Differentially Private Sparse Vectors with Low Error, Optimal Space, and Fast Access

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ABSTRACT

Representing a sparse histogram, or more generally a sparse vector, is a fundamental task in differential privacy. An ideal solution would use space close to information-theoretical lower bounds, have an error distribution that depends optimally on the desired privacy level, and allow fast random access to entries in the vector. However, existing approaches have only achieved two of these three goals.

In this paper we introduce the Approximate Laplace Projection (ALP) mechanism for approximating \( k \)-sparse vectors. This mechanism is shown to simultaneously have information-theoretically optimal space (up to constant factors), fast access to vector entries, and error of the same magnitude as the Laplace mechanism applied to dense vectors. A key new technique is a unary representation of small integers, which is shown to be robust against "randomized response" noise. This representation is combined with hashing, in the spirit of Bloom filters, to obtain a space-efficient, differentially private representation. Our theoretical performance bounds are complemented by simulations which show that the constant factors on the main performance parameters are quite small, suggesting practicality of the technique.

CCS CONCEPTS

- Security and privacy → Privacy-preserving protocols.

KEYWORDS

Algorithms, Differential Privacy, Sparse Vector

1 INTRODUCTION

One of the fundamental results in differential privacy is that a histogram can be made differentially private by adding noise from the Laplace distribution to each entry of the histogram before it is released [7]. The expected magnitude of the noise on each histogram entry is \( O(1/\varepsilon) \), where \( \varepsilon \) is the privacy parameter, and this is known to be optimal [12]. In fact, there is a sense in which the Laplace mechanism is optimal [14]. However, some histograms of interest are extremely sparse, and cannot be represented in explicit form.

Consider, for example, a histogram of the number of HTTP requests to various servers. Already the IPv4 address space has over 4 billion entries, and though hash collisions can lead to overestimates, they do not influence the error asymptotically. Finally, privacy is achieved by perturbing each bit in the data structure using randomized response [15]. This is where the unary representation is important: It is redundant enough to allow accurate estimation on the expected per-entry error becomes \( O\left(\frac{\log(1/\delta)}{\varepsilon}\right) \), which is significantly worse than the Laplace mechanism for small \( \delta \). Cormode, Procopiuc, Srivastava, and Tran [5] showed how to achieve pure \( \varepsilon \)-differential privacy with expected per-entry error bounded by \( O\left(\frac{\log(d)}{\varepsilon}\right) \), where \( d \) is the dimension of the histogram (number of columns). While both these methods sacrifice accuracy they are very fast, allowing access to entries of the private histogram in constant time. If access time is not of concern, it is possible to combine small space with small per-entry error, as shown by Balcer and Vadhan [2]. They achieve an error distribution that is comparable to the Laplace mechanism (up to constant factors) and space proportional to the sum \( n \) of all histogram entries — but the time to access a single entry is \( O(n/\varepsilon) \), which is excessive for large data sets.

1.1 Our results

Our contribution is a mechanism that achieves optimal error and space (up to constant factors) with only a small increase in access time. The mechanism works for either approximate or pure differential privacy, with the former providing faster access time. Our main results are summarized in Theorem 1.1.

**Theorem 1.1 (Informal Version of Theorems 5.10 and 5.11).**

Given privacy parameters \( \varepsilon > 0 \) and \( \delta \geq 0 \), there exists an \( (\varepsilon, \delta) \)-differentially private algorithm to represent a \( k \)-sparse histogram using \( O(k \log(d + u)) \) bits with per-entry error matching the Laplace mechanism up to constant factors. The access time is \( \log(1/\delta) \frac{1}{\varepsilon} \) when \( \delta > 0 \) and \( \log(d) \) when \( \delta = 0 \).

Here we assume that \( k = \Omega(\log(d)) \). Otherwise the mechanism has an additional term of \( O(\log^2(d)) \) or \( O(\log(d) \log(1/\delta)) \) bits in its space usage for pure and approximate differential privacy, respectively.

1.2 Techniques

We first use a thresholding technique developed in [5, 13] to handle all "large" histogram entries that have at least logarithmic size. To encode the small entries of the histogram, we conceptually switch to a unary encoding. In order to pack all unary representations into a small space, we use hashing to randomize the position of each bit in the unary representation of a given entry. The access time is logarithmic, and though hash collisions can lead to overestimates, they do not influence the error asymptotically.
We use definitions and results as presented by Dwork and Roth [8]. We define the $\delta$:

\[ x, x' \in \mathbb{R}^d \]

and can now define differential privacy for neighboring vectors.

Problem Setup. In this work, we consider $d$-dimensional $k$-sparse vectors of non-negative real values. We say that a vector $x \in \mathbb{R}^d_+$ is $k$-sparse if it contains at most $k$ non-zero entries. We assume that $k = \Omega(\log(d))$. All entries are bounded from above by a value $u \in \mathbb{R}$, i.e., $\max_{i \in [d]} x_i =: \|x\|_\infty \leq u$. Here $[d]$ is the set of integers $\{0, \ldots, d-1\}$. We consider the problem of constructing an algorithm $M$ that releases a differentially private representation of $x$, i.e., $\tilde{x} := M(x)$. Note that $\tilde{x}$ does not itself need to be $k$-sparse.

Utility Measurements. We use two measures for the utility of $\tilde{x} = M(x)$. We define the per-entry error as $|x_i - \tilde{x}_i|$ for any $i \in [d]$. We define the maximum error as $\max_{i \in [d]} |x_i - \tilde{x}_i| = \|x - \tilde{x}\|_\infty$. We compare the utility of algorithms using the expected per-entry and maximum error, and compare the tail probabilities of the per-entry error of our algorithm with the Laplace mechanism introduced below.

Differential Privacy. Differential privacy is a constraint to limit privacy loss introduced by Dwork, McSherry, Nissim, and Smith [7]. We use definitions and results as presented by Dwork and Roth [8]. Intuitively, a differentially private algorithm ensures that a slight change in the input does not significantly impact the probability of seeing any particular output. We measure the distance between inputs using their $l_1$-distance.

In this work, two vectors are neighbors iff their $l_1$-distance is at most 1. That is for all neighboring vectors $x, x' \in \mathbb{R}^d_+$ we have $\|x - x'\|_1 := \sum_{i \in [d]} |x_i - x'_i| \leq 1$. We can now define differential privacy for neighboring vectors.

Definition 2.1 (Differential privacy [8, Def 2.4]). Given $\epsilon > 0$ and $\delta \geq 0$, a randomized algorithm $M : \mathbb{R}^d_+ \rightarrow \mathcal{R}$ is $(\epsilon, \delta)$-differentially private if for all subsets of outputs $S \subseteq \mathcal{R}$ and pairs of $k$-sparse input vectors $x, x' \in \mathbb{R}^d_+$ such that $\|x - x'\|_1 \leq 1$ it holds that:

\[ \Pr[M(x) \in S] \leq e^\epsilon \cdot \Pr[M(x') \in S] + \delta. \]

$M$ satisfies approximate differential privacy when $\delta > 0$ and pure differential privacy when $\delta = 0$. In particular, a pure differentially private algorithm satisfies $\epsilon$-differential privacy. The following properties of differential privacy are useful in this paper.

Lemma 2.2 (Post-processing [8, Proposition 2.1]). Let $M : \mathbb{R}^d_+ \rightarrow \mathcal{R}$ be an $(\epsilon, \delta)$-differentially private algorithm and let $f : \mathcal{R} \rightarrow \mathcal{R}'$ be any randomized mapping. Then $f \circ M : \mathbb{R}^d_+ \rightarrow \mathcal{R}'$ is $(\epsilon, \delta)$-differentially private.

Lemma 2.3 (Composition [8, Theorem 3.16]). Let $M_1 : \mathbb{R}^d_+ \rightarrow \mathcal{R}_1$ and $M_2 : \mathbb{R}^d_+ \rightarrow \mathcal{R}_2$ be randomized algorithms such that $M_1$ is $(\epsilon_1, \delta_1)$-differentially private and $M_2$ is $(\epsilon_2, \delta_2)$-differentially private. Then the algorithm $M$ where $M(x) = (M_1(x), M_2(x))$ is $(\epsilon_1 + \epsilon_2, \delta_1 + \delta_2)$-differentially private.

Throughout this paper, we clamp the output of all algorithms to the interval $[0, \ldots, u]$. An estimate outside this interval is due to noise and clamping outputs cannot increase the error. It follows from Lemma 2.2 that clamping the output does not affect privacy.

We clamp the output implicitly to simplify presentation.

Probabilistic Tools. The Laplace Mechanism introduced by Dwork et al. [7] satisfies pure differential privacy by adding noise calibrated to the $l_1$-distance to each entry. For completeness, Algorithm 1 provides a formulation of the Laplace mechanism in the context of releasing an $\epsilon$-differentially private representation of a sparse vector.

Algorithm 1: The Laplace Mechanism

**Parameters:** $\epsilon > 0$.

**Input:** $k$-sparse vector $x \in \mathbb{R}^d_+$.

**Output:** $\epsilon$-differentially private approximation of $x$.

1. Let $\tilde{x}_i = x_i + \eta_i$ for all $i \in [d]$, where $\eta_i \sim \text{Lap}(1/\epsilon)$.
2. Release $\tilde{x}$.

Here Lap$(1/\epsilon)$ is the Laplace distribution with scale parameter $1/\epsilon$. The PDF and CDF of the distribution are presented in Definitions 2.4 and 2.5 and the expected error and tail bound of the mechanism are shown in Propositions 2.6 and 2.7. The Laplace mechanism works well for dense vectors and serves as a baseline for our work. However, it is impractical or even infeasible in the setting of $k$-sparse vectors, as the space scales linearly in the input dimensionality $d$.

Definition 2.4. The probability density function of the Laplace distribution centered around 0 with scale parameter $1/\epsilon$ is $\frac{\lambda}{2} e^{-|x|/\lambda}$.

Definition 2.5. The cumulative distribution function of the Laplace distribution centered around 0 with scale parameter $1/\epsilon$ is:

\[ \Pr[\text{Lap}(1/\epsilon) \leq x] = \begin{cases} \frac{1}{2} e^{x / \epsilon}, & \text{if } x < 0 \\ 1 - \frac{1}{2} e^{-x / \epsilon}, & \text{if } x \geq 0 \end{cases} \]

Proposition 2.6 (Expected Error [8, Theorem 3.8]). The expected per-entry and maximum error of the Laplace mechanism are $\mathbb{E}[|x_i - \tilde{x}_i|] = O(1/\epsilon)$ and $\mathbb{E}[\|x - \tilde{x}\|_{\infty}] = O\left(\frac{\log(d)}{\epsilon}\right)$ respectively.
Proposition 2.7 (Tail bound [8, Theorem 3.8]). With probability at least $1 - \psi$ we have:

$$|\text{Lap}(1/\varepsilon)| \leq \frac{1}{\varepsilon} \ln \frac{1}{\psi}$$

Random rounding or stochastic rounding is used for rounding a real value probabilistically based on its fractional part. We define random rounding for any real $r \in \mathbb{R}$ as follows:

$$\text{RandRound}(r) = \begin{cases} [r] & \text{with probability } r - [r] \\ [r] & \text{with probability } 1 - (r - [r]) \end{cases}$$

Lemma 2.8. The expected error of random rounding is maximized when $r - [r] = 0.5$. For any $r$ we have:

$$\mathbb{E}[|r - \text{RandRound}(r)|] \leq \frac{1}{2}$$

Randomized response was first introduced by Warner [15]. The purpose of the mechanism is to achieve plausible deniability by changing one’s answer to some question with probability $p$ and answer truthfully with probability $q = 1 - p$. We define randomized response for a boolean value $b \in \{0, 1\}$ as follows:

$$\text{RandResponse}(b, p) = \begin{cases} 1 - b & \text{with probability } p \\ b & \text{with probability } q \end{cases}$$

Universal Hashing. A hash family is a collection of functions $\mathcal{H}$ mapping keys from a universe $U$ to a range $R$. A family $\mathcal{H}$ is called universal, if each pair of different keys collides with probability at most $1/|R|$, where the randomness is taken over the random choice of $h \in \mathcal{H}$. A particular efficient construction that uses $O(\log |U|)$ bits and constant evaluation time is presented in [6].

Model of Computation. We use the $w$-bit word RAM model defined by Hagerup [11] where $w = \Theta(\log(d) + \log(u))$. This model allows constant time memory access and basic operations on $w$-bit words. As such, we can store a $k$-sparse vector using $O(k \log(d+u))$ bits with constant lookup time using a hash table. We assume that the privacy parameter $\varepsilon$ and $\delta$ can be represented in a single word.

Negative values. In this paper, we consider non-negative real values, but the mechanism can be generalized for negative values using the following reduction. Let $v \in \mathbb{R}^d$ be a real valued $k$-sparse vector. Construct $x \in \mathbb{R}_2^{2d}$ from $v$ such that

$$x_i = \begin{cases} \max(v_i, 0), & \text{if } i < d \\ -\min(v_{i-d}, 0), & \text{if } i \geq d \end{cases}$$

By construction $x$ is $k$-sparse and the $\ell_1$-distance between vectors is preserved. We can access elements in $v_i$ as $v_i = x_i - x_{i+d}$. As such, any differential private representation of $x$ can be used as a differential private representation of $v$ with at most twice the error.

3 RELATED WORK

Previous work on releasing differentially private sparse vectors primarily focused on the special case of discrete vectors in the context of releasing the histogram of a dataset.

Korolova, Kenthapadi, Mishra, and Ntoulas [13] first introduced an approximate differentially private mechanism for the release of a sparse histogram. A similar mechanism was later introduced independently by Bun, Nissim, and Stemmer [3] in another context. The mechanism adds noise to non-zero entries and removes those with a noisy value below a threshold $t = O\left(\frac{\log(1/\delta)}{\varepsilon}\right)$. The threshold is chosen such that the probability of releasing an entry with true value 1 is at most $\delta$. The expected maximum error is $O\left(\frac{\log(\max\{k,1/\delta\})}{\varepsilon}\right)$. Since $\delta$ is usually chosen to be negligible in the input size, we assume that $\delta = O(1/k)$. As such, the expected maximum error is $O\left(\frac{\log(1/\delta)}{\varepsilon}\right)$. We discuss the per-entry error below. Their technique satisfies differential privacy for discrete data. We extend their technique to real-valued data as part of Section 5, where we combine it with the ALP mechanism.

Cormode, Procopiuc, Srivastava, and Tran [5] introduced a pure differential privacy mechanism in their work on range queries for sparse data. The mechanism adds noise to all entries and removes those with a noisy value below a threshold $t = O\left(\frac{\log(d)}{\varepsilon}\right)$. Here the threshold is used to reduce the expected output size. The number of noisy entries above $t$ is $O(k)$ with high probability. The running time of a naive implementation of their technique scales linearly in $d$. They improve on this by sampling a binomial distribution to determine the number of zero entries to store. They show that their approach has the same distribution as a naive implementation that adds noise to every zero entry. Their mechanism works for real-valued data in a straightforward way.

Since the expected number of non-zero values in the output is $O(k)$ for both mechanisms above, the memory requirement is $O(k \log(d+u))$ bits using a hash table. An entry is accessed in constant time. The expected per-entry error depends on the true value of the entry. The expected error is $O(1/e)$ for an entry whose noisy value is above the threshold with high probability. However, this does not hold for entries that are likely removed. Consider an entry with a true value exactly at the threshold that is $x_i = t$. This entry is removed for any negative noise added. As such the expected per-entry error is $O(1/e)$ for worst-case input. That is $O\left(\frac{\log(1/\delta)}{\varepsilon}\right)$ and $O\left(\frac{\log(d)}{\varepsilon}\right)$ for the two mechanisms, respectively.

In their work on differentially private on finite computers, Balcer and Vadhan [2] introduced several algorithms including some with similar utility as the mechanisms described above. They provided a lower bound of $O\left(\frac{\min(\log(d), \log(1/\delta))}{\varepsilon}\right)$ for the expected per-entry error of any algorithm that always outputs a sparse histogram. They bypass this bound by producing a compact representation of a dense histogram. Their representation has expected per-entry and maximum error of $O(1/e)$ and $O\left(\frac{\log(d)}{\varepsilon}\right)$. The representation requires $O\left(\frac{\log(d)}{\varepsilon}\right)$ bits and they access the value of an entry in time $O\left(\frac{d}{\varepsilon}\right)$. Here $n$ is the number of rows in the dataset, i.e., the sum of all entries of the histogram. Note that their problem setup differs from ours in that each entry is bounded only by $n$ such that $\|x\|_0 \leq n$. That is, $n$ serves a similar purpose as $u$ does in our setup. We do not know how to extend their approach to our setup with real-valued input.

In light of the results achieved in previous work, our motivation is to design a mechanism that achieves three properties simultaneously: $O(1/e)$ expected per-entry error for any input, fast access, and (asymptotically) optimal space. Previous approaches only
We start by considering the special case of we want the per-entry error to match the tail bounds of the Laplace representation. It also celebrates the mountains, whose silhouette plays a role in a certain random walk considered in the analysis of the ALP mechanism.

Theorem 5.10 (this work) $O(k \log (d + u))$ $O(\log (d))$ $O(\frac{\log (1/\delta)}{\varepsilon})$

Korolova et al.* $O(k \log (d + u))$ $O(\log (d))$ $O(\frac{\log (1/\delta)}{\varepsilon})$

Theorem 5.11* (this work) $O(k \log (d + u))$ $O(\log (d))$ $O(\frac{\log (1/\delta)}{\varepsilon})$

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Table 1: Comparison with previous work of expected values for worst-case input.

Previous work is listed by the authors that first introduced the technique.

*Approximate differential privacy.

achieved at most two of these properties simultaneously. Moreover, we want the per-entry error to match the tail bounds of the Laplace mechanism up to constant factors. We construct a compact representation of a dense vector to bypass the lower bound found by Balcer and Vadhan. The access time of our mechanism is $O(\log (d))$ and $O(\log (1/\delta))$ for pure and approximate differential privacy, respectively. Table 1 summarizes the results of previous work and our approach.

## 4 THE ALP MECHANISM

In this section, we introduce the Approximate Laplace Projection (ALP) mechanism and give an upper bound on the expected per-entry error. The ALP mechanism consists of two algorithms. The first algorithm constructs a differentially private representation of a $k$-sparse vector and the second estimates the value of an entry based on its representation.

### 4.1 A 1-differentially private algorithm

We start by considering the special case of $\varepsilon = 1$ and later generalize to all values of $\varepsilon > 0$. Moreover, the mechanism works well only for entries bounded by a parameter $\beta$. In general, this would mean that we had to set $\beta = u$ if we only were to use the ALP mechanism. However, in Section 5 we will discuss how to set $\beta$ smaller and still perform well for all entries.

In the first step of the projection algorithm, we apply random rounding to a scaled version of every non-zero entry such that every entry maps to an integer. We then store the unary representation of these integers in a two-dimensional bit-array using a sequence of universal hash functions [4]. We call this bit-array the embedding. Lastly, we apply randomized response on the embedding to achieve privacy. The pseudocode of the algorithm is given in Algorithm 2 and we discuss it next.

Figure 1 shows an example of an embedding before applying randomized response. The input is a vector $x$ where the $i$th entry $x_i$ is the only non-zero value. The result of evaluating $i$ for each hash function is shown in the table at the bottom and the $m = 8$ bits representing the $i$th entry in the bit-array have been highlighted. In

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Algorithm 2: ALP1-projection

**Parameters:** $\beta > 0$, $\alpha > 1$, and $s \in N$.

**Input** $k$-sparse vector $x \in \mathbb{R}^d$ where $s > 2k$.

Sequence of hash functions from domain $\{d\}$ to $\{s\}$, $h = (h_0, \ldots, h_{m-1})$, where $m = \left\lceil \frac{\beta}{\pi - 1} \right\rceil$.

**Output** $1$-differentially private representation of $x$.

1. Apply random rounding to a scaled version of each non-zero entry of $x$ such that $y_i = \text{RandRound}(\frac{x_i}{\beta})$.
2. Construct $z \in \{0, 1\}^m$ by hashing the unary representations of $y$ such that:

$$z_{a,b} = \begin{cases} 1, & \text{if } a < y_i \text{ and } h_a(i) = b \\ 0, & \text{otherwise} \end{cases}$$

3. Apply randomized response to each bit of $z$ such that $\tilde{z}_{a,b} = \text{RandResponse}(z_{a,b}, \frac{1}{\alpha - 1})$.
4. Release $h$ and $\tilde{z}$.

Step (1) of the algorithm, $x_i$ is scaled by $1/(\alpha - 1)$ and randomized rounding is applied to the scaled value. This results in $y_i = 5$. Using the hash functions, we represent this value in unary encoding by setting the first five bits to 1 in Step (4), where the $i$th bit is selected by evaluating the hash function $h_j$ on $i$. The final three bits are unaffected by the entry. Finally, we apply randomized response in each cell of the bit-array. The bit-array after applying randomized response is not shown here, but we present it later in Figure 2. Both the bit-array and the hash functions are the differentially private representation of the input vector $x$. We use this construction when estimating the value of $x_i$ later.

The algorithm takes three parameters $\alpha$, $\beta$, and $s$. The parameters $\alpha$ and $s$ are adjustable.

We discuss these parameters later as part of the error analysis. In Section 6 we discuss how to select values for $\alpha$ and $s$. Throughout the paper we sometimes assume that $\alpha$ is a constant and $s$ is a constant multiple of $k$ that is $\alpha = \Theta(1)$ and $s = \Theta(k)$. The parameter $\beta$ bounds the values stored in the embedding. We discuss $\beta$ as part of the error analysis as well.

**Lemma 4.1.** Algorithm 2 satisfies 1-differentially privacy.
Let $x, x' \in \mathbb{R}_d^q$ denote two neighboring vectors. We proof the lemma in several steps. First, the vectors differ only in their $i$th entry. In this case, we start by assuming that only a single bit of $z$ is affected by changing $x$ to $x'$ and that there are no hash collisions. We then allow them to differ in several bits and include hash collisions. Finally, we generalize to the case that they differ in more than one entry.

Assume that $z$ differs only in a single bit for $x$ and $x'$. Let $Y$ denote the event that the affected bit is set to one after running the algorithm. Let $p = \frac{1}{\alpha+1}$ be the parameter of the randomized response step and let $q = 1 - p$. Then the probability of $Y$ occurring with input $x$ is $\Pr[Y \mid x] = (1 - r) \cdot p + r \cdot q$, where $r = \frac{x_i}{\alpha+1} - \frac{\min(x_i, x'_i)}{\alpha-1}$. The minimum term is needed when $\frac{\max(x_i, x'_i)}{\alpha-1}$ is an integer such that the probability should be 1. We can now find the difference in the probability of $Y$ occurring for $x$ and $x'$ as:

$$\Pr[Y \mid x] - \Pr[Y \mid x'] = ((1 - r)p + rq) - ((1 - r')p + r'q) = (r - r') \cdot (q - p) = \frac{x_i - x'_i}{\alpha+1} \cdot \frac{\alpha - 1}{\alpha + 1} = \frac{x_i - x'_i}{\alpha + 1}.$$ 

By symmetry, the absolute difference in probability for setting the bit to either zero or one is $\frac{|x_i - x'_i|}{\alpha+1}$.

Next, we remove the assumption that only a single bit is affected by composing probabilities. We provide the following inductive construction. Let $x, x' \in \mathbb{R}_d^q$ be vectors that differ in the $i$th entry such that exactly two bits are affected. We consider the case of $x_i < x'_i$ and fix a vector $x'' \in \mathbb{R}_d^q$ with $x_i < x''_i < x'_i$ such that the differences affects exactly one bit each. Again, let $Z$ be an arbitrary output of Algorithm 2. Applying the upper bound from above twice, we may bound the change in probabilities by:

$$\Pr[\text{ALP1-projection}(x'') = Z] \leq e^{x_i-x'_i} \cdot e^{x''_i-x'_i} = e^{x_i-x'_i} .$$

which can be applied inductively if changing an entry affects more than two bits.

We are now ready to generalize to any vectors $x, x' \in \mathbb{R}_d^q$, i.e., where vectors may differ in more than a single position. Using the bound from above, we can bound the ratio of probabilities by:

$$\frac{\Pr[\text{ALP1-projection}(x'') = Z]}{\Pr[\text{ALP1-projection}(x) = Z]} \leq \prod_{i \in [d]} e^{x_i-x'_i} = e^{\|x-x''\|_1} .$$

The privacy loss is thus bounded by the $l_1$-distance of the vectors for any output. Recall that the $l_1$-distance is upper bounded by 1 for two neighboring vectors. As such the algorithm is 1-differentially private as for any pair of neighboring vectors $x$ and $x'$ and any subset of outputs $S$ we have:

$$\Pr[\text{ALP1-projection}(x) \in S] \leq e^{\|x-x''\|_1} \cdot \Pr[\text{ALP1-projection}(x') \in S] \leq e \cdot \Pr[\text{ALP1-projection}(x') \in S] .$$

The following lemma summarizes the space complexity of storing the bit-array and the collection of hash functions.
Algorithm 3: ALP1-estimator

Parameters: \( \alpha > 1 \).

Input: Embedding \( \tilde{z} \in \{0,1\}^{m \times s} \), Sequence of hash functions \( h = (h_0, \ldots, h_{m-1}) \). Index \( i \in [d] \).

Output: \( \tilde{y}_i \) - Estimate of \( y_i \). The pseudocode for the algorithm is given as Algorithm 3.

1. Define the function \( f: [m] \to \{1,-1\} \) as:
   \[
   f(a) = 2\tilde{z}_a(h_a(i)) - 1
   \]

2. Let \( \tilde{y}_i = \text{average(peaks)} \), where peaks is a set of indices maximizing \( \sum_{a=0}^{m-1} f(a) \), that is:
   \[\text{peaks} = \arg \max_{a \in [m+1]} \sum_{a=0}^{m-1} f(a)\]

3. Return \( \tilde{y}_i \cdot (\alpha - 1) \).

\[\text{Lemma 4.2.} \quad \text{The number of bits required to store } h \text{ and } \tilde{z} \text{ is} \]
\[O\left( \frac{(s + \log d) \cdot \beta}{\alpha - 1} \right). \]

\[\text{Proof.} \quad \text{By definition } m = O\left( \frac{\beta}{\alpha - 1} \right) \text{ and } m \cdot s = O\left( \frac{\beta d}{\alpha - 1} \right) \text{ bits are used to store } \tilde{z}. \text{ Each hash function uses } O(\log(d)) \text{ bits for a total of } \frac{O(\log(d)\beta)}{\alpha - 1} \text{ bits to store } h. \]

4.2 Estimating an entry

We now introduce the algorithm to estimate an entry based on the embedding from Algorithm 2. When accessing the \( i \)th entry, we estimate the value of \( y_i \) and multiply by \( (\alpha - 1) \). As stated above, the \( i \)th entry is represented by \( m \) bits in the embedding. The estimate of \( y_i \) is chosen to maximize a partial sum. If multiple values maximize the sum we chose use their average.

\[\text{Intuition.} \quad \text{The first } y_i \text{ bits representing the } i \text{th entry are set to one before applying noise in Algorithm 2, cf. Figure 1. The last } m - y_i \text{ bits are zero, except if there are hash collisions. Some bits might be flipped due to randomized response, but we expect the majority of the first } y_i \text{ bits to be ones and the majority of the remaining } m - y_i \text{ bits to be zeros. As such the estimate of } y_i \text{ is based on indices maximizing the difference between ones and zeros prior to the index. The pseudocode for the algorithm is given as Algorithm 3.} \]

Figure 2 shows an example of Algorithm 3. The example is based on the embedding from Figure 1 after adding noise. The plot shows the value of \( \tilde{z}_a(h_a(i)) - 1 \) for all candidate estimates. This sum is maximized at index 3 and 5. This is visualized as the \text{peaks} in the plot. The estimate is the average of those indices.

\[\text{Lemma 4.3.} \quad \text{The evaluation time of Algorithm 3 is } O\left( \frac{\beta}{\alpha - 1} \right). \]

\[\text{Proof.} \quad \text{We can compute all partial sums by evaluating each bit } (\tilde{z}_0(h_0(i)), \ldots, \tilde{z}_m(h_m(i))) \text{ once using dynamic programming. As such the evaluation time is } O(m) \text{ with } m = \left\lceil \frac{\beta}{\alpha - 1} \right\rceil. \text{ We have } m = O(\beta) \text{ when } \alpha = \Theta(1). \]

We now analyze the per-entry error of Algorithm 3. We first analyze the expected error based on the parameters of the algorithm.

The results is presented in Lemma 4.7. In Lemmas 4.8 and 4.9 we bound the tail distribution of the per-entry error of the algorithm.

\[\text{Lemma 4.4.} \quad \text{The expected per-entry error of Algorithm 3 is bounded by } \left( \frac{1}{4} + \mathbb{E}[|y_i - \tilde{y}_i|]\right) \cdot (\alpha - 1) \text{ for entries with a value of at most } \beta. \]

\[\text{Proof.} \quad \text{It is clear that the error of the } i \text{th entry is } (\alpha - 1) \text{ times the difference between } \tilde{y}_i \text{ and the scaled true value } \frac{x_i}{\alpha - 1}. \text{ The expected difference is bounded by:} \]

\[\mathbb{E}[|\frac{x_i}{\alpha - 1} - \tilde{y}_i|] \leq \mathbb{E}[|\frac{x_i}{\alpha - 1} - y_i|] + \mathbb{E}[|y_i - \tilde{y}_i|] \]

\[\leq \frac{1}{2} + \mathbb{E}[|y_i - \tilde{y}_i|]. \]

The last inequality follows from Lemma 2.8.

We find an upper bound on \( \mathbb{E}[|y_i - \tilde{y}_i|] \) by analyzing simple random walks. A simple random walk is a stochastic process such that \( S_0 = 0 \) and \( S_n = \sum_{t=1}^{n} X_t \), where \( X \) are independent and identically distributed random variables with \( \text{Pr}[X = 1] = p \) and \( \text{Pr}[X = -1] = 1 - p = q \). Alm [1] summarized several properties of simple random walks. For our analysis we are concerned with the largest \( n \) such that \( S_n \geq 0 \). For an infinite random walk where \( p < q \) such an \( n \) exists with probability 1.

\[\text{Lemma 4.5.} \quad \text{Let } S \text{ be a random walk with } p < q. \text{ The expected last non-negative step of } S \text{ is } \mathbb{E}[\max_n : S_n \geq 0] = \frac{3pq}{(q-p)^2}. \]
Proof. At any step $n$ the probability that there exists a later step $\ell > n$ such that $S_{\ell} > S_n$ is $\frac{q}{q}$ [1]. We use this to find the probability that $S_n$ is the unique maximum in $\{S_n, \ldots, S_m\}$ as follows:

$$
\Pr[S_n > \max(S_{n+1}, \ldots, S_m)] = \Pr[X_{n+1} = -1]:
$$

$$
\Pr[S_{n+1} = \max(S_{n+1}, \ldots, S_m)] = q \cdot \frac{1 - p}{q} = 1 - p.
$$

The last non-negative step must have value exactly zero and as such must be at an even numbered step. The probability that step $2i$ is the last non-negative is:

$$
\Pr[(\max : S_n \geq 0) = 2i] = \Pr[S_{2i} = 0].
$$

We are now ready to find the expected last non-negative step of an infinite simple random walk as:

$$
\mathbb{E}[(\max : S_n \geq 0) = 2i] = \sum_{i=0}^{\infty} 2i \cdot \Pr[(\max : S_n \geq 0) = 2i]
$$

$$
= \sum_{i=0}^{\infty} 2i \cdot \left(\frac{2i}{i} \right) (pq)^i (q - p).
$$

The last equality follows from the identity $\sum_{i=0}^{\infty} \binom{2i}{i} (pq)^i = \frac{2pq}{(q-p)^2}$. See Appendix A for a proof of this identity.

We are now ready to bound $\mathbb{E}[|y_i - \tilde{y}_i|]$. We consider entries with value at most $\beta$, i.e., $y_i \leq m$.

Lemma 4.6. Let $y_i \leq m$ and $\gamma = \frac{\alpha + 1}{1 + (\alpha - 1) \gamma} - 1$. Then we can upper bound the expected value of $|y_i - \tilde{y}_i|$ by

$$
\mathbb{E}[|y_i - \tilde{y}_i|] \leq \frac{4\alpha}{(\alpha - 1)^2} + \frac{4\gamma}{(\gamma - 1)^2}.
$$

Proof. Recall the definition peaks = $\max_{n \in [m+1]} \sum_{a=0}^{n-1} f(a)$ from Algorithm 3. Let $\tilde{y}_i \in y_i$ denotes the element furthest from $y_i$ that is $\forall a \in \text{peaks}: |y_i - a| \leq |y_i - \tilde{y}_i|$. It follows that it is sufficient to consider $\tilde{y}_i$ for the proof since $|y_i - \tilde{y}_i| \leq |y_i - y_i|$. We first consider the case of $\tilde{y}_i \leq y_i$. It follows from the definition of $\tilde{y}_i$ as a maximum that $\sum_{j=0}^{y_i} f(j) \leq 0$. As such at least half the bits $(\tilde{y}_i, y_i, h_{y_i}, i), \ldots, (\tilde{y}_i, 1, h_{y_i-1}, i)$ must be zero, that is they were flipped as part of the randomized response step of Algorithm 2. So as such the value $y_i - \tilde{y}_i$ is bounded by the expected last non-negative step of a simple random walk with $p = \frac{1}{1+\gamma}$. It follows from Lemma 4.5 that:

$$
\mathbb{E}[y_i - \tilde{y}_i | \tilde{y}_i \leq y_i] \leq \frac{4\alpha}{(\alpha - 1)^2} + \frac{4\gamma}{(\gamma - 1)^2}.
$$

We can use a similar argument when $y_i \geq \tilde{y}_i$ to show that at least half the bits in $(\tilde{y}_i, y_i, h_{y_i}, i), \ldots, (\tilde{y}_i, 1, h_{y_i-1}, i)$ must be 1 since $\tilde{y}_i$ is a maximum. Here we have to consider the possibility of a hash collision. Each hash function maps to $[s]$ and at most $k$ entries result in a hash collision. The probability of a hash collision is at most $\frac{k}{s}$ using a union bound. As such for $j \geq y_i$ we have $\Pr[\tilde{y}_j, h_j(i)] = 1$ $\leq \frac{1 + (\alpha - 1) \gamma}{(\alpha - 1) \gamma}$. We let $\frac{1 + (\alpha - 1) \gamma}{(\alpha - 1) \gamma} = \frac{1}{\gamma}$ such that $\mathbb{E}[y_i - \tilde{y}_i | \tilde{y}_i \geq y_i] \leq \frac{4\alpha}{(\alpha - 1)^2} + \frac{4\gamma}{(\gamma - 1)^2}$ by Lemma 4.5 and the calculation above. We isolate $\gamma$ to find:

$$
\gamma = \frac{1 + \alpha}{1 + (\alpha - 1) \gamma}.
$$

Note that $\gamma > 1$ due to the requirement $s > 2k$ of Algorithm 2. By conditional expectation, we may upper bound the total expected error by

$$
\mathbb{E}[|y_i - \tilde{y}_i|] \leq \mathbb{E}[|y_i - \tilde{y}_i|]
$$

$$
\leq \mathbb{E}[y_i - \tilde{y}_i | \tilde{y}_i \leq y_i] + \mathbb{E}[y_i - \tilde{y}_i | \tilde{y}_i \geq y_i]
$$

$$
\leq \frac{4\alpha}{(\alpha - 1)^2} \frac{4\gamma}{(\gamma - 1)^2}.
$$

As such we can bound the expected per-entry error for entries with a true value of at most $\beta$ by a function of the parameters $\alpha$ and $s$. In Section 6 we discuss the choice of these parameters based on the upper bound and experiments. For any fixed value of $\alpha$ and the fraction $\frac{k}{s}$ we bound the expected per-entry error as follows:

Lemma 4.7. Let $\alpha = \Theta(1)$ and $s = \Theta(k)$. Then the expected per-entry error of Algorithm 3 is $\mathbb{E}[|x_i - \tilde{x}_i|] \leq \max(0, x_i - \beta) + O(1)$.

Proof. It follows from Lemmas 4.4 and 4.6 that the expected error for any entry bounded by $\beta$ is:

$$
\mathbb{E}[|x_i - \tilde{x}_i| | x_i \leq \beta] \leq \beta \left(\frac{1}{2} + \frac{4\alpha}{(\alpha - 1)^2} + \frac{4\gamma}{(\gamma - 1)^2}\right),
$$

where $\gamma = \frac{\alpha + 1}{1 + (\alpha - 1) \gamma} - 1$. Entries above $\beta$ have an additional error of up to $x_i - \beta$, since $y_i = m$ and $y_i > m$ are represented identically in the embedding by Algorithm 2. Since $\alpha$ and $\frac{k}{s}$ are constants we have:

$$
\mathbb{E}[|x_i - \tilde{x}_i|] \leq \max(0, x_i - \beta) + O(1).
$$

Next, we bound the tail probabilities for the per-entry error of the mechanism. We bound the error of the estimate $\tilde{y}_i$, which implies bounds on the error of the mechanism.
Lemma 4.8. Let $\gamma = \frac{a+1}{1+\alpha (a-1)^{1/2}} - 1$ and $\tau \geq 0$. Let $p = \frac{1}{\tau + 1}$ and $q = 1 - p$. Then for Algorithm 3 we have:

$$\Pr\{ |y_i - \tilde{y}_i| \geq \tau \} \leq \frac{2}{(q-p)\sqrt{\pi}} (4pq)^{\tau/2},$$

Proof. Let $S$ be a simple random walk. We find an upper bound on the probability that the last non-negative step in $S$ is at least the $r$th step:

$$\Pr\{ \max_k S_k \geq 0 \geq \tau \} = \sum_{j=\lfloor \tau/2 \rfloor}^{\infty} \binom{2j}{j} (pq)^j (q-p) \leq \frac{q-p}{\sqrt{\pi}} \sum_{j=\lfloor \tau/2 \rfloor}^{\infty} (4pq)^j \leq \frac{q-p}{\sqrt{\pi}} (4pq)^{\tau/2},$$

where the first inequality follows from $\binom{2j}{j} \leq \frac{4j}{\sqrt{\pi}}$ when $j \geq 1$ [9]. The last step follows from $1 - 4pq = (q-p)^2$. As discussed in the proof of Lemma 4.6, $|y_i - \tilde{y}_i|$ can be bounded by two random walks each with $p$ at most $\frac{1}{\tau + 1}$. \qed

Lemma 4.9. Let $\gamma = \frac{a+1}{1+\alpha (a-1)^{1/2}} - 1$, $p = \frac{1}{\tau + 1}$ and $q = 1 - p$. With probability at least $1 - \psi$ for Algorithm 3 we have:

$$\Pr\{|y_i - \tilde{y}_i| \leq 2 \log \frac{2(q-p)\sqrt{\pi}}{\log (1/(4pq))} \leq \tau \} \geq 1 - \psi.$$

Proof. We set $\psi = \frac{2}{(q-p)\sqrt{\pi}} (4pq)^{\tau/2}$ and isolate $\tau$ as follows:

$$\frac{2(q-p)\sqrt{\pi}}{2} = (4pq)^{\tau/2} \log \left( \frac{2(q-p)\sqrt{\pi}}{2} \right) = \log (1/(4pq)) \cdot \frac{\tau}{2} \leq 2 \log \frac{1}{(4pq)} = \tau.$$

By Lemma 4.8 we have: $\Pr\{|y_i - \tilde{y}_i| \leq \tau \} \geq 1 - \psi$. \qed

Up to constant factors, the tail probabilities of our mechanism are similar to the properties of the Laplace mechanism summarized in Proposition 2.7. The probabilities depend on the parameters of the mechanism. In Section 6 we fix the parameters and evaluate the error in practice. We summarize the tail probabilities for $|x_i - \tilde{x}_i|$ in Lemma 4.10.

Lemma 4.10. Let $\gamma = \frac{a+1}{1+\alpha (a-1)^{1/2}} - 1$, $p = \frac{1}{\tau + 1}$, $q = 1 - p$, $x_i \leq \beta$, and $\tau \geq (\alpha - 1)$. Then for Algorithm 3 we have:

$$\Pr\{ |x_i - \tilde{x}_i| \geq \tau \} \leq \frac{2}{(q-p)\sqrt{\pi}} (4pq)^{\tau/(2\alpha - 2) - 1/2},$$

Algorithm 4: ALP-projection

Parameters: $\epsilon, \beta > 0$, $\alpha > 1$, and $s \in \mathbb{N}$.
Input: $k$-sparse vector $x \in \mathbb{R}^d$, where $s > 2k$.
Sequence of hash functions from domain $[d]$ to $[s]$, $h = (h_0, \ldots, h_{m-1})$, where $m = \left\lceil \frac{\beta}{\alpha - 1} \right\rceil$.
Output: $\epsilon$-differentially private representation of $x$.
(1) Scale the entries of $x$ such that $\tilde{x}_i = x_i \cdot \epsilon$.
(2) Let $h, \tilde{z} = \text{ALP1-projection}_{\alpha, \beta, \epsilon, s} (x, h)$.
(3) Release $\tilde{h}$ and $\tilde{z}$.

Algorithm 5: ALP-estimator

Parameters: $\epsilon > 0$ and $\alpha > 1$.
Input: Embedding $\tilde{z} \in [0,1]^m$. Sequence of hash functions $h = (h_0, \ldots, h_{m-1})$. Index $i \in [d]$.
Output: Estimate of $x_i$.
(1) Let $\tilde{x}_i = \text{ALP1-estimator}_{\epsilon} (\tilde{z}, h, i)$.
(2) Return $\frac{\tilde{x}_i}{\epsilon}$.

With probability at least $1 - \psi$ we have:

$$|\tilde{x}_i - x_i| < \left( 1 + \frac{2 \log \left( \frac{2}{\log (1/(4pq))} \right)}{\log (1/(4pq))} \right) \cdot (\alpha - 1)$$

Proof. It is easy to see that $|\tilde{x}_i - x_i| < (1 + |y_i - \tilde{y}_i|) \cdot (\alpha - 1)$ holds, as the error of random rounding is strictly less than 1. The bounds follow from Lemmas 4.8 and 4.9. \qed

4.3 From 1-differential privacy to $\epsilon$-differential privacy

We now generalize the ALP mechanism to satisfy $\epsilon$-differential privacy. A natural approach is to use a function of $\epsilon$ as the parameter for randomized response in Algorithm 2. The projection algorithm is $\epsilon$-differentially private if we remove the scaling step and set $p = \frac{1}{(\epsilon+1)^2}$. However, the expected per-entry error would be bounded by $O \left( \frac{1}{(\epsilon+1)^2} \right)$ by Equation 1 (without considering hash collisions), which is as large as $O \left( \frac{1}{\epsilon^2} \right)$ for small values of $\epsilon$. Other approaches modifying the value of $p$ have a similar expectation.

In the following, we use a simple pre-processing and post-processing step to achieve optimal error. The idea is to scale the input vector as well as the parameter $\beta$ by $\epsilon$ before running Algorithm 2. We scale back the estimates from Algorithm 3 by $1/\epsilon$. These generalizations are given as Algorithm 4 and Algorithm 5, respectively.

Lemma 4.11. Algorithm 4 satisfies $\epsilon$-differential privacy.

Proof. It follows from the proof of Lemma 4.1 that for any subset of outputs $S$ we have $\Pr[|\text{ALP-projection}(x')| \in S] \leq e^{\epsilon \cdot |x-x'|_1}$. As such for any pair of neighboring vectors $x$ and $x'$ we have:

$$\Pr[|\text{ALP-projection}(x') - \text{ALP-projection}(x)| \in S] \leq e^{\epsilon \cdot |x-x'|_1} = e^{\epsilon \cdot |x-x'|_1} \leq e^\epsilon.$$
Algorithm 6: Threshold

Parameters: \( \varepsilon, t > 0 \).
Input: \( k \)-sparse vector \( x \in \mathbb{R}^d \).
Output: \( \varepsilon \)-differentially private representation of \( x \).

\( 1 \) Let \( \tilde{v}_i = x_i + \eta_i \) for all \( i \in [d] \), where \( \eta_i \sim \text{Lap}(1/\varepsilon^2) \).

\( 2 \) Remove entries below \( t \) such that:
\[
\tilde{v}_i = \begin{cases} 
\eta_i, & \text{if } y_i \geq t \\
0, & \text{otherwise}
\end{cases}
\]

\( 3 \) Return \( \tilde{v} \).

Algorithm 7: Threshold ALP-projection

Parameters: \( \varepsilon_1, \varepsilon_2 > 0 \), \( \alpha > 1 \), and \( s \in \mathbb{N} \).
Input: \( k \)-sparse vector \( x \in \mathbb{R}^d \), where \( s > 2k \).
Sequence of hash functions from domain \( [d] \) to \( [s] \), \( h = (h_0, \ldots, h_{m-1}) \), where \( m = \left\lceil \frac{\beta s}{\alpha-1} \right\rceil \).
Output: \( (\varepsilon_1 + \varepsilon_2) \)-differentially private representation of \( x \).

\( 1 \) Let \( t = \frac{\ln(d/2)}{\varepsilon_1} \).

\( 2 \) Let \( \tilde{x} \) be Threshold\(,t(x) \).

\( 3 \) Let \( h, \tilde{z} \) be ALP-projection\(,\alpha,t,s(x,h) \).

\( 4 \) Return \( \tilde{v}, h \) and \( \tilde{z} \).

Lemma 5.1. Algorithm 6 satisfies \( \varepsilon \)-differential privacy.

Proof. The algorithm is equivalent to the Laplace mechanism followed by post-processing. It follows directly from Lemma 2.2 that the algorithm satisfies \( \varepsilon \)-differential privacy.

Lemma 5.2. Let \( t = \frac{\ln(d/2)}{\varepsilon_1} \). Then Algorithm 6 returns a \( O(k) \)-sparse vector with high probability.

Proof. Using Definition 2.5 we find that the probability of storing a zero entry is:
\[
\text{Pr}[\text{Lap}(1/\varepsilon) \geq t] = \text{Pr}[\text{Lap}(1/\varepsilon) \leq -t] = \frac{1}{2} e^{-t \varepsilon} = \frac{1}{d}.
\]
By linearity of expectation, the expected number of stored true zero entries at most one and as such the vector is \( O(k) \)-sparse with high probability.

As discussed in Section 3, the expected per-entry error for the Algorithm 6 is \( O\left( \frac{\log(d)}{\varepsilon} \right) \) for worst-case input. We combine the algorithm with the ALP mechanism from the previous section to achieve \( O(1/\varepsilon) \) expected per-entry error for any input. We use the threshold parameter \( t \) as value for parameter \( \beta \) in Algorithm 4. The algorithm is presented in Algorithm 7.

Lemma 5.3. Algorithm 7 satisfies \( (\varepsilon_1 + \varepsilon_2) \)-differential privacy.

Proof. The two parts of the algorithm are independent as there is no shared randomness. The first part of the algorithm satisfies \( \varepsilon_1 \)-differential privacy by Lemma 5.1 and the second part satisfies \( \varepsilon_2 \)-differential privacy by Lemma 4.11. As such it follows directly from composition (Lemma 2.3) that Algorithm 7 satisfies \( (\varepsilon_1 + \varepsilon_2) \)-differential privacy.
Algorithm 8: Threshold ALP-estimator

**Parameters:** $\varepsilon > 0$ and $\alpha > 1$.

**Input:**
- Vector $\hat{\delta} \in \mathbb{R}^d$. Embedding $\hat{z} \in \{0, 1\}^{m \times n}$.
- Sequence of hash functions $h = (h_0, \ldots, h_{m-1})$. Index $i \in [d]$.

**Output:** Estimate of $\hat{x}_i$.

1. Estimate the entry using either the vector or the embedding such that:
   \[
   \hat{x}_i = \begin{cases} 
   \hat{\delta}_i, & \text{if } \hat{\delta}_i \neq 0 \\
   \text{ALP-estimator}_{\varepsilon, \alpha}(\hat{z}, h, i), & \text{otherwise}
   \end{cases}
   \]

2. Return $\hat{x}_i$.

**Proof.** We can store a $O(k)$-sparse vector using $O(k \log (d + u))$ bits. It follows from Lemma 5.2 that we can store $\hat{\delta}$ using $O(k \log (d + u))$ bits with high probability. Since $\beta = t$ it follows from Lemma 4.12 that we can store $h$ and $z$ using $O(k \log (d))$ bits.

To estimate an entry, we use $\hat{\delta}$ when possible and the ALP embedding otherwise. This algorithm is presented in Algorithm 8.

Lemma 5.4. Let $\alpha = \Theta(1)$, $s = \Theta(k)$, and $\varepsilon_1 = \Theta(\varepsilon_2)$. Then the output of Algorithm 7 is stored using $O(k \log (d + u))$ bits with high probability.

**Proof.** The evaluation time follows from Lemma 4.13. That is, the evaluation time is $O(\beta t) = O(t \varepsilon) = O(\log (d))$.

The error depends on both parts of the algorithm. The expected per-entry error for the $i$th entry is $\max(0, x_i - \beta) + O(1/\varepsilon_2)$ when $\hat{x}_i = 0$ by Lemma 4.13. That is, when $\eta_i$ is less than $\beta - x_i$ in Algorithm 6. When $\hat{x}_i \neq 0$ the error is the absolute value of $\eta_i$. That is, we can analyze it using the probability density function of the Laplace distribution from Definition 2.4.

The expected maximum error of Algorithm 6 is $O\left(\frac{\log (d)}{\varepsilon} \right)$ and the output of the ALP mechanism is at most $\beta$. Since $\beta = O\left(\frac{\log (d)}{\varepsilon} \right)$ the expected maximum error is $O\left(\frac{\log (d)}{\varepsilon} \right)$. The expected maximum error of Algorithm 6 is $O\left(\frac{\log (d)}{\varepsilon} \right)$ and the output of the ALP mechanism is at most $\beta$. Since $\beta = O\left(\frac{\log (d)}{\varepsilon} \right)$ the expected maximum error is $O\left(\frac{\log (d)}{\varepsilon} \right)$.

Algorithm 9: Threshold2

**Parameters:** $\varepsilon, \delta > 0$.

**Input:** $k$-sparse vector $x \in \mathbb{R}^d$.

**Output:** $(\varepsilon, \delta)$-differentially private approximation of $x$.

Applying random rounding to non-zero entries below 1 such that:

\[
y_i = \begin{cases} 
\text{RandRound}(x_i), & \text{if } 0 < x_i < 1 \\
x_i, & \text{otherwise}
\end{cases}
\]

(2) Let $y_i = x_i + \eta_i$ for all non-zero entries, where $\eta_i \sim \text{Lap}(1/\varepsilon)$.

(3) Let $t = \frac{\ln (1/(2\delta))}{\varepsilon} + 1$.

(4) Remove entries below $t$ such that:

\[
y_i = \begin{cases} 
y_i, & \text{if } y_i \neq 0 \text{ and } y_i \geq t \\
0, & \text{otherwise}
\end{cases}
\]

(5) Return $\hat{v}$.

5.1 An approximate differentially private version

To improve access time, we can turn to approximate differential privacy. This allows us to use a smaller threshold in the initial thresholding approach, which in turn results in smaller values for $\beta$ in the ALP mechanism.

The following algorithm is similar to that introduced by Korolova et al. [13], which we discussed in Section 3. It removes zero entries, adds noise to non-zero entries, and uses a threshold to satisfy approximate differential privacy. Our algorithm differs from the work of Korolova et al. by using a random rounding step. This step is not needed in a discrete setting, where at most a single zero-valued entry is changed to a non-zero entry for neighboring vectors. However, in the real-valued context, several zero entries can change.

Lemma 5.6. Algorithm 9 satisfies $(\varepsilon, \delta)$-differential privacy.

**Proof.** Let $x$ and $x'$ be neighboring vectors. We consider two additional vectors $\hat{x}$ and $\hat{x}'$ such that:

\[
\hat{x} = \begin{cases} 
\min(1, x_i), & \text{if } x_i \leq 1 \\
x_i, & \text{otherwise}
\end{cases}
\]

\[
\hat{x}' = \begin{cases} 
1, & \text{if } x_i > 1 \land x_i' < 1 \\
x_i', & \text{otherwise}
\end{cases}
\]

The vectors are constructed such that $x$ and $\hat{x}$ can only differ for entries at most 1 in both vectors. The same holds for $x'$ and $\hat{x}'$. The expected maximum error is $O\left(\frac{\log (d)}{\varepsilon} \right)$.
Additionally, the $\ell_1$-distance is still at most 1 between any pair of vectors.

We can find the probability of outputting any entry at most 1 as:

\[
\Pr[\hat{y}_i \neq 0 | x_i \leq 1] = x_i \cdot \Pr[\text{Lap}(1/\epsilon) \geq t - 1] = x_i \cdot \Pr[\text{Lap}(1/\epsilon) \leq -(t-1)] = x_i \cdot \frac{1}{2} e^{-(t-1)\epsilon} = x_i \cdot \frac{1}{2} e^{-\ln(1/(2\delta)) - 2\epsilon} = x_i \cdot \frac{\delta}{2 \cdot e^{2\epsilon}}.
\]

Since $x$ and $\hat{x}$ only differ for entries at most 1 we have for any subset of outputs $S$:

\[
\Pr[\text{Threshold2}(x) \in S] \leq \Pr[\text{Threshold2}(\hat{x}) \in S] + \sum_{i \in [d]} |\hat{x}_i - x_i| \cdot \frac{\delta}{2 \cdot e^{2\epsilon}} \leq \Pr[\text{Threshold2}(\hat{x}) \in S] + \frac{\delta}{2 \cdot e^{2\epsilon}}.
\]

The inequality holds in both directions and for the pair of $x'$ and $\hat{x}$ as well.

By definition $\hat{x}$ and $x'$ only differ for entries of at least 1. As such we can ignore the random rounding step and we have:

\[
\Pr[\text{Threshold2}(\hat{x}) \in S] \leq e^{\|x' - \hat{x}\|_1} \cdot \epsilon \cdot \Pr[\text{Threshold2}(\hat{x}') \in S] \leq e^{\epsilon} \cdot \epsilon \cdot \Pr[\text{Threshold2}(x) \in S].
\]

Using the results above we have:

\[
\Pr[\text{Threshold2}(x) \in S] \leq \Pr[\text{Threshold2}(\hat{x}) \in S] + \frac{\delta}{2 \cdot e^{2\epsilon}} \leq e^{\epsilon} \cdot \Pr[\text{Threshold2}(\hat{x}') \in S] + \frac{\delta}{2 \cdot e^{2\epsilon}} \leq e^{\epsilon} \cdot \Pr[\text{Threshold2}(x') \in S] + \frac{\delta}{2 \cdot e^{2\epsilon}} + \frac{\delta}{2 \cdot e^{2\epsilon}} \leq e^{\epsilon} \cdot \Pr[\text{Threshold2}(x') \in S] + \delta.
\]

\[\square\]

**Lemma 5.7.** Let $\delta = O(1/\epsilon)$. Then the expected maximum error of Algorithm 9 is $O\left(\frac{\log(1/\delta)}{\epsilon}\right)$.

**Proof.** The expected maximum error added by the Laplace noise is $O\left(\frac{\log(k)}{\epsilon}\right)$, since we add noise to at most $k$ entries. By removing entries we add error of $O\left(\frac{\log(1/\delta)}{\epsilon}\right)$ for worst case input. As such the expected maximum error is at most:

\[
\mathbb{E}[\|x - \hat{x}\|] \leq O\left(\frac{\log(k)}{\epsilon}\right) + O\left(\frac{\log(1/\delta)}{\epsilon}\right) = O\left(\frac{\log(1/\delta)}{\epsilon}\right).
\]

\[\square\]

In the following, we use Algorithm 9 instead of Algorithm 6 in Algorithm 7.

\[\]
each experiment and the same probability is used for all bits. This simulates worst-case input in which all other non-zero entries have a true value of at least \( \beta \). We increment \( \alpha \) by steps of 0.1 in the interval \([1.1, \ldots, 10]\) and the probability of a hash collision by 0.05 in the interval \([0, \ldots, 0.2]\). The probability of 0 serves only as a baseline, as it is not achievable in general for \( k > 1 \). The experiment was repeated \( 10^5 \) times for every data point.

Figure 3(b) shows plots of the mean absolute error of the experiments. As \( \alpha \) is increased, the error drops off at first and slowly climbs. The estimates of \( y_i \) are more accurate for large values of \( \alpha \). However, any inaccuracy is more significant, as \( y_i \) is scaled back by a larger value. There the error from the random rounding step also increases. The plots of the upper bound and observed error follow similar trajectories. However, the upper bound is approximately twice as large for most parameters.

**Fixed parameter.** The experiments show how different values of \( \alpha \) and \( \beta \) affect the expected per-entry error. However, the parameters also determine constant factors for space usage and access time. The space requirements scale linearly in \( \frac{1}{\alpha} \) and the access time scales linearly in \( \frac{1}{\alpha^2} \). As such, the optimal parameter choice differs depending on use cases due to space, access time, and error trade-offs.

To evaluate the error distribution of the ALP1-estimator algorithm we fixed the parameters of an experiment. We set \( \alpha = 4 \) and the hash collision probability to 0.1. We repeated the experiment \( 10^6 \) times.

The error distribution is shown in Figure 3(c). The mean absolute error of the experiment is 6.4 and the standard deviation is 11.

Plugging in the parameters in Lemma 4.10, with probability at least 90% the error is at most

\[
|x_i - \hat{x}_i| < 3 + \frac{6}{\log \left( \frac{25}{1.2 \sqrt{0.1}} \right)} \log \left( \frac{5}{1.2 \sqrt{0.1}} \right) \approx 75.33.
\]

The error of the observed 90th percentile is 15.78, which is shown in Figure 3(c) using vertical lines. Again, this shows that the upper bounds are pessimistic.

For comparison, the plots include the Laplace distribution with scale parameters 1 and 6. Note that the Laplace distribution with parameter 1 is optimal for the privacy budget.

The distribution is slightly off-center, and the mean error is 2.33. This is expected due to hash collisions. The effect of hash collisions is also apparent for the largest observed errors. The lowest observed error was \(-114\), while the highest was 274. There is a clear trade-off between space usage and per-entry error. We reran the experiment with hash collision probability 0.01 using the same value for \( \alpha \). The error improved for all the metrics mentioned above. The absolute error is 4.8, the standard deviation is 7.8, the mean error is 0.18, the 90th percentile is 11.5, and the largest observed error is 147.

**REFERENCES**

A CLOSED-FORM PROOF OF LEMMA 4.5

Here we provide a closed-form expression used in the proof of Lemma 4.5.

In the proof, we will make use of general binomial coefficient([10, Equation 5.1]):

\[
\binom{r}{k} = \frac{r(r-1)\ldots(r-k+2)(r-k+1)}{k!},
\]

and the binomial theorem ([10, Equation 5.12]):

\[
(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} (z)^k.
\]

Starting from an infinite series with \(z < 1/4\), we simplify as follows:

\[
\sum_{k=0}^{\infty} k^{2k} \binom{2k}{k} (z)^k = \sum_{k=1}^{\infty} \frac{(2k)!}{k!} z^k
\]

\[
= \sum_{k=1}^{\infty} k(k-\frac{1}{2})(k-1)\ldots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) \left(\frac{1}{2}\right) \left(\frac{1}{2}\right) 2^{2k} z^k
\]

\[
= \sum_{k=1}^{\infty} (k-\frac{1}{2})(k-\frac{3}{2})\ldots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) (4z)^k
\]

\[
= \frac{4z}{2} \sum_{k=1}^{\infty} (k-\frac{1}{2})(k-\frac{3}{2})\ldots\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) (4z)^{k-1}
\]

\[
= 2z \sum_{k=1}^{\infty} \left(-\frac{3}{2}\right)\ldots\left(-k+\frac{3}{2}\right)(-k+\frac{1}{2}) (4z)^{k-1}
\]

\[
= 2z \sum_{k=0}^{\infty} \left(-\frac{3}{2}\right)\ldots\left(-k+\frac{3}{2}\right)(-k+\frac{1}{2}) (4z)^{k-1}
\]

\[
= 2z \sum_{k=0}^{\infty} \left(-\frac{3}{k}\right) (4z)^k
\]

\[
= \frac{2z}{(1-4z)^{3/2}}.
\]

Let \(p = \frac{a}{a+b}\) and \(q = \frac{b}{a+b}\). Then we have:

\[
1 - 4pq = \frac{(a+b)^2}{(a+b)^2} - \frac{4ab}{(a+b)^2}
\]

\[
= \frac{a^2 + b^2 - 2ab}{(a+b)^2}
\]

\[
= \frac{(b-a)^2}{(a+b)^2}
\]

\[
= (q-p)^2.
\]
Finally, let $p < q$ and let $z = pq$. This gives us the closed-form expression:

\[
\sum_{k=0}^{\infty} \binom{2k}{k} (pq)^k = \frac{2pq}{(1 - 4pq)^{3/2}}
\]

\[
= \frac{2pq}{((q - p)^2)^{3/2}}
\]

\[
= \frac{2pq}{(q - p)^3}.
\]