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Overlaps for matrix product states of arbitrary bond dimension in ABJM theory

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\section*{A R T I C L E   I N F O}

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\section*{A B S T R A C T}

We find a closed formula for the overlap of Bethe eigenstates of an alternating SU(4) spin chain, describing the scalar sector of ABJM theory, and matrix product states of any bond dimension representing 1/2 BPS co-dimension one domain walls in the field theory. One point functions of the defect CFTs involved, being directly expressible in terms of these overlaps, are hence completely determined.

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\section*{1. Introduction}

Matrix product states, a tool from quantum information theory, were put to use in holography in the computation of one-point functions in defect CFTs being domain wall versions of $\mathcal{N} = 4$ SYM theory \cite{1,2}. The matrix product states could be of any bond dimension, $d$, and the matrices involved were matrices generating an irreducible $d$-dimensional representation of either SU(2), SU(2) $\times$ SU(2) or SO(5) where the dimension of the representation was related to a jump in the rank of the gauge group of the underlying field theory across the domain wall \cite{3–5}. One-point functions of conformal operators being given by the overlap between the matrix product state and Bethe eigenstates of the integrable spin chain underlying $\mathcal{N} = 4$ SYM \cite{6} turned out to be computable in closed form in many cases \cite{1,2,7–12} and entailed new developments in the understanding of boundary integrability from the statistical mechanical point of view \cite{13,14}. Furthermore, certain matrix product states were understood to encode, via overlaps, the correlation function between giant gravitons and a single trace operator in $\mathcal{N} = 4$ SYM, and exact results were found for the case of two giant gravitons which corresponds to a matrix product state of bond dimension two \cite{15,16}.

ABJM theory, while being a close cousin of $\mathcal{N} = 4$ SYM as the field theory occurring in a lower dimensional version of the AdS/CFT correspondence \cite{17}, symmetry wise differs from the latter by being less than maximally supersymmetric, having its fields in a bi-fundamental rather than adjoint representation of the gauge group and by having an unknown interpolating function in its integrability description \cite{18}. In particular, the integrable spin chain encoding the spectral problem of the theory is a chain where the representation alternates between even and odd sites. The investigation of boundary integrability of ABJM theory and its associated spin chain via the use of matrix product states is much less developed than for $\mathcal{N} = 4$ SYM and is likely to provide us with novel examples of integrable boundary states and further insight on the interplay between defects and integrability. The program of calculating correlation functions between giant gravitons and single trace operators by means of matrix product states has been set up in \cite{19}, see also \cite{20} and exact results have been presented for the case of two giant gravitons \cite{19}. ABJM theory also allows for a defect set-up with a 1/2 BPS co-dimension one domain wall \cite{21,22} and scalar one-point functions in this set-up can be calculated using matrix product states of an alternating integrable SU(4) spin chain \cite{23}. A derivation of the relevant overlap formula for a matrix product state of bond dimension 2 was presented in \cite{24}. In the present paper we consider matrix product states of arbitrary bond dimension and find a closed expression for the overlaps, thereby providing a whole new class of integrable matrix product states. We note that the ability of treating matrix product states

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0370-2693\textcopyright{} 2022 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY license (http://creativecommons.org/licenses/by/4.0/). Funded by SCOAP\textsuperscript{3}. 
of arbitrary bond dimension is useful for making connection with string theory computations, as the bond dimension translates via the holographic dictionary to a string theory flux which can be taken large and allow for a semi-classical approach [25,5,26,27].

We begin in section 2 by introducing the integrable alternating SU(4) spin chain and the matrix product states representing the 1/2 BPS co-dimension one domain walls in ABJM theory. Subsequently, in section 3 we show that these matrix product states constitute integrable boundary states according to the criterion of [13]. We then, in section 4, derive a closed overlap formula for any such matrix product state, i.e. in particular for any bond dimension, by following a recursive strategy suggested in [12]. As this strategy builds on a number of assumptions which cannot immediately be verified we validate our result by a detailed numerical check in section 5. Finally, section 6 contains our conclusion.

2. The model

As it is the case for AdS5/CFT4, the planar spectral problem of the AdS4/CFT3 correspondence is solved by an integrable super spin chain [28,29]. In particular, at the two-loop level the conformal single trace scalar operators of the 3-dimensional field theory involved, i.e. ABJM theory, constitute the eigenstates of an alternating SU(4) spin chain whose Hamiltonian reads [28]

$$H = \lambda^2 \sum_{l=1}^{2L} \left( 1 - P_{l,l+2} + \frac{1}{2} P_{l,l+2} K_{l,l+1} + \frac{1}{2} K_{l,l+1} P_{l,l+2} \right),$$

(1)

where $K_{l,m}$ and $P_{l,m}$ are the trace and permutation operators and where $\lambda$ is the ’t Hooft coupling of ABJM theory.

The odd and even sites of the chain carry respectively the representations $4$ and $4$ of SU(4) corresponding to the following composition of the single trace operators

$$O = \Psi A_{1}^{A_{2} \ldots A_{2L-1}} \tr Y_{A_{1}} Y_{A_{2}} \cdots Y_{A_{2L-1}} Y_{A_{2L}} \equiv |\Psi\rangle,$$

(2)

with $A_{i} \in \{1, 2, 3, 4\}$.

The field theory has a classical solution where a subset of fields acquires a vacuum expectation value different from zero on one side (say $x > 0$) of a co-dimension one domain wall [21], a solution which can conveniently be described as a Nahm pole solution for certain composite fields [23]

$$\langle Y_{\alpha} Y_{\beta} \rangle = \frac{M_{\alpha \beta}}{x^2}, \quad x > 0, \quad \alpha, \beta \in \{1, 2\},$$

where

$$M_{\alpha \beta} = \frac{q}{2} \Omega_{q-1} (\sigma^0)^{\alpha \beta} + t^{q-1}_{i} (\sigma^i)^{\alpha \beta},$$

(3)

with the $t^{q-1}_{i}$ constituting a $q-1$ dimensional representation of SU(2), i.e.

$$[t^i, t^j] = i e^{ij} t^k.$$

(4)

These vevs break the SU(4) symmetry of the scalar sub-sector to SU(2) × SU(2) × U(1). The domain wall is conveniently described as a boundary state in the form of a matrix product state of bond dimension $q - 1$. More precisely,

$$\text{MPS}^{(1 \ldots q_{2L-1})}_{(1 \ldots q_{2L-1})} = \tr M_{p_{1}}^{q_{1}} \cdots M_{p_{2L-1}}^{q_{2L-1}},$$

where the trace is over the $q-1$-dimensional auxiliary space. The one-point functions of the field theory (for $x > 0$) in the presence of the domain wall can then be expressed as the following spin chain overlap [1,2,1]

$$\langle O(x) \rangle = \frac{1}{x^2} \frac{\langle \text{MPS}_{q} \mid \Psi \rangle}{\langle \Psi \mid \Psi \rangle}, \quad x > 0,$$

(5)

where we can write the matrix product state as, cf. eqn (2)

$$\langle \text{MPS}_{q} \rangle = \tr M_{\alpha}^{1} M_{\beta}^{3} \cdots M_{\beta}^{2L-1},$$

(6)

with the subscript $q$ referring to the representation of dimension $q-1$. Following standard conventions we chose the pseudo vacuum of our spin chain to correspond to the operator

$$O_{\text{vac}} = \tr (Y_{1} Y_{4})^L.$$  

(7)

As this state does not have any overlap with the matrix product state defined above, and our method for deriving the overlap formula relies on this overlap being non-zero, we will perform a rotation of the matrices $M$ by a certain angle $\theta$ that we will eventually send to zero. First, we extend the definition of the matrices $M^{\alpha \beta}$ to $\alpha, \beta \in \{1, 2, 3, 4\}$ by simply setting matrix elements not defined above to zero. Secondly, we define a rotated version of the matrix product state in terms of rotated versions of the $M$-matrices given by

$$M^{\alpha \beta} = g(\theta) M^{\alpha \beta} g(-\theta),$$

(8)

where

$$g(\theta) = \begin{pmatrix} \cos \theta & 0 & 0 & -\sin \theta \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ \sin \theta & 0 & 0 & \cos \theta \end{pmatrix}. $$

(9)

We denote the corresponding rotated version of the matrix product state as $|\text{MPS}_{\theta,\alpha}\rangle$.

3. The integrability of the MPS

In this section we show that the MPS (6) preserves “half” of the conserved charges, i.e. the MPS is integrable in the sense of [13].

The transfer matrices of the alternating SU(4) spin chain can be obtained from the following $K$-matrix

$$R_{ab}(u) = u - P_{ab},$$

(10)

and its crossed version, $\tilde{R}_{ab}(u) = R_{ab}(2 - u)$, more precisely

$$T(u) = Tr_{q} R_{a1}(u) R_{a2}(u) \ldots R_{a,2L-1}(u) R_{a,2L}(u),$$

(11)

$$\tilde{T}(u) = Tr_{q} \tilde{R}_{a1}(u) R_{a2}(u) \ldots \tilde{R}_{a,2L-1}(u) R_{a,2L}(u).$$

(12)

These transfer matrices commute with each other

$$[T(u), \tilde{T}(v)] = [T(u), \tilde{T}(u)] = [\tilde{T}(u), T(u)] = 0,$$

(13)

and they generate the Hamiltonian (1).

The $K$-matrix

$$K^{q, \alpha}(u) = \begin{pmatrix} a(u) + u t^{(q-1)}_{-2} & a(u) - u t^{(q-1)}_{-3} & 0 & 0 \\ 0 & a(u) + u t^{(q-1)}_{-3} & 0 & 0 \\ 0 & 0 & a(u) + u t^{(q-1)}_{-2} & 0 \\ 0 & 0 & 0 & a(u) + u t^{(q-1)}_{-3} \end{pmatrix}$$

(14)

1 We note that there exists an equally valid way of expressing the one-point function in terms of a matrix product state, MPS, built from matrices encoding the vevs of composite fields of the type $Y^{\alpha}_{\beta} Y^{\nu}$ [23].
satisfies the following reflection equation
\[ R_{ab}(u-v)K_b^b(Y)(u)R_{ab}(u+v)K_b^b(Y)(v) = \]
\[ \gamma \left( t(q-1) \right) Y \right) \] (15)
where \( t(q-1) = t(q-1) \pm it(q-1) \) and
\[ a(u) = -u^2 + u + \gamma^2 \pm \frac{1}{4}Q(q-2) \] (16)
\[ b(u) = u^2 - (2\gamma + 1)u + \gamma^2 + \frac{1}{4}Q(q-2) \] (17)
Taking \( \gamma = \frac{q}{2} \) the \( K \)-matrix at \( u = 1 \) simplifies as
\[ \gamma \left( t \right) \] (18)
which is just the matrix \( M \) of eqn. (3). Using the argument of [14],
appropriately adapted to our case [11], we obtain
\[ \langle MP(N) \mid T(u) \rangle = T \langle 2 - u \rangle = 0, \] (19)
which proves the integrability of the MPS.

4. The overlap formula

A closed formula for the overlap between the matrix product state and Bethe eigenstates with paired root configurations (to be specified below) is expected to take the form
\[ \langle MP_{\theta,\phi}(u) \rangle = \left( \sum_{k=1}^{\theta-1} \prod_{a=0}^{N_a/2} h_{q,a}^{(q)}(u) \right) \sqrt{\frac{\det G_{\phi}}{\det G_{\theta}}} \] (20)
This follows from the hand-on from concrete investigations [2,7–9] and the other hand from a proposed iteration procedure for the determination of such overlaps [12] which builds on a number of assumptions. The upper limit of the sum is equal to the bond dimension of the matrix product state and in the iterative process it originates from the fact that one cuts open the trace appearing in the matrix product state and sums over diagonal contributions. The constants \( A_k \) encode the overlap of the vacuum state of the initial homogeneous spin chain with the matrix product state. We define the open version of the matrix product state as
\[ \langle MP_{\theta,\phi}(u) \rangle = \left( \sum_{k=1}^{\theta} \prod_{a=0}^{N_a/2} h_{q,a}^{(q)}(u) \right) \sqrt{\frac{\det G_{\phi}}{\det G_{\theta}}} \] (21)
and choosing a concrete parametrization of the \( SU(2) \) generators we compute
\[ A_k = \langle MP_{\theta,\phi}(u) \rangle = \left( \frac{\sin(2\theta)}{2} \right)^L \] (22)
where \( |0 \rangle \) is the state corresponding to the operator (7). The remaining factors in the overlap formula depend on the Bethe roots of the eigenstate in question and trivialize to unity for the vacuum state. In particular, \( G \) is the Gaudin matrix whose determinant factorizes as \( \det G = \det G_{\theta} \det G_{\phi} \) for Bethe states with paired root configurations [30]. For the alternating \( SU(4) \) spin chain of ABM theory there are three types of Bethe roots of which two are momentum carrying [28]. The \( b \)-factors in the overlap formula can be found by an iterative procedure that relies on the nested Bethe ansatz. The procedure builds on the assumption that at each level of nesting the boundary state can be written as a two-site product state expressed in terms of a reflection matrix \( K_{ab}(p) \) that is given as the asymptotic overlap between the boundary state and a two-particle state with particle momenta \( p \) and \(-p \) and quantum numbers \( a \) and \( b \). For each level of nesting the overlap formula picks up a reflection factor corresponding to the vacuum state at that level and at the last level one is left with an overlap of a \( SU(2) \) spin chain which is well-known.

The nested Bethe ansatz in the present case involves two levels of nesting. At the first level, at the odd sites of the spin chain, one replaces a field of type \( Y_1 \) with an excitation in the form of a field of type \( Y_2 \) or \( Y_3 \). Similarly, at the even sites of the spin chain one replaces a field of type \( Y_2 \) with a field of type \( Y_1 \) or \( Y_3 \). Level one excitations at the odd sites carry Bethe rapidities \( u_{1j} \), \( j = 1, \ldots, N_1 \) and level one rapidities at the even sites carry Bethe rapidities \( u_{2j} \), \( j = 1, \ldots, N_2 \). At the second level, where there are four different neighbor configurations, one effectively has an inhomogeneous \( SU(2) \) spin chain of length \( N_1 + N_3 \) and inhomogeneties \( u_{1j} \), \( u_{3j} \). The excitations of this chain carry rapidities \( u_{1j} \), \( j = 1, \ldots, N_2 \). As argued in [23] the selection rule for a Bethe eigenstate to have a non-vanishing overlap with the matrix product states under consideration is the following type of pairing of roots
\[ \langle u_{1j}^{N_1} u_{2k}^{N_2} \rangle \rightarrow \langle -u_{1j}^{N_1} u_{2k}^{N_2} \rangle \] (23)
where we note that in particular \( N_3 = N_1 \). The achiral pair structure is also a consequence of the untwisted integrability condition (18), see the argument in [11,24].

In the following we use the mapping of local operators to spin chain states
\[ Y_{A_1}^A \cdots Y_{A_{2L-1}}^A Y_{A_{2L}}^A \rightarrow |A_1, A_2, \ldots, A_{2L-1}, A_{2L} \rangle, \] (24)
therefore the pseudo-vacuum is
\[ |0 \rangle = |1, 4, 1, 4, \ldots, 1, 4 \rangle. \] (25)
We also define the \( gl_4 \) generators \( (E_{a,b})_n \) which perform the state replacement \( A \rightarrow B \) at site \( n \). At the first level of the nested coordinate Bethe Ansatz we can write the two-particle Bethe state as
\[ |u_{11}, u_{33} \rangle_{a,b} = \sum_{n,m=1}^{L} \right) |2n - 1, 2m \rangle_{c,d} \] (26)
and when we replace a \( Y^1 \) or a \( Y^4 \) of the pseudovacuum by \( Y^1 \) or \( Y^4 \) we define the states as
\[ |2n - 1, 2m \rangle_{c,d} = (-1)^{d-1} |E_{1-1,1}^{2n-1}(E_{4-d-4})_{2m}|0 \rangle \] (27)
and when we replace a \( Y^1 \) or a \( Y^4 \) of the pseudovacuum by \( Y^4 \) or \( Y^1 \) we define the states as
\[ |2n - 1, 2m \rangle_{c,d} = (-1)^{d-1} |E_{1-1,1}^{2n-1}|0 \rangle, \] (28)
\[ |2n \rangle = |E_{1,4}^{2n}|0 \rangle. \] (29)

The expansion coefficients are found to be
\[ \gamma_{a,b}^{c,d}(u) = \left\{ \begin{array}{ll}
\delta_{a,b}^{c,d}, & 2n - 1 < 2m \\
\delta_{a,b}^{c,d}, & 2n - 1 > 2m 
\end{array} \right. \] (30)
where
\[ R_{a,b}^d(u) = \frac{u \delta_{a,b}^d - i \delta_{a,b}^d \epsilon}{u - i} \]
and
\[ \epsilon^{(1)}_{a,b}(u, v) = \epsilon_{a,b} \frac{v + i/2}{u - v - i}, \quad \epsilon^{(2)}_{a,b}(u, v) = \epsilon_{a,b} \frac{u + i/2}{u - v - i}. \]

Finally, the momentum is related to the rapidity as
\[ e^\theta = \frac{u + i/2}{u - i/2}. \]  

(25)

From the concrete expression for the two-particle state at the first level of nesting we can find the expression for the reflection factor at this level
\[ K_{a,b}(u) = \lim_{L \to \infty} \frac{A^L}{L} \left( \langle \text{MPS}_{q,\theta,k,k} | u, -u \rangle_{a,b} \right)^2 \]
\[ = \frac{(u + i/2)(u + i(q - 1/2))}{(u + i(k - 1/2))(u + i(k + 1/2))} \times \left( 1 - \frac{\cot(2\theta)}{\sin(2\theta)} - \frac{1}{u - i/2} \right). \]

As announced earlier, at the second level of nesting we now effectively have a reflection matrix characteristic of a \( SU(2) \) XXX spin chain whose vacuum has a non-trivial overlap with the matrix product state. Choosing the vacuum at this level of nesting to be the 1-state pre-factor of the matrix in eqn. (26) encodes the overlap of the vacuum state with the matrix product state, i.e.
\[ h_{q,k}^{(1)}(u) = k_{1,1}^{(q,k)}(u) = \frac{(u + i/2)(u + i(q - 1/2))}{(u + i(k - 1/2))(u + i(k + 1/2))}, \]

(27)

whereas the matrix itself leads to the following overlap function \[ h_{q,k}^{(2)}(u) = \frac{u^2 + k^2}{\sin(2\theta)^2 u, u^2 + 1/4}. \]

(28)

Under the assumption that the matrix product state decomposes into two-site product states at both levels of nesting the overlap will receive \((N_1 + N_2)/2 = N_1\) factors of type \( h_{q,k}^{(1)}(u) \) and \( N_2/2 \) factors of \( h_{q,k}^{(2)}(u) \).

Summarizing, we get for the complete overlap formula
\[ \langle \text{MPS}_{q,\theta} | u \rangle_{\text{MPS}_{q,\theta}} = \left( \frac{\sin(2\theta)}{2} \right)^{N_1} \langle \text{MPS}_{q,\theta} | u \rangle_{\text{MPS}_{q,\theta}} \]
\[ = \prod_{k=1}^{q-1} \frac{Q_1(-i/2)Q_1(-i(q - 1/2))}{Q_1(-i(k - 1/2))Q_1(-i(k + 1/2))} Q_2(-i k). \]

(29)

where all \( \theta \) dependence is explicitly exposed. We are interested in the limit \( \theta \to 0 \) and we immediately see that the requirement of a non-vanishing result imposes the selection rule \( N_1 = L \) which is in accordance with the analysis carried out for \( q = 2 \) in \( [24, 23] \). We note, however, that the \( \theta \) independent part of the overlap is well-defined not only for \( N_1 = L \) but for any Bethe state with a paired root configuration. More precisely, it reads in terms of Baxter polynomials
\[ \frac{\langle \text{MPS}_{q,\theta} | u \rangle_{\text{MPS}_{q,\theta}}}{\sqrt{\langle u | u \rangle}} = \prod_{k=1}^{q-1} \frac{Q_1(-i/2)Q_2(-i(k + 1/2))}{Q_1(-i(k - 1/2))Q_1(-i(k + 1/2))} \times \left( \frac{k}{2} \right)^L \frac{Q_2(-i k)}{\sqrt{Q_2(0)Q_2(-i/2)}} \frac{\det G_+}{\det G_-}. \]

Here we have introduced \( Q_2(u) \) which is the Baxter polynomial \( Q_2 \) with possible zero roots excluded. The rationale behind this kind of regularization was explained in \( [32] \). We notice that for \( q = 2 \) the sum reduces to a single term corresponding to \( k = 1 \) for which the first line of the expression becomes equal to one. Thus, the \( q = 2 \) result of \( [24, 23] \) is correctly recovered. As mentioned earlier, the parameter \( q \) has the interpretation of a flux in the string theory language, and large \( q \) in a combination with large \( \lambda \) allows for a semi-classical approximation \( [23] \). The leading term in the large-\( q \) limit of the tree level overlap for the excited, singlet states in eqn. (30) is of order \( q^{L+2} \). The problem of computing one-point functions in string theory for non-protected operators has so far resisted resolution.

5. Validation

The recursive derivation carried out above relies on the assumption that the matrix product state has an asymptotic factorization into two site states at each level of nesting \( [12] \). Instead of engaging into formally proving this we will simply perform a detailed numerical test of our formula. As \( [30] \) was proven analytically for \( q = 2 \) \( [24] \) it suffices to validate the ratio
\[ \frac{\langle \text{MPS}_{q,\theta} | u \rangle_{\text{MPS}_{q,\theta}}}{\langle \text{MPS}_{q,\theta} | u \rangle_{\text{MPS}_{q,\theta}}} = \prod_{k=1}^{q-1} \frac{Q_1(-i/2)Q_1(-i(q - 1/2))}{Q_1(-i(k - 1/2))Q_1(-i(k + 1/2))} Q_2(-i k). \]

As already pointed out, the ratio is well-defined for any Bethe state with paired root configurations. We validated the formula for all Bethe states with paired root configurations with \( L = 2, 3 \).

The relevant Bethe eigenstates are most conveniently described by associating the following \( gl(4) \) weights to the possible spin states at the odd, respectively even sites of the spin chain
\[ Y_1, Y_1^+ \to (\pm 1, 0, 0, 0), \quad Y_2, Y_2^+ \to (0 \pm 1, 0, 0). \]

(32)
\[ Y_3, Y_3^+ \to (0, 0, \pm 1, 0), \quad Y_4, Y_4^+ \to (0, 0, 0, \pm 1). \]

(33)

The \( gl(4) \) weights of a Bethe state with root numbers \( N_1, N_2, N_3 \) are then \( (L - N_1, N_1 - N_2, N_2 - N_3, N_3 - L) \). The full Hilbert space is \((\pm 1, 0, 0, 0) \otimes (0, 0, 0, -1)\). The fusion rule for \( L = 2 \) reads
\[ ((1, 0, 0, 0) \otimes (0, 0, 0, -1))^{\otimes 2} = \]
\[ = (2, 0, 0, -2) \oplus (1, 1, 0, -2) \oplus (2, 0, -1, -1) \oplus (1, -1, -1) \oplus 4(1, 0, 0, -1) \oplus 2(0, 0, 0, 0), \]
and for \( L = 3 \) we obtain

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<th>( Q_3(u) )</th>
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(34)
Table 2

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<th>(Q_2(u))</th>
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<tbody>
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<td>1</td>
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</table>
| (1, 0, 1)         | \(u + \frac{1}{2} \)
|                   | \(-\frac{1}{2} u\)           | \(u\)   |
| (1, 1, 1)         | \(u + \frac{1}{2} \)
|                   | \(-\frac{1}{2} u\)           | \(u\)   |
| (2, 1, 2)         | \(u^2 - \frac{1}{12} (u^2 + 4)\)
|                   | \(-\frac{1}{12} (u^2 - 4)\)  | \(u^2\) |
| (2, 2, 2)         | \(u^2 - \frac{1}{12} (67 - 7\sqrt{73}) u + \frac{1}{12} (7 - \sqrt{73})\)
|                   | \(-\frac{1}{12} (67 + 7\sqrt{73}) u + \frac{1}{12} (7 + \sqrt{73})\) | \(u^2\) |
| (3, 3, 3)         | \(u^2 + \frac{1}{12} (67 + 7\sqrt{73}) u + \frac{1}{12} (7 + \sqrt{73})\)
|                   | \(-\frac{1}{12} (67 - 7\sqrt{73}) u + \frac{1}{12} (7 - \sqrt{73})\) | \(u^2\) |
| \((1, 0, 0) \otimes (0, 0, 0, -1)\) | \(= (3, 0, 0, -3) \oplus 2(2, 1, 0, -3) \oplus 2(3, 0, -1, -2) \oplus 4(2, 1, -1, -2) \oplus 9(2, 0, 0, -2) \oplus 9(1, 1, 0, -2) \oplus 9(2, 0, -1, -1) \oplus (1, 1, 1, -3) \oplus (3, -1, -1, -1) \oplus 8(1, 1, -1, -1) \oplus 18(1, 0, 0, -1) \oplus 6(0, 0, 0, 0)\) | \((35)\) |

From the explicit solutions of the Bethe equations we observe that there are 6 Bethe vectors with non-vanishing overlap for \(L = 2\) and 17 such vectors for \(L = 3\). We have checked the overlap formula for all of these vectors. The Tables 1 and 2 summarize the corresponding \(Q\)-functions. The recursive strategy has once again proven its worth and this time in deriving a formula that had not been presented beforehand.

6. Conclusion

We have found a closed formula for the overlaps of the Bethe eigenstates of the alternating integrable \(SU(4)\) spin chain, describing the scalar operators of ABJM theory, and particular matrix product states of arbitrary bond dimension which represent co-dimension one 1/2 BPS domain walls in the field theory. This demonstrates that the domain walls in question constitute integrable boundary states which is in agreement with the analysis of [27] where a similar conclusion was reached from the string theory side when considering a semi-classical limit corresponding to the limit of large bond dimension. With the derivation of the overlap formula we have determined the value of the one-point functions of all scalar operators in the presence of particular domain walls, at least at the leading order in perturbation theory. An interesting question is whether the formula can be extended to the full field theory by exploiting the idea of covariance under fermionic duality transformations [33], as implemented for the matrix product state of bond dimension 2 in [23]. It appears that the arguments of [23] carry over to the present case since the \(Q\)-functions in the first line of eqn. (30) are unaffected by the duality transformations considered. It would likewise be interesting if the leading order result could be bootstrapped to higher loop orders as it was the case for \(\mathcal{N} = 4\) SYM theory and its string theory dual [34,35,10–12]. In this connection, it might also be useful or necessary for consistency checks to set up the perturbative program for defect ABJM theory as it was done for the various versions of defect \(\mathcal{N} = 4\) SYM theory [36–39].

Our novel overlap formula is the latest addition to the growing list of exact overlap formulas for integrable boundary states of various spin chains. Recently, a novel type of integrable boundary states, cross-cap states [40], were identified in 1+1 dimensional field theories and in the rank one integrable \(SU(2)\) and \(SL(2, R)\) spin chains. Cross-cap states should also have an analogue for nested spin chains and in particular for the alternating spin chain considered here with possible implications for ABJM theory and its dual type IIA string theory. Finally, one could imagine that the yet lower dimensional versions of the AdS/CFT correspondence have more integrable boundary states waiting to be discovered.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

All data already in the article.

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