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Protected Hybrid Superconducting Qubit in an Array of Gate-Tunable Josephson Interferometers

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We propose a protected qubit based on a modular array of superconducting islands connected by semiconductor Josephson interferometers. The individual interferometers realize effective $\cos 2\phi$ elements that exchange “pairs of Cooper pairs” between the superconducting islands when gate tuned into balance and frustrated by a half-flux quantum. If a large capacitor shunts the ends of the array, the circuit forms a protected qubit because its degenerate ground states are robust to offset-charge and magnetic field fluctuations for a sizable window around zero offset charge and half-flux quantum. This protection window broadens upon increasing the number of interferometers if the individual elements are balanced. We use an effective spin model to describe the system and show that a quantum phase transition sets the critical flux value at which protection is destroyed.

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A promising route to creating protected qubits is to rely on circuits with underlying symmetries and encode the qubit into distinct eigenstates of the corresponding symmetry operator [1–5]. Such circuits satisfy the requirements of a protected qubit because (1) the vanishing transition matrix elements between states with different symmetries prevent bit-flip errors, and (2) the near degeneracy of the qubit states suppresses phase-flip errors. However, a challenge faced by symmetry-protected qubits is the fragility of the underlying symmetry. If the relevant symmetry is broken by detuning or noise, protection is lost. Reducing the susceptibility of protected circuits to a symmetry-breaking environment is the focus of this work.

A paradigmatic symmetry-protected superconducting qubit is the Cooper-pair-parity qubit (or $\cos 2\phi$ qubit), which encodes the qubit state onto the parity of the number of Cooper pairs on a superconducting island [6–11]. The ideal $\cos 2\phi$ circuit preserves the Cooper-pair parity symmetry via a special type of Josephson junction that only permits the tunneling of pairs of Cooper pairs. Considerable progress has been made towards effectively realizing double Cooper-pair junctions by a variety of approaches [12–18]. Improving robustness of the $\cos 2\phi$ qubit by constructing arrays that average uncorrelated local noise has also been investigated [2,19,20].

In this paper we introduce and analyze a novel protected qubit based on an array of superconducting islands coupled via semiconductor Josephson interferometers, as illustrated in Fig. 1. By taking advantage of the naturally occurring higher harmonics in semiconductor Josephson junctions [21], the individual interferometers in the array can readily realize $\cos 2\phi$ elements when gate tuned into balance and frustrated by a half-flux quantum [18]. If a large capacitor shunts the ends of the array, the ground states of the system are twofold degenerate, carry opposite total Cooper-pair parity, and are robust to offset-charge and magnetic field fluctuations. While the array offers protection from noise and offsets that improves with the number of elements, we find that an array as short as two elements considerably improves immunity to flux offset and noise compared to a single interferometer.

Besides protection against energy relaxation as well as fluctuations of charge and flux, two additional features are noteworthy. (1) The gate tunability of the semiconductor Josephson junctions allows the interferometers to be tuned into balance using gate voltages rather than additional fluxes. This feature is critical because the proposed concatenation approach enhances the protection only if the elements are pairwise balanced. (2) The superconducting-semiconductor platform facilitates integration with other hybrid qubits such as gate-mon [22–24] or Majorana qubits [25–28].
FIG. 1. Hybrid nanowire-array protected qubit. The qubit consists of pairs of nanowires (brown) proximitized by superconducting shells and contacted by superconducting islands (gray). The Josephson tunnel barriers are formed by breaking the superconducting shell. In the figure, \( N = 3 \) Josephson interferometers form an array. The two islands at the ends of the array are shunted by a big capacitor with capacitance \( C_B \), while the islands within the array are coupled by small capacitances \( C_S \ll C_B \). When the Josephson junctions are balanced by local gate electrodes the array are coupled by small capacitances \( C_S \).

The rest of the paper is organized as follows. In Sec. I, we review the \( \cos 2\phi \) qubit and describe its circuit in terms of a tight-binding model for a particle moving in the \( \cos 2\phi \) potential. In Sec. II, we discuss the realization of a \( \cos 2\phi \) qubit based on a single semiconductor Josephson interferometer and analyze errors due to flux noise and unbalanced junctions. In Sec. III, we introduce the modular array of semiconductor Josephson interferometers and investigate its operation as a protected qubit. We discuss the origin of protection against flux and charge noise, and derive an effective spin Hamiltonian that describes the low-energy spectrum of the array. Finally, in Sec. IV, we introduce a “giant spin” representation of the effective Hamiltonian, which is useful to compute the critical flux value above which flux noise protection is lost. We associate this critical flux with a quantum phase transition. Finally, in Sec. V, we give an overview on the operation of the \( \cos 2\phi \) qubit, such as single-, two-qubit gates, initialization, and readout.

I. THE \( \cos 2\phi \) QUBIT

A. Hamiltonian

Before considering the interferometer array, where qubit states are encoded in the overall Cooper-pair parity on multiple islands, we first review the single \( \cos 2\phi \) qubit. As shown in Fig. 2(a), the \( \cos 2\phi \) qubit consists of a pair of superconducting islands with capacitance \( C \) connected via a modified Josephson-like junction that only passes pairs of Cooper pairs (denoted by a Josephson junction symbol with an extra line). The Hamiltonian of this circuit includes a single phase degree of freedom, \( \phi \), the superconducting phase difference across the junction,

\[
H_{ng} = 4E_C(n - n_g)^2 - E_{J,2} \cos 2\phi,
\]

where \( n \) is the number operator of Cooper pairs on the capacitor, \( E_{J,2} \) is the tunneling amplitude of double Cooper pairs across the junction, \( E_C = e^2/2C \) is the charging energy, \( e \) is the electron charge, and \( n_g \) is the offset charge. We display the wave functions of the ground and first excited states in both charge and phase space in Figs. 2(c) and 2(d). In charge space, the qubit eigenstates are superpositions of even or odd Cooper-pair parity states, with an envelope function that broadens with increasing \( E_{J,2}/E_C \) ratio, analogous to the transmon qubit [29]. In phase space, qubit states are the symmetric and antisymmetric combinations of states localized in the 0 or \( \pi \) valley of the \( \cos 2\phi \) potential. The offset charge \( n_g \) tunes the qubit transition frequency, which is maximal at \( n_g = 0 \) and zero at \( n_g = 0.5 \), where the qubit states are degenerate; see Fig. 2(b).

Protection against energy relaxation in the \( \cos 2\phi \) qubit results from the symmetry of Cooper-pair parity, which prohibits transitions between the qubit states, \( \langle 1|O|0 \rangle = 0 \) for noise operators \( O \) that do not induce single Cooper-pair tunneling. The qubit is protected against dephasing because the eigenstates have support over many charge states, \( E_{J,2}/E_C \gg 1 \), leading to exponentially reduced charge dispersion and near degeneracy. Finally, separation of the qubit states from higher-lying excited states prevents leakage errors.

B. Effective “tight-binding” model

It is useful to describe the \( \cos 2\phi \) qubit in terms of a spin model that can shed light on the benefits of using multiple interferometers. We focus on the regime of \( E_{J,2}/E_C \gg 1 \) and use Bloch’s theorem to define “atomic” Wannier functions that are localized in the 0 or \( \pi \) valleys of the phase lattice formed by the \( \cos 2\phi \) potential.

As a first step, we recall that a qubit with a compact phase degree of freedom is equivalent to a particle moving in a one-dimensional lattice [30–32]. To show this, we eliminate the offset-charge dependence in \( H_{ng} \) via a unitary rotation, \( H = e^{i\theta_0}H_{ng}e^{-i\theta_0} \), yielding a qubit Hamiltonian of the form

\[
H = 4E_Cn^2 - E_{J,2} \cos 2\phi,
\]

which is equivalent to the Hamiltonian of a particle with mass proportional to \( C \) moving in a \( V(\phi) = -E_{J,2} \cos 2\phi \) potential. The eigenstates of the transformed Hamiltonian \( H \) are Bloch waves \( \psi_{\ell,n_g}(\phi + 2\pi) = e^{in_g2\pi}\psi_{\ell,n_g}(\phi) \) with
FIG. 2. Spectrum and eigenstates of the cos 2φ qubit. (a) Circuit layout of the cos 2φ qubit with a unique Josephson element that only permits tunneling of “pairs of Cooper pairs” with charge 4e and shunted with capacitance C. (b) Energy levels as a function of the gate charge ng. Circuit parameters are Ec/h = 0.2 GHz and Ejz/h = 10 GHz. The two lowest-energy levels (red and blue) correspond to the two qubit states |0⟩ and |1⟩. The higher-lying states (gray) are well separated from the qubit subspace. (c) Qubit eigenstates in charge space corresponding to superpositions of even charge states for |0⟩ and odd charge states for |1⟩. (d) Qubit eigenstates in phase space corresponding to symmetric and antisymmetric combinations of wave functions that are localized in the 0 and π valleys of the cos 2φ Josephson potential (gray). (e) Spin model of the cos 2φ qubit. The |0⟩ and |1⟩ states correspond to spins pointing along the +x and −x directions, respectively.

energy-band index ℓ. For the case of the ground and first excited states, we combine the Bloch waves in a symmetric and antisymmetric way, ψ1,ng(φ) = (ψ0,ng ± ψ1,ng) / √2. The resulting states are concentrated in either the 0 or π valleys of the extended cos 2φ potential; see the filled curves in Fig. 3. Within the spin model, we associate these Bloch states with effective spin states, ψ1,ng(φ) ↔ | ↑⟩ and ψ1,ng(φ) ↔ | ↓⟩. Note that in this spin basis, the qubit eigenstates are symmetric and antisymmetric combinations of spin-up and spin-down states, |0⟩ = (| ↑⟩ + | ↓⟩) / √2 and |1⟩ = (| ↑⟩ − | ↓⟩) / √2; see Fig. 2(e).

To derive the Hamiltonian of the spin model, we now introduce Wannier functions on the phase lattice, similar to solid-state systems. These Wannier functions are localized in a particular phase unit cell m and they are the combination of Bloch states at different offset charges, 

\[ w_s(φ - 2π m) = \frac{1}{\sqrt{M}} \sum_{n_g} e^{-i n_g 2π m} ψ_{s,n_g}(φ). \]  

(3)

Here, s = ↑, ↓ labels the effective spin degree of freedom and M is a normalization constant. We depict examples of the Wannier functions in Fig. 3.

We next introduce a tight-binding representation of H on the qubit subspace [33]. First, we project the Hamiltonian H onto the low-energy subspace spanned by \{ψ1,ng, ψ1,ng\}. This yields the following matrix elements:

\[ (H_{\text{eff}})_{ss'} = \int dφ w_s(φ)Hψ_{s',ng}(φ) \]

\[ = \sum_m e^{i2π m n_g} \int dφ w_s(φ)H_{ss'}(φ - 2π m). \]  

(4)

Since longer-range hybridizations between Wannier functions in more distant unit cells are negligible (Ejz/Ec ≫ 1), we only keep matrix elements that connect Wannier functions that are in the same or nearest-neighbor unit

FIG. 3. Tight-binding representation of the cos 2φ qubit. The cos 2φ qubit is analogous to a phase particle moving in a periodic lattice formed by the cos 2φ Josephson potential (gray). The unit cells consist of two potential minima, the 0 and π valleys. The qubit wave functions are Bloch waves and their symmetric and antisymmetric combinations ψ1,ng(φ) (shown in red and blue) have finite support in one of the two valleys. The Wannier functions \( w_+ (φ) \) (shown as black lines) are exponentially localized in a particular unit cell of the lattice. The hopping amplitudes \( t_{\text{in}} \), and \( t_{\text{out}} \), outside of the unit cell, \( t_{\text{out}} \), are given by the hybridization of neighboring Wannier functions.
cells. In this tight-binding approximation, there are two types of hybridizations (see Fig. 3): tunneling inside a unit cell, \( t_{\text{in}} \), and tunneling outside of the unit cell, \( t_{\text{out}} \), where

\[
\begin{align*}
    t_{\text{in}} &= \int d\phi \, w_{\text{in}}^* (\phi) H w_{\text{in}} (\phi), \\
    t_{\text{out}} &= \int d\phi \, w_{\text{out}}^* (\phi) H w_{\text{out}} (\phi + 2\pi).
\end{align*}
\]

In case of the \( \cos 2\phi \) qubit, \( t_{\text{in}} = t_{\text{out}} \equiv t < 0 \). However, we show below that single Cooper-pair tunneling terms can introduce asymmetries in these tunneling amplitudes.

After collecting all the dominant nearest-neighbor hoppings, we arrive at the following spin Hamiltonian of the \( \cos 2\phi \) qubit:

\[
H_{\text{eff}} \approx 2t \cos (\pi n_g) \{ \cos (\pi n_g) \sigma_x + \sin (\pi n_g) \sigma_y \} = 2t \cos (\pi n_g) \sigma_x.
\]

Here we have introduced the offset-charge-dependent rotated Pauli matrices \( \sigma_x = R_R^y \sigma_x R_R^z \), with \( R_R^z = e^{-i\pi n_g \sigma_z/2} \). At \( n_g = 0 \), we denote the ground state of the spin model in Eq. (6) as \( |0\rangle = |\rightarrow\rangle \) and the first-excited state as \( |1\rangle = |\leftarrow\rangle \), indicating that these states point along the \( +x \) and \( -x \) spin directions; see Fig. 2(e). In terms of the Cooper-pair parity, \( |\rightarrow\rangle \) corresponds to the even parity state, while \( |\leftarrow\rangle \) corresponds to the odd parity state. Away from \( n_g = 0 \), the two eigenstates still point in opposite direction in the \( x \)-\( y \) plane but the tunnel splitting is reduced. The transition energy of the qubit is \( E_{01} = |4t \cos (\pi n_g)| \), which is in agreement with the exact result shown in Fig. 2(b).

II. SINGLE-INTERFEROMETER QUBIT

A. Josephson energy

We now turn to the realization of a single \( \cos 2\phi \) qubit using nanowire Josephson junctions, as discussed recently in Ref. [18]. In contrast to conventional Al/AlO\(_x\)/Al Josephson junctions with Josephson energy \( E_J(\phi) = -E_{J,1} \cos \phi \), the Josephson energy of semiconductor nanowire junctions contains higher harmonics, \( E_J(\phi) = -E_{J,1} \cos \phi + E_{J,2} \cos 2\phi + \cdots \). These higher harmonics originate from a few high-transmission channels that mediate the Cooper-pair tunneling via Andreev bound states in the junction. For junctions shorter than the superconducting coherence length, this yields a Josephson energy [21,34,35],

\[
E_{1}(\phi) = -\Delta \sum_m \sqrt{1 - T_m \sin^2 \phi/2}.
\]

Here, the transmission coefficient \( T_m \) characterizes the \( m \)th Andreev bound state and \( \Delta \) is the superconducting gap. Although the second harmonic, \( E_{J,2} \cos 2\phi \), can be sizable in a few-channel junctions [18], the first harmonic, \( -E_{J,1} \cos \phi \), remains the dominant contribution to the Josephson energy unless deliberately removed.

One approach to ensuring that the leading term in the Josephson energy is the second harmonic is to make use of Aharonov-Bohm interference [18,19] by considering a symmetric Josephson interferometer composed of two semiconductor Josephson junctions enclosing a flux \( \Phi_{\text{ext}} \); see Fig. 4(a). When Cooper pairs follow the two different paths of the interferometer, they acquire a relative Aharonov-Bohm phase, \( 2\pi q \Phi_{\text{ext}}/\hbar \), proportional to the charge \( q \). At one half quantum of external flux, \( \Phi_{\text{ext}} = \Phi_0/2 = h/4e \), single Cooper pairs interfere destructively with a relative Aharonov-Bohm phase \( \pi \) because they carry charge \( q = 2e \). In contrast, double Cooper pairs interfere constructively with relative Aharonov-Bohm phase \( 2\pi \) because they carry charge \( q = 4e \). As a result, the leading term in the Josephson energy is given by double

![Image](338x330 to 374x346)

![Image](338x391 to 374x408)

![Image](338x453 to 375x470)

![Image](403x423)

![Image](338x397)
Cooper-pair hopping and the circuit realizes an effective 
\[ \cos 2 \phi \] element.

To formalize this argument, we note that the Josephson energies of the individual junctions are well approximated by 
\[ E^{(k)}_j(\phi) \approx -E^{(k)}_{j,1} \cos \phi + E^{(k)}_{j,2} \cos 2\phi \]
with 
\[ k = 1, 2 \] labeling the interferometer arms. In the presence of an external flux \( \Phi_{\text{ext}} \), the total Josephson energy of the circuit reads
\[ E^{(\text{tot})}_j(\phi) = E^{(1)}_j(\phi - \pi \Phi_{\text{ext}}/\Phi_0) + E^{(2)}_j(\phi + \pi \Phi_{\text{ext}}/\Phi_0). \] (8)

After substitution, we see that, when the first-order terms are equal, \( E^{(1)}_{j,1} = E^{(2)}_{j,1} \), and the flux is half-flux quantum, \( \Phi_{\text{ext}} = \Phi_0/2 \), single Cooper-pair tunneling vanishes, and \( E^{(\text{tot})}_j(\phi) = (E^{(1)}_{j,2} + E^{(2)}_{j,2}) \cos 2\phi \).

### B. Error sources

To achieve complete suppression of single Cooper-pair tunneling events (complete destructive interference), the circuit needs to satisfy two requirements. First, the external flux through the loop needs to be exactly biased at half-flux quantum, \( \Phi_{\text{ext}} = \Phi_0/2 \). Second, the single Cooper-pair tunneling amplitudes across the two junctions need to be equal, \( E^{(1)}_{j,1} = E^{(2)}_{j,1} \). If the circuit fails to satisfy any of these two requirements, there will be finite tunneling contributions of single Cooper pairs across the device. Although in both cases the Cooper-pair parity protection is destroyed, the two types of error are fundamentally different, and protection against them requires different strategies.

The first type of error results from unbalanced transmission amplitudes, \( \delta E_{j,1} = E^{(1)}_{j,1} - E^{(2)}_{j,1} \neq 0 \); see Fig. 4(b). This imbalance leads to a sinusoidal error contribution in the Josephson energy, \( \delta E_j(\phi) = \delta E_{j,1} \sin \phi \). This type of error changes the tunnel barrier between the 0 and \( \pi \) valleys without inducing a tilt between the two valleys. As a result, the inter- and intra-unit-cell hybridizations of the Wannier functions become asymmetric. In the modified spin Hamiltonian, this asymmetric hybridization leads to \( \sigma_z \)-type errors
\[ \delta H_{\text{eff}} = (t_m - t_{\text{out}}) \sigma_z. \] (9)

The second type of error results from noise or offset in magnetic flux away from one half-flux quantum; see Fig. 4(c). For balanced junctions, \( E^{(1)}_{j,1} = E^{(2)}_{j,1} \), but flux detuned from half-flux quantum, \( \Phi_{\text{ext}} = \Phi_0/2 + \delta \Phi_{\text{ext}} \) with \( \delta \Phi_{\text{ext}}/\Phi_0 \ll 1 \), the Josephson energy acquires a cosinusoidal error contribution, \( \delta E_j(\phi) = -2E_{j,1}(\delta \Phi_{\text{ext}}/\Phi_0) \cos \phi \). Such an error induces a tilt between the 0 and the \( \pi \) valleys, i.e., different on-site energies for the Wannier functions \( w_j(\phi) \) and \( w_{j+1}(\phi) \). In the spin model, these different on-site energies lead to a \( \sigma_z \) error term
\[ \delta H_{\text{eff}} = -\epsilon \sigma_z. \] (10)

Here, the amplitude of the \( \sigma_z \) term is determined by the flux detuning away from half-flux quantum, \( \epsilon \approx E_{j,1}(\delta \Phi_{\text{ext}}/\Phi_0) \).

In the following, we show that the flux errors can be effectively eliminated with multi-interferometer \( \cos 2\phi \) qubits, while errors due to unequal junction transmissions can be prevented by \textit{in situ} gate tuning of the junctions.

### III. MULTI-INTERFEROMETER QUBITS

#### A. Hamiltonian

With the motivation of reducing the effect of flux offset and noise (\( \sigma_z \) errors), we now extend our spin model to multi-interferometer qubits (see Fig. 1 for the case of three connected interferometers). Each pair of junctions in the interferometers is gate tuned into balance and the interferometer loops are frustrated with half-flux quanta. The superconducting islands of the array are capacitively coupled, such that the islands within the array are coupled by a small capacitance \( C_S \), while the two islands at the ends of the array are coupled by a big capacitance \( C_B \gg C_S \). The Hamiltonian of the setup with \( N \) interference loops is
\[ H_{n_g}^{(N)} = \sum_{i,j=1}^{N} 4E_C^{(ij)}(n_i - n_{g,i}^{(i)})(n_j - n_{g,j}^{(j)}). \] (11)

Here, \( n_i \) is the charge operator with offset charge \( n_{g,i}^{(i)} \) of the \( i \)th loop, \( \phi_i \) is the conjugate phase operator, \( E^{(ij)}_{C} \) is the double Cooper-pair tunneling amplitude, and the charging energies are \( E_C^{(ij)} = e^2C_{ij}^{-1}/2 \), where \( C_{ij} \) is the capacitance matrix (see Appendix A for details).

Similar to the single-interferometer case, we can eliminate the offset-charge dependence of the Hamiltonian in Eq. (11) by a unitary rotation, \( H^{(N)} = e^{-in_{g}^{T}\phi_{H_0}^{(N)}}e^{-in_{g}^{(N)}}H_{n_g}^{(N)} \) with \( n_g = (n_{g,1}^{(1)}, \ldots, n_{g,N}^{(N)})^{T} \) and \( \phi = (\phi_1, \ldots, \phi_N)^{T} \), yielding
\[ H^{(N)} = \sum_{i,j=1}^{N} 4E_C^{(ij)}n_in_j - \sum_{i=1}^{N} E_{j,2}^{(i)} \cos 2\phi_i. \] (12)

The offset-charge dependence appears in the boundary conditions on the eigenfunctions \( \psi_{\ell,n_{g}}(\phi) \), which are Bloch waves with quasiperiodicity, \( \psi_{\ell,n_{g}}(\phi + 2\pi e_{\ell}) = e^{2\pi i n_{g}e_{\ell}}\psi_{\ell,n_{g}}(\phi) \), where \( e_{\ell} = (0, \ldots, 1, \ldots, 0)^{T} \).
B. Two-interferometer qubit

We can gain insight into the origin of protection against flux errors by examining the simplest extension, with $N = 2$ interferometer loops. In this case, the Josephson energy of the setup, $-E_{J2} \cos 2\phi_1 - E_{J2} \cos 2\phi_2$, has four minima in the unit cell at positions $\{(0,0), (0,\pi), (\pi,0), (\pi,\pi)\}$. We denote the linear combinations of the four lowest-energy Bloch waves with support in each of the four minima by $\{\psi_{\uparrow,\uparrow}, \psi_{\downarrow,\downarrow}, \psi_{\uparrow,\downarrow}, \psi_{\downarrow,\uparrow}\}$, and use them to define Wannier functions that are localized in a particular valley of the two-dimensional phase lattice,

$$w_{s_1s_2}(\phi - 2\pi m) = \frac{1}{\sqrt{M}} \sum_{n_g} e^{-i2\pi m n_g} \psi_{s_1s_2n_g}(\phi). \quad (13)$$

Here, $s_1, s_2 = \uparrow, \downarrow$ denote the effective spin degrees of freedom, $M$ is a normalization factor, and $m = (m_1, m_2)$ is a two-dimensional matrix of integers.

After projecting the Hamiltonian of Eq. (11) onto the $\{\psi_{s_1s_2n_g}\}$ subspace and expressing the matrix elements in terms of these Wannier functions, we obtain an effective tight-binding Hamiltonian,

$$(H^{(2)}_{\text{eff}})_{s_1s_2s_3s_4} = \sum_m e^{i2\pi m n_g} t_{s_1s_2s_3s_4}(m), \quad (14)$$

where

$$t_{s_1s_2s_3s_4}(m) = \int d\phi w_{s_1s_2}(\phi) H^{(2)} w_{s_3s_4}(\phi - 2\pi m).$$

In this representation, we can associate the Bloch wave functions $\{\psi_{s_1s_2n_g}\}$ with the states of a double spin-1/2 system $\{s_1s_2\}$ and the interaction between the spins with the hybridizations $t_{s_1s_2s_3s_4}(m)$ between Wannier functions. We consider two limiting cases, $C_B \ll C_S$ and $C_B \gg C_S$, and focus on the situation $n_g^{(i)} = 0$ for simplicity. We discuss the effects of offset charges later.

For $C_B \ll C_S$, coupling between the ends of the array is negligible, $E^{(12)}_c \approx 0$; see Fig. 5(a). In the Wannier picture, this implies that nearest-neighbor hopping along the $\phi_1$ and $\phi_2$ directions give the dominant contributions to the tight-binding Hamiltonian; see Fig. 5(b). By retaining only such nearest-neighbor hybridizations, the tight-binding Hamiltonian for $C_B \ll C_S$ takes on the simplified form

$$H^{(2)}_{\text{eff}} \approx 2t (\sigma^{(1)}_x + \sigma^{(2)}_x), \quad (15)$$

which describes two separate spin-1/2 systems with no spin-spin interaction. The eigenstates and energy levels are [Fig. 5(e)]

$$|\rightarrow\rightarrow\rangle \propto (|\uparrow\uparrow^{(1)}\rangle + |\downarrow\downarrow^{(1)}\rangle)(|\uparrow^{(2)}\rangle + |\downarrow^{(2)}\rangle), \quad E_{\rightarrow\rightarrow} = 0,$$

$$|\leftarrow\leftarrow\rangle \propto (|\uparrow\downarrow^{(1)}\rangle - |\downarrow\uparrow^{(1)}\rangle)(|\uparrow^{(2)}\rangle + |\downarrow^{(2)}\rangle), \quad E_{\rightarrow\leftarrow} = 4t,$$

$$|\rightarrow\leftarrow\rangle \propto (|\uparrow\downarrow^{(1)}\rangle - |\downarrow\uparrow^{(1)}\rangle)(|\uparrow^{(2)}\rangle - |\downarrow^{(2)}\rangle), \quad E_{\rightarrow\leftarrow} = 4t,$$

$$|\leftarrow\rightarrow\rangle \propto (|\uparrow^{(1)}\rangle - |\downarrow^{(1)}\rangle)(|\uparrow^{(2)}\rangle - |\downarrow^{(2)}\rangle), \quad E_{\rightarrow\rightarrow} = 8t.$$  

Here, the ground state, $|\rightarrow\rightarrow\rangle$, and the highest-energy state, $|\leftarrow\leftarrow\rangle$, are both ferromagnetic with both spins pointing along the $+x$ or $-x$ direction, and correspond to states of opposite total Cooper-pair parity. These two configurations are good candidates for encoding a protected qubit since local magnetic field errors ($\sigma^{(i)}$ error terms) are unable to mix them, $(\rightarrow\rightarrow |\sigma^{(i)}\rangle |\leftarrow\leftarrow) = 0$. Thus, the suggested qubit encoding scheme provides protection against flux noise. However, in the limit of $C_B \ll C_S$, the ferromagnetic configurations do not form a low-energy subspace that is well separated from the remaining states. Hence, the qubit encoding in the $\{|\rightarrow\rightarrow\rangle,|\leftarrow\leftarrow\rangle\}$ subspace requires some modification.

One modification that leads to the desired arrangement, namely the two ferromagnetic configurations becoming a low-lying pair of nearly degenerate states, can be realized by adding a large capacitive coupling between the end two islands, $C_B \gg C_S$; see Fig. 5(d). In the Wannier picture, this enhances diagonal next-nearest-neighbor hybridization [Fig. 5(e)],

$$H^{(2)}_{\text{eff}} \approx 2t (\sigma^{(1)}_x + \sigma^{(2)}_x) - 2J \sigma^{(1)}_x \sigma^{(2)}_x. \quad (17)$$

Here, the coupling $J \equiv -t_{\uparrow\downarrow,\downarrow\uparrow}(0) > 0$ arises from the elongated Wannier function overlap along one of the diagonals; see the derivation in Appendix B. For $J \gg |t|$, the two ferromagnetic states $\{|\rightarrow\rightarrow\rangle,|\leftarrow\leftarrow\rangle\}$ form a pair of well-separated low-energy states protected against flux noise. In terms of qubit energy flux dispersion, this manifests as a broadening of the “sweet spot” around $\Phi_{\text{ext}} = \Phi_0/2$, as shown in Fig. 6.

Including offset charges (Appendix B) generalizes the coupling term in Eq. (17) in the $J \gg |t|$ regime to

$$-2J \sigma^{(1)}_x \sigma^{(2)}_x \rightarrow -2J \cos(\pi[n_g^{(1)} - n_g^{(2)}])\hat{\sigma}^{(1)}_x \hat{\sigma}^{(2)}_x, \quad (18)$$

where the $\hat{\sigma}^{(i)}$ are the rotated Pauli matrices defined above. Thus, the finite offset charges have two effects. First, the
offset charges induce a spin rotation around the $z$ axis by an angle $\pi n^{(i)}_g / 2$. Critically, in the rotated spin basis, the ferromagnetic configurations remain protected from mixing due to local $\sigma_{z}^{(i)}$ error because $\tilde{\sigma}^{(i)}_z = (R^{(i)}_z) \sigma_{z}^{(i)} R^{(i)}_z = \sigma_{z}^{(i)}$. Second, the offset charges give rise to a modulation of the coupling $J$ by a factor of $\cos(\pi [n^{(i)}_g - n^{(i)}_g])$. Thus, the ferromagnetic configuration remains the ground states as long as $|n^{(i)}_g - n^{(i)}_g|$ is small; see Appendix C.

A requirement for realizing the ferromagnetic eigenstates, and thus the robust parity protected states, is the balanced transmission amplitudes in both loops. If the single Cooper-pair tunneling amplitudes are unbalanced, additional $\sigma_{x}^{(1)}\sigma_{x}^{(2)}$ and $\sigma_{z}^{(1)}\sigma_{z}^{(2)}$ coupling terms arise. Such coupling terms introduce mixing between the ferromagnetic states and destroy the protection, as, for example, $\langle \rightarrow \rightarrow | \sigma_{z}^{(1)}\sigma_{z}^{(2)} | \leftarrow \leftarrow \rangle \neq 0$. This leads to the enlarged sweet spot disappearing; see Fig. 6(d), which shows the flux dependence of the eigenstates in the presence of unbalanced junctions. This requirement further highlights the benefits of semiconductor-based tunnel junctions, where the junction transmission amplitudes are in situ tunable.

We note that asymmetries in the Josephson energies of the second harmonics, $E^{(1)}_{J,2} \neq E^{(2)}_{J,2}$, and asymmetries in the small capacitance values across the two interferometers, $C^{(1)}_S \neq C^{(2)}_S$, preserve the Cooper-pair symmetry. These imperfections only introduce asymmetries of the tunneling amplitudes associated with the single spin terms, $t^{(i)}$ $(i = 1, 2)$, but leave the interaction between the spins as $\sigma_{x}^{(1)}\sigma_{x}^{(2)}$ coupling. Thus, the flux-noise-protected ferromagnetic states $\{| \rightarrow \rightarrow \rangle, | \leftarrow \leftarrow \rangle \}$ remain the two low-energy states of the asymmetric Hamiltonian, $H^{(2)}_{ef} \approx 2t^{(1)}\sigma_{x}^{(1)} + 2t^{(2)}\sigma_{x}^{(2)} - 2J\sigma_{x}^{(1)}\sigma_{x}^{(2)}$.

C. Multi-interferometer qubit

Finally, we now comment on the extension to $N \geq 2$ unit cells in the array. For this generalized scenario, we again focus on the case of zero offset charges, and interpret Eq. (12) as the Hamiltonian of a particle hopping between...
against flux noise is absent. The capacitance values are seen upon appending additional loops to the interferometer array.

noise has the most detrimental effect on the qubit degeneracy half-flux quantum. This is the direction in flux space, where flux

interferometer qubit with state degeneracy exists only if $\delta/\Phi_1(0) \leq 1/\Phi_0$. The Hamiltonian written in the form

\[ H_{\text{eff}}(N) = \frac{1}{N} \sum_{i=1}^{N} \sigma_i^{(0)} - \frac{2}{N} \sum_{i<j} \sigma_i^{(0)} \sigma_j^{(0)} - \sum_{i=1}^{N} \epsilon_i \sigma_i^{(0)}. \]  

Here, $t$ and $J$ denote the overlap integrals for nearest- and next-nearest-neighbor Wannier functions, while the $\epsilon_i$ are the flux offsets. The factor $1/N$ in the term $\propto J$ ensures that the ground-state energy per effective spin remains finite in the thermodynamic limit. Such an array exhibits a near-ground-state degeneracy for a sizable window around half-flux quantum that broadens upon increasing the length $N$ of the array; see Fig. 6. This robust degeneracy for the multi-interferometer devices with $N > 1$ is notably different from the fragile degeneracy for the single-interferometer device with $N = 1$ and permits the encoding of a protected qubit. In terms of the effective model in Eq. (19), the degeneracy implies that the ferromagnetic coupling $J$ also remains the dominant energy scale for longer arrays provided that $C_B \gg C_S$. In the next section, we derive a specific upper bound for the flux noise protection window in terms of the effective parameters $(t, J, \epsilon_i)$.

FIG. 6. Flux-noise protection in arrays. (a) Example for the energy dispersion $E_0(\delta \Phi^{(1)}_{\text{ext}}, \delta \Phi^{(2)}_{\text{ext}})/h$ for an unprotected two-interferometer qubit with $C_B = 0$, where the $\delta \Phi^{(i)}_{\text{ext}}$ are the flux offsets from half-flux quantum in the two loops. The ground-state degeneracy exists only if $\delta \Phi^{(1)}_{\text{ext}} = 0$ or $\delta \Phi^{(2)}_{\text{ext}} = 0$. For $C_B \gg C_S$, the qubit remains protected within a broader window around half-flux quantum. (b) Energy dispersion $E_0/h$ for an array of $N = 1 \ldots 4$ symmetric interferometer loops as a function of the correlated (global) deviation $\delta \Phi^{(i)}_{\text{ext}} \equiv \delta \Phi^{(i)}_{\text{ext}}$ away from half-flux quantum. This is the direction in flux space, where flux noise has the most detrimental effect on the qubit degeneracy [dashed lines in (a) and (b)]. The protection window broadens upon appending additional loops to the interferometer array. (d) In the case of unbalanced junctions, the protection window against flux noise is absent. The capacitance values are $C_S = 10 \text{ fF}$ for all cases, and $C_B = 100 \text{ fF}$ ($N = 1$), $C_B = 190 \text{ fF}$ ($N = 2$), $C_B = 350 \text{ fF}$ ($N = 3$), $C_B = 620 \text{ fF}$ ($N = 4$); the tunneling energy is $E_{f1}/h = 10 \text{ GHz}$; the slope of the asymmetry term due to flux noise is $E_{f1}/\Phi_0 = 250 \text{ GHz}/\Phi_0$ (similar to Ref. [18]); the asymmetry term due to unbalanced junctions in (d) is $E_{f1}/h = 2 \text{ GHz}$. The $2^N$ minima of the potential $-\sum_{i=1}^{N} E_{f1}^{(i)} \cos 2\phi_i$ with an inverse effective mass tensor $E_{C}^{(0)}$. The inverse effective mass of the particle along the $\phi_i$ axes and the diagonals are uniform, $E_{C}^{(0)} \equiv E_{C}^{(\text{diag})} > 0$ for $i = j$ and $E_{C}^{(0)} \equiv E_{C}^{(\text{off-diag})} < 0$ for $i \neq j$ (see Appendix A). Hence, if we retain only nearest-neighbor hopping along the main axes and next-nearest-neighbor hopping along the diagonals, the effective tight-binding Hamiltonian takes the form

IV. “GIANT SPIN” REPRESENTATION

We next use the above results to derive a specific upper bound for the protection window against flux noise and offset. We find that the critical flux at which protection is destroyed corresponds to a quantum phase transition.

A. Hamiltonian

To begin, note that the Hamiltonian in Eq. (19) involves an all-to-all interaction between effective spins proportional to $\sigma_i^{(0)} \sigma_j^{(0)}$. This allows us to introduce an $N$-spin representation (or “giant spin”) $S_{\alpha i i} = \sum_{i=1}^{N} \sigma_{\alpha i i}^{(i)}/2$ with $[S_\alpha, S_\beta] = i\epsilon_{\alpha\beta\gamma} S_\gamma$ and rewrite the Hamiltonian in the form

\[ H_{\text{eff}}^{(N)} = -2(\epsilon S_z - 2t S_x) - (4J/N) S_x^2. \]  

Here, terms proportional to $S_z$ and $S_x$ describe effective magnetic fields along the $x$ and $z$ directions, while the term proportional to $S_x^2$ describes a magnetic “easy axis” along the $x$ direction. In comparison to Eq. (19), we consider correlated flux noise or offset, $\epsilon \equiv \epsilon_i$, due to fluctuations of the global magnetic field. The Hamiltonian written in the form of Eq. (20) with $t = 0$ is the Lipkin-Meshkov-Glick Hamiltonian [36–38]. In the following, we analyze this Hamiltonian through the lens of the interferometer-array-protected qubit.
B. Symmetries

We begin with a discussion on the symmetries of the Hamiltonian in Eq. (20) and the structure of its eigenstates.

First, the Hamiltonian preserves the magnitude of the total giant spin $S = (S_x, S_y, S_z)$ so that

$$[H^{(N)}_{\text{eff}}, S^2] = 0. \quad (21)$$

This symmetry implies that the eigenstate sectors with different total giant spin $S$ decouple. Consequently, we can focus on the $S = N/2$ sector that contains the low-energy states of our system.

Second, the Hamiltonian with $t = 0$ also has a spin-flip symmetry, prohibiting coupling between states with a different number of spins pointing along the $z$ direction,

$$\left[ H^{(N)}_{\text{eff}}, \prod_{i=1}^N \sigma_z^{(i)} \right] = 0 \quad (t = 0). \quad (22)$$

In a common eigenbasis of $H^{(N)}_{\text{eff}}$ and $\prod_{i=1}^N \sigma_z^{(i)}$, this symmetry implies that $\langle S_z \rangle = \langle S_z' \rangle = 0$ and $\langle S_x S_z \rangle = \langle S_x S_z' \rangle = 0$.

C. Phase diagram

We next derive a phase diagram for the Hamiltonian of Eq. (20) that characterizes the parameter space regions for which the ground-state subspace is twofold degenerate and permits the encoding of a protected qubit. We initially focus on the limit when the two ends of the device are shunted by a large capacitor, $C_B \gg C_S$. This ensures that the effective magnetic field $\propto t$ in Eq. (20) is small compared to the ferromagnetic coupling $\propto J$ and the flux noise $\propto \epsilon$. Moreover, we also assume the limiting case of a device with many interference loops, $N \gg 1$. This large-$N$ limit allows us to approximate the ground states by “spin coherent states” of the form [39]

$$|\theta, \chi\rangle = \bigotimes_{i=1}^N \left[ \cos \left( \frac{\theta}{2} \right) |\uparrow\rangle^{(i)} + e^{i\chi} \sin \left( \frac{\theta}{2} \right) |\downarrow\rangle^{(i)} \right] \quad (23)$$

with mean spin direction that is given by a point on the unit sphere, $n_0 = |\theta, \chi| S|\theta, \chi\rangle = (\sin \theta \cos \chi, \sin \theta \sin \chi, \cos \theta)$.

To find the variational parameters $\theta$ and $\chi$ of this mean field ansatz, we minimize the variational energy

$$E^{(N)}_{\text{var}}(\theta, \chi) \equiv \langle \theta, \chi | H^{(N)}_{\text{eff}} | \theta, \chi \rangle$$

$$= -N \cos \theta - J(N - 1)(\sin \theta \cos \chi)^2. \quad (24)$$

From the minimization, we identify two phases with different ground-state degeneracies.

---

FIG. 7. “Giant spin” representation. The effective Hamiltonian for an $N$-interferometer device can be written in terms of an $N$-spin representation (or “giant spin”). In the large-$N$ limit, two phases can be distinguished as a function of the flux noise amplitude $\epsilon$ and ferromagnetic exchange coupling $J$: in the protected regime for $\epsilon < 2J$ (left panel), the ground-state subspace $\{|0\rangle, |1\rangle\}$ is twofold degenerate and the mean spin direction of the ground states is in the $x$-$z$ plane with angles $\pm \theta_0 \approx \arccos(\epsilon/2J)$ relative to the $z$ axis. In the unprotected regime for $\epsilon > 2J$ (right panel) the ground state $|0\rangle$ is nondegenerate with a mean spin direction along the $z$ axis. The protected and unprotected regimes are separated by a quantum phase transition that, in a mean field picture, is located at $\epsilon/2J = 1$ (vertical red line). From a numerical exact diagonalization, we can compute the energy splitting $E_{10}$ between the two lowest-energy states as a function of $\epsilon/2J$ (solid black curve). We find that the mean field value of the quantum phase transition is approached from below by increasing the number of interferometers. The plot shows $E_{10}/4J$ for $N = 2000$ interferometers.

The first phase occurs for the parameter range $|\epsilon| < 2J$. In this phase, the ground-state subspace is doubly degenerate. The two spin coherent states with $\theta_0 \approx \arccos(\epsilon/2J)$ and $\chi_0 = 0, \pi$ form a basis of the subspace and realize the two qubit states. For vanishing flux noise, $\epsilon \to 0$, we note that the two qubit states correspond to the ferromagnetic configurations, $|\rightarrow, \ldots, \rightarrow\rangle$, $|\leftarrow, \ldots, \leftarrow\rangle$, that we already encountered for the $N = 2$ loop device in the previous section. For finite flux noise, $\epsilon > 0$, the two qubit states acquire a small rotation angle in the $x$-$z$ plane so that $\langle S_z \rangle = \epsilon/2J$; see the left panel of Fig. 7. However, despite the rotation, the two states remain degenerate as long as $|\epsilon| < 2J$, which sets an upper bound for the flux noise protection of our qubit in the limit $N \gg 1$.

The second phase occurs for the parameter ranges $|\epsilon| \gtrsim 2J$. For our multiloop device, this parameter range corresponds to the situation of substantial flux noise. As expected for such a scenario, the ground state is nondegenerate and spanned by the spin-coherent state with $\theta_0 = 0$, which points along the $z$ axis with $\langle S_z \rangle = 1$; see the right
panel of Fig. 7. The encoding of a protected qubit is not possible in this regime.

We further note that the “mean field” transition point that separates the two phases at $|\epsilon| = 2J$ marks a quantum phase transition point, since the twofold degenerate ground-state subspace abruptly changes to a single nondegenerate ground state. From a numerical diagonalization of the Hamiltonian in Eq. (20), we find that the quantum phase transition point of the “mean field” picture is approached from below upon increasing the length of the interferometer array; see Fig. 7. These findings are consistent with the results of the full model, for which we also find a broadening of the flux noise protection window upon increasing the length of the interferometer array; recall Fig. 6(c).

V. QUBIT OPERATION

After having discussed the realization of the cos $2\phi$ qubit in the hybrid nanowire platform, we give an overview on the operation of the cos $2\phi$ qubit, focusing on the single-loop device introduced in Secs. I and II.

A. Qubit initialization and readout

We begin by introducing a protocol for initializing and measuring the states of the cos $2\phi$ qubit based on a “conditional fluorescence technique” that has been previously employed for the readout of fluxonium qubits [40].

The setup comprises a cos $2\phi$ qubit, operated at $n_g = 0$, and coupled capacitively to a microwave transmission line. Assuming that the qubit is in some unknown state, $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, we apply a coherent drive to the transmission line that is resonant with the transition frequency between the $|0\rangle$ and $|3\rangle$ states of the cos $2\phi$ circuit; see Fig. 8(a). As the Cooper-pair parities of the $|0\rangle$ and $|3\rangle$ states are the same, the coupling between the electromagnetic waves in the transmission line and the qubit (proportional to the charge operator matrix element $\langle 0|n|3\rangle$) is infinite and, through the cycling of the $|0\rangle \rightarrow |3\rangle$ transition, will generate continuous fluorescence [40].

This continuous fluorescence does not arise for the $|1\rangle \rightarrow |3\rangle$ transition that has a frequency that is comparable to the $|0\rangle \rightarrow |3\rangle$ transition; see Fig. 8(b). The reason is that the states $|1\rangle$ and $|3\rangle$ have opposite Cooper-pair parities so that the dipole moment $\langle 1|n|3\rangle$ vanishes. Because of the same Cooper-pair parity selection rule, we note that all other transitions in the $\{|0,1,2,3\}\rangle$ low-energy subspace are also prohibited with the exception of the $|1\rangle \rightarrow |2\rangle$ transition. However, the frequency of this transition can be sufficiently detuned from our drive frequency, as shown for a typical parameter set in Fig. 8(b), and can be further suppressed by engineering the microwave environment [40].

From the previous consideration, we are now in the position to formulate our readout protocol for the cos $2\phi$ qubit. If, upon driving the $|0\rangle \rightarrow |3\rangle$ transition, fluorescence is observed, then the qubit has been projected from the unknown state, $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$, onto the $|0\rangle$ state. If, under the same conditions, fluorescence has not been observed, then the qubit has been projected onto the $|1\rangle$ state. Regarding the initialization of the qubit, if the fluorescence is observed, the qubit is already initialized in the $|0\rangle$ state. If the qubit is projected onto the $|1\rangle$ state (no fluorescence observed), an additional single-qubit gate is required to initialize the qubit.

B. Qubit gates

Besides the initialization and measurement, another important requirement for the operation of the cos $2\phi$ qubit is the realization of a universal set of logical gates. We note that, based on Eqs. (9) and (10), unprotected single-qubit gates can be implemented by the unitary time-evolution operator when temporarily unbalancing the Josephson interferometer or tuning the flux away from half-flux quantum. Our focus in this subsection will thus be on the detailed description of a maximally entangling controlled-$Z$ gate inspired by similar protocols that have previously been devised for fluxonium qubits [41,42].

The setup for the proposed protocol comprises two capacitively coupled cos $2\phi$ qubits ($i = A, B$); see Fig. 9(a).
A two-qubit gate protocol is given by

$$U_0(t^*) = P_0 e^{-i H_{\text{low}} t^* / \hbar} P_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

which corresponds to the desired controlled-Z gate.

VI. CONCLUSIONS

We have proposed a protected superconducting qubit realized in an array of superconducting islands connected by semiconductor Josephson interferometers. Such interferometers can realize \( \cos 2\phi \) elements when gate tuned into balance and frustrated by a half-flux quantum \([18]\). When an array of \( \cos 2\phi \) elements is shunted by a large capacitance, the qubit encoded in the degenerate ground-state subspace is robust to offset charge and flux noise for a window around zero offset charge and half-flux quantum. By introducing an effective spin model, we showed that flux noise protection broadens upon increasing the length of the array. In the long-array limit, a giant-spin model yielded a quantum phase transition as a function of flux offset between protected and unprotected regimes. The construction of interferometer array protected qubits can be realized using existing semiconductor-superconductor hybrid materials based on semiconductor nanowires \([18]\) or two-dimensional heterostructures \([23,43,44]\).

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APPENDIX A: CHARGING ENERGIES

In this appendix, we derive explicit expressions for the charging energies of the nanowire array qubit.

To start, we write the Hamiltonian of the device in the form

\[ H = \frac{1}{2} (\mathbf{n} - \mathbf{n}_g)^T \mathbf{C}^{-1} (\mathbf{n} - \mathbf{n}_g) + V(\phi), \quad (A1) \]

where \( \mathbf{n} \) and \( \mathbf{n}_g \) are vectors that contain the Cooper-pair number operators and the offset charges of the \( N+1 \) superconducting islands, and \( \phi \) is the vector of phase operators. Moreover, \( \mathbf{C} \) denotes the \( (N+1) \times (N+1) \) capacitance matrix given by

\[
\mathbf{C} = \begin{bmatrix}
C_B + C_S & -C_S & 0 & 0 & 0 & -C_B \\
-C_S & 2C_B - C_S & \ldots & 0 & 0 & 0 \\
0 & -C_S & 2C_S & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 2C_S & -C_S & 0 \\
0 & 0 & 0 & -C_S & 2C_S & -C_S \\
-C_B & 0 & 0 & 0 & -C_S & C_B + C_S
\end{bmatrix}. \quad (A2)
\]

As a next step, we move from a description of charges on individual islands ("node charges") to a description of relative charges between neighboring islands ("branch charges"), as used in the main text. We achieve this change with the transformations

\[
\mathbf{n} = (\mathbf{R}^T)^{-1} \mathbf{n}, \quad (A3a)
\]

\[
\phi = (\mathbf{R}^T)^{-1} \phi, \quad (A3b)
\]

\[
\mathbf{n}_g = (\mathbf{R}^T)^{-1} \mathbf{n}_g, \quad (A3c)
\]

\[
\mathbf{C}^{-1} \equiv \mathbf{R} \cdot \tilde{\mathbf{C}}^{-1} \cdot \mathbf{R}^T, \quad (A3d)
\]

where we have introduced the \( (N+1) \times (N+1) \) transformation matrix

\[
\mathbf{R} = \begin{bmatrix}
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1
\end{bmatrix}. \quad (A4)
\]

After removing the free mode in the circuit [45], the resulting transformed Hamiltonian takes the form

\[
H = \frac{1}{2} (\mathbf{n} - \mathbf{n}_g)^T \mathbf{C}^{-1} (\mathbf{n} - \mathbf{n}_g) + V(\phi) \quad (A5)
\]

with the transformed capacitance matrix

\[
\mathbf{C} = \begin{bmatrix}
C_B + C_S & C_B & \ldots & C_B & C_B \\
C_B & C_B + C_S & \ldots & C_B & C_B \\
\vdots & \ddots & \ddots & \ddots & \ddots \\
C_B & C_B & \ldots & C_B & C_B + C_S \\
C_B & C_B & \ldots & C_B & C_B + C_S
\end{bmatrix}. \quad (A6)
\]

To evaluate \( \mathbf{C}^{-1} \) and obtain the relevant charging energies, we note that it can be written in the form

\[
\mathbf{C} = C_B \mathbf{Q} + C_S \mathbf{1}, \quad (A7)
\]

where we have defined the \( N \times N \) matrices \( \mathbf{Q}_{ij} = 1 \) and \( \mathbf{1}_{ij} = \delta_{ij} \). We note that \( \mathbf{Q}^2 = N \mathbf{Q} \). For the inverted capacitance matrix \( \mathbf{C}^{-1} \), we make the ansatz

\[
\mathbf{C}^{-1} = \kappa \mathbf{Q} + \frac{1}{C_S} \mathbf{1}, \quad (A8)
\]

where \( \kappa \) is a to-be-determined parameter. We then require that

\[
\mathbf{1} \equiv \mathbf{C} \cdot \mathbf{C}^{-1} = \left[ C_B \mathbf{Q} + C_S \mathbf{1} \right] \left[ \kappa \mathbf{Q} + \frac{1}{C_S} \mathbf{1} \right] = \left[ C_B \kappa N + \frac{C_B}{C_S} + C_S \kappa \right] \mathbf{Q} + \mathbf{1}. \quad (A9)
\]

This condition is equivalent to

\[
C_B \kappa N + \frac{C_B}{C_S} + C_S \kappa = 0, \quad (A10)
\]

which leads to

\[
\kappa = -\frac{C_B}{C_S(C_S + C_B N)}. \quad (A11)
\]

Having derived the explicit form of \( \kappa \), we find that the inverse capacitance matrix of our setup is given by

\[
\mathbf{C}^{-1} = -\frac{C_B}{C_S(C_S + C_B N)} \mathbf{Q} + \frac{1}{C_S} \mathbf{1}, \quad (A12)
\]

or, equivalently,

\[
(C^{-1})_{ij} = \begin{cases} 
\frac{1}{C_S} - \frac{C_B}{C_S(C_S + C_B N)} & \text{for } i = j, \\
-\frac{C_B}{C_S(C_S + C_B N)} & \text{otherwise.}
\end{cases} \quad (A13)
\]
APPENDIX B: TIGHT-BINDING MODELS

Now, we provide more details on the derivation of the tight-binding models for the single- and two-interferometer qubits.

We begin with the circuit Hamiltonian of the single-interferometer qubit,
\[
H = 4E_C \hbar^2 - E_{J,3} \cos 2\phi. \tag{B1}
\]

We recall that the eigenfunctions are Bloch states \(\psi_{\ell,n_g}(\phi)\), which obey the quasiperiodic boundary conditions
\[
\psi_{\ell,n_g}(\phi + 2\pi) = e^{i n_g 2\pi} \psi_{\ell,n_g}(\phi), \tag{B2}
\]
where \(\ell\) is the band index.

For the derivation of the tight-binding model of the single-interferometer qubit, we consider the eigenfunctions of the two lowest-energy states, \(\psi_{0,n_g}(\phi)\) and \(\psi_{1,n_g}(\phi)\), and combine them as
\[
\psi_{1,n_g}(\phi) = \frac{1}{\sqrt{2}} \{ \psi_{0,n_g}(\phi) - \psi_{1,n_g}(\phi) \},
\]
\[
\psi_{1,n_g}(\phi) = \frac{1}{\sqrt{2}} \{ \psi_{0,n_g}(\phi) + \psi_{1,n_g}(\phi) \}. \tag{B3}
\]

These combinations of the eigenfunctions have finite support in the 0 or \(\pi\) valleys of the \(\cos 2\phi\) potential. We associate these Bloch states with the spin-up and spin-down states of the effective spin model, and introduce the Dirac notation \(\langle \phi | \downarrow \rangle = \psi_{1,n_g}(\phi)\) and \(\langle \phi | \uparrow \rangle = \psi_{1,n_g}(\phi)\).

We can use the Wannier functions to express the Bloch states,
\[
| \downarrow \rangle = \frac{1}{\sqrt{M}} \sum_m e^{i n_g 2\pi m} | w_{1,m} \rangle, \tag{B4}
\]
\[
| \uparrow \rangle = \frac{1}{\sqrt{M}} \sum_m e^{i n_g 2\pi m} | w_{\uparrow,m} \rangle,
\]
where \(M\) is a normalization factor, and for simplicity, we again introduced the Dirac notation for the Wannier states \(\langle \phi | w_{\uparrow,m} \rangle = w_{\uparrow}(\phi - 2\pi m)\) and \(\langle \phi | w_{\downarrow,m} \rangle = w_{\downarrow}(\phi - 2\pi m)\).

We are now in the position to write down the effective tight-binding Hamiltonian by projecting the Hamiltonian onto the subspace spanned by the spin states \(| \uparrow \rangle, | \downarrow \rangle \), which yields
\[
H_{\text{eff}} = \sum_{s,s'=\uparrow,\downarrow} \langle s | H | s' \rangle | s \rangle \langle s' |. \tag{B5}
\]

Next, we can compute the effective Hamiltonian by evaluating the following matrix elements in a tight-binding
approximation ($E_{J,2} \gg E_C$):

$$
\langle \uparrow | H | \downarrow \rangle = \frac{1}{M} \sum_{m,m'} e^{i g 2\pi s(m-m')} \langle w_{1,m'} | H | w_{1,m} \rangle 
= \frac{1}{M} \sum_{m} e^{-i g 2\pi m} \langle w_{1,m} | H | w_{1,0} \rangle 
\approx e^{i 2\pi n g} \langle w_{1,-1} | H | w_{1,0} \rangle + \langle w_{1,0} | H | w_{1,0} \rangle. \quad (B6)
$$

We also note that $\langle \uparrow | H | \uparrow \rangle = \langle \downarrow | H | \downarrow \rangle$, so that the diagonal terms in the effective Hamiltonian only give a constant offset.

It is now helpful to introduce the inter- and intra-unit-cell tunnelling amplitudes

$$
t_{\text{in}} = \langle w_{1,0} | H | w_{1,0} \rangle, 
\quad t_{\text{out}} = \langle w_{1,-1} | H | w_{1,0} \rangle. \quad (B7)
$$

By noting that $t_{\text{in}} = t_{\text{out}} \equiv t$, we can write the effective Hamiltonian of Eq. (B11) compactly as

$$
H_{\text{eff}} = t(1 + e^{i 2\pi n g}) | \uparrow \rangle \langle \downarrow | + \text{H.c.} 
= \begin{bmatrix} 0 & t(1 + e^{i 2\pi n g}) \\
 t(1 + e^{-i 2\pi n g}) & 0 \end{bmatrix}. \quad (B8)
$$

After introducing the spin-space Pauli matrices $\sigma_{x,y,z}$ as in the main text, we find that this form of effective Hamiltonian reads

$$
H_{\text{eff}} = 2t \cos(\pi n g) \{ \cos(\pi n g) \sigma^x + \sin(\pi n g) \sigma^y \}, \quad (B9)
$$

which is the result that we presented in Eq. (6).

Now, we follow a similar approach to obtain the spin model for two coupled interferometers. Using Dirac notation, we express the two-dimensional Bloch functions $|s_1, s_2\rangle$ with the two-dimensional Wannier functions $|w_{s_1, s_2, \ell, m}\rangle$.

$$
|s_1, s_2\rangle = \frac{1}{\sqrt{M}} \sum_{\ell, m} e^{i h_{s_1}^{(1)} N \ell} e^{i h_{s_2}^{(2)} 2\pi m} |w_{s_1, s_2, \ell, m}\rangle. \quad (B10)
$$

To get the effective spin Hamiltonian, we first project the Hamiltonian onto the subspace of the four lowest-lying Bloch states, $|\downarrow \downarrow\rangle, |\downarrow \uparrow\rangle, |\uparrow \downarrow\rangle, |\uparrow \uparrow\rangle$:

$$
H_{\text{eff}}^{(2)} = \sum_{s_1, s_2} \langle s_1, s_2 | H | s_1', s_2' \rangle \langle s_1, s_2 \rangle \langle s_1', s_2' \rangle. \quad (B11)
$$

When the potential is symmetric (the loops are biased at half-flux quantum and the junctions are balanced), there are three different types of hybridization: parallel to the $\phi_1$ axes, $t_1$, along the $\phi_1 + \phi_2$ direction, $t_2^x$, and along the $\phi_1 - \phi_2$ direction, $t_2^z$; see Fig. 11. In the protected regime, the wave functions are elongated along the $\phi_1 - \phi_2$ direction, thus, $t_2^x \ll t_2^z$. After taking into account all possible terms, using the tight-binding approximation, and expressing the result with Pauli matrices, we arrive at the effective Hamiltonian of two coupled interferometers,

$$
H_{\text{eff}}^{(2)} = 2t [ \cos(\pi n g^{(1)}) \sigma_x^{(1)} + \cos(\pi n g^{(2)}) \sigma_x^{(2)} ] + 2t_2^x \cos(\pi n g^{(1)} + n g^{(2)}) \sigma_x^{(1)} \sigma_x^{(2)} + 2t_2^z \cos(\pi n g^{(1)} - n g^{(2)}) \sigma_x^{(1)} \sigma_x^{(2)}. \quad (B12)
$$

which is equivalent to the result in Eq. (18), after taking into account the facts that $t_2^x \ll t_2^z$ and $J = -t_2^z$.

**APPENDIX C: OFFSET-CHARGE SENSITIVITY**

In Fig. 12, we provide numerical results on the charge sensitivity of the unprotected and protected $N = 2$ arrays. In the protected regime, the energy is a complicated function of the offset charges because of the offset-charge dependence of the interaction term, $2t_2^x \cos(\pi n_g^{(1)} - n_g^{(2)}) \sigma_x^{(1)} \sigma_x^{(2)}$. The qubit needs to be biased around the $n_g^{(1)} = n_g^{(2)} = 0$ regime.
FIG. 12. Energy dispersion as a function of offset charges. (a) The qubit energy in the unprotected regime shows a maximum and a sweet spot at $n_g^{(1)} = n_g^{(2)} = 0$, and gradually decreases as the offset charges are increased. (b) The offset-charge dependence in the protected regime displays a more complicated pattern due to the offset-charge dependence of the coefficient of $\sigma^{(1)}(\Phi_{ext})$ interaction. The qubit can be operated around the $n_g^{(1)} = n_g^{(2)} = 0$ sweet spot. Parameters are the same as in Fig. 6.

APPENDIX D: GATE-NOISE SENSITIVITY

Although the primary goal of the proposed concatenation strategy is to reduce the sensitivity of the device to flux fluctuations, it is important to investigate the response of the array to symmetry-breaking terms due to imbalance in the junction transmissions. In a nanowire-based Josephson junction, the gate voltages determine the transmission amplitudes and hence the magnitude of the first and second harmonics of the Josephson energy in an interferometer. Thus, voltage noise leads to imbalance in the transmission amplitudes and increases the magnitude of $E_{J,1}$. We numerically explored the effects of increasing $E_{J,1}$ on the qubit energy around the frustration point in arrays with different sizes up to $N = 4$; see Fig. 13. Similar to the studies of flux noise, we consider the worst-case scenario, when the gate noise affecting the interferometers is correlated (each interferometer experience the same $E_{J,1}$ offset). While for $N = 2$, the correlated gate noise is worse than that of a single loop device; we find that further increasing the number of interferometers remarkably reduces the effect of gate noise close to the level of a single loop device.

FIG. 13. Imbalanced interferometers. (a) The qubit energy at half-flux quantum as a function of the symmetry-breaking first harmonics term for arrays with different numbers of loops. We assume that the asymmetries are correlated, i.e., each loop has the same imbalance. The parameters are the same as in Fig. 6. (b) The qubit energies when the interferometers biased slightly away from the frustration point.


