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STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE
C*-ALGEBRAS

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Abstract. We show that certain extensions of classifiable C*-algebra are strongly classified by the associated six-term exact sequence in K-theory together with the positive cone of K0-groups of the ideal and quotient. We apply our result to give a complete classification of graph C*-algebras with exactly one ideal.

1. Introduction

The classification program for C*-algebras has for the most part progressed independently for the classes of infinite and finite C*-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered K0-group. In the infinite case, there are a number of results for purely infinite C*-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite C*-algebras up to stable isomorphism by the K0-group together with the K1-group. For nonsimple purely infinite C*-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C*-algebras with exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K-groups [34], the second named author has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered K-theory [31, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite C*-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated K-theory [28, Theorem 4.14]. However, in all of these situations the nonsimple C*-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Recently, the authors have provided a framework for classifying nonsimple C*-algebras that are not necessarily AF-algebras or purely infinite C*-algebras. In particular, the authors have shown in [16] that certain extensions of classifiable C*-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. This has allowed for the classification of certain nonsimple C*-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [16] was then used by the first named author and Tomforde in [18] to classify a certain class of non-simple graph C*-algebras, showing that graph C*-algebras with exactly one non-trivial ideal can be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. The authors in [15] then showed that all non-unital graph C*-algebras with exactly one
non-trivial ideal can be classified up to isomorphism by their associated six-term exact sequence in $K$-theory. In this paper, we complete the classification of graph $C^*$-algebras with exactly one non-trivial ideal by classifying those that are unital. Our methods here differ rather dramatically from the methods in [18] and [15]. In particular, we use the traditional methods of classification via existence and uniqueness theorems. As a consequence, for unital graph $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with exactly one non-trivial ideal, then any isomorphism between the associated six-term exact sequence in $K$-theory which preserves the unit lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

2. Preliminaries

2.1. $C^*$-algebras over topological spaces. Let $X$ be a topological space and let $O(X)$ be the set of open subsets of $X$, partially ordered by set inclusion $\subseteq$. A subset $Y$ of $X$ is called locally closed if $Y = U \setminus V$ where $U, V \in O(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathbb{LC}(X)$. The set of all connected, non-empty, locally closed subsets of $X$ will be denoted by $\mathbb{LC}(X)^*$.

The partially ordered set $(O(X), \subseteq)$ is a complete lattice, that is, any subset $S$ of $O(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset $S$ of $O(X)$,

$$\bigwedge_{U \in S} U = \left( \bigcap_{U \in S} U \right)^{\circ} \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.$$  

For a $C^*$-algebra $\mathfrak{A}$, let $\mathbb{I}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{I}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset $S$ of $\mathbb{I}(\mathfrak{A})$,

$$\bigwedge_{\mathfrak{J} \in S} \mathfrak{J} = \bigcap_{\mathfrak{J} \in S} \mathfrak{J} \quad \text{and} \quad \bigvee_{\mathfrak{J} \in S} \mathfrak{J} = \bigcup_{\mathfrak{J} \in S} \mathfrak{J}.$$  

Definition 2.1. Let $\mathfrak{A}$ be a $C^*$-algebra. Let Prim($\mathfrak{A}$) denote the primitive ideal space of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let $X$ be a topological space. A $C^*$-algebra over $X$ is a pair $(\mathfrak{A}, \psi)$ consisting of a $C^*$-algebra $\mathfrak{A}$ and a continuous map $\psi : \text{Prim}(\mathfrak{A}) \to X$. A $C^*$-algebra over $X$, $(\mathfrak{A}, \psi)$, is separable if $\mathfrak{A}$ is a separable $C^*$-algebra. We say that $(\mathfrak{A}, \psi)$ is tight if $\psi$ is a homeomorphism.

We always identify $O(\text{Prim}(\mathfrak{A}))$ and $\mathbb{I}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p.$$  

Let $(\mathfrak{A}, \psi)$ be a $C^*$-algebra over $X$. Then we get a map $\psi^* : O(X) \to O(\text{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$ defined by

$$U \mapsto \{ p \in \text{Prim}(\mathfrak{A}) : \psi(p) \in U \} = \mathfrak{A}(U)$$  

For $Y = U \setminus V \in \mathbb{LC}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [27], $\mathfrak{A}(Y)$ does not depend on $U$ and $V$.

Example 2.2. For any $C^*$-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_\text{Prim}(\mathfrak{A}))$ is a tight $C^*$-algebra over Prim($\mathfrak{A}$). For each $U \in O(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$. 
Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip $X_n$ with the Alexandrov topology, so the non-empty open subsets are

$$[a, n] = \{x \in X : a \leq x \leq n\}$$

for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathcal{A}, \phi)$ be a $C^*$-algebra over $X_n$. We will use the following notation throughout the paper:

$$\mathcal{A}[k] = \mathcal{A}(\{k\}), \mathcal{A}[a, b] = \mathcal{A}([a, b]), \text{ and } \mathcal{A}(i, j) = \mathcal{A}[i + 1, j].$$

Using the above notation we have ideals $\mathcal{A}[a, n]$ such that

$$\{0\} \leq \mathcal{A}[n] \leq \mathcal{A}[n - 1, n] \leq \cdots \leq \mathcal{A}[2, n] \leq \mathcal{A}[1, n] = \mathcal{A}.$$

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras over $X$. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is $X$-equivariant if $\phi(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ for all $U \in \mathcal{O}(X)$. Hence, for every $Y = U \setminus V$, $\phi$ induces a homomorphism $\phi_Y : \mathcal{A}(Y) \to \mathcal{B}(Y)$. Let $C^*$-$alg(X)$ be the category whose objects are $C^*$-algebras over $X$ and whose morphisms are $X$-equivariant homomorphisms.

An $X$-equivariant homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a full $X$-equivariant homomorphism if for all $Y \in \mathcal{L}(X)$, $\phi_Y(a)$ is norm-full in $\mathcal{B}(Y)$ for all norm-full elements $a \in \mathcal{A}(Y)$, i.e., the closed ideal of $\mathcal{B}(Y)$ generated by $\phi_Y(a)$ is $\mathcal{B}(Y)$ whenever the closed ideal of $\mathcal{A}(Y)$ generated by $a$ is $\mathcal{A}(Y)$.

Remark 2.5. Suppose $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_n$. Then it is clear that $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\phi$ is a $X_n$-equivariant isomorphism.

It is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_2$, then $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism if and only if $\phi$ is an $X_2$-equivariant homomorphism and $\phi_{[1]}$ and $\phi_{[2]}$ are injective. Also, if $\mathcal{A}$ and $\mathcal{A}[2]$ have non-zero projections $p$ and $q$ respectively, then there exists $\epsilon > 0$ such that if $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism and $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that

$$\|\phi(p) - \psi(p)\| < 1 \quad \|\phi(q) - \psi(q)\| < 1,$$

then $\psi$ is a full $X_2$-equivariant homomorphism.

Remark 2.6. Let $\epsilon_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathcal{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathcal{E}_i$ can be considered as a $C^*$-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathcal{B}_i$ in $\mathcal{E}_i$, and $\{1, 2\}$ to $\mathcal{E}_i$. Hence, there exists a one-to-one correspondence between $X_2$-equivariant homomorphisms $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and homomorphisms from $\epsilon_1$ and $\epsilon_2$.

2.2. The ideal related $K$-theory of $\mathcal{A}$.

Definition 2.7. Let $X$ be a topological space and let $\mathcal{A}$ be a $C^*$-algebra over $X$. For open subsets $U_1, U_2, U_3$ of $X$ with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_1 \in \mathcal{L}(X)$. Then the diagram

$$\begin{array}{ccc}
K_0(\mathcal{A}(Y_1)) & \xrightarrow{\imath_*} & K_0(\mathcal{A}(Y_2)) \xrightarrow{\pi_*} K_0(\mathcal{A}(Y_3)) \\
\partial_* \downarrow & & \partial_* \\
K_1(\mathcal{A}(Y_3)) & \xrightarrow{\pi_*} K_1(\mathcal{A}(Y_2)) & \xrightarrow{\imath_*} K_1(\mathcal{A}(Y_1))
\end{array}$$
is an exact sequence. The ideal related $K$-theory of $\mathcal{A}$, $K_X(\mathcal{A})$, is the collection of all $K$-groups thus occurring and the natural transformations \{\text{id}, \pi, \partial\}. The ideal related, ordered $K$-theory of $\mathcal{A}$, $K^\times_X(\mathcal{A})$, is $K_X(\mathcal{A})$ of $\mathcal{A}$ together with $K_0(\mathcal{A}(Y))_+$ for all $Y \in \mathbb{L}(X)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^\ast$-algebras over $X$, we will say that $\alpha : K_X(\mathcal{A}) \rightarrow K_X(\mathcal{B})$ is an isomorphism if for all $Y \in \mathbb{L}(X)$, there exists a graded group isomorphism

$$\alpha_Y : K_*(\mathcal{A}(Y)) \rightarrow K_*(\mathcal{B}(Y))$$

preserving all natural transformations. We say that $\alpha : K^\times_X(\mathcal{A}) \rightarrow K^\times_X(\mathcal{B})$ is an isomorphism if there exists an isomorphism $\alpha : K_X(\mathcal{A}) \rightarrow K_X(\mathcal{B})$ in such a way that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathbb{L}(X)$.

**Remark 2.8.** Meyer-Nest in [28] defined a similar functor $FK_X(-)$ which they called filtrated $K$-theory. For all known cases in which there exists a UCT, the natural transformation from $FK_X(-)$ to $K_X(-)$ is an equivalence. In particular, this is true for the space $X_n$.

If $Y \in \mathbb{L}(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in 0(Y) \subseteq \mathbb{L}(X)$, then $\mathcal{A}(Y) \cong \mathcal{A}(Y_1) \oplus \mathcal{A}(Y_2)$ for any $C^\ast$-algebra over $X$. Moreover, there is a natural isomorphism $K_*(\mathcal{A}(Y))$ to $K_*(\mathcal{A}(Y_1)) \oplus K_*(\mathcal{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathcal{A}(Y))$ to $K_0(\mathcal{A}(Y_1)) \oplus K_0(\mathcal{A}(Y_2))$. If $X$ is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathbb{L}(X)$ by the subset $\mathbb{L}(X)^\ast$.

**Notation 2.9.** Let $\mathcal{N}$ be the bootstrap category of Rosenberg and Schochet in [37].

Let $\mathcal{R}^\times(X)$ be the category whose objects are separable $C^\ast$-algebras over $X$ and the set of morphisms is $KK(X; \mathcal{A}, \mathcal{B})$. For a finite topological space $X$, let $\mathcal{B}(X) \subseteq \mathcal{R}^\times(X)$ be the bootstrap category of Meyer and Nest in [27]. By Corollary 4.13 of [27], if $\mathcal{A}$ is a nuclear $C^\ast$-algebra over $X$, then $\mathcal{A} \in \mathcal{B}(X)$ if and only if $\mathcal{A}([\{x\}) \in \mathcal{N}$ for all $x \in X$.

**Theorem 2.10.** (Bonkat [4] and Meyer-Nest [28]) Let $\mathcal{A}$ and $\mathcal{B}$ be in $\mathcal{R}^\times(X_n)$ such that $\mathcal{A}$ is in $\mathcal{B}(X_n)$, then the sequence

$$0 \rightarrow \text{Ext}^1_{\mathcal{N}^\ast}(FK_{X_n}(\mathcal{A}), \mathcal{B}) \rightarrow KK(X_n; \mathcal{A}, \mathcal{B}) \rightarrow \text{Hom}_{\mathcal{N}^\ast}(FK_{X_n}(\mathcal{A}), FK_{X_n}(\mathcal{B})) \rightarrow 0$$

is exact. Consequently, if $\mathcal{B}$ is in $\mathcal{B}(X_n)$, then an isomorphism from $FK_{X_n}(\mathcal{A})$ to $FK_{X_n}(\mathcal{B})$ lifts to an invertible element in $KK(X_n; \mathcal{A}, \mathcal{B})$.

**Corollary 2.11.** Let $\mathcal{A}$ and $\mathcal{B}$ be in $\mathcal{B}(X_n)$. Then an isomorphism from $K_{X_n}(\mathcal{A})$ to $K_{X_n}(\mathcal{B})$ lifts to an invertible element in $KK(X_n; \mathcal{A}, \mathcal{B})$.

**Proof.** This follows from Remark 2.8 and Theorem 2.10.

**Remark 2.12.** Let $x \in KK(X_n; \mathcal{A}, \mathcal{B})$ be an invertible element. Then $K_{X_n}(x)$ will denote the isomorphism from $K_{X_n}(\mathcal{A})$ to $K_{X_n}(\mathcal{B})$ given by $\Gamma(x)$ where we have identified $K_{X_n}(\mathcal{A})$ with $FK_{X_n}(\mathcal{A})$ and $K_{X_n}(\mathcal{B})$ with $FK_{X_n}(\mathcal{B})$.

2.3. **Functors.** We now define some functors that will be used throughout the rest of the paper. Let $X$ and $Y$ be topological spaces. For every continuous function $f : X \rightarrow Y$ we have a functor

$$f : C^\ast\text{-alg}(X) \rightarrow C^\ast\text{-alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi).$$
(1) Define $g_X^1 : X \to X_1$ by $g_X^1(x) = 1$. Then $g_X^1$ is continuous. Note that the induced functor $g_X^1 : \mathcal{C}^*-\text{alg}(X) \to \mathcal{C}^*-\text{alg}(X_1)$ is the forgetful functor.

(2) Let $U$ be an open subset of $X$. Define $g_{U,X}^2 : X \to X_2$ by $g_{U,X}^2(x) = 1$ if $x \notin U$ and $g_{U,X}^2(x) = 2$ if $x \in U$. Then $g_{U,X}^2$ is continuous. Thus the induced functor

$$g_{U,X}^2 : \mathcal{C}^*-\text{alg}(X) \to \mathcal{C}^*-\text{alg}(X_2)$$

is just specifying the extension $0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0$.

(3) We can generalize (2) to finitely many ideals. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X$ be open subsets of $X$. Define $g_{U_1,U_2,\ldots,U_n,X}^n : X \to X_n$ by $g_{U_1,U_2,\ldots,U_n,X}^n(x) = n - k + 1$ if $x \in U_k \setminus U_{k-1}$. Then $g_{U_1,U_2,\ldots,U_n,X}^n$ is continuous. Therefore, any $C^*$-algebra with ideals $0 \leq I_1 \leq I_2 \leq \cdots \leq I_n = \mathfrak{A}$ can be made into a $C^*$-algebra over $X_n$.

(4) For all $Y \in \mathbb{L}C(X)$, $r_X^Y : \mathcal{C}^*-\text{alg}(X) \to \mathcal{C}^*-\text{alg}(Y)$ is the restriction functor defined in Definition 2.19 of [27].

(5) If $f : X \to Y$ is an embedding of a subset with the subspace topology, we write

$$i_X^Y = f_* : \mathcal{C}^*-\text{alg}(X) \to \mathcal{C}^*-\text{alg}(Y).$$

By Proposition 3.4 of [27], the functors defined above induce functors from $\mathfrak{R}(X)$ to $\mathfrak{R}(Z)$, where $Z = Y, X_1, X_n$.

### 2.4. Graph $C^*$-algebras

A graph $(E^0, E^1, r, s)$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. If $E$ is a graph, the graph $C^*$-algebra $C^*(E)$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

1. $s_e^* s_e = p_r(e)$ for all $e \in E^1$
2. $s_e s_e^* \leq p_s(e)$ for all $e \in E^1$
3. $p_v = \sum_{e \in E^1 \mid s(e) = v} s_e s_e^*$ for all $v$ with $0 < |s^{-1}(v)| < \infty$.

### 3. Meta-theorems

In many cases one can obtain a classification result for a class of unital $C^*$-algebras $\mathcal{C}$ by obtaining a classification result for the class $\mathcal{C} \otimes \mathfrak{K}$, where each object in $\mathcal{C} \otimes \mathfrak{K}$ is the stabilization of an object in $\mathcal{C}$. A meta-theorem of this sort was proved by the first and second named authors in [13] Theorem 11. It was shown there that if $\mathcal{C}$ is a subcategory of the category of $C^*$-algebras, $\mathcal{C}^*-\text{alg}$, and if $F$ is a functor from $\mathcal{C}$ to an abelian category such that an isomorphism $F(\mathfrak{A} \otimes \mathfrak{K}) \cong F(\mathfrak{B} \otimes \mathfrak{K})$ lifts to an isomorphism in $\mathcal{C} \otimes \mathfrak{K} \cong \mathfrak{B} \otimes \mathfrak{K}$, then under suitable conditions, we have that $F(\mathfrak{A}) \cong F(\mathfrak{B})$ implies $\mathfrak{A} \cong \mathfrak{B}$. In [31], the second and third named authors improved this result by showing that the isomorphism $F(\mathfrak{A}) \cong F(\mathfrak{B})$ lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

In this section, we improve these results in order to deal with cases when $\mathcal{C}$ is a category (not necessarily a subcategory of $\mathcal{C}^*-\text{alg}$) and there exists a functor from $\mathcal{C}$ to $\mathcal{C}^*-\text{alg}$. An example of such a category is the category of $C^*$-algebras over \{1, 2\}, where \{1, 2\} is given the discrete topology. Then $\mathcal{C}$ is not a subcategory of $\mathcal{C}^*-\text{alg}$ but the forgetful functor (forgetting the \{1, 2\}-structure) is a functor from $\mathcal{C}$ to $\mathcal{C}^*-\text{alg}$. We also replace the condition of proper pure infiniteness by the stable weak cancellation property.
Definition 3.1. A $C^*$-algebra $\mathfrak{A}$ is said to have the weak cancellation property if $p$ is Murray-von Neumann equivalent to $q$ whenever $p$ and $q$ generate the same ideal $\mathfrak{J}$ and $[p] = [q]$ in $K_0(\mathfrak{J})$. A $C^*$-algebra is said to have the stable weak cancellation property if $M_n(\mathfrak{A})$ has the weak cancellation property for all $n \in \mathbb{N}$.

Theorem 3.2. (cf. [13] Theorem 11) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{C}^*$-alg be the category of $C^*$-algebras, and let $\mathbb{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F: \mathcal{C} \to \mathcal{C}^*$-alg, $G: \mathcal{C} \to \mathcal{D}$, and $H: \mathcal{D} \to \mathbb{Ab}$ such that

1. $H \circ G = K_0 \circ F$.
2. For objects $\mathfrak{A}$ in $\mathcal{C}$, there exist an object $\mathfrak{A}_K$ and a morphism $\kappa_{\mathfrak{A}}: \mathfrak{A} \to \mathfrak{A}_K$ such that $G(\kappa_{\mathfrak{A}})$ is an isomorphism in $\mathcal{D}$, $F(\mathfrak{A}_K) = F(\mathfrak{A}) \otimes \mathbb{K}$, and $F(\kappa_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})} \otimes e_{11}$.
3. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$, every isomorphism $G(\mathfrak{A}_K) \to G(\mathfrak{B}_K)$ is induced by an isomorphism from $\mathfrak{A}_K$ to $\mathfrak{B}_K$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be given such that $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are unital $C^*$-algebras. Let $\rho: G(\mathfrak{A}) \to G(\mathfrak{B})$ be an isomorphism such that $H(\rho)([1_{F(\mathfrak{A})}]) = [1_{F(\mathfrak{B})}]$. If $F(\mathfrak{B})$ has the stable weak cancellation property, then $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Proof. Note that $G(\kappa_{\mathfrak{A}})$ and $G(\kappa_{\mathfrak{B}})$ are isomorphisms. Therefore $G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}$ is an isomorphism from $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$. Thus, there exists an isomorphism $\phi: \mathfrak{A}_K \to \mathfrak{B}_K$ such that $G(\phi) = G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}$.

Set $\psi = F(\phi)$. Then $\psi: F(\mathfrak{A}) \otimes \mathbb{K} \to F(\mathfrak{B}) \otimes \mathbb{K}$ is a $*$-isomorphism such that $K_0(\psi) = K_0(F(\phi)) = H(G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}) = H(G(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ H(G(\kappa_{\mathfrak{A}})^{-1}) = K_0(F(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ K_0(F(\kappa_{\mathfrak{A}}))^{-1} = K_0(\text{id}_{F(\mathfrak{B})}) \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}$.

Hence,

$$K_0(\psi)([1_{F(\mathfrak{A})} \otimes e_{11}]) = K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}([1_{F(\mathfrak{A})} \otimes e_{11}])$$

$$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathfrak{A})}])$$

$$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11})([1_{F(\mathfrak{B})}])$$

$$= [1_{F(\mathfrak{B})} \otimes e_{11}].$$

Stable weak cancellation implies that there exists $v \in F(\mathfrak{B}) \otimes \mathbb{K}$ such that $v^*v = \psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathfrak{B})} \otimes e_{11}$ since $\psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $1_{F(\mathfrak{B})} \otimes e_{11}$ are full projections in $F(\mathfrak{B}) \otimes \mathbb{K}$. Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [13] Theorem 11, $\gamma$ is an isomorphism from $F(\mathfrak{A}) \otimes e_{11}$ to $F(\mathfrak{B}) \otimes e_{11}$. Hence, $F(\mathfrak{A}) \cong F(\mathfrak{B})$. □

Theorem 3.3. (cf. [32] Theorem 2.1) Let $\mathcal{C}$ be a subcategory of $\mathcal{C}^*$-alg($\mathfrak{X}$). Moreover, $\mathcal{C}$ is assumed to be closed under tensoring by $\mathcal{M}_2(\mathfrak{C})$ and $\mathbb{K}$ and contains the canonical embeddings $\kappa_1: \mathfrak{A} \to \mathcal{M}_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ as morphisms for every object $\mathfrak{A}$ in $\mathcal{C}$. Assume there is a functor $F: \mathcal{C} \to \mathcal{D}$ satisfying

1. For $\mathfrak{A}$ in $\mathcal{C}$, the embeddings $\kappa_1: \mathfrak{A} \to \mathcal{M}_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.
2. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$ that are stable $C^*$-algebras, every isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
3. There exists a functor $G$ from $\mathcal{D}$ to $\mathbb{Ab}$ such that $G \circ F = K_0$.       
Assume that every $X$-equivariant isomorphism between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ and that for objects $\mathfrak{A}$ in $\mathcal{C}$, $F(\text{Ad}(u)\mathfrak{A}) = \text{id}_{F(\mathfrak{A})}$ for every unitary $u \in M(\mathfrak{A})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are objects $\mathcal{C}$ that are unital $C^*$-algebras and if $\mathfrak{A}$ and $\mathfrak{B}$ have the stable weak cancellation property and there is an isomorphism $\alpha : F(\mathfrak{A}) \to F(\mathfrak{B})$ such that $G(\alpha)([1_\mathfrak{A}]) = [1_\mathfrak{B}]$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ in $\mathcal{C}$ such that $F(\phi) = \alpha$.

**Proof.** The difference between the statement of Theorem 2.1 of [32] and statement of the theorem are

(i) $\mathcal{C}$ is assumed to be a subcategory of $\mathcal{C}^\star_{\mathfrak{Alg}}(X)$ instead of a subcategory of $\mathcal{C}^\star_{\mathfrak{Alg}}$.

(ii) $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to have the stable weak cancellation property instead of being properly infinite.

In the proof of Theorem 2.1 of [32], properly infinite was needed to insure that $\psi(1_\mathfrak{A} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_\mathfrak{B} \otimes e_{11}$, where $\psi : \mathfrak{A} \otimes K \to \mathfrak{B} \otimes K$ is the isomorphism from (2) that lifts the isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ that is induced by $\alpha$. As in the proof of Theorem 3.2 we get that $\psi(1_\mathfrak{A} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_\mathfrak{B} \otimes e_{11}$. Arguing as in the proof of Theorem 2.1 of [32], we get the desired result.

\(\square\)

4. Classification results

In this section, we show that $K^+_X(-)$ is a strong classification functor for a class of $C^*$-algebras with exactly one proper nontrivial ideal containing $C^*$-algebras associated to finite graphs. The results of this section will be used in the next section to show that $K^+_X(-)$ together with the appropriate scale is a complete isomorphism invariant for $C^*$-algebras associated to graphs. Moreover, in a forthcoming paper, we use these results to solve the following extension problem: If $\mathfrak{A}$ fits into the following exact sequence

$$0 \to C^*(E) \otimes K \to \mathfrak{A} \to C^*(G) \to 0,$$

where $C^*(E)$ and $C^*(G)$ are simple $C^*$-algebras, then when is $\mathfrak{A} \cong C^*(F)$ for some graph $F$?

**Theorem 4.1.** (Existence Theorem) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is a positive isomorphism for all $Y \in \mathbb{L}C(X_2)$. Suppose $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension, $\mathfrak{A}_i[2]$ is a stable $C^*$-algebra, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra with real rank zero, and either

(i) $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra; or

(ii) $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple $C^*$-algebra.

Then there exists an $X_2$-equivariant homomorphism $\phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \phi) = KK(X_2; id_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; id_{\mathfrak{A}_2} \otimes e_{11})$, and $\phi(2)$ and $\phi(1)$ are injective, where $\{e_{ij}\}$ is a system of matrix units for $K$.

**Proof.** Set $y = KK(X_2; id_{\mathfrak{A}_2} \otimes e_{11})^{-1} \times x \times KK(X_2; id_{\mathfrak{A}_2} \otimes e_{11})$. Note that by Lemma 3.10 and Theorem 3.8 of [14], $\mathfrak{A}_i[2] \otimes K$ satisfies the corona factorization property (see [21] for the definition of the corona factorization property). Since $\mathfrak{A}_i[2]$ is an AF-algebra or an Kirchberg algebra, $\mathfrak{A}_i[2]$ has the stable weak cancellation. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation. Let $e_i$ be the extension

$$0 \to \mathfrak{A}_i[2] \otimes K \to \mathfrak{A}_i \otimes K \to \mathfrak{A}_i[1] \otimes K \to 0.$$
By Corollary 3.24 of [15], $\epsilon_i$ is a full extension since $A_i[1]$ has cancellation of projections (in the AF case) and $A_i[1]$ is properly infinite (in the purely infinite case).

Case (i): $A_i[2]$ is a purely infinite simple $C^*$-algebra and $A_i[1]$ is an AF-algebra. By Theorem 3.3 of [14], $r^{(1)}_{X_2}(y) \times [r_{X_2}] = \{r_{X_2}\} \times r^{(2)}_{X_2}(y)$ in $KK^1(A_1[1] \otimes K, A_2[2] \otimes K)$. Since $y$ is invertible in $KK(X_2, A_1 \otimes K, A_2 \otimes K)$, we have that $r^{(1)}_{X_2}(y)$ is invertible in $KK(A_1[1] \otimes K, A_2[1] \otimes K)$ and $\Gamma(r^{(1)}_{X_2}(y)) = \Gamma(x)_{\{2\}}$ is a positive isomorphism. Thus, by Elliott's classification [19], there exists an isomorphisms $\psi : A_1[1] \otimes K \rightarrow A_2[1] \otimes K$ such that $KK(\psi) = r^{(1)}_{X_2}(y)$.

Since $y$ is invertible in $KK(X_2, A_1 \otimes K, A_2 \otimes K)$, we have that $r^{(2)}_{X_2}(y)$ is invertible in $KK(A_1[2] \otimes K, A_2[2] \otimes K)$. Thus, by Kirchberg-Phillips classification (see 20 and 29), there exists an isomorphism $\psi_0 : A_1[2] \otimes K \rightarrow A_2[2] \otimes K$ such that $KK(\psi_0) = r^{(2)}_{X_2}(y)$. By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in M(A_2[2] \otimes K)$ such that $\psi = (Ad(u) \circ \psi_0, Ad(u) \circ \psi_0, \psi_0)$ is an $X_2$-equivariant isomorphism from $A_1[2] \otimes K$ to $A_2[2] \otimes K$, where $\psi_0 : M(A_1[2] \otimes K) \rightarrow M(A_1[2] \otimes K)$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_{\{1\}}) = r^{(1)}_{X_2}(y)$ for $k = 1, 2$.

Note that

$$0 \rightarrow i^{X_2}_{\{2\}}((A_1 \otimes K)[2]) \xrightarrow{\lambda_2} A_1 \otimes K \xrightarrow{\beta_1} i^{X_2}_{\{1\}}((A_1 \otimes K)[1]) \rightarrow 0$$

is a semi-split extension of $C^*$-algebras over $X_2$ (see Definition 3.5 of [27]). Set

$$J_i = i^{X_2}_{\{2\}}((A_i \otimes K)[2]) \quad \text{and} \quad \mathcal{B}_i = i^{X_2}_{\{1\}}((A_i \otimes K)[1]).$$

By Theorem 3.6 of [27] (see also Korollar 3.4.6 of [4]),

$$KK(X_2; A_1 \otimes K, J_2) \xrightarrow{(\lambda_2)^*} KK(X_2; A_1 \otimes K, A_2 \otimes K) \xrightarrow{(\beta_2)^*} KK(X_2; A_1 \otimes K, B_2)$$

is exact. By Proposition 3.12 of [27], $KK(X_2; A_1 \otimes K, J_2)$ and $KK(A_1[1] \otimes K, A_2[2] \otimes K)$ are naturally isomorphic. Hence, there exists $z \in KK(X_2; A_1 \otimes K, J_2)$ such that $y - KK(X_2; \psi) = z \times KK(X_2; \lambda_2)$ since $KK(\psi_{\{1\}}) = r^{(1)}_{X_2}(y)$.

By Proposition 3.13 of [27], $KK(X_2; A_1 \otimes K, J_2)$ and $KK(A_1[1] \otimes K, A_2[2] \otimes K)$ are isomorphic. By Theorem 8.3.3 of [36] (see also Hauptsatz 4.2 of [20]), there exists a $*$-homomorphism $\eta : A_1 \otimes K \rightarrow (A_2 \otimes K)[2]$ such that $KK(\eta) = \pi$, where $\pi$ is the image of $z$ under the isomorphism $KK(X_2; A_1 \otimes K, J_2) \cong KK(A_1 \otimes K, (A_2 \otimes K)[2])$. Note that $\eta$ induces an $X_2$-equivariant homomorphism $\eta : A_1 \otimes K \rightarrow J_2$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\lambda_2 \circ \eta)$, where the sum is the Cuntz sum in $M(A_2 \otimes K)$. Then $\phi : A_1 \otimes K \rightarrow A_2 \otimes K$ is an $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective homomorphisms.

Case (ii): $A_i[2]$ is an AF-algebra and $A_i[1]$ is a purely infinite simple $C^*$-algebra. By Theorem 3.3 of [14], $r^{(1)}_{X_2}(y) \times [r_{X_2}] = \{r_{X_2}\} \times r^{(2)}_{X_2}(y)$ in $KK^1(A_1[1] \otimes K, A_2[2] \otimes K)$. Since $y$ is invertible in $KK(X_2, A_1 \otimes K, A_2 \otimes K)$, we have that $r^{(2)}_{X_2}(y)$ is invertible in $KK(A_1[2] \otimes K, A_2[2] \otimes K)$ and $\Gamma(r^{(2)}_{X_2}(y)) = \Gamma(x)_{\{2\}}$ is an order isomorphism. Thus, by Elliott's classification [19], there exists an isomorphism $\psi_0 : A_1[2] \otimes K \rightarrow A_2[2] \otimes K$ such that $KK(\psi_0) = r^{(2)}_{X_2}(y)$.

Since $y$ is invertible in $KK(X_2, A_1 \otimes K, A_2 \otimes K)$, we have that $r^{(1)}_{X_2}(y)$ is invertible in
there exists an isomorphism $\psi : \mathcal{A}_1 \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(\psi_1) = r_{X_2}^{(1)}(y)$. By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in \mathcal{M}(\mathcal{A}_2[2] \otimes \mathbb{K})$ such that $\psi = (\text{Ad}(u) \circ \psi_0, \text{Ad}(u) \circ \psi_0, \psi_1)$ is an $X_2$-equivariant isomorphism from $\mathcal{A}_1 \otimes \mathbb{K}$ to $\mathcal{A}_2 \otimes \mathbb{K}$, where $\psi_0 : \mathcal{M}(\mathcal{A}_1[2] \otimes \mathbb{K}) \rightarrow \mathcal{M}(\mathcal{A}_1[2] \otimes \mathbb{K})$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_1) = r_{X_2}^{(k)}(y)$ for $k = 1, 2$.

Note that

$$0 \rightarrow i_{X_2}^{(2)}((\mathcal{A}_i \otimes \mathbb{K})[2]) \xrightarrow{\lambda_i} \mathcal{A}_i \otimes \mathbb{K} \xrightarrow{\beta_i} i_{X_2}^{(1)}((\mathcal{A}_i \otimes \mathbb{K})[1]) \rightarrow 0$$

is a semi-split extension of $C^*$-algebras over $X_2$ (see Definition 3.5 of [27]). Set

$$\mathcal{J}_i = i_{X_2}^{(2)}((\mathcal{A}_i \otimes \mathbb{K})[2]) \quad \text{and} \quad \mathcal{B}_i = i_{X_2}^{(1)}((\mathcal{A}_i \otimes \mathbb{K})[1]).$$

By Theorem 3.6 of [27] (see also Korollar 3.4.6 [4])

$$KK(X_2; \mathcal{B}_1, \mathcal{A}_2 \otimes \mathbb{K}) \xrightarrow{(\beta_1)^*} KK(X_2; \mathcal{A}_1 \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K}) \xrightarrow{(\lambda_1)^*} KK(X_2; \mathcal{J}_1, \mathcal{A}_2 \otimes \mathbb{K})$$

is exact. By Proposition 3.12 of [27], $KK(X_2; \mathcal{J}_1, \mathcal{A}_2 \otimes \mathbb{K})$ and $KK(\mathcal{A}_1[2] \otimes \mathbb{K}, \mathcal{A}_2[2] \otimes \mathbb{K})$ are naturally isomorphic. Hence, there exists $z \in KK(X_2; \mathcal{B}_1, \mathcal{A}_2 \otimes \mathbb{K})$ such that $y - KK(X; \psi) = KK(X_2; \beta_1) \times z$ since $KK((\mathcal{A}_1 \otimes \mathbb{K})[1], \mathcal{A}_2 \otimes \mathbb{K})$ are isomorphic. Therefore, by Theorem 8.3.3 of [36], there exists a homomorphism $\eta : (\mathcal{A}_1 \otimes \mathbb{K})[1] \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(\eta) = \tilde{z}$, where $\tilde{z}$ is the image of $z$ under the isomorphism $KK(X_2; \mathcal{B}_1, \mathcal{A}_2 \otimes \mathbb{K}) \cong KK((\mathcal{A}_1 \otimes \mathbb{K})[1], \mathcal{A}_2 \otimes \mathbb{K})$ (the existence of the homomorphism uses the fact that $\mathcal{A}_2 \otimes \mathbb{K}$ is a properly infinite $C^*$-algebra which follows from Proposition 3.21 and Theorem 3.22 of [15]). Note that $\eta$ induces an $X_2$-equivariant homomorphism $\eta : \mathcal{B}_1 \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\eta \circ \beta_1)$, where the sum is the Cuntz sum in $\mathcal{M}(\mathcal{A}_2 \otimes \mathbb{K})$. Then $\phi$ is an $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{[2]}$ and $\psi_{[1]}$ are injective homomorphisms, $\phi_{[2]}$ and $\phi_{[1]}$ are injective homomorphisms. \hfill \Box

### 4.1. Strong classification of extensions of AF-algebras by purely infinite $C^*$-algebras.

**Definition 4.2.** Let $\mathcal{A}$ and $\mathcal{B}$ be separable $C^*$-algebras over $X$. Two $X$-equivariant homomorphisms $\phi, \psi : \mathcal{A} \rightarrow \mathcal{B}$ are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathcal{B})$ such that

$$\lim_{n \rightarrow \infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0$$

for all $a \in \mathcal{A}$.

We now recall the definition of $KL(\mathcal{A}, \mathcal{B})$ from [33].

**Definition 4.3.** Let $\mathcal{A}$ be a separable, nuclear $C^*$-algebra in $\mathcal{N}$ and let $\mathcal{B}$ be a $\sigma$-unital $C^*$-algebra. Let

$$\text{Ext}^1_2(K_0(\mathcal{A}), K_{*+1}(\mathcal{B})) = \text{Ext}^1_{KL}(K_0(\mathcal{A}), K_1(\mathcal{B})) \oplus \text{Ext}^1_{KL}(K_1(\mathcal{B}, K_0(\mathcal{B})).$$

Since $\mathcal{A}$ is in $\mathcal{N}$, by [37], $\text{Ext}^1_{KL}(K_0(\mathcal{A}), K_{*+1}(\mathcal{B}))$ can be identified as a sub-group of the group $KK(\mathcal{A}, \mathcal{B})$. 

For abelian groups, $G$ and $H$, let $\text{Pext}^1_{\mathbb{Z}}(G,H)$ be the subgroup of $\text{Ext}^1_{\mathbb{Z}}(G,H)$ of all pure extensions of $G$ by $H$. Set

$$\text{KGL}(\mathfrak{A}, \mathfrak{B}) = \text{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) = \text{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Pext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Define $\text{KGL}(\mathfrak{A}, \mathfrak{B})$ as the quotient

$$\text{KGL}(\mathfrak{A}, \mathfrak{B}) = K\text{K}(\mathfrak{A}, \mathfrak{B}) / \text{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})).$$

Rørdam in [33] proved that if $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $\text{KGL}(\phi) = \text{KGL}(\psi)$.

**Notation 4.4.** Let $x \in K\text{K}(\mathfrak{A}, \mathfrak{B})$. Then the element $x + \text{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B}))$ in $\text{KGL}(\mathfrak{A}, \mathfrak{B})$ will be denoted by $\text{KGL}(x)$.

A nuclear, purely infinite, separable, simple $C^*$-algebra will be called a Kirchberg algebra.

**Theorem 4.5.** (Uniqueness Theorem 1) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, $C^*$-algebras over $X_\mathcal{D}$ such that $\mathfrak{A}_i$ has real rank zero, $\mathfrak{A}_i$ is stable, $\mathfrak{A}_i[2]$ is a Kirchberg algebra in $N$, $\mathfrak{A}_i[1]$ is an AF-algebra, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Suppose $\phi, \psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be $X_\mathcal{D}$-equivariant homomorphism such that $K\text{K}(X_\mathcal{D}; \phi) = K\text{K}(X_\mathcal{D}; \psi)$, and $\phi[2], \psi[2], \phi'[2], \psi'[2]$, and $\psi[1]$ are injective homomorphisms. Then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** Since $\mathfrak{A}_i[1]$ is an AF algebra, every finitely generated subgroup of $K_0(\mathfrak{A}_i[1])$ is torsion free (hence free) and every finitely generated subgroup of $K_1(\mathfrak{A}_i[1])$ is zero. Thus,

$$\text{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}_i[1]), K_1(\mathfrak{Q}(\mathfrak{A}_j[2]))) = \text{Ext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}_i[1]), K_1(\mathfrak{Q}(\mathfrak{A}_j[2])))$$

which implies that $\text{KGL}(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_j[2])) \cong \text{Hom}(K_0(\mathfrak{A}_i[1]), K_1(\mathfrak{Q}(\mathfrak{A}_j[2])))$.

Let $e_i$ denote the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$. Since $\mathfrak{A}_i$ has real rank zero and $K_1(\mathfrak{A}_i[1]) = 0$, we have that $K_j(\tau_{e_i}) = 0$, where $\tau_{e_i}$ is the Busby invariant of $e_i$. Hence, $[\tau_{e_i}] = 0$ in $\text{KGL}(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_i[2]))$. By Corollary 6.7 of [24], $e_i$ is quasi-diagonal. Thus, there exists an approximate identity of $\mathfrak{A}_i[2]$ consisting of projections $\{e_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \|e_k x - xe_k\| = 0$$

for all $x \in \mathfrak{A}_i$.

Since $\mathfrak{A}_i[1]$ is an AF-algebra and $\mathfrak{A}_i$ has real rank zero, as in the proof of Lemma 9.8 of [10], there exists a sequence of finite dimensional sub-$C^*$-algebras $\{\mathfrak{B}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_i$ such that $\mathfrak{B}_k \cap \mathfrak{A}_i[2] = \{0\}$ and for each $x \in \mathfrak{A}_i$, there exist $y_1 \in \bigcup_{k=1}^{\infty} \mathfrak{B}_k$ and $y_2 \in \mathfrak{A}_i[2]$ such that $x = y_1 + y_2$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Note that we may assume $\mathcal{F}$ is the union of the generators of $\mathfrak{B}_m$, for some $m \in \mathbb{N}$ and $\mathcal{G}$, for some finite subset $\mathcal{G}$ of $\mathfrak{A}_1[2]$. Since $\mathfrak{B}_m$ is a finite dimensional $C^*$-algebra,

$$\lim_{k \to \infty} \|e_k x - xe_k\| = 0$$

for all $x \in \mathfrak{A}_i$, and $\{e_k\}_{k \in \mathbb{N}}$ is an approximate identity for $\mathfrak{A}_i[2]$ consisting of projections, there exist $k \in \mathbb{N}$, a finite dimensional sub-$C^*$-algebra $\mathfrak{D}$ of $\mathfrak{A}_i$ with $\mathfrak{D} \subseteq (1_{M(\mathfrak{A}_i)} - e_k)\mathfrak{A}_1(1_{M(\mathfrak{A}_i)} - e_k)$ and $\mathfrak{D} \cap \mathfrak{A}_i[2] = \{0\}$, and there exists a finite subset $\mathcal{H}$ of $e_k \mathfrak{A}_i[2]e_k$ such that for all $x \in \mathcal{F}$, there exist $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H}$

$$\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}.$$
Set $\mathcal{D} = \bigoplus_{\ell=1}^s M_{n_{\ell}}$ and let $\{f_{ij}^\ell\}_{i,j=1}^{n_{\ell}}$ be a system of matrix units for $M_{n_{\ell}}$. Let $\mathfrak{I}_\ell$ be the ideal in $\mathfrak{A}_\ell$ generated by $f_{11}^\ell$. Since $\mathfrak{A}_i[2]$ is simple and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, we have that $\mathfrak{A}_i[2] \subseteq \mathfrak{I}$ for all nonzero ideal $\mathfrak{I}$ of $\mathfrak{A}_i$. Thus, $\mathfrak{A}_1[2] \subseteq \mathfrak{I}_\ell$ since $\mathcal{D} \cap \mathfrak{A}_1[2] = 0$.

Let $\mathfrak{I}_\ell^\phi$ be the ideal in $\mathfrak{A}_2$ generated by $\phi(f_{11}^\ell)$ and let $\mathfrak{I}_\ell^\psi$ be the ideal in $\mathfrak{A}_2$ generated by $\psi(f_{11}^\ell)$. Since $\phi$ and $\psi$ are $X_2$-equivariant homomorphisms and since $\phi(1)$ and $\psi(1)$ are injective homomorphisms, we have that $\phi(f_{11}^\ell) \notin \mathfrak{A}_2[2]$ and $\psi(f_{11}^\ell) \notin \mathfrak{A}_2[2]$. Therefore, $\mathfrak{A}_2[2] \subseteq \mathfrak{I}_\ell^\phi$ and $\mathfrak{A}_2[2] \subseteq \mathfrak{I}_\ell^\psi$. Since $K_0(\phi(1)) = K_0(\psi(1))$ and since $\mathfrak{A}_2[1]$ is an AF-algebra, we have that $\phi(1)(\mathfrak{I}_1)$ is Murray-von Neumann equivalent to $\psi(1)(\mathfrak{I}_1)$, where $\mathfrak{I}_1$ is the image of $f_{11}^\ell$ in $\mathfrak{A}_1[1]$. Thus, they generate the same ideal in $\mathfrak{A}_2[1]$. Since $\mathfrak{A}_2[2] \subseteq \mathfrak{I}_\ell^\phi$ and $\mathfrak{A}_2[2] \subseteq \mathfrak{I}_\ell^\psi$ and since $\psi(1)(\mathfrak{I}_1)$ and $\phi(1)(\mathfrak{I}_1)$ generate the same ideal in $\mathfrak{A}_2[1]$, we have that $\mathfrak{I} = \mathfrak{I}_\ell^\phi = \mathfrak{I}_\ell^\psi$.

Note that the following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_0(\mathfrak{A}_2[2]) \\
\downarrow & & \downarrow K_0(\mathfrak{I}) \\
0 & \longrightarrow & K_0(\mathfrak{A}_2) \\
\end{array}
\]

is commutative, the rows are exact, and $\iota$ and $\pi$ are the canonical embeddings. Since $\mathfrak{A}_2[1]$ is an AF-algebra, $K_0(\mathfrak{I})$ is injective. A diagram chase shows that $K_0(\iota)$ is injective. Since $KK(\mathfrak{X}_2; \phi) = KK(\mathfrak{X}_2; \psi)$, we have that $[\phi(f_{11}^\ell)] = [\psi(f_{11}^\ell)]$ in $K_0(\mathfrak{A}_2)$. Since $\phi(f_{11}^\ell)$ and $\psi(f_{11}^\ell)$ are elements of $\mathfrak{I}$ and $K_0(\iota)$ is injective, we have that $[\phi(f_{11}^\ell)] = [\psi(f_{11}^\ell)]$ in $K_0(\mathfrak{I})$. Since $\mathfrak{A}_i[1]$ is an AF-algebra and $\mathfrak{A}_i[2]$ is a Kirchberg algebra, they both have stable weak cancellation. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation. Thus, $\phi(f_{11}^\ell)$ is Murray-von Neumann equivalent to $\psi(f_{11}^\ell)$. Hence, there exists $v_\ell \in \mathfrak{A}_2$ such that

\[
v_\ell^* v_\ell = \phi(f_{11}^\ell) \quad \text{and} \quad v_\ell v_\ell^* = \psi(f_{11}^\ell).
\]

Set

\[
u_1 = \sum_{\ell=1}^s \sum_{i=1}^{n_{\ell}} \psi(f_{ii}^\ell) v_\ell \phi(f_{ii}^\ell)
\]

Then, $u_1$ is a partial isometry in $\mathfrak{A}_1$ such that $u_1^* u_1 = \phi(1)$, $u_1 u_1^* = \psi(1)$, and $u_1 \phi(x) u_1^* = \psi(x)$ for all $x \in \mathcal{D}$.

Let $\beta : e_k \mathfrak{A}_1[2] e_k \to \mathfrak{A}_1[2]$ be the usual embedding. Note that $KK(\phi(2) \circ \beta) = KK(\psi(2) \circ \beta)$ and $\phi(2) \circ \beta$, $\psi(2) \circ \beta$ are monomorphisms. Therefore, by Theorem 6.7 of [23], there exists a partial isometry $u_2 \in \mathfrak{A}_2[2]$ such that $u_2^* u_2 = \phi(e_k)$, $u_2 u_2^* = \psi(e_k)$, and

\[
\|u_2 \phi(x) u_2^* - \psi(x)\| < \frac{\epsilon}{3}
\]

for all $x \in \mathcal{H}$.

Since $\mathfrak{A}_2$ is stable, there exists $u_3 \in \mathcal{M}(\mathfrak{A}_2)$ such that $u_3^* u_3 = 1_{\mathcal{M}(\mathfrak{A}_2)} - (u_1 + u_2)^* (u_1 + u_2)$ and $u_3 u_3^* = 1_{\mathcal{M}(\mathfrak{A}_2)} - (u_1 + u_2)(u_1 + u_2)^*$. Set $u = u_1 + u_2 + u_3 \in \mathcal{M}(\mathfrak{A}_2)$. Then $u$ is a unitary in $\mathcal{M}(\mathfrak{A}_2)$. 

Lemma 4.6. Let $\mathfrak{A}$ be a separable $C^*$-algebra over a finite topological space $X$. Let $u$ be unitary in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$. Then $K_X(\text{Ad}(u)_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$.

Proof. Since $\mathfrak{A} \otimes \mathbb{K}$ is stable, we have that there exists a norm continuous path of unitaries $\{u_t\}$ in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ such that $u_0 = u$ and $u_1 = 1_{\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})}$. It follows that $K_X(\text{Ad}(u)_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$.

Theorem 4.7. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is an order isomorphism for all $Y \in \mathbb{LC}(X_2)$. Suppose $\mathfrak{A}_i[2]$ is a Kirchberg algebra, $\mathfrak{A}_i[1]$ is an AF-algebra, $\mathfrak{A}_i$ has real rank zero, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Then there exists an $X_2$-equivariant isomorphism $\phi: \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KL(\phi) = KL(g^{1}_{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_n; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$.

Proof. Since $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra, $\mathfrak{A}_i[2]$ is either unital or stable. Since $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, $\mathfrak{A}_i[2]$ is non-unital else $\mathfrak{A}_i[2]$ is isomorphic to a direct summand of $\mathfrak{A}_i$ which would contradict the essential assumption. Therefore, $\mathfrak{A}_i[2]$ is stable. Moreover, $\mathcal{Q}(\mathfrak{A}_i[2])$ is simple which implies that $0 \rightarrow \mathfrak{A}_i[2] \rightarrow \mathfrak{A}_i \rightarrow \mathfrak{A}_i[1] \rightarrow 0$ is a full extension. Since $\mathfrak{A}_i[2]$ and $\mathfrak{A}_i[1]$ are nuclear $C^*$-algebras, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra.

Let $z \in KK(X_2; \mathfrak{A}_2 \otimes \mathbb{K}, \mathfrak{A}_1 \otimes \mathbb{K})$ such that $y \times z = [\text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}]$ and $y \times z = [\text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}}]$. By Theorem 4.4, there exists an $X_2$-equivariant homomorphism $\psi_1: \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \psi_1) = x$, and $(\psi_1)_{(2)}$ and $(\psi_1)_{(1)}$ are injective homomorphisms. By Theorem 4.4, there exists an $X_2$-equivariant homomorphism $\psi_2: \mathfrak{A}_2 \otimes \mathbb{K} \rightarrow \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \psi_2) = y$, and $(\psi_2)_{(2)}$ and $(\psi_2)_{(1)}$ are injective homomorphisms. Using Theorem 4.5 and a typical approximate intertwining argument, there exists an isomorphism $\phi: \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K}$ such that $\phi$ and $\psi_1$ are approximately unitarily equivalent.

Let $\pi_2: \mathfrak{A}_2 \rightarrow \mathfrak{A}_2[1]$ be the canonical quotient map. Then $\pi_2 \circ \phi|_{\mathfrak{A}_2[2]}$ is either zero or injective since $\mathfrak{A}_2[2]$ is simple. Since $\mathfrak{A}_2[2]$ is purely infinite and $\mathfrak{A}_2[1]$ is an AF-algebra, we must have that $\pi_2 \circ \phi|_{\mathfrak{A}_2[2]} = 0$. Thus, $\phi$ is an $X_2$-equivariant homomorphism. Similarly, $\phi^{-1}$ is an $X_2$-equivariant homomorphism. Hence, $\phi$ is an $X_2$-equivariant isomorphism. By construction, $KL(\phi) = KL(\psi_1) = KL(g^{1}_{X_2}(y))$. By Lemma 4.6, $K_{X_2}(\phi) = K_{X_2}(x)$.

Corollary 4.8. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is an order isomorphism for all $Y \in \mathbb{LC}(X_2)$. Suppose $\mathfrak{A}_i[2]$ is a Kirchberg algebra, $\mathfrak{A}_i[1]$ is an AF-algebra, $\mathfrak{A}_i$ has real rank zero, $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, and $K_i(\mathfrak{A}[Y])$ and $K_i(\mathfrak{B}[Y])$ are finitely generated for all $Y \in \mathbb{LC}(X_2)$. Then there exists an $X_2$-equivariant isomorphism $\phi: \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\phi) = KK(g^{1}_{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_n; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$.
4.2. **Strong classification of extensions of purely infinite by $\mathbb{K}$**. We recall the following from [1] p. 341. Let $\psi : \mathfrak{A} \to B(H)$ be a representation of $\mathfrak{A}$. Let $H_e$ denote the subspace of $H$ spanned by the ranges of all compact operators in $\psi(\mathfrak{A})$. Since $\psi(\mathfrak{A}) \cap \mathbb{K}$ is an ideal of $\psi(\mathfrak{A})$, we have that $H_e$ reduces $\pi(\mathfrak{A})$, and so the decomposition $H = H_e \oplus H_e^\perp$ induces a decomposition of $\psi$ into sub-representations $\psi = \psi_e \oplus \psi'$. The summand $\psi_e$, considered as a representation of $\mathfrak{A}$ on $H_e$, will be called the **essential part of $\psi$** and $H_e$ is called the **essential subspace** for $\psi$.

Let $\mathfrak{B}$ be a tight $C^*$-algebra over $X_2$. Consider the essential extension

$$
\epsilon_{\mathfrak{B}} : 0 \to \mathfrak{B}[2] \to \mathfrak{B} \to \mathfrak{B}[1] \to 0.
$$

If $\tau_{\mathfrak{B}} : \mathfrak{B}[1] \to \mathcal{Q}(\mathfrak{B}[2])$ is the Busby invariant of $\epsilon$, then there exists an injective homomorphism $\sigma_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{M}(\mathfrak{B}[2])$ such that the diagram

$$
\begin{array}{c}
0 \to \mathfrak{B}[2] \to \mathfrak{B} \oplus \mathfrak{B}[1] \to 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \to \mathfrak{B}[2] \to \mathcal{M}(\mathfrak{B}[2]) \to \mathcal{Q}(\mathfrak{B}[2]) \to 0
\end{array}
$$

If $\mathfrak{B}[2] \cong \mathbb{K}$, let $\eta_{\mathfrak{B}} : \mathcal{M}(\mathfrak{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathfrak{B}[2] \cong \mathbb{K}$ and let $\overline{\eta}_{\mathfrak{B}} : \mathcal{Q}(\mathfrak{B}[2]) \to B(\ell^2)/\mathbb{K}$ be the induced isomorphism.

**Lemma 4.9.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}[2] \cong \mathfrak{B}[2] \cong \mathbb{K}$. Let $\psi_1, \psi_2 : \mathfrak{A} \to \mathfrak{B}$ be two, full $X_2$-equivariant homomorphisms such that $K_0(\{\psi_1\}_{\{2\}}) = K_0(\{\psi_2\}_{\{2\}})$ and $\eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1$ is a non-degenerate representation of $\mathfrak{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[2])$ such that

$$
U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a) \in \mathfrak{B}[2]
$$

for all $a \in \mathfrak{A}$ and for all $n \in \mathbb{N}$, and

$$
\lim_{n \to \infty} \|U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a)\| = 0
$$

for all $a \in \mathfrak{A}$.

**Proof.** We argue as in the proof of Lemma 2.8 of [22]. Set $\sigma_i = \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_i$. By assumption, $\sigma_i : \mathfrak{A} \to B(\ell^2)$ is a non-degenerated representation of $\mathfrak{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^{\infty}$ in $B(\ell^2)$ such that $V_n \sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K}$ for all $n \in \mathbb{N}$ and

$$
\lim_{n \to \infty} \|V_n \sigma_1(a)V_n^* - \sigma_2(a)\| = 0
$$

for all $a \in \mathfrak{A}$. This will be a consequence of Theorem 5(iii) of [1].

Let $\rho : \mathfrak{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathfrak{A}[2] \cong \mathbb{K}$. Since $\psi_i, \sigma_{\mathfrak{B}}, \eta_{\mathfrak{B}}$ are injective homomorphisms, $\sigma_i$ is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi : B(\ell^2) \to B(\ell^2)/\mathbb{K}$ be the natural projection. Note that

$$
\pi \circ \sigma_i = \pi \circ \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_i = \overline{\eta}_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_i = \overline{\eta}_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \tau_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \psi_i = \overline{\eta}_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \psi_i \{1\} \circ \pi_{\mathfrak{A}}.
$$
It now follows that \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = \mathfrak{A}[2] \) since \( \pi_{\mathfrak{M}} \), \( \tau_{\mathfrak{E}_\beta} \), and \( (\psi_1)_{\{1\}} \) are injective homomorphisms.

Let \( H_1 \) be the essential subspace of \( \sigma_1 \). Since \( \sigma_1(\mathfrak{A}[2]) \subseteq \mathbb{K} \) and for each \( x \notin \mathfrak{A}[2] \), we have that \( \sigma_1(x) \notin \mathbb{K} \), we have that \( H_1 = \sigma_1(\mathfrak{A}[2])\ell^2 \). Similarly, we have that \( H_2 = \sigma_2(\mathfrak{A}[2])\ell^2 \), where \( H_2 \) is the essential subspace of \( \sigma_2 \). Let \( e \) be a minimal projection of \( \mathfrak{A}[2] \cong \mathbb{K} \). Suppose \( \sigma_1(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_1(-)|_{H_1} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \). Since \( K_0((\psi_1)_{\{2\}}) = K_0((\psi_2)_{\{2\}}) \), we have that \( \sigma_1(e) \) is Murray-von Neumann equivalent to \( \sigma_2(e) \). Hence, \( \sigma_2(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_2(-)|_{H_2} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \).

The above paragraph implies that \( \sigma_2(-)|_{H_2} \) and \( \sigma_1(-)|_{H_1} \) are unitarily equivalent. Since \( \ker(\sigma_1) = \ker(\sigma_2) \) and \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) \) by Theorem 5(iii) of [1], there exists a sequence of unitaries \( \{V_n\}_{n=1}^\infty \) in \( B(\ell^2) \) such that \( V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K} \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and

\[
\lim_{n \to \infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

Set \( U_n = \eta_{\mathfrak{M}}^{-1}(V_n) \). Then \( \{U_n\}_{n=1}^\infty \) is a sequence of unitaries in \( M(\mathfrak{M}[2]) \) such that \( U_n(\sigma_{\mathfrak{E}_\beta} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{E}_\beta} \circ \psi_2)(a) \in \mathfrak{M}[2] \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and

\[
\lim_{n \to \infty} \|U_n(\sigma_{\mathfrak{E}_\beta} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{E}_\beta} \circ \psi_2)(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

Definition 4.10. A C*-algebra \( \mathfrak{A} \) is called weakly semiprojective if we can always solving the *-homomorphism lifting problem

\[
\mathfrak{A} \xrightarrow{\phi} \prod_{n=1}^\infty \mathfrak{B}_n \xrightarrow{\rho_N} \prod_{n=1}^\infty \mathfrak{B}_n / \bigoplus_{n=1}^\infty \mathfrak{B}_n
\]

and \( \mathfrak{A} \) is called semiprojective if we can always solve the lifting problem

\[
\mathfrak{A} \xrightarrow{\phi} \mathfrak{B} / \mathfrak{I}_N \xrightarrow{\rho_N} \prod_{n=1}^\infty \mathfrak{B}_n / \bigoplus_{n=1}^\infty \mathfrak{I}_n
\]

Lemma 4.11. Let \( \mathfrak{A}_0 \) be a unital, separable, nuclear, tight C*-algebra over \( X_2 \) such that \( \mathfrak{A}_0[2] \cong \mathbb{K} \) and \( \mathfrak{A}_0 \) has the stable weak cancellation property. Set \( \mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K} \). Suppose \( \beta : \mathfrak{A} \to \mathfrak{A} \) is a full \( X_2 \)-equivariant homomorphism such that \( K_{X_2}(\beta) = K_{X_2}(\text{id}_\mathfrak{A}) \) and \( \beta_{\{1\}} = \text{id}_\mathfrak{A}^{\{1\}} \). Then there exists a sequence of contractive, completely positive, linear maps \( \{\alpha_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^\infty \) such that

1. \( \alpha_n|_{\mathfrak{A}_{\mathfrak{E}_\beta}} \) is a homomorphism for all \( n \in \mathbb{N} \) and
2. for all \( a \in \mathfrak{A} \),

\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(a) - a\| = 0
\]
where \( e_n = \sum_{k=1}^{n} 1_{\mathcal{A}_0} \otimes e_{kk} \) and \( \{ e_{ij} \}_{i,j} \) is a system of matrix units for \( \mathcal{K} \). If, in addition, \( \mathcal{A} \) is assumed to be weakly semiprojective, then \( \alpha_n \) can be chosen to be a homomorphism for all \( n \in \mathbb{N} \).

**Proof.** Since \( \beta \) is a full \( X_2 \)-equivariant homomorphism and the ideal in \( \mathcal{A} \) generated by \( e_n \) is \( \mathcal{A} \), we have that the ideal in \( \mathcal{A} \) generated by \( \beta(e_n) \) is \( \mathcal{A} \). Since \( K_{X_2}(\beta) = K_{X_2}(\mathcal{K}_0) \), we have that \( [\beta(e_n)] = [e_n] \) in \( K_0(\mathcal{A}) \). It now follows that \( \beta(e_n) \) and \( e_n \) are Murray-von Neumann equivalent since \( \mathcal{A}_0 \) has the stable weak cancellation property. Since \( \mathcal{A} \) is stable, there exists a unitary \( v_n \) in the unitization of \( \mathcal{A} \) such that \( v_n \beta(e_n) v_n^* = e_n \).

Fix \( n \in \mathbb{N} \). Let \( e_n \) be the extension \( 0 \to e_n \mathcal{A}[2] e_n \to e_n \mathcal{A} e_n \to \mathcal{T}_n \mathcal{A}[1] \mathcal{T}_n \to 0 \). By Lemma 1.5 of [16], \( e_n \) is a full extension. Therefore, \( \sigma(e_n) \) is Murray-von Neumann equivalent to \( 1_{\mathcal{M}(\mathcal{A}[2])} \). Hence, \( e_n \mathcal{A}[2] e_n \cong \mathcal{A}[2] \cong \mathcal{K} \). Set \( \mathcal{A}_n = e_n \mathcal{A} e_n \) and define \( \beta_n : \mathcal{A}_n \to \mathcal{A}_n \) by \( \beta_n(x) = Ad(v_n) \circ \beta(x) \). Then \( \beta_n \) is a unital, full \( X_2 \)-equivariant homomorphism. Since \( \eta_{\mathcal{A}_n} \circ \sigma_{\mathcal{A}_n} \circ \beta_n \) is a unitary representation of \( \mathcal{A}_n \), the closed subspace of \( \ell^2 \) generated by \( \{ (\eta_{\mathcal{A}_n} \circ \sigma_{\mathcal{A}_n} \circ \beta_n)(x) \xi : x \in \mathcal{A}_n, \xi \in \ell^2 \} \) is \( \ell^2 \). Therefore, \( \eta_{\mathcal{A}_n} \circ \sigma_{\mathcal{A}_n} \circ \beta_n \) is a non-degenerate representation of \( \mathcal{A}_n \).

Since \( K_{X_2}(\beta) = K_{X_2}(\mathcal{K}_0) \) and the \( X_2 \)-equivariant embedding of \( \mathcal{A}_n \) as a sub-algebra of \( \mathcal{A} \) induces an isomorphism in ideal related \( K \)-theory, we have that \( K_{X_2}(\beta_n) = K_{X_2}(\mathcal{K}_n) \). By Lemma 4.9 there exists a sequence of unitaries \( W_{k,n} \in \mathcal{M}(\mathcal{A}_n[2]) \) such that

\[
(Ad(W_{k,n}) \circ \sigma_{\mathcal{A}_n} \circ \beta_n)(x) - \sigma_{\mathcal{A}_n}(x) \in \mathcal{A}_n[2]
\]

for all \( x \in \mathcal{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (Ad(W_{k,n}) \circ \sigma_{\mathcal{A}_n} \circ \beta_n)(x) - \sigma_{\mathcal{A}_n}(x) \| = 0
\]

for all \( x \in \mathcal{A}_n \).

Note that \( \mathcal{M}(\mathcal{A}_n[2]) \cong \sigma_\epsilon(e_n) \mathcal{M}(\mathcal{A}[2]) \sigma_\epsilon(e_n) \) with an isomorphism mapping \( \mathcal{A}_n[2] \) onto \( e_n \mathcal{A}[2] e_n \). Thus, we get a partial isometry \( \tilde{W}_{k,n} \in \mathcal{M}(\mathcal{A}[2]) \) such that \( \tilde{W}_{k,n}^* \tilde{W}_{k,n} = W_{k,n} \) and

\[
(Ad(\tilde{W}_{k,n}) \circ \sigma_\epsilon \circ Ad(v_n) \circ \beta)(x) - \sigma_\epsilon(x) \in \mathcal{A}[2]
\]

for all \( x \in \mathcal{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (Ad(\tilde{W}_{k,n}) \circ \sigma_\epsilon \circ Ad(v_n) \circ \beta)(x) - \sigma_\epsilon(x) \| = 0
\]

for all \( x \in \mathcal{A}_n \).

Set \( V_{k,n} = (W_{k,n} + 1_{\mathcal{M}(\mathcal{A}[2])} - \sigma_\epsilon(e_n)) \sigma_\epsilon(v_n) \). Then \( V_{k,n} \) is a unitary in \( \mathcal{M}(\mathcal{A}[2]) \) such that

\[
(Ad(V_{k,n}) \circ \sigma_\epsilon \circ \beta)(x) - \sigma_\epsilon(x) \in \mathcal{A}[2]
\]

for all \( x \in e_n \mathcal{A} e_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (Ad(V_{k,n}) \circ \sigma_\epsilon \circ \beta)(x) - \sigma_\epsilon(x) \| = 0
\]

for all \( x \in e_n \mathcal{A} e_n \). A consequence of the first part is that \( (Ad(V_{k,n}) \circ \sigma_\epsilon \circ \beta)(x) \in \sigma_\epsilon(e_n \mathcal{A} e_n) + \mathcal{A}[2] \) for all \( x \in e_n \mathcal{A} e_n \). Since \( \beta_{(1)} = \mathcal{K}_n[2] \), we have that \( x - \beta(x) \in \mathcal{A}[2] \) for all \( x \in e_n \mathcal{A} e_n \). Therefore,

\[
Ad(V_{k,n})(\sigma_\epsilon(x)) = Ad(V_{k,n}) \circ \sigma_\epsilon(x - \beta(x)) + Ad(V_{k,n}) \circ \beta(x) \in \sigma_\epsilon(e_n \mathcal{A} e_n) + \mathcal{A}[2]
\]

Thus, \( \alpha_{k,n} = \sigma_\epsilon^{-1} \circ (Ad(V_{k,n}) \circ \sigma_\epsilon \circ Ad(v_n))|_{e_n \mathcal{A} e_n} \) is a homomorphism from \( e_n \mathcal{A} e_n \) to \( \mathcal{A} \).
Since
\[
\lim_{k \to \infty} \|(\text{Ad}(V_{k,n}) \circ \sigma_t \circ \beta)(x) - \sigma_t(x)\| = 0
\]
for all \(x \in e_n \mathfrak{A} e_n\) and \(e_n \mathfrak{A} e_n \subseteq e_{n+1} \mathfrak{A} e_{n+1}\), there exists a strictly increasing sequence \(\{k(n)\}_{n=1}^\infty\) of positive integers such that
\[
\lim_{n \to \infty} \|\alpha_{k(n),n} \circ \beta(x) - x\| = 0
\]
for all \(x \in \bigcup_{n=1}^\infty e_n \mathfrak{A} e_n\). Let \(\alpha_n\) be a completely, contractive, positive linear extension of \(\alpha_{k(n),n}\). Since \(\bigcup_{n=1}^\infty e_n \mathfrak{A} e_n\) is dense in \(\mathfrak{A}\), we have that
\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(x) - x\| = 0
\]
for all \(x \in \mathfrak{A}\). We have just proved the first part of the lemma.

We now show that \(\mathfrak{A}\) can be chosen to be a homomorphism provide that \(\mathfrak{A}\) is weakly semiprojective. Suppose \(\mathfrak{A}\) is weakly semiprojective. Let \(\epsilon > 0\) and \(F\) be a finite subset of \(\mathfrak{A}\). By Theorem 2.4 of [23] (see also Definition 2.1 and Theorem 2.3 of [25], and Theorem 19.1.3 of [26]), there exist a \(\delta > 0\) and a finite subset \(G\) of \(\mathfrak{A}\) such that for any \(C^*\)-algebra \(\mathfrak{B}\) and any contractive, completely positive, linear map \(L : \mathfrak{A} \to \mathfrak{B}\) such that
\[
\|L(ab) - L(a)L(b)\| < \delta
\]
for all \(a, b \in G\), there exists a homomorphism \(h : \mathfrak{A} \to \mathfrak{B}\) such that
\[
\|h(x) - L(x)\| < \frac{\epsilon}{2}
\]
for all \(x \in \beta(F)\).

Without loss of generality, we may assume that \(\epsilon < 1\) and \(\delta < 1\). Set
\[
M = 1 + \max (\{\|a\| : a \in G\} \cup \{\|x\| : x \in F\})
\]
Since \(e_n \mathfrak{A} e_n \subseteq e_{n+1} \mathfrak{A} e_{n+1}\) and \(\bigcup_{n=1}^\infty e_n \mathfrak{A} e_n\) is dense in \(\mathfrak{A}\), there exist \(n \in \mathbb{N}\) and a finite subset \(H \subseteq e_n \mathfrak{A} e_n\) such that for each \(a \in G\), there exists \(y \in H\) such that \(\|a - y\| < \frac{\delta}{4M}\) and
\[
\|\alpha_n \circ \beta(x) - x\| < \frac{\epsilon}{2}
\]
for all \(x \in F\). Let \(a, b \in G\). Choose \(x, y \in H \subseteq e_n \mathfrak{A} e_n\) such that \(\|a - x\| < \frac{\delta}{4M}\) and \(\|b - y\| < \frac{\delta}{4M}\). Note that \(\|x\| \leq 1 + \|\alpha\| \leq M\) and \(\|y\| \leq 1 + \|b\| \leq M\). Then
\[
\|\alpha_n(ab) - \alpha_n(a)\alpha_n(b)\| = \|\alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b)\|
\leq \|b\|\|a - x\| + \|x\|\|b - y\|
\leq 2M\|a - x\| + 2M\|b - y\|
< 4M\frac{\delta}{4M} = \delta.
\]
By the choice of \(\delta\) and \(G\), there exists a homomorphism \(\psi : \mathfrak{A} \to \mathfrak{A}\) such that
\[
\|\psi(t) - \alpha_n(t)\| < \frac{\epsilon}{2}
\]
for all $t \in \beta(\mathcal{F})$. Let $x \in \mathcal{F}$. Then

$$\|\psi \circ \beta(x) - x\| \leq \|\psi(\beta(x)) - \alpha_n(\beta(x))\| + \|\alpha_n(\beta(x)) - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$  

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{A}$, there exists a homomorphism $\psi : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|\psi \circ \beta(x) - x\| < \epsilon$$

for all $x \in \mathcal{F}$. Consequently, there exists a sequence of endomorphisms $\{\psi_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \|\psi_n \circ \beta(x) - x\| = 0$$

for all $x \in \mathfrak{A}$ since $\mathfrak{A}$ is separable. 

To prove a uniqueness theorem involving tight $C^*$-algebras $\mathfrak{A}$ over $X_2$, we require that $\mathfrak{A}[1]$ belongs to a class of $C^*$-algebras whose injective homomorphisms between two objects in this class are classified by $KK$.

**Definition 4.12.** We will be interested in classes $\mathcal{C}$ of separable, nuclear, simple $C^*$-algebras satisfying the following property that if $\mathfrak{A}, \mathfrak{B} \in \mathcal{C}$ and $\phi, \psi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$ are two injective homomorphisms such that $KK(\phi) = KK(\psi)$, then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Remark 4.13.**

1. By Theorem 4.1.3 of [29] if $\mathcal{C}$ is the class of Kirchberg algebras, then $\mathcal{C}$ satisfies the property in Definition 4.12.

2. Let $\mathcal{C}$ be the class of unital, separable, nuclear, simple tracially AF $C^*$-algebras in $\mathcal{N}$. Then $\mathcal{C}$ satisfies the property in Definition 4.12.

**Theorem 4.14.** (Uniqueness Theorem 2) Let $\mathcal{C}$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathfrak{A}$ be a unital, separable, nuclear, tight $C^*$-algebra over $X_2$ such that $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1] \in \mathcal{C}$. Suppose $\mathfrak{A} \otimes \mathbb{K}$ is semiprojective and $\mathfrak{A}$ has the stable weak cancellation property. Let $\phi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}$ be a full $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \id_{\mathfrak{A} \otimes \mathbb{K}})$. Then there exists a sequence of full $X_2$-equivariant endomorphisms $\{\alpha_n : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \id_{\mathfrak{A} \otimes \mathbb{K}})$ and

$$\lim_{n \to \infty} \|\alpha_n \circ \phi(x) - x\| = 0$$

for all $x \in \mathfrak{A} \otimes \mathbb{K}$.

**Proof.** Set $\mathfrak{B} = \mathfrak{A} \otimes \mathbb{K}$. Note that $\mathfrak{B}$ is a tight $C^*$-algebra over $X_2$ with $\mathfrak{B}[2] \cong \mathbb{K}$. Throughout the proof, $\pi : \mathfrak{B} \to \mathfrak{B}[1]$ will denote the canonical projection. Note that $KK(\phi_{[1]}) = KK(\id_{\mathfrak{B}[1]})$ since $KK(X_2; \phi) = KK(X_2; \id_{\mathfrak{B}})$. Since $\mathfrak{A}[1] \in \mathcal{C}$, there exists a sequence of unitaries $\{z_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[1])$ such that

$$\lim_{k \to \infty} \|z_k \phi_{[1]}(\pi(b)) z_k^* - \pi(b)\| = 0$$

for all $b \in \mathfrak{B}$. Using the fact that $\phi$ is an $X_2$-equivariant homomorphism, we have that $\pi \circ \phi = \phi_{[1]} \circ \pi$, and hence

$$\lim_{k \to \infty} \|z_k (\pi \circ \phi(b)) z_k^* - \pi(b)\| = 0$$
for all \( b \in \mathfrak{B} \).

Let \( \pi : \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{B}[1]) \) be the surjective homomorphism induced by \( \pi \). Since \( \mathfrak{B} \) is stable, by Corollary 2.3 of [35], we have that \( \mathfrak{B}[1] \) is stable. Thus, the unitary group of \( \mathcal{M}(\mathfrak{B}[1]) \) is path-connected, which implies that every unitary in \( \mathcal{M}(\mathfrak{B}[1]) \) lifts to a unitary in \( \mathcal{M}(\mathfrak{B}) \). Hence, there exists a sequence of unitaries \( \{ w_k \}_{k=1}^{\infty} \) in \( \mathcal{M}(\mathfrak{B}) \) such that \( \pi(w_k) = z_k \).

Since \( \mathfrak{B} \) is semi-projective, by Proposition 2.2 of [7] (see [26]), there exists a sequence of homomorphisms \( \{ \beta_\ell : \mathfrak{B} \to \mathfrak{B} \}_{\ell=1}^{\infty} \) and a strictly increasing sequence \( \{ k(\ell) \}_{\ell=1}^{\infty} \) of positive integers such that \( \pi \circ \beta_\ell = \pi \) and

\[
\lim_{\ell \to \infty} \| \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b) \| = 0
\]

for all \( b \in \mathfrak{B} \).

By Remark 2.3 there exists \( N_1 \in \mathbb{N} \) such that \( \beta_\ell \) is a full \( X_2\)-equivariant homomorphism for all \( \ell \geq N_1 \). By Proposition 2.3 of [7], we may choose \( N_2 \geq N_1 \) such that for all \( \ell \geq N_2 \), we have that \( \beta_\ell \) and \( \text{Ad}(w_{k(\ell)}) \circ \phi \) is homotopic. It follows from Theorem 5.5 of [8] that \( KK(X_2; \beta_\ell) = KK(X_2; \text{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \text{id}_{\mathfrak{B}}) \).

Let \( \ell \geq N_2 \). Note that \( (\beta_\ell)_{[1]} = \text{id}_{\mathfrak{B}[1]} \) since \( \pi \circ \beta_\ell = \pi \). Since \( \mathfrak{B} \) is semi-projective, by Corollary 3.6 of [6] (also see Chapter 19 of [26]), \( \mathfrak{B} \) is weakly semi-projective. Hence, by Lemma 4.11 there exists a sequence of homomorphisms \( \{ \alpha_{m,\ell} : \mathfrak{B} \to \mathfrak{B} \}_{m=1}^{\infty} \) such that

\[
\lim_{m \to \infty} \| \alpha_{m,\ell} \circ \beta_\ell(x) - x \| = 0
\]

for all \( x \in \mathfrak{B} \). Since \( \beta_\ell \) and \( \text{id}_{\mathfrak{B}} \) are full \( X_2\)-equivariant homomorphisms, by Remark 2.5 there exists \( N_3 \) such that, for all \( m \geq N_3 \), we have that \( \alpha_{m,\ell} \) is a full \( X_2\)-equivariant homomorphism. Moreover, by Proposition 2.3 of [7], we can choose \( N_3 \geq N_2 \) such that \( \alpha_{m,\ell} \circ \beta_\ell \) and \( \text{id}_{\mathfrak{B}} \) are homotopic. It follows from Theorem 5.5 of [8] that \( KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \text{id}_{\mathfrak{B}}) \) for all \( m \geq N_3 \). Consequently, \( KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_{\mathfrak{B}}) \) for all \( m \geq N_3 \) since \( KK(X_2; \beta_\ell) = KK(X_2; \text{id}_{\mathfrak{B}}) \).

Let \( F \) be a finite subset of \( \mathfrak{B} \) and \( \epsilon > 0 \). Then there exists \( \ell \geq N_2 \) such that

\[
\| \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b) \| < \frac{\epsilon}{2}
\]

for all \( b \in F \). Moreover, there exists \( m \geq N_3 \) such that

\[
\| \alpha_{m,\ell} \circ \beta_\ell(b) - b \| < \frac{\epsilon}{2}
\]

for all \( b \in F \). Set \( \alpha_1 = \text{Ad}(w_{k(\ell)}) \circ \phi \) and \( \alpha = \alpha_{m,\ell} \circ \alpha_1 \). Since \( w_{k(\ell)} \) is a unitary in \( \mathcal{M}(\mathfrak{B}) \), we have that \( \alpha_1 \) is an automorphism of \( \mathfrak{B} \) and \( KK(X_2; \alpha_1) = KK(X_2; \text{id}_{\mathfrak{B}}) \). Therefore, \( \alpha \) is a full \( X_2\)-equivariant homomorphism. Since \( \ell \geq N_2 \) and \( m \geq N_3 \), we have that \( KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_{\mathfrak{B}}) \). Therefore, \( KK(X_2; \alpha) = KK(X_2; \text{id}_{\mathfrak{B}}) \). Let \( b \in F \). Then

\[
\| \alpha \circ \phi(b) - b \| = \| \alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - b \|
\leq \| \alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b) \| + \| \alpha_{m,\ell} \circ \beta_\ell(b) - b \|
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

We have just shown that for every \( \epsilon > 0 \) and for every finite subset \( F \) of \( \mathfrak{B} \), there exists a full \( X_2\)-equivariant homomorphism \( \alpha : \mathfrak{B} \to \mathfrak{B} \) such that \( KK(X_2; \alpha) = KK(X_2; \text{id}_{\mathfrak{B}}) \) and

\[
\| \alpha \circ \phi(b) - b \| < \epsilon
\]
for all \( b \in \mathcal{B} \). Since \( \mathcal{B} \) is a separable \( C^* \)-algebra, there exists a sequence of full \( X_2 \)-equivariant homomorphisms \( \{\alpha_n : \mathcal{B} \to \mathcal{B}\}_{n=1}^{\infty} \) such that \( KK(X_2;\alpha_n) = KK(X_2;\mathrm{id}_\mathcal{B}) \) and
\[
\lim_{n \to \infty} \|\alpha_n \circ \phi(b) - b\| = 0
\]
for all \( b \in \mathcal{B} \).

**Theorem 4.15.** Let \( C \) be a class of \( C^* \)-algebras satisfying the property in Definition \( \ref{def:strong-classification} \) and let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be unital, separable, nuclear, tight \( C^* \)-algebras over \( X_2 \) such that \( \mathfrak{A}_i[2] \cong K \) and \( \mathfrak{A}_i[1] \in \mathcal{C} \). Suppose \( \mathfrak{A}_i \otimes K \) is semiprojective and \( \mathfrak{A}_i \) has the stable weak cancellation property. If there exist full \( X_2 \)-equivariant homomorphisms, \( \phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) and \( \psi : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K \), such that \( KK(X_2;\phi \circ \psi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2 \otimes K}) \) and \( KK(X_2;\psi \circ \phi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_1 \otimes K}) \), then for any finite subset \( \mathcal{F} \) and \( \epsilon > 0 \), there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) such that \( KK(X_2;\gamma) = KK(\phi) \) and
\[
\|\gamma(x) - \phi(x)\| < \epsilon
\]
for all \( x \in \mathcal{F} \).

**Proof.** Let \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) be a sequence of finite subsets of \( \mathfrak{A}_1 \otimes K \) such that \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes K \) and let \( \{\mathcal{G}_n\}_{n=1}^{\infty} \) be a sequence of finite subsets of \( \mathfrak{A}_2 \otimes K \) such that \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \) and \( \bigcup_{n=1}^{\infty} \mathcal{G}_n \) is dense in \( \mathfrak{A}_2 \otimes K \).

Let \( \epsilon > 0 \) and \( \mathcal{F} \) be a finite subset of \( \mathfrak{A}_1 \). Set \( \mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_1 \) and choose \( m_1 \in \mathbb{N} \) such that \( \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon \). By Theorem \( \ref{thm:strong-classification} \) there exists a full \( X_2 \)-equivariant homomorphism \( \alpha_1 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_1 \otimes K \) such that \( KK(X_2;\alpha_1) = KK(X_2;\mathrm{id}_{\mathfrak{A}_1 \otimes K}) \) and
\[
\|\alpha_1 \circ \psi \circ \phi(a) - a\| < \frac{1}{2^{m_1+1}}
\]
for all \( a \in \mathcal{F}_1 \). Set \( \phi_1 = \phi \) and \( \psi_1 = \alpha_1 \circ \psi \). Then \( KK(X_2;\psi_1) = KK(X_2;\psi) \) and
\[
\|\psi_1 \circ \phi_1(a) - a\| < \frac{1}{2^{m_1+1}}
\]
for all \( a \in \mathcal{F}_1 \).

Set \( \mathcal{G}_1 = \mathcal{G}_1 \cup \phi(\mathcal{F}_1) \). Note that \( KK(X_2;\phi \circ \psi_1) = KK(X_2;\phi \circ \psi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2 \otimes K}) \). Hence, by Theorem \( \ref{thm:strong-classification} \) there exists a full \( X_2 \)-equivariant homomorphism \( \beta_1 : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_2 \otimes K \) such that \( KK(X_2;\beta_1) = KK(X_2;\mathrm{id}_{\mathfrak{A}_2 \otimes K}) \) and
\[
\|\beta_1 \circ \phi \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}
\]
for all \( x \in \mathcal{G}_1 \). Set \( \phi_2 = \beta_1 \circ \phi \). Then \( KK(X_2;\phi_2) = KK(X_2;\phi) \) and
\[
\|\phi_2 \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}
\]
for all \( x \in \mathcal{G}_1 \). Note that for all \( x \in \mathcal{F}_1 \), then
\[
\|\phi(x) - \phi_2(x)\| \leq \|\phi_1(x) - \phi_2 \circ \psi_1(\phi_1(x))\| + \|\phi_2 \circ \psi_1(\phi_1(x)) - \phi_2(x)\|
\]
\[
< \frac{1}{2^{m_1+1}} + \|\psi_1 \circ \phi_1(x) - x\| < \frac{1}{2^{m_1}}.
\]

Set \( \mathcal{F}_2 = \mathcal{F}_2 \cup \phi_2(\mathcal{G}_1) \). Note that \( KK(X_2;\psi \circ \phi_2) = KK(X_2;\psi \circ \phi) = KK(X_2;\mathrm{id}_{\mathfrak{A}_1 \otimes K}) \). Hence, by Theorem \( \ref{thm:strong-classification} \) there exists a full \( X_2 \)-equivariant homomorphism \( \alpha_2 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \).
Hence, by Theorem 4.14, there exists a full morphisms \( \{ F \} \) for all \( x \) such that
\[
\| \alpha_2 \circ \psi \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}}
\]
for all \( a \in \mathcal{F}_2 \). Set \( \psi_2 = \alpha_2 \circ \psi \). Then \( KK(X_2; \psi_2) = KK(X_2; \psi) \) and
\[
\| \psi_2 \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{F}_2 \).

Set \( \mathcal{G}_2 = \overline{\mathcal{F}_2} \cup \phi_2(\mathcal{F}_2) \). Note that \( KK(X_2; \phi \circ \psi_2) = KK(X_2; \phi \circ \psi) = KK(X_2; id_{A_2} \otimes K) \).

Hence, by Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \beta_2 : A_2 \otimes K \to A_2 \otimes K \) such that \( KK(X_2; \beta_2) = KK(X_2; id_{A_2} \otimes K) \) and
\[
\| \beta_2 \circ \phi \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{G}_2 \). Set \( \phi_3 = \beta_2 \circ \phi \). Then \( KK(X_2; \phi_3) = KK(X_2; \phi) \) and
\[
\| \phi_3 \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{G}_2 \). Note that for all \( x \in \mathcal{F}_2 \), we have that
\[
\| \phi_2(x) - \phi_3(x) \| \leq \| \phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x)) \| + \| \phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x) \|
\leq \frac{1}{2^{m_1+2}} + \| \psi_2(\phi_2(x)) - x \| < \frac{1}{2^{m_1+1}}.
\]

Continuing this process, we have constructed a sequence \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) of finite subsets of \( A_1 \otimes K \), a sequence \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \) of finite subsets of \( A_2 \otimes K \), a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \phi_n : A_1 \otimes K \to A_2 \otimes K \}_{n=1}^{\infty} \), and a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \psi_n : A_2 \otimes K \to A_1 \otimes K \}_{n=1}^{\infty} \) such that

1. \( KK(X_2; \phi_n) = KK(X_2; \phi) \) for all \( n \in \mathbb{N} \) and \( \phi_1 = \phi \);
2. \( KK(X_2; \psi_n) = KK(X_2; \psi) \) for all \( n \in \mathbb{N} \);
3. \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \mathcal{F}_{n+1} \subseteq \mathcal{F}_n \);
4. \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \) and \( \mathcal{G}_{n+1} \subseteq \mathcal{G}_n \);
5. for each \( x \in \mathcal{F}_n \) and for each \( x \in \mathcal{G}_n \)
\[
\| \psi_n \circ \phi_n(x) - x \| < \frac{1}{2^{m_1+n}} \quad \text{and} \quad \| \phi_{n+1} \circ \psi_n(x) - x \| < \frac{1}{2^{m_1+n}}
\]
6. for each \( x \in \mathcal{F}_n \),
\[
\| \phi_n(x) - \phi_{n+1}(x) \| < \frac{1}{2^{m_1+n-1}}
\]

Since \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( A_1 \otimes K \) and \( \overline{\mathcal{F}_n} \subseteq \mathcal{F}_n \), we have that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( A_1 \otimes K \). Similarly, \( \bigcup_{n=1}^{\infty} \mathcal{G}_n \) is dense in \( A_2 \otimes K \). Therefore, there exists an isomorphism \( \gamma : A_1 \otimes K \to A_2 \otimes K \) such that
\[
\| \gamma(a) - \phi_n(a) \| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k}
\]
for all \( a \in \mathcal{F}_n \). Since \( \mathcal{F} \subseteq \mathcal{F}_1 \), we have that
\[
\| \phi(x) - \gamma(x) \| = \| \phi_1(x) - \gamma(x) \| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon.
\]
Since
\[
\lim_{n \to \infty} \sum_{k=m_1 + n - 1}^{\infty} \frac{1}{2^k} = 0,
\]
we have that
\[
\lim_{n \to \infty} \| \gamma(a) - \phi_n(a) \| = 0
\]
for all \( a \in \mathfrak{A}_1 \otimes \mathbb{K} \). Since \( \mathfrak{A}_1 \otimes \mathbb{K} \) is semiprojective, by Proposition 2.3 of [7], there exists \( N \in \mathbb{N} \) such that \( \gamma \) and \( \phi_N \) are homotopic. Hence, by Theorem 5.5 of [8], \( \text{KK}(X_2; \gamma) = \text{KK}(X_2; \phi_N) = x \).

4.3. Unital Classification. We know combine the above results with the Meta-theorem of Section 3 (see Theorem 3.3) to get a strong classification for a class of unital \( C^* \)-algebras which includes all unital graph \( C^* \)-algebras with exactly one non-trivial ideal.

**Corollary 4.16.** Let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be unital, tight \( C^* \)-algebras over \( X_n \) such that \( \mathfrak{A}_n \) has real rank zero, \( \mathfrak{A}_i \) is a Kirchberg algebra in \( \mathcal{N} \), and \( \mathfrak{A}_i[1, n - 1] \) is an AF-algebra. Let \( x \in \text{KK}(X_2; \mathfrak{A}_1, \mathfrak{A}_2) \) be an invertible such that \( K_{X_n}(x) \) is an order isomorphism for each \( Y \in \mathcal{L}(X_n) \) and \( K_{X_n}(x)_{X_n}(1_{\mathfrak{A}_1}) = 1_{\mathfrak{A}_2} \) in \( K_0(\mathfrak{A}_2) \). Then there exists an isomorphism \( \phi : \mathfrak{A}_1 \rightarrow \mathfrak{B} \) such that \( K_{X_n}(\phi) = K_{X_n}(x) \).

**Proof.** Since \( \mathfrak{A}_i[1] \) and \( \mathfrak{A}_i[2] \) are separable and nuclear, we have that \( \mathfrak{A}_i \) is separable and nuclear. Since \( \mathfrak{A}_i[1, n - 1] \) is an AF-algebra and \( \mathfrak{A}_i[n] \) is a Kirchberg algebra, they both have the stable weak cancellation property. By Lemma 3.15 of [15], \( \mathfrak{A}_i \) has stable weak cancellation property. By Lemma 4.6 for each tight \( C^* \)-algebra \( \mathfrak{A} \) over \( X_n \), we have that \( K_{X_n}(\mathfrak{A} u)_{|\mathfrak{A}} \) for each unitary \( u \in \mathcal{M}(\mathfrak{A}) \). A computation shows that \( K_{X_n}(\gamma) \) satisfies (1), (2), and (3) of Theorem 3.3 since \( K_{X_n}(\gamma) \) does. The corollary now follows from Theorem 3.3 and Theorem 4.7.

**Corollary 4.17.** Let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be unital, tight \( C^* \)-algebras over \( X_2 \) such that \( \mathfrak{A}_i[2] \cong \mathbb{K} \) and \( \mathfrak{A}_i[1] \) is a Kirchberg algebra in \( \mathcal{N} \). Let \( x \in \text{KK}(X_2; \mathfrak{A}_1, \mathfrak{A}_2) \) be an invertible such that \( K_{X_2}(x) \) is an order isomorphism for each \( Y \in \mathcal{L}(X_2) \) and \( K_{X_2}(x)_{X_2}(1_{\mathfrak{A}_1}) = 1_{\mathfrak{A}_2} \) in \( K_0(\mathfrak{A}_2) \). If \( \mathfrak{A}_i \otimes \mathbb{K} \) is semiprojective, then there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( \text{KK}(X_2; \gamma) = x \).

**Proof.** Since \( \mathfrak{A}_i[1] \) and \( \mathfrak{A}_i[2] \) are separable and nuclear, we have that \( \mathfrak{A}_i \) is separable and nuclear. Since \( \mathfrak{A}_i[2] \) and \( \mathfrak{A}_i[1] \) have real rank zero and \( K_1(\mathfrak{A}_i[2]) = 0 \), we have that \( \mathfrak{A} \) has real rank zero. Since \( \mathfrak{A}_i[2] \) is an AF-algebra and \( \mathfrak{A}_i[1] \) is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by Lemma 3.15 of [15], \( \mathfrak{A} \) has the stable weak cancellation property.

By Lemma 1.5 of [16], the extension \( 0 \rightarrow \mathfrak{A}_i[2] \rightarrow \mathfrak{A}_i \rightarrow \mathfrak{A}_i[1] \rightarrow 0 \) is full, and hence by Proposition 1.6 of [16], \( 0 \rightarrow \mathfrak{A}_i[2] \otimes \mathbb{K} \rightarrow \mathfrak{A}_i \otimes \mathbb{K} \rightarrow \mathfrak{A}_i[1] \otimes \mathbb{K} \rightarrow 0 \) is full. The corollary now follows from Theorem 4.1(iii), Theorem 4.15 and Theorem 3.3.
It is an open question to determine if every unital, separable, nuclear, tight C*-algebra \( \mathfrak{A} \) over \( X_2 \) whose unique proper nontrivial ideal is isomorphic to \( \mathbb{K} \) and quotient is a Kirchberg algebra in \( \mathcal{N} \) with finitely generated \( K \)-theory is semiprojective. The following results show that under some \( K \)-theoretical conditions, \( \mathfrak{A} \) is semiprojective.

**Lemma 4.18.** Let \( E \) be a graph with finitely many vertices such that \( C^\ast(E) \) is a tight C*-algebra over \( X_2 \) with \( C^\ast(E)[1] \) being purely infinite. Then \( C^\ast(E) \) and \( C^\ast(E) \otimes \mathbb{K} \) are semiprojective.

**Proof.** The fact that \( C^\ast(E) \) is semiprojective follows from the results of \([12]\). By Proposition 6.4 of \([18]\), \( C^\ast(E)[2] \) is stable. Since \( C^\ast(E) \) is a unital C*-algebra, by Lemma 1.5 of \([16]\), the extension \( \varepsilon : 0 \to C^\ast(E)[2] \to C^\ast(E) \to C^\ast(E)[1] \to 0 \) is a full extension. By Proposition 3.21 and Corollary 3.22 of \([15]\), \( C^\ast(E) \) is properly infinite. Therefore, by Theorem 4.1 of \([3]\), \( C^\ast(E) \otimes \mathbb{K} \) is semiprojective. \( \square \)

**Proposition 4.19.** Let \( \mathfrak{A} \) be unital, separable, nuclear, tight C*-algebras over \( X_2 \). If \( \mathfrak{A}[2] \cong \mathbb{K} \) and \( \mathfrak{A}[1] \) is a Kirchberg algebra in \( \mathcal{N} \) such that \( \text{rank}(K_1(\mathfrak{A}[1])) \leq \text{rank}(K_0(\mathfrak{A}[1])) \), \( K_1(\mathfrak{A}[1]) \) is free, and the \( K \)-groups of \( \mathfrak{A}[i] \) are finitely generated, then \( \mathfrak{A} \) and \( \mathfrak{A} \otimes \mathbb{K} \) are semiprojective. Consequently, \( \mathfrak{A} \) semiprojective.

**Proof.** By Lemma 1.5 of \([16]\), \( \varepsilon : 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0 \) is a full extension. By Corollary 3.22 of \([15]\), \( K_0(\mathfrak{A}) = K_0(\mathfrak{A}) \). By Theorem 6.4 of \([11]\), there exists a graph \( E \) with finitely many vertices such that \( K_+^\ast(\mathfrak{A}) \cong K_+^\ast(C^\ast(E)) \) such that \( C^\ast(E) \) is a tight C*-algebra over \( X_2 \). Since \( E \) has finitely many vertices, \( C^\ast(E) \) is unital. Since \( K_+^\ast(\mathfrak{A}) \cong K_+^\ast(C^\ast(E)) \), we have that \( C^\ast(E)[1] \) is a Kirchberg algebra. By Theorem 3.9 of \([16]\), we have that \( \mathfrak{A} \otimes \mathbb{K} \cong C^\ast(E) \otimes \mathbb{K} \). By Lemma 4.18, \( C^\ast(E) \) and \( C^\ast(E) \otimes \mathbb{K} \) are semiprojective. Hence, by Proposition 2.7 of \([3]\), \( \mathfrak{A} \) and \( \mathfrak{A} \otimes \mathbb{K} \) are semiprojective. \( \square \)

**Corollary 4.20.** Let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be unital, tight C*-algebras over \( X_2 \) such that \( \mathfrak{A}_1[2] \cong \mathbb{K} \) and \( \mathfrak{A}_2[1] \) is a Kirchberg algebra in \( \mathcal{N} \) such that \( \text{rank}(K_1(\mathfrak{A}_1[1])) \leq \text{rank}(K_0(\mathfrak{A}_1[1])) \), \( K_1(\mathfrak{A}_1[1]) \) is free, and the \( K \)-groups of \( \mathfrak{A}_1 \) are finitely generated. Let \( x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2) \) be an invertible such that \( K_{X_2}(x) \gamma \text{ is an order isomorphism for each } Y \in \mathbb{L}(X_2) \) and \( K_{X_2}(x) \gamma (1_{\mathfrak{A}_1}) = 1_{\mathfrak{A}_2} \) in \( K_0(\mathfrak{A}_2) \). Then there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(X_2; \gamma) = x \).

**Proof.** This follows from Proposition 4.19 and Corollary 4.17 \( \square \)

5. **Applications**

Let \( E \) be a graph satisfying Condition (K) (in particular, if \( C^\ast(E) \) has finitely many ideals, then \( E \) satisfies Condition (K)). Let \( \mathfrak{J}_1, \mathfrak{J}_2 \) be ideals of \( C^\ast(E) \) such that \( \mathfrak{J}_1 \subseteq \mathfrak{J}_2 \) and \( \mathfrak{J}_2/\mathfrak{J}_1 \) is simple. Then by Theorem 5.1 of \([38]\) and Corollary 3.5 of \([2]\), \( \mathfrak{J}_2/\mathfrak{J}_1 \) is a simple graph C*-algebra. Hence, \( \mathfrak{J}_2/\mathfrak{J}_1 \) is either a Kirchberg algebra or an AF algebra.

5.1. **Classification of graph C*-algebras with exactly one ideal.**

**Lemma 5.1.** Let \( E \) be a graph with finitely many vertices such that \( C^\ast(E) \) is a simple AF-algebra. Then \( C^\ast(E) \otimes \mathbb{K} \cong \mathbb{K} \). Consequently, if \( F \) is a graph with finitely many vertices such that \( C^\ast(F) \) is a tight C*-algebra over \( X_2 \) and \( C^\ast(F)[2] \) is an AF-algebra, then \( C^\ast(F)[2] \cong \mathbb{K} \).
Proof. We claim that \( E \) is a finite graph. By Corollary 2.13 and Corollary 2.15 of [9], \( E \) has no cycles, and for every vertex \( v_0 \) that emits infinitely many edges and for each vertex \( v \), there exists a path from \( v \) to \( v_0 \). Since \( E \) has no cycles, we have that every vertex of \( E \) emits only finitely many edges. Hence, \( E \) is a finite graph. By Proposition 1.18 of [30], \( C^*(E) \cong M_n \).

We now prove the second statement. First note that \( C^*(F)[2] \) is a simple AF-algebra. Since \( C^*(F)[2] \) is stably isomorphic to a subgraph of \( E \), \( C^*(F)[2] \otimes \mathbb{K} \cong C^*(E) \) for some graph \( E \) with finitely many vertices. Since \( C^*(E) \) is a simple AF-algebra, we have that \( C^*(E) \otimes \mathbb{K} \cong \mathbb{K} \). Hence, \( C^*(F)[2] \otimes \mathbb{K} \cong \mathbb{K} \) which implies that \( C^*(F)[2] \cong M_n \) or \( C^*(F)[2] \cong \mathbb{K} \). Since \( C^*(F)[2] \) is a non-unital \( C^*(E) \)-algebra (\( C^*(E) \) is a tight \( C^* \)-algebra over \( X_2 \)), we have that \( C^*(F)[2] \cong \mathbb{K} \).

\( \Box \)

**Definition 5.2.** For a \( C^* \)-algebra \( \mathfrak{A} \), set

\[
\Sigma \mathfrak{A} = \{ x \in K_0(\mathfrak{A}) : x = [p] \text{ for some projection } p \text{ in } \mathfrak{A} \}.
\]

Let \( \mathfrak{B} \) be a \( C^* \)-algebra. An order isomorphism \( \alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B}) \) is scale preserving if one of the following holds:

1. \( \mathfrak{A} \) is unital if and only if \( \mathfrak{B} \) unital and \( \alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}] \).
2. \( \mathfrak{A} \) is non-unital if and only if \( \mathfrak{B} \) is non-unital and \( \alpha(\Sigma \mathfrak{A}) = \Sigma \mathfrak{B} \).

**Theorem 5.3.** Let \( E_1 \) and \( E_2 \) be graphs with finitely many vertices and \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_2 \). If \( \alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2)) \) is an isomorphism such that \( \alpha Y \) is scale preserving for all \( Y \in \mathcal{L}(\mathcal{C}(X_2)) \), then there exists an isomorphism \( \phi : C^*(E_1) \to C^*(E_2) \) such that \( K_{X_2}(\phi) = \alpha \).

**Proof.** Since \( E_i \) has finitely many vertices, \( C^*(E_1) \) and \( C^*(E_2) \) are unital \( C^* \)-algebras.

**Case 1:** Suppose \( C^*(E_1) \) is an AF-algebra. Then \( C^*(E_2) \) is an AF-algebra. Hence, the result follows from Elliott’s classification of AF-algebras [19].

**Case 2:** Suppose \( C^*(E_1) \) is not an AF-algebra. Then \( C^*(E_2) \) is not an AF-algebra.

**Subcase 2.1:** Suppose \( C^*(E_1)[1] \) is an AF-algebra. Then \( C^*(E_2)[1] \) is an AF-algebra. By Corollary 4.10 and Corollary 2.11 there exists an isomorphism \( \phi : C^*(E_1) \to C^*(E_2) \) such that \( K_{X_2}(\phi) = \alpha \).

**Subcase 2.2:** Suppose \( C^*(E_1)[1] \) is a Kirchberg algebra. Then \( C^*(E_2)[1] \) is a Kirchberg algebra. Since \( C^*(E_i) \) is not an AF-algebra, either \( C^*(E_i)[2] \) is Kirchberg algebra or an AF-algebra.

Suppose \( C^*(E_i)[2] \) is a Kirchberg algebra. By Theorem 2.4 of [32], there exists an isomorphism \( \phi : C^*(E_1) \to C^*(E_2) \) such that \( K_{X_2}(\phi) = \alpha \). Suppose \( C^*(E_i)[2] \) is an AF-algebra. Then, by Lemma 5.1 \( C^*(E_i)[2] \cong \mathbb{K} \). By Corollary 4.20 and Corollary 2.11 there exists an isomorphism \( \phi : C^*(E_1) \to C^*(E_2) \) such that \( K_{X_2}(\phi) = \alpha \).

\( \Box \)

The following theorem completes the classification of graph \( C^* \)-algebras with exactly one non-trivial ideal.

**Corollary 5.4.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_2 \). Then \( C^*(E_1) \cong C^*(E_2) \) if and only if there exists an isomorphism \( \alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2)) \) such that \( \alpha Y \) is a scale preserving isomorphism for all \( Y \in \mathcal{L}(\mathcal{C}(X_2)) \).
Proof. The only case that is not covered by Theorem 4.9 of [15] is the case that $C^*(E_i)$ is unital. The unital case follows from Theorem 5.3 because of Theorem 3.3. □

5.2. Classification of graph $C^*$-algebras with more than one ideal. For a tight $C^*$-algebra $A$ over $X_n$, the finite and infinite simple sub-quotients of $A$ are separated if there exists $U \in O(X_n)$ such that either

1. $A(U)$ is an AF-algebra and $A(X_n \setminus U) \otimes O_\infty \cong A(X_n \setminus U)$ or
2. $A(X_n \setminus U)$ is an AF-algebra and $A(U) \otimes O_\infty \cong A(U)$.

In [14], the authors proved that if $A_1$ and $A_2$ are graph $C^*$-algebras that are tight $C^*$-algebras over $X_n$ such that the finite and infinite simple sub-quotients are separated, then $A_1 \otimes K \cong A_2 \otimes K$ if and only if $K_{X_n}^+(A_1) \cong K_{X_n}^+(A_2)$. We will show in this section that under mild $K$-theoretical conditions, we may remove the separated condition for the case $n = 3$.

Lemma 5.5. Let $E$ be a graph such that $C^*(E)$ is a tight $C^*$-algebra over $X_n$.

(i) If $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra, then

$$e_1 : 0 \to C^*(E)[2, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1] \otimes K \to 0$$

is a full extension.

(ii) If $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite, then

$$e_2 : 0 \to C^*(E)[k, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1, k - 1] \otimes K \to 0$$

is a full extension.

Proof. Suppose $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra. Note that $C^*(E)[1, n - 1] / C^*(E)[2, n - 1] \cong C^*(E)[1]$ and $C^*(E)[2, n - 1]$ is the largest ideal of $C^*(E)[1, n - 1]$ which is an AF-algebra. Since $C^*(E)[1, n - 1]$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[2, n - 1] \otimes K \to C^*(E)[1, n - 1] \otimes K \to C^*(E)[1] \otimes K \to 0$$

is a full extension. Since $C^*(E)[n] \otimes K$ is a purely infinite simple $C^*$-algebra, we have that

$$0 \to C^*(E)[n] \otimes K \to C^*(E)[2, n] \otimes K \to C^*(E)[2, n - 1] \otimes K \to 0$$

is a full extension. Hence, by Proposition 3.2 of [17], $e_1$ is a full extension.

Suppose $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite. Note that $C^*(E)[k, n]$ is the largest ideal of $C^*(E)[k - 1, n]$ such that $C^*(E)[k, n]$ is an AF-algebra and $C^*(E)[k - 1, n] / C^*(E)[k, n] \cong C^*(E)[k - 1]$ is purely infinite. Since $C^*(E)[k - 1, n] \otimes K$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[k, n] \otimes K \to C^*(E)[k - 1, n] \otimes K \to C^*(E)[k - 1] \otimes K \to 0$$

is a full extension. By Proposition 5.4 of [14], $e_2$ is a full extension. □

Theorem 5.6. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_n$.

Suppose

(i) $C^*(E_1)[n]$ and $C^*(E_1)[1]$ are purely infinite;

(ii) $C^*(E_1)[2, n - 1]$ is an AF-algebra; and
(iii) $KK^{1}(C^{*}(E_{1})[1], C^{*}(E_{2})[2, n]) = KL^{1}(C^{*}(E_{1})[1], C^{*}(E_{2})[2, n])$.

Then $C^{*}(E_{1}) \otimes K \cong C^{*}(E_{2}) \otimes K$ if and only if $K_{X_{n}}^{+}(C^{*}(E_{1}) \otimes K) \cong K_{X_{n}}^{+}(C^{*}(E_{2}) \otimes K)$.

**Proof.** Let $\epsilon_{i}$ be the extension

$$0 \to C^{*}(E_{i})[2, n] \otimes K \to C^{*}(E_{i}) \otimes K \to C^{*}(E_{i})[1] \otimes K \to 0.$$ 

By Lemma 5.5(i), $\epsilon_{i}$ is a full extension. Suppose $\alpha : K_{X_{n}}^{+}(C^{*}(E_{1}) \otimes K) \to K_{X_{n}}^{+}(C^{*}(E_{2}) \otimes K)$. Lift $\alpha$ to an invertible element $x \in KK(X_{n}; C^{*}(E_{1}) \otimes K, C^{*}(E_{2}) \otimes K)$. Note that $r_{X_{n}}^{[2, n]}(x)$ is invertible in $KK([2, n]; C^{*}(E_{1})[2, n] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$ and $r_{X_{n}}^{[1]}(x)$ is invertible in $KK(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[1] \otimes K)$. By Theorem 4.7, there exists an isomorphism $\phi_{0} : C^{*}(E_{1})[2, n] \otimes K \to C^{*}(E_{2})[2, n] \otimes K$ such that $KL(\phi_{0}) = z$, where $z$ is the invertible element of $KL(C^{*}(E_{1})[2, n] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$ induced by $r_{X_{n}}^{[2, n]}(x)$. By the Kirchberg-Phillips classification ([21] and [29]), there exists an isomorphism $\phi_{2} : C^{*}(E_{1})[1] \otimes K \to C^{*}(E_{2})[1] \otimes K$ such that $KK(\phi_{2}) = r_{X_{n}}^{[1]}(x)$.

Consider $C^{*}(E_{i})$ as a $C^{*}$-algebra over $X_{2}$ by setting $C^{*}(E_{i})[2] = C^{*}(E_{i})[2, n]$ and $C^{*}(E_{i})[1, 2] = C^{*}(E_{i})$. Let $y$ be the invertible element in $KK(X_{2}, C^{*}(E_{1}), C^{*}(E_{2}))$ induced by $x$. Note that $r_{X_{2}}^{[1]}(y) = r_{X_{n}}^{[1]}(x)$ and $KL(r_{X_{2}}^{[1]}(y)) = z = KL(\phi_{0})$ in $KL(C^{*}(E_{1})[2, n], C^{*}(E_{2})[2, n])$.

By Theorem 3.7 of [13],

$$r_{X_{2}}^{[1]}(y) \times [\tau_{\epsilon_{2}}] = [\tau_{\epsilon_{1}}] \times r_{X_{n}}^{[2]}(y)$$

in $KK^{1}(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$, where $\epsilon_{i}$ is the extension

$$0 \to C^{*}(E_{i})[2, n] \otimes K \to C^{*}(E_{i}) \otimes K \to C^{*}(E_{i})[1] \otimes K \to 0.$$ 

Thus,

$$KL(\phi_{2}) \times [\tau_{\epsilon_{2}}] = [\tau_{\epsilon_{1}}] \times KL(\phi_{0})$$

in $KL^{1}(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$. Since $KL^{1}(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[2, n] \otimes K) = KK^{1}(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$,

$$KK(\phi_{2}) \times [\tau_{\epsilon_{2}}] = [\tau_{\epsilon_{1}}] \times KK(\phi_{0})$$

in $KK^{1}(C^{*}(E_{1})[1] \otimes K, C^{*}(E_{2})[2, n] \otimes K)$. By Lemma 4.5 of [11], $C^{*}(E_{1}) \otimes K \cong C^{*}(E_{2}) \otimes K$. □

**Theorem 5.7.** Let $E_{1}$ and $E_{2}$ be graphs such that $C^{*}(E_{i})$ is a tight $C^{*}$-algebra over $X_{n}$. Suppose

(i) $C^{*}(E_{i})[k, n]$ and $C^{*}(E_{i})[1, k - 2]$ are AF-algebras;

(ii) $C^{*}(E_{i})[k - 1]$ is purely infinite; and

(iii) $KK^{1}(C^{*}(E_{1})[1, k - 1], C^{*}(E_{2})[k, n]) = KL^{1}(C^{*}(E_{1})[1, k - 1], C^{*}(E_{2})[k, n]).$

Then $C^{*}(E_{1}) \otimes K \cong C^{*}(E_{2}) \otimes K$ if and only if $K_{X_{n}}^{+}(C^{*}(E_{1}) \otimes K) \cong K_{X_{n}}^{+}(C^{*}(E_{2}) \otimes K)$.

**Proof.** Let $\epsilon_{i}$ be the extension $0 \to C^{*}(E_{i})[k, n] \otimes K \to C^{*}(E_{i}) \otimes K \to C^{*}(E_{i})[1, k - 1] \otimes K \to 0$. By Lemma 5.5(ii), $\epsilon_{i}$ is a full extension. Suppose $\alpha : K_{X_{n}}^{+}(C^{*}(E_{1}) \otimes K) \to K_{X_{n}}^{+}(C^{*}(E_{2}) \otimes K)$. Lift $\alpha$ to an invertible element $x \in KK(X_{n}; C^{*}(E_{1}) \otimes K, C^{*}(E_{2}) \otimes K)$. Note that $r_{X_{n}}^{[k, n]}(x)$ is invertible in $KK([k, n]; C^{*}(E_{1})[k, n] \otimes K, C^{*}(E_{2})[k, n] \otimes K)$ and $r_{X_{n}}^{[1, k - 1]}(x)$ is invertible in $KK(C^{*}(E_{1})[1, k - 1], C^{*}(E_{2})[1, k - 1])$. By Theorem 4.7, there exists an isomorphism
\[ \phi_2 : C^*(E_1)[1, k-1] \otimes K \to C^*(E_2)[1, k-1] \otimes K \] such that \( KL(\phi_2) = z_2 \), where \( z_2 \) is the invertible element in \( KL(C^*(E_1)[1, k-1], C^*(E_2)[1, k-1]) \) induced by \( r_{X_1}^{[1, k-1]}(x) \). By Elliott’s classification \([19]\), there exists an isomorphism \( \phi_0 : C^*(E_1)[k, n] \otimes K \to C^*(E_2)[k, n] \otimes K \) such that \( KK(\phi_0) = z_0 \), where \( z_0 \) is the invertible element in \( KK(C^*(E_1)[k, n] \otimes K, C^*(E_2)[k, n] \otimes K) \) induced by \( r_{X_1}^{[k, n]}(x) \).

Consider \( C^*(E_1) \) as a \( C^* \)-algebra over \( X_2 \) by setting \( C^*(E_1)[2] = C^*(E_1)[k, n] \) and \( C^*(E_1)[1, 2] = C^*(E_1) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( KL(r_{X_2}^{[1]}(y)) = z_2 = KL(\phi_2) \) and \( r_{X_2}^{[2]}(y) = z_0 = KK(\phi_0) \). By Theorem 3.7 of \([14]\),

\[
KL(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KL(\phi_0)
\]

in \( KK^1(C^*(E_1)[1, k-1] \otimes K, C^*(E_2)[k, n] \otimes K) \), where \( e \) is the extension

\[
0 \to C^*(E_1)[k, n] \otimes K \to C^*(E_1) \otimes K \to C^*(E_1)[1, k-1] \otimes K \to 0.
\]

Thus,

\[
KL(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KL(\phi_0)
\]

in \( KL^1(C^*(E_1)[1, k-1] \otimes K, C^*(E_2)[k, n] \otimes K) \). Since \( KL^1(C^*(E_1)[1, k-1] \otimes K, C^*(E_2)[k, n] \otimes K) = KK^1(C^*(E_1)[1, k-1] \otimes K, C^*(E_2)[k, n] \otimes K) \),

\[
KK(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KK(\phi_0)
\]

in \( KK^1(C^*(E_1)[1, k-1] \otimes K, C^*(E_2)[k, n] \otimes K) \). By Lemma 4.5 of \([14]\), \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \).

**Theorem 5.8.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_1) \) is a tight \( C^* \)-algebra over \( X_3 \). Suppose \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is purely infinite and \( K_0(C^*(E_1)[1, 2]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is an AF-algebra. Then \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \) if and only if \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \).

**Proof.** The “only if” direction is clear. Suppose \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \). Suppose \( C^*(E_1)[1] \) is purely infinite. Then \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups. Thus, \( Pext^1_0(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0 \). Since \( K_1(C^*(E_1)[1]) \) is a free group, \( Pext^1_0(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0 \). Hence,

\[
KK^1(C^*(E_1)[1], C^*(E_2)[2, 3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, 3]).
\]

Suppose \( C^*(E_1)[1] \) is an AF-algebra. Then \( K_0(C^*(E_1)[1, 2]) \) is the direct sum of cyclic groups. Thus, \( Pext^1_0(K_0(C^*(E_1)[1, 2]), K_0(C^*(E_2)[3])) = 0 \). Since \( K_1(C^*(E_1)[1, 2]) \) is a free group, \( Pext^1_0(K_1(C^*(E_1)[1, 2]), K_1(C^*(E_2)[3])) = 0 \). Therefore,

\[
KK^1(C^*(E_1)[1, 2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1, 2], C^*(E_2)[3]).
\]

**Case 1:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are separated. Hence, by Theorem 6.9 of \([14]\), \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \).

**Case 2:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are not separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are not separated.
Subcase 2.1: Suppose \( C^*(E_1)[3] \) and \( C^*(E_1)[1] \) are purely infinite and \( C^*(E_1)[2] \) is an AF-algebra. Then \( C^*(E_2)[3] \) and \( C^*(E_2)[1] \) are purely infinite and \( C^*(E_2)[2] \) is an AF-algebra. Then by the above paragraph we have that
\[
KK^1(C^*(E_1)[1], C^*(E_2)[2, 3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, 3]).
\]
Hence, by Theorem 5.6, \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \).

Subcase 2.2: Suppose \( C^*(E_1)[3] \) and \( C^*(E_1)[1] \) are AF-algebras and \( C^*(E_1)[2] \) is purely infinite. Then \( C^*(E_2)[3] \) and \( C^*(E_2)[1] \) are AF-algebras and \( C^*(E_2)[2] \) is purely infinite. Then by the above paragraph we have that
\[
KK^1(C^*(E_1)[1, 2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1, 2], C^*(E_2)[3]).
\]
Hence, by Theorem 5.7, \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \).

\[ \square \]

Corollary 5.9. Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_3 \). Suppose that \( K_0(C^*(E_i)) \) is finitely generated. Then \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \) if and only if \( K^+_X(C^*(E_i)) \cong K^+_X(C^*(E_2)) \).

Proof. Since \( C^*(E_i) \) is real rank zero, the canonical projection \( \pi : C^*(E_i) \rightarrow C^*(E_i)[1] \) induces a surjective homomorphism \( \pi : K_0(C^*(E_i)) \rightarrow K_0(C^*(E_i)[1]) \). Hence, \( K_0(C^*(E_i)[1]) \) is finitely generated since \( K_0(C^*(E_i)) \) is finitely generated. The corollary now follows from Theorem 5.8.

\[ \square \]

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