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STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE C*-ALGEBRAS

SØREN EILERS, GUNNAR RESTORFF, AND EFREN RUIZ

Abstract. We show that certain extensions of classifiable C*-algebra are strongly classified by the associated six-term exact sequence in K-theory together with the positive cone of K0-groups of the ideal and quotient. We apply our result to give a complete classification of graph C*-algebras with exactly one ideal.

1. Introduction

The classification program for C*-algebras has for the most part progressed independently for the classes of infinite and finite C*-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered K0-group. In the infinite case, there are a number of results for purely infinite C*-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite C*-algebras up to stable isomorphism by the K0-group together with the K1-group. For nonsimple purely infinite C*-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C*-algebras with exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K-groups [34], the second named author has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered K-theory [31, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite C*-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated K-theory [28, Theorem 4.14]. However, in all of these situations the nonsimple C*-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Recently, the authors have provided a framework for classifying nonsimple C*-algebras that are not necessarily AF-algebras or purely infinite C*-algebras. In particular, the authors have shown in [16] that certain extensions of classifiable C*-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. This has allowed for the classification of certain nonsimple C*-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [16] was then used by the first named author and Tomforde in [18] to classify a certain class of non-simple graph C*-algebras, showing that graph C*-algebras with exactly one non-trivial ideal can be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. The authors in [15] then showed that all non-unital graph C*-algebras with exactly one
non-trivial ideal can be classified up to isomorphism by their associated six-term exact sequence in $K$-theory. In this paper, we complete the classification of graph $C^*$-algebras with exactly one non-trivial ideal by classifying those that are unital. Our methods here differ rather dramatically from the methods in [13] and [15]. In particular, we use the traditional methods of classification via existence and uniqueness theorems. As a consequence, for unital graph $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with exactly one non-trivial ideal, then any isomorphism between the associated six-term exact sequence in $K$-theory which preserves the unit lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

2. Preliminaries

2.1. $C^*$-algebras over topological spaces. Let $X$ be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of $X$, partially ordered by set inclusion $\subseteq$. A subset $Y$ of $X$ is called \textit{locally closed} if $Y = U \setminus V$ where $U, V \in \mathcal{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathbb{L}(X)$. The set of all connected, non-empty, locally closed subsets of $X$ will be denoted by $\mathbb{L}(X)^\ast$.

The partially ordered set $(\mathcal{O}(X), \subseteq)$ is a \textit{complete lattice}, that is, any subset $S$ of $\mathcal{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset $S$ of $\mathcal{O}(X)$,

$$\bigwedge_{U \in S} U = \left( \bigcap_{U \in S} U \right)^{\circ} \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.$$ 

For a $C^*$-algebra $\mathfrak{A}$, let $\mathbb{I}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{I}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset $S$ of $\mathbb{I}(\mathfrak{A})$,

$$\bigwedge_{\mathfrak{J} \in S} \mathfrak{J} = \bigcap_{\mathfrak{J} \in S} \mathfrak{J} \quad \text{and} \quad \bigvee_{\mathfrak{J} \in S} \mathfrak{J} = \bigcup_{\mathfrak{J} \in S} \mathfrak{J}.$$ 

\textbf{Definition 2.1.} Let $\mathfrak{A}$ be a $C^*$-algebra. Let $\text{Prim}(\mathfrak{A})$ denote the \textit{primitive ideal space} of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let $X$ be a topological space. A $C^*$-\textit{algebra over} $X$ is a pair $(\mathfrak{A}, \psi)$ consisting of a $C^*$-algebra $\mathfrak{A}$ and a continuous map $\psi : \text{Prim}(\mathfrak{A}) \to X$. A $C^*$-algebra over $X$, $(\mathfrak{A}, \psi)$, is \textit{separable} if $\mathfrak{A}$ is a separable $C^*$-algebra. We say that $(\mathfrak{A}, \psi)$ is \textit{tight} if $\psi$ is a homeomorphism.

We always identify $\mathcal{O}(\text{Prim}(\mathfrak{A}))$ and $\mathbb{I}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p.$$ 

Let $(\mathfrak{A}, \psi)$ be a $C^*$-algebra over $X$. Then we get a map $\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(\mathfrak{A})) \cong \mathbb{I}(\mathfrak{A})$ defined by

$$U \mapsto \{ p \in \text{Prim}(\mathfrak{A}) : \psi(p) \in U \} = \mathfrak{A}(U).$$ 

For $Y = U \setminus V \in \mathbb{L}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [27], $\mathfrak{A}(Y)$ does not depend on $U$ and $V$.

\textbf{Example 2.2.} For any $C^*$-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_{\text{Prim}(\mathfrak{A})})$ is a tight $C^*$-algebra over $\text{Prim}(\mathfrak{A})$. For each $U \in \mathcal{O}(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$. 

\[ \]
Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip $X_n$ with the Alexandrov topology, so the non-empty open subsets are
$$[a, n] = \{x \in X : a \leq x \leq n\}$$
for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathcal{A}, \phi)$ be a $C^*$-algebra over $X_n$. We will use the following notation throughout the paper:
$$\mathcal{A}[k] = \mathcal{A}([k]), \mathcal{A}[a, b] = \mathcal{A}([a, b]), \text{ and } \mathcal{A}(i, j) = \mathcal{A}[i + 1, j].$$
Using the above notation we have ideals $\mathcal{A}[a, n]$ such that
$$\{0\} \leq \mathcal{A}[n] \leq \mathcal{A}[n - 1, n] \leq \cdots \leq \mathcal{A}[2, n] \leq \mathcal{A}[1, n] = \mathcal{A}.
$$

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras over $X$. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is $X$-equivariant if $\phi(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ for all $U \in \mathcal{O}(X)$. Hence, for every $Y = U \setminus V$, $\phi$ induces a homomorphism $\phi_Y : \mathcal{A}(Y) \to \mathcal{B}(Y)$. Let $\mathcal{C}^{*}\text{-alg}(X)$ be the category whose objects are $C^*$-algebras over $X$ and whose morphisms are $X$-equivariant homomorphisms.

An $X$-equivariant homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a full $X$-equivariant homomorphism if for all $Y \in \mathcal{L}(X)$, $\phi_Y(a)$ is norm-full in $\mathcal{B}(Y)$ for all norm-full elements $a \in \mathcal{A}(Y)$, i.e., the closed ideal of $\mathcal{B}(Y)$ generated by $\phi_Y(a)$ is $\mathcal{B}(Y)$ whenever the closed ideal of $\mathcal{A}(Y)$ generated by $a$ is $\mathcal{A}(Y)$.

Remark 2.5. Suppose $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_n$. Then it is clear that $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\phi$ is a $X_n$-equivariant isomorphism.

It is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_2$, then $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism if and only if $\phi$ is an $X_2$-equivariant homomorphism and $\phi_{[1]}$ and $\phi_{[2]}$ are injective. Also, if $\mathcal{A}$ and $\mathcal{A}[2]$ have non-zero projections $p$ and $q$ respectively, then there exists $\epsilon > 0$ such that if $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism and $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that
$$\|\phi(p) - \psi(p)\| < 1 \quad \|\phi(q) - \psi(q)\| < 1,$$
then $\psi$ is a full $X_2$-equivariant homomorphism.

Remark 2.6. Let $\xi_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathcal{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathcal{E}_i$ can be considered as a $C^*$-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathcal{B}_i$ in $\mathcal{E}_i$, and $\{1, 2\}$ to $\mathcal{E}_i$. Hence, there exists a one-to-one correspondence between $X_2$-equivariant homomorphisms $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and homomorphisms from $\xi_1$ and $\xi_2$.

2.2. The ideal related $K$-theory of $\mathcal{A}$.

Definition 2.7. Let $X$ be a topological space and let $\mathcal{A}$ be a $C^*$-algebra over $X$. For open subsets $U_1, U_2, U_3$ of $X$ with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1$, $Y_2 = U_3 \setminus U_1$, $Y_3 = U_3 \setminus U_1 \in \mathcal{L}(X)$. Then the diagram
$$
\begin{array}{ccc}
K_0(\mathcal{A}(Y_1)) & \xrightarrow{i_*} & K_0(\mathcal{A}(Y_2)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}(Y_3)) \\
\downarrow{\partial_*} & & \downarrow{\partial_*} & & \\
K_1(\mathcal{A}(Y_3)) & \xrightarrow{\pi_*} & K_1(\mathcal{A}(Y_2)) & \xrightarrow{i_*} & K_1(\mathcal{A}(Y_1))
\end{array}
$$

where $\pi_* : K_0(\mathcal{A}(Y_2)) \to K_0(\mathcal{A}(Y_3))$ and $i_* : K_1(\mathcal{A}(Y_1)) \to K_1(\mathcal{A}(Y_2))$.
is an exact sequence. The ideal related $K$-theory of $\mathfrak{A}$, $K_X(\mathfrak{A})$, is the collection of all $K$-groups thus occurring and the natural transformations $\{i_*, \pi_*, \partial_*\}$. The ideal related, ordered $K$-theory of $\mathfrak{A}$, $K_X^+(\mathfrak{A})$, is $K_X(\mathfrak{A})$ of $\mathfrak{A}$ together with $K_0(\mathfrak{A}(Y))_+$ for all $Y \in \mathcal{L}(X)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras over $X$, we will say that $\alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B})$ is an isomorphism if for all $Y \in \mathcal{L}(X)$, there exists a graded group isomorphism

$$\alpha_{Y,*} : K_*(\mathfrak{A}(Y)) \to K_*(\mathfrak{B}(Y))$$

preserving all natural transformations. We say that $\alpha : K_X^+(\mathfrak{A}) \to K_X^+(\mathfrak{B})$ is an isomorphism if there exists an isomorphism $\alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B})$ in such a way that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathcal{L}(X)$.

**Remark 2.8.** Meyer-Nest in [28] defined a similar functor $FK_X(-)$ which they called filtrated $K$-theory. For all known cases in which there exists a UCT, the natural transformation from $FK_X(-)$ to $K_X(-)$ is an equivalence. In particular, this is true for the space $X_n$.

If $Y \in \mathcal{L}(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \subseteq \overline{Y}(Y) \subseteq \mathcal{L}(X)$, then $\mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \oplus \mathfrak{A}(Y_2)$ for any $C^*$-algebra over $X$. Moreover, there is a natural isomorphism $K_*(\mathfrak{A}(Y))$ to $K_*(\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathfrak{A}(Y))$ to $K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2))$. If $X$ is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathcal{L}(X)$ by the subset $\mathcal{L}(X)^*$.

**Notation 2.9.** Let $\mathcal{N}$ be the bootstrap category of Rosenberg and Schochet in [37].

Let $\mathfrak{R}(X)$ be the category whose objects are separable $C^*$-algebras over $X$ and the set of morphisms is $KK(X; \mathfrak{A}, \mathfrak{B})$. For a finite topological space $X$, let $\mathcal{B}(X) \subseteq \mathfrak{R}(X)$ be the bootstrap category of Meyer and Nest in [27]. By Corollary 4.13 of [27], if $\mathfrak{A}$ is a nuclear $C^*$-algebra over $X$, then $\mathfrak{A} \in \mathcal{B}(X)$ if and only if $\mathfrak{A}(\{x\}) \in \mathcal{N}$ for all $x \in X$.

**Theorem 2.10.** (Bonkat [4] and Meyer-Nest [28]) Let $\mathfrak{A}$ and $\mathfrak{B}$ be in $\mathfrak{R}(X_n)$ such that $\mathfrak{A}$ is in $\mathcal{B}(X_n)$, then the sequence

$$0 \to \text{Ext}_{X}^1(FK_{X_n}(\mathfrak{A})[1],FK_{X_n}(\mathfrak{B})) \xrightarrow{\delta} KK(X_n; \mathfrak{A}, \mathfrak{B}) \xrightarrow{\Gamma} \text{Hom}_{X}^1(FK_{X_n}(\mathfrak{A}),FK_{X_n}(\mathfrak{B})) \to 0$$

is exact. Consequently, if $\mathfrak{B}$ is in $\mathcal{B}(X_n)$, then an isomorphism from $FK_{X_n}(\mathfrak{A})$ to $FK_{X_n}(\mathfrak{B})$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

**Corollary 2.11.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be in $\mathcal{B}(X_n)$. Then an isomorphism from $K_{X_n}(\mathfrak{A})$ to $K_{X_n}(\mathfrak{B})$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

**Proof.** This follows from Remark 2.8 and Theorem 2.10.

**Remark 2.12.** Let $x \in KK(X_n; \mathfrak{A}, \mathfrak{B})$ be an invertible element. Then $K_{X_n}(x)$ will denote the isomorphism from $K_{X_n}(\mathfrak{A})$ to $K_{X_n}(\mathfrak{B})$ given by $\Gamma(x)$ where we have identified $K_{X_n}(\mathfrak{A})$ with $FK_{X_n}(\mathfrak{A})$ and $K_{X_n}(\mathfrak{B})$ with $FK_{X_n}(\mathfrak{B})$.

**2.3. Functors.** We now define some functors that will be used throughout the rest of the paper. Let $X$ and $Y$ be topological spaces. For every continuous function $f : X \to Y$ we have a functor

$$f : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(Y), \ (A, \psi) \mapsto (A, f \circ \psi).$$
(1) Define \( g_X^1 : X \to X_1 \) by \( g_X^1(x) = 1 \). Then \( g_X^1 \) is continuous. Note that the induced functor \( g_X^1 : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(X_1) \) is the forgetful functor.

(2) Let \( U \) be an open subset of \( X \). Define \( g_{U, X}^2 : X \to X_2 \) by \( g_{U, X}^2(x) = 1 \) if \( x \notin U \) and \( g_{U, X}^2(x) = 2 \) if \( x \in U \). Then \( g_{U, X}^2 \) is continuous. Thus the induced functor

\[
g_{U, X}^2 : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(X_2)
\]

is just specifying the extension \( 0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0 \).

(3) We can generalize (2) to finitely many ideals. Let \( U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X \) be open subsets of \( X \). Define \( g_{U_1, U_2, \ldots, U_n, X}^n : X \to X_n \) by \( g_{U_1, U_2, \ldots, U_n, X}^n(x) = n - k + 1 \) if \( x \in U_k \setminus U_{k-1} \). Then \( g_{U_1, U_2, \ldots, U_n, X}^n \) is continuous. Therefore, any \( C^*\)-algebra with ideals \( 0 \leq \mathfrak{I}_1 \leq \mathfrak{I}_2 \leq \cdots \leq \mathfrak{I}_n = \mathfrak{A} \) can be made into a \( C^*\)-algebra over \( X_n \).

(4) For all \( Y \in \mathbb{L}(\mathcal{C}(X)) \), \( i_X^Y : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(Y) \) is the restriction functor defined in Definition 2.19 of [27].

(5) If \( f : X \to Y \) is an embedding of a subset with the subspace topology, we write

\[
i_X^Y = f_* : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(Y).
\]

By Proposition 3.4 of [27], the functors defined above induce functors from \( \mathfrak{R}(X) \) to \( \mathfrak{R}(Z) \), where \( Z = Y, X_1, X_n \).

2.4. Graph \( C^*\)-algebras. A graph \((E^0, E^1, r, s)\) consists of a countable set \( E^0 \) of vertices, a countable set \( E^1 \) of edges, and maps \( r : E^1 \to E^0 \) and \( s : E^1 \to E^0 \) identifying the range and source of each edge. If \( E \) is a graph, the graph \( C^*\)-algebra \( C^*(E) \) is the universal \( C^*\)-algebra generated by mutually orthogonal projections \( \{ p_v : v \in E^0 \} \) and partial isometries \( \{ s_e : e \in E^1 \} \) with mutually orthogonal ranges satisfying

1. \( s_e^*s_e = p_r(e) \) for all \( e \in E^1 \)
2. \( s_e s_e^* \leq p_s(e) \) for all \( e \in E^1 \)
3. \( p_v = \sum_{e \in E^1 : s(e) = v} s_es_e^* \) for all \( v \) with \( 0 < |s^{-1}(v)| < \infty \).

3. Meta-theorems

In many cases one can obtain a classification result for a class of unital \( C^*\)-algebras \( \mathcal{C} \) by obtaining a classification result for the class \( \mathcal{C} \otimes \mathbb{K} \), where each object in \( \mathcal{C} \otimes \mathbb{K} \) is the stabilization of an object in \( \mathcal{C} \). A meta-theorem of this sort was proved by the first and second named authors in [13] Theorem 11. It was shown there that if \( \mathcal{C} \) is a subcategory of the category of \( C^*\)-algebras, \( \mathcal{C}^*\text{-alg} \), and if \( F \) is a functor from \( \mathcal{C} \) to an abelian category such that an isomorphism \( F(\mathfrak{A} \otimes \mathbb{K}) \cong F(\mathfrak{B} \otimes \mathbb{K}) \) lifts to an isomorphism in \( \mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K} \), then under suitable conditions, we have that \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) implies \( \mathfrak{A} \cong \mathfrak{B} \). In [31], the second and third named authors improved this result by showing that the isomorphism \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) lifts to an isomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \).

In this section, we improve these results in order to deal with cases when \( \mathcal{C} \) is a category (not necessarily a subcategory of \( \mathcal{C}^*\text{-alg} \)) and there exists a functor from \( \mathcal{C} \) to \( \mathcal{C}^*\text{-alg} \). An example of such a category is the category of \( C^*\)-algebras over \( \{1, 2\} \), where \( \{1, 2\} \) is given the discrete topology. Then \( \mathcal{C} \) is not a subcategory of \( \mathcal{C}^*\text{-alg} \) but the forgetful functor (forgetting the \( \{1, 2\}\)-structure) is a functor from \( \mathcal{C} \) to \( \mathcal{C}^*\text{-alg} \). We also replace the condition of proper pure infiniteness by the stable weak cancellation property.
Definition 3.1. A $C^*$-algebra $\mathfrak{A}$ is said to have the weak cancellation property if $p$ is Murray-von Neumann equivalent to $q$ whenever $p$ and $q$ generate the same ideal $\mathfrak{I}$ and $[p] = [q]$ in $K_0(\mathfrak{I})$. A $C^*$-algebra is said to have the stable weak cancellation property if $M_n(\mathfrak{A})$ has the weak cancellation property for all $n \in \mathbb{N}$.

Theorem 3.2. (cf. [13] Theorem 11) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathfrak{C}^*{\text{-alg}}$ be the category of $C^*$-algebras, and let $\mathfrak{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F: \mathcal{C} \to \mathfrak{C}^*{\text{-alg}}$, $G: \mathcal{C} \to \mathcal{D}$, and $H: \mathcal{D} \to \mathfrak{Ab}$ such that

1. $H \circ G = K_0 \circ F$.
2. For objects $\mathfrak{A}$ in $\mathcal{C}$, there exist an object $\mathfrak{A}_K$ and a morphism $\kappa_{\mathfrak{A}}: \mathfrak{A} \to \mathfrak{A}_K$ such that $G(\kappa_{\mathfrak{A}}) = 1$ is an isomorphism in $\mathcal{D}$, $F(\mathfrak{A}_K) = F(\mathfrak{A}) \otimes K$, and $F(\kappa_{\mathfrak{A}}) = 1_F \otimes e_{11}$.
3. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$, every isomorphism $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$ is induced by an isomorphism from $\mathfrak{A}_K$ to $\mathfrak{B}_K$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be given such that $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are unital $C^*$-algebras. Let $\rho: G(\mathfrak{A}) \to G(\mathfrak{B})$ be an isomorphism such that $H(\rho)\{[1_F(\mathfrak{A})]\} = [1_F(\mathfrak{B})]$. If $F(\mathfrak{B})$ has the stable weak cancellation property, then $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Proof. Note that $G(\kappa_{\mathfrak{A}})$ and $G(\kappa_{\mathfrak{B}})$ are isomorphisms. Therefore $G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}$ is an isomorphism from $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$. Thus, there exists an isomorphism $\phi: \mathfrak{A}_K \to \mathfrak{B}_K$ such that $G(\phi) = G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}$.

Set $\psi = F(\phi)$. Then $\psi: F(\mathfrak{A}) \otimes K \to F(\mathfrak{B}) \otimes K$ is an $*$-isomorphism such that

\[ K_0(\psi) = K_0(F(\phi)) = H(G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}) = H(G(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ H(G(\kappa_{\mathfrak{A}})^{-1}) = K_0(F(\mathfrak{B})) \circ H(\rho) \circ K_0(F(\mathfrak{A}))^{-1} = K_0(id_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(id_{F(\mathfrak{A})} \otimes e_{11})^{-1}. \]

Hence,

\[ K_0(\psi)[1_{F(\mathfrak{A})} \otimes e_{11}] = K_0(id_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(id_{F(\mathfrak{A})} \otimes e_{11})^{-1}([1_{F(\mathfrak{A})} \otimes e_{11}]) = K_0(id_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathfrak{A})}]) = K_0(id_{F(\mathfrak{B})} \otimes e_{11})([1_{F(\mathfrak{B})}]) = [1_{F(\mathfrak{B})} \otimes e_{11}]. \]

Stable weak cancellation implies that there exists $v \in F(\mathfrak{B}) \otimes K$ such that $v^*v = \psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathfrak{B})} \otimes e_{11}$ since $\psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $1_{F(\mathfrak{B})} \otimes e_{11}$ are full projections in $F(\mathfrak{B}) \otimes K$. Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [13] Theorem 11], $\gamma$ is an isomorphism from $F(\mathfrak{A}) \otimes e_{11}$ to $F(\mathfrak{B}) \otimes e_{11}$. Hence, $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Theorem 3.3. (cf. [32] Theorem 2.1) Let $\mathcal{C}$ be a subcategory of $\mathfrak{C}^*{\text{-alg}}(X)$. Moreover, $\mathcal{C}$ is assumed to be closed under tensoring by $\mathbb{M}_2(\mathbb{C})$ and $K$ and contains the canonical embeddings $\kappa_1: \mathfrak{A} \to \mathbb{M}_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes K$ as morphisms for every object $\mathfrak{A}$ in $\mathcal{C}$. Assume there is a functor $F: \mathcal{C} \to \mathcal{D}$ satisfying

1. For $\mathfrak{A}$ in $\mathcal{C}$, the embeddings $\kappa_1: \mathfrak{A} \to \mathbb{M}_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes K$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.
2. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$ that are stable $C^*$-algebras, every isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
3. There exists a functor $G$ from $\mathcal{D}$ to $\mathfrak{Ab}$ such that $G \circ F = K_0$. 

\[ \min(\mathfrak{A}) \to \max(\mathfrak{A}) \]
Assume that every $X$-equivariant isomorphism between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ and that for objects $\mathfrak{A}$ in $\mathcal{C}$, $F(\text{Ad}(u)|_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})}$ for every unitary $u \in M(\mathfrak{A})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are objects $\mathcal{C}$ that are unital $C^*$-algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ have the stable weak cancellation property and there is an isomorphism $\alpha : F(\mathfrak{A}) \rightarrow F(\mathfrak{B})$ such that $G(\alpha)([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$, then there exists an isomorphism $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ in $\mathcal{C}$ such that $F(\phi) = \alpha$.

Proof. The difference between the statement of Theorem 2.1 of [32] and statement of the theorem are

(i) $\mathcal{C}$ is assumed to be a subcategory of $C^*$-alg$(X)$ instead of a subcategory of $C^*$-alg.

(ii) $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to have the stable weak cancellation property instead of being properly infinite.

In the proof of Theorem 2.1 of [32], properly infinite was needed to insure that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$, where $\psi : \mathfrak{A} \otimes K \rightarrow \mathfrak{B} \otimes K$ is the isomorphism from (2) that lifts the isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ that is induced by $\alpha$. As in the proof of Theorem 3.2 we get that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$. Arguing as in the proof of Theorem 2.1 of [32], we get the desired result.

\[ \square \]

4. Classification results

In this section, we show that $K^+_X(-)$ is a strong classification functor for a class of $C^*$-algebras with exactly one proper nontrivial ideal containing $C^*$-algebras associated to finite graphs. The results of this section will be used in the next section to show that $K^+_X(-)$ together with the appropriate scale is a complete isomorphism invariant for $C^*$-algebras associated to graphs. Moreover, in a forthcoming paper, we use these results to solve the following extension problem: If $\mathfrak{A}$ fits into the following exact sequence

\[ 0 \rightarrow C^*(E) \otimes K \rightarrow \mathfrak{A} \rightarrow C^*(G) \rightarrow 0, \]

where $C^*(E)$ and $C^*(G)$ are simple $C^*$-algebras, then when is $\mathfrak{A} \cong C^*(F)$ for some graph $F$?

**Theorem 4.1.** (Existence Theorem) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X)$ and let $x \in KK(X; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is a positive isomorphism for all $Y \in \mathfrak{L}(X)$. Suppose $0 \rightarrow \mathfrak{A}_i[2] \rightarrow \mathfrak{A}_i \rightarrow \mathfrak{A}_i[1] \rightarrow 0$ is a full extension, $\mathfrak{A}_i[2]$ is a stable $C^*$-algebra, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra with real rank zero, and either

(i) $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra; or

(ii) $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple $C^*$-algebra.

Then there exists an $X_2$-equivariant homomorphism $\phi : \mathfrak{A}_1 \otimes K \rightarrow \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$, and $\phi[2]$ and $\phi[1]$ are injective, where $\{e_{ij}\}$ is a system of matrix units for $K$.

Proof. Set $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$. Note that by Lemma 3.10 and Theorem 3.8 of [14], $\mathfrak{A}_i[2] \otimes K$ satisfies the corona factorization property (see [21] for the definition of the corona factorization property). Since $\mathfrak{A}_i[k]$ is an AF-algebra or an Kirchberg algebra, $\mathfrak{A}_i[k]$ has the stable weak cancellation. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation. Let $\mathfrak{c}_i$ be the extension

\[ 0 \rightarrow \mathfrak{A}_i[2] \otimes K \rightarrow \mathfrak{A}_i \otimes K \rightarrow \mathfrak{A}_i[1] \otimes K \rightarrow 0. \]
By Corollary 3.24 of [15], ε is a full extension since A_i[1] has cancellation of projections (in the AF case) and A_i[1] is properly infinite (in the purely infinite case).

Case (i): A_2[2] is a purely infinite simple C*-algebra and A_i[1] is an AF-algebra. By Theorem 3.3 of [14], r^{(1)}_{\chi_2}(y) × [τ_{ε2}] = [r_{ε2}] × r^{(2)}_{\chi_2}(y) in KK^1(A_i[1]⊗ K, A_2[2]⊗ K). Since y is invertible in KK(X_2, A_i⊗ K, A_2⊗ K), we have that r^{(1)}_{\chi_2}(y) is invertible in KK(A_i[1]⊗ K, A_2[1]⊗ K) and Γ(r^{(1)}_{\chi_2}(y)) = Γ(x)_{i1} is a positive isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism ψ_i : A_i[1]⊗ K → A_2[1]⊗ K such that KK(ψ_i) = r^{(1)}_{\chi_2}(y).

By Corollary 3.24 of [15], ε is a full extension since A_i[1] has cancellation of projections (in the AF case) and A_i[1] is properly infinite (in the purely infinite case).

Since y is invertible in KK(X_2, A_i⊗ K, A_2⊗ K), we have that r^{(2)}_{\chi_2}(y) is invertible in KK(A_i[1]⊗ K, A_2[1]⊗ K) and Γ(r^{(2)}_{\chi_2}(y)) = Γ(x)_{i2} is a positive isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism ψ_i : A_i[1]⊗ K → A_2[1]⊗ K such that KK(ψ_i) = r^{(2)}_{\chi_2}(y).

By Theorem 3.6 of [27] (see also Hauptsatz 4.2 of [20]), there exists a η-isomorphic to KK(A_i[2]⊗ K) such that KK(η) = r^{(1)}_{\chi_2}(y). By Lemma 4.5 of [14] and its proof, there exists a unitary u ∈ M(A_2[2]⊗ K) such that ψ = (Ad(u) ◦ ψ_0, Ad(u) ◦ ψ_0, ψ_i) is an X_2-equivariant isomorphism from A_i[1]⊗ K to A_2⊗ K, where ψ_0 : M(A_i[2]⊗ K) → M(A_i[1]⊗ K) is the unique isomorphism extending ψ_0. Note that KK(ψ_{i,k}) = r^{(k)}_{\chi_2}(y) for k = 1, 2.

Note that

0 → i^{X_2}_{(2)}((A_i⊗ K)[2]) → A_i⊗ K[2] → i^{X_2}_{(1)}((A_i⊗ K)[1]) → 0

is a semi-split extension of C*-algebras over X_2 (see Definition 3.5 of [27]). Set

J_i = i^{X_2}_{(2)}((A_i⊗ K)[2]) and B_i = i^{X_2}_{(1)}((A_i⊗ K)[1]).

By Theorem 3.6 of [27] (see also Korollar 3.4.6 of [4]),

KK(X_2, A_i⊗ K, J_i) → KK(X_2, A_i⊗ K, B_i) → KK(X_2, A_i⊗ K, B_i) → ...

is exact. By Proposition 3.12 of [27], KK(X_2, A_i⊗ K, J_i) and KK(A_i[1]⊗ K, A_2[1]⊗ K) are naturally isomorphic. Hence, there exists z ∈ KK(X_2, A_i⊗ K, J_i) such that y = KK(X_2, ψ) = z × KK(X_2, λ_2) since KK(ψ_{i1}) = r^{(1)}_{\chi_2}(y).

Case (ii): A_i[2] is an AF-algebra and A_i[1] is a purely infinite simple C*-algebra. By Theorem 3.3 of [14], r^{(1)}_{\chi_2}(y) × [τ_{ε2}] = [r_{ε2}] × r^{(2)}_{\chi_2}(y) in KK^1(A_i[1]⊗ K, A_2[2]⊗ K). Since y is invertible in KK(X_2, A_i⊗ K, A_2⊗ K), we have that r^{(1)}_{\chi_2}(y) is invertible in KK(A_i[2]⊗ K, A_2[2]⊗ K) and Γ(r^{(1)}_{\chi_2}(y)) = Γ(x)_{i2} is an order isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism ψ_i : A_i[2]⊗ K → A_2[2]⊗ K such that KK(ψ_i) = r^{(2)}_{\chi_2}(y).

Since y is invertible in KK(X_2, A_i⊗ K, A_2⊗ K), we have that r^{(1)}_{\chi_2}(y) is invertible in...
\( KK(\mathfrak{A}_1[1] \otimes \mathbb{K}, \mathfrak{A}_2[1] \otimes \mathbb{K}) \). Thus, by Kirchberg-Phillips classification (see [20] and [29]), there exists an isomorphism \( \psi_1 : \mathfrak{A}_1[1] \otimes \mathbb{K} \to \mathfrak{A}_2[1] \otimes \mathbb{K} \) such that \( KK(\psi_1) = r_{X_2}^{(2)}(y) \). By Lemma 4.5 of [14] and its proof, there exists a unitary \( u \in \mathcal{M}(\mathfrak{A}_2[2] \otimes \mathbb{K}) \) such that 

\[
\psi = (\text{Ad}(u) \circ \psi_0, \text{Ad}(u) \circ \tilde{\psi}_0, \psi_1) \text{ is an } X_2\text{-equivariant isomorphism from } \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K},
\]

where \( \psi_0 : \mathcal{M}(\mathfrak{A}_1[2] \otimes \mathbb{K}) \to \mathcal{M}(\mathfrak{A}_1[2] \otimes \mathbb{K}) \) is the unique isomorphism extending \( \psi_0 \). Note that \( KK(\psi_{\{k\}}) = r_{X_2}^{(k)}(y) \) for \( k = 1,2 \).

Note that 

\[
0 \to i_{\{2\}}^{X_2}( (\mathfrak{A}_i \otimes \mathbb{K})[2] ) \xrightarrow{\lambda} \mathfrak{A}_i \otimes \mathbb{K} \xrightarrow{\beta_i} i_{\{1\}}^{X_2}( (\mathfrak{A}_i \otimes \mathbb{K})[1] ) \to 0
\]

is a semi-split extension of \( C^*\)-algebras over \( X_2 \) (see Definition 3.5 of [27]). Set 

\[
\mathcal{J}_i = i_{\{2\}}^{X_2}( (\mathfrak{A}_i \otimes \mathbb{K})[2] ) \quad \text{and} \quad \mathcal{B}_i = i_{\{1\}}^{X_2}( (\mathfrak{A}_i \otimes \mathbb{K})[1] ) .
\]

By Theorem 3.6 of [27] (see also Korollar 3.4.6 [4])

\[
KK(X_2; \mathcal{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\beta_i)^*} KK(X_2; \mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\lambda_i)^*} KK(X_2; \mathcal{J}_1, \mathfrak{A}_2 \otimes \mathbb{K})
\]

is exact. By Proposition 3.12 of [27], \( KK(X_2; \mathcal{J}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) and \( KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K}) \) are naturally isomorphic. Hence, there exists \( z \in KK(X_2; \mathcal{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) such that \( y = \text{Ad}(\psi_0, \psi_1) = KK(X_2; \psi) = KK(X_2; \psi_1) \times z \). By Proposition 3.13 of [27], \( KK(X_2; \mathcal{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) and \( KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K}) \) are isomorphic. Therefore, by Theorem 8.3.3 of [36], there exists a homomorphism \( \eta : (\mathfrak{A}_1 \otimes \mathbb{K})[1] \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(\eta) = \bar{z} \), where \( \bar{z} \) is the image of \( z \) under the isomorphism \( KK(X_2; \mathcal{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \cong KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K}) \) (the existence of the homomorphism uses the fact that \( \mathfrak{A}_2 \otimes \mathbb{K} \) is a properly infinite \( C^*\)-algebra which follows from Proposition 3.21 and Theorem 3.22 of [15]). Note that \( \eta \) induces an \( X_2\)-equivariant homomorphism \( \eta : \mathcal{B}_1 \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(X_2; \eta) = z \).

Set \( \phi = \psi + (\eta \circ \beta_1) \), where the sum is the Cuntz sum in \( \mathcal{M}(\mathfrak{A}_2 \otimes \mathbb{K}) \). Then \( \phi \) is an \( X_2\)-equivariant homomorphism such that \( KK(X_2; \phi) = y \). Since \( \psi_{\{2\}} \) and \( \psi_{\{1\}} \) are injective homomorphisms, \( \phi_{\{2\}} \) and \( \phi_{\{1\}} \) are injective homomorphisms. \( \square \)

4.1. **Strong classification of extensions of AF-algebras by purely infinite \( C^*\)-algebras.**

**Definition 4.2.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be separable \( C^*\)-algebras over \( X \). Two \( X\)-equivariant homomorphisms \( \phi, \psi : \mathfrak{A} \to \mathfrak{B} \) are said to be **approximately unitarily equivalent** if there exists a sequence of unitaries \( \{u_n\}_{n=1}^{\infty} \) in \( \mathcal{M}(\mathfrak{B}) \) such that 

\[
\lim_{n \to \infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

We now recall the definition of \( KL(\mathfrak{A}, \mathfrak{B}) \) from [33].

**Definition 4.3.** Let \( \mathfrak{A} \) be a separable, nuclear \( C^*\)-algebra in \( \mathcal{N} \) and let \( \mathfrak{B} \) be a \( \sigma\)-unital \( C^*\)-algebra. Let 

\[
\text{Ext}^1_2(K_s(\mathfrak{A}), K_{s+1}(\mathfrak{B})) = \text{Ext}^1_2(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Ext}^1_2(K_1(\mathfrak{A}), K_0(\mathfrak{B})).
\]

Since \( \mathfrak{A} \) is in \( \mathcal{N} \), by [37], \( \text{Ext}^1_2(K_s(\mathfrak{A}), K_{s+1}(\mathfrak{B})) \) can be identified as a sub-group of the group \( KK(\mathfrak{A}, \mathfrak{B}) \).
For abelian groups, $G$ and $H$, let $\text{Pext}^1_\mathbb{Z}(G,H)$ be the subgroup of $\text{Ext}^1_\mathbb{Z}(G,H)$ of all pure extensions of $G$ by $H$. Set

$$\text{Pext}^1_\mathbb{Z}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \text{Pext}^1_\mathbb{Z}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Pext}^1_\mathbb{Z}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Define $KL(\mathfrak{A}, \mathfrak{B})$ as the quotient

$$KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})/\text{Pext}^1_\mathbb{Z}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})).$$

Rørdam in [33] proved that if $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $KL(\phi) = KL(\psi)$.

**Notation 4.4.** Let $x \in KK(\mathfrak{A}, \mathfrak{B})$. Then the element $x + \text{Pext}^1_\mathbb{Z}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ in $KL(\mathfrak{A}, \mathfrak{B})$ will be denoted by $KL(x)$.

A nuclear, purely infinite, separable, simple $C^*$-algebra will be called a *Kirchberg algebra.*

**Theorem 4.5.** *(Uniqueness Theorem 1)* Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i$ is real rank zero, $\mathfrak{A}_i$ is stable, $\mathfrak{A}_i[2]$ is a Kirchberg algebra in $N$, $\mathfrak{A}_i[1]$ is an AF-algebra, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Suppose $\phi, \psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be $X_2$-equivariant homomorphisms such that $KK(X_2; \phi) = KK(X_2; \psi)$, and $\phi_{(2)}$, $\phi_{(1)}$, $\psi_{(2)}$, and $\psi_{(1)}$ are injective homomorphisms. Then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** Since $\mathfrak{A}_i[1]$ is an AF algebra, every finitely generated subgroup of $K_0(\mathfrak{A}_i[1])$ is torsion free (hence free) and every finitely generated subgroup of $K_1(\mathfrak{A}_i[1])$ is zero. Thus, $\text{Pext}^1_\mathbb{Z}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2]))) = \text{Ext}^1_\mathbb{Z}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2])))$ which implies that $KL(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_j[2])) = \text{Hom}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2])))$.

Let $e_i$ denote the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$. Since $\mathfrak{A}_i$ has real rank zero and $K_1(\mathfrak{A}_i[1]) = 0$, we have that $K_j(\tau_{e_i}) = 0$, where $\tau_{e_i}$ is the Busby invariant of $e_i$. Hence, $[\tau_{e_i}] = 0$ in $KL(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_j[2]))$. By Corollary 6.7 of [24], $e_i$ is quasi-diagonal. Thus, there exists an approximate identity of $\mathfrak{A}_i[2]$ consisting of projections $\{e_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \|e_k x - xe_k\| = 0$$

for all $x \in \mathfrak{A}_i$.

Since $\mathfrak{A}_i[1]$ is an AF-algebra and $\mathfrak{A}_1$ has real rank zero, as in the proof of Lemma 9.8 of [10], there exists a sequence of finite dimensional sub-$C^*$-algebras $\{\mathfrak{B}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_1$ such that $\mathfrak{B}_k \cap \mathfrak{A}_i[2] = \{0\}$ and for each $x \in \mathfrak{A}_1$, there exist $y_1 \in \bigcup_{k=1}^{\infty} \mathfrak{B}_k$ and $y_2 \in \mathfrak{A}_i[2]$ such that $x = y_1 + y_2$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Note that we may assume $\mathcal{F}$ is the union of the generators of $\mathfrak{B}_m$, for some $m \in \mathbb{N}$ and $\mathcal{G}$, for some finite subset $\mathcal{G}$ of $\mathfrak{A}_1[2]$ . Since $\mathfrak{B}_m$ is a finite dimensional $C^*$-algebra,

$$\lim_{k \to \infty} \|e_k x - xe_k\| = 0$$

for all $x \in \mathfrak{A}_1$, and $\{e_k\}_{k \in \mathbb{N}}$ is an approximate identity for $\mathfrak{A}_i[2]$ consisting of projections, there exist $k \in \mathbb{N}$, a finite dimensional sub-$C^*$-algebra $\mathfrak{D}$ of $\mathfrak{A}_1$ with $\mathfrak{D} \subseteq (1_{\mathfrak{M}(\mathfrak{A}_1)} - e_k\mathfrak{A}_1(1_{\mathfrak{M}(\mathfrak{A}_1)} - e_k))$ and $\mathfrak{D} \cap \mathfrak{A}_i[2] = \{0\}$, and there exists a finite subset $\mathcal{H}$ of $e_k\mathfrak{A}_i[2]e_k$ such that for all $x \in \mathcal{F}$, there exist $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H}$

$$\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}.$$
Set $D = \bigoplus_{\ell = 1}^s M_{n_{\ell}}$ and let $\{f_{\ell i j}^{\ell} \}_{i, j = 1}^{n_{\ell}}$ be a system of matrix units for $M_{n_{\ell}}$. Let $I_{\ell}$ be the ideal in $A_i$ generated by $f_{11}^{\ell}$. Since $A_i[2]$ is simple and $A_i[2]$ is an essential ideal of $A_i$, we have that $A_i[2] \subseteq I$ for all nonzero ideal $I$ of $A_i$. Thus, $A_1[2] \subseteq I$ since $D \cap A_1[2] = 0$.

Let $J^\psi_{\ell}$ be the ideal in $A_2$ generated by $\phi(f_{11}^{\ell})$ and let $J^\psi_{\ell}$ be the ideal in $A_2$ generated by $\psi(f_{11}^{\ell})$. Since $\phi$ and $\psi$ are $X_2$-equivariant homomorphisms and since $\phi(1_1)$ and $\psi(1_1)$ are injective homomorphisms, we have that $\phi(f_{11}^{\ell}) \notin A_2[2]$ and $\psi(f_{11}^{\ell}) \notin A_2[2]$. Therefore, $A_2[2] \subseteq J^\psi_{\ell}$ and $A_2[2] \subseteq J^\psi_{\ell}$. Since $K_0(\phi(1_1)) = K_0(\psi(1_1))$ and since $A_2[1]$ is an AF-algebra, we have that $\phi(1_1)(J^\psi_{11})$ is Murray-von Neumann equivalent to $\psi(1_1)(J^\psi_{11})$, where $J^\psi_{11}$ is the image of $f_{11}^{\ell}$ in $A_1[1]$. Thus, they generate the same ideal in $A_2[1]$. Since $A_2[2] \subseteq J^\psi_{\ell}$ and $A_2[2] \subseteq J^\psi_{\ell}$ and since $\psi(1_1)(J^\psi_{11})$ and $\phi(1_1)(J^\psi_{11})$ generate the same ideal in $A_2[1]$, we have that $I = J^\psi_{\ell} = J^\psi_{\ell}$.

Note that the following diagram

$$
\begin{array}{ccc}
0 & \longrightarrow & K_0(A_2[2]) \\
& & \bigg| \\
& & K_0(I) \\
0 & \longrightarrow & K_0(A_2[2]) \\
\end{array}
\begin{array}{ccc}
& & K_0(I) \\
& & \bigg| \\
& & K_0(I) \\
0 & \longrightarrow & K_0(A_2[2]) \\
\end{array}
\begin{array}{ccc}
\longrightarrow & \longrightarrow & K_0(A_2) \\
& & K_0(I) \\
\end{array}
$$

is commutative, the rows are exact, and $\iota$ and $\tau$ are the canonical embeddings. Since $A_2[1]$ is an AF-algebra, $K_0(I)$ is injective. A diagram chase shows that $K_0(I)$ is injective. Since $KK(X_2; \phi) = KK(X_2; \psi)$, we have that $[\phi(f_{11}^{\ell})] = [\psi(f_{11}^{\ell})]$ in $K_0(A_2)$. Since $\phi(f_{11}^{\ell})$ and $\psi(f_{11}^{\ell})$ are elements of $I$ and $K_0(I)$ is injective, we have that $[\phi(f_{11}^{\ell})] = [\psi(f_{11}^{\ell})]$ in $K_0(I)$. Since $A_i[1]$ is an AF-algebra and $A_i[2]$ is a Kirchberg algebra, they both have stable weak cancellation. By Lemma 3.15 of [15], $A_i$ has stable weak cancellation. Thus, $\phi(f_{11}^{\ell})$ is Murray-von Neumann equivalent to $\psi(f_{11}^{\ell})$. Hence, there exists $v_{\ell} \in A_2$ such that $v_{\ell}^* v_{\ell} = \phi(f_{11}^{\ell})$ and $v_{\ell} v_{\ell}^* = \psi(f_{11}^{\ell})$.

Set

$$u_1 = \sum_{\ell = 1}^s \sum_{i = 1}^{n_{\ell}} \psi(f_{11}^{\ell}) v_{\ell} \phi(f_{11}^{\ell})$$

Then, $u_1$ is a partial isometry in $A_1$ such that $u_1^* u_1 = \phi(1_D)$, $u_1 u_1^* = \psi(1_D)$, and $u_1 \phi(x) u_1^* = \psi(x)$ for all $x \in D$.

Let $\beta : e_k A_1[2] e_k \to A_1[2]$ be the usual embedding. Note that $KK(\phi(2) \circ \beta) = KK(\psi(2) \circ \beta)$ and $\phi(2) \circ \beta$, $\psi(2) \circ \beta$ are monomorphisms. Therefore, by Theorem 6.7 of [23], there exists a partial isometry $u_2 \in A_2[2]$ such that $u_2^* u_2 = \phi(e_k)$, $u_2 u_2^* = \psi(e_k)$, and

$$||u_2 \phi(x) u_2^* - \psi(x)|| < \frac{\epsilon}{3}$$

for all $x \in H$.

Since $A_2$ is stable, there exists $u_3 \in M(A_2)$ such that $u_3^* u_3 = 1_{M(A_2)} - (u_1 + u_2)^* (u_1 + u_2)$ and $u_3 u_3^* = 1_{M(A_2)} - (u_1 + u_2)((u_1 + u_2)^*$. Set $u = u_1 + u_2 + u_3 \in M(A_2)$. Then $u$ is a unitary in $M(A_2)$. 
Lemma 4.6. Let $\mathfrak{A}$ be a separable $C^*$-algebra over a finite topological space $X$. Let $u$ be unitary in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$. Then $K_X (\text{Ad}(u)|_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$. 

Proof. Since $\mathfrak{A} \otimes \mathbb{K}$ is stable, we have that there exists a norm continuous path of unitaries $\{u_t\}$ in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ such that $u_0 = u$ and $u_1 = 1_{\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})}$. It follows that $K_X (\text{Ad}(u)|_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$. 

Theorem 4.7. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is an order isomorphism for all $Y \in \mathbb{L}(X_2)$. Suppose $\mathfrak{A}_i[2]$ is a Kirchberg algebra, $\mathfrak{A}_i[1]$ is an AF-algebra, $\mathfrak{A}_i$ has real rank zero, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KL(\phi) = KL(g^1_{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; id_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times KK(X_n; id_{\mathfrak{A}_1} \otimes e_{11})$. 

Proof. Since $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra, $\mathfrak{A}_i[2]$ is either unital or stable. Since $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, $\mathfrak{A}_i[2]$ is non-unital else $\mathfrak{A}_i[2]$ is isomorphic to a direct summand of $\mathfrak{A}_i$ which would contradict the essential assumption. Therefore, $\mathfrak{A}_i[2]$ is stable. Moreover, $Q(\mathfrak{A}_i[2])$ is simple which implies that $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension. Since $\mathfrak{A}_i[2]$ and $\mathfrak{A}_i[1]$ are nuclear $C^*$-algebras, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra.

Let $z \in KK(X_2; \mathfrak{A}_2 \otimes \mathbb{K}, \mathfrak{A}_1 \otimes \mathbb{K})$ such that $y \times z = [id_{\mathfrak{A}_1 \otimes \mathbb{K}}]$ and $y \times z = [id_{\mathfrak{A}_2 \otimes \mathbb{K}}]$. By Theorem 4.4 there exists an $X_2$-equivariant homomorphism $\psi_1 : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \psi_1) = x$, and $(\psi_1)_{(2)}$ and $(\psi_1)_{(1)}$ are injective homomorphisms. By Theorem 4.4 there exists an $X_2$-equivariant homomorphism $\psi_2 : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \psi_2) = y$, and $(\psi_2)_{(2)}$ and $(\psi_2)_{(1)}$ are injective homomorphisms. Using Theorem 4.5 and a typical approximate intertwining argument, there exists an isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $\phi$ and $\psi_1$ are approximately unitarily equivalent.

Let $\pi_2 : \mathfrak{A}_2 \to \mathfrak{A}_2[1]$ be the canonical quotient map. Then $\pi_2 \circ \phi|_{\mathfrak{A}_1[2]}$ is either zero or injective since $\mathfrak{A}_1[2]$ is simple. Since $\mathfrak{A}_2[1]$ is purely infinite and $\mathfrak{A}_2[1]$ is an AF-algebra, we must have that $\pi_2 \circ \phi|_{\mathfrak{A}_1[2]} = 0$. Thus, $\phi$ is an $X_2$-equivariant homomorphism. Similarly, $\phi^{-1}$ is an $X_2$-equivariant homomorphism. Hence, $\phi$ is an $X_2$-equivariant isomorphism. By construction, $KL(\phi) = KL(\psi_1) = KL(g^1_{X_2}(y))$. By Lemma 4.6 $K_{X_2}(\phi) = K_{X_2}(y)$. 

Corollary 4.8. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is an order isomorphism for all $Y \in \mathbb{L}(X_2)$. Suppose $\mathfrak{A}_i[2]$ is a Kirchberg algebra, $\mathfrak{A}_i[1]$ is an AF-algebra, $\mathfrak{A}_i$ has real rank zero, $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$, and $K_i(\mathfrak{A}[Y])$ and $K_i(\mathfrak{A}[Y])$ are finitely generated for all $Y \in \mathbb{L}(X_2)$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\phi) = KK(g^1_{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; id_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times KK(X_n; id_{\mathfrak{A}_1} \otimes e_{11})$.
Strong classification of extensions of purely infinite by $K$. We recall the following from [1] p. 341. Let $\psi : \mathcal{A} \to \mathcal{B}(H)$ be a representation of $\mathcal{A}$. Let $H_e$ denote the subspace of $H$ spanned by the ranges of all compact operators in $\psi(\mathcal{A})$. Since $\psi(\mathcal{A}) \cap K$ is an ideal of $\psi(\mathcal{A})$, we have that $H_e$ reduces $\pi(\mathcal{A})$, and so the decomposition $H = H_e \oplus H_{e}^\perp$ induces a decomposition of $\psi$ into sub-representations $\psi = \psi_e \oplus \psi'$. The summand $\psi_e$, considered as a representation of $\mathcal{A}$ on $H_e$, will be called the essential part of $\psi$ and $H_e$ is called the essential subspace for $\psi$.

Let $\mathcal{B}$ be a tight $C^*$-algebra over $X_2$. Consider the essential extension

$$e_{\mathcal{B}} : 0 \to \mathcal{B}[2] \to \mathcal{B} \to \mathcal{B}[1] \to 0.$$  

If $\tau_{\mathcal{B}} : \mathcal{B}[1] \to Q(\mathcal{B}[2])$ is the Busby invariant of $e$, then there exists an injective homomorphism $\sigma_{\mathcal{B}} : \mathcal{B} \to M(\mathcal{B}[2])$ such that the diagram

$$\begin{array}{cccc}
0 & \to & \mathcal{B}[2] & \to \mathcal{B} & \to & \mathcal{B}[1] & \to 0 \\
\| & \| & \sigma_{\mathcal{B}} & \downarrow & \tau_{\mathcal{B}} & \downarrow & \| \\
0 & \to & \mathcal{B}[2] & \to M(\mathcal{B}[2]) & \to & Q(\mathcal{B}[2]) & \to 0
\end{array}$$

If $\mathcal{B}[2] \cong K$, let $\eta_{\mathcal{B}} : M(\mathcal{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathcal{B}[2] \cong K$ and let $\overline{\eta}_{\mathcal{B}} : Q(\mathcal{B}[2]) \to B(\ell^2)/K$ be the induced isomorphism.

Lemma 4.9. Let $\mathcal{A}$ and $\mathcal{B}$ be separable, tight $C^*$-algebras over $X_2$ such that $\mathcal{A}[2] \cong \mathcal{B}[2] \cong K$. Let $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{B}$ be two, full $X_2$-equivariant homomorphisms such that $K_0((\psi_1)_2) = K_0((\psi_2)_2)$ and $\eta_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi_i$ is a non-degenerate representation of $\mathcal{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $M(\mathcal{B}[2])$ such that

$$U_n(\sigma_{\mathcal{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathcal{B}} \circ \psi_2)(a) \in \mathcal{B}[2]$$

for all $a \in \mathcal{A}$ and for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} \|U_n(\sigma_{\mathcal{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathcal{B}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathcal{A}$.

Proof. We argue as in the proof of Lemma 2.8 of [22]. Set $\sigma_i = \eta_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi_i$. By assumption, $\sigma_i : \mathcal{A} \to B(\ell^2)$ is a non-degenerated representation of $\mathcal{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^{\infty}$ in $B(\ell^2)$ such that $V_n \sigma_1(a) V_n^* - \sigma_2(a) \in K$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|V_n \sigma_1(a) V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathcal{A}$. This will be a consequence of Theorem 5(iii) of [1].

Let $\rho : \mathcal{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathcal{A}[2] \cong K$. Since $\psi_i, \sigma_{\mathcal{B}}, \eta_{\mathcal{B}}$ are injective homomorphisms, $\sigma_i$ is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi : B(\ell^2) \to B(\ell^2)/K$ be the natural projection. Note that

$$\pi \circ \sigma_1 = \pi \circ \eta_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi_1 = \overline{\eta}_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi_1 = \overline{\eta}_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ \psi_1 = \overline{\eta}_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ (\psi_1)_{\{1\}} \circ \pi_{\mathcal{A}}.$$
It now follows that \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = A[2] \) since \( \tau_{\mathfrak{B}}, \tau_{\mathfrak{B}} \), and \( (\psi_1)_{(1)} \) are injective homomorphisms.

Let \( H_1 \) be the essential subspace of \( \sigma_1 \). Since \( \sigma_1(A[2]) \subseteq \mathbb{K} \) and for each \( x \not\in A[2] \), we have that \( \sigma_1(x) \not\in \mathbb{K} \), we have that \( H_1 = \sigma_1(A[2])\ell^2 \). Similarly, we have that \( H_2 = \sigma_2(A[2])\ell^2 \), where \( H_2 \) is the essential subspace of \( \sigma_2 \). Let \( e \) be a minimal projection of \( A[2] \cong \mathbb{K} \). Suppose \( \sigma_1(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_1(-)|_{H_1} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \). Since \( K_0((\psi_1)_{(2)}) = K_0((\psi_2)_{(2)}) \), we have that \( \sigma_1(e) \) is Murray-von Neumann equivalent to \( \sigma_2(e) \). Hence, \( \sigma_2(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_2(-)|_{H_2} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \).

The above paragraph imply that \( \sigma_2(-)|_{H_2} \) and \( \sigma_1(-)|_{H_1} \) are unitarily equivalent. Since \( \ker(\sigma_1) = \ker(\sigma_2) \) and \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) \) by Theorem 5(iii) of [1], there exists a sequence of unitaries \( \{V_n\}_{n=1}^\infty \) in \( B(\ell^2) \) such that \( V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K} \) for all \( n \in \mathbb{N} \) and for all \( a \in A \), and

\[
\lim_{n \to \infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0
\]

for all \( a \in A \).

Set \( U_n = \eta_{\mathfrak{B}}^{-1}(V_n) \). Then \( \{U_n\}_{n=1}^\infty \) is a sequence of unitaries in \( \mathcal{M}(\mathfrak{B}[2]) \) such that \( U_n(\sigma_{\mathfrak{B}_n} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}_n} \circ \psi_2)(a) \in \mathfrak{B}[2] \) for all \( n \in \mathbb{N} \) and for all \( a \in A \), and

\[
\lim_{n \to \infty} \|U_n(\sigma_{\mathfrak{B}_n} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}_n} \circ \psi_2)(a)\| = 0
\]

for all \( a \in A \).

**Definition 4.10.** A \( C^* \)-algebra \( A \) is called **weakly semiprojective** if we can always solving the \(*\)-homomorphism lifting problem

\[
\begin{array}{ccc}
\prod_{n=1}^\infty \mathfrak{B}_n & \xrightarrow{\rho_N} & (b_N, b_{N+1}, \ldots) \\
\mathfrak{A} & \xrightarrow{\phi} & \prod_{n=1}^\infty \mathfrak{B}_n / \bigoplus_{n=1}^\infty \mathfrak{B}_n \\
& & [(0, \ldots, 0, b_N, b_{N+1}, \ldots)]
\end{array}
\]

and \( \mathfrak{A} \) is called **semiprojective** if we can always solve the lifting problem

\[
\begin{array}{ccc}
\mathfrak{B}/I_N & \xleftarrow{\phi} & \prod_{n=1}^\infty \mathfrak{B}_n / \bigoplus_{n=1}^\infty I_n \\
& & (I_1 \subseteq I_2 \subseteq \cdots \subseteq \mathfrak{B})
\end{array}
\]

**Lemma 4.11.** Let \( \mathfrak{A}_0 \) be a unital, separable, nuclear, tight \( C^* \)-algebra over \( X_2 \) such that \( \mathfrak{A}_0[2] \cong \mathbb{K} \) and \( \mathfrak{A}_0 \) has the stable weak cancellation property. Set \( \mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K} \). Suppose \( \beta : \mathfrak{A} \to \mathfrak{A} \) is a full \( X_2 \)-equivariant homomorphism such that \( K_{X_2}(\beta) = K_{X_2}(\text{id}_{\mathfrak{A}}) \) and \( \beta_{(1)} = \text{id}_{\mathfrak{A}_{(1)}} \). Then there exists a sequence of contractive, completely positive, linear maps \( \{\alpha_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^\infty \) such that

1. \( \alpha_n|_{\mathfrak{A} \cap \mathfrak{A}_n} \) is a homomorphism for all \( n \in \mathbb{N} \) and
2. for all \( a \in \mathfrak{A} \),

\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(a) - a\| = 0
\]
where $e_n = \sum_{k=1}^{n} 1_{A_k} \otimes e_{kk}$ and $\{ e_{ij} \}_{i,j}$ is a system of matrix units for $\mathbb{K}$. If, in addition, $\mathfrak{A}$ is assumed to be weakly semiprojective, then $\alpha_n$ can be chosen to be a homomorphism for all $n \in \mathbb{N}$.

**Proof.** Since $\beta$ is a full $X_2$-equivariant homomorphism and the ideal in $\mathfrak{A}$ generated by $e_n$ is $\mathfrak{A}$, we have that the ideal in $\mathfrak{A}$ generated by $\beta(e_n)$ is $\mathfrak{A}$. Since $K_{X_2}(\beta) = K_{X_2}(\mathrm{id}_{\mathfrak{A}})$, we have that $[\beta(e_n)] = [e_n]$ in $K_0(\mathfrak{A})$. It now follows that $\beta(e_n)$ and $e_n$ are Murray-von Neumann equivalent since $\mathfrak{A}_0$ has the stable weak cancellation property. Since $\mathfrak{A}$ is stable, there exists a unitary $v_n$ in the unitization of $\mathfrak{A}$ such that $v_n \beta(e_n) v_n^* = e_n$.

Fix $n \in \mathbb{N}$. Let $e_n$ be the extension $0 \to e_n \mathfrak{A}[2] e_n \to e_n \mathfrak{A} e_n \to \mathfrak{T}_n \mathfrak{A}[1] \mathfrak{T}_n \to 0$. By Lemma 1.5 of [16], $e_n$ is a full extension. Therefore, $\sigma_{\epsilon}(e_n)$ is Murray-von Neumann equivalent to $1_{\mathcal{M}(\mathfrak{A}[2])}$. Hence, $e_n \mathfrak{A}[2] e_n \cong \mathfrak{A}[2] \cong \mathbb{K}$. Set $\mathfrak{A}_n = e_n \mathfrak{A} e_n$ and define $\beta_n : \mathfrak{A}_n \to \mathfrak{A}_n$ by

$$
\beta_n(x) = \Ad(v_n) \circ \beta(x).
$$

Then $\beta_n$ is a unital, full $X_2$-equivariant homomorphism. Since $\eta_{\mathfrak{A}_n} \circ \sigma_{\epsilon_n} \circ \beta_n$ is a unital representation of $\mathfrak{A}_n$, the closed subspace of $l^2$ generated by $\{(\eta_{\mathfrak{A}_n} \circ \sigma_{\epsilon_n} \circ \beta_n)(x) : x \in \mathfrak{A}_n, \xi \in l^2\}$ is $l^2$. Therefore, $\eta_{\mathfrak{A}_n} \circ \sigma_{\epsilon_n} \circ \beta_n$ is a non-degenerate representation of $\mathfrak{A}_n$.

Since $K_{X_2}(\beta) = K_{X_2}(\mathrm{id}_{\mathfrak{A}})$ and the $X_2$-equivariant embedding of $\mathfrak{A}_n$ as a sub-algebra of $\mathfrak{A}$ induces an isomorphism in ideal related $K$-theory, we have that $K_{X_2}(\beta_n) = K_{X_2}(\mathrm{id}_{\mathfrak{A}_n})$.

By Lemma 4.9 there exists a sequence of unitaries $W_{k,n} \in \mathcal{M}(\mathfrak{A}_n[2])$ such that

$$(\Ad(W_{k,n}) \circ \sigma_{\epsilon_n} \circ \beta_n)(x) - \sigma_{\epsilon_n}(x) \in \mathfrak{A}_n[2]$$

for all $x \in \mathfrak{A}_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} \| (\Ad(W_{k,n}) \circ \sigma_{\epsilon_n} \circ \beta_n)(x) - \sigma_{\epsilon_n}(x) \| = 0$$

for all $x \in \mathfrak{A}_n$.

Note that $\mathcal{M}(\mathfrak{A}_n[2]) \cong \sigma_{\epsilon}(e_n) \mathcal{M}(\mathfrak{A}[2]) \sigma_{\epsilon}(e_n)$ with an isomorphism mapping $\mathfrak{A}_n[2]$ onto $e_n \mathfrak{A}[2] e_n$. Thus, we get a partial isometry $\tilde{W}_{k,n}$ in $\mathcal{M}(\mathfrak{A}[2])$ such that $\tilde{W}_{k,n}^* \tilde{W}_{k,n} = \tilde{W}_{k,n} \tilde{W}_{k,n} = \sigma_{\epsilon}(e_n)$ and

$$(\Ad(\tilde{W}_{k,n}) \circ \sigma_{\epsilon} \circ \Ad(v_n) \circ \beta_n)(x) - \sigma_{\epsilon}(x) \in \mathfrak{A}[2]$$

for all $x \in \mathfrak{A}_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} \| (\Ad(\tilde{W}_{k,n}) \circ \sigma_{\epsilon} \circ \Ad(v_n) \circ \beta_n)(x) - \sigma_{\epsilon}(x) \| = 0$$

for all $x \in \mathfrak{A}_n$.

Set $V_{k,n} = (\tilde{W}_{k,n} + 1_{\mathcal{M}(\mathfrak{A}[2])} - \sigma_{\epsilon}(e_n))\sigma_{\epsilon}(e_n)$. Then $V_{k,n}$ is a unitary in $\mathcal{M}(\mathfrak{A}[2])$ such that

$$\Ad(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta(x) - \sigma_{\epsilon}(x) \in \mathfrak{A}[2]$$

for all $x \in e_n \mathfrak{A} e_n$ and for all $k \in \mathbb{N}$, and

$$\lim_{k \to \infty} \| (\Ad(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta(x) - \sigma_{\epsilon}(x) \| = 0$$

for all $x \in e_n \mathfrak{A} e_n$. A consequence of the first part is that $(\Ad(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta(x) \in \sigma_{\epsilon}(e_n \mathfrak{A} e_n) + \mathfrak{A}[2]$ for all $x \in e_n \mathfrak{A} e_n$. Since $\beta_{(1)} = \mathrm{id}_{\mathfrak{A}[2]}$, we have that $x - \beta(x) \in \mathfrak{A}[2]$ for all $x \in e_n \mathfrak{A} e_n$. Therefore,

$$\Ad(V_{k,n})(\sigma_{\epsilon}(x)) = \Ad(V_{k,n}) \circ \sigma_{\epsilon}(x - \beta(x)) + \Ad(V_{k,n}) \circ \beta(x) \in \sigma_{\epsilon}(e_n \mathfrak{A} e_n) + \mathfrak{A}[2]$$

Thus, $\alpha_{k,n} = \sigma_{\epsilon}^{-1} \circ (\Ad(V_{k,n}) \circ \sigma_{\epsilon} \circ \Ad(v_n))|_{e_n \mathfrak{A} e_n}$ is a homomorphism from $e_n \mathfrak{A} e_n$ to $\mathfrak{A}$. 
Since
\[ \lim_{k \to \infty} \| (\text{Ad}(V_{k,n}) \circ \sigma \circ \beta)(x) - \sigma(x) \| = 0 \]
for all \( x \in e_n \mathfrak{A}e_n \) and \( e_n \mathfrak{A}e_n \subseteq e_{n+1} \mathfrak{A}e_{n+1} \), there exists a strictly increasing sequence \( \{k(n)\}_{n=1}^{\infty} \) of positive integers such that
\[ \lim_{n \to \infty} \| \alpha_{k(n),n} \circ \beta(x) - x \| = 0 \]
for all \( x \in \bigcup_{n=1}^{\infty} e_n \mathfrak{A}e_n \). Let \( \alpha_n \) be a completely, contractive, positive linear extension of \( \alpha_{k(n),n} \). Since \( \bigcup_{n=1}^{\infty} e_n \mathfrak{A}e_n \) is dense in \( \mathfrak{A} \), we have that
\[ \lim_{n \to \infty} \| \alpha_n \circ \beta(x) - x \| = 0 \]
for all \( x \in \mathfrak{A} \). We have just proved the first part of the lemma.

We now show that \( \mathfrak{A} \) is weakly semiprojective. Suppose \( \mathfrak{A} \) is weakly semiprojective. Let \( \epsilon > 0 \) and \( \mathcal{F} \) be a finite subset of \( \mathfrak{A} \). By Theorem 2.4 of [23] (see also Definition 2.1 and Theorem 2.3 of [25], and Theorem 19.1.3 of [26]), there exist a \( \delta > 0 \) and a finite subset \( \mathcal{G} \) of \( \mathfrak{A} \) such that for any \( C^\ast \)-algebra \( \mathfrak{B} \) and any contractive, completely positive, linear map \( L : \mathfrak{A} \to \mathfrak{B} \) such that
\[ \| L(ab) - L(a)L(b) \| < \delta \]
for all \( a, b \in \mathcal{G} \), there exists a homomorphism \( h : \mathfrak{A} \to \mathfrak{B} \) such that
\[ \| h(x) - L(x) \| < \frac{\epsilon}{2} \]
for all \( x \in \beta(\mathcal{F}) \).

Without loss of generality, we may assume that \( \epsilon < 1 \) and \( \delta < 1 \). Set
\[ M = 1 + \max \left( \{ \| a \| : a \in \mathcal{G} \} \cup \{ \| x \| : x \in \mathcal{F} \} \right) \]
Since \( e_n \mathfrak{A}e_n \subseteq e_{n+1} \mathfrak{A}e_{n+1} \) and \( \bigcup_{n=1}^{\infty} e_n \mathfrak{A}e_n \) is dense in \( \mathfrak{A} \), there exist \( n \in \mathbb{N} \) and a finite subset \( \mathcal{H} \subseteq e_n \mathfrak{A}e_n \) such that for each \( a \in \mathcal{G} \), there exists \( y \in \mathcal{H} \) such that \( \| a - y \| < \frac{\delta}{4M} \) and
\[ \| \alpha_n \circ \beta(x) - x \| < \frac{\epsilon}{2} \]
for all \( x \in \mathcal{F} \). Let \( a, b \in \mathcal{G} \). Choose \( x, y \in \mathcal{H} \subseteq e_n \mathfrak{A}e_n \) such that \( \| a - x \| < \frac{\delta}{4M} \) and \( \| b - y \| < \frac{\delta}{4M} \). Note that \( \| x \| \leq 1 + \| a \| \leq M \) and \( \| y \| \leq 1 + \| b \| \leq M \). Then
\[
\| \alpha_n(ab) - \alpha_n(a)\alpha_n(b) \| = \| \alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b) \| \\
\leq \| (\alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b)) \| \\
+ \| \alpha_n(x)\alpha_n(y) - \alpha_n(x)\alpha_n(b) \| \\
+ \| \alpha_n(x)\alpha_n(b) - \alpha_n(a)\alpha_n(b) \| \\
\leq 2M \| a - x \| + 2M \| b - y \| \\
< 4M \frac{\delta}{4M} = \delta.
\]
By the choice of \( \delta \) and \( \mathcal{G} \), there exists a homomorphism \( \psi : \mathfrak{A} \to \mathfrak{A} \) such that
\[ \| \psi(t) - \alpha_n(t) \| < \frac{\epsilon}{2} \]
for all $t \in \beta(\mathcal{F})$. Let $x \in \mathcal{F}$. Then

$$\|\psi \circ \beta(x) - x\| \leq \|\psi(\beta(x)) - \alpha_n(\beta(x))\| + \|\alpha_n(\beta(x)) - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$ 

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{A}$, there exists a homomorphism $\psi : \mathfrak{A} \to \mathfrak{A}$ such that

$$\|\psi \circ \beta(x) - x\| < \epsilon$$

for all $x \in \mathcal{F}$. Consequently, there exists a sequence of endomorphisms $\{\psi_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} \|\psi_n \circ \beta(x) - x\| = 0$$

for all $x \in \mathfrak{A}$ since $\mathfrak{A}$ is separable. \hfill \qed

To prove a uniqueness theorem involving tight $C^*\text{-algebras}$ $\mathfrak{A}$ over $X_2$, we require that $\mathfrak{A}[1]$ belongs to a class of $C^*\text{-algebras}$ whose injective homomorphisms between two objects in this class are classified by $KK$.

**Definition 4.12.** We will be interested in classes $C$ of separable, nuclear, simple $C^*$-algebras satisfying the following property that if $\mathfrak{A}, \mathfrak{B} \in C$ and $\phi, \psi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K}$ are two injective homomorphisms such that $KK(\phi) = KK(\psi)$, then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Remark 4.13.**

1. By Theorem 4.1.3 of [29] if $C$ is the class of Kirchberg algebras, then $C$ satisfies the property in Definition 4.12.
2. Let $C$ be the class of unital, separable, nuclear, simple tracially AF $C^*$-algebras in $\mathcal{N}$. Then $C$ satisfies the property in Definition 4.12.

**Theorem 4.14. (Uniqueness Theorem 2)** Let $C$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathfrak{A}$ be a unital, separable, nuclear, tight $C^*$-algebra over $X_2$ such that $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1] \in C$. Suppose $\mathfrak{A} \otimes \mathbb{K}$ is semiprojective and $\mathfrak{A}$ has the stable weak cancellation property. Let $\phi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}$ be a full $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \id_{\mathfrak{A} \otimes \mathbb{K}})$. Then there exists a sequence of full $X_2$-equivariant endomorphisms $\{\alpha_n : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \id_{\mathfrak{A} \otimes \mathbb{K}})$ and

$$\lim_{n \to \infty} \|\alpha_n \circ \phi(x) - x\| = 0$$

for all $x \in \mathfrak{A} \otimes \mathbb{K}$.

**Proof.** Set $\mathfrak{B} = \mathfrak{A} \otimes \mathbb{K}$. Note that $\mathfrak{B}$ is a tight $C^*$-algebra over $X_2$ with $\mathfrak{B}[2] \cong \mathbb{K}$. Throughout the proof, $\pi : \mathfrak{A} \to \mathfrak{B}[1]$ will denote the canonical projection. Note that $KK(\phi_{\{1\}}) = KK(\id_{\mathfrak{B}[1]})$ since $KK(X_2; \phi) = KK(X_2; \id_{\mathfrak{B}})$. Since $\mathfrak{A}[1] \in C$, there exists a sequence of unitaries $\{z_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[1])$ such that

$$\lim_{k \to \infty} \|z_k \phi_{\{1\}}(\pi(b))z_k^* - \pi(b)\| = 0$$

for all $b \in \mathfrak{B}$. Using the fact that $\phi$ is an $X_2$-equivariant homomorphism, we have that $\pi \circ \phi = \phi_{\{1\}} \circ \pi$, and hence

$$\lim_{k \to \infty} \|z_k(\pi \circ \phi(b))z_k^* - \pi(b)\| = 0$$
for all \( b \in \mathcal{B} \).

Let \( \pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B}[1]) \) be the surjective homomorphism induced by \( \pi \). Since \( \mathcal{B} \) is stable, by Corollary 2.3 of [35], we have that \( \mathcal{B}[1] \) is stable. Thus, the unitary group of \( \mathcal{M}(\mathcal{B}[1]) \) is path-connected, which implies that every unitary in \( \mathcal{M}(\mathcal{B}[1]) \) lifts to a unitary in \( \mathcal{M}(\mathcal{B}) \). Hence, there exists a sequence of unitaries \( \{ w_k \}_{k=1}^\infty \) in \( \mathcal{M}(\mathcal{B}) \) such that \( \pi(w_k) = z_k \).

Since \( \mathcal{B} \) is semiprojective, by Proposition 2.2 of [7] (see [26]), there exists a sequence of homomorphisms \( \{ \beta_\ell : \mathcal{B} \to \mathcal{B} \}_{\ell=1}^\infty \) and a strictly increasing sequence \( \{ k(\ell) \}_{\ell=1}^\infty \) of positive integers such that \( \pi \circ \beta_\ell = \pi \) and

\[
\lim_{\ell \to \infty} \| \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b) \| = 0
\]

for all \( b \in \mathcal{B} \).

By Remark 2.5, there exists \( N_1 \in \mathbb{N} \) such that \( \beta_\ell \) is a full \( X_2 \)-equivariant homomorphism for all \( \ell \geq N_1 \). By Proposition 2.3 of [7], we may choose \( N_2 \geq N_1 \) such that for all \( \ell \geq N_2 \), we have that \( \beta_\ell \) and \( \text{Ad}(w_{k(\ell)}) \circ \phi \) is homotopic. It follows from Theorem 5.5 of [8] that \( KK(X_2; \beta_\ell) = KK(X_2; \text{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \text{id}_\mathcal{B}) \).

Let \( \ell \geq N_2 \). Note that \( \{ \beta_\ell \}_{\ell=1}^\infty \) is strictly increasing, and by Proposition 2.3 of [7], we may choose \( \alpha \).\( \in \mathcal{B} \) such that \( \alpha \) is semiprojective, by Proposition 2.2 of [7] (see [26]), there exists a sequence of homomorphisms \( \{ \alpha_{m,\ell} : \mathcal{B} \to \mathcal{B} \}_{m=1}^\infty \) such that

\[
\lim_{m \to \infty} \| \alpha_{m,\ell} \circ \beta_\ell(x) - x \| = 0
\]

for all \( x \in \mathcal{B} \). Since \( \beta_\ell \) and \( \text{id}_\mathcal{B} \) are full \( X_2 \)-equivariant homomorphisms, by Remark 2.5, there exists \( N_3 \) such that, for all \( m \geq N_3 \), we have that \( \alpha_{m,\ell} \) is a full \( X_2 \)-equivariant homomorphism. Moreover, by Proposition 2.3 of [7], we can choose \( N_3 \geq N_2 \) such that \( \alpha_{m,\ell} \circ \beta_\ell \circ \beta_\ell \circ \beta_\ell \) are homotopic. It follows from Theorem 5.5 of [8] that \( KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \text{id}_\mathcal{B}) \) for all \( m \geq N_3 \). Consequently, \( KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B}) \) for all \( m \geq N_3 \), since \( KK(X_2; \beta_\ell) = KK(X_2; \text{id}_\mathcal{B}) \).

Let \( \mathcal{F} \) be a finite subset of \( \mathcal{B} \) and \( \epsilon > 0 \). Then there exists \( \ell \geq N_2 \) such that

\[
\| \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b) \| < \frac{\epsilon}{2}
\]

for all \( b \in \mathcal{F} \). Moreover, there exists \( m \geq N_3 \) such that

\[
\| \alpha_{m,\ell} \circ \beta_\ell(b) - b \| < \frac{\epsilon}{2}
\]

for all \( b \in \mathcal{F} \). Set \( \alpha_1 = \text{Ad}(w_{k(\ell)})|_{\mathcal{B}} \) and \( \alpha = \alpha_{m,\ell} \circ \alpha_1 \). Since \( w_{k(\ell)} \) is a unitary in \( \mathcal{M}(\mathcal{B}) \), we have that \( \alpha_1 \) is an automorphism of \( \mathcal{B} \) and \( KK(X_2; \alpha_1) = KK(X_2; \text{id}_\mathcal{B}) \). Therefore, \( \alpha \) is a full \( X_2 \)-equivariant homomorphism. Since \( \ell \geq N_2 \) and \( m \geq N_3 \), we have that \( KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B}) \). Therefore, \( KK(X_2; \alpha) = KK(X_2; \text{id}_\mathcal{B}) \). Let \( b \in \mathcal{F} \). Then

\[
\| \alpha \circ \phi(b) - b \| = \| \alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - b \|
\leq \| \alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b) \| + \| \alpha_{m,\ell} \circ \beta_\ell(b) - b \|
\leq \epsilon + \frac{\epsilon}{2} = \epsilon.
\]

We have just shown that for every \( \epsilon > 0 \) and for every finite subset \( \mathcal{F} \) of \( \mathcal{B} \), there exists a full \( X_2 \)-equivariant homomorphism \( \alpha : \mathcal{B} \to \mathcal{B} \) such that \( KK(X_2; \alpha) = KK(X_2; \text{id}_\mathcal{B}) \) and

\[
\| \alpha \circ \phi(b) - b \| < \epsilon
\]
for all $b \in \mathcal{B}$. Since $\mathcal{B}$ is a separable $C^*$-algebra, there exists a sequence of full $X_2$-equivariant homomorphisms $\{\alpha_n : \mathcal{B} \to \mathcal{B}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \text{id}_{\mathcal{B}})$ and

$$\lim_{n \to \infty} \|\alpha_n \circ \phi(b) - b\| = 0$$

for all $b \in \mathcal{B}$. 

**Theorem 4.15.** Let $C$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i[2] \cong K$ and $\mathfrak{A}_i[1] \in C$. Suppose $\mathfrak{A}_i \otimes K$ is semiprojective and $\mathfrak{A}_i$ has the stable weak cancellation property. If there exist full $X_2$-equivariant homomorphisms, $\phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ and $\psi : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K$, such that $KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$ and $KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K})$, then for any finite subset $\mathcal{F}$ and $\epsilon > 0$, there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \gamma) = KK(\phi)$ and

$$\|\gamma(x) - \phi(x)\| < \epsilon$$

for all $x \in \mathcal{F}$.

**Proof.** Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_1 \otimes K$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes K$ and let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_2 \otimes K$ such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $\mathfrak{A}_2 \otimes K$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Set $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_1$ and choose $m_1 \in \mathbb{N}$ such that $\sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon$. By Theorem 4.14 there exists a full $X_2$-equivariant homomorphism $\alpha_1 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_1 \otimes K$ such that $KK(X_2; \alpha_1) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K})$ and

$$\|\alpha_1 \circ \psi \circ \phi(a) - a\| < \frac{1}{2^{m_1+1}}$$

for all $a \in \mathcal{F}_1$. Set $\phi_1 = \phi$ and $\psi_1 = \alpha_1 \circ \psi$. Then $KK(X_2; \psi_1) = KK(X_2; \psi)$ and $\|\psi_1 \circ \phi_1(a) - a\| < \frac{1}{2^{m_1+1}}$ for all $a \in \mathcal{F}_1$.

Set $\mathcal{G}_1 = \mathcal{G}_1 \cup \phi(\mathcal{F}_1)$. Note that $KK(X_2; \phi \circ \psi_1) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$. Hence, by Theorem 4.14 there exists a full $X_2$-equivariant homomorphism $\beta_1 : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \beta_1) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$ and

$$\|\beta_1 \circ \phi \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Set $\phi_2 = \beta_1 \circ \phi$. Then $KK(X_2; \phi_2) = KK(X_2; \phi)$ and

$$\|\phi_2 \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Note that for all $x \in \mathcal{F}_1$, then

$$\|\phi(x) - \phi_2(x)\| \leq \|\phi_1(x) - \phi_2 \circ \psi_1(\phi_1(x))\| + \|\phi_2 \circ \psi_1(\phi_1(x)) - \phi_2(x)\|$$

$$< \frac{1}{2^{m_1+1}} + \|\psi_1 \circ \phi_1(x) - x\| < \frac{1}{2^{m_1}}.$$

Set $\mathcal{F}_2 = \mathcal{F}_2 \cup \phi_2(\mathcal{G}_1)$. Note that $KK(X_2; \psi \circ \phi_2) = KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$. Hence, by Theorem 4.14 there exists a full $X_2$-equivariant homomorphism $\alpha_2 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$. 

\( \mathfrak{A}_1 \otimes K \) such that \( KK(X_2; \alpha_2) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K}) \) and

\[ \| \alpha_2 \circ \psi \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}} \]

for all \( a \in \mathcal{F}_2 \). Set \( \psi_2 = \alpha_2 \circ \psi \). Then \( KK(X_2; \psi_2) = KK(X_2; \psi) \) and

\[ \| \psi_2 \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}} \]

for all \( x \in \mathcal{F}_2 \).

Set \( \mathcal{G}_2 = \overline{\mathcal{G}_2} \cup \phi_2(\mathcal{F}_2) \). Note that \( KK(X_2; \phi \circ \psi_2) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \).

Hence, by Theorem 1.14 there exists a full \( X_2 \)-equivariant homomorphism \( \beta_2 : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_2 \otimes K \) such that \( KK(X_2; \beta_2) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \) and

\[ \| \beta_2 \circ \phi \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}} \]

for all \( x \in \mathcal{G}_2 \). Set \( \phi_3 = \beta_2 \circ \phi \). Then \( KK(X_2; \phi_3) = KK(X_2; \phi) \) and

\[ \| \phi_3 \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}} \]

for all \( x \in \mathcal{G}_2 \). Note that for all \( x \in \mathcal{F}_2 \), we have that

\[ \| \phi_2(x) - \phi_3(x) \| \leq \| \phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x)) \| + \| \phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x) \| \]

\[ < \frac{1}{2^{m_1+2}} + \| \phi_2(\phi_2(x)) - x \| \leq \frac{1}{2^{m_1+1}}. \]

Continuing this process, we have constructed a sequence \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_1 \otimes K \), a sequence \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_2 \otimes K \), a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \phi_n : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \}_{n=1}^{\infty} \), and a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \psi_n : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K \}_{n=1}^{\infty} \) such that

1. \( KK(X_2; \phi_n) = KK(X_2; \phi) \) for all \( n \in \mathbb{N} \) and \( \phi_1 = \phi \);
2. \( KK(X_2; \psi_n) = KK(X_2; \psi) \) for all \( n \in \mathbb{N} \);
3. \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \overline{\mathcal{F}_n} \subseteq \mathcal{F}_n \);
4. \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \) and \( \overline{\mathcal{G}_n} \subseteq \mathcal{G}_n \);
5. for each \( x \in \mathcal{F}_n \) and for each \( x \in \mathcal{G}_n \),

\[ \| \psi_n \circ \phi_n(x) - x \| < \frac{1}{2^{m_1+n}} \quad \text{and} \quad \| \phi_{n+1} \circ \psi_n(x) - x \| < \frac{1}{2^{m_1+n}} \]

6. for each \( x \in \mathcal{F}_n \),

\[ \| \phi_n(x) - \phi_{n+1}(x) \| < \frac{1}{2^{m_1+n-1}} \]

Since \( \bigcup_{n=1}^{\infty} \overline{\mathcal{F}_n} \) is dense in \( \mathfrak{A}_1 \otimes K \) and \( \overline{\mathcal{F}_n} \subseteq \mathcal{F}_n \), we have that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes K \).

Similarly, \( \bigcup_{n=1}^{\infty} \mathcal{G}_n \) is dense in \( \mathfrak{A}_2 \otimes K \). Therefore, there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) such that

\[ \| \gamma(a) - \phi_n(a) \| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} \]
for all \( a \in \mathcal{F}_n \). Since \( \mathcal{F} \subseteq \mathcal{F}_1 \), we have that

\[
\| \phi(x) - \gamma(x) \| = \| \phi_1(x) - \gamma(x) \| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon.
\]

Since

\[
\lim_{n \to \infty} \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} = 0,
\]

we have that

\[
\lim_{n \to \infty} \| \gamma(a) - \phi_n(a) \| = 0
\]

for all \( a \in \mathcal{A}_1 \otimes \mathbb{K} \). Since \( \mathcal{A}_1 \otimes \mathbb{K} \) is semiprojective, by Proposition 2.3 of \([7]\), there exists \( N \in \mathbb{N} \) such that \( \gamma \) and \( \phi_N \) are homotopic. Hence, by Theorem 5.5 of \([8]\), \( KK(X_2; \gamma) = KK(X_2; \phi_N) = x \).

4.3. Unital Classification. We know combine the above results with the Meta-theorem of Section 3.1 (see Theorem 3.3) to get a strong classification for a class of unital \( C^* \)-algebras which includes all unital graph \( C^* \)-algebras with exactly one non-trivial ideal.

**Corollary 4.16.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be unital, tight \( C^* \)-algebras over \( X_n \) such that \( \mathcal{A}_i \) has real rank zero, \( \mathcal{A}_i[n] \) is a Kirchberg algebra in \( \mathcal{N} \), and \( \mathcal{A}_i[1,n-1] \) is an AF-algebra. Let \( x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2) \) be an invertible such that \( K_{X_n}(x)_Y \) is an order isomorphism for each \( Y \in \mathcal{L}(X_n) \) and \( K_{X_n}(x)_{X_n}([1_{\mathcal{A}_1}]) = [1_{\mathcal{A}_2}] \) in \( K_0(\mathcal{A}_2) \). Then there exists an isomorphism \( \phi : \mathcal{A} \to \mathcal{B} \) such that \( K_{X_n}(\phi) = K_{X_n}(x) \).

**Proof.** Since \( \mathcal{A}_i[1] \) and \( \mathcal{A}_i[2] \) are separable and nuclear, we have that \( \mathcal{A}_i \) is separable and nuclear. Since \( \mathcal{A}_i[1,n-1] \) is an AF-algebra and \( \mathcal{A}_i[n] \) is a Kirchberg algebra, they both have the stable weak cancellation property. By Lemma 3.15 of \([15]\), \( \mathcal{A}_i \) has stable weak cancellation property. By Lemma 4.6, for each tight \( C^* \)-algebra \( \mathcal{A} \) over \( X_n \), we have that \( K_{X_n}(\text{Ad}(u)|_{\mathcal{A}}) \) for each unitary \( u \in \mathcal{M}(\mathcal{A}) \). A computation shows that \( K_{X_n}(-) \) satisfies (1), (2), and (3) of Theorem 3.3 since \( K_n(-) \) does. The corollary now follows from Theorem 3.3 and Theorem 4.7.

**Corollary 4.17.** Let \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) be unital, tight \( C^* \)-algebras over \( X_2 \) such that \( \mathcal{A}_i[2] \cong \mathbb{K} \) and \( \mathcal{A}_i[1] \) is a Kirchberg algebra in \( \mathcal{N} \). Let \( x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2) \) be an invertible such that \( K_{X_2}(x)_Y \) is an order isomorphism for each \( Y \in \mathcal{L}(X_2) \) and \( K_{X_2}(x)_{X_2}([1_{\mathcal{A}_1}]) = [1_{\mathcal{A}_2}] \) in \( K_0(\mathcal{A}_2) \). If \( \mathcal{A}_i \otimes \mathbb{K} \) is semiprojective, then there exists an isomorphism \( \gamma : \mathcal{A}_1 \otimes \mathbb{K} \to \mathcal{A}_2 \otimes \mathbb{K} \) such that \( KK(X_2; \gamma) = x \).

**Proof.** Since \( \mathcal{A}_i[1] \) and \( \mathcal{A}_i[2] \) are separable and nuclear, we have that \( \mathcal{A}_i \) is separable and nuclear. Since \( \mathcal{A}_i[2] \) and \( \mathcal{A}_i[1] \) have real rank zero and \( K_1(\mathcal{A}_i[2]) = 0 \), we have that \( \mathcal{A} \) has real rank zero. Since \( \mathcal{A}_i[2] \) is an AF-algebra and \( \mathcal{A}_i[1] \) is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by Lemma 3.15 of \([15]\), \( \mathcal{A} \) has the stable weak cancellation property.

By Lemma 1.5 of \([16]\), the extension \( 0 \to \mathcal{A}_i[2] \to \mathcal{A}_i \to \mathcal{A}_i[1] \to 0 \) is full, and hence by Proposition 1.6 of \([16]\), \( 0 \to \mathcal{A}_i[2] \otimes \mathbb{K} \to \mathcal{A}_i \otimes \mathbb{K} \to \mathcal{A}_i[1] \otimes \mathbb{K} \to 0 \) is full. The corollary now follows from Theorem 4.1(iii), Theorem 4.15 and Theorem 3.3.
It is an open question to determine if every unital, separable, nuclear, tight $C^*$-algebra $\mathfrak{A}$ over $X_2$ whose unique proper nontrivial ideal is isomorphic to $\mathbb{K}$ and quotient is a Kirchberg algebra in $\mathcal{N}$ with finitely generated $K$-theory is semiprojective. The following results show that under some $K$-theoretical conditions, $\mathfrak{A}$ is semiprojective.

**Lemma 4.18.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$ with $C^*(E)[1]$ being purely infinite. Then $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semiprojective.

**Proof.** The fact that $C^*(E)$ is semiprojective follows from the results of [12]. By Proposition 6.4 of [18], $C^*(E)[2]$ is stable. Since $C^*(E)$ is a unital $C^*$-algebra, by Lemma 1.5 of [16], the extension $e : 0 \to C^*(E)[2] \to C^*(E) \to C^*(E)[1] \to 0$ is a full extension. By Proposition 3.21 and Corollary 3.22 of [15], $C^*(E)$ is properly infinite. Therefore, by Theorem 4.1 of [3], $C^*(E) \otimes \mathbb{K}$ is semiprojective.

**Proposition 4.19.** Let $\mathfrak{A}$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$. If $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}[1])) \leq \text{rank}(K_0(\mathfrak{A}[1]))$, $K_1(\mathfrak{A}[1])$ is free, and the $K$-groups of $\mathfrak{A}[i]$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semiprojective. Consequently, $\mathfrak{A}$ semiprojective.

**Proof.** By Lemma 1.5 of [16], $e : 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$ is a full extension. By Corollary 3.22 of [15], $K_0(\mathfrak{A}) = K_0(\mathfrak{A})$. By Theorem 6.4 of [11], there exists a graph $E$ with finitely many vertices such that $K_{X_2}^+(\mathfrak{A}) \cong K_{X_2}^+(C^*(E))$ such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$. Since $E$ has finitely many vertices, $C^*(E)$ is unital. Since $K_{X_2}^+(\mathfrak{A}) \cong K_{X_2}^+(C^*(E))$, we have that $C^*(E)[1]$ is a Kirchberg algebra. By Theorem 3.9 of [16], we have that $\mathfrak{A} \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. By Lemma 4.18, $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semiprojective. Hence, by Proposition 2.7 of [3], $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semiprojective.

**Corollary 4.20.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_1[2] \cong \mathbb{K}$ and $\mathfrak{A}_1[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}_1[1])) \leq \text{rank}(K_0(\mathfrak{A}_1[1]))$, $K_1(\mathfrak{A}_1[1])$ is free, and the $K$-groups of $\mathfrak{A}_1$ are finitely generated. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{L}(X_2)$ and $K_{X_2}(x)[1_{\mathfrak{A}_2}] = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \gamma) = x$.

**Proof.** This follows from Proposition 4.19 and Corollary 4.17.

5. Applications

Let $E$ be a graph satisfying Condition (K) (in particular, if $C^*(E)$ has finitely many ideals, then $E$ satisfies Condition (K)). Let $\mathfrak{I}_1, \mathfrak{I}_2$ be ideals of $C^*(E)$ such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ and $\mathfrak{I}_2/\mathfrak{I}_1$ is simple. Then by Theorem 5.1 of [38] and Corollary 3.5 of [2], $\mathfrak{I}_2/\mathfrak{I}_1$ is a simple graph $C^*$-algebra. Hence, $\mathfrak{I}_2/\mathfrak{I}_1$ is either a Kirchberg algebra or an AF algebra.

5.1. Classification of graph $C^*$-algebras with exactly one ideal.

**Lemma 5.1.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a simple AF-algebra. Then $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Consequently, if $F$ is a graph with finitely many vertices such that $C^*(F)$ is a tight $C^*$-algebra over $X_2$ and $C^*(F)[2]$ is an AF-algebra, then $C^*(F)[2] \cong \mathbb{K}$.
Proof. We claim that $E$ is a finite graph. By Corollary 2.13 and Corollary 2.15 of [9], $E$ has no cycles, and for every vertex $v_0$ that emits infinitely many edges and for each vertex $v$, there exists a path from $v$ to $v_0$. Since $E$ has no cycles, we have that every vertex of $E$ emits only finitely many edges. Hence, $E$ is a finite graph. By Proposition 1.18 of [30], $C^*(E) \cong M_n$.

We now prove the second statement. First note that $C^*(F)[2]$ is a simple AF-algebra. Since $C^*(F)[2]$ is stably isomorphic to a subgraph of $E$, $C^*(F)[2] \otimes \mathbb{K} \cong C^*(E)$ for some graph $E$ with finitely many vertices. Since $C^*(E)$ is a simple AF-algebra, we have that $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Hence, $C^*(F)[2] \otimes \mathbb{K} \cong \mathbb{K}$ which implies that $C^*(F)[2] \cong M_n$ or $C^*(F)[2] \cong \mathbb{K}$. Since $C^*(F)[2]$ is a non-unital $C^*$-algebra ($C^*(E)$ is a tight $C^*$-algebra over $X_2$), we have that $C^*(F)[2] \cong \mathbb{K}$. \hfill \Box

Definition 5.2. For a $C^*$-algebra $\mathfrak{A}$, set

$$\Sigma\mathfrak{A} = \{ x \in K_0(\mathfrak{A}) : x = [p] \text{ for some projection } p \text{ in } \mathfrak{A} \}.$$ 

Let $\mathfrak{B}$ be a $C^*$-algebra. An order isomorphism $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is \textit{scale preserving} if one of the following holds:

1. $\mathfrak{A}$ is unital if and only if $\mathfrak{B}$ unital and $\alpha([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$.
2. $\mathfrak{A}$ is non-unital if and only if $\mathfrak{B}$ is non-unital and $\alpha(\Sigma\mathfrak{A}) = \Sigma\mathfrak{B}$.

Theorem 5.3. Let $E_1$ and $E_2$ be graphs with finitely many vertices and $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. If $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ is an isomorphism such that $\alpha_Y$ is scale preserving for all $Y \in \mathcal{I}(X_2)$, then there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Proof. Since $E_i$ has finitely many vertices, $C^*(E_1)$ and $C^*(E_2)$ are unital $C^*$-algebras.

Case 1: Suppose $C^*(E_1)$ is an AF-algebra. Then $C^*(E_2)$ is an AF-algebra. Hence, the result follows from Elliott’s classification of AF-algebras [19].

Case 2: Suppose $C^*(E_1)$ is not an AF-algebra. Then $C^*(E_2)$ is not an AF-algebra.

Subcase 2.1: Suppose $C^*(E_1)[1]$ is an AF-algebra. Then $C^*(E_2)[1]$ is an AF-algebra. By Corollary 2.10 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Subcase 2.2: Suppose $C^*(E_1)[1]$ is a Kirchberg algebra. Then $C^*(E_2)[1]$ is a Kirchberg algebra. Since $C^*(E_i)$ is not an AF-algebra, either $C^*(E_i)[2]$ is Kirchberg algebra or an AF-algebra.

Suppose $C^*(E_i)[2]$ is a Kirchberg algebra. By Theorem 2.4 of [32], there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. Suppose $C^*(E_i)[2]$ is an AF-algebra. Then, by Lemma 5.1 $C^*(E_i)[2] \cong \mathbb{K}$. By Corollary 4.20 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. \hfill \Box

The following theorem completes the classification of graph $C^*$-algebras with exactly one non-trivial ideal.

Corollary 5.4. Let $E_1$ and $E_2$ be graphs such that $C^*(E_1)$ is a tight $C^*$-algebra over $X_2$. Then $C^*(E_1) \cong C^*(E_2)$ if and only if there exists an isomorphism $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ such that $\alpha_Y$ is a scale preserving isomorphism for all $Y \in \mathcal{I}(X_2)$. 


Proof. The only case that is not covered by Theorem 4.9 of [15] is the case that \( C^*(E_i) \) is unital. The unital case follows from Theorem 5.3 because of Theorem 3.3.

5.2. Classification of graph \( C^* \)-algebras with more than one ideal. For a tight \( C^* \)-algebra \( \mathfrak{A} \) over \( X_n \), the finite and infinite simple sub-quotients of \( \mathfrak{A} \) are separated if there exists \( U \in O(X_n) \) such that either

1. \( \mathfrak{A}(U) \) is an AF-algebra and \( \mathfrak{A}(X_n \setminus U) \otimes O_\infty \cong \mathfrak{A}(X_n \setminus U) \) or
2. \( \mathfrak{A}(X_n \setminus U) \) is an AF-algebra and \( \mathfrak{A}(U) \otimes O_\infty \cong \mathfrak{A}(U) \).

In [14], the authors proved that if \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) are graph \( C^* \)-algebras that are tight \( C^* \)-algebras over \( X_n \) such that the finite and infinite simple sub-quotients are separated, then \( \mathfrak{A}_1 \otimes K \cong \mathfrak{A}_2 \otimes K \) if and only if \( K^+_X(\mathfrak{A}_1) \cong K^+_X(\mathfrak{A}_2) \). We will show in this section that under mild \( K \)-theoretical conditions, we may remove the separated condition for the case \( n = 3 \).

Lemma 5.5. Let \( E \) be a graph such that \( C^*(E) \) is a tight \( C^* \)-algebra over \( X_n \).

(i) If \( C^*(E)[n] \) and \( C^*(E)[1] \) are purely infinite and \( C^*(E)[2, n-1] \) is an AF-algebra, then

\[
e_1 : 0 \to C^*(E)[2,n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1] \otimes K \to 0
\]

is a full extension.

(ii) If \( C^*(E)[k,n] \) and \( C^*(E)[1,k-2] \) are AF-algebras and \( C^*(E)[k-1] \) is purely infinite, then

\[
e_2 : 0 \to C^*(E)[k,n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1,k-1] \otimes K \to 0
\]

is a full extension.

Proof. Suppose \( C^*(E)[n] \) and \( C^*(E)[1] \) are purely infinite and \( C^*(E)[2,n-1] \) is an AF-algebra. Note that \( C^*(E)[1,n-1]/C^*(E)[2,n-1] \cong C^*(E)[1] \) and \( C^*(E)[2,n-1] \) is the largest ideal of \( C^*(E)[1,n-1] \) which is an AF-algebra. Since \( C^*(E)[1,n-1] \) is isomorphic to a graph \( C^* \)-algebra, by Proposition 3.10 of [15],

\[
0 \to C^*(E)[2,n-1] \otimes K \to C^*(E)[1,n-1] \otimes K \to C^*(E)[1] \otimes K \to 0
\]

is a full extension. Since \( C^*(E)[n] \otimes K \) is a purely infinite simple \( C^* \)-algebra, we have that

\[
0 \to C^*(E)[n] \otimes K \to C^*(E)[2,n] \otimes K \to C^*(E)[2,n-1] \otimes K \to 0
\]

is a full extension. Hence, by Proposition 3.2 of [17], \( e_1 \) is a full extension.

Suppose \( C^*(E)[k,n] \) and \( C^*(E)[1,k-2] \) are AF-algebras and \( C^*(E)[k-1] \) is purely infinite. Note that \( C^*(E)[k,n] \) is the largest ideal of \( C^*(E)[k-1,n] \) such that \( C^*(E)[k,n] \) is an AF-algebra and \( C^*(E)[k-1,n]/C^*(E)[k,n] \cong C^*(E)[k-1] \) is purely infinite. Since \( C^*(E)[k-1,n] \otimes K \) is isomorphic to a graph \( C^* \)-algebra, by Proposition 3.10 of [18],

\[
0 \to C^*(E)[k,n] \otimes K \to C^*(E)[k-1,n] \otimes K \to C^*(E)[k-1] \otimes K \to 0
\]

is a full extension. By Proposition 5.4 of [14], \( e_2 \) is a full extension.

Theorem 5.6. Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_n \).

Suppose

(i) \( C^*(E_i)[n] \) and \( C^*(E_i)[1] \) are purely infinite; and

(ii) \( C^*(E_i)[2,n-1] \) is an AF-algebra; and
Theorem 5.7. \( KK^1(C^*(E_1)[1], C^*(E_2)[2, n]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, n]) \).

Then \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \) if and only if \( K_{X_n}^+(C^*(E_1) \otimes \mathbb{K}) \cong K_{X_n}^+(C^*(E_2) \otimes \mathbb{K}) \).

Proof. Let \( e_i \) be the extension

\[
0 \to C^*(E_i)[2, n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1] \otimes \mathbb{K} \to 0.
\]

By Lemma 5.5(i), \( e_i \) is a full extension. Suppose \( \alpha : K_{X_n}^+(C^*(E_1) \otimes \mathbb{K}) \to K_{X_n}^+(C^*(E_2) \otimes \mathbb{K}) \).

Let \( \alpha \) to an invertible element \( x \in KK(X_n; C^*(E_1) \otimes \mathbb{K}, C^*(E_2) \otimes \mathbb{K}) \). Note that \( r_{X_n}^{[2, n]}(x) \) is invertible in \( KK((2, n); C^*(E_1)[2, n] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \) and \( r_{X_n}^{[1]}(x) \) is invertible in \( KK(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[1] \otimes \mathbb{K}) \). By Theorem 4.7 there exists an isomorphism \( \phi_0 : C^*(E_1)[2, n] \otimes \mathbb{K} \to C^*(E_2)[2, n] \otimes \mathbb{K} \) such that \( KL(\phi_0) = z \), where \( z \) is the invertible element of \( KL(C^*(E_1)[2, n] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \) induced by \( r_{X_n}^{[2, n]}(x) \). By the Kirchberg-Phillips classification \([21]\) and \([29]\), there exists an isomorphism \( \phi_2 : C^*(E_1)[1] \otimes \mathbb{K} \to C^*(E_2)[1] \otimes \mathbb{K} \) such that \( KK(\phi_2) = r_{X_n}^{[1]}(x) \).

Consider \( C^*(E_i) \) as a \( C^* \)-algebra over \( X_2 \) by setting \( C^*(E_i)[2] = C^*(E_i)[2, n] \) and \( C^*(E_i)[1, 2] = C^*(E_i) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( r_{X_2}^{[1]}(y) = r_{X_n}^{[1]}(x) = KK(\phi_2) \) and \( KL(r_{X_2}^{[2]}(y)) = z = KL(\phi_0) \) in \( KL(C^*(E_1)[2, n], C^*(E_2)[2, n]) \).

By Theorem 3.7 of \([13]\),

\[
r_{X_2}^{[1]}(y) \times [\tau_2] = [\tau_1] \times r_{X_2}^{[2]}(y)
\]

in \( KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \), where \( e_i \) is the extension

\[
0 \to C^*(E_i)[2, n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1] \otimes \mathbb{K} \to 0.
\]

Thus,

\[
KL(\phi_2) \times [\tau_2] = [\tau_1] \times KL(\phi_0)
\]

in \( KL^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \). Since \( KL^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) = KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \),

\[
KK(\phi_2) \times [\tau_2] = [\tau_1] \times KK(\phi_0)
\]

in \( KK^1(C^*(E_1)[1] \otimes \mathbb{K}, C^*(E_2)[2, n] \otimes \mathbb{K}) \). By Lemma 4.5 of \([13]\), \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \). □

Theorem 5.7. Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_n \).

Suppose

(i) \( C^*(E_i)[k, n] \) and \( C^*(E_i)[1, k - 2] \) are AF-algebras;

(ii) \( C^*(E_i)[k - 1] \) is purely infinite; and

(iii) \( KK^1(C^*(E_1)[1, k - 1], C^*(E_2)[k, n]) = KL^1(C^*(E_1)[1, k - 1], C^*(E_2)[k, n]) \).

Then \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \) if and only if \( K_{X_n}^+(C^*(E_1) \otimes \mathbb{K}) \cong K_{X_n}^+(C^*(E_2) \otimes \mathbb{K}) \).

Proof. Let \( e_i \) be the extension \( 0 \to C^*(E_i)[k, n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1, k - 1] \otimes \mathbb{K} \to 0 \).

By Lemma 5.5(ii), \( e_i \) is a full extension. Suppose \( \alpha : K_{X_n}^+(C^*(E_1) \otimes \mathbb{K}) \to K_{X_n}^+(C^*(E_2) \otimes \mathbb{K}) \).

Let \( \alpha \) to an invertible element \( x \in KK(X_n; C^*(E_1) \otimes \mathbb{K}, C^*(E_2) \otimes \mathbb{K}) \). Note that \( r_{X_n}^{[k, n]}(x) \) is invertible in \( KK([k, n]; C^*(E_1)[k, n] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \) and \( r_{X_n}^{[1, k - 1]}(x) \) is invertible in \( KK(C^*(E_1)[1, k - 1], C^*(E_2)[1, k - 1]) \). By Theorem 4.7 there exists an isomorphism
\[ \phi_2 : C^*(E_1)[1, k-1] \otimes \mathbb{K} \to C^*(E_2)[1, k-1] \otimes \mathbb{K} \text{ such that } KL(\phi_2) = z_2, \text{ where } z_2 \text{ is the invertible element in } KL(C^*(E_1)[1, k-1], C^*(E_2)[1, k-1]) \text{ induced by } r^{[1,k-1]}_{X_3}(x). \]

By Elliott’s classification \cite{Elliott}, there exists an isomorphism \( \phi_0 : C^*(E_1)[k, n] \otimes \mathbb{K} \to C^*(E_2)[k, n] \otimes \mathbb{K} \) such that \( KK(\phi_0) = z_0 \), where \( z_0 \) is the invertible element in \( KK(C^*(E_1)[k, n] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \) induced by \( r^{[k,n]}_{X_3}(x) \).

Consider \( C^*(E_i) \) as a C*-algebra over \( X_2 \) by setting \( C^*(E_i)[2] = C^*(E_i)[k, n] \) and \( C^*(E_i)[1, 2] = C^*(E_i) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( KL(r^{[1]}_{X_3}(y)) = z_2 = KL(\phi_2) \) and \( r^{[2]}_{X_3}(y) = z_0 = KK(\phi_0) \). By Theorem 3.7 of \cite{Elliott},

\[
\tau_{e_3} \times [\tau_{e_3}] \times [\tau_{e_3}] = [\tau_{e_3}] \times [\tau_{e_3}] \times [\tau_{e_3}]
\]

in \( KK^1(C^*(E_1)[1, k-1] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \), where \( e_3 \) is the extension

\[
0 \to C^*(E_i)[k, n] \otimes \mathbb{K} \to C^*(E_i) \otimes \mathbb{K} \to C^*(E_i)[1, k-1] \otimes \mathbb{K} \to 0.
\]

Thus,

\[
KL(\phi_2) \times [\tau_{e_3}] = [\tau_{e_3}] \times KL(\phi_0)
\]

in \( KL^1(C^*(E_1)[1, k-1] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \). Since \( KL^1(C^*(E_1)[1, k-1] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) = KK^1(C^*(E_1)[1, k-1] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \),

\[
KK(\phi_2) \times [\tau_{e_3}] = [\tau_{e_3}] \times KK(\phi_0)
\]

in \( KK^1(C^*(E_1)[1, k-1] \otimes \mathbb{K}, C^*(E_2)[k, n] \otimes \mathbb{K}) \). By Lemma 4.5 of \cite{Elliott}, \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \).

**Theorem 5.8.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight C*-algebra over \( X_3 \). Suppose \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is purely infinite and \( K_0(C^*(E_1)[1, 2]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1, 2] \) is an AF-algebra. Then \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \) if and only if \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \).

**Proof.** The “only if” direction is clear. Suppose \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \). Suppose \( C^*(E_1)[1] \) is purely infinite. Then \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_2^1(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0 \). Since \( K_1(C^*(E_1)[1]) \) is a free group, \( \text{Pext}_2^1(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0 \). Hence,

\[
KK^1(C^*(E_1)[1], C^*(E_2)[2, 3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, 3]).
\]

Suppose \( C^*(E_1)[1] \) is an AF-algebra. Then \( K_0(C^*(E_1)[1, 2]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_2^1(K_0(C^*(E_1)[1, 2]), K_0(C^*(E_2)[3])) = 0 \). Since \( K_1(C^*(E_1)[1, 2]) \) is a free group, \( \text{Pext}_2^1(K_1(C^*(E_1)[1, 2]), K_1(C^*(E_2)[3])) = 0 \). Therefore,

\[
KK^1(C^*(E_1)[1, 2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1, 2], C^*(E_2)[3]).
\]

**Case 1:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are separated. Hence, by Theorem 6.9 of \cite{Elliott}, \( C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K} \).

**Case 2:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are not separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are not separated.
Subcase 2.1: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are purely infinite and $C^*(E_1)[2]$ is an AF-algebra. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are purely infinite and $C^*(E_2)[2]$ is an AF-algebra. Then by the above paragraph we have that $KK^1(C^*(E_1)[1], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,3])$. Hence, by Theorem 5.6, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

Subcase 2.2: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are AF-algebras and $C^*(E_1)[2]$ is purely infinite. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are AF-algebras and $C^*(E_2)[2]$ is purely infinite. Then by the above paragraph we have that

$$KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).$$

Hence, by Theorem 5.7, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

**Corollary 5.9.** Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_3$. Suppose that $K_0(C^*(E_i))$ is finitely generated. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K^+_X(C^*(E_i)) \cong K^+_X(C^*(E_2))$.

**Proof.** Since $C^*(E_1)$ is real rank zero, the canonical projection $\pi : C^*(E_1) \to C^*(E_1)[1]$ induces a surjective homomorphism $\pi : K_0(C^*(E_1)) \to K_0(C^*(E_1)[1])$. Hence, $K_0(C^*(E_1)[1])$ is finitely generated since $K_0(C^*(E_i))$ is finitely generated. The corollary now follows from Theorem 5.8.

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