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Maiani, Andrea; Kjaergaard, Morten; Schrade, Constantin

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Entangling Transmons with Low-Frequency Protected Superconducting Qubits

Andrea Maiani, Morten Kjaergaard, and Constantin Schrade
Center for Quantum Devices, Niels Bohr Institute, University of Copenhagen, Copenhagen 2100, Denmark

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Novel qubits with intrinsic noise protection constitute a promising route for improving the coherence of quantum information in superconducting circuits. However, many protected superconducting qubits exhibit relatively low transition frequencies, which could make their integration with conventional transmon circuits challenging. In this work, we propose and study a scheme for entangling a tunable transmon with a Cooper-pair parity-protected qubit, a paradigmatic example of a low-frequency protected qubit that stores quantum information in opposite Cooper-pair parity states on a superconducting island. By tuning the external flux on the transmon, we show that noncomputational states can mediate a two-qubit entangling gate that preserves the Cooper-pair parity independent of the detailed pulse sequence. Interestingly, the entangling gate bears similarities to a controlled-phase gate in conventional transmon devices. Hence, our results suggest that standard high-precision gate calibration protocols could be repurposed for operating heterogeneous quantum processors.

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Superconducting transmon qubits [1] are a highly promising platform for noisy intermediate-scale quantum (NISQ) devices [2] and error-corrected quantum computers [3–6] with applications ranging from quantum simulations [7–10] to the first experiments on quantum advantage [11,12]. Among the most attractive features of the transmon circuit are its reproducibility, insensitivity to charge-noise-induced dephasing, and coherence times that have seen steady improvements over the past decade [13]. Interestingly though, despite notable advances in prolonging the coherence of transmons, the transmon circuit does not exhibit intrinsic protection to qubit-relaxation errors. It is thus an important question how transmon devices can be further optimized with complementary qubit technologies to accelerate the path to fault-tolerant quantum computation.

Motivated by the challenge of exploring complementary qubit modalities, several alternative qubit encodings have been proposed [14–36] and experimentally studied [37–49]. A particular class of such novel qubit encodings are Cooper-pair parity-protected qubits (PPQs) [34–38], which rely on a special Josephson element that only permits tunneling of pairs of Cooper pairs. Similar to the transmon qubit, the two nearly degenerate ground states of the PPQ have a nearly flat charge dispersion, which makes them insensitive to charge-noise-induced dephasing. Similar to the fluxonium qubit [43–48], the two qubit states also have disjoint support, since they carry opposite Cooper-pair parity. This disjoint support ensures that if the qubit-environment coupling conserves the Cooper-pair parity, relaxation errors between the computational states are prevented.

While considerable efforts have been devoted to the development of a gate set for protected superconducting qubits [16,50–54] for their use as an independent quantum computing platform, a different approach is to integrate protected qubits as memory elements in a conventional transmon-based quantum computing architecture. Such heterogeneous superconducting quantum processing architectures have recently gained increased attention [34,55]. In such a scheme, the qubit state would be stored on the protected qubit during idle times and transferred to the transmon qubits for fast high-fidelity operations, using the full machinery of well-established high-fidelity transmon operation. However, many protected qubits exhibit relatively low qubit transition frequencies, which could make this integration with transmon devices challenging. This motivates the question of efficiently generating entanglement between protected superconducting qubits and transmon qubits.

In this work, we propose and study a capacitive coupling scheme for entangling a tunable transmon with a PPQ, a paradigmatic example of a protected superconducting qubit featuring a nearly flat charge dispersion and qubit states with disjoint support. By tuning the external flux on the transmon, we show that noncomputational states can
mediate an entangling gate that preserves the Cooper-pair parity irrespective of the detailed pulse sequence. Besides opening the way to coherent state transfer, the proposed entangling gate also bears similarities with a controlled-phase gate in conventional capacitively coupled transmon qubits. Consequently, our results suggest standard high-precision two-qubit calibration protocols could be repurposed for the operation of hybrid qubit devices.

I. SETUP

As depicted in Fig. 1(a), we consider a direct capacitively coupling between a frequency-tunable transmon qubit and a PPQ, realized by a capacitively shunted $\cos(2\phi)$ Josephson element for the tunneling of pairs of Cooper pairs. The individual Hamiltonians of the transmon, $H_t$, and of the PPQ, $H_p$, are given by

$$H_t = 4E_{C,t}(n_t - n_{g,t})^2 - E_{J,t} \cos(\phi_t),$$

$$H_p = 4E_{C,p}(n_p - n_{g,p})^2 - E_{J,p} \cos(2\phi_p).$$

(1)

Here, $(n_t, \phi_t)$ and $(n_p, \phi_p)$ denote Cooper-pair charge and phase degrees of freedom of the transmon and PPQ. Moreover, $E_{J,t}$ is the transmon Josephson energy and $E_{J,p}$ is the two-Cooper-pair tunneling amplitude of the PPQ. The charging energies of the two qubit circuits are $E_{C,t} = e^2/2C_t$ and $E_{C,p} = e^2/2C_p$, with the shunt capacitances $C_t$ and $C_p$.

Both Hamiltonians in Eq. (1) can be diagonalized exactly by rewriting the eigenvalue problems as Mathieu equations. For the transmon [1], the energy splitting between the ground and first excited state, which form the qubit basis $|0\rangle_t$ and $|1\rangle_t$, is $\omega_t = \sqrt{8E_{J,t}E_{C,t}} + \delta\omega_t$ with $\delta\omega_t \propto \exp(-\sqrt{8E_{J,t}/E_{C,t}}) \cos(2\pi n_{g,t})/E_{J,t} \gg E_{C,t}$ [see the left panel of Fig. 1(b)]. Here and in the following discussions, we put $\hbar = 1$. For the PPQ [35], the qubit basis is given by the lowest-energy states with even and odd Cooper-pair parity, $|0\rangle_p$ and $|1\rangle_p$. These states have an exponentially suppressed energy splitting, $\omega_p \propto \exp(-\sqrt{2E_{J,p}/E_{C,p}}) \cos(\pi n_{g,p})/E_{J,p} \gg E_{C,p}$ [see the right panel of Fig. 1(b)]. Unlike the transmon, the PPQ is thus a low-frequency qubit with the computational states exhibiting an exact degeneracy if $\cos(\pi n_{g,p}) = 0$ and a near-degeneracy otherwise. However, like the transmon, the energy splitting of the PPQ is insensitive to variations in $n_{g,p}$ if $E_{J,p} \gg E_{C,p}$, which ensures insensitivity to charge-noise dephasing.

To explain the protection of the PPQ against parity-preserving relaxation errors, we consider the wave functions of the computational basis. In phase space, these wave functions are symmetric or antisymmetric combinations of states that are localized in the 0 and $\pi$ valleys of the Josephson potential [see Fig. 1(c)]. In charge space, the same wave functions are superpositions of states with even or odd Cooper-pair numbers. Due to this disjoint support of the charge-space wave functions, $\langle 0_p | \tilde{O} | 1_p \rangle = 0$ for any operator $\tilde{O}$ that preserves the Cooper-pair parity, which is the condition for protection against parity-preserving relaxation errors [34].

Having introduced the two decoupled qubit circuits with the associated computational subspace $\mathcal{P}_0 = \{|1_t, 1_p\}, |1_t, 0_p\}, |0_t, 1_p\}, |0_t, 0_p\}$, we proceed by coupling the qubits via a standard capacitive coupling [see also Fig. 1(a)] corresponding to a coupling Hamiltonian given by

$$H_c = 4E_{C,\phi}(n_p - n_{g,p})(n_t - n_{g,t}).$$

(2)

Here, $E_{C,\phi} = e^2C_c/(C_pC_t)$, with the coupling capacitance $C_c$. We note that there are two main advantages of the proposed capacitive coupling: (1) it always conserves the Cooper-pair parity on the PPQ, ensuring the protection of the PPQ against relaxation errors; and (2) it is compatible with standard transmon technology and is routinely used for coupling transmons in state-of-the-art architectures [13]. Despite these possible advantages of a capacitive coupling, we acknowledge that inductive couplings

![Image](https://example.com/image.png)
have also been studied for entangling transmons, such as the “gmon” circuit [56]. However, since the inductive coupling in the “gmon” arises from a conventional Josephson junction, we expect that maintaining the Cooper-pair parity conservation could be a challenge when generalizing such a scheme to a PPQ-transmon hybrid system.

To conclude, the full Hamiltonian of our setup is $H = H_p + H_t + H_c$. In the next section, we derive the effective qubit interaction due to this direct capacitive coupling.

II. EFFECTIVE HAMILTONIAN

To motivate the derivation of the effective qubit interaction, we first recall the case of two capacitively coupled transmon qubits, $r_1$ and $r_2$, which are both “high-frequency” qubits. In this example, the capacitive coupling mediates a $\sigma^y_1 \sigma^y_2$ interaction when projected onto the computational subspace and a $\sigma^z_1 \sigma^z_2$ interaction due to the mixing of computational and noncomputational states [57]. In our setup, which involves the coupling of a “high-frequency” transmon qubit and a “low-frequency” PPQ, we show that the couplings to noncomputational states play an even more essential role.

We anticipate this result, we note that the coupling Hamiltonian of Eq. (2) at $n_{g,p} = 0$ vanishes exactly when projected onto the computational subspace, $\langle s, s_p' | H_c | s, s_p'' \rangle = 0$ for any two states $|s, s_p'\rangle, |s, s_p''\rangle$ in $\mathcal{P}_0$, since $\langle 0 | n_l | 0 \rangle = \langle 1 | n_l | 1 \rangle = 0$ and $\langle 0 | n_p | 1 \rangle = 0$. A direct coupling within the computational subspace is thus fully absent at $n_{g,p} = 0$ and any qubit interaction, if present, is necessarily mediated by virtual transitions through noncomputational states.

A. Special case: $n_{g,p} = 0$

To identify the origin of such virtual transitions, we initially compare two special cases with the offset charge on the PPQ set to either $n_{g,p} = 0$ or $n_{g,p} = 0.5$. Starting with the $n_{g,p} = 0$ case, we show the low-energy spectrum as a function of external magnetic flux $\Phi^{ext}_0$ of the tunable transmon in Fig. 2(a). The spectrum comprises not only the four qubit levels of $\mathcal{P}_0$ but also two additional levels corresponding to the $|0, 3_p\rangle$ state of the uncoupled system. Interestingly, the noncomputational states exhibit two anticrossings with the computational states, $|1, 1_p\rangle \leftrightarrow |0, 2_p\rangle$ and $|1, 0_p\rangle \leftrightarrow |0, 3_p\rangle$, at certain values of external flux. These anticrossings arise because the respective couplings preserve the Cooper-pair parity on the PPQ. On the other hand, anticrossings between $|1, 0_p\rangle \leftrightarrow |0, 2_p\rangle$ and $|1, 1_p\rangle \leftrightarrow |0, 3_p\rangle$ are absent from the spectrum in Fig. 2(a), as such couplings violate the Cooper-pair parity conservation on the PPQ. We now show that in the vicinity of the two anticrossings, virtual transitions in and out of the computational subspace are enhanced and, consequently, can induce a sizable effective qubit interaction between the transmon and the PPQ.

For computing the effective qubit interaction at $n_{g,p} = 0$, we initially project our setup Hamiltonian $H$ onto the four qubit states of $\mathcal{P}_0$ and on the additional $|0, 2_p\rangle$ and $|0, 3_p\rangle$ states. This yields the following low-energy Hamiltonian:

$$H_{low}^{(n_{g,p}=0)} = \begin{pmatrix}
\omega_{11} & 0 & 0 & 0 & \lambda' & 0 \\
0 & \omega_{10} & 0 & 0 & 0 & -\lambda'' \\
0 & 0 & \omega_{01} & 0 & 0 & 0 \\
0 & 0 & 0 & \omega_{00} & 0 & 0 \\
\lambda' & 0 & 0 & 0 & \omega_{02} & 0 \\
0 & -\lambda'' & 0 & 0 & 0 & \omega_{03}
\end{pmatrix}.$$  

Here, the frequency of the $|s, s_p'\rangle$ state in the uncoupled system is denoted by $\omega_{ss'} = \omega_s(\Phi_{ext}^0) + \omega_{p,s'}$. 

FIG. 2. The low-energy spectrum of the hybrid qubit setup and the CZ gate. (a) The low-energy spectrum of the hybrid qubit setup for $n_{g,p} = 0$ and $(E_{1,1}, E_{1,0}, E_{C,1}, E_{C,0}, E_{C,C}) = 2\pi(12, 2.7, 0.2, 0.18, 0.025)$ GHz as a function of the external flux $\Phi_{ext}$ of the tunable transmon. The CZ gate is realized by a rapid excursion from $\Phi_{ext} = 0$ to the vicinity of the $|1, 0_p\rangle \leftrightarrow |0, 3_p\rangle$ anticrossing at $\Phi_{ext} = \Phi_{ext}$. The bottom panel shows the coupling strengths, $g^{zz}$ and $g^{zz'}$, upon approaching the anticrossing. (b) The same as (a) but for $n_{g,p} = 0.5$. Each energy level shown is now exactly twofold degenerate.
Moreover, the coupling matrix elements are given by \( \lambda' = \langle 1, 1_p | H_c | 0, 2_p \rangle \) and \( \lambda'' = \langle 1, 0_p | H_c | 0, 3_p \rangle \), where we pick a wave-function gauge for which \( (\lambda', \lambda'') \) are real valued. We point out that the low-energy Hamiltonian of Eq. (3) is different from that of capacitively coupled transmons [57], because conservation of the Cooper-pair parity prohibits a coupling of the |0, 1_p⟩ to the |1, 0_p⟩ state. Also, for two coupled transmons, only the highest-energy computational state exhibits crossing with noncomputational states. In our case, the two computational states, |1, 0_p⟩ and |1, 1_p⟩, both cross with noncomputational states, albeit at different values of external flux.

Next, we integrate out the noncomputational states to second order in \( \lambda' \) and \( \lambda'' \) by a Schrieffer-Wolff transformation [58]. Provided that \( \lambda'^2 \ll |\omega_{02} - \omega_{11}| \) and \( \lambda'' \ll |\omega_{03} - \omega_{10}| \), we find that the effective qubit Hamiltonian reads (see the Supplemental Material [59])

\[
H_{\text{eff}}^{(ng,p=0)}(0) = \left( \omega_t + \frac{\gamma_z^+}{2} \right) \sigma_t^+ + \left( \omega_p + \frac{\gamma_z^-}{2} \right) \sigma_p^+ \\
+ \frac{\gamma_z^+}{2} \sigma_t^+ \sigma_p^+,
\]

(4)

where \( \omega_{0/t} = \omega_{p/t,1} - \omega_{p/t,0} \) denote the bare qubit frequencies. The key insight from Eq. (4) is that the interaction between the two qubits is of \( \sigma_t^+ \sigma_p^+ \) type. As anticipated, this interaction arises from a two-step perturbative sequence involving virtual transitions in and out of the \( \{ 0, 2_p \} \) and \( \{ 0, 3_p \} \) states. For example, in a perturbative sequence close to the \( |1, 1_p⟩ \leftrightarrow |0, 2_p⟩ \) anticrossing, the system exhibits a first virtual transition from the computational state \( |1, 1_p⟩ \) to the noncomputational state \( |0, 2_p⟩ \) and, subsequently, a second virtual transition back to \( |1, 1_p⟩ \). Such a sequence preserves the state of the transmon, which explains why the interaction is \( \propto \sigma_t^+ \). The dependence of the interaction on \( \sigma_t^+ \) arises because the coupling Hamiltonian of Eq. (2) preserves the Cooper-pair parity.

B. Special case: \( n_{g,p} = 0.5 \)

Having derived the qubit interaction at \( n_{g,p} = 0 \), we want to compare the results of Eq. (4) with the \( n_{g,p} = 0.5 \) case. We therefore plot the low-energy spectrum at \( n_{g,p} = 0.5 \) in Fig. 2(b). Unlike in the previous case, we find that each depicted energy level exhibits an exact twofold degeneracy, corresponding to opposite Cooper-pair parity sectors. This finding is consistent with our results of Fig. 1(b), where we point out that the levels on the uncoupled PPQ are exactly degenerate at \( n_{g,p} = 0.5 \). In particular, the anticrossings \( |1, 1_p⟩ \leftrightarrow |0, 2_p⟩ \) and \( |1, 0_p⟩ \leftrightarrow |0, 3_p⟩ \) now occur at the same value of the external flux and overlap exactly. Couplings between \( |1, 0_p⟩ \leftrightarrow |0, 2_p⟩ \) and \( |1, 1_p⟩ \leftrightarrow |0, 3_p⟩ \) remain absent (they are forbidden since the states belong to different parity sectors). We now show that this new scenario at \( n_{g,p} = 0.5 \) leads to a different effective qubit Hamiltonian compared to Eq. (4).

We again begin by projecting the setup Hamiltonian \( H \) onto the qubit subspace \( P_0 \) and onto the states \( |0, 2_p⟩ \) and \( |0, 3_p⟩ \). The resulting low-energy Hamiltonian reads

\[
H_{\text{low}}^{(ng,p=0.5)} = \begin{pmatrix}
\omega_{11} & 0 & -i\eta & 0 \\
0 & \omega_{10} & 0 & i\eta \\
-i\eta & 0 & \omega_{01} & 0 \\
0 & 0 & 0 & \omega_{03}
\end{pmatrix}
\]

(5)

Here, we have \( \lambda = \langle 1, 1_p | H_c | 0, 2_p \rangle = -\langle 1, 0_p | H_c | 0, 3_p \rangle \) and \( \eta = i \langle 1, 1_p | H_c | 0, 1_p \rangle = -i \langle 1, 0_p | H_c | 0, 0_p \rangle \) in a wavefunction gauge for which \( (\lambda, \eta) \) are real valued. By inserting the coupling Hamiltonian in the expressions for the matrix elements, we note that \( \eta \propto \langle s_p | n_p | s_p \rangle - n_{g,p} \). In the previous case when \( n_{g,p} = 0 \), we have \( \langle s_p | n_p | s_p \rangle = 0 \) and, consequently, \( \eta \) vanishes. In the present case when \( n_{g,p} = 0.5 \) and \( E_{J,p} \gg E_{C,p} \), we have \( \langle s_p | n_p | s_p \rangle \neq n_{g,p} \) so that \( \eta \) is finite and yet gets successively smaller upon increasing \( E_{J,p} \). In particular, for \( E_{J,p} \gg E_{C,p} \), we have \( \langle s_p | n_p | s_p \rangle \rightarrow n_{g,p} \), so that the contribution of \( \eta \) to the low-energy Hamiltonian is negligible. Lastly, we note that due to the degeneracy of the PPQ levels, \( \omega_{0,0} = \omega_{0,1} \) and \( \omega_{0,2} = \omega_{0,3} \). Hence, the frequencies of the hybrid setup satisfy \( \omega_{11} = \omega_{10}, \omega_{01} = \omega_{03}, \) and \( \omega_{03} = \omega_{02} \).

We now proceed by integrating out the effects of the noncomputational states, \( |0, 2_p⟩ \) and \( |0, 3_p⟩ \), with a Schrieffer-Wolff transformation. The resulting effective Hamiltonian is of the form

\[
H_{\text{eff}}^{(ng,p=0.5)} = \left( \omega_t + \frac{g_{zz}}{2} \right) \sigma_t^+ + g_{zz} \sigma_t^+ \sigma_p^+,
\]

(6)

Contrasting this result with Eq. (4), we note that both terms \( \propto \sigma_t^+ \) and \( \propto \sigma_t^+ \sigma_p^+ \) vanish, because \( \omega_{0} = 0 \) and \( g_{zz}^2 = 0 \). As a result, the effective qubit interaction is not of \( \sigma_t^+ \sigma_p^+ \) but, rather, of \( \sigma_t^+ \sigma_p^+ \) type. The physical origin of the interaction at \( n_{g,p} = 0.5 \) is different from the \( n_{g,p} = 0 \) case, since it arises directly from the finite matrix elements of the charge operators of the parity-protected qubit and the transmon qubits. The noncomputational states induce only a renormalization of the transmon frequency through the \( g_{zz}^2 \) contribution in the coefficient of \( \sigma_t^+ \).
C. General case

So far, we have seen that the capacitive coupling between the transmon and the PPQ induces a qubit interaction that is substantially different for \( n_{g,p} = 0 \) and \( n_{g,p} = 0.5 \). In a last step, we want to interpolate between those two representative cases. This interpolation is achieved by studying the dependence on the offset charge \( n_{g,p} \) of the various matrix elements. Since the procedure for obtaining the effective interaction is otherwise identical to the special cases, we only note that for generic values of \( n_{g,p} \), the effective Hamiltonian acquires both a \( \sigma^z \sigma^z_p \) and a \( \sigma^x \sigma^p \) interaction term (see the Supplemental Material [59]):

\[
H_{\text{eff}}^{(n_{g,p})} = \left( \omega_p + \frac{g_{\text{eff}}^{zz}}{2} \right) \sigma^z_p + \left( \omega_p + \frac{g_{\text{eff}}^{xy}}{2} \right) \sigma^y_p + g_{\text{eff}}^y \sigma^y_p.
\]

Here, the couplings are given by

\[
g_{\text{eff}}^{zz} = \frac{\lambda^2}{\omega_{10} - \omega_{12}} \pm \frac{\lambda'^2}{\omega_{10} - \omega_{30}}
\]

\[
g_{\text{eff}}^{xy} = (\eta' + \eta'')/2,
\]

\[
g_{\text{eff}}^y = (\eta' - \eta'')/2,
\]

where \( \lambda = |1, 1_p|H_i|0, 2_p\rangle, \lambda'' = |1, 0_p|H_i|0, 3_p\rangle, \eta' = i|1, 1_p|H_i|0, 1_p\rangle, \) and \( \eta'' = -i|1, 0_p|H_i|0, 0_p\rangle \). With the help of Eq. (8), we are now in the position to numerically evaluate the couplings as a function of \( n_{g,p} \). As shown in Fig. 3, the transition from a pure \( \sigma^z \sigma^z_p \) at \( n_{g,p} = 0 \) to a pure \( \sigma^x \sigma^p \) at \( n_{g,p} = 0.5 \) is gradual. Moreover, while the functional form of \( g_{\text{eff}}^{zz}(n_{g,p}) \) is more complicated, we find that the functional form of \( g_{\text{eff}}^{xy}(n_{g,p}) \) is approximately sinusoidal, \( g_{\text{eff}}^{xy}(n_{g,p}) \approx g_{0} \sin(\pi n_{g,p}) \). As for the dependence on the transmon offset charge, we remark that in the deep-transmon regime, \( E_{11} \gg E_{C,t} \), the qubit interaction is almost independent of \( n_{g,t} \).

So far, we have derived the effective qubit interaction and demonstrated that it depends on the anticrossings with the noncomputational states, \( |0, 2_p\rangle \) and \( |0, 3_p\rangle \). To realize the respective anticrossings, we note that it is essential that

\[
\omega_{12} < \omega_{10}.
\]

The transmon energy levels are approximated by \( \omega_{10,p} \approx \sqrt{8E_{11}E_{C,t}(n + 1/2)} - E_{11}, \) while the PPQ energy levels are approximated by \( (\omega_{p,2} + \omega_{p,3})/2 \approx 2\sqrt{8E_{11}E_{C,p} - 4E_{11}p} \). Neglecting the anharmonicity corrections on both qubits, we find that the necessary condition in Eq. (9) simplifies to \( 2\sqrt{E_{11}E_{C,p}} < \sqrt{E_{11}E_{C,t}} \). This condition is satisfied for the parameters chosen in Fig. 2.

III. QUANTUM GATES

We now use the effective Hamiltonian for the hybrid PPQ-transmon setup to implement a controlled-phase gate (CZ_\phi), which will preserve the Cooper-pair parity irrespective of the detailed pulse sequence. In addition, we also discuss a complete set of single-qubit gates realized by controllably driving the system in and out of protection [41]. In combination with the CZ_\phi gate, these single-qubit gates will permit the coherent state transfer, a SWAP operation, between the transmon and PPQ.

A. CZ_\phi gate

For deriving the CZ_\phi gate protocol, we initially move to the frame that rotates with the bare qubit frequencies, \( \tilde{H}_{\text{eff}}^{(n_{g,p})} = U^\dagger(t) H^{(n_{g,p})} U(t) - iU^\dagger(t) \dot{U}(t), \) with \( U(t) = e^{i\text{cos}^2 \sigma^z + i\text{cos}^2 \sigma^y \phi^\prime}\). Within this rotating frame, the effective Hamiltonian reads

\[
\tilde{H}_{\text{eff}}^{(n_{g,p})} = \frac{g_{\text{eff}}^{zz} \sigma^z_p}{2} + \frac{g_{\text{eff}}^{xy} \sigma^x_p}{2} + \frac{g_{\text{eff}}^y \sigma^y_p}{2} + \frac{[\text{-i}e^{\text{i}\theta^\prime} g_{\text{eff}}^{xy} \sigma^z_p + g_{\text{eff}}^y \sigma^x_p] + \text{H.c.}}{2}
\]

where we introduce \( \sigma^\pm = (\sigma^x \pm i\sigma^y)/2 \).

To derive the \( \text{CZ}_\phi^{10} \) gate protocol, we initially move to the frame that rotates with the bare qubit frequencies, \( \tilde{H}_{\text{eff}}^{(n_{g,p})} = U^\dagger(t) H^{(n_{g,p})} U(t) - iU^\dagger(t) \dot{U}(t), \) with \( U(t) = e^{i\text{cos}^2 \sigma^z + i\text{cos}^2 \sigma^y \phi^\prime}\). Within this rotating frame, the effective Hamiltonian reads

\[
\tilde{H}_{\text{eff}}^{(n_{g,p})} = \frac{g_{\text{eff}}^{zz} \sigma^z_p}{2} + \frac{g_{\text{eff}}^{xy} \sigma^x_p}{2} + \frac{g_{\text{eff}}^y \sigma^y_p}{2} + \frac{[\text{-i}e^{\text{i}\theta^\prime} g_{\text{eff}}^{xy} \sigma^z_p + g_{\text{eff}}^y \sigma^x_p] + \text{H.c.}}{2}
\]

where we introduce \( \sigma^\pm = (\sigma^x \pm i\sigma^y)/2 \). We note that the terms \( \propto g_{\text{eff}}^{xy} \sigma^z_p \sigma^x_p \) and \( \propto g_{\text{eff}}^{xy} \sigma^z_p \sigma^y_p \) vanish if \( n_{g,p} = 0 \). In this situation, the free evolution of the effective Hamiltonian can implement a \( \text{CZ}_\phi^{10} \) gate. To execute this \( \text{CZ}_\phi^{10} \) gate, we carry out a rapid excursion from \( \Phi_{\text{ext}}^{\text{out}} \approx 0 \) to a flux \( \Phi_{\text{ext}}^{\text{in}} \approx \Phi_{\text{ext}}^{\text{out}} \) close to the anticrossing \( |1, 0_p\rangle \leftrightarrow |0, 3_p\rangle \). We then let the system evolve freely for a time \( t_s = \phi \). This

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free evolution gives rise to a rotation in the space of $|1, 0\rangle$ and $|0, 3\rangle$. After the time $t_*$, the $|1, 0\rangle$ state will have acquired a finite phase factor and we rapidly return to the idle configuration at $\Phi_t^{\text{ext}} \approx 0$. Because $g_{\phi z}^z \approx -g_{\phi z}^z$ near the anticrossing, the result of this rapid excursion is a $CZ_{\phi}^{10}$ gate of the form

$$CZ_{\phi}^{10} = |0\rangle\langle 0| \otimes I_p + |1\rangle\langle 1| \otimes P_p,$$

$$P_p = e^{-\phi}|0\rangle\langle 0| + |1\rangle\langle 1|.$$  \hspace{1cm} (11)

Unlike for the case of two capacitively coupled transmons $t1$ and $t2$, we remark that the phase factor is not acquired by the $|1, 1\rangle$ state but by the $|1, 0\rangle$ state. Also, as stated at the beginning of this section, we highlight that the Cooper-pair parity is preserved for the full duration of the $CZ_{\phi}^{10}$ gate.

In the protocol for the $CZ_{\phi}^{10}$ gate, we assume that the offset charge on the PPQ is gate-tuned to $n_{g,p} = 0$. Such a tuning is beneficial as it maximizes the coefficient of the $\sigma_p^z \sigma_p^z$ terms, thereby allowing for improved gate speed. Furthermore, the tuning should always be achievable because higher levels of the PPQ are strongly offset charge sensitive and can be used for adjusting $n_{g,p}$. However, the fine tuning to $n_{g,p} = 0$ is not essential for the gate protocol. To see this, we note that the terms $\propto g_{\phi z}^z \sigma_p^z \sigma_p^z$ and $\propto g_{\phi z}^z \sigma_p^z \sigma_p^z$ in Eq. (10), which appear when $n_{g,p}$ is detuned from zero, share a fast-oscillating prefactor $\propto e^{it\phi}$. This fast-oscillating prefactor suggests that such terms are average to zero when invoking a “rotating-wave approximation.” To make this argument more precise, we integrate out the fast-oscillating terms to second order in $g_{\phi z}^x$ and $g_{\phi z}^y$ with a time-dependent Schrieffer-Wolff transformation [60, 61]. The resulting modified effective Hamiltonian reads (see the Supplemental Material [59])

$$\tilde{H}_{\text{eff}}(n_{g,p}) (t) \approx \left( \frac{g_{\phi z}^x}{2} + \frac{4\tilde{g}(t)^2 + \tilde{g}(t)^2}{\omega_i} \right) \sigma_p^z + \left( \frac{g_{\phi z}^y}{2} + \frac{16\tilde{g}(t)^2}{\omega_i} \right) \sigma_p^z,$$

$$+ \frac{16\tilde{g}(t)^2}{\omega_i} \sigma_p^z,$$

where $\tilde{g}(t) = g \sin(\omega_i t/2)$. Provided that $g_{\phi z}^y \ll \omega_i$ and $g_{\phi z}^x \ll \omega_i$, we see that the correction terms to the effective Hamiltonian are indeed negligibly small. For the realistic parameters chosen in Fig. 2, we have $g_{\phi z}^y/(2\pi) = 345$ kHz and $g_{\phi z}^x/(2\pi) = 3.88$ MHz if $n_{g,p} = 0.1$.

**B. Single-qubit gates**

Having introduced the $CZ_{\phi}^{10}$ gate, we now discuss the implementation of single-qubit gates on the PPQ. To implement these single-qubit gates, we consider the generalized circuit for a PPQ depicted in Fig. 4(a). The circuit comprises not only a $\cos(2\phi_p)$ element for the tunneling of pairs of Cooper pairs but also $\cos(\phi_p)$ and $\sin(\phi_p)$ elements that describe the tunneling of single Cooper pairs. The Hamiltonians for these additional circuit elements are given by

$$H_x = -\delta \omega_p \cos(\pi n_{g,p}) \sigma_p^x / 2 + \delta h^+ \sigma_p^x,$$

$$H_y = -\delta \omega_p \cos(\pi n_{g,p}) \sigma_p^y / 2 + \delta h^+ \sin(\pi n_{g,p}) \sigma_p^y.$$  \hspace{1cm} (14)

While both additional circuit elements permit single Cooper-pair tunnelings and temporarily lift the qubit protection, they are typically tuned by different control parameters, depending on the experimental implementation of the PPQ [37, 38]. For example, if the PPQ is realized in a nanowire Josephson interferometer, then the sinusoidal term arises if the interferometer junctions are tuned out of balance by the local gate electrodes. In contrast, the cosinusoidal term arises when the interferometer magnetic flux is biased away from half flux quantum [36, 37].

We now project the Hamiltonians $H_p + H_x$ and $H_p + H_y$ onto the computational subspace of the PPQ. The resulting qubit Hamiltonians read

$$H_{\text{eff}}^x = \delta \omega_p \cos(\pi n_{g,p}) \sigma_p^x / 2 + \delta h^+ \sigma_p^x,$$

$$H_{\text{eff}}^y = \delta \omega_p \cos(\pi n_{g,p}) \sigma_p^y / 2 + \delta h^+ \sin(\pi n_{g,p}) \sigma_p^y.$$  \hspace{1cm} (14)
From this result, we see that the $\cos(\phi_p)$ and $\sin(\phi_p)$ elements induce rotations around the $x$ and $y$ axes of the Bloch sphere. The respective matrix elements are given by $\delta H_0 = \{0_p|H^x|1_p\}$ and $\delta H_0 \sin(\pi n_{g,p}) = \{0_p|H^y|1_p\}$. The dependence of these matrix elements on the offset charge $n_{g,p}$ is shown in Fig. 4(b). Since we can reach any point on the Bloch sphere through a combined rotation around the $x$ and $y$ axes, we conclude that the free time evolution of the Hamiltonians in Eq. (14) can implement a complete set of single-qubit gates. However, we also emphasize that these single-qubit gates break the Cooper-pair parity conservation, so that the PPQ is prone to relaxation errors during the operation time of the single-qubit gates.

C. CNOT and SWAP gates

We now combine the proposed method for single-qubit gates with the $CZ^0_{\phi}$ (with $\phi = \pi$) gate to realize a controlled-NOT (CNOT$_{tp}$) gate with the transmon as control and the PPQ as target, by the gate sequence

$$CNOT_{tp} = \begin{array}{c} 0 \end{array} = \begin{array}{c} \text{Y}_\frac{\pi}{2} \\ \text{CZ}^{10} \\ \text{Y}_\frac{\pi}{2} \end{array}$$

A CNOT$_{tp}$ gate that uses the PPQ as control and the transmon as target is similarly given by $CNOT_{pt} = H_t \cdot H_p \cdot CNOT_{tp} \cdot H_t \cdot H_p$, with the Hadamards $H_{tp} = (\sigma_{1p}^+ + \sigma_{1p}^-)/\sqrt{2}$. Most notably, the CNOT$_{tp}$ and CNOT$_{pt}$ gate can now be combined to realize a SWAP = CNOT$_{tp} \cdot$ CNOT$_{pt}$ CNOT$_{tp}$ operation. The SWAP operation enables the coherent transfer of quantum information between the transmon and the PPQ. Interestingly, this coherent state transfer also gives a novel readout method for the PPQ by swapping the quantum information onto the transmon and performing the readout on the latter.

IV. ERRORS ON THE PARITY-PROTECTED QUBIT

In the previous sections, we have focused on deriving a scheme for a $CZ_{\phi}$ gate within our hybrid qubit setup. For our scheme, we assume that the Cooper-pair parity on the PPQ is conserved during the gate operation time. An interesting question is whether the gate protocol is modified if errors due to unintentional single Cooper-pair tunneling terms, as given by Eq. (13), are present on the PPQ.

A. $\sin(\phi_p)$ errors

To address this question, we consider the PPQ at its $n_{g,p} = 0$ operation point for optimal gate speed. We initially consider an error term, $H_{\text{err}} = -\epsilon^x \sin(\phi_p)$, with an amplitude $\epsilon^x$ that is small compared to the remaining energy scales of the setup. This $\sin(\phi_p)$ error arises in a PPQ realized by a nanowire Josephson interferometer if the two interferometers junctions are not in balance [36]. Due to the error term, we find that the low-energy Hamiltonian of Eq. (3) changes to

$$H_{\text{low}}^{(n_{g,p}=0)} \rightarrow \begin{pmatrix} \omega_{11} & 0 & 0 & 0 \\ 0 & \omega_{10} & 0 & 0 \\ 0 & 0 & \omega_{01} & 0 \\ 0 & 0 & 0 & \omega_{00} \end{pmatrix} + \begin{pmatrix} \lambda' & 0 \\ 0 & \lambda'' \end{pmatrix} \begin{pmatrix} 0 & \kappa \\ -\kappa & 0 \end{pmatrix} \begin{pmatrix} \omega_{02} & 0 \\ 0 & \omega_{03} \end{pmatrix}$$

(16)

Here, we introduce the real-valued matrix element $\kappa = \{0_p|H^z|0_p\} = \{0_p|H^z|0_p\} = \{0_p|H^z|0_p\}$. Moreover, in accordance with Eq. (14), couplings of states with opposite Cooper-pair parity within the qubit subspace $P_0$ are found to be absent at $n_{g,p} = 0$.

Next, we integrate out the noncomputational states, $|0_p, 2_p\rangle$ and $|0_p, 3_p\rangle$, with a Schrieffer-Wolff transformation and move to the rotating frame of the bare qubit frequencies. The effective rotating-frame Hamiltonian of Eq. (10) is then modified to

$$H_{\text{eff}}^{(n_{g,p}=0)} \rightarrow \frac{g_{xz}^+ \sigma_x^x \sigma_z^x}{2} + \frac{g_{xz}^- \sigma_x^- \sigma_z^-}{2} + \frac{g_{yz}^+ \sigma_y^+ \sigma_z^+}{2} + \frac{g_{yz}^- \sigma_y^- \sigma_z^-}{2} + (\epsilon^{(\phi_p)}) g^{zz} \sigma_z^+ \sigma_z^- + (\epsilon^{(\phi_p)}) g^{zz} \sigma_z^+ \sigma_z^- + \text{H.c.},$$

(17a)
with the coefficients

\[ g^{++} = \frac{\kappa \lambda'}{2(\omega_{11} - \omega_{02})}, \quad g^{+-} = \frac{\kappa \lambda''}{2(\omega_{03} - \omega_{10})}. \]  

(17b)

It is now instructive to compare this result to the case of two capacitively coupled transmons, \( r1 \) and \( r2 \), near the operation point of the iSWAP gate [57]. In the latter case, the effective Hamiltonian comprises similar terms, \( \propto \sigma_{i1}^{+} \sigma_{i2}^{-} \) and \( \propto \sigma_{i1}^{+} \sigma_{i2}^{+} \), that are “rotating” with a factor \( e^{i(\omega_{i1} - \omega_{i2})t} \) and “counter-rotating” with a factor \( e^{i(\omega_{i1} + \omega_{i2})t} \), respectively. For \( \omega_{i1} \approx \omega_{i2} \), the “counter-rotating” terms, which are fast oscillating, average to zero within a “rotating-wave approximation.” Only the “rotating” terms, which oscillate slowly, are thus retained in the effective qubit Hamiltonian. In our case, the situation is very different. Because \( \omega_{i1} \gg \omega_{p} \), both factors, \( e^{i(\omega_{i1} + \omega_{p})t} \) and \( e^{i(\omega_{i1} - \omega_{p})t} \), are fast oscillating. Within a “rotating-wave approximation,” we thus expect that both error terms average to zero.

To formalize this “rotating-wave approximation” argument, we integrate out the fast-oscillating terms with a time-dependent Schrieffer-Wolff transformation. To second order in \( g^{++} \) and \( g^{-+} \), we find that (see the Supplemental Material [59])

\[
\hat{H}_{\text{eff}}^{(n_{g,p})} \approx \left( \frac{g^{zz}}{2} + \frac{2[\tilde{g}^{zz}(t) - \tilde{g}^{xy}(t)]^2}{\omega_{t}} \right) \sigma_{t}^{x} + \left( \frac{g^{zz}}{2} + \frac{2[\tilde{g}^{xz}(t) - \tilde{g}^{yz}(t)]^2}{\omega_{t}} \right) \sigma_{t}^{y} + \left( g^{zz} - \frac{4[\tilde{g}^{xx}(t) + \tilde{g}^{yy}(t)]^2}{\omega_{t}} \right) \frac{\sigma_{t}^{x} \sigma_{t}^{y}}{2},
\]

(18)

where \( \tilde{g}(t) = g \sin(\omega_{t}t/2), g^{xx} = (g^{++} + g^{-+})/2 \), and \( g^{yy} = (g^{++} - g^{-+})/2 \). From this expression for the effective rotating-frame Hamiltonian, we conclude that the mitigation of the effects of \( \sin(\phi_{p}) \) errors requires us to operate the setup in the regime when \( g^{xx} \ll \omega_{t} \) and \( g^{yy} \ll \omega_{t} \).

### B. \( \cos(\phi_{p}) \) Errors

It is now interesting to compare our results for \( \sin(\phi_{p}) \) errors with \( \cos(\phi_{p}) \) errors that are described by an error term \( H^{x} = -\epsilon^{x} \cos(\phi_{p}) \) in the Hamiltonian. Such an error term can arise in an implementation of the PPQ with a nanowire Josephson interferometer if the external flux that is threading the interferometer loop is detuned from half flux quantum [36]. In this situation, the low-energy Hamiltonian of Eq. (3) is modified to

\[
H_{\text{low}}(n_{g,p}=0) = \begin{pmatrix}
\omega_{11} & \delta h^{x} & 0 & 0 \\
\delta h^{x} & \omega_{01} & 0 & 0 \\
0 & 0 & \omega_{03} & \delta h^{z} \\
\lambda' & 0 & 0 & 0
\end{pmatrix} \begin{pmatrix}
\lambda' & 0 \\
0 & -\lambda'' \\
0 & 0 & \omega_{02} \\
\omega_{03}
\end{pmatrix},
\]

(19)

with the matrix element \( \chi = \langle 0, 2_p | H^{x} | 0, 3_p \rangle \). Importantly, we see that the \( \cos(\phi_{p}) \) errors do not lead to off-diagonal terms that couple the matrix blocks representing the qubit subspace \( |\text{P}\rangle \) and the noncomputational subspace \( \{|0, 2_p\}, |0, 3_p\rangle \rangle \). Consequently, we note that the \( \cos(\phi_{p}) \) errors primarily induce mixing of opposite-parity states on the PPQ as described by \( H_{\text{eff}} \) in Eq. (14), see also Fig. 5.

In summary, we find that the nature of \( \sin(\phi_{p}) \) errors and \( \cos(\phi_{p}) \) errors is different in our hybrid qubit. While the \( \sin(\phi_{p}) \) errors lead primarily to additional two-qubit interactions that become less relevant in the limit when \( g^{xx} \ll \omega_{t} \) and \( g^{yy} \ll \omega_{t} \), the \( \cos(\phi_{p}) \) errors lead primarily to additional single-qubit terms. Finding strategies of mitigating such flux errors—for example, by concatenating multiple imperfect PPQs [36,39]—is an important open challenge of the field.

### V. ERRORS ON THE TRANSMON QUBIT

Besides the possible errors on the PPQ, it is essential to note that the performance of the \( \text{CZ}^{10}_\phi \) gate in our hybrid setup can also be affected by errors on the transmon qubit. One source of such errors is \( 1/f \) flux noise [62,63], which can give rise to fluctuations in the transmon qubit frequency, \( \omega_{t}(\Phi_{t}) \), and thus induce qubit dephasing. In its idle configuration at \( \Phi_{t}^{\text{ext}} = 0 \), the flux-tunable transmon is always first-order protected against flux noise, \( \partial \omega_{t}/\partial \Phi_{t}^{\text{ext}} = 0 \) at \( \Phi_{t}^{\text{ext}} = 0 \). However, when tuning the transmon away from \( \Phi_{t}^{\text{ext}} \) to realize the \( \text{CZ}^{10}_\phi \) gate, it becomes susceptible to flux noise, because (in general) \( \partial \omega_{t}/\partial \Phi_{t}^{\text{ext}} \neq 0 \) when \( \Phi_{t}^{\text{ext}} \neq 0 \). In this section, we would like to understand how our proposed \( \text{CZ}^{10}_\phi \) gate performs in the presence of realistic \( 1/f \) flux-noise amplitudes, which are of the order of a few \( \mu \Phi_{0} \) at 1Hz [64].

As a starting point of our analysis, we assume that the PPQ is in its protected regime, as described by \( H_{p} \) in Eq. (1), and tuned to \( n_{g,p} = 0 \) for optimal gate speed. Following our previously outlined protocol, the \( \text{CZ}^{10}_\phi \) gate is then realized via a rapid excursion from the idle configuration at \( \Phi_{t}^{\text{ext}} = 0 \) to a flux \( \Phi_{t}^{\text{ext}} = \Phi_{t}^{\text{ext}} \) close to the anticrossing \( |1, 0_p\rangle \leftrightarrow |0, 3_p\rangle \) and back. We parametrize this excursion within the time interval \( [0, t_{\text{s}} + t_{\text{f}}] \) through...
The system parameters are the same as in (b). (d) The optimized gate error, $1 - F$, in the absence of $1/f$ flux noise and qubit-relaxation errors obtained from Eq. (21) versus the coupling capacitance, $E_{CC}$. The system parameters are the same as in (b). The noise parameters are $\Gamma_1^{\text{even}} = \Gamma_1^{\text{odd}} = 1/T_1$, with $T_1 = 20 \mu s$ [62]. Moreover, $A_{1/f} = 5 \mu t \Phi_0$ [64] and $\lambda_{1/f} = 4$ [65].

FIG. 6. The performance of the CZ$^{10}_\phi$ gate. (a) A typical flux pulse, $\Phi_t^{\text{ext}}(t)$, for the CZ$^{10}_\phi$ gate as described by Eq. (20). The wait time near the anticrossing $|1, 0_p\rangle \leftrightarrow |0, 3_p\rangle$ is $t_s$. The ramp up or down time of the flux pulse is $t_r/2$. (b) The optimized gate error, $1 - F$, in the absence of $1/f$ flux noise and qubit-relaxation errors obtained from Eq. (21) versus the coupling capacitance, $E_{CC}$. The system parameters are $(E_J, E_d, E_{CC}, E_{Cp}) = 2\pi (12.2, 2.7, 0.2, 0.15)$ GHz, and $\eta_{GP} = 0$. (c) The optimized gate time, $t_t + t_s$, versus the coupling capacitance, $E_{CC}$. The system parameters are the same as in (b). The noise parameters are $\Gamma_1^{\text{even}} = \Gamma_1^{\text{odd}} = 1/T_1$, with $T_1 = 20 \mu s$ [62]. Moreover, $A_{1/f} = 5 \mu t \Phi_0$ [64] and $\lambda_{1/f} = 4$ [65].

the following pulse shape [66]:

$$\Phi_t^{\text{ext}}(t) = \Phi_t^{\text{ext}} = \begin{cases} C(e^{-f(t)} - 1), & t > \frac{t_r}{2} + t_s \\ C(e^{-f(t-t_s)} - 1), & t < \frac{t_r}{2} \\ 1, & \text{else.} \end{cases}$$ (20)

Here, $t_r/2$ denotes the rise or decay time of the pulse and $t_s$ is the wait time near the anticrossing. Moreover, we introduce the constant $C = 1/(e^{-A/4} - 1)$ and the function $f(t) = A(t - t_r)/t_r^2$ with the parameter $A$ that sets the curvature of the rising or decaying pulse. An example of the pulse shape is shown in Fig. 6(a).

With the help of Eq. (20), it is instructive to first look into errors of the unitary time evolution. To assess the importance of such unitary errors, we numerically solve $i \partial_t U(t) = \tilde{H}_{l_{\text{low}}(n_{GP} = 0)}^{(\text{ext})}(\Phi_t^{\text{ext}}(t))U(t)$, where $\tilde{H}_{l_{\text{low}}(n_{GP} = 0)}^{(\text{ext})}$ is represented in the rotating frame of the bare frequencies at $\Phi_t^{\text{ext}} = 0$, and project the resulting time-evolution operator, $U(t_t + t_s)$, onto the computational subspace. The projected operator is (in general) nonunitary due to leakage to the noncomputational states and is of the form $U_c = \text{diag}[a_{11} e^{i \phi_1}, a_{10} e^{i \phi_0}, 1, 1]$. To compare $U_c$ to the CZ$^{10}_\phi$ gate, we apply a single-qubit $Z$ operation, yielding $U'_c = U_Z U_c$ with $U_Z = \text{diag}[e^{-i \phi_1}, e^{-i \phi_1}, 1, 1]$. We then use $U'_c$ to compute the gate fidelity [67,68]:

$$F = |\text{Tr}(U'^{\dagger}_c U_c) + |\text{Tr}(U'^{\dagger}_c U'_c)|^2]/20,$$ (21)

where $U'^{10}_{CZ\phi} = \text{diag}[1, e^{-i \phi_1}, 1, 1]$. In our simulations, we focus on $\phi = \pi$ and optimized over the pulse parameters $(\Phi_t^{\text{ext}}, A, t_r, t_s)$. The resulting gate errors, $1 - F$, and gate times are shown in Figs. 6(b) and 6(c) versus the coupling charging energy, $E_{CC}$. While the gate time is reduced for stronger couplings, we find that the gate errors increase upon increasing $E_{CC}$. We attribute this increase in $1 - F$ to an increase in both the phase error, $e^{i(\phi_{10} - \phi_{11})} \neq 1$, and the error due to leakage to noncomputational states, $a_{11} \neq 1$ or $a_{10} \neq 1$.

Having discussed the effect of errors in the unitary time evolution, we now proceed by analyzing the performance of the CZ$^{10}_\phi$ gate in the presence of incoherent errors, including $1/f$ flux noise. To determine the time evolution of the density matrix in the presence of $1/f$ flux noise, we follow the approach of Ref. [69] and consider a phenomenological master equation of the form

$$\partial_t \rho = [\tilde{H}_{l_{\text{low}}(n_{GP} = 0)}^{\phi}, \rho] + D[L_{1,1}] \rho + D[L_{1,0}] \rho + D[L_{1,0}]^{\dagger} \rho,$$ (22)

where $D[L] \rho = L \rho L^\dagger - (L^\dagger L \rho + \rho L L^\dagger)/2$.

For the time-independent collapse operators describing relaxation errors, we use

$$L_{1}^{(\text{even})} = \sqrt{\Gamma_{1}^{(\text{even})}} (|0, 0_p\rangle \langle 1, 0_p | + |0, 0_p\rangle \langle 0, 3_p | + |0, 3_p\rangle \langle 1, 0_p |),$$

$$L_{1}^{(\text{odd})} = \sqrt{\Gamma_{1}^{(\text{odd})}} (|0, 1_p\rangle \langle 1, 1_p | + |0, 1_p\rangle \langle 0, 2_p | + |0, 2_p\rangle \langle 1, 1_p |).$$ (23)

Here, $\Gamma_{1}^{(\text{even} or odd)}$ are the decay rates within the even or odd Cooper-pair parity sectors. We assume, for simplicity, that within a particular Cooper-pair parity sector, all decay channels are characterized by the same decay rate. Moreover, we assume that due to the conservation of Cooper-pair parity on the PPQ, any decay channels that connect the two Cooper-pair parity sectors are suppressed. For the time-dependent collapse operators accounting for $1/f$ flux noise, we use

$$L_{\psi}^{(10)}(t) = 2 \sqrt{\Gamma_{\psi}^{(10)}(t)} |1_t, 0_p\rangle \langle 1_t, 0_p |,$$

$$L_{\psi}^{(11)}(t) = 2 \sqrt{\Gamma_{\psi}^{(11)}(t)} |1_t, 1_p\rangle \langle 1_t, 1_p |,$$ (24)
with the $1/f$ flux-noise dephasing rates

$$\Gamma_\varphi^{(ss)}(t) = \lambda_{1/f} \left| \frac{\partial \omega^{ss}_\varphi}{\partial \Phi^c_\varphi} \right| A_{1/f, \varphi}. \quad (25)$$

Here, $\lambda_{1/f}$ is a dimensionless numerical prefactor and $A_{1/f, \varphi}$ denotes the amplitude of the $1/f$ flux-noise power spectral density, $S(\omega) = 2\pi A^2_{1/f, \varphi}/|\omega|$. We assume, again for simplicity, that the dephasing arises primarily from the flux dependence of the $|1,0_p\rangle$ and $|1,1_p\rangle$ levels.

To estimate the gate error, $1 - F$, in the presence of $1/f$ flux noise and the decay channels, we follow closely the procedure in Ref. [68]. For a given initial state, $|\psi_0\rangle$, we first compute the time evolution of the density matrix, $\rho$, from Eq. (22) using the QuTip package [70]. Subsequently, we compute the state-dependent gate fidelity,

$$F_\rho = \text{Tr}[\rho \rho_{\text{ideal}}]. \quad (26)$$

Here, $\rho_{\text{ideal}} = |\psi_{\text{ideal}}\rangle \langle \psi_{\text{ideal}}|$ with $|\psi_{\text{ideal}}\rangle = U^\dagger_Z U^\dagger_{\text{CZ10}} |\psi_0\rangle$ and $U_Z$ is obtained from the calculation of the unitary error. We repeat this procedure for 36 initial two-qubit states obtained by combining the single-qubit states $\{|0_{1/p}\rangle, |1_{1/p}\rangle, (|0_{1/p}\rangle \pm |1_{1/p}\rangle)/\sqrt{2}, (|0_{1/p}\rangle \pm i|1_{1/p}\rangle)/\sqrt{2}\}$. By averaging the resulting values for $1 - F_\rho$, we arrive at an estimate for $1 - F$.

Our results from the aforementioned procedure are shown in Fig. 6(d) for a typical set of system and noise parameters. We find that the gate error depends strongly on the magnitude of the coupling charging energy, $E_{CC}$. For smaller capacitive couplings, $E_{CC}/2\pi = 5$ MHz, corresponding to longer gate times, 152 ns, we find a rather substantial reduction of the fidelity to $F \approx 98.6\%$. In contrast, for stronger couplings, $E_{CC}/2\pi = 20$ MHz, the shorter gate times, 51 ns, reduce the exposure to low-frequency flux noise and the decay channels. As a result, the theoretical gate fidelity can reach $F \approx 99.7\%$, which is comparable to entangling gates between transmons [13]. However, we acknowledge that additional factors may further degrade the theoretical gate fidelity values in experiments. For example, it is to be expected that the effects of $1/f$ flux noise become more acute when many qubits are operated on the same chip. In this scenario, the realization of accurate qubit calibration and high-fidelity gate operations will become more difficult. An interesting challenge for future works will be to further optimize gate protocols for hybrid PPQ-transmon devices, for example, by using dynamical decoupling techniques [71] or optimal control [72].

VI. CONCLUSIONS

To conclude, we propose a coupling scheme for entangling a parity-protected superconducting qubit with a conventional transmon qubit and discuss coherent state transfer as an application. While our scheme may open the way for using PPQs as quantum memories in a transmon architecture, it may also allow for a comparison of coherence times of the two qubit types within the same device.

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