STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE C*-ALGEBRAS

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Abstract. We show that certain extensions of classifiable C*-algebra are strongly classified by the associated six-term exact sequence in K-theory together with the positive cone of K_0-groups of the ideal and quotient. We apply our result to give a complete classification of graph C*-algebras with exactly one ideal.

1. Introduction

The classification program for C*-algebras has for the most part progressed independently for the classes of infinite and finite C*-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered K_0-group. In the infinite case, there are a number of results for purely infinite C*-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite C*-algebras up to stable isomorphism by the K_0-group together with the K_1-group. For nonsimple purely infinite C*-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite C*-algebras with exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of K-groups [34], the second named author has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered K-theory [31, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite C*-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated K-theory [28, Theorem 4.14]. However, in all of these situations the nonsimple C*-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Recently, the authors have provided a framework for classifying nonsimple C*-algebras that are not necessarily AF-algebras or purely infinite C*-algebras. In particular, the authors have shown in [16] that certain extensions of classifiable C*-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. This has allowed for the classification of certain nonsimple C*-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [16] was then used by the first named author and Tomforde in [18] to classify a certain class of non-simple graph C*-algebras, showing that graph C*-algebras with exactly one non-trivial ideal can be classified up to stable isomorphism by their associated six-term exact sequence in K-theory. The authors in [15] then showed that all non-unital graph C*-algebras with exactly one
non-trivial ideal can be classified up to isomorphism by their associated six-term exact sequence in $K$-theory. In this paper, we complete the classification of graph $C^\ast$-algebras with exactly one non-trivial ideal by classifying those that are unital. Our methods here differ rather dramatically from the methods in [13] and [15]. In particular, we use the traditional methods of classification via existence and uniqueness theorems. As a consequence, for unital graph $C^\ast$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with exactly one non-trivial ideal, then any isomorphism between the associated six-term exact sequence in $K$-theory which preserves the unit lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

2. Preliminaries

2.1. $C^\ast$-algebras over topological spaces. Let $X$ be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of $X$, partially ordered by set inclusion $\subseteq$. A subset $Y$ of $X$ is called locally closed if $Y = U \setminus V$ where $U, V \in \mathcal{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathbb{L}(X)$. The set of all connected, non-empty, locally closed subsets of $X$ will be denoted by $\mathbb{C}(X)^\ast$.

The partially ordered set $(\mathcal{O}(X), \subseteq)$ is a complete lattice, that is, any subset $S$ of $\mathcal{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset $S$ of $\mathcal{O}(X)$,

$$\bigwedge_{U \in S} U = \left( \bigcap_{U \in S} U \right)$$

and

$$\bigvee_{U \in S} U = \bigcup_{U \in S} U.$$

For a $C^\ast$-algebra $\mathfrak{A}$, let $\mathbb{l}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathbb{l}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset $S$ of $\mathbb{l}(\mathfrak{A})$,

$$\bigwedge_{\mathfrak{J} \in S} \mathfrak{J} = \bigcap_{\mathfrak{J} \in S} \mathfrak{J}$$

and

$$\bigvee_{\mathfrak{J} \in S} \mathfrak{J} = \bigvee_{\mathfrak{J} \in S} \mathfrak{J}.$$

Definition 2.1. Let $\mathfrak{A}$ be a $C^\ast$-algebra. Let $\text{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let $X$ be a topological space. A $C^\ast$-algebra over $X$ is a pair $(\mathfrak{A}, \psi)$ consisting of a $C^\ast$-algebra $\mathfrak{A}$ and a continuous map $\psi : \text{Prim}(\mathfrak{A}) \to X$. A $C^\ast$-algebra over $X$, $(\mathfrak{A}, \psi)$, is separable if $\mathfrak{A}$ is a separable $C^\ast$-algebra. We say that $(\mathfrak{A}, \psi)$ is tight if $\psi$ is a homeomorphism.

We always identify $\mathcal{O}(\text{Prim}(\mathfrak{A}))$ and $\mathbb{l}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p.$$

Let $(\mathfrak{A}, \psi)$ be a $C^\ast$-algebra over $X$. Then we get a map $\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(\mathfrak{A})) \cong \mathbb{l}(\mathfrak{A})$ defined by

$$U \mapsto \{ p \in \text{Prim}(\mathfrak{A}) : \psi(p) \in U \} = \mathfrak{A}(U).$$

For $Y = U \setminus V \in \mathbb{L}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [27], $\mathfrak{A}(Y)$ does not depend on $U$ and $V$.

Example 2.2. For any $C^\ast$-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_{\text{Prim}(\mathfrak{A})})$ is a tight $C^\ast$-algebra over $\text{Prim}(\mathfrak{A})$. For each $U \in \mathcal{O}(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$. 
Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip $X_n$ with the Alexandrov topology, so the non-empty open subsets are
\[ [a, n] = \{x \in X : a \leq x \leq n \}\]
for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathcal{A}, \phi)$ be a $C^*$-algebra over $X_n$. We will use the following notation throughout the paper:
\[ \mathcal{A}[k] = \mathcal{A}([k]), \mathcal{A}[a, b] = \mathcal{A}([a, b]), \text{ and } \mathcal{A}(i, j) = \mathcal{A}[i + 1, j]. \]
Using the above notation we have ideals $\mathcal{A}[a, n]$ such that
\[ \{0\} \leq \mathcal{A}[n] \leq \mathcal{A}[n - 1, n] \leq \cdots \leq \mathcal{A}[2, n] \leq \mathcal{A}[1, n] = \mathcal{A}. \]

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be $C^*$-algebras over $X$. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is $X$-equivariant if $\phi(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ for all $U \subseteq \mathcal{O}(X)$. Hence, for every $Y = U \setminus V$, $\phi$ induces a homomorphism $\phi_Y : \mathcal{A}(Y) \to \mathcal{B}(Y)$. Let $C^*\text{-alg}(X)$ be the category whose objects are $C^*$-algebras over $X$ and whose morphisms are $X$-equivariant homomorphisms.

An $X$-equivariant homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a full $X$-equivariant homomorphism if for all $Y \in \mathcal{L}_C(X)$, $\phi_Y(a)$ is norm-full in $\mathcal{B}(Y)$ for all norm-full elements $a \in \mathcal{A}(Y)$, i.e., the closed ideal of $\mathcal{B}(Y)$ generated by $\phi_Y(a)$ is $\mathcal{B}(Y)$ whenever the closed ideal of $\mathcal{A}(Y)$ generated by $a$ is $\mathcal{A}(Y)$.

Remark 2.5. Suppose $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_n$. Then it is clear that $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\phi$ is a $X_n$-equivariant isomorphism.

It is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are tight $C^*$-algebras over $X_2$, then $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism if and only if $\phi$ is an $X_2$-equivariant homomorphism and $\phi_{[1]}$ and $\phi_{[2]}$ are injective. Also, if $\mathcal{A}$ and $\mathcal{A}[2]$ have non-zero projections $p$ and $q$ respectively, then there exists $\epsilon > 0$ such that if $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism and $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that
\[ \|\phi(p) - \psi(p)\| < 1 \quad \|\phi(q) - \psi(q)\| < 1, \]
then $\psi$ is a full $X_2$-equivariant homomorphism.

Remark 2.6. Let $\varepsilon_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathcal{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathcal{E}_i$ can be considered as a $C^*$-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathcal{B}_i$ in $\mathcal{E}_i$, and $\{1, 2\}$ to $\mathcal{E}_i$. Hence, there exists a one-to-one correspondence between $X_2$-equivariant homomorphisms $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and homomorphisms from $\varepsilon_1$ and $\varepsilon_2$.

2.2. The ideal related $K$-theory of $\mathcal{A}$.

Definition 2.7. Let $X$ be a topological space and let $\mathcal{A}$ be a $C^*$-algebra over $X$. For open subsets $U_1, U_2, U_3$ of $X$ with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_1 \in \mathcal{L}_C(X)$. Then the diagram
\[
\begin{array}{cccc}
K_0(\mathcal{A}(Y_1)) & \xrightarrow{i_*} & K_0(\mathcal{A}(Y_2)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}(Y_3)) \\
\downarrow{\partial_*} & & & & \downarrow{\partial_*} \\
K_1(\mathcal{A}(Y_3)) & \xrightarrow{\pi_*} & K_1(\mathcal{A}(Y_2)) & \xrightarrow{i_*} & K_1(\mathcal{A}(Y_1))
\end{array}
\]
is an exact sequence. The *ideal related $K$-theory of $\mathfrak{A}$, $K_X(\mathfrak{A})$, is the collection of all $K$-groups thus occurring and the natural transformations $\{\iota_*, \pi_*, \partial_*\}$. The *ideal related, ordered $K$-theory of $\mathfrak{A}$, $K^X_\Delta(\mathfrak{A})$, is $K_X(\mathfrak{A})$ of $\mathfrak{A}$ together with $K_0(\mathfrak{A}(Y))_+$ for all $Y \in \mathcal{L}C(X)$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be $C^*$-algebras over $X$, we will say that $\alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B})$ is an isomorphism if for all $Y \in \mathcal{L}C(X)$, there exists a graded group isomorphism

$$\alpha_{Y,*} : K_*(\mathfrak{A}(Y)) \to K_*(\mathfrak{B}(Y))$$

preserving all natural transformations. We say that $\alpha : K_X^\Delta(\mathfrak{A}) \to K_X^\Delta(\mathfrak{B})$ is an isomorphism if there exists an isomorphism $\alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B})$ in such a way that $\alpha_{Y,0}$ is an order isomorphism for all $Y \in \mathcal{L}C(X)$.

**Remark 2.8.** Meyer-Nest in [28] defined a similar functor $FK_X(-)$ which they called filtrated $K$-theory. For all known cases in which there exists a UCT, the natural transformation from $FK_X(-)$ to $K_X(-)$ is an equivalence. In particular, this is true for the space $X_n$.

If $Y \in \mathcal{L}C(X)$ such that $Y = Y_1 \sqcup Y_2$ with two disjoint relatively open subsets $Y_1, Y_2 \in \mathcal{O}(Y) \subseteq \mathcal{L}C(X)$, then $\mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \oplus \mathfrak{A}(Y_2)$ for any $C^*$-algebra over $X$. Moreover, there is a natural isomorphism $K_*(\mathfrak{A}(Y))$ to $K_*(\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2))$ which is a positive isomorphism from $K_0(\mathfrak{A}(Y))$ to $K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2))$. If $X$ is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace $\mathcal{L}C(X)$ by the subset $\mathcal{L}C(X)^*$.

**Notation 2.9.** Let $\mathcal{N}$ be the bootstrap category of Rosenberg and Schochet in [37].

Let $\mathfrak{R}(X)$ be the category whose objects are separable $C^*$-algebras over $X$ and the set of morphisms is $KK(X; \mathfrak{A}, \mathfrak{B})$. For a finite topological space $X$, let $\mathcal{B}(X) \subseteq \mathfrak{R}(X)$ be the bootstrap category of Meyer and Nest in [27]. By Corollary 4.13 of [27], if $\mathfrak{A}$ is a nuclear $C^*$-algebra over $X$, then $\mathfrak{A} \in \mathcal{B}(X)$ if and only if $\mathfrak{A}(\{x\}) \in \mathcal{N}$ for all $x \in X$.

**Theorem 2.10.** (Bonkat [4] and Meyer-Nest [28]) Let $\mathfrak{A}$ and $\mathfrak{B}$ be in $\mathfrak{R}(X_n)$ such that $\mathfrak{A}$ is in $\mathcal{B}(X_n)$, then the sequence

$$0 \to \text{Ext}_\mathcal{N}^1(KX_n(\mathfrak{A})[1], FX_n(\mathfrak{B})) \xrightarrow{\delta} KK(X_n; \mathfrak{A}, \mathfrak{B}) \xrightarrow{\Gamma} \text{Hom}_\mathcal{N}(FX_n(\mathfrak{A}), FX_n(\mathfrak{B})) \to 0$$

is exact. Consequently, if $\mathfrak{B}$ is in $\mathcal{B}(X_n)$, then an isomorphism from $FK_{X_n}(\mathfrak{A})$ to $FK_{X_n}(\mathfrak{B})$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

**Corollary 2.11.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be in $\mathcal{B}(X_n)$. Then an isomorphism from $K_{X_n}(\mathfrak{A})$ to $K_{X_n}(\mathfrak{B})$ lifts to an invertible element in $KK(X_n; \mathfrak{A}, \mathfrak{B})$.

**Proof.** This follows from Remark 2.8 and Theorem 2.10 $\square$

**Remark 2.12.** Let $x \in KK(X_n; \mathfrak{A}, \mathfrak{B})$ be an invertible element. Then $K_{X_n}(x)$ will denote the isomorphism from $K_{X_n}(\mathfrak{A})$ to $K_{X_n}(\mathfrak{B})$ given by $\Gamma(x)$ where we have identified $K_{X_n}(\mathfrak{A})$ with $FK_{X_n}(\mathfrak{A})$ and $K_{X_n}(\mathfrak{B})$ with $FK_{X_n}(\mathfrak{B})$.

### 2.3. Functors

We now define some functors that will be used throughout the rest of the paper. Let $X$ and $Y$ be topological spaces. For every continuous function $f : X \to Y$ we have a functor

$$f : \mathcal{C}^*\text{-alg}(X) \to \mathcal{C}^*\text{-alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi).$$
(1) Define $g_X^1 : X \to X_1$ by $g_X^1(x) = 1$. Then $g_X^1$ is continuous. Note that the induced functor $g_X^1 : \mathfrak{C}^*\mathrm{-alg}(X) \to \mathfrak{C}^*\mathrm{-alg}(X_1)$ is the forgetful functor.

(2) Let $U$ be an open subset of $X$. Define $g_{U,X}^2 : X \to X_2$ by $g_{U,X}^2(x) = 1$ if $x \notin U$ and $g_{U,X}^2(x) = 2$ if $x \in U$. Then $g_{U,X}^2$ is continuous. Thus the induced functor

$$g_{U,X}^2 : \mathfrak{C}^*\mathrm{-alg}(X) \to \mathfrak{C}^*\mathrm{-alg}(X_2)$$

is just specifying the extension $0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0$.

(3) We can generalize (2) to finitely many ideals. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X$ be open subsets of $X$. Define $g_{U_1,\ldots,U_n,X}^n : X \to X_n$ by $g_{U_1,\ldots,U_n,X}^n(x) = n - k + 1$ if $x \in U_k \setminus U_{k-1}$. Then $g_{U_1,\ldots,U_n,X}^n$ is continuous. Therefore, any $C^*$-algebra with ideals $0 \subseteq \mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \cdots \subseteq \mathfrak{I}_n = \mathfrak{A}$ can be made into a $C^*$-algebra over $X_n$.

(4) For all $Y \in \mathfrak{C}(X)$, $r_Y^X : \mathfrak{C}^*\mathrm{-alg}(X) \to \mathfrak{C}^*\mathrm{-alg}(Y)$ is the restriction functor defined in Definition 2.19 of [27].

(5) If $f : X \to Y$ is an embedding of a subset with the subspace topology, we write

$$i_X^Y = f_* : \mathfrak{C}^*\mathrm{-alg}(X) \to \mathfrak{C}^*\mathrm{-alg}(Y).$$

By Proposition 3.4 of [27], the functors defined above induce functors from $\mathfrak{R}(X)$ to $\mathfrak{R}(Z)$, where $Z = Y, X_1, X_n$.

2.4. Graph $C^*$-algebras. A graph $(E^0, E^1, r, s)$ consists of a countable set $E^0$ of vertices, a countable set $E^1$ of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. If $E$ is a graph, the graph $C^*$-algebra $C^*_r(E)$ is the universal $C^*$-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ with mutually orthogonal ranges satisfying

1. $s_e^*s_e = p_{r(e)}$ for all $e \in E^1$
2. $s_es_e^* \leq p_{s(e)}$ for all $e \in E^1$
3. $p_v = \sum_{e \in E^1 : s(e) = v} s_es_e^*$ for all $v$ with $0 < |s^{-1}(v)| < \infty$.

3. Meta-theorems

In many cases one can obtain a classification result for a class of unital $C^*$-algebras $\mathcal{C}$ by obtaining a classification result for the class $\mathcal{C} \otimes \mathbb{K}$, where each object in $\mathcal{C} \otimes \mathbb{K}$ is the stabilization of an object in $\mathcal{C}$. A meta-theorem of this sort was proved by the first and second named authors in [13] Theorem 11. It was shown there that if $\mathcal{C}$ is a subcategory of the category of $C^*$-algebras, $\mathfrak{C}^*\mathrm{-alg}$, and if $F$ is a functor from $\mathcal{C}$ to an abelian category such that an isomorphism $F(\mathfrak{A} \otimes \mathbb{K}) \cong F(\mathfrak{B} \otimes \mathbb{K})$ lifts to an isomorphism in $\mathfrak{C}^*\mathrm{-alg}$, then under suitable conditions, we have that $F(\mathfrak{A}) \cong F(\mathfrak{B})$ implies $\mathfrak{A} \cong \mathfrak{B}$. In [31], the second and third named authors improved this result by showing that the isomorphism $F(\mathfrak{A}) \cong F(\mathfrak{B})$ lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

In this section, we improve these results in order to deal with cases when $\mathcal{C}$ is a category (not necessarily a subcategory of $\mathfrak{C}^*\mathrm{-alg}$) and there exists a functor from $\mathcal{C}$ to $\mathfrak{C}^*\mathrm{-alg}$. An example of such a category is the category of $C^*$-algebras over $\{1,2\}$, where $\{1,2\}$ is given the discrete topology. Then $\mathcal{C}$ is not a subcategory of $\mathfrak{C}^*\mathrm{-alg}$ but the forgetful functor (forgetting the $\{1,2\}$-structure) is a functor from $\mathcal{C}$ to $\mathfrak{C}^*\mathrm{-alg}$. We also replace the condition of proper pure infiniteness by the stable weak cancellation property.
Definition 3.1. A $C^*$-algebra $\mathfrak{A}$ is said to have the weak cancellation property if $p$ is Murray-von Neumann equivalent to $q$ whenever $p$ and $q$ generate the same ideal $\mathcal{I}$ and $[p] = [q]$ in $K_0(\mathcal{I})$. A $C^*$-algebra is said to have the stable weak cancellation property if $M_n(\mathfrak{A})$ has the weak cancellation property for all $n \in \mathbb{N}$.

Theorem 3.2. (cf. [13, Theorem 11]) Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $\mathcal{C}^*\text{-alg}$ be the category of $C^*$-algebras, and let $\text{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F: \mathcal{C} \to \mathcal{C}^*\text{-alg}$, $G: \mathcal{C} \to \mathcal{D}$, and $H: \mathcal{D} \to \text{Ab}$ such that

1. $H \circ G = K_0 \circ F$.

2. For objects $\mathfrak{A}$ in $\mathcal{C}$, there exist an object $\mathfrak{A}_K$ and a morphism $\kappa_\mathfrak{A}: \mathfrak{A} \to \mathfrak{A}_K$ such that $G(\kappa_\mathfrak{A})$ is an isomorphism in $\mathcal{D}$, $F(\mathfrak{A}_K) = F(\mathfrak{A}) \otimes \mathbb{K}$, and $F(\kappa_\mathfrak{A}) = \text{id}_{F(\mathfrak{A})} \otimes e_{11}$.

3. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$, every isomorphism $G(\kappa_\mathfrak{A})$ to $G(\mathfrak{B}_K)$ is induced by an isomorphism from $\mathfrak{A}_K$ to $\mathfrak{B}_K$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be given such that $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are unital $C^*$-algebras. Let $\rho: G(\mathfrak{A}) \to G(\mathfrak{B})$ be an isomorphism such that $H(\rho)\left([1_{F(\mathfrak{A})}]\right) = [1_{F(\mathfrak{B})}]$. If $F(\mathfrak{B})$ has the stable weak cancellation property, then $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Proof. Note that $G(\kappa_\mathfrak{A})$ and $G(\kappa_\mathfrak{B})$ are isomorphisms. Therefore $G(\kappa_\mathfrak{A}) \circ \rho \circ G(\kappa_\mathfrak{B})^{-1}$ is an isomorphism from $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$. Thus, there exists an isomorphism $\phi: \mathfrak{A}_K \to \mathfrak{B}_K$ such that $G(\phi) = G(\kappa_\mathfrak{A}) \circ \rho \circ G(\kappa_\mathfrak{B})^{-1}$.

Set $\psi = F(\phi)$. Then $\psi: F(\mathfrak{A}) \otimes \mathbb{K} \to F(\mathfrak{B}) \otimes \mathbb{K}$ is a $*$-isomorphism such that

$$K_0(\psi) = K_0(F(\phi)) = H(G(\kappa_\mathfrak{A}) \circ \rho \circ G(\kappa_\mathfrak{B})^{-1}) = H(G(\kappa_\mathfrak{B})) \circ H(\rho) \circ H(G(\kappa_\mathfrak{B})^{-1}) = K_0(F(\kappa_\mathfrak{B})) \circ H(\rho) \circ K_0(F(\kappa_\mathfrak{A}))^{-1} = K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}.$$

Hence,

$$K_0(\psi)([1_{F(\mathfrak{A})} \otimes e_{11}]) = K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}([1_{F(\mathfrak{A})} \otimes e_{11}])$$

$$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathfrak{A})}])$$

$$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11})([1_{F(\mathfrak{B})}])$$

$$= [1_{F(\mathfrak{A})} \otimes e_{11}].$$

Stable weak cancellation implies that there exists $v \in F(\mathfrak{B}) \otimes \mathbb{K}$ such that $v^*v = \psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathfrak{B})} \otimes e_{11}$ since $\psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $1_{F(\mathfrak{B})} \otimes e_{11}$ are full projections in $F(\mathfrak{B}) \otimes \mathbb{K}$. Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [13, Theorem 11], $\gamma$ is an isomorphism from $F(\mathfrak{A}) \otimes e_{11}$ to $F(\mathfrak{B}) \otimes e_{11}$. Hence, $F(\mathfrak{A}) \cong F(\mathfrak{B})$. \qed

Theorem 3.3. (cf. [32, Theorem 2.1]) Let $\mathcal{C}$ be a subcategory of $\mathcal{C}^*\text{-alg}(X)$. Moreover, $\mathcal{C}$ is assumed to be closed under tensoring by $M_2(\mathbb{C})$ and $\mathbb{K}$ and contains the canonical embeddings $\kappa_1: \mathfrak{A} \to M_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ as morphisms for every object $\mathfrak{A}$ in $\mathcal{C}$. Assume there is a functor $F: \mathcal{C} \to \mathcal{D}$ satisfying

1. For $\mathfrak{A}$ in $\mathcal{C}$, the embeddings $\kappa_1: \mathfrak{A} \to M_2(\mathfrak{A})$ and $\kappa: \mathfrak{A} \to \mathfrak{A} \otimes \mathbb{K}$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.

2. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $\mathcal{C}$ that are stable $C^*$-algebras, every isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

3. There exists a functor $G$ from $\mathcal{D}$ to $\text{Ab}$ such that $G \circ F = K_0$. 

□
Assume that every \( X \)-equivariant isomorphism between objects in \( \mathcal{C} \) is a morphism in \( \mathcal{C} \) and that for objects \( \mathfrak{A} \) in \( \mathcal{C} \), \( F(\text{Ad}(u)|_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})} \) for every unitary \( u \in \mathcal{M}(\mathfrak{A}) \). If \( \mathfrak{A} \) and \( \mathfrak{B} \) are objects \( \mathcal{C} \) that are unital \( C^* \)-algebras such that \( \mathfrak{A} \) and \( \mathfrak{B} \) have the stable weak cancellation property and there is an isomorphism \( \alpha : F(\mathfrak{A}) \to F(\mathfrak{B}) \) such that \( G(\alpha)([1_{\mathfrak{A}}]) = [1_{\mathfrak{B}}] \), then there exists an isomorphism \( \phi : \mathfrak{A} \to \mathfrak{B} \) in \( \mathcal{C} \) such that \( F(\phi) = \alpha \).

**Proof.** The difference between the statement of Theorem 2.1 of [32] and statement of the theorem are

(i) \( \mathcal{C} \) is assumed to be a subcategory of \( \mathcal{C}^*\text{-alg}(X) \) instead of a subcategory of \( \mathcal{C}^*\text{-alg} \).

(ii) \( \mathfrak{A} \) and \( \mathfrak{B} \) are assumed to have the stable weak cancellation property instead of being properly infinite.

In the proof of Theorem 2.1 of [32], properly infinite was needed to insure that \( \psi(1_{\mathfrak{A}} \otimes e_{11}) \) is Murray-von Neumann equivalent to \( 1_{\mathfrak{B}} \otimes e_{11} \), where \( \psi : \mathfrak{A} \otimes \mathfrak{K} \to \mathfrak{B} \otimes \mathfrak{K} \) is the isomorphism from (2) that lifts the isomorphism from \( F(\mathfrak{A}) \) to \( F(\mathfrak{B}) \) that is induced by \( \alpha \). As in the proof of Theorem 3.2 we get that \( \psi(1_{\mathfrak{A}} \otimes e_{11}) \) is Murray-von Neumann equivalent to \( 1_{\mathfrak{B}} \otimes e_{11} \). Arguing as in the proof of Theorem 2.1 of [32], we get the desired result. \( \Box \)

4. Classification results

In this section, we show that \( K_{X_2}^+(-) \) is a strong classification functor for a class of \( C^* \)-algebras with exactly one proper nontrivial ideal containing \( C^* \)-algebras associated to finite graphs. The results of this section will be used in the next section to show that \( K_{X_3}^+(-) \) together with the appropriate scale is a complete isomorphism invariant for \( C^* \)-algebras associated to graphs. Moreover, in a forthcoming paper, we use these results to solve the following extension problem: If \( \mathfrak{A} \) fits into the following exact sequence

\[
0 \to C^*(E) \otimes \mathfrak{K} \to \mathfrak{A} \to C^*(G) \to 0,
\]

where \( C^*(E) \) and \( C^*(G) \) are simple \( C^* \)-algebras, then when is \( \mathfrak{A} \cong C^*(F) \) for some graph \( F \)?

**Theorem 4.1.** (Existence Theorem) Let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be in \( \mathcal{B}(X_2) \) and let \( x \in \mathcal{K}(X_2; \mathfrak{A}_1; \mathfrak{A}_2) \) be an invertible element such that \( \Gamma(x)_Y \) is a positive isomorphism for all \( Y \in \mathbb{L} \mathcal{C}(X_2) \).

Suppose \( 0 \to \mathfrak{A}_1[2] \to \mathfrak{A}_1 \to \mathfrak{A}_1[1] \to 0 \) is a full extension, \( \mathfrak{A}_1[2] \) is a stable \( C^* \)-algebra, \( \mathfrak{A}_1 \) is a nuclear \( C^* \)-algebra with real rank zero, and either

(i) \( \mathfrak{A}_1[2] \) is a purely infinite simple \( C^* \)-algebra and \( \mathfrak{A}_1[1] \) is an \( AF \)-algebra; or

(ii) \( \mathfrak{A}_1[2] \) is an \( AF \)-algebra and \( \mathfrak{A}_1[1] \) is a purely infinite simple \( C^* \)-algebra.

Then there exists an \( X_2 \)-equivariant homomorphism \( \phi : \mathfrak{A}_1 \otimes \mathfrak{K} \to \mathfrak{A}_2 \otimes \mathfrak{K} \) such that

\[
\mathcal{K}(X_2; \phi) = \mathcal{K}(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times \mathcal{K}(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11}),
\]

and \( \phi[2] \) and \( \phi[1] \) are injective, where \( \{e_{ij}\} \) is a system of matrix units for \( \mathfrak{K} \).

**Proof.** Set \( y = \mathcal{K}(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times \mathcal{K}(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11}) \). Note that by Lemma 3.10 and Theorem 3.8 of [14], \( \mathfrak{A}_1[2] \otimes \mathfrak{K} \) satisfies the corona factorization property (see [21] for the definition of the corona factorization property). Since \( \mathfrak{A}_1[k] \) is an \( AF \)-algebra or an Kirchberg algebra, \( \mathfrak{A}_1[k] \) has the stable weak cancellation. By Lemma 3.15 of [15], \( \mathfrak{A}_1 \) has stable weak cancellation. Let \( \mathfrak{c} \) be the extension

\[
0 \to \mathfrak{A}_1[2] \otimes \mathfrak{K} \to \mathfrak{A}_1 \otimes \mathfrak{K} \to \mathfrak{A}_1[1] \otimes \mathfrak{K} \to 0.
\]
By Corollary 3.24 of [15], $\epsilon_i$ is a full extension since $\mathcal{A}_i[1]$ has cancellation of projections (in the AF case) and $\mathcal{A}_i[1]$ is properly infinite (in the purely infinite case).

Case (i): $\mathcal{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathcal{A}_i[1]$ is an AF-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) \times (\tau_{e_2}) = [\tau_{e_2}] \times r_{X_2}^{(2)}(y)$ in $KK^1(\mathcal{A}_i[1] \otimes \mathbb{K}, \mathcal{A}_i[2] \otimes \mathbb{K})$. Since $y$ is invertible in $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{(1)}(y)$ is invertible in $KK(\mathcal{A}_i[1] \otimes \mathbb{K}, \mathcal{A}_2[1] \otimes \mathbb{K})$ and $\Gamma(r_{X_2}^{(1)}(y)) = \Gamma(x)_{\{1\}}$ is a positive isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_1 : \mathcal{A}_i[1] \otimes \mathbb{K} \rightarrow \mathcal{A}_2[1] \otimes \mathbb{K}$ such that $KK(\psi_1) = r_{X_2}^{(1)}(y)$.

Since $y$ is invertible in $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK(\mathcal{A}_i[2] \otimes \mathbb{K}, \mathcal{A}_2[1] \otimes \mathbb{K})$. Thus, by Kirchberg-Phillips classification (see [20] and [29]), there exists an isomorphism $\psi_0 : \mathcal{A}_i[2] \otimes \mathbb{K} \rightarrow \mathcal{A}_2[2] \otimes \mathbb{K}$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$.

By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in \mathcal{M}(\mathcal{A}_2[2] \otimes \mathbb{K})$ such that $\psi = (Ad(u) \circ \psi_0, Ad(u) \circ \psi_0, \psi_1)$ is an $X_2$-equivariant isomorphism from $\mathcal{A}_i \otimes \mathbb{K}$ to $\mathcal{A}_2 \otimes \mathbb{K}$, where $\psi_0 : \mathcal{M}(\mathcal{A}_i[2] \otimes \mathbb{K}) \rightarrow \mathcal{M}(\mathcal{A}_i[1] \otimes \mathbb{K})$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_{\{1\}}) = r_{X_2}^{(k)}(y)$ for $k = 1, 2$.

Note that

$$0 \rightarrow i_{\{2\}}^X((\mathcal{A}_i \otimes \mathbb{K})[2]) \xrightarrow{\lambda_2} \mathcal{A}_1 \otimes \mathbb{K} \xrightarrow{\beta_2} i_{\{1\}}^X((\mathcal{A}_i \otimes \mathbb{K})[1]) \rightarrow 0$$

is a semi-split extension of $C^*$-algebras over $X_2$ (see Definition 3.5 of [27]). Set

$$\mathcal{I}_i = i_{\{2\}}^X((\mathcal{A}_i \otimes \mathbb{K})[2]) \quad \text{and} \quad \mathcal{B}_i = i_{\{1\}}^X((\mathcal{A}_i \otimes \mathbb{K})[1]).$$

By Theorem 3.6 of [27] (see also Korollar 3.4.6 of [4]),

$$KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{I}_2) \xrightarrow{(\lambda_2)_*} KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K}) \xrightarrow{(\beta_2)_*} KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{B}_2)$$

is exact. By Proposition 3.12 of [27], $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{B}_2)$ and $KK(\mathcal{A}_i[1] \otimes \mathbb{K}, \mathcal{A}_2[1] \otimes \mathbb{K})$ are naturally isomorphic. Hence, there exists $z \in KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{I}_2)$ such that $y - KK(X_2, \psi) = z \times KK(X_2; \lambda_2)$ since $KK(\psi_{\{1\}}) = r_{X_2}^{(1)}(y)$.

By Proposition 3.13 of [27], $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{I}_2)$ and $KK(\mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})[2]$ are isomorphic. By Theorem 8.3.3 of [36] (see also Hauptsatz 4.2 of [20]), there exists a $*$-homomorphism $\eta : \mathcal{A}_i \otimes \mathbb{K} \rightarrow (\mathcal{A}_2 \otimes \mathbb{K})[2]$ such that $KK(\eta) = \tau$, where $\tau$ is the image of $z$ under the isomorphism $KK(X_2; \mathcal{A}_i \otimes \mathbb{K}, \mathcal{I}_2) \cong KK(\mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})[2]$. Note that $\eta$ induces an $X_2$-equivariant homomorphism $\eta : \mathcal{A}_i \otimes \mathbb{K} \rightarrow \mathcal{I}_2$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\lambda_2 \circ \eta)$, where the sum is the Cuntz sum in $\mathcal{M}(\mathcal{A}_2 \otimes \mathbb{K})$. Then $\phi : \mathcal{A}_i \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ is an $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective homomorphisms.

Case (ii): $\mathcal{A}_i[2]$ is an AF-algebra and $\mathcal{A}_i[1]$ is a purely infinite simple $C^*$-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) \times (\tau_{e_2}) = [\tau_{e_2}] \times r_{X_2}^{(2)}(y)$ in $KK^1(\mathcal{A}_i[1] \otimes \mathbb{K}, \mathcal{A}_2[2] \otimes \mathbb{K})$. Since $y$ is invertible in $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK(\mathcal{A}_i[2] \otimes \mathbb{K}, \mathcal{A}_2[2] \otimes \mathbb{K})$ and $\Gamma(r_{X_2}^{(2)}(y)) = \Gamma(x)_{\{2\}}$ is an order isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_0 : \mathcal{A}_i[2] \otimes \mathbb{K} \rightarrow \mathcal{A}_2[2] \otimes \mathbb{K}$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$.

Since $y$ is invertible in $KK(X_2, \mathcal{A}_i \otimes \mathbb{K}, \mathcal{A}_2 \otimes \mathbb{K})$, we have that $r_{X_2}^{(1)}(y)$ is invertible in
there exists an isomorphism \( \psi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K} \). Thus, by Kirchberg-Phillips classification (see \([20]\) and \([29]\)), there exists an isomorphism

\[
\psi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}
\]

such that \( KK(\psi_1) = r_{X_2}^{(k)}(y) \). By Lemma 4.5 of \([14]\) and its proof, there exists a unitary \( u \in \mathcal{M}(\mathfrak{A}_2[2] \otimes \mathbb{K}) \) such that \( \psi = (\text{Ad}(u) \circ \psi_0, \text{Ad}(u) \circ \widetilde{\psi}_0, \psi_1) \) is an \( X_2 \)-equivariant isomorphism from \( \mathfrak{A}_1 \otimes \mathbb{K} \) to \( \mathfrak{A}_2 \otimes \mathbb{K} \), where \( \psi_0 : \mathcal{M}(\mathfrak{A}_1[2] \otimes \mathbb{K}) \to \mathcal{M}(\mathfrak{A}_1[2] \otimes \mathbb{K}) \) is the unique isomorphism extending \( \psi_0 \). Note that \( KK(\psi_{\{k\}}) = r_{X_2}^{(k)}(y) \) for \( k = 1, 2 \).

Note that

\[
0 \to i^{X_2}_{\{2\}}((\mathfrak{A}_1 \otimes \mathbb{K})[2]) \xrightarrow{\lambda} \mathfrak{A}_1 \otimes \mathbb{K} \xrightarrow{\beta} i^{X_2}_{\{1\}}((\mathfrak{A}_1 \otimes \mathbb{K})[1]) \to 0
\]

is a semi-split extension of \( C^* \)-algebras over \( X_2 \) (see Definition 3.5 of \([27]\)). Set

\[
\mathfrak{J}_i = i^{X_2}_{\{2\}}((\mathfrak{A}_i \otimes \mathbb{K})[2]) \quad \text{and} \quad \mathfrak{B}_i = i^{X_2}_{\{1\}}((\mathfrak{A}_i \otimes \mathbb{K})[1]).
\]

By Theorem 3.6 of \([27]\) (see also Korollar 3.4.6 \([1]\) )

\[
KK(X_2; \mathfrak{B}_1, \mathfrak{A}_1 \otimes \mathbb{K}) \xrightarrow{(\beta_1)^*} KK(X_2; \mathfrak{A}_1 \otimes \mathbb{K}, \mathfrak{A}_2 \otimes \mathbb{K}) \xrightarrow{(\lambda_1)^*} KK(X_2; \mathfrak{J}_1, \mathfrak{A}_2 \otimes \mathbb{K})
\]

is exact. By Proposition 3.12 of \([27]\), \( KK(X_2; \mathfrak{J}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) and \( KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K}) \) are naturally isomorphic. Hence, there exists \( z \in KK(X_2; \mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) such that \( y - KK(X; \psi) = KK(X_2; \beta_1) \times z \) since \( KK(\psi_{\{2\}}) = r_{X_2}^{(2)}(y) \). By Proposition 3.13 of \([27]\), \( KK(X_2; \mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \) and \( KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K}) \) are isomorphic. Therefore, by Theorem 8.3.3 of \([36]\), there exists a homomorphism \( \eta : (\mathfrak{A}_1 \otimes \mathbb{K})[1] \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(\eta) = z \), where \( z \) is the image of \( z \) under the isomorphism \( KK(X_2; \mathfrak{B}_1, \mathfrak{A}_2 \otimes \mathbb{K}) \cong KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K}) \) (the existence of the homomorphism uses the fact that \( \mathfrak{A}_2 \otimes \mathbb{K} \) is a properly infinite \( C^* \)-algebra which follows from Proposition 3.21 and Theorem 3.22 of \([15]\)). Note that \( \eta \) induces an \( X_2 \)-equivariant homomorphism \( \eta : \mathfrak{B}_1 \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(X_2; \eta) = z \).

Set \( \phi = \psi + (\eta \circ \beta_1) \), where the sum is the Cuntz sum in \( \mathcal{M}(\mathfrak{A}_2 \otimes \mathbb{K}) \). Then \( \phi \) is an \( X_2 \)-equivariant homomorphism such that \( KK(X_2; \phi) = y \). Since \( \psi_{\{2\}} \) and \( \psi_{\{1\}} \) are injective homomorphisms, \( \phi_{\{2\}} \) and \( \phi_{\{1\}} \) are injective homomorphisms.

### 4.1. Strong classification of extensions of AF-algebras by purely infinite \( C^* \)-algebras.

**Definition 4.2.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be separable \( C^* \)-algebras over \( X \). Two \( X \)-equivariant homomorphisms \( \phi, \psi : \mathfrak{A} \to \mathfrak{B} \) are said to be *approximately unitarily equivalent* if there exists a sequence of unitaries \( \{u_n\}_{n=1}^\infty \) in \( \mathcal{M}(\mathfrak{B}) \) such that

\[
\lim_{n \to \infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

We now recall the definition of \( KL(\mathfrak{A}, \mathfrak{B}) \) from \([33]\).

**Definition 4.3.** Let \( \mathfrak{A} \) be a separable, nuclear \( C^* \)-algebra in \( \mathcal{N} \) and let \( \mathfrak{B} \) be a \( \sigma \)-unital \( C^* \)-algebra. Let

\[
\text{Ext}^1_\mathbb{Z}(K_n(\mathfrak{A}), K_{n+1}(\mathfrak{B})) = \text{Ext}^1_\mathbb{Z}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Ext}^1_\mathbb{Z}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).
\]

Since \( \mathfrak{A} \) is in \( \mathcal{N} \), by \([37]\), \( \text{Ext}^1_\mathbb{Z}(K_n(\mathfrak{A}), K_{n+1}(\mathfrak{B})) \) can be identified as a sub-group of the group \( KK(\mathfrak{A}, \mathfrak{B}) \).
For abelian groups, $G$ and $H$, let $\text{Pext}^1_{\mathbb{Z}}(G, H)$ be the subgroup of $\text{Ext}^1_{\mathbb{Z}}(G, H)$ of all pure extensions of $G$ by $H$. Set

$$\text{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \text{Pext}^1_{\mathbb{Z}}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Pext}^1_{\mathbb{Z}}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Define $KL(\mathfrak{A}, \mathfrak{B})$ as the quotient

$$KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})/\text{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})).$$

Rørdam in [33] proved that if $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $KL(\phi) = KL(\psi)$.

**Notation 4.4.** Let $x \in KK(\mathfrak{A}, \mathfrak{B})$. Then the element $x + \text{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ in $KL(\mathfrak{A}, \mathfrak{B})$ will be denoted by $KL(x)$.

A nuclear, purely infinite, separable, simple $C^*$-algebra will be called a Kirchberg algebra.

**Theorem 4.5.** (Uniqueness Theorem 1) Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i$ has real rank zero, $\mathfrak{A}_i$ is stable, $\mathfrak{A}_i[2]$ is a Kirchberg algebra in $N$, $\mathfrak{A}_i[1]$ is an AF-algebra, and $\mathfrak{A}_i[2]$ is an essential ideal of $\mathfrak{A}_i$. Suppose $\phi, \psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \psi)$, and $\phi(1), \psi(1)$, $\phi(2)$, and $\psi(2)$, are injective homomorphisms. Then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** Since $\mathfrak{A}_i[1]$ is an AF algebra, every finitely generated subgroup of $K_0(\mathfrak{A}_i[1])$ is torsion free (hence free) and every finitely generated subgroup of $K_1(\mathfrak{A}_i[1])$ is zero. Thus, $\text{Pext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2]))) = \text{Ext}^1_{\mathbb{Z}}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2])))$ which implies that $KL(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_j[2])) = \text{Hom}(K_*(\mathfrak{A}_i[1]), K_{*+1}(\mathfrak{Q}(\mathfrak{A}_j[2])))$.

Let $e_i$ denote the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$. Since $\mathfrak{A}_i$ has real rank zero and $K_1(\mathfrak{A}_i[1]) = 0$, we have that $K_j(\tau_{e_i}) = 0$, where $\tau_{e_i}$ is the Busby invariant of $e_i$. Hence, $[\tau_{e_i}] = 0$ in $KL(\mathfrak{A}_i[1], \mathfrak{Q}(\mathfrak{A}_j[2]))$. By Corollary 6.7 of [24], $e_i$ is quasi-diagonal. Thus, there exists an approximate identity of $\mathfrak{A}_i[2]$ consisting of projections $\{e_k\}_{k \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_i$.

Since $\mathfrak{A}_i[1]$ is an AF-algebra and $\mathfrak{A}_i$ has real rank zero, as in the proof of Lemma 9.8 of [10], there exists a sequence of finite dimensional sub-$C^*$-algebras $\{\mathfrak{B}_k\}_{k=1}^{\infty}$ of $\mathfrak{A}_1$ such that $\mathfrak{B}_k \cap \mathfrak{A}_i[2] = \{0\}$ and for each $x \in \mathfrak{A}_i$, there exist $y_1 \in \bigcup_{k=1}^{\infty} \mathfrak{B}_k$ and $y_2 \in \mathfrak{A}_i[2]$ such that $x = y_1 + y_2$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_i$. Note that we may assume $\mathcal{F}$ is the union of the generators of $\mathfrak{B}_m$, for some $m \in \mathbb{N}$ and $\mathcal{G}$, for some finite subset $\mathcal{G}$ of $\mathfrak{A}_i[2]$ . Since $\mathfrak{B}_m$ is a finite dimensional $C^*$-algebra,

$$\lim_{k \to \infty} \|e_k x - x e_k\| = 0$$

for all $x \in \mathfrak{A}_i$, and $\{e_k\}_{k \in \mathbb{N}}$ is an approximate identity for $\mathfrak{A}_i[2]$ consisting of projections, there exist $k \in \mathbb{N}$, a finite dimensional sub-$C^*$-algebra $\mathcal{D}$ of $\mathfrak{A}_i$ with $\mathcal{D} \subseteq (1_{\mathfrak{M}(\mathfrak{A}_i)} - e_k)\mathfrak{A}_i(1_{\mathfrak{M}(\mathfrak{A}_i)} - e_k)$ and $\mathcal{D} \cap \mathfrak{A}_i[2] = \{0\}$, and there exists a finite subset $\mathcal{H}$ of $e_k \mathfrak{A}_i[2] e_k$ such that for all $x \in \mathcal{F}$, there exist $y_1 \in \mathcal{D}$ and $y_2 \in \mathcal{H}$

$$\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}.$$
Set $\mathcal{D} = \bigoplus_{\ell=1}^s M_{n_{\ell}}$ and let $\{f_{ij}^\ell\}_{i,j=1}^{n_{\ell}}$ be a system of matrix units for $M_{n_{\ell}}$. Let $\mathcal{I}_\ell$ be the ideal in $\mathcal{A}_\ell$ generated by $f_{11}^\ell$. Since $\mathcal{A}_\ell[2]$ is simple and $\mathcal{A}_\ell[2]$ is an essential ideal of $\mathcal{A}_\ell$, we have that $\mathcal{A}_\ell[2] \subseteq \mathcal{I}$ for all nonzero ideal $\mathcal{I}$ of $\mathcal{A}_\ell$. Thus, $\mathcal{A}_1[2] \subseteq \mathcal{I} \subseteq \mathcal{A}_\ell[2]$ since $\mathcal{D} \cap \mathcal{A}_1[2] = 0$.

Let $\mathcal{J}_\ell^\phi$ be the ideal in $\mathcal{A}_\ell$ generated by $\phi(f_{11}^\ell)$ and let $\mathcal{J}_\ell^\psi$ be the ideal in $\mathcal{A}_\ell$ generated by $\psi(f_{11}^\ell)$. Since $\phi$ and $\psi$ are $X_2$-equivariant homomorphisms and since $\phi(1)$ and $\psi(1)$ are injective homomorphisms, we have that $\phi(f_{11}^\ell) \notin \mathcal{A}_\ell[2]$ and $\psi(f_{11}^\ell) \notin \mathcal{A}_\ell[2]$. Therefore, $\mathcal{A}_\ell[2] \subseteq \mathcal{J}_\ell^\phi$ and $\mathcal{A}_\ell[2] \subseteq \mathcal{J}_\ell^\psi$. Since $K_0(\phi(1)) = K_0(\psi(1))$ and since $\mathcal{A}_\ell[1]$ is an AF-algebra, we have that $\phi(1)(\mathcal{J}_1^\ell)$ is Murray-von Neumann equivalent to $\psi(1)(\mathcal{J}_1^\ell)$, where $\mathcal{J}_1^\ell$ is the image of $f_{11}^\ell$ in $\mathcal{A}_\ell[1]$. Thus, they generate the same ideal in $\mathcal{A}_\ell[1]$. Since $\mathcal{A}_2[2] \subseteq \mathcal{J}_\ell^\phi$ and $\mathcal{A}_2[2] \subseteq \mathcal{J}_\ell^\psi$ and since $\psi(1)(\mathcal{J}_1^\ell)$ and $\phi(1)(\mathcal{J}_1^\ell)$ generate the same ideal in $\mathcal{A}_2[1]$, we have that $\mathcal{J} = \mathcal{J}_\ell^\phi = \mathcal{J}_\ell^\psi$.

Note that the following diagram

\[
\begin{array}{c}
0 \rightarrow K_0(\mathcal{A}_2[2]) \rightarrow K_0(\mathcal{I}) \rightarrow K_0(\mathcal{I}/\mathcal{A}_2[2]) \\
\downarrow \quad \downarrow K_0(\mathcal{I}) \quad \quad \downarrow K_0(\mathcal{I}) \\
0 \rightarrow K_0(\mathcal{A}_2[2]) \rightarrow K_0(\mathcal{A}_2) \rightarrow K_0(\mathcal{A}_2[1])
\end{array}
\]

is commutative, the rows are exact, and $\iota$ and $7$ are the canonical embeddings. Since $\mathcal{A}_2[1]$ is an AF-algebra, $K_0(7)$ is injective. A diagram chase shows that $K_0(\iota)$ is injective. Since $KK(X_2; \phi) = KK(X_2; \psi)$, we have that $[\phi(f_{11}^1)] = [\psi(f_{11}^1)]$ in $K_0(\mathcal{A}_2)$. Since $\phi(f_{11}^1)$ and $\psi(f_{11}^1)$ are elements of $\mathcal{J}$ and $K_0(\iota)$ is injective, we have that $[\phi(f_{11}^1)] = [\psi(f_{11}^1)]$ in $K_0(\mathcal{I})$. Since $\mathcal{A}_1[1]$ is an AF-algebra and $\mathcal{A}_1[2]$ is a Kirchberg algebra, they both have stable weak cancellation. By Lemma 3.15 of [15], $\mathcal{A}_1$ has stable weak cancellation. Thus, $\phi(f_{11}^1)$ is Murray-von Neumann equivalent to $\psi(f_{11}^1)$. Hence, there exists $v_\ell \in \mathcal{A}_2$ such that $v_\ell^*v_\ell = \phi(f_{11}^1)$ and $v_\ell v_\ell^* = \psi(f_{11}^1)$.

Set

\[ u_1 = \sum_{\ell=1}^s \sum_{i=1}^{n_{\ell}} \psi(f_{1i}^\ell) v_\ell \phi(f_{1i}^\ell) \]

Then, $u_1$ is a partial isometry in $\mathcal{A}_1$ such that $u_1^*u_1 = \phi(1_D)$, $u_1u_1^* = \psi(1_D)$, and $u_1 \phi(x) u_1^* = \psi(x)$ for all $x \in \mathcal{D}$.

Let $\beta : e_k \mathcal{A}_1[2] e_k \rightarrow \mathcal{A}_1[2]$ be the usual embedding. Note that $KK(\phi(2) \circ \beta) = KK(\psi(2) \circ \beta)$ and $\phi(2) \circ \beta$, $\psi(2) \circ \beta$ are monomorphisms. Therefore, by Theorem 6.7 of [23], there exists a partial isometry $u_2 \in \mathcal{A}_2[2]$ such that $u_2^*u_2 = \phi(e_k)$, $u_2u_2^* = \psi(e_k)$, and

\[ \|u_2 \phi(x) u_2^* - \psi(x)\| < \frac{\epsilon}{3} \]

for all $x \in \mathcal{H}$.

Since $\mathcal{A}_2$ is stable, there exists $u_3 \in \mathcal{M}(\mathcal{A}_2)$ such that $u_3^*u_3 = 1_{\mathcal{M}(\mathcal{A}_2)} - (u_1 + u_2)^*(u_1 + u_2)$ and $u_3u_3^* = 1_{\mathcal{M}(\mathcal{A}_2)} - (u_1 + u_2)(u_1 + u_2)^*$. Set $u = u_1 + u_2 + u_3 \in \mathcal{M}(\mathcal{A}_2)$. Then $u$ is a unitary in $\mathcal{M}(\mathcal{A}_2)$. 
Let $x \in \mathcal{F}$. Choose $y_1 \in \mathcal{D}$ and $y_2 \in \mathcal{H} \subseteq e_k \mathcal{A}_1[e_k]e_k$ such that $\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}$. Then

$$\|u\phi(x)u^* - \psi(x)\| \leq \|u\phi(x)u^* - u\phi(y_1 + y_2)u^*\| + \|u_1\phi(y_1)u_1^* + u_2\phi(y_2)u_2^* - \psi(y_1) - \psi(y_2)\| + \|\psi(y_1 + y_2) - \psi(x)\| < \epsilon.$$

We have just shown that for each $\epsilon > 0$ and for each finite subset $\mathcal{F}$ of $\mathcal{A}_1$, there exists a unitary $u \in \mathcal{M}(\mathcal{A}_2)$ such that $\|u\phi(x)u^* - \psi(x)\| < \epsilon$ for all $x \in \mathcal{F}$. Since $\mathcal{A}_1$ is a separable $C^*$-algebra, we have that $\phi$ is approximately unitarily equivalent to $\psi$.

Lemma 4.6. Let $\mathcal{A}$ be a separable $C^*$-algebra over a finite topological space $X$. Let $u$ be unitary in $\mathcal{M}(\mathcal{A} \otimes \mathbb{K})$. Then $K_X(\text{Ad}(u)_{\mathcal{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathcal{A})}$.

Proof. Since $\mathcal{A} \otimes \mathbb{K}$ is stable, we have that there exists a norm continuous path of unitaries $\{u_t\}$ in $\mathcal{M}(\mathcal{A} \otimes \mathbb{K})$ such that $u_0 = u$ and $u_1 = 1_{\mathcal{M}(\mathcal{A} \otimes \mathbb{K})}$. It follows that $K_X(\text{Ad}(u)_{\mathcal{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathcal{A})}$.

Theorem 4.7. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2)$ be an invertible element such that $\Gamma(x)Y$ is an order isomoprhism for all $Y \in \mathcal{L}(X_2)$. Suppose $\mathcal{A}_i[2]$ is a Kirchberg algebra, $\mathcal{A}_i[1]$ is an AF-algebra, $\mathcal{A}_i$ has real rank zero, and $\mathcal{A}_i[2]$ is an essential ideal of $\mathcal{A}_i$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathcal{A}_2 \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(\phi) = KL(g_1^{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathcal{A}_1} \otimes e_1)\times K(X_n; \text{id}_{\mathcal{A}_2} \otimes e_1)$.

Proof. Since $\mathcal{A}_i[2]$ is a purely infinite simple $C^*$-algebra, $\mathcal{A}_i[2]$ is either unital or stable. Since $\mathcal{A}_i[2]$ is an essential ideal of $\mathcal{A}_i$, $\mathcal{A}_i[2]$ is non-unital else $\mathcal{A}_i[2]$ is isomorphic to a direct summand of $\mathcal{A}_i$ which would contradict the essential assumption. Therefore, $\mathcal{A}_i[2]$ is stable. Moreover, $\mathcal{Q}(\mathcal{A}_i[2])$ is simple which implies that $0 \rightarrow \mathcal{A}_i[2] \rightarrow \mathcal{A}_i \rightarrow \mathcal{A}_i[1] \rightarrow 0$ is a full extension. Since $\mathcal{A}_i[2]$ and $\mathcal{A}_i[1]$ are nuclear $C^*$-algebras, $\mathcal{A}_i$ is a nuclear $C^*$-algebra.

Let $z \in KK(X_2; \mathcal{A}_2 \otimes \mathbb{K}, \mathcal{A}_1 \otimes \mathbb{K})$ such that $y \times z = [\text{id}_{\mathcal{A}_1 \otimes \mathbb{K}}] \text{ and } y = [\text{id}_{\mathcal{A}_2 \otimes \mathbb{K}}]$. By Theorem 4.1 there exists an $X_2$-equivariant homomorphism $\psi_1 : \mathcal{A}_1 \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \psi_1) = x$, and $(\psi_1)_{(2)}$ and $(\psi_1)_{(1)}$ are injective homomorphisms. By Theorem 4.1 there exists an $X_2$-equivariant homomorphism $\psi_2 : \mathcal{A}_2 \otimes \mathbb{K} \rightarrow \mathcal{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \psi_2) = y$, and $(\psi_2)_{(2)}$ and $(\psi_2)_{(1)}$ are injective homomorphisms. Using Theorem 4.5 and a typical approximate intertwining argument, there exists an isomorphism $\phi : \mathcal{A}_1 \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $\phi$ and $\psi_1$ are approximately unitarily equivalent.

Let $\pi_2 : \mathcal{A}_2 \rightarrow \mathcal{A}_2[1]$ be the canonical quotient map. Then $\pi_2 \circ \phi|_{\mathcal{A}_2[2]}$ is either zero or injective. Since $\mathcal{A}_2[2]$ is simple. Since $\mathcal{A}_2[2]$ is purely infinite and $\mathcal{A}_2[1]$ is an AF-algebra, we must have that $\pi_2 \circ \phi|_{\mathcal{A}_2[2]} = 0$. Thus, $\phi$ is an $X_2$-equivariant homomorphism. Similarly, $\phi^{-1}$ is an $X_2$-equivariant homomorphism. Hence, $\phi$ is an $X_2$-equivariant isomorphism. By construction, $KL(\phi) = KL(\psi_1) = KL(g_1^{X_2}(y))$. By Lemma 4.6 $K_{X_2}(\phi) = K_{X_2}(y)$. □

Corollary 4.8. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathcal{A}_1, \mathcal{A}_2)$ be an invertible element such that $\Gamma(x)Y$ is an order isomorphism for all $Y \in \mathcal{L}(X_2)$. Suppose $\mathcal{A}_i[2]$ is a Kirchberg algebra, $\mathcal{A}_i[1]$ is an AF-algebra, $\mathcal{A}_i$ has real rank zero, $\mathcal{A}_i[2]$ is an essential ideal of $\mathcal{A}_i$, and $K_i(\mathcal{A}[Y])$ and $K_i(\mathcal{B}[Y])$ are finitely generated for all $Y \in \mathcal{L}(X_2)$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathcal{A}_1 \otimes \mathbb{K} \rightarrow \mathcal{A}_2 \otimes \mathbb{K}$ such that $KK(\phi) = KK(g_1^{X_2}(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathcal{A}_1} \otimes e_1)\times K(X_n; \text{id}_{\mathcal{A}_2} \otimes e_1)$.
4.2. Strong classification of extensions of purely infinite by $K$. We recall the following from [1] p. 341. Let $\psi : \mathcal{A} \to B(H)$ be a representation of $\mathcal{A}$. Let $\mathcal{H}_e$ denote the subspace of $\mathcal{H}$ spanned by the ranges of all compact operators in $\psi(\mathcal{A})$. Since $\psi(\mathcal{A}) \cap K$ is an ideal of $\psi(\mathcal{A})$, we have that $\mathcal{H}_e$ reduces $\pi(\mathcal{A})$, and so the decomposition $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_e^\perp$ induces a decomposition of $\psi$ into sub-representations $\psi = \psi_e \oplus \psi'_e$. The summand $\psi_e$, considered as a representation of $\mathcal{A}$ on $\mathcal{H}_e$, will be called the essential part of $\psi$ and $\mathcal{H}_e$ is called the essential subspace for $\psi$.

Let $\mathcal{B}$ be a tight $C^*$-algebra over $X_2$. Consider the essential extension

$$e_{\mathcal{B}} : 0 \to \mathcal{B}[2] \to \mathcal{B} \to \mathcal{B}[1] \to 0.$$ 

If $\tau_{\mathcal{B}} : \mathcal{B}[1] \to Q(\mathcal{B}[2])$ is the Busby invariant of $e$, then there exists an injective homomorphism $\sigma_{e_{\mathcal{B}}} : \mathcal{B} \to M(\mathcal{B}[2])$ such that the diagram

$$
\begin{array}{ccc}
0 & \xrightarrow{\sigma_{e_{\mathcal{B}}}} & \mathcal{B}[2] \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\tau_{e_{\mathcal{B}}}} & M(\mathcal{B}[2]) \\
\end{array}
\quad
\begin{array}{ccc}
\mathcal{B}[1] & \xrightarrow{\pi_{e_{\mathcal{B}}}} & \mathcal{B} \\
\pi_{e_{\mathcal{B}}} & & \pi_{e_{\mathcal{B}}} \\
\downarrow & & \downarrow \\
0 & \xrightarrow{\tau_{e_{\mathcal{B}}}} & Q(\mathcal{B}[2]) \\
\end{array}
\quad
\begin{array}{ccc}
0 & \xrightarrow{\pi_{\mathcal{B}}} & 0 \\
\end{array}
$$

If $\mathcal{B}[2] \cong K$, let $\eta_{\mathcal{B}} : M(\mathcal{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathcal{B}[2] \cong K$ and let $\overline{\eta}_{\mathcal{B}} : Q(\mathcal{B}[2]) \to B(\ell^2)/K$ be the induced isomorphism.

**Lemma 4.9.** Let $\mathcal{A}$ and $\mathcal{B}$ be separable, tight $C^*$-algebras over $X_2$ such that $\mathcal{A}[2] \cong \mathcal{B}[2] \cong K$. Let $\psi_1, \psi_2 : \mathcal{A} \to \mathcal{B}$ be two, full $X_2$-equivariant homomorphisms such that $K_0((\psi_1)_1) = K_0((\psi_2)_1)$ and $\eta_{\mathcal{B}} \circ \sigma_{e_{\mathcal{B}}} \circ \psi_i$ is a non-degenerate representation of $\mathcal{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^\infty$ in $M(\mathcal{B}[2])$ such that

$$U_n(\sigma_{e_{\mathcal{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{e_{\mathcal{B}}} \circ \psi_2)(a) \in \mathcal{B}[2]$$

for all $a \in \mathcal{A}$ and for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} \|U_n(\sigma_{e_{\mathcal{B}}} \circ \psi_1)(a)U_n^* - (\sigma_{e_{\mathcal{B}}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathcal{A}$.

**Proof.** We argue as in the proof of Lemma 2.8 of [22]. Set $\sigma_i = \eta_{\mathcal{B}} \circ \sigma_{e_{\mathcal{B}}} \circ \psi_i$. By assumption, $\sigma_i : \mathcal{A} \to B(\ell^2)$ is a non-degenerated representation of $\mathcal{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^\infty$ in $B(\ell^2)$ such that $V_n\sigma_1(a)V_n^* - \sigma_2(a) \in K$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathcal{A}$. This will be a consequence of Theorem 5(iii) of [1].

Let $\rho : \mathcal{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathcal{A}[2] \cong K$. Since $\psi_i, \sigma_{e_{\mathcal{B}}}, \eta_{\mathcal{B}}$ are injective homomorphisms, $\sigma_i$ is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi : B(\ell^2) \to B(\ell^2)/K$ be the natural projection. Note that

$$\pi \circ \sigma_i = \pi \circ \eta_{\mathcal{B}} \circ \sigma_{e_{\mathcal{B}}} \circ \psi_i = \overline{\eta}_{\mathcal{B}} \circ \pi_{\mathcal{B}} \circ \sigma_{\mathcal{B}} \circ \psi_i = \overline{\eta}_{\mathcal{B}} \circ \tau_{e_{\mathcal{B}}} \circ \pi_{\mathcal{B}} \circ \psi_i = \overline{\eta}_{\mathcal{B}} \circ \tau_{e_{\mathcal{B}}} \circ (\psi_i)_1 \circ \pi_{\mathcal{A}}.$$
It now follows that $\ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = \mathfrak{A}[2]$ since $\pi_\mathfrak{B}$, $\tau_\mathfrak{B}$, and $(\psi_1)_1$ are injective homomorphisms.

Let $H_1$ be the essential subspace of $\sigma_1$. Since $\sigma_1(\mathfrak{A}[2]) \subseteq \mathbb{K}$ and for each $x \notin \mathfrak{A}[2]$, we have that $\sigma_1(x) \notin \mathbb{K}$, we have that $H_1 = \sigma_1(\mathfrak{A}[2])\ell^2$. Similarly, we have that $H_2 = \sigma_2(\mathfrak{A}[2])\ell^2$, where $H_2$ is the essential subspace of $\sigma_2$. Let $e$ be a minimal projection of $\mathfrak{A}[2] \cong \mathbb{K}$. Suppose $\sigma_1(e)$ has rank $k$. Standard representation theory now implies that $\sigma_1(-)|_{H_1}$ is unitarily equivalent to the direct sum of $k$ copies of $\rho$. Since $K_0(\sigma_1(\mathfrak{A}[2])) = K_0(\sigma_2(\mathfrak{A}[2]))$, we have that $\sigma_1(e)$ is Murray-von Neumann equivalent to $\sigma_2(e)$. Hence, $\sigma_2(e)$ has rank $k$. Standard representation theory now implies that $\sigma_2(-)|_{H_2}$ is unitarily equivalent to the direct sum of $k$ copies of $\rho$.

The above paragraph imply that $\sigma_2(-)|_{H_2}$ and $\sigma_1(-)|_{H_1}$ are unitarily equivalent. Since $\ker(\sigma_1) = \ker(\sigma_2)$ and $\ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2)$ by Theorem 5(iii) of [1], there exists a sequence of unitaries $\{V_n\}_{n=1}^\infty$ in $B(\ell^2)$ such that $V_n\sigma_1(a)V_n^* - \sigma_2(a) \in \mathbb{K}$ for all $n \in \mathbb{N}$ and for all $a \in \mathfrak{A}$, and

$$\lim_{n \to \infty} \|V_n\sigma_1(a)V_n^* - \sigma_2(a)\| = 0$$

for all $a \in \mathfrak{A}$.

Set $U_n = \eta_{\mathfrak{B}}^{-1}(V_n)$. Then $\{U_n\}_{n=1}^\infty$ is a sequence of unitaries in $\mathcal{M}(\mathfrak{B}[2])$ such that $U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a) \in \mathfrak{B}[2]$ for all $n \in \mathbb{N}$ and for all $a \in \mathfrak{A}$, and

$$\lim_{n \to \infty} \|U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a)\| = 0$$

for all $a \in \mathfrak{A}$. □

**Definition 4.10.** A $C^*$-algebra $\mathfrak{A}$ is called weakly semiprojective if we can always solving the $*$-homomorphism lifting problem

$$\xymatrix{ & \prod_{n=N}^\infty \mathfrak{B}_n \ar[d]^{\rho_N} \ar@{.>}[dl]_{\tilde{\phi}} \ar@{-->}[d]^{\phi} \ar[r] & (b_N, b_{N+1}, \ldots) \ar[d] \ar@{.>}[dl]_{\bar{\phi}} \ar@{-->}[d]^{\phi} \\
\mathfrak{A} \ar[r]_{\phi} & \prod_{n=1}^\infty \mathfrak{B}_n / \bigoplus_{n=1}^\infty \mathfrak{B}_n & \ar[r] & \{(0, \ldots, 0, b_N, b_{N+1}, \ldots)\}
}$$

and $\mathfrak{A}$ is called semiprojective if we can always solve the lifting problem

$$\xymatrix{ & \mathfrak{B}/\mathcal{I}_N \ar[d]^{\rho_N} \ar@{.>}[dl]_{\tilde{\phi}} \ar@{-->}[d]^{\phi} \ar[r] & (\mathcal{I}_1 \subseteq \mathcal{I}_2 \subseteq \cdots \subseteq \mathfrak{B}) \ar[d] \\
\mathfrak{A} \ar[r]_{\phi} & \mathfrak{B}/\bigcup_{n=1}^\infty \mathcal{I}_n
}$$

**Lemma 4.11.** Let $\mathfrak{A}_0$ be a unital, separable, nuclear, tight $C^*$-algebra over $X_2$ such that $\mathfrak{A}_0[2] \cong \mathbb{K}$ and $\mathfrak{A}_0$ has the stable weak cancellation property. Set $\mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K}$. Suppose $\beta : \mathfrak{A} \to \mathfrak{A}$ is a full $X_2$-equivariant homomorphism such that $K_{X_2}(\beta) = K_{X_2}(\text{id}_\mathfrak{A})$ and $\beta_{1}[1] = \text{id}_\mathfrak{A}[1]$. Then there exists a sequence of contractive, completely positive, linear maps $\{\alpha_n : \mathfrak{A} \to \mathfrak{A}\}_{n=1}^\infty$ such that

1. $\alpha_n|_{\mathfrak{A}_n \mathfrak{A}_n}$ is a homomorphism for all $n \in \mathbb{N}$ and
2. for all $a \in \mathfrak{A}$,

$$\lim_{n \to \infty} \|\alpha_n \circ \beta(a) - a\| = 0$$
where \( e_n = \sum_{k=1}^n 1_{A_0} \otimes e_{k,k} \) and \( \{e_{i,j}\}_{i,j} \) is a system of matrix units for \( K \). If, in addition, \( A \) is assumed to be weakly semiprojective, then \( \alpha_n \) can be chosen to be a homomorphism for all \( n \in \mathbb{N} \).

**Proof.** Since \( \beta \) is a full \( X_2 \)-equivariant homomorphism and the ideal in \( A \) generated by \( e_n \) is \( A \), we have that the ideal in \( A \) generated by \( \beta(e_n) \) is \( A \). Since \( K_{X_2}(\beta) = K_{X_2}(id_A) \), we have that \( [\beta(e_n)] = [e_n] \) in \( K_0(A) \). It now follows that \( \beta(e_n) \) and \( e_n \) are Murray-von Neumann equivalent since \( A_0 \) has the stable weak cancellation property. Since \( A \) is stable, there exists a unitary \( v_n \) in the unitization of \( A \) such that \( v_n \beta(e_n) v_n^* = e_n \).

Fix \( n \in \mathbb{N} \). Let \( e_n \) be the extension \( 0 \rightarrow e_n A[2]e_n \rightarrow e_n A e_n \rightarrow \tau_n A[1] \tau_n \rightarrow 0 \). By Lemma 1.5 of [16], \( e \) is a full extension. Therefore, \( \sigma_e(e_n) \) is Murray-von Neumann equivalent to \( 1_M(A[2]) \). Hence, \( e_n A[2]e_n \cong A[2] \cong K \). Set \( \mathfrak{A}_n = e_n \mathfrak{A} e_n \) and define \( \beta_n : \mathfrak{A}_n \rightarrow \mathfrak{A}_n \) by \( \beta_n(x) = \operatorname{Ad}(v_n) \circ \beta(x) \). Then \( \beta_n \) is a unital, full \( X_2 \)-equivariant homomorphism. Since \( \eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n \) is a unital representation of \( \mathfrak{A}_n \), the closed subspace of \( \ell^2 \) generated by \( \{(\eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n)(x) : x \in \mathfrak{A}_n, \xi \in \ell^2 \} \) is \( \ell^2 \). Therefore, \( \eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n \) is a non-degenerate representation of \( \mathfrak{A}_n \).

Since \( K_{X_2}(\beta) = K_{X_2}(id_A) \) and the \( X_2 \)-equivariant embedding of \( A_0 \) as a sub-algebra of \( A \) induces an isomorphism in ideal related \( K \)-theory, we have that \( K_{X_2}(\beta_n) = K_{X_2}(id_{\mathfrak{A}_n}) \).

By Lemma 4.9 there exists a sequence of unitaries \( W_{k,n} \in M(\mathfrak{A}_n[2]) \) such that
\[
(\operatorname{Ad}(W_{k,n}) \circ \sigma_{e_n} \circ \beta_n)(x) \in \mathfrak{A}_n[2]
\]
for all \( x \in \mathfrak{A}_n \) and for all \( k \in \mathbb{N} \), and
\[
\lim_{k \to \infty} \| (\operatorname{Ad}(W_{k,n}) \circ \sigma_{e_n} \circ \beta_n)(x) - \sigma_{e_n}(x) \| = 0
\]
for all \( x \in \mathfrak{A}_n \).

Note that \( M(\mathfrak{A}_n[2]) \cong \sigma_e(e_n)M(\mathfrak{A}[2])\sigma_e(e_n) \) with an isomorphism mapping \( \mathfrak{A}_n[2] \) onto \( e_n \mathfrak{A}[2]e_n \). Thus, we get a partial isometry \( W_{k,n} \) in \( M(\mathfrak{A}[2]) \) such that \( W_{k,n}^* W_{k,n} = \mathbf{1}_n \mathbf{1}_k \mathbf{1}_n^* \mathbf{1}_k^* \). Therefore, \( \eta_{\mathfrak{A}_n} \circ \sigma_{e_n} \circ \beta_n \) is a non-degenerate representation of \( \mathfrak{A}_n \).

Set \( V_{k,n} = (W_{k,n} + 1_{M(\mathfrak{A}[2])}) - \sigma_e(e_n) \sigma_e(e_n) \). Then \( V_{k,n} \) is a unitary in \( M(\mathfrak{A}[2]) \) such that
\[
(\operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \beta)(x) \in \mathfrak{A}[2]
\]
for all \( x \in \mathfrak{A}_n \) and for all \( k \in \mathbb{N} \), and
\[
\lim_{k \to \infty} \| (\operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \beta)(x) - \sigma_e(x) \| = 0
\]
for all \( x \in \mathfrak{A}_n \).

Set \( V_{k,n} = W_{k,n} + 1_{M(\mathfrak{A}[2])} - \sigma_e(e_n) \sigma_e(e_n) \). Then \( V_{k,n} \) is a unitary in \( M(\mathfrak{A}[2]) \) such that
\[
(\operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \beta)(x) \in \mathfrak{A}[2]
\]
for all \( x \in \mathfrak{A}_n \) and for all \( k \in \mathbb{N} \), and
\[
\lim_{k \to \infty} \| (\operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \beta)(x) - \sigma_e(x) \| = 0
\]
for all \( x \in \mathfrak{A}_n \).

A consequence of the first part is that \( \operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \beta \in \sigma(e_n \mathfrak{A} e_n) + \mathfrak{A}[2] \) for all \( x \in e_n \mathfrak{A} e_n \). Since \( \beta_{[1]} = id_{\mathfrak{A}[2]} \), we have that \( x - \beta(x) \in \mathfrak{A}[2] \) for all \( x \in e_n \mathfrak{A} e_n \). Therefore,
\[
\operatorname{Ad}(V_{k,n})(\sigma_e(x)) = \operatorname{Ad}(V_{k,n}) \circ \sigma_e(x - \beta(x)) + \operatorname{Ad}(V_{k,n}) \circ \beta(x) \in \sigma(e_n \mathfrak{A} e_n) + \mathfrak{A}[2]
\]
Thus, \( \alpha_{k,n} = \sigma_e^{-1} \circ (\operatorname{Ad}(V_{k,n}) \circ \sigma_e \circ \operatorname{Ad}(v_n)) \in e_n \mathfrak{A} e_n \) is a homomorphism from \( e_n \mathfrak{A} e_n \) to \( A \).
Since
\[ \lim_{k \to \infty} \|(\operatorname{Ad}(V_{k,n}) \circ \sigma_\varepsilon \circ \beta)(x) - \sigma_\varepsilon(x)\| = 0 \]
for all \( x \in e_nAe_n \) and \( e_nAe_n \subseteq e_{n+1}Ae_{n+1} \), there exists a strictly increasing sequence \( \{k(n)\}_{n=1}^\infty \) of positive integers such that
\[ \lim_{n \to \infty} \|\alpha_{k(n),n} \circ \beta(x) - x\| = 0 \]
for all \( x \in \bigcup_{n=1}^\infty e_nAe_n \). Let \( \alpha_n \) be a completely, contractive, positive linear extension of \( \alpha_{k(n),n} \). Since \( \bigcup_{n=1}^\infty e_nAe_n \) is dense in \( A \), we have that
\[ \lim_{n \to \infty} \|\alpha_n \circ \beta(x) - x\| = 0 \]
for all \( x \in A \). We have just proved the first part of the lemma.

We now show that \( \alpha_n \) can be chosen to be a homomorphism provide that \( A \) is weakly semiprojective. Suppose \( A \) is weakly semiprojective. Let \( \varepsilon > 0 \) and \( F \) be a finite subset of \( A \).

By Theorem 2.4 of [23] (see also Definition 2.1 and Theorem 2.3 of [25], and Theorem 19.1.3 of [26]), there exist a \( \delta > 0 \) and a finite subset \( G \) of \( A \) such that for any \( C^* \)-algebra \( B \) and any contractive, completely positive, linear map \( L : A \to B \) such that
\[ \|L(ab) - L(a)L(b)\| < \delta \]
for all \( a, b \in G \), there exists a homomorphism \( h : A \to B \) such that
\[ \|h(x) - L(x)\| < \frac{\varepsilon}{2} \]
for all \( x \in \beta(F) \).

Without loss of generality, we may assume that \( \varepsilon < 1 \) and \( \delta < 1 \). Set
\[ M = 1 + \max (\{|a| : a \in G\} \cup \{|x| : x \in F\}) \]
Since \( e_nAe_n \subseteq e_{n+1}Ae_{n+1} \) and \( \bigcup_{n=1}^\infty e_nAe_n \) is dense in \( A \), there exist \( n \in \mathbb{N} \) and a finite subset \( H \subseteq e_nAe_n \) such that for each \( a \in G \), there exists \( y \in H \) such that \( \|a - y\| < \frac{\delta}{4M} \) and
\[ \|\alpha_n \circ \beta(x) - x\| < \frac{\varepsilon}{2} \]
for all \( x \in F \). Let \( a, b \in G \). Choose \( x, y \in H \subseteq e_nAe_n \) such that \( \|a - x\| < \frac{\delta}{4M} \) and \( \|b - y\| < \frac{\delta}{4M} \). Note that \( \|x\| \leq 1 + \|a\| \leq M \) and \( \|y\| \leq 1 + \|b\| \leq M \). Then
\[ \|\alpha_n(ab) - \alpha_n(a)\alpha_n(b)\| = \|\alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b)\|
\leq \|b\|\|a - x\| + \|x\|\|b - y\|
+ \|\alpha_n(x)\alpha_n(y) - \alpha_n(x)\alpha_n(b)\|
+ \|\alpha_n(x)\alpha_n(b) - \alpha_n(a)\alpha_n(b)\|
\leq 2M\|a - x\| + 2M\|b - y\|
< 4M\frac{\delta}{4M} = \delta. \]

By the choice of \( \delta \) and \( G \), there exists a homomorphism \( \psi : A \to A \) such that
\[ \|\psi(t) - \alpha_n(t)\| < \frac{\varepsilon}{2} \]
for all $t \in \beta(\mathcal{F})$. Let $x \in \mathcal{F}$. Then
\[ \| \psi \circ \beta(x) - x \| \leq \| \psi(\beta(x)) - \alpha_n(\beta(x)) \| + \| \alpha_n(\beta(x)) - x \| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathbb{A}$, there exists a homomorphism $\psi : \mathbb{A} \to \mathbb{A}$ such that
\[ \| \psi \circ \beta(x) - x \| < \epsilon \]
for all $x \in \mathcal{F}$. Consequently, there exists a sequence of endomorphisms $\{ \psi_n : \mathbb{A} \to \mathbb{A} \}_{n=1}^\infty$ such that
\[ \lim_{n \to \infty} \| \psi_n \circ \beta(x) - x \| = 0 \]
for all $x \in \mathbb{A}$ since $\mathbb{A}$ is separable.

To prove a uniqueness theorem involving tight $C^*$-algebras $\mathbb{A}$ over $X_2$, we require that $\mathbb{A}[1]$ belongs to a class of $C^*$-algebras whose injective homomorphisms between two objects in this class are classified by $KK$.

**Definition 4.12.** We will be interested in classes $\mathcal{C}$ of separable, nuclear, simple $C^*$-algebras satisfying the following property that if $\mathbb{A}, \mathbb{B} \in \mathcal{C}$ and $\phi, \psi : \mathbb{A} \otimes \mathbb{K} \to \mathbb{B} \otimes \mathbb{K}$ are two injective homomorphisms such that $KK(\phi) = KK(\psi)$, then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Remark 4.13.**

1. By Theorem 4.1.3 of [29], if $\mathcal{C}$ is the class of Kirchberg algebras, then $\mathcal{C}$ satisfies the property in Definition 4.12.
2. Let $\mathcal{C}$ be the class of unital, separable, nuclear, simple tracially AF $C^*$-algebras in $\mathcal{N}$. Then $\mathcal{C}$ satisfies the property in Definition 4.12.

**Theorem 4.14.** (Uniqueness Theorem 2) Let $\mathcal{C}$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathbb{A}$ be a unital, separable, nuclear, tight $C^*$-algebra over $X_2$ such that $\mathbb{A}[2] \cong \mathbb{K}$ and $\mathbb{A}[1] \in \mathcal{C}$. Suppose $\mathbb{A} \otimes \mathbb{K}$ is semiprojective and $\mathbb{A}$ has the stable weak cancellation property. Let $\phi : \mathbb{A} \otimes \mathbb{K} \to \mathbb{A} \otimes \mathbb{K}$ be a full $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \text{id}_{\mathbb{A} \otimes \mathbb{K}})$. Then there exists a sequence of full $X_2$-equivariant endomorphisms $\{ \alpha_n : \mathbb{A} \otimes \mathbb{K} \to \mathbb{A} \otimes \mathbb{K} \}_{n=1}^\infty$ such that $KK(X_2; \alpha_n) = KK(X_2; \text{id}_{\mathbb{A} \otimes \mathbb{K}})$ and
\[ \lim_{n \to \infty} \| (\alpha_n \circ \phi)(x) - x \| = 0 \]
for all $x \in \mathbb{A} \otimes \mathbb{K}$.

**Proof.** Set $\mathcal{B} = \mathbb{A} \otimes \mathbb{K}$. Note that $\mathcal{B}$ is a tight $C^*$-algebra over $X_2$ with $\mathcal{B}[2] \cong \mathbb{K}$. Throughout the proof, $\pi : \mathcal{B} \to \mathcal{B}[1]$ will denote the canonical projection. Note that $KK(\phi_{\{1\}}) = KK(\text{id}_{\mathcal{B}[1]})$ since $KK(X_2; \phi) = KK(X_2; \text{id}_{\mathcal{B}})$. Since $\mathbb{A}[1] \in \mathcal{C}$, there exists a sequence of unitaries $\{ z_k \}_{k=1}^\infty$ in $\mathcal{M}(\mathcal{B}[1])$ such that
\[ \lim_{k \to \infty} \| z_k \phi_{\{1\}}(\pi(b)) z_k^* - \pi(b) \| = 0 \]
for all $b \in \mathcal{B}$. Using the fact that $\phi$ is an $X_2$-equivariant homomorphism, we have that $\pi \circ \phi = \phi_{\{1\}} \circ \pi$, and hence
\[ \lim_{k \to \infty} \| z_k (\pi \circ \phi)(b) z_k^* - \pi(b) \| = 0 \]
for all $b \in \mathfrak{B}$.

Let $\pi : \mathcal{M}(\mathfrak{B}) \to \mathcal{M}(\mathfrak{B}[1])$ be the surjective homomorphism induced by $\pi$. Since $\mathfrak{B}$ is stable, by Corollary 2.3 of [35], we have that $\mathfrak{B}[1]$ is stable. Thus, the unitary group of $\mathcal{M}(\mathfrak{B}[1])$ is path-connected, which implies that every unitary in $\mathcal{M}(\mathfrak{B}[1])$ lifts to a unitary in $\mathcal{M}(\mathfrak{B})$. Hence, there exists a sequence of unitaries $\{w_k\}_{k=1}^\infty$ in $\mathcal{M}(\mathfrak{B})$ such that $\pi(w_k) = z_k$.

Since $\mathfrak{B}$ is semiprojective, by Proposition 2.2 of [7] (see [26]), there exists a sequence of homomorphisms $\{\beta_\ell : \mathfrak{B} \to \mathfrak{B}\}_{\ell=1}^\infty$ and a strictly increasing sequence $\{k(\ell)\}_{\ell=1}^\infty$ of positive integers such that $\pi \circ \beta_\ell = \pi$ and

$$\lim_{\ell \to \infty} \|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| = 0$$

for all $b \in \mathfrak{B}$.

By Remark 2.5 there exists $N_1 \in \mathbb{N}$ such that $\beta_\ell$ is a full $X_2$-equivariant homomorphism for all $\ell \geq N_1$. By Proposition 2.3 of [7], we may choose $N_2 \geq N_1$ such that for all $\ell \geq N_2$, we have that $\beta_\ell$ and $\text{Ad}(w_{k(\ell)}) \circ \phi$ is homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \beta_\ell) = KK(X_2; \text{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \text{id}_\mathfrak{B})$.

Let $\ell \geq N_2$. Note that $(\beta_\ell)_{\{1\}} = \text{id}_\mathfrak{B}[1]$ since $\pi \circ \beta_\ell = \pi$. Since $\mathfrak{A}$ is semiprojective, by Corollary 3.6 of [6] (also see Chapter 19 of [26]), $\mathfrak{A}$ is weakly semiprojective. Hence, by Lemma 4.11 there exists a sequence of homomorphisms $\{\alpha_{m,\ell} : \mathfrak{B} \to \mathfrak{B}\}_{m=1}^\infty$ such that

$$\lim_{m \to \infty} \|\alpha_{m,\ell} \circ \beta_\ell(x) - x\| = 0$$

for all $x \in \mathfrak{B}$. Since $\beta_\ell$ and $\text{id}_\mathfrak{B}$ are full $X_2$-equivariant homomorphisms, by Remark 2.5 there exists $N_3$ such that, for all $m \geq N_3$, we have that $\alpha_{m,\ell}$ is a full $X_2$-equivariant homomorphism. Moreover, by Proposition 2.3 of [7], we can choose $N_3 \geq N_2$ such that $\alpha_{m,\ell} \circ \beta_\ell$ and $\text{id}_\mathfrak{B}$ are homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \text{id}_\mathfrak{B})$ for all $m \geq N_3$. Consequently, $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathfrak{B})$ for all $m \geq N_3$ since $KK(X_2; \beta_\ell) = KK(X_2; \text{id}_\mathfrak{B})$.

Let $\mathcal{F}$ be a finite subset of $\mathfrak{B}$ and $\epsilon > 0$. Then there exists $\ell \geq N_2$ such that

$$\|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Moreover, there exists $m \geq N_3$ such that

$$\|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Set $\alpha_1 = \text{Ad}(w_{k(\ell)})|_{\mathfrak{B}}$ and $\alpha = \alpha_{m,\ell} \circ \alpha_1$. Since $w_{k(\ell)}$ is a unitary in $\mathcal{M}(\mathfrak{B})$, we have that $\alpha_1$ is an automorphism of $\mathfrak{B}$ and $KK(X_2; \alpha_1) = KK(X_2; \text{id}_\mathfrak{B})$. Therefore, $\alpha$ is a full $X_2$-equivariant homomorphism. Since $\ell \geq N_2$ and $m \geq N_3$, we have that $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathfrak{B})$. Therefore, $KK(X_2; \alpha) = KK(X_2; \text{id}_\mathfrak{B})$. Let $b \in \mathcal{F}$. Then

$$\|\alpha \circ \phi(b) - b\| = \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - b\| \\
\leq \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b)\| + \|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| \\
< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathfrak{B}$, there exists a full $X_2$-equivariant homomorphism $\alpha : \mathfrak{B} \to \mathfrak{B}$ such that $KK(X_2; \alpha) = KK(X_2; \text{id}_\mathfrak{B})$ and

$$\|\alpha \circ \phi(b) - b\| < \epsilon$$
for all $b \in \mathcal{B}$. Since $\mathcal{B}$ is a separable $C^*$-algebra, there exists a sequence of full $X_2$-equivariant homomorphisms $\{\alpha_n : \mathcal{B} \to \mathcal{B}\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; \text{id}_{\mathcal{B}})$ and

$$\lim_{n \to \infty} \|\alpha_n \circ \phi(b) - b\| = 0$$

for all $b \in \mathcal{B}$. 

**Theorem 4.15.** Let $C$ be a class of $C^*$-algebras satisfying the property in Definition 4.14 and let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i[2] \cong \mathbb{K}$ and $\mathfrak{A}_i[1] \in C$. Suppose $\mathfrak{A}_i \otimes \mathbb{K}$ is semiprojective and $\mathfrak{A}_i$ has the stable weak cancellation property. If there exist full $X_2$-equivariant homomorphisms, $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ and $\psi : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$, such that $KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$ and $KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}})$, then for any finite subset $\mathcal{F}$ and $\epsilon > 0$, there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \gamma) = KK(\phi)$ and

$$\|\gamma(x) - \phi(x)\| < \epsilon$$

for all $x \in \mathcal{F}$.

**Proof.** Let $\{\mathcal{F}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_1 \otimes \mathbb{K}$ such that $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes \mathbb{K}$ and let $\{\mathcal{G}_n\}_{n=1}^{\infty}$ be a sequence of finite subsets of $\mathfrak{A}_2 \otimes \mathbb{K}$ such that $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $\mathfrak{A}_2 \otimes \mathbb{K}$.

Let $\epsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Set $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_1$ and choose $m_1 \in \mathbb{N}$ such that $\sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon$. By Theorem 4.14, there exists a full $X_2$-equivariant homomorphism $\alpha_1 : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \alpha_1) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}})$ and

$$\|\alpha_1 \circ \psi \circ \phi(a) - a\| < \frac{1}{2^{m_1+1}}$$

for all $a \in \mathcal{F}_1$. Set $\phi_1 = \phi$ and $\psi_1 = \alpha_1 \circ \psi$. Then $KK(X_2; \psi_1) = KK(X_2; \psi)$ and $\|\psi_1 \circ \phi_1(a) - a\| < \frac{1}{2^{m_1+1}}$ for all $a \in \mathcal{F}_1$.

Set $\mathcal{G}_1 = \mathcal{G}_1 \cup \phi(\mathcal{F}_1)$. Note that $KK(X_2; \phi \circ \psi_1) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$. Hence, by Theorem 4.14, there exists a full $X_2$-equivariant homomorphism $\beta_1 : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \beta_1) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$ and

$$\|\beta_1 \circ \phi \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Set $\phi_2 = \beta_1 \circ \phi$. Then $KK(X_2; \phi_2) = KK(X_2; \phi)$ and

$$\|\phi_2 \circ \psi_1(x) - x\| < \frac{1}{2^{m_1+1}}$$

for all $x \in \mathcal{G}_1$. Note that for all $x \in \mathcal{F}_1$, then

$$\|\phi(x) - \phi_2(x)\| \leq \|\phi_1(x) - \phi_2 \circ \psi_1(\phi_1(x))\| + \|\phi_2 \circ \psi_1(\phi_1(x)) - \phi_2(x)\| < \frac{1}{2^{m_1+1}} + \|\phi_1 \circ \phi_1(x) - x\| < \frac{1}{2^{m_1}}.$$

Set $\mathcal{F}_2 = \mathcal{F}_2 \cup \phi_2(\mathcal{G}_1)$. Note that $KK(X_2; \phi \circ \phi_2) = KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}})$. Hence, by Theorem 4.14, there exists a full $X_2$-equivariant homomorphism $\alpha_2 : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ and

$$\|\alpha_2 \circ \psi \circ \phi_2(b) - b\| = 0$$

for all $b \in \mathcal{B}$.

\[\square\]
\( \mathfrak{A}_1 \otimes \mathbb{K} \) such that \( KK(X_2; \alpha_2) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}) \) and
\[
\| \alpha_2 \circ \psi \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}}
\]
for all \( a \in \mathcal{F}_2 \). Set \( \psi_2 = \alpha_2 \circ \psi \). Then \( KK(X_2; \psi_2) = KK(X_2; \psi) \) and
\[
\| \psi_2 \circ \phi_2(a) - a \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{F}_2 \).

Set \( \mathcal{G}_2 = \overline{\mathcal{G}_2} \cup \phi_2(\mathcal{F}_2) \). Note that \( KK(X_2; \phi \circ \psi_2) = KK(X_2; \phi \circ \psi_2) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}) \).

Hence, by Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \beta_2 : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that \( KK(X_2; \beta_2) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}) \) and
\[
\| \beta_2 \circ \phi \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{G}_2 \). Set \( \phi_3 = \beta_2 \circ \phi \). Then \( KK(X_2; \phi_3) = KK(X_2; \phi) \) and
\[
\| \phi_3 \circ \psi_2(x) - x \| < \frac{1}{2^{m_1+2}}
\]
for all \( x \in \mathcal{G}_2 \). Note that for all \( x \in \mathcal{F}_2 \), we have that
\[
\| \phi_2(x) - \phi_3(x) \| \leq \| \phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x)) \| + \| \phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x) \| < \frac{1}{2^{m_1+2}} + \| \phi_2(\phi_2(x)) - x \| < \frac{1}{2^{m_1+1}}.
\]

Continuing this process, we have constructed a sequence \( \{\mathcal{F}_n\}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_1 \otimes \mathbb{K} \), a sequence \( \{\mathcal{G}_n\}_{n=1}^{\infty} \) of finite subsets of \( \mathfrak{A}_2 \otimes \mathbb{K} \), a sequence of full \( X_2 \)-equivariant homomorphisms \( \{\psi_n : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}\}_{n=1}^{\infty} \), and a sequence of full \( X_2 \)-equivariant homomorphisms \( \{\phi_n : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}\}_{n=1}^{\infty} \) such that
1. \( KK(X_2; \phi_n) = KK(X_2; \phi) \) for all \( n \in \mathbb{N} \) and \( \phi_1 = \phi \);
2. \( KK(X_2; \psi_n) = KK(X_2; \psi) \) for all \( n \in \mathbb{N} \);
3. \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \);
4. \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \) and \( \mathcal{G}_n \subseteq \mathcal{G}_{n+1} \);
5. for each \( x \in \mathcal{F}_n \) and for each \( x \in \mathcal{G}_n \)
\[
\| \psi_n \circ \phi_n(x) - x \| < \frac{1}{2^{m_1+n}} \quad \text{and} \quad \| \phi_{n+1} \circ \psi_n(x) - x \| < \frac{1}{2^{m_1+n}}
\]
6. for each \( x \in \mathcal{F}_n \),
\[
\| \phi_n(x) - \phi_{n+1}(x) \| < \frac{1}{2^{m_1+n-1}}
\]
Since \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes \mathbb{K} \) and \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \), we have that \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes \mathbb{K} \).

Similarly, \( \bigcup_{n=1}^{\infty} \mathcal{G}_n \) is dense in \( \mathfrak{A}_2 \otimes \mathbb{K} \). Therefore, there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K} \) such that
\[
\| \gamma(a) - \phi_n(a) \| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k}.
\]
Since $\mathcal{F} \subseteq \mathcal{F}_1$, we have that
\[ \|\phi(x) - \gamma(x)\| = \|\phi_1(x) - \gamma(x)\| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon. \]

Since
\[ \lim_{n \to \infty} \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} = 0, \]
we have that
\[ \lim_{n \to \infty} \|\gamma(a) - \phi_n(a)\| = 0 \]
for all $a \in \mathfrak{A}_1 \otimes \mathbb{K}$. Since $\mathfrak{A}_1 \otimes \mathbb{K}$ is semiprojective, by Proposition 2.3 of [7], there exists $N \in \mathbb{N}$ such that $\gamma$ and $\phi_N$ are homotopic. Hence, by Theorem 5.5 of [8], $KK(X_2; \gamma) = KK(X_2; \phi_N) = x$. \(\square\)

4.3. Unital Classification. We know combine the above results with the Meta-theorem of Section [3] (see Theorem 3.3) to get a strong classification for a class of unital $C^*$-algebras which includes all unital graph $C^*$-algebras with exactly one non-trivial ideal.

**Corollary 4.16.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_n$ such that $\mathfrak{A}_i$ has real rank zero, $\mathfrak{A}_i[n]$ is a Kirchberg algebra in $\mathcal{N}$, and $\mathfrak{A}_i[1, n-1]$ is an AF-algebra. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_n}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{L}(\mathcal{C}(X_n))$ and $K_{X_n}(x)_{X_n([1_{\mathfrak{A}_1}])} = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\phi: \mathfrak{A}_1 \rightarrow \mathfrak{A}_2$ such that $K_{X_n}(\phi) = K_{X_n}(x)$.

**Proof.** Since $\mathfrak{A}_i[1]$ and $\mathfrak{A}_i[2]$ are separable and nuclear, we have that $\mathfrak{A}_i$ is separable and nuclear. Since $\mathfrak{A}_i[1, n-1]$ is an AF-algebra and $\mathfrak{A}_i[n]$ is a Kirchberg algebra, they both have the stable weak cancellation property. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation property. By Lemma 4.16, for each tight $C^*$-algebra $\mathfrak{A}$ over $X_n$, we have that $K_{X_n}(Ad(u)|_{\mathfrak{A}})$ for each unitary $u \in \mathcal{M}(\mathfrak{A})$. A computation shows that $K_{X_n}(-)$ satisfies (1), (2), and (3) of Theorem 3.3 since $K_n(-)$ does. The corollary now follows from Theorem 3.3 and Theorem 4.7. \(\square\)

**Corollary 4.17.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i[2] \cong \mathbb{K}$ and $\mathfrak{A}_i[1]$ is a Kirchberg algebra in $\mathcal{N}$. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{L}(\mathcal{C}(X_2))$ and $K_{X_2}(x)_{X_2([1_{\mathfrak{A}_1}])} = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. If $\mathfrak{A}_i \otimes \mathbb{K}$ is semiprojective, then there exists an isomorphism $\gamma: \mathfrak{A}_1 \otimes \mathbb{K} \rightarrow \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \gamma) = x$.

**Proof.** Since $\mathfrak{A}_i[1]$ and $\mathfrak{A}_i[2]$ are separable and nuclear, we have that $\mathfrak{A}_i$ is separable and nuclear. Since $\mathfrak{A}_i[2]$ and $\mathfrak{A}_i[1]$ have real rank zero and $K_1(\mathfrak{A}_i[2]) = 0$, we have that $\mathfrak{A}$ has real rank zero. Since $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by Lemma 3.15 of [15], $\mathfrak{A}$ has the stable weak cancellation property.

By Lemma 1.5 of [16], the extension $0 \rightarrow \mathfrak{A}_i[2] \rightarrow \mathfrak{A}_i \rightarrow \mathfrak{A}_i[1] \rightarrow 0$ is full, and hence by Proposition 1.6 of [16], $0 \rightarrow \mathfrak{A}_i[2] \otimes \mathbb{K} \rightarrow \mathfrak{A}_i \otimes \mathbb{K} \rightarrow \mathfrak{A}_i[1] \otimes \mathbb{K} \rightarrow 0$ is full. The corollary now follows from Theorem 4.1(ii), Theorem 4.15 and Theorem 3.3. \(\square\)
It is an open question to determine if every unital, separable, nuclear, tight $C^*$-algebra $\mathfrak{A}$ over $X_2$ whose unique proper nontrivial ideal is isomorphic to $K$ and quotient is a Kirchberg algebra in $\mathcal{N}$ with finitely generated $K$-theory is semiprojective. The following results show that under some $K$-theoretical conditions, $\mathfrak{A}$ is semiprojective.

**Lemma 4.18.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$ with $C^*(E)[1]$ being purely infinite. Then $C^*(E)$ and $C^*(E) \otimes K$ are semiprojective.

**Proof.** The fact that $C^*(E)$ is semiprojective follows from the results of [12]. By Proposition 6.4 of [18], $C^*(E)[2]$ is stable. Since $C^*(E)$ is a unital $C^*$-algebra, by Lemma 1.5 of [15], the extension $\mathfrak{e} : 0 \rightarrow C^*(E)[2] \rightarrow C^*(E) \rightarrow C^*(E)[1] \rightarrow 0$ is a full extension. By Proposition 3.21 and Corollary 3.22 of [15], $C^*(E)$ is properly infinite. Therefore, by Theorem 4.1 of [9], $C^*(E) \otimes K$ is semiprojective. □

**Proposition 4.19.** Let $\mathfrak{A}$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$. If $\mathfrak{A}[2] \cong K$ and $\mathfrak{A}[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}[1])) \leq \text{rank}(K_0(\mathfrak{A}[1]))$, $K_1(\mathfrak{A}[1])$ is free, and the $K$-groups of $\mathfrak{A}[i]$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{A} \otimes K$ are semiprojective. Consequently, $\mathfrak{A}$ is semiprojective.

**Proof.** By Lemma 1.5 of [15], $\mathfrak{e} : 0 \rightarrow \mathfrak{A}[2] \rightarrow \mathfrak{A} \rightarrow \mathfrak{A}[1] \rightarrow 0$ is a full extension. By Corollary 3.22 of [15], $K_0(\mathfrak{A}) = K_0(\mathfrak{A})$. By Theorem 6.4 of [11], there exists a graph $E$ with finitely many vertices such that $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$ such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$. Since $E$ has finitely many vertices, $C^*(E)$ is unital. Since $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$, we have that $C^*(E)[1]$ is a Kirchberg algebra. By Theorem 3.9 of [15], we have that $\mathfrak{A} \otimes K \cong C^*(E) \otimes K$. By Lemma 4.18, $C^*(E)$ and $C^*(E) \otimes K$ are semiprojective. Hence, by Proposition 2.7 of [9], $\mathfrak{A}$ and $\mathfrak{A} \otimes K$ are semiprojective. □

**Corollary 4.20.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_i[2] \cong K$ and $\mathfrak{A}_i[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}_i[1])) \leq \text{rank}(K_0(\mathfrak{A}_i[1]))$, $K_1(\mathfrak{A}_i[1])$ is free, and the $K$-groups of $\mathfrak{A}_i$ are finitely generated. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathcal{L}(C(X_2))$ and $K_{X_2}(x)_{X_2}[1_{\mathfrak{A}_2}] = [1_{\mathfrak{A}_2}]$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes K \rightarrow \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \gamma) = x$.

**Proof.** This follows from Proposition 4.19 and Corollary 4.17. □

5. Applications

Let $E$ be a graph satisfying Condition (K) (in particular, if $C^*(E)$ has finitely many ideals, then $E$ satisfies Condition (K)). Let $\mathfrak{I}_1, \mathfrak{I}_2$ be ideals of $C^*(E)$ such that $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ and $\mathfrak{I}_2/\mathfrak{I}_1$ is simple. Then by Theorem 5.1 of [38] and Corollary 3.5 of [2], $\mathfrak{I}_2/\mathfrak{I}_1$ is a simple graph $C^*$-algebra. Hence, $\mathfrak{I}_2/\mathfrak{I}_1$ is either a Kirchberg algebra or an AF algebra.

5.1. Classification of graph $C^*$-algebras with exactly one ideal.

**Lemma 5.1.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a simple AF-algebra. Then $C^*(E) \otimes K \cong K$. Consequently, if $F$ is a graph with finitely many vertices such that $C^*(F)$ is a tight $C^*$-algebra over $X_2$ and $C^*(F)[2]$ is an AF-algebra, then $C^*(F)[2] \cong K$. □
Proof. We claim that $E$ is a finite graph. By Corollary 2.13 and Corollary 2.15 of \cite{9}, $E$ has no cycles, and for every vertex $v_0$ that emits infinitely many edges and for each vertex $v$, there exists a path from $v$ to $v_0$. Since $E$ has no cycles, we have that every vertex of $E$ emits only finitely many edges. Hence, $E$ is a finite graph. By Proposition 1.18 of \cite{30}, $C^*(E) \cong M_n$.

We now prove the second statement. First note that $C^*(F)[2]$ is a simple AF-algebra. Since $C^*(F)[2]$ is stably isomorphic to a subgraph of $E$, $C^*(F)[2] \otimes \mathbb{K} \cong C^*(E)$ for some graph $E$ with finitely many vertices. Since $C^*(E)$ is a simple AF-algebra, we have that $C^*(E) \otimes \mathbb{K} \cong \mathbb{K}$. Hence, $C^*(F)[2] \otimes \mathbb{K} \cong \mathbb{K}$ which implies that $C^*(F)[2] \cong M_n$ or $C^*(F)[2] \cong \mathbb{K}$. Since $C^*(E)[2]$ is a non-unital $C^*$-algebra ($C^*(E)$ is a tight $C^*$-algebra over $X_2$), we have that $C^*(F)[2] \cong \mathbb{K}$. □

Definition 5.2. For a $C^*$-algebra $\mathcal{A}$, set

$$\Sigma \mathcal{A} = \{x \in K_0(\mathcal{A}) : x = [p] \text{ for some projection } p \text{ in } \mathcal{A}\}.$$  

Let $\mathcal{B}$ be a $C^*$-algebra. An order isomorphism $\alpha : K_0(\mathcal{A}) \to K_0(\mathcal{B})$ is scale preserving if one of the following holds:

1. $\mathcal{A}$ is unital if and only if $\mathcal{B}$ unital and $\alpha([1_\mathcal{A}]) = [1_\mathcal{B}]$.
2. $\mathcal{A}$ is non-unital if and only if $\mathcal{B}$ is non-unital and $\alpha(\Sigma \mathcal{A}) = \Sigma \mathcal{B}$.

Theorem 5.3. Let $E_1$ and $E_2$ be graphs with finitely many vertices and $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. If $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ is an isomorphism such that $\alpha Y$ is scale preserving for all $Y \in \mathcal{L}C(X_2)$, then there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Proof. Since $E_i$ has finitely many vertices, $C^*(E_1)$ and $C^*(E_2)$ are unital $C^*$-algebras.

Case 1: Suppose $C^*(E_1)$ is an AF-algebra. Then $C^*(E_2)$ is an AF-algebra. Hence, the result follows from Elliott’s classification of AF-algebras \cite{19}.

Case 2: Suppose $C^*(E_1)$ is not an AF-algebra. Then $C^*(E_2)$ is not an AF-algebra.

Subcase 2.1: Suppose $C^*(E_1)[1]$ is an AF-algebra. Then $C^*(E_2)[1]$ is an AF-algebra. By Corollary 4.10 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$.

Subcase 2.2: Suppose $C^*(E_1)[1]$ is a Kirchberg algebra. Then $C^*(E_2)[1]$ is a Kirchberg algebra. Since $C^*(E_i)$ is not an AF-algebra, either $C^*(E_i)[2]$ is Kirchberg algebra or an AF-algebra.

Suppose $C^*(E_i)[2]$ is a Kirchberg algebra. By Theorem 2.4 of \cite{32}, there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. Suppose $C^*(E_i)[2]$ is an AF-algebra. Then, by Lemma 5.1 $C^*(E_i)[2] \cong \mathbb{K}$. By Corollary 4.20 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \to C^*(E_2)$ such that $K_{X_2}(\phi) = \alpha$. □

The following theorem completes the classification of graph $C^*$-algebras with exactly one non-trivial ideal.

Corollary 5.4. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. Then $C^*(E_1) \cong C^*(E_2)$ if and only if there exists an isomorphism $\alpha : K_{X_2}^+(C^*(E_1)) \to K_{X_2}^+(C^*(E_2))$ such that $\alpha Y$ is a scale preserving isomorphism for all $Y \in \mathcal{L}C(X_2)$. 

Proof. The only case that is not covered by Theorem 4.9 of [15] is the case that \( C^*(E_i) \) is unital. The unital case follows from Theorem 5.3 because of Theorem 3.3.

5.2. Classification of graph \( C^* \)-algebras with more than one ideal. For a tight \( C^* \)-algebra \( A \) over \( X_n \), the finite and infinite simple sub-quotients of \( A \) are separated if there exists \( U \in \mathcal{O}(X_n) \) such that either

1. \( \mathcal{A}(U) \) is an AF-algebra and \( \mathcal{A}(X_n \setminus U) \otimes \mathcal{O}_\infty \cong \mathcal{A}(X_n \setminus U) \) or
2. \( \mathcal{A}(X_n \setminus U) \) is an AF-algebra and \( \mathcal{A}(U) \otimes \mathcal{O}_\infty \cong \mathcal{A}(U) \).

In [14], the authors proved that if \( \mathcal{A}_1 \) and \( \mathcal{A}_2 \) are graph \( C^* \)-algebras that are tight \( C^* \)-algebras over \( X_n \) such that the finite and infinite simple sub-quotients are separated, then \( \mathcal{A}_1 \otimes K \cong \mathcal{A}_2 \otimes K \) if and only if \( K_{X_n}^+(\mathcal{A}_1) \cong K_{X_n}^+(\mathcal{A}_2) \). We will show in this section that under mild \( K \)-theoretical conditions, we may remove the separated condition for the case \( n = 3 \).

Lemma 5.5. Let \( E \) be a graph such that \( C^*(E) \) is a tight \( C^* \)-algebra over \( X_n \).

(i) If \( C^*(E)[n] \) and \( C^*(E)[1] \) are purely infinite and \( C^*(E)[2, n - 1] \) is an AF-algebra, then

\[
\epsilon_1 : 0 \to C^*(E)[2, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1] \otimes K \to 0
\]

is a full extension.

(ii) If \( C^*(E)[k, n] \) and \( C^*(E)[1, k - 2] \) are AF-algebras and \( C^*(E)[k - 1] \) is purely infinite, then

\[
\epsilon_2 : 0 \to C^*(E)[k, n] \otimes K \to C^*(E) \otimes K \to C^*(E)[1, k - 1] \otimes K \to 0
\]

is a full extension.

Proof. Suppose \( C^*(E)[n] \) and \( C^*(E)[1] \) are purely infinite and \( C^*(E)[2, n - 1] \) is an AF-algebra. Note that \( C^*(E)[1, n - 1] \) and \( C^*(E)[2, n - 1] \) are isomorphic to a graph \( C^* \)-algebra, by Proposition 3.10 of [18],

\[
0 \to C^*(E)[2, n - 1] \otimes K \to C^*(E)[1, n - 1] \otimes K \to C^*(E)[1] \otimes K \to 0
\]

is a full extension. Since \( C^*(E)[n] \otimes K \) is a purely infinite simple \( C^* \)-algebra, we have that

\[
0 \to C^*(E)[n] \otimes K \to C^*(E)[2, n] \otimes K \to C^*(E)[2, n - 1] \otimes K \to 0
\]

is a full extension. Hence, by Proposition 3.2 of [17], \( \epsilon_1 \) is a full extension.

Suppose \( C^*(E)[k, n] \) and \( C^*(E)[1, k - 2] \) are AF-algebras and \( C^*(E)[k - 1] \) is purely infinite. Note that \( C^*(E)[k, n] \) is the largest ideal of \( C^*(E)[k - 1, n] \) such that \( C^*(E)[k, n] \) is an AF-algebra and \( C^*(E)[k - 1, n] \) is purely infinite. Since \( C^*(E)[k - 1, n] \otimes K \) is isomorphic to a graph \( C^* \)-algebra, by Proposition 3.10 of [18],

\[
0 \to C^*(E)[k, n] \otimes K \to C^*(E)[k - 1, n] \otimes K \to C^*(E)[k - 1] \otimes K \to 0
\]

is a full extension. By Proposition 5.4 of [14], \( \epsilon_2 \) is a full extension.

Theorem 5.6. Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight \( C^* \)-algebra over \( X_n \).

(i) \( C^*(E_1)[n] \) and \( C^*(E_1)[1] \) are purely infinite;

(ii) \( C^*(E_1)[2, n - 1] \) is an AF-algebra; and
(iii) $KK^1(C^*(E_1)[1], C^*(E_2)[2, n]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, n])$.

Then $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$ if and only if $K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K)$.

Proof. Let $\epsilon_i$ be the extension

$0 \to C^*(E_1)[2, n] \otimes K \to C^*(E_1) \otimes K \to C^*(E_1)[1] \otimes K \to 0$.

By Lemma 5.5(i), $\epsilon_i$ is a full extension. Suppose $\alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K)$.

Lift $\alpha$ to an invertible element $x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K)$. Note that $r_{X_n}^{[2,n]}(x)$ is invertible in $KK([2,n]; C^*(E_1)[2, n] \otimes K, C^*(E_2)[2, n] \otimes K)$ and $r_{X_n}^{[1]}(x)$ is invertible in $KK(C^*(E_1)[1] \otimes K, C^*(E_2)[1] \otimes K)$.

By Theorem 4.7 there exists an isomorphism $\phi_0 : C^*(E_1)[2, n] \otimes K \to C^*(E_2)[2, n] \otimes K$ such that $KL(\phi_0) = z$, where $z$ is the invertible element of $KL(C^*(E_1)[2, n] \otimes K, C^*(E_2)[2, n] \otimes K)$ induced by $r_{X_n}^{[2,n]}(x)$. By the Kirchberg-Phillips classification (21 and 29), there exists an isomorphism $\phi_2 : C^*(E_1)[1] \otimes K \to C^*(E_2)[1] \otimes K$ such that $KK(\phi_2) = r_{X_n}^{[1]}(x)$.

Consider $C^*(E_i)$ as a $C^*$-algebra over $X_2$ by setting $C^*(E_i)[2] = C^*(E_i)[2, n]$ and $C^*(E_i)[1, 2] = C^*(E_i)$. Let $y$ be the invertible element in $KK(X_2, C^*(E_1), C^*(E_2))$ induced by $x$. Note that $r_{X_2}^{[1]}(y) = r_{X_n}^{[1]}(x) = KK(\phi_2)$ and $KL(r_{X_2}^{[2]}(y)) = z = KL(\phi_0)$ in $KL(C^*(E_1)[2, n], C^*(E_2)[2, n])$.

By Theorem 3.7 of [21],

$$r_{X_2}^{[1]}(y) \times [\tau_{e_2}] = [\tau_{e_1}] \times r_{X_2}^{[2]}(y)$$

in $KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$, where $\epsilon_i$ is the extension

$0 \to C^*(E_i)[2, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0$.

Thus,

$$KL(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KL(\phi_0)$$

in $KL(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$. Since $KL(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) = KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$,

$$KK(\phi_2) \times [\tau_{e_2}] = [\tau_{e_1}] \times KK(\phi_0)$$

in $KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$. By Lemma 4.5 of [21], $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$. \qed

Theorem 5.7. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_n$.

Suppose

(i) $C^*(E_i)[k, n]$ and $C^*(E_i)[1, k - 2]$ are AF-algebras;

(ii) $C^*(E_i)[k - 1]$ is purely infinite; and

(iii) $KK^1(C^*(E_i)[1], C^*(E_2)[k, n]) = KL^1(C^*(E_1)[1, k - 1], C^*(E_2)[k, n])$.

Then $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$ if and only if $K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K)$.

Proof. Let $\epsilon_i$ be the extension $0 \to C^*(E_0)[k, n] \otimes K \to C^*(E_1) \otimes K \to C^*(E_1)[1, k - 1] \otimes K \to 0$.

By Lemma 5.5(ii), $\epsilon_i$ is a full extension. Suppose $\alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K)$.

Lift $\alpha$ to an invertible element $x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K)$. Note that $r_{X_n}^{[k,n]}(x)$ is invertible in $KK([k, n]; C^*(E_1)[k, n] \otimes K, C^*(E_2)[k, n] \otimes K)$ and $r_{X_n}^{[1,k-1]}(x)$ is invertible in $KK(C^*(E_1)[1, k - 1], C^*(E_2)[1, k - 1])$.

By Theorem 4.7 there exists an isomorphism
\( \phi_2 : C^*(E_1)[1,k-1] \otimes K \to C^*(E_2)[1,k-1] \otimes K \) such that \( KL(\phi_2) = z_2 \), where \( z_2 \) is the invertible element in \( KL(C^*(E_1)[1,k-1], C^*(E_2)[1,k-1]) \) induced by \( r_{X_2}^{[1]}(x) \). By Elliott's classification [19], there exists an isomorphism \( \phi_0 : C^*(E_1)[k,n] \otimes K \to C^*(E_2)[k,n] \otimes K \) such that \( KK(\phi_0) = z_0 \), where \( z_0 \) is the invertible element in \( KK(C^*(E_1)[k,n] \otimes K, C^*(E_2)[k,n] \otimes K) \) induced by \( r_{X_2}^{[k,n]}(x) \).

Consider \( C^*(E_i) \) as a C*-algebra over \( X_2 \) by setting \( C^*(E_i)[2] = C^*(E_i)[k,n] \) and \( C^*(E_i)[1,2] = C^*(E_i) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( KL(r_{X_2}^{[1]}(y)) = z_2 = KL(\phi_2) \) and \( r_{X_2}^{[2]}(y) = z_0 = KK(\phi_0) \). By Theorem 3.7 of [14],

\[
 r_{X_2}^{[1]}(y) \times [\tau_{\epsilon_1}] \times r_{X_2}^{[2]}(y)
\]
in \( KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \), where \( \epsilon_1 \) is the extension

\[
0 \to C^*(E_i)[k,n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1,k-1] \otimes K \to 0.
\]

Thus,

\[
KL(\phi_2) \times [\tau_{\epsilon_1}] \times KL(\phi_0)
\]
in \( KL^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \). Since \( KL^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) = KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \),

\[
KK(\phi_2) \times [\tau_{\epsilon_1}] \times KK(\phi_0)
\]
in \( KK^1(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \). By Lemma 4.5 of [14], \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \).

**Theorem 5.8.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight C*-algebra over \( X_3 \). Suppose \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is purely infinite and \( K_0(C^*(E_1)[1,2]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1,2] \) is an AF-algebra. Then \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \) if and only if \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \).

**Proof.** The “only if” direction is clear. Suppose \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)) \). Suppose \( C^*(E_1)[1] \) is purely infinite. Then \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_2^1(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0 \). Since \( K_1(C^*(E_1)[1]) \) is a free group, \( \text{Pext}_2^1(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0 \). Hence,

\[
KK^1(C^*(E_1)[1,2], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[2,3]).
\]

Suppose \( C^*(E_1)[1] \) is an AF-algebra. Then \( K_0(C^*(E_1)[1,2]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_2^1(K_0(C^*(E_1)[1,2]), K_0(C^*(E_2)[3])) = 0 \). Since \( K_1(C^*(E_1)[1,2]) \) is a free group, \( \text{Pext}_2^1(K_1(C^*(E_1)[1,2]), K_1(C^*(E_2)[3])) = 0 \). Therefore,

\[
KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]).
\]

**Case 1:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are separated. Hence, by Theorem 6.9 of [14], \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \).

**Case 2:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are not separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are not separated.
Subcase 2.1: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are purely infinite and $C^*(E_1)[2]$ is an AF-algebra. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are purely infinite and $C^*(E_2)[2]$ is an AF-algebra. Then by the above paragraph we have that $KK_1(C^*(E_1)[1], C^*(E_2)[3]) = KL(C^*(E_1)[1], C^*(E_2)[2, 3])$. Hence, by Theorem 5.6, $C^*(E_1)[1] \cong C^*(E_2)[1] \cong C^*(E_1)[2, 3]$.

Subcase 2.2: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are AF-algebra and $C^*(E_1)[2]$ is purely infinite. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are AF-algebras and $C^*(E_2)[2]$ is purely infinite. Then by the above paragraph we have that

$$KK_1(C^*(E_1)[1], C^*(E_2)[3]) = KL(C^*(E_1)[1], C^*(E_2)[2, 3]).$$

Hence, by Theorem 5.7, $C^*(E_1) \cong C^*(E_2) \cong C^*(E_1)[2, 3]$. \hfill $\square$

**Corollary 5.9.** Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_3$. Suppose that $K_0(C^*(E_i))$ is finitely generated. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2))$.

**Proof.** Since $C^*(E_1)$ is real rank zero, the canonical projection $\pi : C^*(E_1) \to C^*(E_1)[1]$ induces a surjective homomorphism $\pi : K_0(C^*(E_1)) \to K_0(C^*(E_1)[1])$. Hence, $K_0(C^*(E_1)[1])$ is finitely generated since $K_0(C^*(E_i))$ is finitely generated. The corollary now follows from Theorem 5.8. \hfill $\square$

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