Hjelmslev’s Geometry of Reality.

Abstract.

During the first half of the 20th century the Danish geometer Johannes Hjelmslev developed what he called a geometry of reality. It was presented as an alternative to the idealized Euclidean paradigm that had recently been completed by Hilbert. Hjelmslev argued that his geometry of reality was superior to the Euclidean geometry both didactically, scientifically and in practice: Didactically, because it was closer to experience and intuition, in practice because it was in accordance with the real geometrical drawing practice of the engineer, and scientifically because it was based on a smaller axiomatic basis than Hilbertian Euclidean geometry but still included the important theorems of ordinary geometry. In this paper, I shall primarily analyze the scientific aspect of Hjelmslev’s new approach to geometry that gave rise to the so-called Hjelmslev (incidence) geometry or ring geometry.


Keywords: Johannes Hjelmslev, Geometry of reality, Didactics, Axiomatization, Descriptive Geometry, Geometric constructions.

Classification codes: 01A60, 51-03, 51CXX, 00A30

Johannes Hjelmslev (born Petersen) (1873-1950)

When the principal character of this paper was born in 1873 in the district (herred) Hjelmslev in Jutland, Denmark, he was baptized Johannes Trolle Petersen. After completing high school in 1890 he moved to Copenhagen to study mathematics at the university. When he graduated in 1894, he began making a living as a teacher in a high school and as a tutor at the university. Parallel with his teaching duties he worked on a doctoral dissertation on infinitesimal methods in descriptive geometry that he defended in 1897. He also began publishing papers on various subjects in geometry, most of them in Danish but from 1898 also in French and German. One of these [Petersen/Hjelmslev 1898] contains one of his most famous results, the so-called Petersen-Morley theorem concerning three skew lines in space. According to Wikipedia [Wikipedia 2020] “the theorem is named after Julius Petersen and Frank Morley”. This incorrect attribution of the theorem to Johannes Petersen’s professor Julius Petersen (1839-1910) is widespread. Already the young Johannes Petersen realized that his papers published under the name J. Petersen risked being attributed to his more famous teacher and so in 1904 he decided to change his last name to Hjelmslev after his birthplace.
In 1903 Petersen (Hjelmslev) was appointed docent of descriptive geometry at the Polytechnic College in Copenhagen. He was promoted to professor in 1905. In 1917 he obtained the more prestigious chair of mathematics at the University of Copenhagen where he remained until he retired in 1942. He was the leading Danish mathematician in the generation between Hieronymus Georg Zeuthen (1839-1920) and Harald Bohr (1887-1951).

The context of Hjelmslev's geometry of reality

a. Inspiration from his peers

Hjelmslev published his first paper on his geometry of reality in 1913. However, one can trace its roots back to some of his earlier experiences. As a student, Hjelmslev followed courses with the above-mentioned Julius Petersen as well as Thorvald Nicolai Thiele (1838-1910) and Zeuthen. The latter in particular had a decisive influence on Hjelmslev, kindling his interests in both geometry and history of mathematics. Also Julius Petersen's widely published approach to geometrical constructions became important for Hjelmslev as a contrast to his own experimental approach to the subject. But among the senior Danish mathematicians, Christian Sophus Juel (1855-1935), 18 years his senior, had the most direct influence on Hjelmslev's realist views on geometry. An early student of Zeuthen, Juel developed a synthetic theory of curves and surfaces in projective space, inspired by the ideas of Karl Georg Christian von Staudt (1798-1867). Hjelmslev's early works on differential geometry of curves and surfaces (e.g. [Hjelmslev 1911]) borrowed many ideas from Juel's papers, in particular his realist concepts of curves, surfaces and tangents. In Juel's earliest Introduction to the theory of graphical curves [Juel 1899] the real and material point of view is particularly evident. For example, Juel defined a point to be “a material point” typically the dot of a pencil [Lützen 2020b]. In an obituary of Juel, Hjelmslev approvingly characterized Juel’s approach to the theory of curves as an “immediate description of reality” [Hjelmslev 1935, 8].

b. Non-Danish precursors

The relation between theoretical geometry and physical reality has been a key philosophical question since the times of the ancient Greeks. From his studies of Greek mathematics, Hjelmslev was aware of early oppositions to Plato’s and Euclid’s idealizing view of geometry. In particular, he called attention to Protagoras’ observation that a real circle has a whole line segment rather than a point in common with its tangent [Hjelmslev 1923, 1]. In the 18th century, Kant famously claimed that (Euclidean) geometry was an a priori and synthetic condition for our experience of the world. In explicit opposition to this dogma, many 19th century geometers advanced an empirical view of geometry. In connection with the debate on the status of the parallel postulate and the nature of space at large (astronomically), Gauss and Lobachevsky

---

1 For obituaries of Hjelmslev see [Bohr 1950], [Fog 1950], [Jessen 1950] and [Nielsen 1950]. Bohr’s is in German the others are in Danish.
2 In Danish Hjelmslev mostly called his new approach "virkelighedsgeometri" but he also used words like "praktisk geometri" or "naturlig geometri". In German he used the terms “Die Natürliche Geometrie” or “Geometrie der Wirklichkeit” and in French “la géométrie sensible”. In this paper, I shall mostly use the English expression “geometry of reality”. He did not distinguish between the different terms.
3 As a historian of mathematics, Hjelmslev continued in the footsteps of Zeuthen with contributions to the history of Greek mathematics. He also edited a republication of Georg Mohr’s until then forgotten Euclides Danicus [Mohr 1928] and made investigations of the life of its author [Hjelmslev 1931].
4 See [Petersen 1866/79]. This book was published in many later editions and was translated into many languages. It is the most published book by a Danish mathematician (see [Lützen, Sabidussi, and Toft 1992]).
emphasized an empirical point of view in the first half of the century. Toward the middle of the century, Riemann had extended the empirical critique to a broader physical context. Hjelmslev himself mentioned the empirical leanings of the later geometers Hermann von Helmholtz (1821-1894), Moritz Pasch (1843-1930), Felix Klein (1849-1925) and Albert Einstein (1879-1955) as well as the conventionalist philosophy of Henri Poincaré (1854-1912) [Hjelmslev 1923, 2-3].

Of these geometers, Pasch and Klein were the two whose aims were closest to Hjelmslev’s (see also Hjelmslev, 1916e, 36). However, Hjelmslev could not endorse the result of Pasch’s investigations:

Pasch has set himself the task to give a basis for projective geometry in close agreement with experience. And in his Vorlesungen über neure Geometrie (1882) he has given valuable contributions to the solution of this task. However, he has formulated an axiom system according to which two straight lines without exception have at most one point in common. Also here one can therefore say that the system considered as an empirical system has beforehand been refuted by Protagoras [Hjelmslev 1923, 2].

Hjelmslev considered Klein’s view on intuitive geometry as closer to his own:

Klein has also several times emphasized the contrast between theoretical geometry and experience. He has made interesting comments on the terms “approximation mathematics” and “precision mathematics” and discusses the difference between the real and the abstract shapes... [Hjelmslev 1923, 3]

Klein already touched on the question of empirical geometry in his Evanston Lectures [Klein 1894, 31] where he described the “naive intuition” of a line as a strip of a certain width. To the exact mathematician who might object that this is not a definition at all, Klein maintained that “in ordinary life we actually operate with such inexact definitions” [Klein 1894, 31]. It is possible that Hjelmslev read this famous book when it was published and it is certain that he read Klein’s autographed lecture notes entitled Anwendung der Differential- und Integralrechnung auf Geometrie (1902 and 1907) published in book form as the third volume of his Elementarmathematik vom höheren Standpunkte aus with the more appropriate title: Präzisions- und Approximationsmathematik [Klein 1928]. Hjelmslev’s use of the phrase “approximation mathematics and precision mathematics” is a clear reference to this work.6

As indicated in the original title, Klein’s book mostly dealt with analysis but he also briefly discussed the meaning of the basic objects in what he called practical geometry:

A point in this discipline is a body that has such a small extension that we can ignore it.

A curve, a straight line in particular, is a sort of strip whose width is insignificant relative to its length.

Two points determine a straight line the more accurately the further they are distanced from each other. If they are close together, they are very unsuited to determine the straight line.

---

5 This printed edition was edited by Fr. Seyfarth after Klein’s death. According to the preface Klein had agreed that the new title was more appropriate than the earlier, and he had during the last two months of his life been involved in discussions about the new edition.

6 Even if these words do not appear in the original title of the autographed lecture notes, they appear prominently in the preface and in the rest of the book.
Two straight lines determine an intersection point the more exactly the less the angle between them differs from a right angle. When the angle between them becomes smaller and smaller, the intersection point becomes more and more undetermined [Klein 1928, 6].

As we shall see below, Hjelslev completely agreed with these principles of practical geometry and might very well have been inspired by Klein’s suggestive presentation. However, as Hjelslev remarked, Klein “does not enter into a systematic treatment of the question” [Hjelslev 1923, 3]; rather, Klein immediately jumped to function theory and other parts of higher mathematics, leaving elementary geometry behind.

c. Hjelslev’s research on Hilbertian foundations of geometry

From his earliest publications, Hjelslev displayed an interest in the foundations of geometry and so when the book Grundlagen der Geometrie by David Hilbert (1862-1943) [Hilbert 1899] was published, he enthusiastically began contributing to the geometric program outlined in this book. In particular, in [Hjelslev 1907], following a hint by Gerhard Hessenberg (1874-1925), he succeeded in proving Pascal’s theorem using only the plane axioms without the use of the parallel axiom (or its negation) or the continuity axioms. He even conjectured that the order axioms are not needed either, a conjecture he proved in 1929 [Hjelslev 1929-49, 2. Mitteilung 1929]. These clarifications of the Hilbertian foundations of geometry became famous through Hilbert’s footsteps in later editions of his Grundlagen (e.g. [Hilbert 1930, 54]). Hjelslev’s intricate axiomatic investigations prepared the way for his later geometry of reality in two ways: First, they emphasized the possibility of studying many different geometries based on different foundations. In particular, he was alerted to the idea that one should strive for a minimal axiom system that would allow a deduction of the most important theorems of geometry, in particular Pascal’s theorem. Second, Hjelslev’s proof of Pascal’s theorem was based on an ingenious theory of transformations, in particular reflections\(^7\) and so-called half-rotations\(^8\) by way of which the entire theory could be limited to a finite part of the Euclidean plane. This also became a characteristic of his geometry of reality.

d. Descriptive geometry

However, the more precise elaboration of Hjelslev’s geometry of reality owed more to his teaching of descriptive geometry to the polytechnic students in the period 1903-1917. This subject was the theoretical foundation for the practical drawing skills that was so important for engineers at the time. According to his predecessor as professor of descriptive geometry at the Polytechnic College, Carl Julius Ludvig Seidelin (1833-1909) “descriptive geometry teaches how to depict spatial quantities exactly…. It demonstrates the reliable way to construct exactly in space” [Seidelin 1895-96, 1]. However, Seidelin and his predecessor Ludvig Stephan Kellner (1796-1883) had not really addressed the problem of accuracy in their textbooks. The question must of course have been taken up during the practical drawing classes but it was handled by subsidiary rules of thumb that were added on top of an Idealized Euclidean theoretical presentation. Some of these rules were mentioned in other texts on descriptive geometry. For example, already the inventor of descriptive geometry, Gaspard Monge (1746-1818), called attention to the problem, that the point of

\(^7\) Hjelslev developed his theory of reflections further in [Hjelslev 1929-49]. After Hjelslev’s death Bachmann used it as a foundational tool [Bachmann 1959]. A posthumous book by that author even carries Hjelslev’s name on the title page: Ebene Spiegelungsgeometrie, eine Vorlesung über Hjelslev-Gruppen [Bachmann 1989]. See [Benz 1990] for more information about the history of reflection-geometry.

\(^8\) A half-rotation (Halbdrehung) is not a rotation of 90° or 180°. Given a pole \(O\) and an angle \(\nu\) the half-rotation around \(O\) maps a point \(P\) into the midpoint of the line segment joining \(P\) and its image \(P'\) under a rotation of an angle \(\nu\) around \(O\). Half-rotations are important because they can be made to map imaginary points in the plane (in the Beltrami-Klein model of hyperbolic geometry they are the points outside the limiting circle) into real points (the points inside the circle).
intersection between two almost parallel lines is poorly determined [Monge 1811, p. 11 (§8)]. This problem, also raised by Klein, was mentioned by other descriptive geometers but it was treated as a problem resulting from the inaccuracy of practical geometry. In principle, the intersection point was assumed to be determined uniquely, but since our drawing tools and skills are imperfect, real geometry only renders an imperfect version of the Euclidean ideal. This Platonic view of geometry was challenged by Hjelmslev who maintained that it was the Euclidean ideal that was an inaccurate model of real geometry rather than the other way round. According to Hjelmslev, the problems of accuracy ought to be reflected in the foundation of geometry rather than being added to it as rules of thumb. The best foundation of a geometry of reality should reflect the way an engineer made his most accurate constructions.

Hjelmslev wrote two textbooks on descriptive geometry [Hjelmslev 1904 and 1918]. They became the last Danish textbooks on a subject that gradually faded away after Hjelmslev left the Polytechnic College in 1917. The first of the books was innovative in its emphasis on projective geometric methods. But it did not challenge the Euclidean ideal. The second book [Hjelmslev 1918] on the other hand, was based on his new ideas on geometry of reality. Published one year after Hjelmslev had left the professorship of descriptive geometry, this book was used by his friend and successor Tommy Bonnesen⁹ (1873-1935) who became the last professor of descriptive geometry in Denmark [Lützen 2019].

_е. Didactics of school geometry_

In the period between the publication of Hjelmslev’s two books on descriptive geometry he was involved in a didactic debate that led him to publish his first controversial ideas on a realist foundation of elementary geometry. This extensive didactic dispute on the best way to teach geometry to students in Danish primary and secondary schools has been studied by H.C. Hansen [Hansen 2002, 106-120]. The two main positions were the Euclidean axiomatic position and a more experimental, material and intuitive position. Several first rank mathematicians opted for the latter alternative. In addition to Hjelmslev, one can mention Poul Heegaard¹⁰ (1871-1940) and Tommy Bonnesen. For example, in 1900 Heegaard published _Rum-anskuelse, Forberedende øvelser (Space intuition, Preliminary exercises)_ [Heegaard 1900] in which he gave a craftsmanlike introduction to the concepts of plane and line similar to the definitions given by Hjelmslev in his later textbooks. Bonnesen wrote a textbook with an experimental approach in 1904 [Bonnesen 1904] and a programmatic paper _Geometrisk-pædagogiske betragtninger_ in 1906 [Bonnesen 1906]. In this paper, he rejected the traditional focus on axioms and deductions of theorems that the student had to learn by heart. Instead, he opted for geometric experiments and suggestive figures that the student could see with his inner eye. “The beginner believes the intuition, does not understand the mathematics but is persuaded and understands by way of experience” [Bonnesen 1906, 3]. In many ways, Hjelmslev and Bonnesen shared the same views on the teaching of geometry. This is also evident from the handwritten note that Hjelmslev wrote on the first page of a copy of his first schoolbook [Hjelmslev 1916a] when he sent it to Bonnesen:¹¹

Dear Dr. Bonnesen. While I inscribe your name in this book, I ask you also here to receive my best thanks for the interest you have in advance shown in this anti-Euclidean enterprise.¹²

According to Hjelmslev, elementary textbooks for schools should “not deal with abstractions but with things that belong to the practice of life” [Hjelmslev 1913a, 50]. In the same vein he was sceptical about the

---

⁹ Now, Bonnesen is primarily famous for the first book on convexity theory that he coauthored with Werner Fenchel.

¹⁰ Now Heegaard is primarily famous for his contributions to topology.

¹¹ This copy of the book is now in the possession of the author of this paper.

¹² “Kære Dr. Bonnesen! Idet jeg indskriver Deres Navn i denne Bog, beder jeg Dem også her modtage min bedste Tak for den Interesse, De paa Forhaand har vist dette anti-Euklidiske Foretagende.”
pedantic deductions in more traditional textbooks “Why do we need all these proofs of things that are often more evident than the axioms that the proofs are built on” [Hjelmslev 1913a, 50].

Hjelmslev engaged himself so seriously in elementary geometry teaching that he wrote a whole series of textbooks for the middle and higher schools in Denmark. The first volume appeared in 1916 and the same year Hjelmslev began teaching mathematics at the State’s Teacher Training College probably trying in this way to spread the gospel of his new geometry of reality to the future schoolteachers. He continued this teaching 10 hours per week for the next ten years, on top of his teaching at the University [Nielsen 1950, 7]. In the Hjelmslev Nachlass in the Archive of the Department of Mathematical Sciences of the University of Copenhagen there is a 153 page handwritten manuscript with the title: Grundlag for Undervisningen i Geometri af Johannes Hjelmslev (Foundation for the teaching of Geometry by Johannes Hjelmslev) [Hjelmslev MS 1916?]. This could be the notes for Hjelmslev’s lectures at the Teacher Training College. The manuscript is not dated, but since he began his teaching in 1916 and since some of the formulations are close to his formulations in his first textbook [Hjelmslev 1916a] and in [Hjelmslev 1916b] it is reasonable to assume that the manuscript was begun that year. The manuscript gives Hjelmslev’s most extensive introduction to his geometry of reality and in particular to its use in school teaching. In this manuscript he also referred back to 18th century reformers of the teaching of geometry, in particular Rousseau and Clairaut and declared confidently: “Euclid’s system belongs to history and the teaching of geometry will - perhaps slowly but quite certainly - gradually develop into different forms” [Hjelmslev MS 1916?, 5].

The empirical foundation of Hjelmslev’s geometry of reality

In order to clarify what Hjelmslev meant when he declared that school geometry should deal with things that belong to the practice of life I shall now explain how he introduced the basic objects of geometry. In general he required that

the definition of the basic concepts: plane, straight line, right angle etc. must be formulated in such a way that they specify a procedure for the manufacture of the thing, a procedure that allows us to check if the thing has the specified property. [Hjelmslev 1923, 3].

To be more specific let us have a look at his definitions of the basic objects in his first elementary textbook [Hjelmslev 1916a]. Having called the students’ attention to their everyday experiences with various box shaped objects such as cigar boxes and bricks, he turned to the most accurate way to manufacture such objects beginning with planes. Accurate planes are produced industrially by beginning with three almost plane iron plates. One then applies two of them on top of each other with some specific dye in between. When they are separated from each other, the dye indicates where the plates have bulges. The bulges are then scraped away, and one continues the process with another pair of plates. Continuing this way one will in the end arrive at three plates, that fits exactly two and two on top of each other. The resulting surfaces (industrially called straightening planes) are (by definition) exact planes after which one can then model other cruder planes such as paper on which one makes geometric constructions.

13 The holdings of the Hjelmslev Nachlass can be seen here: http://web.math.ku.dk/arkivet/jhjelmsl/hjelmslv.htm
Given this definition of a plane, Hjelmslev defined a wedge as a body having two plane sides meeting along an edge. Such an edge is called a straight line. Two wedges are called neighbor wedges if they can be placed in such a way that they each have one of their planes lying on one and the same plane and such that their two other planes fit tightly together (Figure 1, Fig. 3). If three wedges are manufactured in such a way that they are two and two neighbor wedges to each other, the wedges are called normal wedges and the planes of the wedges are said to be orthogonal to each other (Figure 1, Fig. 4). A normal corner is a corner where three planes meet each other orthogonally. The lines of intersection of the planes are said to be orthogonal or to make a right angle with each other.

Finally, a normal block is a block consisting of six exact planes that meet at right angles and having eight normal corners. They are exact versions of the inexact boxes mentioned above. According to Hjelmslev, it is possible to manufacture normal blocks so accurately that if a number of congruent blocks are piled up to form a larger normal block they fit so tightly together, that it requires great force to separate them from each other.

Hjelmslev encouraged the schoolteachers to let the pupils produce wedges and blocks out of either wood or clay, and apparently, he had 10 such wooden models manufactured for his own teaching [Lützen 2020a].

This brief introduction will suffice to illustrate Hjelmslev’s materialist or even industrialist definition of the basic objects of geometry. Note that according to Hjelmslev, the industrially produced planes are not
approximations to exact mathematical planes, they are exact mathematical planes, and similarly for the other objects defined above. The last mentioned experience with congruent blocks exactly filling up a portion of space is an example of how Hjelmslev inferred geometric properties from experience. If the blocks are chosen such that they have all edges equal, one can conclude that it is possible to divide space by a cubic grid and to impose a square grid on a plane. As we will see later, the existence of such a square grid is fundamental to Hjelmslev not just as an important means for geometric constructions and measurements but also because it replaces the parallel postulate in his geometry of reality.

Hjelmslev often used the word exact in connection with his geometry of reality. He knew full well that it would provoke traditional geometers for whom the drawings of the geometrical figures were inexact representations of Platonic idealized figures. Even if the drawing paper was only an approximation of the straightening planes and the straightedge was probably a somewhat imprecise rendering of the industrial exact straight line, Hjelmslev usually maintained that the figures drawn by these means were the exact figures that his geometry of reality dealt with. According to Hjelmslev, the inexactness only appears when we begin to measure (fix) the magnitudes of the drawn figures (see below).

**Geometric constructions or experiments**

The same year Hjelmslev published his first programmatic paper on his new geometry of reality he published a textbook entitled *Geometric Experiments*. According to the preface

This book deals with geometric constructions in the widest sense. It does not deal with approximations but with exact constructions. At the same time, it deals with the practical execution seeking to form the construction with a view to the simplest realization. And through a natural union of these purposes it tries to prepare the way for a wider view of this area than has prevailed since Plato’s time [Hjelmslev 1913b, preface].

The prevailing paradigm that he referred to was the Euclidean insistence on constructions using ruler and compass. Hjelmslev admitted that the question of constructability by ruler and compass was of great theoretical importance [Hjelmslev, 1913a, 57], but for the practical execution of geometrical constructions, he considered insistence on Euclidean means to be an unnecessary and harmful limitation.

The question whether a problem can be solved by ruler and compass has little interest in practice; what matters is that one finds a good useful result and not whether it has been found by this or that method [Hjelmslev 1913a, 57].

He pointed out that even when a construction by ruler and compass was possible it could be unnecessarily complicated and result in great inaccuracy. To replace the traditional ruler and compass constructions his book presented a “systematic introduction of the geometric experiment” [Hjelmslev 1913a, 57]. It taught its readers how to make “exact constructions” by using the ruler and compass in ways that are not covered by Euclid’s construction postulates or by using entirely different instruments such as geometric triangles or square grids (squared paper). Even for a simple task like determining the midpoint of a line segment, Hjelmslev recommended to try with a particular opening of the compass and adjust it until it fits twice on the segment. A similar method could be used to solve the trisection of the angle, a problem that was known to be unsolvable by ruler and compass [Hjelmslev 1913a, 57]. In general Hjelmslev thought it was more important to give rules for checking if a construction was accurate than to give rules for making the construction.
Hjelmslev’s experimental approach to geometric constructions was a controversial proposal in Denmark where Julius Petersen’s *Methods and theories for the solution of Geometric Construction Problems* [Petersen 1866/79] had raised the method of constructions with ruler and compass to a highly developed art form (see [Lützen et al. 1992]).

**The relation between Geometry of reality and Euclidean theoretical geometry: Fixing**

Hjelmslev began his first programmatic paper on the geometry of reality [Hjelmslev 1913a] with a historical survey of the early development of geometry. In particular, he emphasized that Euclid had partially idealized and axiomatized the previous Egyptian material geometry. In doing so, he had gone beyond experience by introducing ideal points without extension, and lines and planes without width. Euclid and the textbook authors who followed him still relied partially on intuition and experience in their deductions; but at the turn of the 20th century

the theoretical geometry has succeeded in freeing itself from the empirical origin such that the concepts it works with need no connection whatsoever with the original real conceptions that gave rise to the erection of the entire theoretical structure. ... In this way all uncertainty has disappeared at least within theoretical geometry [Hjelmslev 1913a, 44].

Here Hjelmslev clearly referred to Hilbert’s *Grundlagen der Geometrie* [Hilbert 1899]. As we saw above, Hjelmslev had himself made important contributions to Hilbert’s geometric program and even after he began developing his geometry of reality, he declared: “It is not my intention to kill the theoretical geometry” [Hjelmslev, 1916b, 16]. However, he wanted to clarify the relation between geometry of real objects and theoretical geometry. The traditional view of the matter was to consider practical geometry as an approximate and inexact version of the Euclidean ideal. Even Klein’s distinction between approximation and precision mathematics supports such a view of the matter. For Hjelmslev it was rather the other way round.14 The traditional mathematician

even goes so far that if one discovers a small deviation from theory, then they put the blame on the experience... With other words one considers the theorems of theoretical geometry as eternal unchangeable truths that must even pose criteria for the greater or lesser accuracy of observations. What right does one have to do that? [Hjelmslev 1916, 6].

While Hilbert had completely axiomatized geometry,

the question concerning its relation to reality has at the same time been entirely left to the side; it has been rejected by pure mathematics as something that does not concern it.... Theoretical geometry works with things that go beyond experience, with mathematical points, lines and planes; the practical geometry must work with things that really exist, real points, lines and planes. The two areas are related, but they are not identical. Some may say that “they are approximately identical, since they follow approximately the same laws”; but as long as one has not accounted for the way in which this approximation holds and how close the approximation is, this remark is only a hypothesis, and it is a hypothesis that cannot hold true in its full extent [Hjelmslev 1913a, 44].

---

14 It is ironic that Jacob Nielsen promoted the traditional view about the relation between Euclidean geometry and Hjelmslev’s geometry of reality in his obituary of Hjelmslev where he wrote: ”...the concepts of the formal geometry are simpler to operate with than the approximations of reality” [Nielsen 1950, 6] (my emphasis).
The relationship between practical geometry and theoretical geometry was not just a philosophical question for Hjelmslev. Indeed, for practical reasons he needed the possibility of making calculations and using the ordinary analytic geometry of $\mathbb{R}^3$ or $\mathbb{R}^2$. Following Hilbert he considered analytical geometry to be a model of theoretical Euclidean geometry. In fact, he often called theoretical geometry the “arithmetic theory”.

In order to account for the precise connection between geometry of reality and the theoretical (arithmetic) theory he used an operation he called fixing (fiksering) [Hjelmslev 1917]. Given a geometric line segment, it is possible to measure its length approximately by applying a sufficiently fine square grid to it. The resulting real (actually rational) number is said to fix the length. In practical geometry, there is a lower bound for the accuracy $\varepsilon$ with which one can fix the length of a line segment. In his textbook on projective geometry, Hjelmslev took this accuracy to be 1/25 mm. This corresponds to the width of a thin line in practical geometry drawn with a very fine pen. In this way, one can fix the coordinates of a geometric point and thus associate an arithmetic point in $\mathbb{R}^2$ to it. In a similar way, one can fix angles. However, this mapping from the real geometric plane (or space) into the arithmetic plane (or space) is not unique. Any number within an interval of length $\varepsilon$ will do as a fixation of a given length.

Conversely, there is an “inverse” mapping that maps the arithmetic plane into the geometric plane at least if one restricts oneself to the part of the arithmetic plane corresponding to the finite limitations of the geometric drawing plane. But this mapping maps many arithmetic points into the same geometric point. If for example one can draw with the accuracy of $\varepsilon=1/25$ mm in the real drawing plane then a line of this length in the arithmetic plane will be mapped into a real geometric point.

However, Hjelmslev maintained “every really existing figure can be fixed in such a way that the theorems of [theoretical] geometry are exactly valid for the fixed figure” [Hjelmslev 1918, 10]. He never presented a proof of this general claim, but argued for it in the case of Pythagoras’s theorem. His argument was based on an analysis of the measuring process [Hjelmslev 1917, 1-4] and the empirically verified existence of a square grid. He first deduced that if a straight line is intersected by a family of equidistant parallels, the intersection points will be equidistant (or rather, they can be fixed in such a way that they are equidistant). This implies that the sides of similar triangles can be fixed in such a way that they are proportional.

![Figure 2. A right angled triangle](image)

He then considered a right-angled triangle ABC (Figure 2) with sides $a$ and $c$. The height cuts off a new triangle with sides $\alpha$ and $\alpha$, which is similar to the original triangle. Thus the sides of the original triangle

---

15 Hjelmslev did not use the word model. In [Hjelmslev MS 1916?] he calls analytic geometry “the full equivalent for the [Hilbert’s] axiom system”.
can be fixed by numbers $a_1$ and $c$ and the sides of the second triangle can be fixed by the numbers $a_2$ and $\alpha$ in such a way that

$$\frac{c}{a_1} = \frac{a_2}{\alpha}$$

But when it [the side $a$] can be fixed by both $a_1$ and $a_2$, it can also be fixed by $a = \sqrt{a_1 a_2}$ which lies between $a_1$ and $a_2$. Then we have $\alpha^2 = ac$, i.e. the side [$a$] can be fixed by a number which is the mean proportional between $\alpha$ and $c$ [Hjelmslev 1917, 7].

From this theorem Hjelmslev deduced in the usual manner that the sides in a right angled triangle can be fixed by numbers $a, b, c$ such that $a^2 + b^2 = c^2$.

Hjelmslev warned his reader against a false interpretation of the theorem. Indeed one might think that if the hypotenuse and one of the other sides of a geometrically given right angled triangle can be fixed by the numbers $c$ and $a$ respectively the last side can be fixed by the number $\sqrt{c^2 - a^2}$. However, that is not true in general. For example if $a = 10$ cm and $c = 10,005$ cm, it is not certain that $\sqrt{c^2 - a^2}$ is a fixing of the last side. As usual, the problem arises because of a badly determined construction. The correct formulation of the theorem goes as follows:

A right angled triangle whose hypotenuse is fixed by the number $c$ and one of its other sides by $a$, can be determined [drawn] such that the third side can be fixed by $b = \sqrt{c^2 - a^2}$. However, it might also be determined such that the third side cannot be fixed by the mentioned number [Hjelmslev 1917, 8].

In order to fix lengths and in particular coordinates, Hjelmslev in fact only needed the rational numbers. In his unpublished lectures [Hjelmslev MS 1916?, first pagination 35-38] he used this idea to revisit the usual story of the discovery of incommensurable geometric quantities. According to Hjelmslev, the usual argument by contradiction does not prove that the side and the diagonal in a square are incommensurable. The contradiction rests on the “correlation between geometry and arithmetic that requires that every geometric quantity corresponds to one determined number. It is here that a false assumption has been introduced” [Hjelmslev MS 1916?, 36]. If one allows more than one fixing of the involved line segments, the contradiction disappears. Indeed if we consider the square with side 1 (Figure 3) the two similar triangles ABC and ACD show that one can fix the diagonal by two numbers $a_1$ and $a_2$ such that

\[16\] In his unpublished manuscript the more detailed proof of Pythagoras’ theorem takes up 11 pages [Hjelmslev MS 1916?, second pagination 49-60].
\[
\frac{a_1}{2} = \frac{1}{a_2} \quad \text{or} \quad a_1 \cdot a_2 = 2.
\]

Of course, it is possible to choose two arbitrarily close rational numbers \(a_1\) and \(a_2\) satisfying this condition. That one introduces the number \(\sqrt{2}\) as a common fixing is then a convenience that Hjelmslev allowed himself.\(^{17}\) “But naturally this does not prove that in geometry there exist incommensurable quantities. And naturally one cannot prove anything like that” [Hjelmslev MS 1916?, 38].

As a consequence of the theory of fixing, two lines in geometry of reality that intersect in a small angle have a line segment in common. If for example the arithmetic lines with equations \(y=0\) and \(y=0.01x\) are mapped into the real geometric plane they will have a line segment of length 8 mm in common if the accuracy (their width) is \(\varepsilon = 1/25\) mm [Hjelmslev 1918, 11-12]. Indeed, when \(x = \pm 4\) mm the arithmetic distance between the two lines is 0.04 mm = 1/25 mm, and so in the interval \([-4, 4]\) along the x-axis the two geometric lines will overlap. This means that both the mentioned lines can be considered as a line through two geometric points lying in this 8 mm intersection of the two lines.

Having thus established the correspondence between the geometry of reality and arithmetic geometry, one can use the latter to make calculations and to introduce ideal elements in the practical geometry. For example if two lines have an arithmetic intersection point that lies outside the drawing plane, one can consider this point as an ideal point in the practical geometry. This is similar to the assignment of an ideal intersecting point to parallel lines in projective geometry and to non-intersecting lines in the Beltrami-Klein model of non-Euclidean geometry. However, one cannot use such points in real constructions in the drawing plane. For example, if one wants to draw a line through a given geometric point and through an ideal intersection point of two given lines, one has to design a construction that can be performed in the

\(^{17}\) Hjelmslev devoted a paper [Hjelmslev 1916d] to the “natural foundation for the theory of real numbers”.  

Figure 3. Hjelmslev’s argument concerning the side and diagonal in a square. 

[Hjelmslev MS 1916?, 37]
finite drawing plane. Hjelmslev’s solution to this problem [Hjelmslev 1917, 17] followed the ideas he had used in his 1907 proof of Pascal’s theorem. In both cases, he had to operate with ideal points.

(Rejected) Axioms

As we saw above, Hjelmslev in his schoolbooks did not deduce his theorems from axioms. He preferred to build on experience and intuition. However, true to his dictum “one shall not begin with abstraction but end with it” [Hjelmslev 1913a, 50], in his scientific papers, he recognized the need for clearing up the axiomatic foundations of his new geometry.

For the time being our problem is principally the same as in the old geometry. We set up a system of axioms. However, we require that this system of requirements satisfies that the axioms can be tested and confirmed through exact investigations (experiments and sensations)... All axioms must express empirical truths, or at least no axiom can be an empirically false statement [Hjelmslev 1923, 3-4].

As we saw above, the objects of Hjelmslev’s geometry of reality were materially defined real objects. The axioms of the geometry of reality should thus be true of these objects. For that reason, the axioms of the geometry of reality need to be different from the axioms of Euclidean (Hilbertian) geometry several of which were in conflict with experience. In his earliest papers about the geometry of reality, Hjelmslev dealt with these axioms or basic properties in a rather informal way, primarily pointing out where they differed from the Euclidean or Hilbertian axioms. However, from 1923 he began to formulate more complete lists of axioms. Let us begin with the Euclidean axioms that Hjelmslev rejected:

a. Infinite extent and parallelism

In the geometry of reality, all planes and lines must have a finite extent. Indeed the industrial procedures for manufacturing planes, lines etc. mentioned above only produce finite planes and lines. Moreover, for the practical geometer the plane is limited to the finite drawing plane. Thus, axioms like Euclid’s postulate 2 or Hilbert’s postulate III1 requiring the indefinite extendibility of straight lines must be rejected in the geometry of reality.

The finiteness of the plane also implies that Euclid’s definition of parallel lines, as lines that do not intersect however far they are prolonged, must be replaced by a local definition. Hjelmslev defined parallel lines as lines that have a common normal, or lines that are opposite sides in a rectangle.

To be sure, we do not know that two lines that are orthogonal to the same third line never intersect each other but we do not care, in particular since this theorem from our point of view makes no sense [Hjelmslev 1913a, 55].

Moreover, the parallel axiom had to be rejected in the geometry of reality because it appealed to the infinite extension of the plane. Hjelmslev replaced it by the local axiom that there exist rectangles, or alternatively that there exists a square grid over the plane (squared paper). In our analysis of the normal blocks, we have already seen how Hjelmslev argued for the empirical justification of this axiom. From the existence of squared paper, Hjelmslev deduced that the angle sum in a triangle is two right angles.

b. The uniqueness of points of intersection
The axiom that Hjelmslev objected to in the most vigorous terms was the uniqueness axiom stating that the line through two given points is unique. This axiom had been used by Euclid and had been formulated explicitly by Hilbert as his second axiom of incidence:

We have not formulated the requirement that 2 arbitrary points determine a straight line. Indeed, this requirement is in its extreme consequences one of the worst assumptions one has ever introduced in geometry since it is the one that can give rise to the greatest errors [Hjelmslev 1913a, 5].

Hjelmslev did assume that the line segment between two points was uniquely determined, but when the points were too close together the extension of this line segment was not necessarily unique.

As a consequence of the uniqueness axiom, Euclid and his modern followers pretend that two lines intersect in a single point. However, if the angle between the lines is very small this point is badly determined because a real line cannot be without thickness as Euclid defined it to be. As Hjelmslev and other descriptive geometers before him had noticed that means that if one makes geometric constructions without paying attention to these inaccuracies the result may end up being entirely wrong. Thus by pretending absolute exactness Euclidean constructions can lead to unverifiable inaccuracies.18 This observation also added to his rejection of the Euclidean requirement of ruler and compass constructions.

c. *Continuity and cardinality*

Also with respect to continuity, Hjelmslev’s geometry differed from Hilbert’s. Hilbert had two continuity axioms: the Archimedean (or Eudoxian) axiom and the completeness axiom. The first excludes infinitesimals and the second guarantees that the line has the same continuity (completeness) as the real numbers. In 1913, Hjelmslev stated: “Finally the practical system includes no requirement of continuity. Also the non-continuous geometries are contained in it” [Hjelmslev 1913, 55]. In 1923, however, he wrote: “In practical geometry one will always recognize the Eudoxian axiom” [Hjelmslev 1923, 13].19 Either he changed his mind in the 10-year period between the two quotes, or he only thought about the completeness axiom when he wrote about continuity in 1913. We shall return to Hjelmslev’s later view on the Archimedean axiom below.

As far as the cardinality of the set of points is concerned, Hjelmslev advanced a finitist and constructivist view of geometric objects:

The practical plane….does not contain infinitely many points. If one asks me how many points it contains I will answer: at the moment none; but if I prick a mark in it with a needle it contains one point and in that way one can get quite a few; but never infinitely many [Hjelmslev 1913a, 51].

**Differential geometry**

Differential geometry gave rise to special considerations concerning the relation between the geometry of reality and arithmetic geometry. In arithmetic geometry, concepts like tangents and osculating circles are defined through differentiation, i.e. through limiting processes. However, in the geometry of reality, there is no continuity axioms and so the limiting procedures make no sense. Differential geometry had interested

---

18 Se e.g. [Hjelmslev 1916b, 13].
19 Here Hjelmslev does not seem to distinguish between Archimedes’ axiom and Eudoxos’ axiom. In his last publication [Hjelmslev 1950] he gave a historically very interesting distinction between the two, insisting on the great novelty and generality of Archimedes’ theory of magnitudes compared to that of Eudoxos.
Hjelmslev since he wrote his doctoral thesis: *Basic principles for the infinitesimal descriptive geometry* in 1897 and it was the first part of his geometry of reality that he developed in some detail [Hjelmslev 1911, 1914]. According to Hjelmslev [1913a, 45], practical geometry always deals with finite differences and an area is always the sum of a finite number of finite quantities.

When we use the sharply defined concepts [the limits] instead of those that the real matter is concerned with, it is first due to the circumstance that one has a preconceived feeling that one will not commit any essential mistake by doing so, but secondly a principle of economy will decide the matter. Indeed, it is much much easier to calculate with differential quotients and integrals than it is to calculate with difference quotients and sums with many terms. In this principle of economy lies the great value of formal mathematics: it simplifies the problems of practical mathematics [Hjelmslev 1913a, 45].

Here we remark again that for Hjelmslev, it is the geometry of reality that provides the true answers. Traditional analysis only gives an approximation. However, according to Hjelmslev one cannot completely rely on these approximations because they do not work in many cases. In particular, Hjelmslev reminded his reader, that modern analysis has produced pathological curves possessing neither tangents nor curve length nor bounding a well-defined area.

Here the “small” encroachments of experience that are at the outset contained in the basic theorems [the axioms of Euclid-Hilbertian geometry] have been added up to produce a deviation which is so essential that the result must be said to contradict experience [Hjelmslev 1913a, 46].

In reality, the definitions of tangent and osculating circle that analysis has introduced are not applicable in practice. The tangent to a curve that appears in practice is a straight line that has as long a line segment in common with the curve as possible; and the osculating circle to a practical curve is a circle that has as long an arc as possible in common with the curve [Hjelmslev 1913a, 48].

With these definitions that point back to Protagoras, a practical curve will have a practical tangent in most points on the curve. However, the practical tangent and the practical radius of curvature may be different from their analytic approximations even if the latter exist. For example in his textbook on descriptive geometry, Hjelmslev showed that:

The practical radius of curvature at the end-points of the axes of an ellipse is somewhat larger than its theoretical value, namely so much larger as the half linear element on the theoretical osculating circle multiplied by the eccentricity of the ellipse [Hjelmslev 1918, 25].

Here the linear element is the length of the line segment that the tangent has in common with the curve. This in turn depends on the accuracy $\varepsilon$ that we mentioned above.

**The flexibility of the axioms**

In his early programmatic papers [Hjelmslev 1913a and 1916b] as well as in his textbooks [Hjelmslev 1916a and 1918] Hjelmslev did not try to formulate a complete set of axioms for his geometry of reality. He often claimed that his geometry of reality was based on one empirical axiom, namely the existence of a square grid in the plane (eg. [Hjelmslev 1913a, 54]. In his first paper in German on the geometry of reality, he first repeated this formulation [Hjelmslev 1916e, 39] but then spelled this requirement out in two plane axioms.

---

20 In [Hjelmslev 1916b, 17] Hjelmslev on the contrary claimed that his real approach made geometry easier (see the conclusion).
and three spacial axioms. It is clear that he did not believe that this set of axioms was complete in any sense and he emphasized

that our practical system finds its simplest expression in its original form. A careful logical analysis into single axioms according to the Euclidean pattern will always prove to be clumsy and unnatural [Hjelmslev 1916e, 42].

In Die Natürliche Geometrie [Hjelmslev, 1923], which was a publication of a manuscript of four lectures he had given in Hamburg the previous year. Hjelmslev began with a more extensive list of axioms. However, he modified them as he went along so that the reader is left with a somewhat unclear impression of the optimal system of axioms for the geometry of reality. Indeed, after having set up the axiom system [Hjelmslev 1823, 4] he admitted that some of the axioms were formulated too sharply for the description of reality. Still he maintained that the system was very useful as a guide on the way to the geometry of reality [Hjelmslev 1923, 5]. Thus, the variability of the system of axioms was apparently a consciously chosen strategy in his geometry of reality.

On the whole, Hjelmslev formulated a rather unconventional view on the role of axioms in his empirical theory:

Let me remark that the wording given here of the foundation is not essential. We do not intend to do dialectics. Whether our axioms are logically independent does not concern us. What matters is only the totality of assumptions [Hjelmslev 1923, 4].

The question of consistency was also a tricky question for Hjelmslev. Having rejected the possibility of proving consistency of a purely theoretical system of axioms he continued:

An axiom system of experience on the other hand, gives immediate certainty about the axioms themselves but no immediate certainty, only a certain probability, for their logical consequences. Thus, at every moment one may face the possibility that the non-worked up [not axiomatized] empirical extra material can be of essential significance. However, one goes on. If one arrives at a contradiction, one tries to manipulate the extra material until everything is again brought into a dialectical order. At last, one may perhaps succeed in gaining a sufficient overview so that all the essential points for the axiomatization of the empirical domain in question stand out. In this case one can set up a proper verbal axiom system [Hjelmslev 1923, 10].

According to Hjelmslev these considerations show that an empirical axiom system is no less certain than an “arbitrary” theoretical axiom system. And the threat of inconsistency is much less serious for an empirical system:

In an empirical system one has the certainty of experience. If a contradiction appears while one attempts to build the system then it does not concern the subject matter; it is only a verbal contradiction and such contradictions can be removed. In a theoretical system, it is different: If a contradiction appears it is of course also a verbal one – indeed there is no subject matter – but this contradiction cannot be removed.

---

21 This is probably his most accessible and complete account of his ideas on the geometry of reality in a foreign language.
22 Remark that this was Hjelmslev’s conviction seven years before Gödel proved it.
23 It is unclear to me whether Hjelmslev believed that he had reached such a sufficient overview.
In an empirical system, a contradiction is only a question of the ordering of the verbal formulation. In a theoretical system it means the difference between sense and nonsense [Hjelmslev 1923, 10-11].

In his “Einleitung in die allgemeine Kongruenzlehre” [Hjelmslev 1929-49], Hjelmslev was more traditional and careful in his axiomatic approach. The first two in this series of papers appeared in 1929 and the last four were published from 1942 to 1949 after Hjelmslev had retired from his heavy administrative duties at the university and at the Carlsberg Foundation. In these papers, Hjelmslev combined his ideas from 1907 on Pascal’s theorem with a scientific and non-material version of his geometry of reality. Starting with a system of axioms for transformations, and in particular reflections, but without the uniqueness axiom, Hjelmslev investigated whether one could deduce (possibly suitably weakened) versions of various theorems of congruence and other theorems of geometry. The axioms vary during the series of papers, but Hjelmslev clearly specified which axioms were used for each deduction. For example in the first of the papers in the series [Hjelmslev 1929-49, 1, 18], Hjelmslev could prove that if two points do not always determine a line uniquely there exist rectangles, and thus the plane is locally Euclidean. From this, he could deduce many of the important theorems of geometry, including Pascal’s theorem.

This series of papers received general recognition even among those who were critical of Hjelmslev’s geometry of reality [Nielsen 1950, 7; Bohr 1950, VIII; Jessen 1950, 241]. Later authors who write on Hjelmslev geometry usually refer back to this series of papers rather than his more philosophically laden materialist introductions to his geometry of reality.

**Infinitesimals**

In [Hjelmslev 1916c] and in his Hamburg lectures, Hjelmslev gave an interesting model of a geometry where two different lines can have a line segment in common, so that the uniqueness axiom does not hold. This was a non-Archimedean geometry build up as the usual analytic (arithmetic) model of geometry, except one must replace the real numbers by the dual numbers, that is numbers of the form $a + \epsilon b, (a, b) \in \mathbb{R}$ where $\epsilon$ is a non-zero symbol (an infinitesimal) such that $\epsilon^2 = 0$. In this geometry, two lines that intersect in an infinitesimal angle have an infinitesimal line segment in common. For example, the two lines with equations: $y = 0$ and $y = \epsilon x$ have the points of the form $(\epsilon k, 0)$ ($k$ real) in common. Moreover, a circle will have an infinitesimal line segment in common with its tangent.

Hjelmslev’s comments to this model sheds a fine light on his untraditional ideas concerning axiomatics of his geometry of reality:

> It is obvious that one can apply this abstract non-Archimedean geometry in practice in the treatment of a problem of our geometry if one goes on to let the quantity $\epsilon$ denote a sufficiently small quantity. How small $\epsilon$ must be, depends on the special case. It is interesting to remark that here one has an application of a non-Archimedian geometry. From a logical point of view, it is interesting that this practical application itself contains a logical contradiction. For in practical geometry one will always recognize the Eudoxian axiom and yet the results of the non-Eudoxian geometry can be applied. An instructive example! An empirical axiom system can be inconsistent in a purely logical sense and yet it

---

24 Already in 1916c, Hjelmslev had announced that he could prove Pascal’s theorem based on the congruence axioms but without the use of the uniqueness axiom. In the case of a circle, Pascal’s theorem would then have to be formulated as follows: “When a hexagon is inscribed in a circle there always exists a straight line that includes a common point of each of the 3 pairs of opposite sides” [Hjelmslev 1916c, 183].
can be consistent in the world of experience. ... The verbal consistency is a convenience when it can be obtained, but it is in no way necessary [Hjelmslev 1923, 13-14].

In his first 1929 paper on congruence Hjelmslev came back to the connection between the Archimedean axiom and the uniqueness axiom. He could now prove [Hjelmslev 1929-49, 1, 17] that the order axioms and the Eudoxian axiom implies the uniqueness of the line through two points. This strengthens the observation above concerning inconsistency. Hjelmslev later published several papers on non-Archimedian geometry [Hjelmslev 1944].

In 1940/41 Dan Barbilian (1895-1961) independently introduced a geometry similar to Hjelmslev’s analytical geometry over the dual numbers [Barbilian 1940/41]. Contrary to Hjelmslev he was not interested in material or real geometry but simply wanted to generalize ordinary geometry to geometries over a ring rather than over a field. The subject of such geometries, often known as Hjelmslev geometry or geometry of Hjelmslev planes or Hjelmslev-spaces developed into an important field after Hjelmslev’s death [Benz 1990, 249-252]. A bibliography of papers in the area before 1976 can be found in Hjelmslevsche Inzidenzgeometrie und verwandte Gebiete [Artman et al. 1976]. In this way, such geometries that violate the uniqueness theorem became well known under Hjelmslev’s name. However, the philosophical connection to Hjelmslev’s ideas of an empirical geometry of reality got lost in the process. What Hjelmslev had presented as a model or an example became the Hjelmslev geometry.

**Geometry of great-points**

The concept of a great-point (Grosspunkt) came to play an important role in Hjelmslev geometry [Benz 1990, 249-50]. Hjelmslev introduced them in a preliminary form called a coarse point (grober Punkt) in [Hjelmslev 1923, 28] in order to illustrate the main ideas behind his geometry of reality:

> A coarse point is a domain in the drawing plane such that every pair of points that lie inside this domain have a distance smaller than 1 cm. The coarse point can be fixed by any of its fine points [Hjelmslev 1923, 28].

Having defined coarse lines in a similar way, he claimed that his geometry of reality holds true of such coarse points and lines.

He refined the idea in his series of papers on the theory of congruence. Already in the conclusion of the first of these papers [Hjelmslev 1929-1949, 1, 35-36], he considered a geometry in which some point-pairs do not determine a straight line uniquely and other point-pairs do. In this case, two points are called neighbor points if they do not determine the line through them uniquely. Hjelmslev then defined a great-point to be the set of all points that are neighbor points to a given point. Similarly, all neighbor points of the points on a straight line make up what he called a great-line. Hjelmslev then claimed that he could develop a great-geometry of such great-points and –lines “where precisely the same axioms hold as in our original geometry. In addition it follows that for these great-points the uniqueness axiom holds” [Hjelmslev 1929-1949, 1, 35-36].

He proved this claim in the third paper of the series from 1942, and also showed that inside each great-point, the geometry would be Euclidean [Hjelmslev 1929-1949, 3, 35-50].

---

25 In the AMS subject classification, number 51CXX is devoted to “Ring Geometry (Hjelmslev, Barbilian etc.).”
Conclusion

The geometry of reality that Hjelmslev developed in the 1910s was primarily inspired by his engagement with descriptive geometry, practical geometric drawing and teaching of elementary geometry to schoolchildren. However, he went much further than other reformers of school geometry developing his approach into a rather well rounded alternative scientific system. “[The materialization of geometry] may at first sight seem to serve exclusively utilitarian purposes. In reality it seeks its first and deepest justification in purely scientific considerations striving to provide an introduction to an exactly executed geometry of reality” [Hjelmslev 1916, 7]. This exactly executed geometry of reality broke with Euclidean geometry, primarily because it did not require lines and planes to be infinite and because it did not require the uniqueness of a line through two points. For that reason, Hjelmslev considered his geometry of reality to be superior to Euclidean geometry also from a theoretical scientific point of view:

Not only is this system the only one that can give us the geometry that is applied in practice and the best basis for education, but it is also in a scientific sense to be preferred to the Euclidean system because its assumptions are logically less extensive and its range therefore considerably larger than the Euclidean system [Hjelmslev 1913a, 54].

As we saw above, he was not the first to suggest an empirical approach to geometry, but his execution of the empirical program was exceptionally thorough. Where most of his 19th century empiricist predecessors had analyzed the Euclidean dogmas from the point of view of astronomy and physics, Hjelmslev’s critique was based on his experience with engineering, practical technical and architectural drawing as well as descriptive geometry. This is reflected in his assault on the uniqueness axiom that only Klein had previously questioned.

Hjelmslev was confident of the superiority of his geometry of reality over the usual Euclidean theoretical geometry:

All in all one can say that the new system both in its design and in all its consequences will turn out to give considerable advantages. At the same time as it makes all the material easier and more natural as a subject for teaching it gives scientific clarification and satisfies the requirements of practical life. And the fact that both pedagogical, practical and scientific interests can be united in this way suggests more than anything else that we are on the right track [Hjelmslev 1916b, 17].

Hjelmslev’s ideas found some resonance among German empirically oriented mathematicians. For example, Otto Hölder (1859-1937) in his epistemological work of 1924 approvingly quoted Hjelmslev’s material introduction of the basic objects of geometry from the Acta paper [Hjelmslev 1916e] [Hölder 1924, 373]. More importantly, Klein26 included a reference to Hjelmslev in the printed version of his book on approximation and precision mathematics:

During recent years the problem of building an empirical geometry as an experimentally verifiable science that does not contradict reality has been treated by the Danish mathematician J. Hjelmslev in several works [Klein 1928, 15].

---

26 It is not clear if the reference to Hjelmslev’s geometry of reality was inserted by Klein himself before his death or by the editor. For the former speaks the fact that the editors noted that their additions would appear in square brackets, and the added reference to Hjelmslev is not.
This quote was followed by detailed references to [Hjelmslev 1916e, 1923 and 1916a]. In her recent biography of Felix Klein, Renate Tobies mentions that in 1915/16 Klein entertained the plan of publishing a fourth volume on geometry for his *Encyklopädie der Mathematischen Wissenschaften* including a chapter “Allgemeine Gestaltenlehre. Hjelmslev”. However, nothing came of that plan [Tobies 2019, 357].

On the other hand Andreas Speiser elevated, Hjelmslev’s first Hamburg lecture on geometry of reality [Hjelmslev 1923] to a mathematical classic when he included it into his book *Klassische Stücke der Mathematik* together with papers by Plato, Euclid, Pascal, Euler, Einstein and other dignitaries.

As mentioned above, it would be another 20 years before what would be called “Hjelmslev geometry” became a fashionable subject, and when that happened the contributors to the field payed little attention to the philosophical, empirical and practical aspects that had been most essential for Hjelmslev.

On the whole, Hjelmslev’s ideas on geometry of reality never really caught on during his lifetime [Fog 1950, 13] and he had no important Danish followers. In the Danish educational system, the experimental and empirical approach to geometry gained some followers in the first half of the 20th century but according to [Hansen 2002, 120] it never became a dominating trend. And many teachers considered Hjelmslev’s take on the matter too radical.

What was the reason for the limited success of Hjelmslev’s geometry of reality?

As far as its use in schools is concerned, his claim in the previous quote that the theory became simpler is questionable and is actually contradicted by Hjelmslev himself in many places (see e.g. the quote above (note 19)).

His complicated account of the relation between his practical geometry and theoretical Euclidean (arithmetic) geometry by way of fixing must have been quite difficult to convey to pupils.

For practical use in connection with descriptive geometry and practical drawing his experimental approach to geometrical constructions was probably a step forward. However, just at the time when Hjelmslev made these advances, the subject of descriptive geometry and precision drawing was losing its importance in the Danish educational system.

As a scientific alternative to Euclidean geometry, Hjelmslev never succeeded in formulating a fixed axiomatic foundation, and his unorthodox views on axiom systems for empirical geometry seems to have had few followers.

When Hjelmslev died, his biographers found various polite ways of calling attention to the limited success of his ideas on the geometry of reality. About Hjelmslev’s system of schoolbooks, Jessen wrote:

    That this system despite its popularity among many pupils, did not find wider distribution must have greatly disappointed its author [Jessen 1950, 241]

Harald Bohr expressed the matter thus:

    Naturally different views concerning Hjelmslev’s way of argumentation in this most difficult field were brought forward and among them were also some rather critical ones [Bohr 1950, VIII]

Fog even suggested that late in life Hjelmslev himself realized the unsatisfactory formulation of his geometry of reality and in particular of the theory of fixing:

    It all dates so long back that one can now venture that in this respect Hjelmslev’s ideas did not catch on. And I think that he gradually became aware of it himself and came to terms with the fact that his geometry was not viable in the form he had given it [Fog 1950, 13].
As far as I can see, Hjelmslev did not convey such doubts in his written works. On the contrary. In a talk he gave to the Scandinavian Congress of Mathematicians in Trondheim half a year before he died, he emphasized that his work on the theory of congruence [Hjelmslev 1929-49] had led to a geometry based on a limited axiom system without the parallel axiom, the continuity axiom and the uniqueness axiom. According to Hjelmslev these excluded axioms are exactly those that in the light of the intuition have been the object of special critique or discussion, while the remaining axioms have a more immediate relation to experience and intuition. In other words we are approaching an apodictic kernel, an apodictic axiom system that could serve as a definition of something that corresponds to Kant’s conception of the pure intuition [Hjelmslev 1949].

If in addition one limits the field of activity to the geometry of the circular disc or the sphere (rather than the infinitely extended space) one will according to Hjelmslev reach a kernel that is common to all synthetic forms of geometry and the empirically given geometry [Hjelmslev 1949, 9]. Hjelmslev even sketched how one could in this way arrive at a “geometry of reality on the sphere”. Thus, from a scientific and philosophical point of view, Hjelmslev continued to believe in his geometry of reality. Whether he, in conversations with Fog, expressed doubts about its didactic merits is another matter.

References
Barbilian, Dan, 1940/41. Zur Axiomatik der projektiven ebenen Ringgeometrien, I. Jahresbericht der DMV 50 (1940), 179-229, II, Jahresbericht der DMV 51 (1941), 34-76.
Barbin, Évelyne; Menghini, Marta; Volkert, Klaus (Eds.), 2019. Descriptive Geometry, The Spread of a Polytechnic Art. The Legacy of Gaspard Monge. Springer, Cham.
Hjelmslev, Johannes, 1897 and 1898. See Petersen, Johannes 1897 and 1898


Hjelmslev, Johannes, MS 1916? Grundlag for Undervisningen i Geometri af Johannes Hjelmslev. Undated handwritten manuscript of 42+64+47 pages in the Archive of the Department of Mathematical Sciences, University of Copenhagen


Hjelmslev, Johannes, 1918. Lærebog i Geometri til Brug ved Den Polytekniske Læreanstalt. Gjellerup, København.


Speiser, Andreas, 1925. *Klassische Stücke der Mathematik*. Orell Füssli, Zürich.