STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE $C^*$-ALGEBRAS

SØREN EILERS, GUNNAR RESTORFF, AND EFREN RUIZ

Abstract. We show that certain extensions of classifiable $C^*$-algebras are strongly classified by the associated six-term exact sequence in $K$-theory together with the positive cone of $K_0$-groups of the ideal and quotient. We apply our result to give a complete classification of graph $C^*$-algebras with exactly one ideal.

1. Introduction

The classification program for $C^*$-algebras has for the most part progressed independently for the classes of infinite and finite $C^*$-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered $K_0$-group. In the infinite case, there are a number of results for purely infinite $C^*$-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite $C^*$-algebras up to stable isomorphism by the $K_0$-group together with the $K_1$-group. For nonsimple purely infinite $C^*$-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite $C^*$-algebras with exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of $K$-groups [34], the second named author has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered $K$-theory [31, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite $C^*$-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated $K$-theory [28, Theorem 4.14]. However, in all of these situations the nonsimple $C^*$-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Recently, the authors have provided a framework for classifying nonsimple $C^*$-algebras that are not necessarily AF-algebras or purely infinite $C^*$-algebras. In particular, the authors have shown in [16] that certain extensions of classifiable $C^*$-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in $K$-theory. This has allowed for the classification of certain nonsimple $C^*$-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [16] were then used by the first named author and Tomforde in [18] to classify a certain class of non-simple graph $C^*$-algebras, showing that graph $C^*$-algebras with exactly one non-trivial ideal can be classified up to stable isomorphism by their associated six-term exact sequence in $K$-theory. The authors in [15] then showed that all non-unital graph $C^*$-algebras with exactly one
non-trivial ideal can be classified up to isomorphism by their associated six-term exact sequence in $K$-theory. In this paper, we complete the classification of graph $C^*$-algebras with exactly one non-trivial ideal by classifying those that are unital. Our methods here differ rather dramatically from the methods in [13] and [15]. In particular, we use the traditional methods of classification via existence and uniqueness theorems. As a consequence, for unital graph $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with exactly one non-trivial ideal, then any isomorphism between the associated six-term exact sequence in $K$-theory which preserves the unit lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

2. Preliminaries

2.1. $C^*$-algebras over topological spaces. Let $X$ be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of $X$, partially ordered by set inclusion $\subseteq$. A subset $Y$ of $X$ is called locally closed if $Y = U \setminus V$ where $U, V \in \mathcal{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathcal{L}(X)$. The set of all connected, non-empty, locally closed subsets of $X$ will be denoted by $\mathcal{LC}(X)$.

The partially ordered set $(\mathcal{O}(X), \subseteq)$ is a complete lattice, that is, any subset $S$ of $\mathcal{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset $S$ of $\mathcal{O}(X)$,

\[
\bigwedge_{U \in S} U = \left( \bigcap_{U \in S} U \right)^{\circ} \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.
\]

For a $C^*$-algebra $\mathfrak{A}$, let $\mathcal{L}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathcal{L}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset $S$ of $\mathcal{L}(\mathfrak{A})$,

\[
\bigwedge_{\mathfrak{J} \in S} \mathfrak{J} = \bigcap_{\mathfrak{J} \in S} \mathfrak{J} \quad \text{and} \quad \bigvee_{\mathfrak{J} \in S} \mathfrak{J} = \sum_{\mathfrak{J} \in S} \mathfrak{J}.
\]

Definition 2.1. Let $\mathfrak{A}$ be a $C^*$-algebra. Let $\text{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let $X$ be a topological space. A $C^*$-algebra over $X$ is a pair $(\mathfrak{A}, \psi)$ consisting of a $C^*$-algebra $\mathfrak{A}$ and a continuous map $\psi : \text{Prim}(\mathfrak{A}) \to X$. A $C^*$-algebra over $X$, $(\mathfrak{A}, \psi)$, is separable if $\mathfrak{A}$ is a separable $C^*$-algebra. We say that $(\mathfrak{A}, \psi)$ is tight if $\psi$ is a homeomorphism.

We always identify $\mathcal{O}(\text{Prim}(\mathfrak{A}))$ and $\mathcal{L}(\mathfrak{A})$ using the lattice isomorphism

\[
U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p.
\]

Let $(\mathfrak{A}, \psi)$ be a $C^*$-algebra over $X$. Then we get a map $\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(\mathfrak{A})) \cong \mathcal{L}(\mathfrak{A})$ defined by

\[
U \mapsto \{ p \in \text{Prim}(\mathfrak{A}) : \psi(p) \in U \} = \mathfrak{A}(U)
\]

For $Y = U \setminus V \in \mathcal{L}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [27], $\mathfrak{A}(Y)$ does not depend on $U$ and $V$.

Example 2.2. For any $C^*$-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_{\text{Prim}(\mathfrak{A})})$ is a tight $C^*$-algebra over Prim($\mathfrak{A}$). For each $U \in \mathcal{O}(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$. 

Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip $X_n$ with the Alexandrov topology, so the non-empty open subsets are
\[ [a, n] = \{x \in X : a \leq x \leq n \} \]
for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathcal{A}, \phi)$ be a $C^*$-algebra over $X_n$. We will use the following notation throughout the paper:
\[ \mathcal{A}[k] = \mathcal{A}(\{k\}), \mathcal{A}[a, b] = \mathcal{A}(\{a, b\}), \text{ and } \mathcal{A}(i, j) = \mathcal{A}[i + 1, j]. \]

Using the above notation we have ideals $\mathcal{A}[a, n]$ such that
\[ \{0\} \subseteq \mathcal{A}[n] \subseteq \mathcal{A}[n - 1, n] \subseteq \cdots \subseteq \mathcal{A}[2, n] \subseteq \mathcal{A}[1, n] = \mathcal{A}. \]

Definition 2.4. Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras over $X$. A homomorphism $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is $X$-equivariant if $\phi(\mathfrak{A}(U)) \subseteq \mathfrak{B}(U)$ for all $U \in \mathcal{O}(X)$. Hence, for every $Y = U \setminus V$, $\phi$ induces a homomorphism $\phi_Y : \mathfrak{A}(Y) \rightarrow \mathfrak{B}(Y)$. Let $\mathcal{C}^*\text{-alg}(X)$ be the category whose objects are $C^*$-algebras over $X$ and whose morphisms are $X$-equivariant homomorphisms.

An $X$-equivariant homomorphism $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is said to be a full $X$-equivariant homomorphism if for all $Y \in \mathcal{L}_C(X)$, $\phi_Y(a)$ is norm-full in $\mathfrak{B}(Y)$ for all norm-full elements $a \in \mathfrak{A}(Y)$, i.e., the closed ideal of $\mathfrak{B}(Y)$ generated by $\phi_Y(a)$ is $\mathfrak{B}(Y)$ whenever the closed ideal of $\mathfrak{A}(Y)$ generated by $a$ is $\mathfrak{A}(Y)$.

Remark 2.5. Suppose $\mathfrak{A}$ and $\mathfrak{B}$ are tight $C^*$-algebras over $X_n$. Then it is clear that $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is an isomorphism if and only if $\phi$ is a $X_n$-equivariant isomorphism.

It is easy to see that if $\mathfrak{A}$ and $\mathfrak{B}$ are tight $C^*$-algebras over $X_2$, then $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a full $X_2$-equivariant homomorphism if and only if $\phi$ is an $X_2$-equivariant homomorphism and $\phi_{[1]}$ and $\phi_{[2]}$ are injective. Also, if $\mathfrak{A}$ and $\mathfrak{A}[2]$ have non-zero projections $p$ and $q$ respectively, then there exists $\epsilon > 0$ such that if $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a full $X_2$-equivariant homomorphism and $\psi : \mathfrak{A} \rightarrow \mathfrak{B}$ is a homomorphism such that
\[ \|\phi(p) - \psi(p)\| < 1 \quad \|\phi(q) - \psi(q)\| < 1, \]
then $\psi$ is a full $X_2$-equivariant homomorphism.

Remark 2.6. Let $\varepsilon_i : 0 \rightarrow \mathfrak{B}_i \rightarrow \mathfrak{E}_i \rightarrow \mathfrak{A}_i \rightarrow 0$ be an extension for $i = 1, 2$. Note that $\mathfrak{E}_i$ can be considered as a $C^*$-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathfrak{B}_i$ in $\mathfrak{E}_i$, and $\{1, 2\}$ to $\mathfrak{E}_i$. Hence, there exists a one-to-one correspondence between $X_2$-equivariant homomorphisms $\phi : \mathfrak{E}_1 \rightarrow \mathfrak{E}_2$ and homomorphisms from $\varepsilon_1$ and $\varepsilon_2$.

2.2. The ideal related $K$-theory of $\mathfrak{A}$.

Definition 2.7. Let $X$ be a topological space and let $\mathfrak{A}$ be a $C^*$-algebra over $X$. For open subsets $U_1, U_2, U_3$ of $X$ with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_1 \subseteq \mathcal{L}_C(X)$. Then the diagram
\[
\begin{array}{c}
K_0(\mathfrak{A}(Y_1)) \xrightarrow{\pi_*} K_0(\mathfrak{A}(Y_2)) \xrightarrow{\pi_*} K_0(\mathfrak{A}(Y_3)) \\
\downarrow \quad \quad \downarrow \quad \quad \downarrow \\
K_1(\mathfrak{A}(Y_3)) \xrightarrow{\pi_*} K_1(\mathfrak{A}(Y_2)) \xrightarrow{\pi_*} K_1(\mathfrak{A}(Y_1))
\end{array}
\]
is an exact sequence. The ideal related $K$-theory of $\mathcal{A}$, $K_X(\mathcal{A})$, is the collection of all $K$-groups thus occurring and the natural transformations $\{\iota_*, \pi_*, \partial_*\}$. The ideal related, ordered $K$-theory of $\mathcal{A}$, $K_X^+(\mathcal{A})$, is $K_X(\mathcal{A})$ of $\mathcal{A}$ together with $K_0(\mathcal{A}(Y))_+$ for all $Y \in \mathbb{L}C(X)$.

Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras over $X$, we will say that $\alpha : K_X(\mathcal{A}) \to K_X(\mathcal{B})$ is an isomorphism if for all $Y \in \mathbb{L}C(X)$, there exists a graded group isomorphism

$$\alpha_{Y,*} : K_*(\mathcal{A}(Y)) \to K_*(\mathcal{B}(Y))$$

preserving all natural transformations. We say that $\alpha : K_X^+(\mathcal{A}) \to K_X^+(\mathcal{B})$ is an isomorphism if there exists an isomorphism $\alpha : K_X(\mathcal{A}) \to K_X(\mathcal{B})$ in such a way that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathbb{L}C(X)$.

**Remark 2.8.** Meyer-Nest in [28] defined a similar functor $FK_X(-)$ which they called filtrated $K$-theory. For all known cases in which there exists a UCT, the natural transformation from $FK_X(-)$ to $K_X(-)$ is an equivalence. In particular, this is true for the space $X_n$.

If $Y \in \mathbb{L}C(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in \mathbb{O}(Y) \subseteq \mathbb{L}C(X)$, then $\mathcal{A}(Y) \cong \mathcal{A}(Y_1) \oplus \mathcal{A}(Y_2)$ for any $C^*$-algebra over $X$. Moreover, there is a natural isomorphism $K_*(\mathcal{A}(Y))$ to $K_*(\mathcal{A}(Y_1)) \oplus K_*(\mathcal{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathcal{A}(Y))$ to $K_0(\mathcal{A}(Y_1)) \oplus K_0(\mathcal{A}(Y_2))$. If $X$ is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathbb{L}C(X)$ by the subset $\mathbb{L}C(X)^*$.

**Notation 2.9.** Let $\mathcal{N}$ be the bootstrap category of Rosenberg and Schochet in [37].

Let $\mathcal{R}(X)$ be the category whose objects are separable $C^*$-algebras over $X$ and the set of morphisms is $KK(X; \mathcal{A}, \mathcal{B})$. For a finite topological space $X$, let $\mathcal{B}(X) \subseteq \mathcal{R}(X)$ be the bootstrap category of Meyer and Nest in [27]. By Corollary 4.13 of [27], if $\mathcal{A}$ is a nuclear $C^*$-algebra over $X$, then $\mathcal{A} \in \mathcal{B}(X)$ if and only if $\mathcal{A}(\{x\}) \in \mathcal{N}$ for all $x \in X$.

**Theorem 2.10.** (Bonkat [4] and Meyer-Nest [28]) Let $\mathcal{A}$ and $\mathcal{B}$ be in $\mathcal{R}(X_n)$ such that $\mathcal{A}$ is in $\mathcal{B}(X_n)$, then the sequence

$$0 \to \text{Ext}^1_{\mathcal{N}^*}(FK_{X_n}(\mathcal{A})[1], FK_{X_n}(\mathcal{B})) \xrightarrow{\delta} KK(X_n; \mathcal{A}, \mathcal{B}) \xrightarrow{\Gamma} \text{Hom}_{\mathcal{N}^*}(FK_{X_n}(\mathcal{A}), FK_{X_n}(\mathcal{B})) \to 0$$

is exact. Consequently, if $\mathcal{B}$ is in $\mathcal{B}(X_n)$, then an isomorphism from $FK_{X_n}(\mathcal{A})$ to $FK_{X_n}(\mathcal{B})$ lifts to an invertible element in $KK(X_n; \mathcal{A}, \mathcal{B})$.

**Corollary 2.11.** Let $\mathcal{A}$ and $\mathcal{B}$ be in $\mathcal{B}(X_n)$. Then an isomorphism from $K_{X_n}(\mathcal{A})$ to $K_{X_n}(\mathcal{B})$ lifts to an invertible element in $KK(X_n; \mathcal{A}, \mathcal{B})$.

**Proof.** This follows from Remark 2.8 and Theorem 2.10 \hfill $\Box$

**Remark 2.12.** Let $x \in KK(X_n; \mathcal{A}, \mathcal{B})$ be an invertible element. Then $K_{X_n}(x)$ will denote the isomorphism from $K_{X_n}(\mathcal{A})$ to $K_{X_n}(\mathcal{B})$ given by $\Gamma(x)$ where we have identified $K_{X_n}(\mathcal{A})$ with $FK_{X_n}(\mathcal{A})$ and $K_{X_n}(\mathcal{B})$ with $FK_{X_n}(\mathcal{B})$.

2.3. **Functors.** We now define some functors that will be used throughout the rest of the paper. Let $X$ and $Y$ be topological spaces. For every continuous function $f : X \to Y$ we have a functor

$$f : C^*\text{-}\text{alg}(X) \to C^*\text{-}\text{alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi).$$
(1) Define \( g^X_X : X \to X_1 \) by \( g^X_X(x) = 1 \). Then \( g^X_X \) is continuous. Note that the induced functor \( g^X_X : C^*\text{-alg}(X) \to C^*\text{-alg}(X_1) \) is the forgetful functor.

(2) Let \( U \) be an open subset of \( X \). Define \( g^2_{U,X} : X \to X_2 \) by \( g^2_{U,X}(x) = 1 \) if \( x \notin U \) and \( g^2_{U,X}(x) = 2 \) if \( x \in U \). Then \( g^2_{U,X} \) is continuous. Thus the induced functor

\[
g^2_{U,X} : C^*\text{-alg}(X) \to C^*\text{-alg}(X_2)
\]

is just specifying the extension \( 0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0 \).

(3) We can generalize (2) to finitely many ideals. Let \( U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X \) be open subsets of \( X \). Define \( g^n_{U_1,U_2,...,U_n,X} : X \to X_n \) by \( g^n_{U_1,U_2,...,U_n,X}(x) = n - k + 1 \) if \( x \in U_k \setminus U_{k-1} \). Then \( g^n_{U_1,U_2,...,U_n,X} \) is continuous. Therefore, any \( C^*\)-algebra with ideals \( 0 \leq \mathfrak{J}_1 \leq \mathfrak{J}_2 \leq \cdots \mathfrak{J}_n = \mathfrak{A} \) can be made into a \( C^*\)-algebra over \( X_n \).

(4) For all \( Y \in \mathcal{L}(\mathcal{C}(X)) \), \( r^Y_X : C^*\text{-alg}(X) \to C^*\text{-alg}(Y) \) is the restriction functor defined in Definition 2.19 of \([27]\).

(5) If \( f : X \to Y \) is an embedding of a subset with the subspace topology, we write

\[
i^Y_X = f_* : C^*\text{-alg}(X) \to C^*\text{-alg}(Y).
\]

By Proposition 3.4 of \([27]\), the functors defined above induce functors from \( \mathfrak{R}(X) \) to \( \mathfrak{R}(Z) \), where \( Z = Y, X_1, X_n \).

2.4. Graph \( C^*\)-algebras. A graph \((E^0,E^1,r,s)\) consists of a countable set \( E^0 \) of vertices, a countable set \( E^1 \) of edges, and maps \( r : E^1 \to E^0 \) and \( s : E^1 \to E^0 \) identifying the range and source of each edge. If \( E \) is a graph, the graph \( C^*\)-algebra \( C^*(E) \) is the universal \( C^*\)-algebra generated by mutually orthogonal projections \( \{p_v : v \in E^0\} \) and partial isometries \( \{s_e : e \in E^1\} \) with mutually orthogonal ranges satisfying

\[
(1) s^*_e s_e = p_r(e) \quad \text{for all } e \in E^1
\]

\[
(2) s^*_e s_e \leq p_s(e) \quad \text{for all } e \in E^1
\]

\[
(3) p_v = \sum_{e \in E^1 : s(e) = v} s^*_e s_e \quad \text{for all } v \text{ with } 0 < |s^{-1}(v)| < \infty.
\]

3. Meta-theorems

In many cases one can obtain a classification result for a class of unital \( C^*\)-algebras \( \mathcal{C} \) by obtaining a classification result for the class \( \mathcal{C} \otimes \mathbb{K} \), where each object in \( \mathcal{C} \otimes \mathbb{K} \) is the stabilization of an object in \( \mathcal{C} \). A meta-theorem of this sort was proved by the first and second named authors in \([13]\) Theorem 11. It was shown there that if \( \mathcal{C} \) is a subcategory of the category of \( C^*\)-algebras, \( C^*\text{-alg} \), and if \( F \) is a functor from \( \mathcal{C} \) to an abelian category such that an isomorphism \( F(\mathfrak{A} \otimes \mathbb{K}) \cong F(\mathfrak{B} \otimes \mathbb{K}) \) lifts to an isomorphism in \( \mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K} \), then under suitable conditions, we have that \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) implies \( \mathfrak{A} \cong \mathfrak{B} \). In \([31]\), the second and third named authors improved this result by showing that the isomorphism \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) lifts to an isomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \).

In this section, we improve these results in order to deal with cases when \( \mathcal{C} \) is a category (not necessarily a subcategory of \( C^*\text{-alg} \)) and there exists a functor from \( \mathcal{C} \) to \( C^*\text{-alg} \). An example of such a category is the category of \( C^*\)-algebras over \( \{1,2\} \), where \( \{1,2\} \) is given the discrete topology. Then \( \mathcal{C} \) is not a subcategory of \( C^*\text{-alg} \) but the forgetful functor (forgetting the \( \{1,2\}\)-structure) is a functor from \( \mathcal{C} \) to \( C^*\text{-alg} \). We also replace the condition of proper pure infiniteness by the stable weak cancellation property.
Definition 3.1. A $C^*$-algebra $\mathcal{A}$ is said to have the weak cancellation property if $p$ is Murray-von Neumann equivalent to $q$ whenever $p$ and $q$ generate the same ideal $\mathcal{I}$ and $[p] = [q]$ in $K_0(\mathcal{I})$. A $C^*$-algebra is said to have the stable weak cancellation property if $M_n(\mathcal{A})$ has the weak cancellation property for all $n \in \mathbb{N}$.

Theorem 3.2. (cf. [13] Theorem 11) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathbf{C}^*-\mathbf{alg}$ be the category of $C^*$-algebras, and let $\mathbf{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F : \mathcal{C} \to \mathbf{C}^*-\mathbf{alg}$, $G : \mathcal{C} \to \mathcal{D}$, and $H : \mathcal{D} \to \mathbf{Ab}$ such that

1. $H \circ G = K_0 \circ F$.
2. For objects $A$ in $\mathcal{C}$, there exist an object $A_K$ and a morphism $\kappa_A : A \to A_K$ such that

   $G(\kappa_A)$ is an isomorphism in $\mathcal{D}$, $F(A_K) = F(A) \otimes K$, and $F(\kappa_A) = \text{id}_{F(A)} \otimes e_{11}$.
3. For all objects $A$ and $B$ in $\mathcal{C}$, every isomorphism $G(A_K) \to G(B_K)$ is induced by an isomorphism from $A_K$ to $B_K$.

Let $\mathcal{A}$ and $\mathcal{B}$ be given such that $F(\mathcal{A})$ and $F(\mathcal{B})$ are unital $C^*$-algebras. Let $\rho : G(\mathcal{A}) \to G(\mathcal{B})$ be an isomorphism such that $H(\rho)([1_{F(\mathcal{A})}]) = [1_{F(\mathcal{B})}]$. If $F(\mathcal{B})$ has the stable weak cancellation property, then $F(\mathcal{A}) \cong F(\mathcal{B})$.

Proof. Note that $G(\kappa_A)$ and $G(\kappa_B)$ are isomorphisms. Therefore $G(\kappa_B) \circ \rho \circ G(\kappa_A)^{-1}$ is an isomorphism from $G(A_K)$ to $G(B_K)$. Thus, there exists an isomorphism $\phi : A_K \to B_K$ such that $G(\phi) = G(\kappa_B) \circ \rho \circ G(\kappa_A)^{-1}$.

Set $\psi = F(\phi)$. Then $\psi : F(\mathcal{A}) \otimes K \to F(\mathcal{B}) \otimes K$ is a $*$-isomorphism such that

$$
K_0(\psi) = K_0(F(\phi)) = H(G(\kappa_B) \circ \rho \circ G(\kappa_A)^{-1}) = H(G(\kappa_B)) \circ H(\rho) \circ H(G(\kappa_A)^{-1})
$$

$$
= K_0(F(\kappa_B)) \circ H(\rho) \circ K_0(F(\kappa_A))^{-1} = K_0(\text{id}_{F(\mathcal{A})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathcal{A})} \otimes e_{11})^{-1}.
$$

Hence,

$$
K_0(\psi)([1_{F(\mathcal{A})} \otimes e_{11}]) = K_0(\text{id}_{F(\mathcal{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathcal{A})} \otimes e_{11})^{-1}([1_{F(\mathcal{A})} \otimes e_{11}])
$$

$$
= K_0(\text{id}_{F(\mathcal{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathcal{A})}])
$$

$$
= K_0(\text{id}_{F(\mathcal{B})} \otimes e_{11})([1_{F(\mathcal{B})}])
$$

$$
= [1_{F(\mathcal{B})} \otimes e_{11}].
$$

Stable weak cancellation implies that there exists $v \in F(\mathcal{B}) \otimes K$ such that $v^*v = \psi(1_{F(\mathcal{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathcal{B})} \otimes e_{11}$ since $\psi(1_{F(\mathcal{A})} \otimes e_{11})$ and $1_{F(\mathcal{B})} \otimes e_{11}$ are full projections in $F(\mathcal{B}) \otimes K$. Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [13] Theorem 11, $\gamma$ is an isomorphism from $F(\mathcal{A}) \otimes e_{11}$ to $F(\mathcal{B}) \otimes e_{11}$. Hence, $F(\mathcal{A}) \cong F(\mathcal{B})$. \hfill \Box

Theorem 3.3. (cf. [32] Theorem 2.1) Let $\mathcal{C}$ be a subcategory of $\mathbf{C}^*-\mathbf{alg}(X)$. Moreover, $\mathcal{C}$ is assumed to be closed under tensoring by $M_2(\mathbf{C})$ and $K$ and contains the canonical embeddings $\kappa_1 : \mathcal{A} \to M_2(\mathcal{A})$ and $\kappa : \mathcal{A} \to \mathcal{A} \otimes K$ as morphisms for every object $\mathcal{A}$ in $\mathcal{C}$. Assume there is a functor $F : \mathcal{C} \to \mathcal{D}$ satisfying

1. For $\mathcal{A}$ in $\mathcal{C}$, the embeddings $\kappa_1 : \mathcal{A} \to M_2(\mathcal{A})$ and $\kappa : \mathcal{A} \to \mathcal{A} \otimes K$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.
2. For all objects $\mathcal{A}$ and $\mathcal{B}$ in $\mathcal{C}$ that are stable $C^*$-algebras, every isomorphism from $F(\mathcal{A})$ to $F(\mathcal{B})$ is induced by an isomorphism from $\mathcal{A}$ to $\mathcal{B}$.
3. There exists a functor $G$ from $\mathcal{D}$ to $\mathbf{Ab}$ such that $G \circ F = K_0$. 

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Assume that every $X$-equivariant isomorphism between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ and that for objects $\mathfrak{A}$ in $\mathcal{C}$, $F(\text{Ad}(u)|_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})}$ for every unitary $u \in \mathcal{M}(\mathfrak{A})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are objects in $\mathcal{C}$ that are unital $C^*$-algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ have the stable weak cancellation property and there is an isomorphism $\alpha : F(\mathfrak{A}) \to F(\mathfrak{B})$ such that $G(\alpha)([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}]$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ in $\mathcal{C}$ such that $F(\phi) = \alpha$.

**Proof.** The difference between the statement of Theorem 2.1 of [32] and statement of the theorem are

(i) $\mathcal{C}$ is assumed to be a subcategory of $\mathcal{C}^*-\text{alg}(X)$ instead of a subcategory of $\mathcal{C}^*-\text{alg}$.

(ii) $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to have the stable weak cancellation property instead of being properly infinite.

In the proof of Theorem 2.1 of [32], properly infinite was needed to insure that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$, where $\psi : \mathfrak{A} \otimes \mathcal{K} \to \mathfrak{B} \otimes \mathcal{K}$ is the isomorphism from (2) that lifts the isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ that is induced by $\alpha$. As in the proof of Theorem 3.2, we get that $\psi(1_{\mathfrak{A}} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_{\mathfrak{B}} \otimes e_{11}$.

Arguing as in the proof of Theorem 2.1 of [32], we get the desired result. $\square$

4. Classification results

In this section, we show that $K^+_X(-)$ is a strong classification functor for a class of $C^*$-algebras with exactly one proper nontrivial ideal containing $C^*$-algebras associated to finite graphs. The results of this section will be used in the next section to show that $K^+_X(-)$ together with the appropriate scale is a complete isomorphism invariant for $C^*$-algebras associated to graphs. Moreover, in a forthcoming paper, we use these results to solve the following extension problem: If $\mathfrak{A}$ fits into the following exact sequence

$$0 \to C^*(E) \otimes \mathcal{K} \to \mathfrak{A} \to C^*(G) \to 0,$$

where $C^*(E)$ and $C^*(G)$ are simple $C^*$-algebras, then when is $\mathfrak{A} \cong C^*(F)$ for some graph $F$?

**Theorem 4.1.** (Existence Theorem) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1; \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)|_Y$ is a positive isomorphism for all $Y \in \mathbb{LC}(X_2)$.

Suppose $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension, $\mathfrak{A}_i[2]$ is a stable $C^*$-algebra, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra with real rank zero, and either

(i) $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra; or

(ii) $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple $C^*$-algebra.

Then there exists an $X_2$-equivariant homomorphism $\phi : \mathfrak{A}_1 \otimes \mathcal{K} \to \mathfrak{A}_2 \otimes \mathcal{K}$ such that $KK(X_2; \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$, and $\phi[2]$ and $\phi[1]$ are injective, where $\{e_{ij}\}$ is a system of matrix units for $\mathcal{K}$.

**Proof.** Set $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$. Note that by Lemma 3.10 and Theorem 3.8 of [14], $\mathfrak{A}_i[2] \otimes \mathcal{K}$ satisfies the corona factorization property (see [21] for the definition of the corona factorization property). Since $\mathfrak{A}_i[k]$ is an AF-algebra or an Kirchberg algebra, $\mathfrak{A}_i[k]$ has the stable weak cancellation. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation. Let $e_i$ be the extension

$$0 \to \mathfrak{A}_i[2] \otimes \mathcal{K} \to \mathfrak{A}_i \otimes \mathcal{K} \to \mathfrak{A}_i[1] \otimes \mathcal{K} \to 0.$$
Case (i): $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) = [r_{X_2}^{(2)}(y)]$ in $KK^1(\mathfrak{A}_i[1] \otimes \mathfrak{K}, \mathfrak{A}_2[2] \otimes \mathfrak{K})$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K})$, we have that $r_{X_2}^{(1)}(y)$ is invertible in $KK(\mathfrak{A}_i[1] \otimes \mathfrak{K}, \mathfrak{A}_2[1] \otimes \mathfrak{K})$ and $\Gamma(r_{X_2}^{(1)}(y)) = \Gamma(x)_{\{1\}}$ is a positive isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_1 : \mathfrak{A}_1[1] \otimes \mathfrak{K} \to \mathfrak{A}_2[1] \otimes \mathfrak{K}$ such that $KK(\psi_1) = r_{X_2}^{(1)}(y)$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K})$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK(\mathfrak{A}_i[2] \otimes \mathfrak{K}, \mathfrak{A}_2[2] \otimes \mathfrak{K})$. Thus, by Kirchberg-Phillips classification (see [20] and [29]), there exists an isomorphism $\psi_0 : \mathfrak{A}_1[2] \otimes \mathfrak{K} \to \mathfrak{A}_2[2] \otimes \mathfrak{K}$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$. By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in M(\mathfrak{A}_2[2] \otimes \mathfrak{K})$ such that $\psi = (Ad(u) \circ \psi_0, Ad(u) \circ \tilde{\psi}_0)$ is an $X_2$-equivariant isomorphism from $\mathfrak{A}_1[1] \otimes \mathfrak{K}$ to $\mathfrak{A}_2[2] \otimes \mathfrak{K}$, where $\tilde{\psi}_0 : M(\mathfrak{A}_1[2] \otimes \mathfrak{K}) \to M(\mathfrak{A}_1[2] \otimes \mathfrak{K})$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_{\{1\}}) = r_{X_2}^{(k)}(y)$ for $k = 1, 2$.

Note that
\[ 0 \to \iota_{\{2\}}^X(\mathfrak{A}_1 \otimes \mathfrak{K}[2]) \xrightarrow{\lambda_2} \mathfrak{A}_1 \otimes \mathfrak{K} \xrightarrow{\beta_1} \iota_{\{1\}}^X(\mathfrak{A}_1 \otimes \mathfrak{K}[1]) \to 0 \]
is a semi-split extension of $C^*$-algebras over $X_2$ (see Definition 3.5 of [27]). Set
\[ J_i = \iota_{\{2\}}^X(\mathfrak{A}_1 \otimes \mathfrak{K}[2]) \quad \text{and} \quad \mathfrak{B}_i = \iota_{\{1\}}^X(\mathfrak{A}_1 \otimes \mathfrak{K}[1]). \]

By Theorem 3.6 of [27] (see also Korollar 3.4.6 of [4]),
\[ KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, J_2) \xrightarrow{\lambda_2} KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K}) \xrightarrow{\beta_2} KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{B}_2) \]
is exact. By Proposition 3.12 of [27], $KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{B}_2)$ and $KK(\mathfrak{A}_1[1] \otimes \mathfrak{K}, \mathfrak{B}_2[1] \otimes \mathfrak{K})$ are naturally isomorphic. Hence, there exists $z \in KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, J_2)$ such that $y - KK(X_2; \psi) = z \times KK(X_2; \lambda_2)$ since $KK(\psi_{\{1\}}) = r_{X_2}^{(1)}(y)$.

By Proposition 3.13 of [27], $KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, J_2)$ and $KK(\mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K}[2])$ are isomorphic. By Theorem 8.3.3 of [36] (see also Hauptsatz 4.2 of [20]), there exists a $*$-homomorphism $\eta : \mathfrak{A}_1 \otimes \mathfrak{K} \to (\mathfrak{A}_2 \otimes \mathfrak{K})[2]$ such that $KK(\eta) = \varpi$, where $\varpi$ is the image of $z$ under the isomorphism $KK(X_2; \mathfrak{A}_1 \otimes \mathfrak{K}, J_2) \cong KK(\mathfrak{A}_1 \otimes \mathfrak{K}, (\mathfrak{A}_2 \otimes \mathfrak{K})[2])$. Note that $\eta$ induces an $X_2$-equivariant homomorphism $\eta : \mathfrak{A}_1 \otimes \mathfrak{K} \to J_2$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\lambda_2 \circ \eta)$, where the sum is the Cuntz sum in $M(\mathfrak{A}_2 \otimes \mathfrak{K})$. Then $\phi : \mathfrak{A}_1 \otimes \mathfrak{K} \to \mathfrak{A}_2 \otimes \mathfrak{K}$ is an $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective homomorphisms.

Case (ii): $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple $C^*$-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) \times [r_{X_2}^{(2)}(y)]$ in $KK^1(\mathfrak{A}_i[1] \otimes \mathfrak{K}, \mathfrak{A}_2[2] \otimes \mathfrak{K})$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K})$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK(\mathfrak{A}_1[2] \otimes \mathfrak{K}, \mathfrak{A}_2[2] \otimes \mathfrak{K})$ and $\Gamma(r_{X_2}^{(2)}(y)) = \Gamma(x)_{\{2\}}$ is an order isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_0 : \mathfrak{A}_1[2] \otimes \mathfrak{K} \to \mathfrak{A}_2[2] \otimes \mathfrak{K}$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes \mathfrak{K}, \mathfrak{A}_2 \otimes \mathfrak{K})$, we have that $r_{X_2}^{(1)}(y)$ is invertible in
By Lemma 4.5 of [14] and its proof, there exists a unitary $A$ such that $KK(A) = 1$. Note that $KK(A) = 1$ is a semi-split extension of $KK$. Hence, there exists a homomorphism $\eta: (A, K) \to (M(A), K)$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_{\{k\}}) = r_{X_2}^{(k)}(y)$ for $k = 1, 2$.

Let $A_1$ be a separable, nuclear $C^*$-algebra over $X_2$. Since $A$ is a properly infinite $C^*$-algebra, by Proposition 3.13 of [27], there exists a homomorphism $\eta: (A, K) \to (M(A), K)$ such that $KK(\psi_{\{1\}}) = \eta(A)$.

By Theorem 3.6 of [27] (see also Korollar 3.4.6 [1]), $KK(X_2; B_1, A_2 \otimes K)$ is exact. By Proposition 3.12 of [27], $KK(X_2; B_1, A_2 \otimes K)$ and $KK(A_1) \otimes K$ are naturally isomorphic. Hence, there exists $z \in KK(X_2; B_1, A_2 \otimes K)$ such that $y = KK(X; \psi) = KK(X_2; \beta_1) \times z$. By Proposition 3.13 of [27], $KK(X_2; B_1, A_2 \otimes K)$ and $KK((A_1 \otimes K)[1], A_2 \otimes K)$ are isomorphic. Therefore, by Theorem 8.3.3 of [36], $KK(X_2; \beta_1) \times z = KK((A_1 \otimes K)[1], A_2 \otimes K)$.

4.1. Strong classification of extensions of AF-algebras by purely infinite $C^*$-algebras.

Definition 4.2. Let $A$ and $B$ be separable $C^*$-algebras over $X$. Two $X$-equivariant homomorphisms $\phi, \psi: A \to B$ are said to be approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\}_{n=1}^{\infty}$ in $M(B)$ such that

$$\lim_{n \to \infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0$$

for all $a \in A$.

We now recall the definition of $KL(A, B)$ from [33].

Definition 4.3. Let $A$ be a separable, nuclear $C^*$-algebra in $\mathcal{N}$ and let $B$ be a $\sigma$-unital $C^*$-algebra. Let

$$\text{Ext}^1_2(K_*(A), K_{*-1}(B)) = \text{Ext}^1_2(K_0(A), K_1(B)) \oplus \text{Ext}^1_2(K_1(A), K_0(B)).$$

Since $A$ is in $N$, by [37], $\text{Ext}^1_2(K_*(A), K_{*-1}(B))$ can be identified as a sub-group of the group $KK(A, B)$.
For abelian groups, $G$ and $H$, let $\text{Pext}_{2}^{1}(G, H)$ be the subgroup of $\text{Ext}_{2}^{1}(G, H)$ of all pure extensions of $G$ by $H$. Set 
$$\text{Pext}_{2}^{1}(K_{*}(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \text{Pext}_{2}^{1}(K_{0}(\mathfrak{A}), K_{1}(\mathfrak{B})) \oplus \text{Pext}_{2}^{1}(K_{1}(\mathfrak{A}), K_{0}(\mathfrak{B})).$$

Define $KL(\mathfrak{A}, \mathfrak{B})$ as the quotient 
$$KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})/\text{Pext}_{2}^{1}(K_{*}(\mathfrak{A}), K_{*+1}(\mathfrak{B})).$$

Rørdam in [33] proved that if $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $KL(\phi) = KL(\psi)$.

**Notation 4.4.** Let $x \in KK(\mathfrak{A}, \mathfrak{B})$. Then the element $x + \text{Pext}_{2}^{1}(K_{*}(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ in $KL(\mathfrak{A}, \mathfrak{B})$ will be denoted by $KL(x)$.

A nuclear, purely infinite, separable, simple $C^*$-algebra will be called a *Kirchberg algebra.*

**Theorem 4.5.** *(Uniqueness Theorem 1)* Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, $C^*$-algebras over $X_2$ such that $\mathfrak{A}_1$ has real rank zero, $\mathfrak{A}_1$ is stable, $\mathfrak{A}_1[2]$ is a Kirchberg algebra in $N$, $\mathfrak{A}_1[1]$ is an AF-algebra, and $\mathfrak{A}_1[2]$ is an essential ideal of $\mathfrak{A}_1$. Suppose $\phi, \psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be $X_2$-equivariant homomorphisms such that $KK(X_2; \phi) = KK(X_2; \psi)$, and $\phi[2], \psi[2]$, and $\psi[1]$ are injective homomorphisms. Then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** Since $\mathfrak{A}_1[1]$ is an AF algebra, every finitely generated subgroup of $K_0(\mathfrak{A}_1[1])$ is torsion free (hence free) and every finitely generated subgroup of $K_1(\mathfrak{A}_1[1])$ is zero. Thus, 
$$\text{Pext}_{2}^{1}(K_{*}(\mathfrak{A}_1[1]), K_{*+1}(Q(\mathfrak{A}_j[2]))) = \text{Ext}_{2}^{1}(K_{*}(\mathfrak{A}_1[1]), K_{*+1}(Q(\mathfrak{A}_j[2])))$$

which implies that 
$$KL(\mathfrak{A}_1[1], Q(\mathfrak{A}_j[2])) \cong \text{Hom}(K_{*}(\mathfrak{A}_1[1]), K_{*}(Q(\mathfrak{A}_j[2])))$$

Let $e_i$ denote the extension $0 \to \mathfrak{A}_1[2] \to \mathfrak{A}_1 \to \mathfrak{A}_1[1] \to 0$. Since $\mathfrak{A}_1$ has real rank zero and $K_1(\mathfrak{A}_1[1]) = 0$, we have that $K_j(\tau_{e_i}) = 0$, where $\tau_{e_i}$ is the Busby invariant of $e_i$. Hence, $[\tau_{e_i}] = 0$ in $KL(\mathfrak{A}_1[1], Q(\mathfrak{A}_1[2]))$. By Corollary 6.7 of [24], $e_i$ is quasi-diagonal. Thus, there exists an approximate identity of $\mathfrak{A}_1[2]$ consisting of projections $\{e_k\}_{k \in \mathbb{N}}$ such that 
$$\lim_{n \to \infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_1$.

Since $\mathfrak{A}_1[1]$ is an AF-algebra and $\mathfrak{A}_1$ has real rank zero, as in the proof of Lemma 9.8 of [10], there exists a sequence of finite dimensional sub-$C^*$-algebras $\{\mathfrak{B}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_1$ such that $\mathfrak{B}_k \cap \mathfrak{A}_1[2] = \{0\}$ and for each $x \in \mathfrak{A}_1$, there exist $y_1 \in \bigcup_{k=1}^{\infty} \mathfrak{B}_k$ and $y_2 \in \mathfrak{A}_1[2]$ such that $x = y_1 + y_2$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Note that we may assume $\mathcal{F}$ is the union of the generators of $\mathfrak{B}_m$, for some $m \in \mathbb{N}$ and $\mathcal{G}$, for some finite subset $\mathcal{G}$ of $\mathfrak{A}_1[2]$ . Since $\mathfrak{B}_m$ is a finite dimensional $C^*$-algebra, 
$$\lim_{k \to \infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_1$, and $\{e_k\}_{k \in \mathbb{N}}$ is an approximate identity for $\mathfrak{A}_1[2]$ consisting of projections, there exist $k \in \mathbb{N}$, a finite dimensional sub-$C^*$-algebra $\mathfrak{D}$ of $\mathfrak{A}_1$ with $\mathfrak{D} \subseteq (1_{M(\mathfrak{A}_1)} - e_k)\mathfrak{A}_1(1_{M(\mathfrak{A}_1)} - e_k)$ and $\mathfrak{D} \cap \mathfrak{A}_1[2] = \{0\}$, and there exists a finite subset $\mathcal{H}$ of $e_k \mathfrak{A}_1[2] e_k$ such that for all $x \in \mathcal{F}$, there exist $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H}$ 
$$\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}.$$
Set $D = \bigoplus_{\ell=1}^{s} M_{n_{\ell}}$ and let $\{f_{ij}^{\ell}\}_{i,j=1}^{n_{\ell}}$ be a system of matrix units for $M_{n_{\ell}}$. Let $I_{\ell}$ be the ideal in $A_{i}$ generated by $f_{ij}^{\ell}$. Since $A_{i}[2]$ is simple and $A_{i}[2]$ is an essential ideal of $A_{i}$, we have that $A_{i}[2] \subseteq I_{\ell}$ for all nonzero ideal $I_{\ell}$ of $A_{i}$. Thus, $A_{i}[2] \subseteq I_{\ell}$ since $D \cap A_{i}[2] = 0$.

Let $I_{\ell}^{\psi}$ be the ideal in $A_{2}$ generated by $\phi(f_{ij}^{\ell})$ and let $I_{\ell}^{\phi}$ be the ideal in $A_{2}$ generated by $\psi(f_{ij}^{\ell})$. Since $\phi$ and $\psi$ are $X_{2}$-equivariant homomorphisms and since $\phi|_{I_{1}}$ and $\psi|_{I_{1}}$ are injective homomorphisms, we have that $\phi(f_{ij}^{\ell}) \notin A_{2}[2]$ and $\psi(f_{ij}^{\ell}) \notin A_{2}[2]$. Therefore, $A_{2}[2] \subseteq I_{\ell}^{\psi}$ and $A_{2}[2] \subseteq I_{\ell}^{\phi}$. Since $K_{0}(\phi|_{I_{1}}) = K_{0}(\psi|_{I_{1}})$ and since $A_{2}[1]$ is an AF-algebra, we have that $\phi|_{I_{1}}(I_{1}^{\phi})$ is Murray-von Neumann equivalent to $\psi|_{I_{1}}(I_{1}^{\phi})$, where $I_{1}^{\phi}$ is the image of $f_{ij}^{\ell}$ in $A_{1}[1]$. Thus, they generate the same ideal in $A_{1}[1]$. Since $A_{2}[2] \subseteq I_{\ell}^{\psi}$ and $A_{2}[2] \subseteq I_{\ell}^{\phi}$ and since $\psi|_{I_{1}}(I_{1}^{\phi})$ and $\phi|_{I_{1}}(I_{1}^{\phi})$ generate the same ideal in $A_{2}[1]$, we have that $I = I_{\ell}^{\psi} = I_{\ell}^{\phi}$.

Note that the following diagram

$$
\begin{CD}
0 @>>> K_{0}(A_{2}[2]) @>>> K_{0}(I) @>>> K_{0}(I/A_{2}[2]) @>>> 0 \\
@. @VV{K_{0}(\iota)}V @VV{K_{0}(\psi)}V @VV{K_{0}(\psi)}V @. \\
0 @>>> K_{0}(A_{2}[2]) @>>> K_{0}(A_{2}) @>>> K_{0}(A_{2}[1]) @. 
\end{CD}
$$

is commutative, the rows are exact, and $\iota$ and $\psi$ are the canonical embeddings. Since $A_{2}[1]$ is an AF-algebra, $K_{0}(I)$ is injective. A diagram chase shows that $K_{0}(\psi)$ is injective. Since $KK(X_{2}; \phi) = KK(X_{2}; \psi)$, we have that $[\phi(f_{ij}^{\ell})] = [\psi(f_{ij}^{\ell})]$ in $K_{0}(A_{2})$. Since $\phi(f_{ij}^{\ell})$ and $\psi(f_{ij}^{\ell})$ are elements of $I$ and $K_{0}(\iota)$ is injective, we have that $[\phi(f_{ij}^{\ell})] = [\psi(f_{ij}^{\ell})]$ in $K_{0}(I)$. Since $A_{i}[1]$ is an AF-algebra and $A_{i}[2]$ is a Kirchberg algebra, they both have stable weak cancellation. By Lemma 3.15 of [16], $A_{i}$ has stable weak cancellation. Thus, $\phi(f_{ij}^{\ell})$ is Murray-von Neumann equivalent to $\psi(f_{ij}^{\ell})$. Hence, there exists $v_{\ell} \in A_{2}$ such that $v_{\ell}^{*}v_{\ell} = \phi(f_{ij}^{\ell})$ and $v_{\ell}v_{\ell}^{*} = \psi(f_{ij}^{\ell})$.

Set

$$u_{1} = \sum_{\ell=1}^{s} \sum_{i=1}^{n_{\ell}} \psi(f_{ij}^{\ell})v_{\ell}\phi(f_{ij}^{\ell})$$

Then, $u_{1}$ is a partial isometry in $A_{1}$ such that $u_{1}^{*}u_{1} = \phi(1_{D})$, $u_{1}u_{1}^{*} = \psi(1_{D})$, and $u_{1}\phi(x)u_{1}^{*} = \psi(x)$ for all $x \in D$.

Let $\beta : e_{k}A_{1}[2]e_{k} \to A_{1}[2]$ be the usual embedding. Note that $KK(\phi|_{I_{1}} \circ \beta) = KK(\psi|_{I_{1}} \circ \beta)$ and $\phi|_{I_{1}} \circ \beta$, $\psi|_{I_{1}} \circ \beta$ are monomorphisms. Therefore, by Theorem 6.7 of [23], there exists a partial isometry $u_{2} \in A_{2}[2]$ such that $u_{2}^{*}u_{2} = \phi(e_{k})$, $u_{2}u_{2}^{*} = \psi(e_{k})$, and

$$\|u_{2}\phi(x)u_{2}^{*} - \psi(x)\| < \frac{\varepsilon}{3}$$

for all $x \in H$.

Since $A_{2}$ is stable, there exists $u_{3} \in M(A_{2})$ such that $u_{3}^{*}u_{3} = 1_{M(A_{2})} - (u_{1} + u_{2})^{*}(u_{1} + u_{2})$ and $u_{3}u_{3}^{*} = 1_{M(A_{2})} - (u_{1} + u_{2})(u_{1} + u_{2})^{*}$. Set $u = u_{1} + u_{2} + u_{3} \in M(A_{2})$. Then $u$ is a unitary in $M(A_{2})$. 


Lemma 4.6. Let \( M \) be a separable \( C^* \)-algebra over a finite topological space \( X \). Let \( u \) be unitary in \( \mathcal{M}(A_2) \) such that \( \| u \phi(x) - \psi(x) \| < \epsilon \) for all \( x \in X \). Since \( A_1 \) is a separable \( C^* \)-algebra, we have that \( \phi \) is approximately unitarily equivalent to \( \psi \). □

**Theorem 4.7.** Let \( A_1 \) and \( A_2 \) be in \( \mathcal{B}(X_2) \) and let \( x \in KK(X_2; A_1, A_2) \) be an invertible element such that \( \Gamma(x)_Y \) is an order isomorphism for all \( Y \in \mathcal{L}(X_2) \). Suppose \( A_1[2] \) is a Kirchberg algebra, \( A_1 \) is an AF-algebra, \( A_1 \) has real rank zero, and \( A_2[2] \) is an essential ideal of \( A_1 \). Then there exists an \( X_2 \)-equivariant isomorphism \( \phi : A_1 \otimes K \to A_2 \otimes K \) such that \( KL(\phi) = KL(g^1_{X_2}(y)) \) and \( K_{X_2}(\phi) = K_{X_2}(y) \), where \( y = KK(X_2; id_{A_1} \otimes e_{11})^{-1} \times KK(X_n; id_{A_2} \otimes e_{11}) \).

**Proof.** Since \( A_1[2] \) is a purely infinite simple \( C^* \)-algebra, \( A_1[2] \) is either unital or stable. Since \( A_1[2] \) is an essential ideal of \( A_1 \), \( A_1[2] \) is non-unital else \( A_1 \) is isomorphic to a direct summand of \( A_1 \), which would contradict the essential assumption. Therefore, \( A_1[2] \) is stable. Moreover, \( Q(A_1[2]) \) is simple which implies that \( 0 \to A_1[2] \to A_1 \to A_1[1] \to 0 \) is a full extension. Since \( A_1[2] \) and \( A_1[1] \) are nuclear \( C^* \)-algebras, \( A_1 \) is a nuclear \( C^* \)-algebra.

Let \( z \in KK(X_2; A_2 \otimes K, A_1 \otimes K) \) such that \( y \times z = [id_{A_1 \otimes K}] \) and \( y \times z = [id_{A_2 \otimes K}] \). By Theorem 4.7 there exists an \( X_2 \)-equivariant homomorphism \( \psi_1 : A_1 \otimes K \to A_2 \otimes K \) such that \( KK(X_2; \psi_1) = x \), and \( (\psi_1)_1 \) and \( (\psi_1)_1 \) are injective homomorphisms. By Theorem 4.7 there exists an \( X_2 \)-equivariant homomorphism \( \psi_2 : A_2 \otimes K \to A_1 \otimes K \) such that \( KK(X_2; \psi_2) = y \), and \( (\psi_2)_2 \) and \( (\psi_2)_2 \) are injective homomorphisms. Using Theorem 4.7 and a typical approximate intertwining argument, there exists an isomorphism \( \phi : A_1 \otimes K \to A_2 \otimes K \) such that \( \phi \) and \( \psi_1 \) are approximately unitarily equivalent.

Let \( \pi_2 : A_2 \to A_2[1] \) be the canonical quotient map. Then \( \pi_2 \circ \phi|_{A_2[2]} \) is either zero or injective since \( A_2[2] \) is simple. Since \( A_2[2] \) is purely infinite and \( A_2[1] \) is an AF-algebra, we must have that \( \pi_2 \circ \phi|_{A_2[2]} = 0 \). Thus, \( \phi \) is an \( X_2 \)-equivariant homomorphism. Similarly, \( \phi^{-1} \) is an \( X_2 \)-equivariant homomorphism. Hence, \( \phi \) is an \( X_2 \)-equivariant isomorphism. By construction, \( KL(\phi) = KL(\psi_1) = KL(g^1_{X_2}(y)) \). By Lemma 4.7, \( K_{X_2}(\phi) = K_{X_2}(y) \). □

**Corollary 4.8.** Let \( A_1 \) and \( A_2 \) be in \( \mathcal{B}(X_2) \) and let \( x \in KK(X_2; A_1, A_2) \) be an invertible element such that \( \Gamma(x)_Y \) is an order isomorphism for all \( Y \in \mathcal{L}(X_2) \). Suppose \( A_1[2] \) is a Kirchberg algebra, \( A_1[1] \) is an AF-algebra, \( A_1 \) has real rank zero, \( A_2[2] \) is an essential ideal of \( A_1 \), and \( K_i(A[Y]) \) and \( K_i(B[Y]) \) are finitely generated for all \( Y \in \mathcal{L}(X_2) \). Then there exists an \( X_2 \)-equivariant isomorphism \( \phi : A_1 \otimes K \to A_2 \otimes K \) such that \( KK(\phi) = KK(g^1_{X_2}(y)) \) and \( K_{X_2}(\phi) = K_{X_2}(y) \), where \( y = KK(X_2; id_{A_1} \otimes e_{11})^{-1} \times KK(X_n; id_{A_2} \otimes e_{11}) \).
Proof. This follows from Theorem 4.7 and the fact that if $G$ is finitely generated, then $\text{Pext}_2^G(G, H) = 0$. \hfill \square

4.2. Strong classification of extensions of purely infinite by $\mathbb{K}$. We recall the following from [1] p. 341. Let $\psi : \mathfrak{A} \to B(H)$ be a representation of $\mathfrak{A}$. Let $\mathcal{H}_e$ denote the subspace of $\mathcal{H}$ spanned by the ranges of all compact operators in $\psi(\mathfrak{A})$. Since $\psi(\mathfrak{A}) \cap \mathbb{K}$ is an ideal of $\psi(\mathfrak{A})$, we have that $\mathcal{H}_e$ reduces $\pi(\mathfrak{A})$, and so the decomposition $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_e^\perp$ induces a decomposition of $\psi$ into sub-representations $\psi = \psi_e \oplus \psi'$. The summand $\psi_e$, considered as a representation of $\mathfrak{A}$ on $\mathcal{H}_e$, will be called the essential part of $\psi$ and $\mathcal{H}_e$ is called the essential subspace for $\psi$.

Let $\mathfrak{B}$ be a tight $C^*$-algebra over $X_2$. Consider the essential extension

$$\varepsilon_{\mathfrak{B}} : 0 \to \mathfrak{B}[2] \to \mathfrak{B} \to \mathfrak{B}[1] \to 0.$$ 

If $\tau_{\varepsilon_{\mathfrak{B}}} : \mathfrak{B}[1] \to \mathcal{Q}(\mathfrak{B}[2])$ is the Busby invariant of $\varepsilon$, then there exists an injective homomorphism $\sigma_{\varepsilon_{\mathfrak{B}}} : \mathfrak{B} \to \mathcal{M}(\mathfrak{B}[2])$ such that the diagram

$$\begin{array}{ccc}
0 & \to & \mathfrak{B}[2] \\
\| & \| & \| \\
0 & \to & \mathfrak{B}[2] \\
\| & \| & \| \\
0 & \to & \mathcal{M}(\mathfrak{B}[2]) \\
\| & \| & \| \\
0 & \to & \mathcal{Q}(\mathfrak{B}[2]) \\
\| & \| & \| \\
0 & \to & 0 \\
\end{array}$$

If $\mathfrak{B}[2] \cong \mathbb{K}$, let $\eta_{\mathfrak{B}} : \mathcal{M}(\mathfrak{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathfrak{B}[2] \cong \mathbb{K}$. Let $\psi_1, \psi_2 : \mathfrak{A} \to \mathfrak{B}$ be two, full $X_2$-equivariant homomorphisms such that $K_0((\psi_1)_1(2)) = K_0((\psi_2)_1(2))$ and $\eta_{\mathfrak{B}} \circ \sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_1$ is a non-degenerate representation of $\mathfrak{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^\infty$ in $\mathcal{M}(\mathfrak{B}[2])$ such that

$$U_n(\sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_2)(a) \in \mathfrak{B}[2]$$

for all $a \in \mathfrak{A}$ and for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} \|U_n(\sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathfrak{A}$.

Proof. We argue as in the proof of Lemma 2.8 of [22]. Set $\sigma_i = \eta_{\mathfrak{B}} \circ \sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_i$. By assumption, $\sigma_1 : \mathfrak{A} \to B(\ell^2)$ is a non-degenerated representation of $\mathfrak{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^\infty$ in $B(\ell^2)$ such that $V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K}$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathfrak{A}$. This will be a consequence of Theorem 5(iii) of [1].

Let $\rho : \mathfrak{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathfrak{A}[2] \cong \mathbb{K}$. Since $\psi_1, \sigma_{\varepsilon_{\mathfrak{B}}}, \eta_{\mathfrak{B}}$ are injective homomorphisms, $\sigma_i$ is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi : B(\ell^2) \to B(\ell^2)/\mathbb{K}$ be the natural projection. Note that

$$\pi \circ \sigma_1 = \pi \circ \eta_{\mathfrak{B}} \circ \sigma_{\varepsilon_{\mathfrak{B}}} \circ \psi_1 = \eta_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1 = \eta_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1 = \eta_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ (\psi_1)_{1} \circ \pi_{\mathfrak{A}}.$$
It now follows that \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = \mathfrak{A}[2] \) since \( \pi |_{\mathfrak{B}_n}, \pi |_{\mathfrak{B}_n} \), and \( \langle \psi_i \rangle \) are injective homomorphisms.

Let \( H_1 \) be the essential subspace of \( \sigma_1 \). Since \( \sigma_1(\mathfrak{A}[2]) \subseteq \mathbb{K} \) and for each \( x \notin \mathfrak{A}[2] \), we have that \( \sigma_1(x) \notin \mathbb{K} \), we have that \( H_1 = \sigma_1(\mathfrak{A}[2])\mathbb{K}^2 \). Similarly, we have that \( H_2 = \sigma_2(\mathfrak{A}[2])\mathbb{K}^2 \), where \( H_2 \) is the essential subspace of \( \sigma_2 \). Let \( e \) be a minimal projection of \( \mathfrak{A}[2] \cong \mathbb{K} \). Suppose \( \sigma_1(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_1(-)|_{H_1} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \). Since \( K_0((\psi_1)_2) = K_0((\psi_2)_2) \), we have that \( \sigma_1(e) \) is Murray-von Neumann equivalent to \( \sigma_2(e) \). Hence, \( \sigma_2(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_2(-)|_{H_2} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \).

The above paragraph imply that \( \sigma_2(-)|_{H_2} \) and \( \sigma_1(-)|_{H_1} \) are unitarily equivalent. Since \( \ker(\sigma_1) = \ker(\sigma_2) \) and \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) \) by Theorem 5(iii) of [1], there exists a sequence of unitaries \( \{ V_n \}_{n=1}^{\infty} \) in \( B(\ell^2) \) such that \( V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K} \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and
\[
\lim_{n \to \infty} \| V_n\sigma_1(a)V_n^* - \sigma_2(a) \| = 0
\]
for all \( a \in \mathfrak{A} \).

Set \( U_n = \eta^{-1}_N(V_n) \). Then \( \{ U_n \}_{n=1}^{\infty} \) is a sequence of unitaries in \( \mathcal{M}(\mathfrak{B}[2]) \) such that \( U_n(\sigma_{\mathfrak{B}_n} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}_n} \circ \psi_2)(a) \in \mathfrak{B}[2] \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and
\[
\lim_{n \to \infty} \| U_n(\sigma_{\mathfrak{B}_n} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}_n} \circ \psi_2)(a) \| = 0
\]
for all \( a \in \mathfrak{A} \).

**Definition 4.10.** A \( C^* \)-algebra \( \mathfrak{A} \) is called weakly semiprojective if we can always solving the \( * \)-homomorphism lifting problem
\[
\begin{array}{ccc}
\mathfrak{A} & \overset{\phi}{\longrightarrow} & \prod_{n=1}^{\infty} \mathfrak{B}_n \\
\end{array}
\]
and \( \mathfrak{A} \) is called semiprojective if we can always solve the lifting problem
\[
\begin{array}{ccc}
\mathfrak{A} & \overset{\phi}{\longrightarrow} & \mathfrak{B}/\mathfrak{I}_N \\
\end{array}
\]

**Lemma 4.11.** Let \( \mathfrak{A}_0 \) be a unital, separable, nuclear, tight \( C^* \)-algebra over \( X_2 \) such that \( \mathfrak{A}_0[2] \cong \mathbb{K} \) and \( \mathfrak{A}_0 \) has the stable weak cancellation property. Set \( \mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K} \). Suppose \( \beta : \mathfrak{A} \to \mathfrak{A} \) is a full \( X_2 \)-equivariant homomorphism such that \( K_{X_2}(\beta) = K_{X_2}(\text{id}_{\mathfrak{A}}) \) and \( \beta_{(1)} = \text{id}_{\mathfrak{A}[1]} \). Then there exists a sequence of contractive, completely positive, linear maps \( \{ \alpha_n : \mathfrak{A} \to \mathfrak{A} \}_{n=1}^{\infty} \) such that
1. \( \alpha_n |_{\mathfrak{A}_n} \) is a homomorphism for all \( n \in \mathbb{N} \) and
2. for all \( a \in \mathfrak{A} \),
\[
\lim_{n \to \infty} \| \alpha_n \circ \beta(a) - a \| = 0
\]
where \( e_n = \sum_{k=1}^{n} 1 \otimes e_{kk} \) and \( \{ e_{ij} \}_{i,j} \) is a system of matrix units for \( K \). If, in addition, \( A \) is assumed to be weakly semiprojective, then \( \alpha_n \) can be chosen to be a homomorphism for all \( n \in \mathbb{N} \).

**Proof.** Since \( \beta \) is a full \( X_2 \)-equivariant homomorphism and the ideal in \( A \) generated by \( e_n \) is \( \mathfrak{A} \), we have that the ideal in \( A \) generated by \( \beta(e_n) \) is \( \mathfrak{A} \). Since \( K_{X_2}(\beta) = K_X(\id_{\mathfrak{A}}) \), we have that \( [\beta(e_n)] = [e_n] \) in \( K_0(\mathfrak{A}) \). It now follows that \( \beta(e_n) \) and \( e_n \) are Murray-von Neumann equivalent since \( \mathfrak{A}_0 \) has the stable weak cancellation property. Since \( \mathfrak{A} \) is stable, there exists a unitary \( v_n \) in the unitization of \( \mathfrak{A} \) such that \( v_n \beta(e_n) v_n^* = e_n \).

Fix \( n \in \mathbb{N} \). Let \( e_n \) be the extension \( 0 \to e_n \mathfrak{A}[2]e_n \to e_n \mathfrak{A}e_n \to \mathfrak{A}e_n[2] \to 0 \). By Lemma 1.5 of [16], \( e_n \) is a full extension. Therefore, \( \sigma_t(e_n) \) is Murray-von Neumann equivalent to \( 1_{\mathcal{M}(\mathfrak{A}[2])} \). Hence, \( e_n \mathfrak{A}[2]e_n \cong \mathfrak{A}[2] \cong K \). Set \( \mathfrak{A}_n = e_n \mathfrak{A}e_n \) and define \( \beta_n : \mathfrak{A}_n \to \mathfrak{A}_n \) by \( \beta_n(x) = \Ad(v_n) \circ \beta(x) \). Then \( \beta_n \) is a unital, full \( X_2 \)-equivariant homomorphism. Since \( \eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n \) is a unital representation of \( \mathfrak{A}_n \), the closed subspace of \( \ell^2 \) generated by \( \{ (\eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n)(x) : x \in \mathfrak{A}_n, x \in \ell^2 \} \) is \( \ell^2 \). Therefore, \( \eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n \) is a non-degenerate representation of \( \mathfrak{A}_n \).

Since \( K_{X_2}(\beta) = K_{X_2}(\id_{\mathfrak{A}}) \) and the \( X_2 \)-equivariant embedding of \( \mathfrak{A}_n \) as a sub-algebra of \( \mathfrak{A} \) induces an isomorphism in ideal related \( K \)-theory, we have that \( K_{X_2}(\beta_n) = K_{X_2}(\id_{\mathfrak{A}_n}) \). By Lemma 4.9 there exists a sequence of unitaries \( W_{k,n} \in \mathcal{M}(\mathfrak{A}_n[2]) \) such that

\[
(\Ad(W_{k,n}) \circ \sigma_{e_n} \circ \beta_n)(x) - \sigma_{e_n}(x) \in \mathfrak{A}_n[2]
\]

for all \( x \in \mathfrak{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (\Ad(W_{k,n}) \circ \sigma_{e_n} \circ \beta_n)(x) - \sigma_{e_n}(x) \| = 0
\]

for all \( x \in \mathfrak{A}_n \).

Note that \( \mathcal{M}(\mathfrak{A}_n[2]) \cong \sigma_t(e_n) \mathcal{M}(\mathfrak{A}[2]) \sigma_t(e_n) \) with an isomorphism mapping \( \mathfrak{A}_n[2] \) onto \( e_n \mathfrak{A}[2]e_n \). Thus, we get a partial isometry \( \widetilde{W}_{k,n} \) in \( \mathcal{M}(\mathfrak{A}[2]) \) such that \( \widetilde{W}_{k,n}^* \widetilde{W}_{k,n} = \widetilde{W}_{k,n} \widetilde{W}_{k,n} = \sigma_t(e_n) \) and

\[
(\Ad(\widetilde{W}_{k,n}) \circ \sigma_t \circ \Ad(v_n) \circ \beta)(x) - \sigma_t(x) \in \mathfrak{A}[2]
\]

for all \( x \in \mathfrak{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (\Ad(\widetilde{W}_{k,n}) \circ \sigma_t \circ \Ad(v_n) \circ \beta)(x) - \sigma_t(x) \| = 0
\]

for all \( x \in \mathfrak{A}_n \).

Set \( V_{k,n} = (\widetilde{W}_{k,n} + 1_{\mathcal{M}(\mathfrak{A}[2])}) - \sigma_t(e_n) \sigma_t(e_n) \). Then \( V_{k,n} \) is a unitary in \( \mathcal{M}(\mathfrak{A}[2]) \) such that

\[
(\Ad(V_{k,n}) \circ \sigma_t \circ \beta)(x) - \sigma_t(x) \in \mathfrak{A}[2]
\]

for all \( x \in e_n \mathfrak{A}e_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \| (\Ad(V_{k,n}) \circ \sigma_t \circ \beta)(x) - \sigma_t(x) \| = 0
\]

for all \( x \in e_n \mathfrak{A}e_n \). A consequence of the first part is that \( (\Ad(V_{k,n}) \circ \sigma_t \circ \beta)(x) \in \sigma_t(e_n) \mathfrak{A}e_n + \mathfrak{A}[2] \) for all \( x \in e_n \mathfrak{A}e_n \). Since \( \beta(1) = \id_{\mathfrak{A}[2]} \), we have that \( x - \beta(x) \in \mathfrak{A}[2] \) for all \( x \in e_n \mathfrak{A}e_n \). Therefore,

\[
\Ad(V_{k,n}) (\sigma_t(x)) = \Ad(V_{k,n}) (\sigma_t(x - \beta(x)) + \Ad(V_{k,n}) \circ \beta(x) \in \sigma_t(e_n) \mathfrak{A}e_n + \mathfrak{A}[2]
\]

Thus, \( \alpha_{k,n} = \sigma_t^{-1} \circ (\Ad(V_{k,n}) \circ \sigma_t \circ \Ad(v_n)) \in e_n \mathfrak{A}e_n \) is a homomorphism from \( e_n \mathfrak{A}e_n \) to \( \mathfrak{A} \).
Since
\[
\lim_{k \to \infty} \|(\text{Ad}(V_{k,n}) \circ \sigma_t \circ \beta)(x) - \sigma_t(x)\| = 0
\]
for all \( x \in e_n \mathcal{A} e_n \) and \( e_n \mathcal{A} e_n \subseteq e_{n+1} \mathcal{A} e_{n+1} \), there exists a strictly increasing sequence \( \{k(n)\}_{n=1}^\infty \) of positive integers such that
\[
\lim_{n \to \infty} \|\alpha_{k(n),n} \circ \beta(x) - x\| = 0
\]
for all \( x \in \bigcup_{n=1}^\infty e_n \mathcal{A} e_n \). Let \( \alpha_n \) be a completely, contractive, positive linear extension of \( \alpha_{k(n),n} \). Since \( \bigcup_{n=1}^\infty e_n \mathcal{A} e_n \) is dense in \( \mathcal{A} \), we have that
\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(x) - x\| = 0
\]
for all \( x \in \mathcal{A} \). We have just proved the first part of the lemma.

We now show that \( \mathcal{A} \) is weakly semiprojective. Suppose \( \mathcal{A} \) is weakly semiprojective. Let \( \epsilon > 0 \) and \( F \) be a finite subset of \( \mathcal{A} \). By Theorem 2.4 of [23] (see also Definition 2.1 and Theorem 2.3 of [25], and Theorem 19.1.3 of [26]), there exist a \( \delta > 0 \) and a finite subset \( G \) of \( \mathcal{A} \) such that for any \( C^* \)-algebra \( \mathcal{B} \) and any contractive, completely positive, linear map \( L : \mathcal{A} \to \mathcal{B} \) such that
\[
\|L(ab) - L(a)L(b)\| < \delta
\]
for all \( a, b \in G \), there exists a homomorphism \( h : \mathcal{A} \to \mathcal{B} \) such that
\[
\|h(x) - L(x)\| < \frac{\epsilon}{2}
\]
for all \( x \in \beta(F) \).

Without loss of generality, we may assume that \( \epsilon < 1 \) and \( \delta < 1 \). Set
\[
M = 1 + \max\left(\{\|a\| : a \in G\} \cup \{\|x\| : x \in F\}\right)
\]
Since \( e_n \mathcal{A} e_n \subseteq e_{n+1} \mathcal{A} e_{n+1} \) and \( \bigcup_{n=1}^\infty e_n \mathcal{A} e_n \) is dense in \( \mathcal{A} \), there exist \( n \in \mathbb{N} \) and a finite subset \( H \subseteq e_n \mathcal{A} e_n \) such that for each \( a \in G \), there exists \( y \in H \) such that \( \|a - y\| < \frac{\delta}{4M} \) and
\[
\|\alpha_n \circ \beta(x) - x\| < \frac{\epsilon}{2}
\]
for all \( x \in F \). Let \( a, b \in G \). Choose \( x, y \in H \subseteq e_n \mathcal{A} e_n \) such that \( \|a - x\| < \frac{\delta}{4M} \) and \( \|b - y\| < \frac{\delta}{4M} \). Note that \( \|x\| \leq 1 + \|a\| \leq M \) and \( \|y\| \leq 1 + \|b\| \leq M \). Then
\[
\|\alpha_n(ab) - \alpha_n(a)\alpha_n(b)\| = \|\alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b)\|
\leq \|\alpha_n\|\|a - x\| + \|x\|\|b - y\|
+ \|\alpha_n(x)\alpha_n(y) - \alpha_n(x)\alpha_n(b)\|
+ \|\alpha_n(x)\alpha_n(b) - \alpha_n(a)\alpha_n(b)\|
\leq 2M\|a - x\| + 2M\|b - y\|
< 4M\frac{\delta}{4M} = \delta.
\]
By the choice of \( \delta \) and \( G \), there exists a homomorphism \( \psi : \mathcal{A} \to \mathcal{A} \) such that
\[
\|\psi(t) - \alpha_n(t)\| < \frac{\epsilon}{2}
\]
for all \( t \in \beta(\mathcal{F}) \). Let \( x \in \mathcal{F} \). Then
\[
\|\psi \circ \beta(x) - x\| \leq \|\psi(\beta(x)) - \alpha_n(\beta(x))\| + \|\alpha_n(\beta(x)) - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\]

We have just shown that for every \( \epsilon > 0 \) and for every finite subset \( \mathcal{F} \) of \( \mathfrak{A} \), there exists a homomorphism \( \psi : \mathfrak{A} \to \mathfrak{A} \) such that
\[
\|\psi \circ \beta(x) - x\| < \epsilon
\]
for all \( x \in \mathcal{F} \). Consequently, there exists a sequence of endomorphisms \( \{\psi_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^\infty \) such that
\[
\lim_{n \to \infty} \|\psi_n \circ \beta(x) - x\| = 0
\]
for all \( x \in \mathfrak{A} \) since \( \mathfrak{A} \) is separable.

To prove a uniqueness theorem involving tight \( C^* \)-algebras \( \mathfrak{A} \) over \( X_2 \), we require that \( \mathfrak{A}[1] \) belongs to a class of \( C^* \)-algebras whose injective homomorphisms between two objects in this class are classified by \( KK \).

**Definition 4.12.** We will be interested in classes \( \mathcal{C} \) of separable, nuclear, simple \( C^* \)-algebras satisfying the following property that if \( \mathfrak{A}, \mathfrak{B} \in \mathcal{C} \) and \( \phi, \psi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K} \) are two injective homomorphisms such that \( \text{KK}(\phi) = \text{KK}(\psi) \), then \( \phi \) and \( \psi \) are approximately unitarily equivalent.

**Remark 4.13.**

1. By Theorem 4.1.3 of [29] if \( \mathcal{C} \) is the class of Kirchberg algebras, then \( \mathcal{C} \) satisfies the property in Definition 4.12.

2. Let \( \mathcal{C} \) be the class of unital, separable, nuclear, simple tracially AF \( C^* \)-algebras in \( \mathcal{N} \). Then \( \mathcal{C} \) satisfies the property in Definition 4.12.

**Theorem 4.14.** *(Uniqueness Theorem 2)* Let \( \mathcal{C} \) be a class of \( C^* \)-algebras satisfying the property in Definition 4.12 and let \( \mathfrak{A} \) be a unital, separable, nuclear, tight \( C^* \)-algebra over \( X_2 \) such that \( \mathfrak{A}[2] \cong \mathbb{K} \) and \( \mathfrak{A}[1] \in \mathcal{C} \). Suppose \( \mathfrak{A} \otimes \mathbb{K} \) is semiprojective and \( \mathfrak{A} \) has the stable weak cancellation property. Let \( \phi : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{B} \otimes \mathbb{K} \) be a full \( X_2 \)-equivariant homomorphism such that \( \text{KK}(X_2; \phi) = \text{KK}(X_2; \text{id}_{\mathfrak{A} \otimes \mathbb{K}}) \). Then there exists a sequence of full \( X_2 \)-equivariant endomorphisms \( \{\alpha_n : \mathfrak{A} \otimes \mathbb{K} \to \mathfrak{A} \otimes \mathbb{K}\}_{n=1}^\infty \) such that \( \text{KK}(X_2; \alpha_n) = \text{KK}(X_2; \text{id}_{\mathfrak{A} \otimes \mathbb{K}}) \) and
\[
\lim_{n \to \infty} \|(\alpha_n \circ \phi)(x) - x\| = 0
\]
for all \( x \in \mathfrak{A} \otimes \mathbb{K} \).

**Proof.** Set \( \mathfrak{B} = \mathfrak{A} \otimes \mathbb{K} \). Note that \( \mathfrak{B} \) is a tight \( C^* \)-algebra over \( X_2 \) with \( \mathfrak{B}[2] \cong \mathbb{K} \). Throughout the proof, \( \pi : \mathfrak{B} \to \mathfrak{B}[1] \) will denote the canonical projection. Note that \( \text{KK}(\phi_{[1]}) = \text{KK}(\text{id}_{\mathfrak{B}[1]}) \) since \( \text{KK}(X_2; \phi) = \text{KK}(X_2; \text{id}_{\mathfrak{B}}) \). Since \( \mathfrak{A}[1] \in \mathcal{C} \), there exists a sequence of unitaries \( \{z_k\}_{k=1}^\infty \) in \( \mathcal{M}(\mathfrak{B}[1]) \) such that
\[
\lim_{k \to \infty} \|z_k \phi_{[1]}(\pi(b))z_k^* - \pi(b)\| = 0
\]
for all \( b \in \mathfrak{B} \). Using the fact that \( \phi \) is an \( X_2 \)-equivariant homomorphism, we have that \( \pi \circ \phi = \phi_{[1]} \circ \pi \), and hence
\[
\lim_{k \to \infty} \|z_k (\pi \circ \phi(b))z_k^* - \pi(b)\| = 0
\]
for all $b \in \mathcal{B}$.

Let $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B}[1])$ be the surjective homomorphism induced by $\pi$. Since $\mathcal{B}$ is stable, by Corollary 2.3 of [35], we have that $\mathcal{B}[1]$ is stable. Thus, the unitary group of $\mathcal{M}(\mathcal{B}[1])$ is path-connected, which implies that every unitary in $\mathcal{M}(\mathcal{B}[1])$ lifts to a unitary in $\mathcal{M}(\mathcal{B})$. Hence, there exists a sequence of unitaries $\{w_k\}_{k=1}^\infty$ in $\mathcal{M}(\mathcal{B})$ such that $\pi(w_k) = z_k$. Since $\mathcal{B}$ is semi-projective, by Proposition 2.2 of [7] (see [26]), there exists a sequence of homomorphisms $\{\beta_\ell : \mathcal{B} \to \mathcal{B}\}_{\ell=1}^\infty$ and a strictly increasing sequence $\{k(\ell)\}_{\ell=1}^\infty$ of positive integers such that $\pi \circ \beta_\ell = \pi$ and

$$\lim_{\ell \to \infty} \|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| = 0$$

for all $b \in \mathcal{B}$.

By Remark 2.5 there exists $N_1 \in \mathbb{N}$ such that $\beta_\ell$ is a full $X_2$-equivariant homomorphism for all $\ell \geq N_1$. By Proposition 2.3 of [7], we may choose $N_2 \geq N_1$ such that for all $\ell \geq N_2$, we have that $\beta_\ell$ and $\text{Ad}(w_{k(\ell)}) \circ \phi$ is homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \beta_\ell) = KK(X_2; \text{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \text{id}_\mathcal{B})$.

Let $\ell \geq N_2$. Note that $\beta_\ell|_{\mathcal{B}[1]} = \text{id}_{\mathcal{B}[1]}$ since $\pi \circ \beta_\ell = \pi$. Since $\mathcal{A}$ is semi-projective, by Corollary 3.6 of [6] (also see Chapter 19 of [25]), $\mathcal{A}$ is weakly semi-projective. Hence, by Lemma 4.11 there exists a sequence of homomorphisms $\{\alpha_{m,\ell} : \mathcal{B} \to \mathcal{B}\}_{m=1}^\infty$ such that

$$\lim_{m \to \infty} \|\alpha_{m,\ell} \circ \beta_\ell(x) - x\| = 0$$

for all $x \in \mathcal{B}$. Since $\beta_\ell$ and $\text{id}_\mathcal{B}$ are full $X_2$-equivariant homomorphisms, by Remark 2.5 there exists $N_3$ such that, for all $m \geq N_3$, we have that $\alpha_{m,\ell}$ is a full $X_2$-equivariant homomorphism. Moreover, by Proposition 2.3 of [7], we can choose $N_3 \geq N_2$ such that $\alpha_{m,\ell} \circ \beta_\ell$ are homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \text{id}_\mathcal{B})$ for all $m \geq N_3$. Consequently, $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B})$ for all $m \geq N_3$.

Let $F$ be a finite subset of $\mathcal{B}$ and $\epsilon > 0$. Then there exists $\ell \geq N_2$ such that

$$\|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| < \frac{\epsilon}{2}$$

for all $b \in F$. Moreover, there exists $m \geq N_3$ such that

$$\|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| < \frac{\epsilon}{2}$$

for all $b \in F$. Set $\alpha_1 = \text{Ad}(w_{k(\ell)})|_{\mathcal{B}}$ and $\alpha = \alpha_{m,\ell} \circ \alpha_1$. Since $w_{k(\ell)}$ is a unitary in $\mathcal{M}(\mathcal{B})$, we have that $\alpha_1$ is an automorphism of $\mathcal{B}$ and $KK(X_2; \alpha_1) = KK(X_2; \text{id}_\mathcal{B})$. Therefore, $\alpha$ is a full $X_2$-equivariant homomorphism. Since $\ell \geq N_2$ and $m \geq N_3$, we have that $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B})$. Therefore, $KK(X_2; \alpha) = KK(X_2; \text{id}_\mathcal{B})$. Let $b \in F$. Then

$$\|\alpha \circ \phi(b) - b\| = \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - b\| \leq \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b)\| + \|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have just shown that for every $\epsilon > 0$ and for every finite subset $F$ of $\mathcal{B}$, there exists a full $X_2$-equivariant homomorphism $\alpha : \mathcal{B} \to \mathcal{B}$ such that $KK(X_2; \alpha) = KK(X_2; \text{id}_\mathcal{B})$ and $\|\alpha \circ \phi(b) - b\| < \epsilon$.
for all $b \in \mathcal{B}$. Since $\mathcal{B}$ is a separable $C^*$-algebra, there exists a sequence of full $X_2$-equivariant homomorphisms $\{\alpha_n : \mathcal{B} \to \mathcal{B}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \text{id}_\mathcal{B})$ and

$$\lim_{n \to \infty} \|\alpha_n \circ \phi - b\| = 0$$

for all $b \in \mathcal{B}$.

**Theorem 4.15.** Let $\mathcal{C}$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i[2] \cong K$ and $\mathfrak{A}_i[1] \in \mathcal{C}$. Suppose $\mathfrak{A}_i \otimes K$ is semiprojective and $\mathfrak{A}_i$ has the stable weak cancellation property. If there exist full $X_2$-equivariant homomorphisms, $\phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ and $\psi : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K$, such that $KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$ and $KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K})$, then for any finite subset $\mathcal{F}$ and $\epsilon > 0$, there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \gamma) = KK(\phi)$ and

$$\|\gamma(x) - \phi(x)\| < \epsilon$$

for all $x \in \mathcal{F}$.

**Proof.** Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_1 \otimes K$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes K$ and let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_2 \otimes K$ such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $\mathfrak{A}_2 \otimes K$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Set $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_1$ and choose $m_1 \in \mathbb{N}$ such that $\sum_{k=m_1}^{\infty} \frac{1}{2k} < \epsilon$. By Theorem 4.14, there exists a full $X_2$-equivariant homomorphism $\alpha_1 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_1 \otimes K$ such that $KK(X_2; \alpha_1) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K})$ and

$$\|\alpha_1 \circ \psi \circ \phi(a) - a\| < \frac{1}{2m_1 + 1}$$

for all $a \in \mathcal{F}_1$. Set $\phi_1 = \phi$ and $\psi_1 = \alpha_1 \circ \psi$. Then $KK(X_2; \psi_1) = KK(X_2; \psi)$ and $\|\psi_1 \circ \phi_1(a) - a\| < \frac{1}{2m_1 + 1}$ for all $a \in \mathcal{F}_1$.

Set $\mathcal{G}_1 = \mathcal{G}_1 \cup \phi(\mathcal{F}_1)$. Note that $KK(X_2; \phi \circ \psi_1) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$. Hence, by Theorem 4.14, there exists a full $X_2$-equivariant homomorphism $\beta_1 : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \beta_1) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K})$ and

$$\|\beta_1 \circ \phi \circ \psi_1(x) - x\| < \frac{1}{2m_1 + 1}$$

for all $x \in \mathcal{G}_1$. Set $\phi_2 = \beta_1 \circ \phi$. Then $KK(X_2; \phi_2) = KK(X_2; \phi)$ and

$$\|\phi_2 \circ \psi_1(x) - x\| < \frac{1}{2m_1 + 1}$$

for all $x \in \mathcal{G}_1$. Note that for all $x \in \mathcal{F}_1$, then

$$\|\phi(x) - \phi_2(x)\| \leq \|\phi_1(x) - \phi_2 \circ \psi_1(x)\| + \|\phi_2 \circ \psi_1(x) - \phi_2(x)\| < \frac{1}{2m_1} + \|\psi_1 \circ \phi_1(x) - x\| < \frac{1}{2m_1}.$$
\[ \mathfrak{A}_1 \otimes K \text{ such that } KK(X_2; \alpha_2) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K}) \text{ and} \]
\[ \| \alpha_2 \circ \psi \circ \phi_2(a) - a \| < \frac{1}{2^{m+2}} \]
for all \( a \in \mathcal{F}_2 \). Set \( \psi_2 = \alpha_2 \circ \psi \). Then \( KK(X_2; \psi_2) = KK(X_2; \psi) \) and
\[ \| \psi_2 \circ \phi_2(a) - a \| < \frac{1}{2^{m+2}} \]
for all \( x \in \mathcal{F}_2 \).

Set \( \mathcal{G}_2 = \mathcal{G}_2 \cup \phi_2(\mathcal{F}_2) \). Note that \( KK(X_2; \phi \circ \psi_2) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \).

Hence, by Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \beta_2 : \mathfrak{A}_2 \otimes K \rightarrow \mathfrak{A}_2 \otimes K \) such that \( KK(X_2; \beta_2) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \) and
\[ \| \beta_2 \circ \phi \circ \psi_2(x) - x \| < \frac{1}{2^{m+2}} \]
for all \( x \in \mathcal{G}_2 \). Set \( \phi_3 = \beta_2 \circ \phi \). Then \( KK(X_2; \phi_3) = KK(X_2; \phi) \) and
\[ \| \phi_3 \circ \psi_2(x) - x \| < \frac{1}{2^{m+2}} \]
for all \( x \in \mathcal{G}_2 \). Note that for all \( x \in \mathcal{F}_2 \), we have that
\[ \| \phi_2(x) - \phi_3(x) \| \leq \| \phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x)) \| + \| \phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x) \| < \frac{1}{2^{m+2}} + \| \phi_3 \circ \psi_2(\phi_2(x)) - x \| < \frac{1}{2^{m+1}}. \]

Continuing this process, we have constructed a sequence \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_1 \otimes K \), a sequence \( \{ \mathcal{G}_n \}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_2 \otimes K \), a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \phi_n : \mathfrak{A}_1 \otimes K \rightarrow \mathfrak{A}_2 \otimes K \}_{n=1}^{\infty} \), and a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \psi_n : \mathfrak{A}_2 \otimes K \rightarrow \mathfrak{A}_1 \otimes K \}_{n=1}^{\infty} \) such that
(1) \( KK(X_2; \phi_n) = KK(X_2; \phi) \) for all \( n \in \mathbb{N} \) and \( \phi_1 = \phi \);
(2) \( KK(X_2; \psi_n) = KK(X_2; \psi) \) for all \( n \in \mathbb{N} \);
(3) \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \overline{\mathcal{F}}_n \subseteq \mathcal{F}_n \);
(4) \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \) and \( \overline{\mathcal{G}}_n \subseteq \mathcal{G}_n \);
(5) for each \( x \in \mathcal{F}_n \) and for each \( x \in \mathcal{G}_n \)
\[ \| \psi_n \circ \phi_n(x) - x \| < \frac{1}{2^{m+n}} \quad \text{and} \quad \| \phi_{n+1} \circ \psi_n(x) - x \| < \frac{1}{2^{m+n}} \]
(6) for each \( x \in \mathcal{F}_n \)
\[ \| \phi_n(x) - \phi_{n+1}(x) \| < \frac{1}{2^{m+n+1}} \]

Since \( \bigcup_{n=1}^{\infty} \overline{\mathcal{F}}_n \) is dense in \( \mathfrak{A}_1 \otimes K \) and \( \overline{\mathcal{F}}_n \subseteq \mathcal{F}_n \), we have that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes K \). Similarly, \( \bigcup_{n=1}^{\infty} \mathcal{G}_n \) is dense in \( \mathfrak{A}_2 \otimes K \). Therefore, there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes K \rightarrow \mathfrak{A}_2 \otimes K \) such that
\[ \| \gamma(a) - \phi_n(a) \| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} \]
for all \(a \in \mathcal{F}_n\). Since \(\mathcal{F} \subseteq \mathcal{F}_1\), we have that
\[
\|\phi(x) - \gamma(x)\| = \|\phi_1(x) - \gamma(x)\| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon.
\]
Since
\[
\lim_{n \to \infty} \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} = 0,
\]
we have that
\[
\lim_{n \to \infty} \|\gamma(a) - \phi_n(a)\| = 0
\]
for all \(a \in \mathcal{A}_1 \otimes \mathbb{K}\). Since \(\mathcal{A}_1 \otimes \mathbb{K}\) is semiprojective, by Proposition 2.3 of [7], there exists \(N \in \mathbb{N}\) such that \(\gamma\) and \(\phi_N\) are homotopic. Hence, by Theorem 5.5 of [8], \(KK(X_2; \gamma) = KK(X_2; \phi_N) = x\).

### 4.3. Unital Classification

We know combine the above results with the Meta-theorem of Section 3 (see Theorem 3.3) to get a strong classification for a class of unital \(C^*\)-algebras which includes all unital graph \(C^*\)-algebras with exactly one non-trivial ideal.

**Corollary 4.16.** Let \(\mathcal{A}_1\) and \(\mathcal{A}_2\) be unital, tight \(C^*\)-algebras over \(X_n\) such that \(\mathcal{A}_i\) has real rank zero, \(\mathcal{A}_i[1]\) is a Kirchberg algebra in \(\mathcal{N}\), and \(\mathcal{A}_i[1, n-1]\) is an AF-algebra. Let \(x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2)\) be an invertible such that \(K_{X_n}(x)Y\) is an order isomorphism for each \(Y \in \mathbb{L}(X_n)\) and \(K_{X_n}(x)X_n([1_{\mathcal{A}_1}]) = [1_{\mathcal{A}_2}]\) in \(K_0(\mathcal{A}_2)\). Then there exists an isomorphism \(\phi: \mathcal{A}_1 \to \mathcal{B}\) such that \(K_{X_n}(\phi) = K_{X_n}(x)\).

**Proof.** Since \(\mathcal{A}_i[1]\) and \(\mathcal{A}_i[2]\) are separable and nuclear, we have that \(\mathcal{A}_i\) is separable and nuclear. Since \(\mathcal{A}_i[1, n-1]\) is an AF-algebra and \(\mathcal{A}_i[1]\) is a Kirchberg algebra, they both have the stable weak cancellation property. By Lemma 3.15 of [15], \(\mathcal{A}_i\) has stable weak cancellation property. By Lemma 4.6, for each tight \(C^*\)-algebra \(\mathcal{A}\) over \(X_n\), we have that \(K_{X_n}(\Ad(u)|\mathcal{A})\) for each unitary \(u \in \mathcal{M}(\mathcal{A})\). A computation shows that \(K_{X_n}(\gamma)\) satisfies (1), (2), and (3) of Theorem 3.3 since \(K_n(-)\) does. The corollary now follows from Theorem 3.3 and Theorem 4.7. \(\square\)

**Corollary 4.17.** Let \(\mathcal{A}_1\) and \(\mathcal{A}_2\) be unital, tight \(C^*\)-algebras over \(X_2\) such that \(\mathcal{A}_i[2] \cong \mathbb{K}\) and \(\mathcal{A}_i[1]\) is a Kirchberg algebra in \(\mathcal{N}\). Let \(x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2)\) be an invertible such that \(K_{X_2}(x)Y\) is an order isomorphism for each \(Y \in \mathcal{L}(X_2)\) and \(K_{X_2}(x)X_2([1_{\mathcal{A}_1}]) = [1_{\mathcal{A}_2}]\) in \(K_0(\mathcal{A}_2)\). If \(\mathcal{A}_i \otimes \mathbb{K}\) is semiprojective, then there exists an isomorphism \(\gamma: \mathcal{A}_1 \otimes \mathbb{K} \to \mathcal{A}_2 \otimes \mathbb{K}\) such that \(KK(X_2; \gamma) = x\).

**Proof.** Since \(\mathcal{A}_i[1]\) and \(\mathcal{A}_i[2]\) are separable and nuclear, we have that \(\mathcal{A}_i\) is separable and nuclear. Since \(\mathcal{A}_i[2]\) and \(\mathcal{A}_i[1]\) have real rank zero and \(K_1(\mathcal{A}_i[2]) = 0\), we have that \(\mathcal{A}\) has real rank zero. Since \(\mathcal{A}_i[2]\) is an AF-algebra and \(\mathcal{A}_i[1]\) is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by Lemma 3.15 of [15], \(\mathcal{A}\) has the stable weak cancellation property.

By Lemma 1.5 of [16], the extension \(0 \to \mathcal{A}_i[2] \to \mathcal{A}_i \to \mathcal{A}_i[1] \to 0\) is full, and hence by Proposition 1.6 of [16], \(0 \to \mathcal{A}_i[2] \otimes \mathbb{K} \to \mathcal{A}_i \otimes \mathbb{K} \to \mathcal{A}_i[1] \otimes \mathbb{K} \to 0\) is full. The corollary now follows from Theorem 4.1(ii), Theorem 4.15 and Theorem 3.3. \(\square\)
Lemma 4.18. Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$ with $C^*(E)[1]$ being purely infinite. Then $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semi-projective.

Proof. The fact that $C^*(E)$ is semi-projective follows from the results of [12]. By Proposition 6.4 of [15], $C^*(E)[2]$ is stable. Since $C^*(E)$ is a unital $C^*$-algebra, by Lemma 1.5 of [16], the extension $\epsilon : 0 \to C^*(E)[2] \to C^*(E) \to C^*(E)[1] \to 0$ is a full extension. By Proposition 3.21 and Corollary 3.22 of [15], $C^*(E)$ is properly infinite. Therefore, by Theorem 4.1 of [3], $C^*(E) \otimes \mathbb{K}$ is semi-projective.

Proposition 4.19. Let $\mathfrak{A}$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$. If $\mathfrak{A}[2] \cong \mathbb{K}$ and $\mathfrak{A}[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that rank($K_1(\mathfrak{A}[1])$) $\leq$ rank($K_0(\mathfrak{A}[1])$), $K_1(\mathfrak{A}[1])$ is free, and the $K$-groups of $\mathfrak{A}[i]$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semi-projective. Consequently, $\mathfrak{A}$ is semi-projective.

Proof. By Lemma 1.5 of [16], $\epsilon : 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$ is a full extension. By Corollary 3.22 of [15], $K_0(\mathfrak{A}[2]) = K_0(\mathfrak{A})$. By Theorem 6.4 of [11], there exists a graph $E$ with finitely many vertices such that $K_{X_2}^+(\mathfrak{A}) \cong K_{X_2}(C^*(E))$ such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$. Since $E$ has finitely many vertices, $C^*(E)$ is unital. Since $K_{X_2}(\mathfrak{A}) \cong K_{X_2}(C^*(E))$, we have that $C^*(E)[1]$ is a Kirchberg algebra. By Theorem 3.9 of [16], we have that $\mathfrak{A} \otimes \mathbb{K} \cong C^*(E) \otimes \mathbb{K}$. By Lemma 4.18, $C^*(E)$ and $C^*(E) \otimes \mathbb{K}$ are semi-projective. Hence, by Proposition 2.7 of [3], $\mathfrak{A}$ and $\mathfrak{A} \otimes \mathbb{K}$ are semi-projective.

Corollary 4.20. Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_1[2] \cong \mathbb{K}$ and $\mathfrak{A}_1[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that rank($K_1(\mathfrak{A}_1[1])$) $\leq$ rank($K_0(\mathfrak{A}_1[1])$), $K_1(\mathfrak{A}_1[1])$ is free, and the $K$-groups of $\mathfrak{A}_1$ are finitely generated. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_2}(x)_{\mathcal{L}(X_2)}$ and $\mathfrak{A}_1 \otimes \mathbb{K}$ is an order isomorphism for each $Y \in \mathcal{L}(X_2)$ and $K_{X_2}(x)_{\mathcal{L}(X_2)}([1_{\mathfrak{A}_1}]) = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \gamma) = x$.

Proof. This follows from Proposition 4.19 and Corollary 4.17.

5. Applications

Let $E$ be a graph satisfying Condition (K) (in particular, if $C^*(E)$ has finitely many ideals, then $E$ satisfies Condition (K)). Let $\mathcal{J}_1, \mathcal{J}_2$ be ideals of $C^*(E)$ such that $\mathcal{J}_1 \subseteq \mathcal{J}_2$ and $\mathcal{J}_2/\mathcal{J}_1$ is simple. Then by Theorem 5.1 of [38] and Corollary 3.5 of [2], $\mathcal{J}_2/\mathcal{J}_1$ is a simple graph $C^*$-algebra. Hence, $\mathcal{J}_2/\mathcal{J}_1$ is either a Kirchberg algebra or an AF algebra.

5.1. Classification of graph $C^*$-algebras with exactly one ideal.

Lemma 5.1. Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a simple AF-algebra. Then $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Consequently, if $F$ is a graph with finitely many vertices such that $C^*(F)$ is a tight $C^*$-algebra over $X_2$ and $C^*(F)[2]$ is an AF-algebra, then $C^*(F)[2] \cong \mathbb{K}$. 

Proof. We claim that $E$ is a finite graph. By Corollary 2.13 and Corollary 2.15 of [9], $E$ has no cycles, and for every vertex $v_0$ that emits infinitely many edges and for each vertex $v$, there exists a path from $v$ to $v_0$. Since $E$ has no cycles, we have that every vertex of $E$ emits only finitely many edges. Hence, $E$ is a finite graph. By Proposition 1.18 of [30], $C^*(E) \cong M_n$.

We now prove the second statement. First note that $C^*(F)[2]$ is a simple AF-algebra. Since $C^*(F)[2]$ is stably isomorphic to a subgraph of $E$, $C^*(F)[2] \otimes K \cong C^*(E)$ for some graph $E$ with finitely many vertices. Since $C^*(E)$ is a simple AF-algebra, we have that $C^*(E) \otimes K \cong K$. Hence, $C^*(F)[2] \otimes K \cong K$ which implies that $C^*(F)[2] \cong M_n$ or $C^*(F)[2] \cong K$. Since $C^*(F)[2]$ is a non-unital $C^*$-algebra ($C^*(E)$ is a tight $C^*$-algebra over $X_2$), we have that $C^*(F)[2] \cong K$. \hfill $\square$

Definition 5.2. For a $C^*$-algebra $\mathfrak{A}$, set

$$\Sigma \mathfrak{A} = \{ x \in K_0(\mathfrak{A}) : x = [p] \text{ for some projection } p \text{ in } \mathfrak{A} \}.$$}

Let $\mathfrak{B}$ be a $C^*$-algebra. An order isomorphism $\alpha : K_0(\mathfrak{A}) \to K_0(\mathfrak{B})$ is scale preserving if one of the following holds:

1. $\mathfrak{A}$ is unital if and only if $\mathfrak{B}$ unital and $\alpha([1_\mathfrak{A}]) = [1_\mathfrak{B}]$.
2. $\mathfrak{A}$ is non-unital if and only if $\mathfrak{B}$ is non-unital and $\alpha(\Sigma \mathfrak{A}) = \Sigma \mathfrak{B}$.

Theorem 5.3. Let $E_1$ and $E_2$ be graphs with finitely many vertices and $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. If $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ is an isomorphism such that $\alpha Y$ is scale preserving for all $Y \in \mathbb{LCl}(X_2)$, then there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Proof. Since $E_i$ has finitely many vertices, $C^*(E_1)$ and $C^*(E_2)$ are unital $C^*$-algebras. 

Case 1: Suppose $C^*(E_1)$ is an AF-algebra. Then $C^*(E_2)$ is an AF-algebra. Hence, the result follows from Elliott’s classification of AF-algebras [19].

Case 2: Suppose $C^*(E_1)$ is not an AF-algebra. Then $C^*(E_2)$ is not an AF-algebra.

Subcase 2.1: Suppose $C^*(E_1)[1]$ is an AF-algebra. Then $C^*(E_2)[1]$ is an AF-algebra. By Corollary 4.10 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Subcase 2.2: Suppose $C^*(E_1)[1]$ is a Kirchberg algebra. Then $C^*(E_2)[1]$ is a Kirchberg algebra. Since $C^*(E_i)$ is not an AF-algebra, either $C^*(E_i)[2]$ is Kirchberg algebra or an AF-algebra.

Suppose $C^*(E_i)[2]$ is a Kirchberg algebra. By Theorem 2.4 of [32], there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. Suppose $C^*(E_i)[2]$ is an AF-algebra. Then, by Lemma 5.1, $C^*(E_i)[2] \cong K$. By Corollary 4.20 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. \hfill $\square$

The following theorem completes the classification of graph $C^*$-algebras with exactly one non-trivial ideal.

Corollary 5.4. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. Then $C^*(E_1) \cong C^*(E_2)$ if and only if there exists an isomorphism $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ such that $\alpha Y$ is a scale preserving isomorphism for all $Y \in \mathbb{LCl}(X_2)$. 

\[ \text{□} \]
5.2. **Classification of graph $C^*$-algebras with more than one ideal.** For a tight $C^*$-algebra $\mathfrak{A}$ over $X_n$, the finite and infinite simple sub-quotients of $\mathfrak{A}$ are separated if there exists $U \in O(X_n)$ such that either

1. $\mathfrak{A}(U)$ is an AF-algebra and $\mathfrak{A}(X_n \setminus U) \otimes O_\infty \cong \mathfrak{A}(X_n \setminus U)$ or
2. $\mathfrak{A}(X_n \setminus U)$ is an AF-algebra and $\mathfrak{A}(U) \otimes O_\infty \cong \mathfrak{A}(U)$.

In [14], the authors proved that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are graph $C^*$-algebras that are tight $C^*$-algebras over $X_n$ such that the finite and infinite simple sub-quotients are separated, then $\mathfrak{A}_1 \otimes K \cong \mathfrak{A}_2 \otimes K$ if and only if $K_{X_n}^+(\mathfrak{A}_1) \cong K_{X_n}^+(\mathfrak{A}_2)$. We will show in this section that under mild $K$-theoretical conditions, we may remove the separated condition for the case $n = 3$.

**Lemma 5.5.** Let $E$ be a graph such that $C^*(E)$ is a tight $C^*$-algebra over $X_n$.

(i) If $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra, then

$$\mathfrak{c}_1 : 0 \to C^*(E)[2, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1] \otimes K \to 0$$

is a full extension.

(ii) If $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite, then

$$\mathfrak{c}_2 : 0 \to C^*(E)[k, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1, k - 1] \otimes K \to 0$$

is a full extension.

**Proof.** Suppose $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra. Note that $C^*(E)[1, n - 1]/C^*(E)[2, n - 1] \cong C^*(E)[1]$ and $C^*(E)[2, n - 1]$ is the largest ideal of $C^*(E)[1, n - 1]$ which is an AF-algebra. Since $C^*(E)[1, n - 1]$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[2, n - 1] \otimes K \to C^*(E)[1, n - 1] \otimes K \to C^*(E)[1] \otimes K \to 0$$

is a full extension. Since $C^*(E)[n] \otimes K$ is a purely infinite simple $C^*$-algebra, we have that

$$0 \to C^*(E)[n] \otimes K \to C^*(E)[2, n] \otimes K \to C^*(E)[2, n - 1] \otimes K \to 0$$

is a full extension. Hence, by Proposition 3.2 of [17], $\mathfrak{c}_1$ is a full extension. Suppose $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite. Note that $C^*(E)[k, n]$ is the largest ideal of $C^*(E)[k - 1, n]$ such that $C^*(E)[k, n]$ is an AF-algebra and $C^*(E)[k - 1, n]/C^*(E)[k, n] \cong C^*(E)[k - 1]$ is purely infinite. Since $C^*(E)[k - 1, n] \otimes K$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[k, n] \otimes K \to C^*(E)[k - 1, n] \otimes K \to C^*(E)[k - 1] \otimes K \to 0$$

is a full extension. By Proposition 5.4 of [14], $\mathfrak{c}_2$ is a full extension.

**Theorem 5.6.** Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_n$.

Suppose

(i) $C^*(E_1)[n]$ and $C^*(E_1)[1]$ are purely infinite;

(ii) $C^*(E_2)[2, n - 1]$ is an AF-algebra; and
(iii) \( KK^1(C^*(E_1)[1], C^*(E_2)[2, n]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, n]) \).
Then \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \) if and only if \( K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K) \).

**Proof.** Let \( \epsilon_i \) be the extension
\[
0 \to C^*(E_i)[2, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0.
\]
By Lemma 5.5(i), \( \epsilon_i \) is a full extension. Suppose \( \alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K) \).
Lift \( \alpha \) to an invertible element \( x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K) \). Note that \( r_{X_n}^{[2, n]}(x) \) is invertible in \( KK([2, n]) \).

Consider \( C^*(E_i) \) as a \( C^* \)-algebra over \( X_2 \) by setting \( C^*(E_i)[2] = C^*(E_i)[2, n] \) and \( C^*(E_i)[1, 2] = C^*(E_i) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( r_{X_2}^{[1]}(y) = r_{X_n}^{[1]}(x) = KK(\phi_2) \) and \( KL(r_{X_2}^{[2]}(y)) = z = KL(\phi_0) \) in \( KL(C^*(E_1)[2, n] \otimes K, C^*(E_2)[2, n] \otimes K) \).

By Theorem 3.7 of [13],
\[
r_{X_2}^{[1]}(y) \otimes [\tau_{e_2}] = [\tau_{e_1}] \times r_{X_2}^{[2]}(y)
\]
in \( KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) \), where \( \epsilon_i \) is the extension
\[
0 \to C^*(E_i)[2, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0.
\]
Thus,
\[
KL(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KL(\phi_0)
\]
in \( KL^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) \). Since \( KL^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) = KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) \),
\[
KK(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KK(\phi_0)
\]
in \( KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) \). By Lemma 4.5 of [13], \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \). □

**Theorem 5.7.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_n \).
Suppose
(i) \( C^*(E_i)[k, n] \) and \( C^*(E_i)[1, k - 2] \) are AF-algebras;
(ii) \( C^*(E_i)[k - 1] \) is purely infinite; and
(iii) \( KK^1(C^*(E_1)[1, k - 1], C^*(E_2)[k, n]) = KL^1(C^*(E_1)[1, k - 1], C^*(E_2)[k, n]) \).
Then \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \) if and only if \( K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K) \).

**Proof.** Let \( \epsilon_i \) be the extension \( 0 \to C^*(E_2)[k, n] \otimes K \to C^*(E_1) \otimes K \to C^*(E_1)[1, k - 1] \otimes K \to 0 \).
By Lemma 5.5(ii), \( \epsilon_i \) is a full extension. Suppose \( \alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K) \).
Lift \( \alpha \) to an invertible element \( x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K) \). Note that \( r_{X_n}^{[k,n]}(x) \) is invertible in \( KK([k, n]; C^*(E_1)[k, n] \otimes K, C^*(E_2)[k, n] \otimes K) \) and \( r_{X_n}^{[1,k-1]}(x) \) is invertible in \( KK(C^*(E_1)[1, k - 1], C^*(E_2)[1, k - 1]) \). By Theorem 4.7, there exists an isomorphism
\(\phi_2 : C^*(E_1)[1,k-1] \otimes K \to C^*(E_2)[1,k-1] \otimes K\) such that \(KL(\phi_2) = z_2\), where \(z_2\) is the invertible element in \(KL(C^*(E_1)[1,k-1], C^*(E_2)[1,k-1])\) induced by \(r^{[1,k-1]}_X(x)\). By Elliott’s classification [19], there exists an isomorphism \(\phi_0 : C^*(E_1)[k,n] \otimes K \to C^*(E_2)[k,n] \otimes K\) such that \(KK(\phi_0) = z_0\), where \(z_0\) is the invertible element in \(KK(C^*(E_1)[k,n] \otimes K, C^*(E_2)[k,n] \otimes K)\) induced by \(r^{[k,n]}_X(x)\).

Consider \(C^*(E_i)\) as a \(C^*\)-algebra over \(X_2\) by setting \(C^*(E_i)[2] = C^*(E_i)[k,n]\) and \(C^*(E_i)[1,2] = C^*(E_i)\). Let \(y\) be the invertible element in \(KK(X_2, C^*(E_1), C^*(E_2))\) induced by \(x\). Note that \(KL(r^{[1]}_{X_2}(y)) = z_2 = KL(\phi_2)\) and \(r^{[2]}_{X_2}(y) = z_0 = KK(\phi_0)\). By Theorem 3.7 of [14],

\[
r^{[1]}_{X_2}(y) \times [\tau_{e_2}] = [\tau_{e_1}] \times r^{[2]}_{X_2}(y)
\]
in \(KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K)\), where \(e_i\) is the extension

\[
0 \to C^*(E_i)[k,n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1,k-1] \otimes K \to 0.
\]

Thus,

\[
KL(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KL(\phi_0)
\]
in \(KL^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K)\). Since \(KL^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) = KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K)\),

\[
KK(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KK(\phi_0)
\]
in \(KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K)\). By Lemma 4.5 of [14], \(C^*(E_1) \otimes K \cong C^*(E_2) \otimes K\).

**Theorem 5.8.** Let \(E_1\) and \(E_2\) be graphs such that \(C^*(E_i)\) is a tight \(C^*\)-algebra over \(X_3\). Suppose \(K_0(C^*(E_1)[1])\) is the direct sum of cyclic groups if \(C^*(E_1)[1]\) is purely infinite and \(K_0(C^*(E_1)[1,2])\) is the direct sum of cyclic groups if \(C^*(E_1)[1]\) is an AF-algebra. Then \(C^*(E_1) \otimes K \cong C^*(E_2) \otimes K\) if and only if \(K^+_X(C^*(E_1)) \cong K^+_X(C^*(E_2))\).

**Proof.** The “only if” direction is clear. Suppose \(K^+_X(C^*(E_1)) \cong K^+_X(C^*(E_2))\). Suppose \(C^*(E_1)[1]\) is purely infinite. Then \(K_0(C^*(E_1)[1])\) is the direct sum of cyclic groups. Thus, \(\text{Pext}^1_2(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0\). Since \(K_1(C^*(E_1)[1])\) is a free group, \(\text{Pext}^1_2(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0\).

\[
KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).
\]

Suppose \(C^*(E_1)[1]\) is an AF-algebra. Then \(K_0(C^*(E_1)[1,2])\) is the direct sum of cyclic groups. Thus, \(\text{Pext}^1_2(K_0(C^*(E_1)[1,2]), K_0(C^*(E_2)[3])) = 0\). Since \(K_1(C^*(E_1)[1,2])\) is a free group, \(\text{Pext}^1_2(K_1(C^*(E_1)[1,2]), K_1(C^*(E_2)[3])) = 0\). Therefore,

\[
KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).
\]

**Case 1:** Suppose the finite and infinite simple sub-quotients of \(C^*(E_1)\) are separated. Then the finite and infinite simple sub-quotients of \(C^*(E_2)\) are separated. Hence, by Theorem 6.9 of [14], \(C^*(E_1) \otimes K \cong C^*(E_2) \otimes K\).

**Case 2:** Suppose the finite and infinite simple sub-quotients of \(C^*(E_1)\) are not separated. Then the finite and infinite simple sub-quotients of \(C^*(E_2)\) are not separated.
Subcase 2.1: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are purely infinite and $C^*(E_1)[2]$ is an AF-algebra. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are purely infinite and $C^*(E_2)[2]$ is an AF-algebra. Then by the above paragraph we have that $KK^1(C^*(E_1)[1], C^*(E_2)[2, 3]) = KL^1(C^*(E_1)[1, 2], C^*(E_2)[2, 3])$. Hence, by Theorem 5.6, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

Subcase 2.2: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are AF-algebras and $C^*(E_1)[2]$ is purely infinite. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are AF-algebras and $C^*(E_2)[2]$ is purely infinite. Then by the above paragraph we have that

$$KK^1(C^*(E_1)[1, 2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1, 2], C^*(E_2)[3]).$$

Hence, by Theorem 5.3, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

\[ \square \]

**Corollary 5.9.** Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_i$. Suppose that $K_0(C^*(E_i))$ is finitely generated. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K^+_X(C^*(E_1)) \cong K^+_X(C^*(E_2))$.

**Proof.** Since $C^*(E_i)$ is real rank zero, the canonical projection $\pi : C^*(E_i) \rightarrow C^*(E_i)[1]$ induces a surjective homomorphism $\pi : K_0(C^*(E_i)) \rightarrow K_0(C^*(E_i)[1])$. Hence, $K_0(C^*(E_i)[1])$ is finitely generated since $K_0(C^*(E_i))$ is finitely generated. The corollary now follows from Theorem 5.8. \[ \square \]

**References**


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