inducing representation \( \mu \) as follows

\[
\mu \begin{pmatrix} a & 0 \\ c & a^{*-1} \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & a^{*-1} \end{pmatrix} (\det a)^2. \tag{5}
\]

**Remark 3.** This inducing representation is equivalent to the one given in [16, (6.7)] with \( d = 2 \) there. Hence, the induced representation (which is guaranteed to be uniquely defined by the inducing one) is also equivalent to the representation in [7]. Recall that a general notion of an induced representation remains somewhat vague in [8, p. 99] and no reference is given there. The fact that vector bundles are mentioned indicates that the notion of “geometric induction” may be partly employed. However, it is clear from the context of [8] and from Segal’s other publications that the Mackey’s concept of induced representation is meant. This concept proved to be a major tool in the modern quantum mechanical description of a particle (see [12]). In the next two paragraphs we provide more (compared to how we have outlined it in Section 1) specific information on the notion of an induced representation and on its specifics as used in our paper.

Segal further developed his Chronometry by introducing spannor and plyor fields [1]. It is mostly Segal [1] who has managed to explain the conformal-invariant method for describing massive states of chronometric fermions (represented by spannors). We have contributed by explicit presentation of the wave functions involved (see our Sections 4–7) and by switching from his massive neutral particle (hypothetical exon) to the (chronometric) proton (see Section 5). In the relativistic limit (this notion belongs to Segal), spannors transform into spinors. To conclude the paragraph, we state that we consider flat-parallelized spannor fields. Since there are different ways to imbed the 4-dimensional vector group into \( SU(2, 2) \), hence there is more than one flat parallelization. We are using the one defined by Eq. (6), see below.

It was shown in [15] that the induced representation, obtained from \( \mu \), yields action \( H \) (see (7) below) on measurable functions on \( M \). Recall (from [15, p. 60]) that the subgroup

\[
\begin{bmatrix} 1 & h \\ 0 & 1 \end{bmatrix}
\]

of \( G \) (where \( h \) goes through \( M \)) is identified with an open dense subset of \( G/P_- \).

**Remark 4.** If one applies conjugation by the matrix \( K \) from our Remark 6 (below), one gets the corresponding to (6) subgroup \( N_0 \) of [7, Corollary 4.3.2].

Let \( g \) be an element of \( G \). The value at \( h \in M \) of the function obtained from \( f \) by the action of the operator \( H(g) \) is calculated as follows

\[
\begin{pmatrix} (ch + d)^* & 0 \\ -c^* & (ch + d)^{-1} \end{pmatrix} \{\det (ch + d)\}^{-2} f ((ah + b)(ch + d)^{-1}). \tag{7}
\]
In that formula, \( g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \) and \( f \) is any measurable function from \( M \) to \( C^4 \).

On the basis of Chapters 2 and 4 of [7], it can be shown that (7) is equivalent to

\[
H(g) = \begin{bmatrix} D^-(g) & 0 \\ c^*U(g) & D^+(g) \end{bmatrix}.
\]

(8)

**Remark 5.** In the above formula, Dirac-type operators \( D^- \), \( D^+ \) enter as ingredients.

Namely, \( D^+(g) \) is for the first entry of [7, (4.1)] (with \( \lambda = 0 \) there) while \( D^-(g) \) is for the second entry of [7, (4.1)] (with \( \lambda = -1 \)). The factor \( c^* \) is the transposed complex conjugate of the lower left corner of the matrix \( g^{-1} \). As regards the operator \( U(g) \), it can be deduced from [7, (2.6)] that

\[
(U(g)f)(h) = (\det (ch + d))^{-2} f(g^{-1}h),
\]

(9)

where \( f \) is any measurable function from \( M \) to \( C^2 \).

3. The action of scaling

The action of scaling will be calculated according to our (8), above. First, let us enter \( D^-(g) \) explicitly,

\[
(D^-(g)f)(h) = (\det (hc^* + d^*))^{-2}(hc^* + d^*)f((ah + b)(ch + d)^{-1}).
\]

(10)

Hereafter, \( g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \).

The expression for \( D^+(g) \) is as follows,

\[
(D^+(g)f)(h) = (\det (ch + d))^{-2}(ch + d)^{-1}f((ah + b)/(ch + d)).
\]

(11)

In each of (10) and (11), \( f \) is any measurable function from \( M \) to \( C^2 \).

**Remark 6.** As it has been indicated in Introduction, to choose generators (and to evaluate elements of corresponding one-parameter transformation groups) we use Table I of [13]. This is the “Segal setting” where \( SU(2,2) \) is defined via the Hermitian form \( \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \). However, to conform to our setting, the entries of that Table are to be conjugated by the following matrix \( K = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix} \). That is, if \( Q \) is a generator (as a matrix) in ‘Segal’s setting’, then we use \( KQK^{-1} \) instead.

In what follows, set \( S = \sinh(t/2), \ C = \cosh(t/2) \) where \( t \) is a real parameter.

The \( 4 \times 4 \) matrix \( \begin{bmatrix} C & S \\ S & C \end{bmatrix} \) represents the inverse of the scale transformation (see line five of [13, Table I]), here \( C, S, \) etc., are the corresponding \( 2 \times 2 \) scalar matrices. When the conjugation by \( K \) is applied to that matrix, one ends up with \( \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix} \). This last matrix is to be entered as \( g^{-1} \) into (7). Let us use \( f_1, f_2 \)
for $C^2$-valued functions on $M$, and consider the totality of columns $\{f_1, f_2\}^T$. That is, we use $T$ in the upper position to denote conversion from a row to a column (as well as from a column to a row). Using (10) and (11), we conclude that when the scaling transformation is applied to the column $\{f_1, f_2\}^T$, then the value of the resulting column at $h$ is as follows,

$$\{e^{(3/2)t} f_1(e^t h), e^{(5/2)t} f_2(e^t h)\}^T.$$  \hspace{1cm} (12)

**Remark 7.** Let us now pay attention to the lower (2-dimensional) component in (12). In our Sections 4, 5 we will see that this component corresponds to an invariant subspace $F_p$ of the entire (induced) representation space $F$. Recall that $P$ denotes an (11-dimensional) Poincaré subgroup of our $G = SU(2, 2)$. Here is an important notion from [14, p. 342]: a representation $R$ of $P$ is said to be of (conformal) weight $w$, $w$ being a given complex number, in case the value of $R(S_A)f_2$ at an argument $h$ equals $(A^w)f_2(Ah)$; for $A$ real and positive, $S_A$ denoting the transformation $h$ goes into $Ah$ in Minkowski space. Later, on the basis of this definition and from (12), we will conclude that the space $F_p$ corresponds to a ‘particle of conformal weight 5/2’, $A$ being equal to $e^t$ in (12). In reproducing this important definition we have Eq. (12) in mind and we use $f_2$ (rather than just $f$) as part of this definition’s notations in order to better relate to our case. Notice that we have hereby corrected a misprint in [14] since if to literally apply [14, p. 342] one then contradicts to the property of the representation being exact. To finish our Remark, let us mention that the notion of the conformal weight has become such a common place in the physics literature that a precise definition seems to be absent in most of the corresponding publications.

4. The choice of the space of special functions

In this section we will make the ultimate choice of the class of special functions (these functions, viewed as vectors, will span the representation space; details follow). Such a choice of the representation space (i.e. the closure of the span of functions (16) in the current section) will stay intact within the remaining body of the paper. In particular (in the next three sections), the three Hilbert spaces of wave functions (for a proton $p$, for a neutrino $\nu$, and for an electron $e$) will be defined.

In what follows, by $D$ we understand the following (8-dimensional, as a real manifold with boundary) domain,

$$D = \{z = x + iy : \quad x \in M, \quad y \in M, \quad y > 0\}.$$  \hspace{1cm} (13)

In the above (13), $y > 0$ means that $y$ is in the solid forward momentum cone $C^+$, see [15, p. 57]. In terms of the matrix $y$, it means that its eigenvalues are strictly positive.

The group $G$ acts naturally on $D$ by fractional linear transformations: $z$ goes into $gz = (az + b)(cz + d)^{-1}$ where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. 

Remark 8. To our mind, a first-time reader of the paper would not experience any logical difficulty by going directly from here to Theorem 1 via the expressions (14)–(16) thus saving time and effort to adequately comprehend both the authors’ main findings and to successfully follow the steps how they were obtained. If interested in a more solid mathematical proof of these findings, one can return to those omitted lines during second reading.

There are two fundamental kernels:

\[ K_F = ((w_1 - w_2^*)/2i)^{-1}, \quad K_F = (w_1 - w_2^*)/(2i), \]

where \( w_1, w_2 \) are in \( D \).

There are two “universal” automorphic factors:

\[ J_F = c z + d, \quad J_F = (zc^* + d^*)^{-1}. \]

They satisfy

\[ J_F(g_1 g_2, z) = J_F(g_1, g_2z) J_F(g_2, z), \quad i = 1, 2. \]

The relation between the kernel and the automorphic factor is

\[ K_F(gz, gw) = J_F(g, z) K_F(z, w) J_F(g, w)^*, \quad i = 1, 2. \]

Now, for the next lemmata, one needs to recall the defining equations for \( SU(2, 2) \).

Specifically we have.

**Lemma 1.** If \( g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \) and \( h \) is from the Minkowski space-time \( M \), then

\[ c^*(g^{-1} h - w^*) = (ch + d)^{-1} - (J_F(g, w)^*)^{-1}. \]

**Proof:** It follows that \( g = \begin{bmatrix} d^* & -b^* \\ -c^* & a^* \end{bmatrix} \). Furthermore, \( c^*(ah + b) = -a^*(ch + d) + 1 \).

Finally, \( -a^* + c^* w^* = (wc - a)^* = -(w(c(g)^* + d(g)^*))^* \), where \( c(g) \) denotes the component of \( g \) in the “\( c \) position”, etc.

Similarly (with proof omitted), we get the following.

**Lemma 2.** \( hc^* + d^* = (h - (gw)^*)c^* + (J_F(g, w)^*)^{-1} \).

Below we will use the following notation (with \( w_1, w_2 \) from \( D \)):

\[ K_2(w_1, w_2) = ((w_1 - w_2^*)/2i)(\det ((w_1 - w_2^*)/2i))^{-2}, \quad (14) \]

\[ K_3(w_1, w_2) = ((w_1 - w_2^*)/2i)^{-1}(\det ((w_1 - w_2^*)/2i))^{-2}, \quad (15) \]

\[ K_0^0(w_1, w_2) = (\det ((w_1 - w_2^*)/2i))^{-2}, \]

where the last expression is for a scalar \( 2 \times 2 \) matrix.

Notice that each of the three functions is holomorphic w.r.t. the first variable, and it is anti-holomorphic w.r.t. the second variable.
We now set
\[
\Psi[w_1, w_2, w_3; v_1, v_2, v_3](h) = \begin{bmatrix}
K_2(h, w_1)v_1 + K_2^0(h, w_2)v_2 \\
-(1/2i)K_2^0(h, w_1)v_1 + K_3(h, w_3)v_3
\end{bmatrix}.
\] (16)

Here \(w_1, w_2, w_3\) are from \(D\), \(v_1, v_2, v_3\) are from \(C^2\), while \(h\) from \(M\) is an argument of such a \(C^4\)-valued function. Also, we set \(w_i = 0\) in \(\Psi\) exactly when the corresponding \(v_i = 0\). Hence, functions (16) are \textit{boundary values of holomorphic functions}.

Clearly,
\[
\Psi[w_1, w_2, w_3; v_1, v_2, v_3](h) = \Psi[w_1, 0, w_3; v_1, 0, v_3](h) + \Psi[0, w_2, 0; 0, v_2, 0](h),
\]
or
\[
\Psi(h) = \Psi_a(h) + \Psi_b(h),
\]
for short.

It follows (from the above equality) that if we find a convenient formula for both \(g\)-transform of \(\Psi_a\) and of \(\Psi_b\), then we will be able to determine the value \((H(g)\Psi)(h)\) at \(h\) of the \(g\)-transform of \(\Psi\); here \(H\) is for the induced representation, and \(g\) is an element of the group \(G = \text{SU}(2, 2)\). Let us recall that the representation \(H\) is the one which we deal with throughout the entire paper (recall (7) from our Section 2).

The next statement (Theorem 1, below) says, in particular, how an element \(g\) acts on parameters of \(\Psi_a\) (respectively, on parameters of \(\Psi_b\)). As several times before, if \(g\) is an element of the group \(G = \text{SU}(2, 2)\), then
\[
g^{-1} = \begin{bmatrix}
a & b \\
c & d
\end{bmatrix}.
\]

One is thus able to calculate each of the terms \(Q_{g, w_1}^0 = (\det (c(gw_i)^* + d))^2\) as soon as an element \(w_i\) of of the domain \(D\) is specified. Those terms are used in formulae (17), (18); see below. We denote by \(\Psi_a = H(g)\Psi\) the \(g\)-transform of \(\Psi_a\), etc. In formulae (17), (18) we use \(T\) in the upper position to denote conversion of a row into a column (and of a column into a row).

**Theorem 1.** The \(g\)-transform of \(\Psi_a\) is
\[
\tilde{\Psi}_a = \Psi_a [gw_1, 0, gw_3; ((Q_a(g))(v_1, 0, v_3)^T)^T],
\] (17)
while the \(g\)-transform of \(\Psi_b\) is
\[
\tilde{\Psi}_b = \Psi_b [gw_2, gw_2, 0; ((Q_b(g))(0, v_2, 0)^T)^T].
\] (18)

The above \(6 \times 6\) matrices \(Q_a, Q_b\) (which we write as a \(3 \times 3\) block matrix with \(2 \times 2\) matrices as entries) are as follows. \(Q_a\) is a diagonal matrix with entries \(Q_{g, w_1}^0 = (c(gw_1)^* + d)^{-1}, 0, Q_{g, w_3}^0((gw_3)^*c^* + d^*)\), in that order. Viewed as a \(3 \times 3\) matrix, \(Q_b\) has just two nonzero entries: the second entry in row 1 is the \(2 \times 2\) matrix \(2ic^*Q_{g, w_2}^0\), while the second entry on the main diagonal is \(Q_{g, w_2}^0((gw_2)^*c^* + d^*)\).
The proof follows from the above observations and from formulae (7) through (11).

Remark 9. Let us clarify that to find the value of the transformed state at \( h \), we simply evaluate \( \tilde{\Psi}_a \) at \( h \) – this is what formula (17) says. Same – about (18).

It is now an easy exercise to show that any \( g \)-transform of a finite linear combination of vectors (16) is in itself a finite linear combination of vectors (16). Denote by \( F \) the closure of the set of all those linear combinations. From our observations it becomes clear that \( F \) has a nontrivial minimal invariant subspace \( F_p \) spanned by vectors \( \Psi \equiv 0 \rightarrow 0 \rightarrow \omega \equiv 0 \rightarrow 0 \equiv \frac{1}{\sqrt{2}} \), \( \frac{1}{\sqrt{2}} \) here is for a proton. The space \( F_p \) is studied in the next section.

5. The proton's representation space

On \( D \times D \) consider the matrix-valued function

\[
K_p(w_1, w_2) = \left((w_1 - w_2^*)/2i\right)\{\det ((w_1 - w_2^*)/2i)\}^{-2}.
\]

In other words, \( K_p \) is \( K_3 \) from expression (15). As we have already mentioned, this function is holomorphic w.r.t. the first variable, and it is anti-holomorphic w.r.t. the second variable. We will use \( K_p \) as the reproducing kernel to make \( F_p \) into a Hilbert space. This kernel can be expressed in terms of the Fourier–Laplace transform from \( C^+ \) (see [7, p. 332] for more details). Recall that the cone \( C^+ \) is defined in Section 4, right after expression (13).

As a vector space, the Hilbert space \( F_p \) will be the completion of the span of the set of following \( \mathbb{C}^4 \)-valued functions on \( M \),

\[
\Psi_1[0, 0, w_1; 0, 0, v_1](h) = \begin{bmatrix} 0 \\ K_3(h, w_1)v_1 \end{bmatrix},
\]

compare with expression (16) from the previous section; here \( w_1 \in D, v_1 \in \mathbb{C}^2 \), and where we use \( 1 \) as a subscript in order to be able to denote such a function as \( \Psi_1 \). As we have already mentioned, \( F_p \) is interpreted as the totality of proton’s (possible) wave functions. To support this claim, we need to supply \( F_p \) with unitary structure, to prove that our representation is unitary in \( F_p \), and to verify that the energy-positivity condition holds. For the definition of (both Poincaré- and Segal-) energy-positivity condition, see our Section 8, below.

The inner product between two (20)-type functions is given as

\[
\langle \Psi_1, \Psi_2 \rangle = \langle K_p(w_1, w_2)v_1, v_2 \rangle,
\]

where, in the right side of this formula, \( \langle \cdot, \cdot \rangle \) is the canonical inner product in \( \mathbb{C}^2 \). The positivity of the inner product (21) has been proven in [15, p. 65]. Due to continuity, linearity (w.r.t. first variable), and anti-linearity (w.r.t. second variable), the formula (21) makes \( F_p \) into a Hilbert space.

Notice that from (12) and (20) it follows that our proton is of conformal weight (which is sometimes called conformal dimension) \( 5/2 \). It is well known that this
representation is of (so called) *Gelfand–Kirillov dimension* 4. We do not think that it is necessary to provide here the definition of that notion (G–K dimension, for short) but it is worth mentioning that Segal (see [1, p. 996]) in his Table I claims that massive spin 1/2 particles (in our case: electron and proton) should have G–K dimension 4, while the *neutrino* should be of G–K dimension 3.

**Remark 10.** The spinor field of conformal dimension 5/2 was studied in [19, pp. 38-39] in the section titled “Massless electron interacting with an electromagnetic 5-potential”. Since their research is performed in the context of the quantum field theory, it is not that easy for us to more accurately relate to their findings. They claim their field is ‘massless’. In our Remarks 13 and 16 we again refer to [19].

We have followed Segal [1] to call our field massive since the Gelfand-Kirillov dimension is 4 (rather than 3) here. This difference shows itself in our (23) vs (30) equations: please notice the 3-dimensional domain of integration in (23) while it is a 4-dimensional domain of integration in (30). Also in [1, p. 996], Segal discusses the interplay between usual relativistic mass operator and the chronometric Hamiltonian. It is our understanding that this discussion has to be continued in order to result in a more rigorous finding about the relation between chronometric and relativistic masses. As a step in this direction, consider our Appendix where we provide a detailed proof that our space $F_p$ corresponds to a massive particle. The proof uses standard mathematical facts of modern quantum mechanics, [12].

That the Hilbert space $F_p$ should be interpreted as describing a particle of spin 1/2 is also well known (see the above mentioned Segal’s Table I).

For the positivity of (both Poincaré- and Segal-) energy, see our Section 8.

6. Where does the chronometric neutrino live?

It can be shown that the above introduced subspace $F_p$ *does not have an invariant complement* in the entire representation space $F$. It means that for an arbitrary vector subspace $V$, if $F$ is a direct sum of $F_p$ with $V$, then $V$ is not invariant w.r.t. the action of the group $G$ in $F$: there exists such $g$ from $G$, and such $v$ from $V$ that $H(g)v$ is outside of $V$. A standard way to proceed in such a case (our $H$ thus being an example of an *indecomposable representation*) is to arrange for the quotient representation space $W = F/F_p$ (as it is suggested on p. 995 of [1], or elsewhere).

On this space $W$ there is no unitarity, but there is a nontrivial minimal invariant subspace $F_\nu$ ($\nu$ here is for a *neutrino*) spanned by (16)-vectors

$$\Psi_1[w_1,0,0;v_1,0,0](h) = \begin{bmatrix} K_2(h,w_1)v_1 \\ -(1/2i)K_2^0(h,w_1)v_1 \end{bmatrix}.\tag{22}$$

In other words, $F_\nu$ can be represented as an invariant subspace of $F$.

As soon as we know the reproducing kernel, we will be able to provide $F_\nu$ with the unitary structure (similarly to how it has been done in the case of $F_p$, see the previous section).
Here is the reproducing kernel
\[ K_\nu(z, w) = \int_{\partial \mathbb{C}^+} e^{i \text{Trace} (z - w^*) y^T c^T (y)} dy. \] (23)

Remark 11. The co-factor \( c(y) \) in (23) is the matrix for which \( c(y) \) times \( y \) (and \( y \) times \( c(y) \)) equals to \( \det(y) \). It is given by the Cramer’s Rule.

The inner product between two (22)-type functions is given as
\[ \langle \Psi_1, \Psi_2 \rangle = \langle K_\nu(w_1, w_2)v_1, v_2 \rangle, \] (24)
where, in the right side of this formula, \( \langle \cdot, \cdot \rangle \) is the canonical inner product in \( \mathbb{C}^2 \). Due to continuity, linearity (w.r.t. first variable), and anti-linearity (w.r.t. second variable), the inner product (24), if positive definite, makes \( F_\nu \) into a Hilbert space. Its positivity follows from [15, p. 75].

For the positivity of (both Poincaré- and Segal-) energy, see our Section 8.

Let us notice that each (22)-type wave function \( \Psi \) satisfies
\[ \nabla \Psi = 0, \] (25)
where the (acting on \( \mathbb{C}^4 \)-valued functions) differential (Dirac-type) operator \( \nabla \) has been defined on p. 311 of [7]. This observation is an indication to interpret \( F_\nu \) as describing the neutrino (compare to discussion on pp. 23–24 of [6]). Also, the Gelfand–Kirillov dimension of this representation is known to be \( 3 \) which shows itself in the above (23): the domain of integration there is 3-dimensional.

Remark 12. It is claimed (without proof) in [1, p. 996] that there are two chronometric neutrinos (\( \nu_e \) and \( \nu_\mu \) of conformal dimensions \( 3/2 \) and \( 5/2 \), respectively). Our current finding is different: there is just one neutrino and it is not of a certain conformal dimension (since our \( \nu \) has both the ‘upper’ and ‘lower’ \( \mathbb{C}^2 \)-components, see formula (12) in Section 3). The conclusion (that there are no other neutrinos as factors of our composition series) will follow after we present the model for electron, \( e \), in the next section. The reason why we prefer to interpret our \( \nu \) as \( \nu_e \) will be given in Section 9.

Remark 13. Let us continue an interesting comparison (which we started in our Remark 10) with findings of [19]. The latter publication deals with certain indecomposable representations of the conformal group (that is, of SU(2, 2), essentially). In [19] these representations are called nondecomposable. Neither the notion of a parallelization, nor that of a composition series is used by the authors of [19]. Also, they start with a Lagrangian, right away [19, p. 1985]. However, such a start can end up with a too restrictive conclusion – this follows from a remark due to Segal – see the last paragraph of our Section 9. Contrarily to our finding of Section 6, no neutrino-type particle is mentioned in [19]. This can be attributed to a ‘fine structure’ of the corresponding (three-step) composition series, as we have detected in our Sections 5, 6, 7.
7. The electron’s representation space

There is now just one more step to proceed through: we factor \( W \) by \( F_\nu \) and announce the result to be the (chronometric) electron’s representation space, \( F_e \).

**Remark 14.** It is an easy exercise to verify that the two invariant subspaces of \( F \), defined by (20) and (22), have only 0 as a common vector. It means that the electron space \( F_e \) could have been obtained as \( F \) factored by the direct sum of the two above subspaces. However, the way how we have already introduced \( F_e \) is in a better compliance with the way how Segal handled this indecomposable representation, [1, p. 995].

Notice that (in our setting) electron is of conformal dimension \( 3/2 \) since only the upper \( C^2 \)-component ‘survives’ the above factoring. The electron’s conformal dimension being \( 3/2 \) is also in compliance with what Segal thought (see [1, p. 996]).

We now notice that our quotient representation is equivalent to \( F_p \) because the (2,2)-entry of the matrix \( Q_b \) (see our (18)) is the same as the (3,3)-entry (except for the label on the \( w \) of the matrix \( Q_a \) (see our (17)). Then the map

\[
\Psi[0, w_2, 0; 0, v_2, 0] \to \Psi[0, 0, w_2; 0, 0, v_2]
\]

proves the electron space \( F_e \) to be equivalent to the proton’s \( F_p \).

**Remark 15.** The map \( L \) (which is defined when (26) is extended linearly to finite sums of corresponding expressions) can be seen to be a bijection. This statement is slightly deeper than it might appear since some finite sums may be zero (the system is “over-determined”). However, using the Fourier–Laplace transform it can be proved that \( L \) is injective – and then clearly bijective.

Since the target space already has a (pre-)Hilbert space structure, we demand that \( L \) is an isometry. This fixes the Hilbert space structure on the electron space \( F_e \).

**Remark 16.** In [19, p. 45] it is indicated that the conformal dimension of \( 3/2 \) is the canonical one for an electron. This matches our conclusion. Also, our Section 7 findings explain why the \( 3/2 \)-field of [19] is not associated to any subspace (while the \( 5/2 \)-field is, see Remark 10).

8. The energy positivity for each of the three particles

We will use (see [8]) a certain basis \( L_{ij} \) (with \( L_{ij} = L_{ji} \)) in the Lie algebra \( su(2, 2) \). Here is the commutation table,

\[
[L_{im}, L_{mk}] = -e_m L_{ik},
\]

where \( (e_{-1}, e_0, e_1, e_2, e_3, e_4) \) stands for \((1, 1, -1, -1, -1, -1)\). Table I of [13] provides, in particular, expressions for each \( L_{ij} \) as a vector field on the Minkowski space-time \( M \). In [13] (as well as in [16, p. 2841]),

\[
E_S = iL_{-10}
\]
is chosen as the operator of the Segal (or curved) energy. Notice that Segal used to call it the Einstein energy.

The operator of the usual Poincaré (or flat) energy in our case should be chosen as

$$\tilde{P}_0 = (i/2)(L_{-10} - L_{04}). \quad (27)$$

Remark 17. In [13] and in [16, p. 2841] the flat energy operator is chosen as $P_0 = (i/2)(L_{-10} + L_{04})$. It does not contradict to our choice since the above two publications deal with an upper Poincaré subgroup in SU(2,2) while we have chosen (see (4) in Section 2) the lower one. This explains tilde above $P_0$ in our (27).

Now, what is left is to show energy positivity for just two cases: $F_p$ and $F_\nu$. Our representation (of the group SU(2,2)) was denoted $H$, so let us use $dH$ to denote the corresponding representation of the Lie algebra $\mathfrak{su}(2,2)$.

We need to show that both $dH(E_S)$ and $dH(\tilde{P}_0)$ are nonnegative self-adjoint operators. However, this is well known: see, for example, [16], starting from page 2841. The proofs there are given in terms of the so-called $K$-types, where $K$ is for the maximal compact (7-dimensional) subgroup of SU(2,2).

9. Conclusions and discussion

One of our main findings is that there are three stable chronometric particles of spin $1/2$ (while Segal stated in [1] that there are four such particles). What we consider to be another main conclusion (technically based on our Theorem 1) is that we can state exactly which particles we associate with the three mathematical objects detected (that is, with the three spaces $F_p$, $F_\nu$, $F_e$); those particles being the three most important fermions: the proton, the neutrino, and the electron (see Sections 5, 6, 7). We are unaware of any research papers where in such an explicit form (see formulae (16), (20), (22)) the states (that is, the wave functions) of these particles have been given. One more new result (being the one contradicting to what Segal claimed) is that our neutrino is not of a certain conformal weight. On the other hand, our work has confirmed the Segal’s viewpoint of these three particles’ spaces entering the space $F$ of the induced representation in a way as if forming a single structure (Segal called this structure a fermionic clan). Also, in several cases (throughout that whole topic) rigorous proofs can only be found in our paper (for example, the proof of Theorem 1). In the next paragraph we list other key properties (of both physical and mathematical nature) of the three particles. In this forthcoming paragraph it is more difficult to distinguish from what has been done before us but these properties can either be proven by explicit calculations or through suitable references. As regards the notion of an elementary particle, itself, we follow the approach of [1] (see the section Indecomposable elementary particle associations there) which is a generalization of Wigner’s one (see his seminal work, [11]). We have detected the three factors of the composition series, which ‘give birth’ to the three representation spaces: $F_p$, $F_\nu$, and $F_e$. Each of these three spaces is provided with an invariant (w.r.t. the action of the group $G = SU(2,2)$) unitary structure as
well as (in each of the three cases) energy positivity condition holds. We have interpreted the corresponding particles as the proton, the neutrino, and the electron (see our Sections 5, 6, 7). With the exception of the case of a proton, we have thus followed Segal [1] in such an association procedure. The second author, Levichev (about 10 years ago), suggested to think of a proton (instead of Segal’s exon). This is neither a big deviation from Segal since the latter considered (see [1, p. 994]) an exon to be the main constituent of both a proton and neutron. Clearly, the proton itself, suits even better for such a role!

In accordance with what is usually thought about relativistic analogues of our (chronometric) particles, it turns out that our proton \( p \) is of conformal weight (or conformal dimension) \( 5/2 \), the electron \( e \) is of conformal weight \( 3/2 \), while the neutrino representation space \( (F_\nu) \) is not of a single conformal weight (which is another significant distinction if to compare to Segal’s findings/conjectures in [1]). Of course, each particle is of spin \( 1/2 \). More details on our three particles have been already presented in the corresponding Sections 5, 6, 7, 8). Here it is worth recalling just one more of these particles’ ‘mathematical’ properties – the Gelfand–Kirillov dimension (G–K dim, for short). Following Segal, we call both the proton, and electron field massive since the G-K dim is 4 there. The neutrino field can be called massless since the G-K dim is 3 there (more on the chronometric mass and on the G–K dim was already mentioned in Remark 10).

What goes next (in Remark 18 and on) is more like a discussion. In particular, we contemplate which resolutions of questions/problems within the Standard Model one could move to, if to combine our paper conclusions with the so called Multi-Level Model.

**Remark 18.** In our ‘neutrino’ Section 6 we stayed close (as much as the math allowed) with how Segal (see [1]) has interpreted his mathematical analysis. However, if we interpret the space \( F_\nu \) as the one corresponding to an anti-neutrino then our findings have a prospective to provide an alternative model of a neutron. Namely, rather than to be modeled as \( udd \), one can now try to return to an ‘old view’ that a neutron consists of a proton, of an electron, and of an anti-neutrino. This seems to provide a new mathematical foundation to Barut’s claim [20]. Besides, one can try to apply such a model to other examples of weak decay.

Our conclusions are mathematically based on the specific formula for the action of our representation in the full representation space \( F_\nu \), see Theorem 1. This formula is presented in terms of interesting functions (as elements of \( F \)) which behave, loosely speaking, like coherent states.

These findings may shed some light on the question “Why are there lepton generations at all?”

A possible answer is related to Levichev’s *multi-level model of quarks*, MLM [3–4] which claims that each quark can be viewed as if being a state of a proton. When (according to the MLM) proton (during a nonelastic interaction) is ‘pushed to a deeper level’ (thus becoming a quark of a certain flavor and color), then the two associated leptons (neutrino and electron) also become ‘species of a deeper
generation': proton becomes an $u$-quark, $\nu_e$ turns into $\nu_\mu$, and electron becomes a muon $\mu$, as an example. This example explains, already, why we prefer to interpret 'our' neutrino as the electron neutrino, $\nu_e$ – compare to [1] which claims both $\nu_e$ and $\nu_\mu$ as ingredients of a single composition series.

We thus feel confident that our research will bring more attention and interest to the use of indecomposable representations of the conformal group in particles and interactions theory. This seems to be a challenge, here is what Segal (in [1][p. 995]) said in this regard: “The elementary particles in chronometric theory are closely integrated into coherent entities …”, he calls them clans, “…Scalar, spinor, and vector elementary particles arise as subunits, and the fundamental interaction is between fermion and boson clans as entities, the total interaction Lagrangian being representable as a sum of interactions between individual elementary particles only in the relativistic limit”.

Appendix

We here give some additional details concerning the massive spin 1/2 particles.

Define $C^+$; the “solid forward cone in energy-momentum space” by

$$C^+ = \{(E, p_1, p_2, p_3) \in \mathbb{R}^4 \mid E^2 - p_1^2 - p_2^2 - p_3^2 > 0, \text{ and } E > 0\}.$$  (28)

We set

$$h = \begin{pmatrix} t + x_1 & x_2 + ix_3 \\ x_2 - ix_3 & t - x_1 \end{pmatrix} \text{ and } k = \begin{pmatrix} E + p_1 & p_2 + ip_3 \\ p_2 - ip_3 & E - p_1 \end{pmatrix},$$  (29)

One has (see [15, Proposition 3.1] (set $\alpha = 0$) or [7, p. 336]),

$$K_3(w_1, w_2) = \beta \int_{C^+} k \cdot e^{i\left(tr(w_1 - w_2^*)k\right)} dk,$$  (30)

where $\beta$ is a positive constant.

So, for $v \in \mathbb{C}^2$, and $w$ any $2 \times 2$ matrix with positive imaginary part, the function $h \rightarrow K_3(h, w) \cdot v$ on Minkwki space $M$ can be written as

$$K_3(h, w)v = \beta \int_{C^+} k \cdot e^{i\left(v(h - w^*)k\right)} \cdot v dk.$$  (31)

Define

$$c(k) = \begin{pmatrix} E - p_1 & -p_2 - ip_3 \\ -p_2 + ip_3 & E + p_1 \end{pmatrix},$$

then

$$k \cdot c(k) = E^2 - p_1^2 - p_2^2 - p_3^2,$$

and

$$\text{tr}(hk) = 2(tE + x_1p_1 + x_2p_2 + x_3p_3).$$
So, \( h \to K_3(h,w) \cdot v \) is the Fourier transform (up to a constant) of the function
\[
(E, p_1, p_2, p_3) \to \begin{pmatrix} E + p_1 & p_2 + i p_3 \\ p_2 - i p_3 & E - p_1 \end{pmatrix} e^{i(w(-w^*)k)} \cdot \chi_{C^+} \cdot v
\]
with \( k \) as above. The term \( \chi_{C^+} \) stands for the characteristic function of \( C^+ \) and makes explicit, that the Fourier transform is supported by \( C^+ \).

Let
\[
\mathbb{D} = \begin{pmatrix} \frac{\partial}{\partial t} + \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_3} \\ \frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \end{pmatrix}
\]
and
\[
c(\mathbb{D}) = \begin{pmatrix} \frac{\partial}{\partial t} - \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_3} \\ -\frac{\partial}{\partial x_2} + i \frac{\partial}{\partial x_3} \end{pmatrix}.
\]

Then \( c(\mathbb{D}) \cdot \mathbb{D} = \mathbb{D} \cdot c(\mathbb{D}) = \Box \cdot I_2 \). The Dirac operator is defined as
\[
\Psi = i \cdot \begin{pmatrix} 0 & c(\mathbb{D}) \\ \mathbb{D} & 0 \end{pmatrix}.
\]
The Fourier transform of the Dirac operator is
\[
\begin{pmatrix} c(k) \\ 0 \end{pmatrix}.
\]

Setting \( h = c = 1 \), the Fourier transform of the Dirac equation is equivalent to the following, which should be thought of as an equation for the mass hyperboloid
\[
\det k = m^2, E > 0,
\]
\[
\begin{pmatrix} 0 & c(k) \\ k & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = m \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}
\]
(The Dirac equation in momentum space). (36)

More generally, we let
\[
\text{Dir}(C^+) = \{ \Psi : C^+ \to \mathbb{C}^4 | \begin{pmatrix} 0 & c(k) \\ k & 0 \end{pmatrix} \Psi(k) = (\det k)^{1/2} \Psi(k) \}.
\]
The inner product on \( \text{Dir}(C^+) \) is given by ([12, p. 358–360])
\[
\langle \phi, \psi \rangle_{\text{Dir}(C^+)} = \int_{C^+} \frac{1}{E} \langle \phi(k), \psi(k) \rangle \, dk.
\]
The reproducing kernel Hilbert space \( F_p \) for which \( K_3 \) is the kernel is unitarily equivalent to the Hilbert space \( \mathcal{L} \) where
\[
\mathcal{L} = \text{span}_\mathbb{C}\{ \phi : C^+ \to \mathbb{C}^4 | \int_{C^+} \langle k \cdot \phi(k), \phi(k) \rangle \, dk < \infty \}.
\]
Indeed, the map
\[
\mathcal{F}(\psi)(h) = \sqrt{\beta} \int_{C^+} k \phi(k) e^{i r((hk))} \, dk
\]
is a linear isometry from $L$ to $F_p$. For more details on this, see [15, Proposition 3.2 p. 71].

We now define a linear map $\mathcal{U}$ from $L$ to $\text{Dir}(\mathbb{C})$,

$$\mathcal{U} \equiv \phi \mapsto \Psi = \mathcal{U}(\phi) = \frac{1}{\sqrt{2}} \begin{pmatrix} (\det k)^{1/2} \phi & 0 \\ k \cdot \phi \\ (\det a)^2 (a^*)^{-1} \end{pmatrix} \in \text{Dir}(\mathbb{C})^+.$$

(41)

It is easy to see that $\Psi$ indeed does take values in $\text{Dir}(\mathbb{C})$. It transforms as usual under the Poincaré group. For instance,

$$\forall a \in \text{GL}(2, \mathbb{C}) : (a * \Psi)(k) = \begin{pmatrix} a \cdot (\det a) & 0 \\ 0 & (\det a)^2 (a^*)^{-1} \end{pmatrix} \Psi(a^* k a).$$

(42)

**Proposition 1.** The map $\mathcal{U}$ above is a linear isometry, and hence the map $\mathcal{F}^{-1} \circ \mathcal{U}$ is a linear isometry from $F_p$ to $\text{Dir}(\mathbb{C})^+$.

**Proof:** That $\mathcal{U}$ is an isometry follows easily from the following,

$$k + c(k) = 2 E \cdot I_2, \quad \text{hence}$$

$$k^2 = 2 E k - \det k.$$}

(43)

(44)

The rest has already been established. □

The map $\mathcal{F}^{-1} \circ \mathcal{U}$ is given on specific functions as

$$K_3(h, w) \mapsto \sqrt{\beta e^{-i \text{tr}(kw^*)}} \chi_{C^+} \cdot v \mapsto \frac{\sqrt{\beta}}{\sqrt{2}} \begin{pmatrix} (\det k)^{1/2} \cdot e^{-i \text{tr}(kw^*)} & \chi_{C^+} \cdot v \\ k \cdot e^{-i \text{tr}(kw^*)} & \chi_{C^+} \cdot v \end{pmatrix}.$$

(45)

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