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STRONG CLASSIFICATION OF EXTENSIONS OF CLASSIFIABLE
$C^*$-ALGEBRAS

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Abstract. We show that certain extensions of classifiable $C^*$-algebra are strongly classified by the associated six-term exact sequence in $K$-theory together with the positive cone of $K_0$-groups of the ideal and quotient. We apply our result to give a complete classification of graph $C^*$-algebras with exactly one ideal.

1. Introduction

The classification program for $C^*$-algebras has for the most part progressed independently for the classes of infinite and finite $C^*$-algebras, and great strides have been made in this program for each of these classes. In the finite case, Elliott’s Theorem classifies all AF-algebras up to stable isomorphism by the ordered $K_0$-group. In the infinite case, there are a number of results for purely infinite $C^*$-algebras. The Kirchberg-Phillips Theorem classifies certain simple purely infinite $C^*$-algebras up to stable isomorphism by the $K_0$-group together with the $K_1$-group. For nonsimple purely infinite $C^*$-algebras many partial results have been obtained: Rørdam has shown that certain purely infinite $C^*$-algebras with exactly one proper nontrivial ideal are classified up to stable isomorphism by the associated six-term exact sequence of $K$-groups [34], the second named author has shown that nonsimple Cuntz-Krieger algebras satisfying Condition (II) are classified up to stable isomorphism by their filtered $K$-theory [31, Theorem 4.2], and Meyer and Nest have shown that certain purely infinite $C^*$-algebras with a linear ideal lattice are classified up to stable isomorphism by their filtrated $K$-theory [28, Theorem 4.14]. However, in all of these situations the nonsimple $C^*$-algebras that are classified have the property that they are either AF-algebras or purely infinite, and consequently all of their ideals and quotients are of the same type.

Recently, the authors have provided a framework for classifying nonsimple $C^*$-algebras that are not necessarily AF-algebras or purely infinite $C^*$-algebras. In particular, the authors have shown in [16] that certain extensions of classifiable $C^*$-algebras may be classified up to stable isomorphism by their associated six-term exact sequence in $K$-theory. This has allowed for the classification of certain nonsimple $C^*$-algebras in which there are ideals and quotients of mixed type (some finite and some infinite). The results in [16] was then used by the first named author and Tomforde in [18] to classify a certain class of non-simple graph $C^*$-algebras, showing that graph $C^*$-algebras with exactly one non-trivial ideal can be classified up to stable isomorphism by their associated six-term exact sequence in $K$-theory. The authors in [15] then showed that all non-unital graph $C^*$-algebras with exactly one
non-trivial ideal can be classified up to isomorphism by their associated six-term exact sequence in K-theory. In this paper, we complete the classification of graph $C^*$-algebras with exactly one non-trivial ideal by classifying those that are unital. Our methods here differ rather dramatically from the methods in [18] and [15]. In particular, we use the traditional methods of classification via existence and uniqueness theorems. As a consequence, for unital graph $C^*$-algebras $\mathfrak{A}$ and $\mathfrak{B}$ with exactly one non-trivial ideal, then any isomorphism between the associated six-term exact sequence in K-theory which preserves the unit lifts to an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.

2. Preliminaries

2.1. $C^*$-algebras over topological spaces. Let $X$ be a topological space and let $\mathcal{O}(X)$ be the set of open subsets of $X$, partially ordered by set inclusion $\subseteq$. A subset $Y$ of $X$ is called locally closed if $Y = U \setminus V$ where $U, V \in \mathcal{O}(X)$ and $V \subseteq U$. The set of all locally closed subsets of $X$ will be denoted by $\mathbb{L}(X)$. The set of all connected, non-empty, locally closed subsets of $X$ will be denoted by $\mathbb{L}_c(X)$.

The partially ordered set $(\mathcal{O}(X), \subseteq)$ is a complete lattice, that is, any subset $S$ of $\mathcal{O}(X)$ has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. More precisely, for any subset $S$ of $\mathcal{O}(X)$,

$$\bigwedge_{U \in S} U = \left( \bigcap_{U \in S} U \right)^\circ \quad \text{and} \quad \bigvee_{U \in S} U = \bigcup_{U \in S} U.$$

For a $C^*$-algebra $\mathfrak{A}$, let $\mathfrak{l}(\mathfrak{A})$ be the set of closed ideals of $\mathfrak{A}$, partially ordered by $\subseteq$. The partially ordered set $(\mathfrak{l}(\mathfrak{A}), \subseteq)$ is a complete lattice. More precisely, for any subset $S$ of $\mathfrak{l}(\mathfrak{A})$,

$$\bigwedge_{J \in S} J = \bigcap_{J \in S} J \quad \text{and} \quad \bigvee_{J \in S} J = \bigcup_{J \in S} J.$$

Definition 2.1. Let $\mathfrak{A}$ be a $C^*$-algebra. Let $\text{Prim}(\mathfrak{A})$ denote the primitive ideal space of $\mathfrak{A}$, equipped with the usual hull-kernel topology, also called the Jacobson topology.

Let $X$ be a topological space. A $C^*$-algebra over $X$ is a pair $(\mathfrak{A}, \psi)$ consisting of a $C^*$-algebra $\mathfrak{A}$ and a continuous map $\psi : \text{Prim}(\mathfrak{A}) \to X$. A $C^*$-algebra over $X$, $(\mathfrak{A}, \psi)$, is separable if $\mathfrak{A}$ is a separable $C^*$-algebra. We say that $(\mathfrak{A}, \psi)$ is tight if $\psi$ is a homeomorphism.

We always identify $\mathcal{O}(\text{Prim}(\mathfrak{A}))$ and $\mathfrak{l}(\mathfrak{A})$ using the lattice isomorphism

$$U \mapsto \bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p.$$

Let $(\mathfrak{A}, \psi)$ be a $C^*$-algebra over $X$. Then we get a map $\psi^* : \mathcal{O}(X) \to \mathcal{O}(\text{Prim}(\mathfrak{A})) \cong \mathfrak{l}(\mathfrak{A})$ defined by

$$U \mapsto \{ p \in \text{Prim}(\mathfrak{A}) : \psi(p) \in U \} = \mathfrak{A}(U).$$

For $Y = U \setminus V \in \mathbb{L}(X)$, set $\mathfrak{A}(Y) = \mathfrak{A}(U)/\mathfrak{A}(V)$. By Lemma 2.15 of [27], $\mathfrak{A}(Y)$ does not depend on $U$ and $V$.

Example 2.2. For any $C^*$-algebra $\mathfrak{A}$, the pair $(\mathfrak{A}, \text{id}_{\text{Prim}(\mathfrak{A})})$ is a tight $C^*$-algebra over $\text{Prim}(\mathfrak{A})$. For each $U \in \mathcal{O}(\text{Prim}(\mathfrak{A}))$, the ideal $\mathfrak{A}(U)$ equals $\bigcap_{p \in \text{Prim}(\mathfrak{A}) \setminus U} p$. 

Example 2.3. Let $X_n = \{1, 2, \ldots, n\}$ partially ordered with $\leq$. Equip $X_n$ with the Alexandrov topology, so the non-empty open subsets are
\[ [a, n] = \{x \in X : a \leq x \leq n \} \]
for all $a \in X_n$; the non-empty closed subsets are $[1, b]$ with $b \in X_n$, and the non-empty locally closed subsets are those of the form $[a, b]$ with $a, b \in X_n$ and $a \leq b$. Let $(\mathcal{A}, \phi)$ be a C*-algebra over $X_n$. We will use the following notation throughout the paper:
\[ \mathcal{A}[k] = \mathcal{A}(\{k\}), \mathcal{A}[a, b] = \mathcal{A}(\{a, b\}), \text{ and } \mathcal{A}(i, j) = \mathcal{A}[i + 1, j]. \]

Using the above notation we have ideals $\mathcal{A}[a, n]$ such that
\[ \{0\} \leq \mathcal{A}[n] \leq \mathcal{A}[n - 1, n] \leq \cdots \leq \mathcal{A}[2, n] \leq \mathcal{A}[1, n] = \mathcal{A}. \]

Definition 2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras over $X$. A homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is $X$-equivariant if $\phi(\mathcal{A}(U)) \subseteq \mathcal{B}(U)$ for all $U \subset \mathcal{O}(X)$. Hence, for every $Y = U \setminus V$, $\phi$ induces a homomorphism $\phi_Y : \mathcal{A}(Y) \to \mathcal{B}(Y)$. Let $C^*\text{-alg}(X)$ be the category whose objects are $C^*$-algebras over $X$ and whose morphisms are $X$-equivariant homomorphisms.

An $X$-equivariant homomorphism $\phi : \mathcal{A} \to \mathcal{B}$ is said to be a full $X$-equivariant homomorphism if for all $Y \in \mathcal{L}(X)$, $\phi_Y(a)$ is norm-full in $\mathcal{B}(Y)$ for all norm-full elements $a \in \mathcal{A}(Y)$, i.e., the closed ideal of $\mathcal{B}(Y)$ generated by $\phi_Y(a)$ is $\mathcal{B}(Y)$ whenever the closed ideal of $\mathcal{A}(Y)$ generated by $a$ is $\mathcal{A}(Y)$.

Remark 2.5. Suppose $\mathcal{A}$ and $\mathcal{B}$ are tight C*-algebras over $X_n$. Then it is clear that $\phi : \mathcal{A} \to \mathcal{B}$ is an isomorphism if and only if $\phi$ is a $X_n$-equivariant isomorphism.

It is easy to see that if $\mathcal{A}$ and $\mathcal{B}$ are tight C*-algebras over $X_2$, then $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism if and only if $\phi$ is an $X_2$-equivariant homomorphism and $\phi_1$ and $\phi_2$ are injective. Also, if $\mathcal{A}$ and $\mathcal{A}[2]$ have non-zero projections $p$ and $q$ respectively, then there exists $\epsilon > 0$ such that if $\phi : \mathcal{A} \to \mathcal{B}$ is a full $X_2$-equivariant homomorphism and $\psi : \mathcal{A} \to \mathcal{B}$ is a homomorphism such that
\[ \|\phi(p) - \psi(p)\| < 1 \quad \|\phi(q) - \psi(q)\| < 1, \]
then $\psi$ is a full $X_2$-equivariant homomorphism.

Remark 2.6. Let $\epsilon_i : 0 \to \mathcal{B}_i \to \mathcal{E}_i \to \mathcal{A}_i \to 0$ be an extension for $i = 1, 2$. Note that $\mathcal{E}_i$ can be considered as a C*-algebra over $X_2 = \{1, 2\}$ by sending $\emptyset$ to the zero ideal, $\{2\}$ to the image of $\mathcal{B}_i$ in $\mathcal{E}_i$, and $\{1, 2\}$ to $\mathcal{E}_i$. Hence, there exists a one-to-one correspondence between $X_2$-equivariant homomorphisms $\phi : \mathcal{E}_1 \to \mathcal{E}_2$ and homomorphisms from $\epsilon_1$ and $\epsilon_2$.

2.2. The ideal related $K$-theory of $\mathcal{A}$.

Definition 2.7. Let $X$ be a topological space and let $\mathcal{A}$ be a C*-algebra over $X$. For open subsets $U_1, U_2, U_3$ of $X$ with $U_1 \subseteq U_2 \subseteq U_3$, set $Y_1 = U_2 \setminus U_1, Y_2 = U_3 \setminus U_1, Y_3 = U_3 \setminus U_1 \in \mathcal{L}(X)$. Then the diagram
\[
\begin{array}{ccc}
K_0(\mathcal{A}(Y_1)) & \xrightarrow{\iota_*} & K_0(\mathcal{A}(Y_2)) & \xrightarrow{\pi_*} & K_0(\mathcal{A}(Y_3)) \\
\downarrow{\partial_*} & & & & \downarrow{\partial_*} \\
K_1(\mathcal{A}(Y_3)) & \xrightarrow{\pi_*} & K_1(\mathcal{A}(Y_2)) & \xrightarrow{\iota_*} & K_1(\mathcal{A}(Y_1))
\end{array}
\]
is an exact sequence. The *ideal related K-theory of \( A \), \( K_X(\mathfrak{A}) \), is the collection of all \( K \)-groups thus occurring and the natural transformations \( \{\iota_*, \pi_*, \partial_*\} \). The *ideal related, ordered K-theory of \( A \), \( K^+_X(\mathfrak{A}) \), is \( K_X(\mathfrak{A}) \) of \( A \) together with \( K_0(\mathfrak{A}(Y))_+ \) for all \( Y \in \mathbb{L}(X) \).

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be \( C^* \)-algebras over \( X \), we will say that \( \alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B}) \) is an *isomorphism* if for all \( Y \in \mathbb{L}(X) \), there exists a graded group isomorphism

\[
\alpha_{Y,*} : K_*(\mathfrak{A}(Y)) \to K_*(\mathfrak{B}(Y))
\]

preserving all natural transformations. We say that \( \alpha : K^+_X(\mathfrak{A}) \to K^+_X(\mathfrak{B}) \) is an *isomorphism* if there exists an isomorphism \( \alpha : K_X(\mathfrak{A}) \to K_X(\mathfrak{B}) \) in such a way that \( \alpha_{Y,0} \) is an order isomorphism for all \( Y \in \mathbb{L}(X) \).

**Remark 2.8.** Meyer-Nest in [28] defined a similar functor \( \text{FK}_X(-) \) which they called filtrated \( K \)-theory. For all known cases in which there exists a UCT, the natural transformation from \( \text{FK}_X(-) \) to \( K_X(-) \) is an equivalence. In particular, this is true for the space \( X_n \).

If \( Y \in \mathbb{L}(X) \) such that \( Y = Y_1 \sqcup Y_2 \) with two disjoint relatively open subsets \( Y_1, Y_2 \subset 0(Y) \subset \mathbb{L}(C) \), then \( \mathfrak{A}(Y) \cong \mathfrak{A}(Y_1) \oplus \mathfrak{A}(Y_2) \) for any \( C^* \)-algebra over \( X \). Moreover, there is a natural isomorphism \( K_*(\mathfrak{A}(Y)) \) to \( K_*(\mathfrak{A}(Y_1)) \oplus K_*(\mathfrak{A}(Y_2)) \) which is a positive isomorphism from \( K_0(\mathfrak{A}(Y)) \) to \( K_0(\mathfrak{A}(Y_1)) \oplus K_0(\mathfrak{A}(Y_2)) \). If \( X \) is finite, then any locally closed subset is a disjoint union of its connected components. Therefore, we lose no information when we replace \( \mathbb{L}(X) \) by the subset \( \mathbb{L}(X)^* \).

**Notation 2.9.** Let \( \mathcal{N} \) be the bootstrap category of Rosenberg and Schochet in [37].

Let \( \mathfrak{A}(X) \) be the category whose objects are separable \( C^* \)-algebras over \( X \) and the set of morphisms is \( \text{KK}(X; \mathfrak{A}, \mathfrak{B}) \). For a finite topological space \( X \), let \( \mathcal{B}(X) \subset \mathfrak{A}(X) \) be the bootstrap category of Meyer and Nest in [27]. By Corollary 4.13 of [27], if \( \mathfrak{A} \) is a nuclear \( C^* \)-algebra over \( X \), then \( \mathfrak{A} \in \mathcal{B}(X) \) if and only if \( \mathfrak{A} \{ \{ x \} \} \in \mathcal{N} \) for all \( x \in X \).

**Theorem 2.10.** (Bonkat [4] and Meyer-Nest [28]) Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be in \( \mathfrak{A}(X_n) \) such that \( \mathfrak{A} \) is in \( \mathcal{B}(X_n) \), then the sequence

\[
0 \to \text{Ext}_{\mathcal{N}}(\text{FK}_X(\mathfrak{A})[\mathfrak{A}], \text{FK}_X(\mathfrak{B})) \to \text{KK}(X_n; \mathfrak{A}, \mathfrak{B}) \to \text{Hom}_{\mathcal{N}}(\text{FK}_X(\mathfrak{A}), \text{FK}_X(\mathfrak{B})) \to 0
\]

is exact. Consequently, if \( \mathfrak{B} \) is in \( \mathcal{B}(X_n) \), then an isomorphism from \( \text{FK}_X(\mathfrak{A}) \) to \( \text{FK}_X(\mathfrak{B}) \) lifts to an invertible element in \( \text{KK}(X_n; \mathfrak{A}, \mathfrak{B}) \).

**Corollary 2.11.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be in \( \mathcal{B}(X_n) \). Then an isomorphism from \( K_{X_n}(\mathfrak{A}) \) to \( K_{X_n}(\mathfrak{B}) \) lifts to an invertible element in \( \text{KK}(X_n; \mathfrak{A}, \mathfrak{B}) \).

**Proof.** This follows from Remark 2.8 and Theorem 2.10

**Remark 2.12.** Let \( x \in \text{KK}(X_n; \mathfrak{A}, \mathfrak{B}) \) be an invertible element. Then \( K_{X_n}(x) \) will denote the isomorphism from \( K_{X_n}(\mathfrak{A}) \) to \( K_{X_n}(\mathfrak{B}) \) given by \( \Gamma(x) \) where we have identified \( K_{X_n}(\mathfrak{A}) \) with \( \text{FK}_X(\mathfrak{A}) \) and \( K_{X_n}(\mathfrak{B}) \) with \( \text{FK}_X(\mathfrak{B}) \).

### 2.3. Functors

We now define some functors that will be used throughout the rest of the paper. Let \( X \) and \( Y \) be topological spaces. For every continuous function \( f : X \to Y \) we have a functor

\[
f : E^*\mathfrak{alg}(X) \to E^*\mathfrak{alg}(Y), \quad (A, \psi) \mapsto (A, f \circ \psi).
\]
(1) Define \( g_X^1 : X \to X_1 \) by \( g_X^1(x) = 1 \). Then \( g_X^1 \) is continuous. Note that the induced functor \( g_X^1 : \mathcal{C}\text{-alg}(X) \to \mathcal{C}\text{-alg}(X_1) \) is the forgetful functor.

(2) Let \( U \) be an open subset of \( X \). Define \( g_{U,X}^2 : X \to X_2 \) by \( g_{U,X}^2(x) = 1 \) if \( x \notin U \) and \( g_{U,X}^2(x) = 2 \) if \( x \in U \). Then \( g_{U,X}^2 \) is continuous. Thus the induced functor

\[
g_{U,X}^2 : \mathcal{C}\text{-alg}(X) \to \mathcal{C}\text{-alg}(X_2)
\]

is just specifying the extension \( 0 \to \mathfrak{A}(U) \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{A}(U) \to 0 \).

(3) We can generalize (2) to finitely many ideals. Let \( U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n = X \) be open subsets of \( X \). Define \( g_{U_1,U_2,\ldots,U_n,X}^n : X \to X_n \) by \( g_{U_1,U_2,\ldots,U_n,X}^n(x) = n - k + 1 \) if \( x \in U_k \setminus U_{k-1} \). Then \( g_{U_1,U_2,\ldots,U_n,X}^n \) is continuous. Therefore, any \( C^* \)-algebra with ideals \( 0 \leq \mathcal{J}_1 \leq \mathcal{J}_2 \leq \cdots \leq \mathcal{J}_n = \mathfrak{A} \) can be made into a \( C^* \)-algebra over \( X_n \).

(4) For all \( Y \in \mathbb{L}\mathcal{C}(X) \), \( r_X^Y : \mathcal{C}\text{-alg}(X) \to \mathcal{C}\text{-alg}(Y) \) is the restriction functor defined in Definition 2.19 of [27].

(5) If \( f : X \to Y \) is an embedding of a subset with the subspace topology, we write

\[
i_X^Y = f_* : \mathcal{C}\text{-alg}(X) \to \mathcal{C}\text{-alg}(Y).
\]

By Proposition 3.4 of [27], the functors defined above induce functors from \( \mathfrak{R}(X) \) to \( \mathfrak{R}(Z) \), where \( Z = Y, X_1, X_n \).

2.4. Graph \( C^* \)-algebras. A graph \((E^0, E^1, r, s)\) consists of a countable set \( E^0 \) of vertices, a countable set \( E^1 \) of edges, and maps \( r : E^1 \to E^0 \) and \( s : E^1 \to E^0 \) identifying the range and source of each edge. If \( E \) is a graph, the graph \( C^* \)-algebra \( C^*\langle E \rangle \) is the universal \( C^* \)-algebra generated by mutually orthogonal projections \( \{p_v : v \in E^0\} \) and partial isometries \( \{s_e : e \in E^1\} \) with mutually orthogonal ranges satisfying

\[
\begin{align*}
(1) & \quad s_es_e = p_r(e) \quad \text{for all} \ e \in E^1 \\
(2) & \quad s_es_e^* \leq p_s(e) \quad \text{for all} \ e \in E^1 \\
(3) & \quad p_v = \sum_{e \in E^1 : s(e) = v} s_es_e^* \quad \text{for all} \ v \ \text{with} \ 0 < |s^{-1}(v)| < \infty.
\end{align*}
\]

3. Meta-theorems

In many cases one can obtain a classification result for a class of unital \( C^* \)-algebras \( \mathcal{C} \) by obtaining a classification result for the class \( \mathcal{C} \otimes \mathbb{K} \), where each object in \( \mathcal{C} \otimes \mathbb{K} \) is the stabilization of an object in \( \mathcal{C} \). A meta-theorem of this sort was proved by the first and second named authors in [13] Theorem 11. It was shown there that if \( \mathcal{C} \) is a subcategory of the category of \( C^* \)-algebras, \( \mathcal{C}\text{-alg} \), and if \( F \) is a functor from \( \mathcal{C} \) to an abelian category such that an isomorphism \( F(\mathfrak{A} \otimes \mathbb{K}) \cong F(\mathfrak{B} \otimes \mathbb{K}) \) lifts to an isomorphism in \( \mathfrak{A} \otimes \mathbb{K} \cong \mathfrak{B} \otimes \mathbb{K} \), then under suitable conditions, we have that \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) implies \( \mathfrak{A} \cong \mathfrak{B} \). In [31], the second and third named authors improved this result by showing that the isomorphism \( F(\mathfrak{A}) \cong F(\mathfrak{B}) \) lifts to an isomorphism from \( \mathfrak{A} \) to \( \mathfrak{B} \).

In this section, we improve these results in order to deal with cases when \( \mathcal{C} \) is a category (not necessarily a subcategory of \( \mathcal{C}\text{-alg} \)) and there exists a functor from \( \mathcal{C} \) to \( \mathcal{C}\text{-alg} \). An example of such a category is the category of \( C^* \)-algebras over \{1, 2\}, where \{1, 2\} is given the discrete topology. Then \( \mathcal{C} \) is not a subcategory of \( \mathcal{C}\text{-alg} \) but the forgetful functor (forgetting the \{1, 2\}-structure) is a functor from \( \mathcal{C} \) to \( \mathcal{C}\text{-alg} \). We also replace the condition of proper pure infiniteness by the stable weak cancellation property.
Definition 3.1. A C*-algebra $\mathfrak{A}$ is said to have the weak cancellation property if $p$ is Murray-von Neumann equivalent to $q$ whenever $p$ and $q$ generate the same ideal $\mathcal{J}$ and $[p] = [q]$ in $K_0(\mathcal{J})$. A C*-algebra is said to have the stable weak cancellation property if $M_n(\mathfrak{A})$ has the weak cancellation property for all $n \in \mathbb{N}$.

Theorem 3.2. (cf. [13] Theorem 11) Let $C$ and $D$ be categories, let $\mathcal{C}^*\text{-alg}$ be the category of C*-algebras, and let $\mathbf{Ab}$ be the category of abelian groups. Suppose we have covariant functors $F : C \to \mathcal{C}^*\text{-alg}$, $G : C \to D$, and $H : D \to \mathbf{Ab}$ such that

1. $H \circ G = K_0 \circ F$.
2. For objects $\mathfrak{A}$ in $C$, there exist an object $\mathfrak{A}_K$ and a morphism $\kappa_{\mathfrak{A}} : \mathfrak{A} \to \mathfrak{A}_K$ such that $G(\kappa_{\mathfrak{A}})$ is an isomorphism in $D$, $F(\mathfrak{A}_K) = F(\mathfrak{A}) \otimes K$, and $F(\kappa_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})} \otimes e_{11}$.
3. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $C$, every isomorphism $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$ is induced by an isomorphism from $\mathfrak{A}_K$ to $\mathfrak{B}_K$.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be given such that $F(\mathfrak{A})$ and $F(\mathfrak{B})$ are unital C*-algebras. Let $\rho : G(\mathfrak{A}) \to G(\mathfrak{B})$ be an isomorphism such that $H(\rho)([1_{F(\mathfrak{A})}]) = [1_{F(\mathfrak{B})}]$. If $F(\mathfrak{B})$ has the stable weak cancellation property, then $F(\mathfrak{A}) \cong F(\mathfrak{B})$.

Proof. Note that $G(\kappa_{\mathfrak{A}})$ and $G(\kappa_{\mathfrak{B}})$ are isomorphisms. Therefore $G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}$ is an isomorphism from $G(\mathfrak{A}_K)$ to $G(\mathfrak{B}_K)$. Thus, there exists an isomorphism $\phi : \mathfrak{A}_K \to \mathfrak{B}_K$ such that $G(\phi) = G(\kappa_{\mathfrak{B}}) \circ \rho \circ G(\kappa_{\mathfrak{A}})^{-1}$.

Set $\psi = F(\phi)$. Then $\psi : F(\mathfrak{A}) \otimes K \to F(\mathfrak{B}) \otimes K$ is a *-isomorphism such that

$K_0(\psi) = K_0(F(\phi)) = H(G(\kappa_{\mathfrak{A}}) \circ \rho \circ G(\kappa_{\mathfrak{B}})^{-1}) = H(G(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ H(G(\kappa_{\mathfrak{A}}))^{-1} = K_0(F(\kappa_{\mathfrak{B}})) \circ H(\rho) \circ K_0(F(\kappa_{\mathfrak{A}}))^{-1} = K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}$.

Hence,

$K_0(\psi)([1_{F(\mathfrak{A})} \otimes e_{11}]) = K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho) \circ K_0(\text{id}_{F(\mathfrak{A})} \otimes e_{11})^{-1}([1_{F(\mathfrak{A})} \otimes e_{11}])$

$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11}) \circ H(\rho)([1_{F(\mathfrak{A})}])$

$= K_0(\text{id}_{F(\mathfrak{B})} \otimes e_{11})([1_{F(\mathfrak{B})}])$

$= [1_{F(\mathfrak{B})} \otimes e_{11}]$.

Stable weak cancellation implies that there exists $v \in F(\mathfrak{B}) \otimes K$ such that $v^*v = \psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $vv^* = 1_{F(\mathfrak{B})} \otimes e_{11}$ since $\psi(1_{F(\mathfrak{A})} \otimes e_{11})$ and $1_{F(\mathfrak{B})} \otimes e_{11}$ are full projections in $F(\mathfrak{B}) \otimes K$. Set $\gamma(x) = v\psi(x \otimes e_{11})v^*$. Arguing as in the proof of [13] Theorem 11, $\gamma$ is an isomorphism from $F(\mathfrak{A}) \otimes e_{11}$ to $F(\mathfrak{B}) \otimes e_{11}$. Hence, $F(\mathfrak{A}) \cong F(\mathfrak{B})$. □

Theorem 3.3. (cf. [32] Theorem 2.1) Let $C$ be a subcategory of $\mathcal{C}^*\text{-alg}(X)$. Moreover, $C$ is assumed to be closed under tensoring by $M_2(\mathfrak{C})$ and $K$ and contains the canonical embeddings $\kappa_1 : \mathfrak{A} \to M_2(\mathfrak{A})$ and $\kappa : \mathfrak{A} \to \mathfrak{A} \otimes K$ as morphisms for every object $\mathfrak{A}$ in $C$. Assume there is a functor $F : C \to D$ satisfying

1. For $\mathfrak{A}$ in $C$, the embeddings $\kappa_1 : \mathfrak{A} \to M_2(\mathfrak{A})$ and $\kappa : \mathfrak{A} \to \mathfrak{A} \otimes K$ induce isomorphisms $F(\kappa_1)$ and $F(\kappa)$.
2. For all objects $\mathfrak{A}$ and $\mathfrak{B}$ in $C$ that are stable C*-algebras, every isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ is induced by an isomorphism from $\mathfrak{A}$ to $\mathfrak{B}$.
3. There exists a functor $G$ from $D$ to $\mathbf{Ab}$ such that $G \circ F = K_0$. 


Assume that every $X$-equivariant isomorphism between objects in $\mathcal{C}$ is a morphism in $\mathcal{C}$ and that for objects $\mathfrak{A}$ in $\mathcal{C}$, $F(\text{Ad}(u)|_{\mathfrak{A}}) = \text{id}_{F(\mathfrak{A})}$ for every unitary $u \in \mathcal{M}(\mathfrak{A})$. If $\mathfrak{A}$ and $\mathfrak{B}$ are objects $\mathcal{C}$ that are unital $C^*$-algebras such that $\mathfrak{A}$ and $\mathfrak{B}$ have the stable weak cancellation property and there is an isomorphism $\alpha : F(\mathfrak{A}) \to F(\mathfrak{B})$ such that $G(\alpha)([1_\mathfrak{A}]) = [1_\mathfrak{B}]$, then there exists an isomorphism $\phi : \mathfrak{A} \to \mathfrak{B}$ in $\mathcal{C}$ such that $F(\phi) = \alpha$.

Proof. The difference between the statement of Theorem 2.1 of [32] and statement of the theorem are

(i) $\mathcal{C}$ is assumed to be a subcategory of $\mathcal{C}^*$-$\text{alg}(X)$ instead of a subcategory of $\mathcal{C}^*$-$\text{alg}$.

(ii) $\mathfrak{A}$ and $\mathfrak{B}$ are assumed to have the stable weak cancellation property instead of being properly infinite.

In the proof of Theorem 2.1 of [32], properly infinite was needed to insure that $\psi(1_\mathfrak{A} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_\mathfrak{B} \otimes e_{11}$, where $\psi : \mathfrak{A} \otimes K \to \mathfrak{B} \otimes K$ is the isomorphism from (2) that lifts the isomorphism from $F(\mathfrak{A})$ to $F(\mathfrak{B})$ that is induced by $\alpha$. As in the proof of Theorem 3.2, we get that $\psi(1_\mathfrak{A} \otimes e_{11})$ is Murray-von Neumann equivalent to $1_\mathfrak{B} \otimes e_{11}$. Arguing as in the proof of Theorem 2.1 of [32], we get the desired result. $\square$

4. Classification results

In this section, we show that $K^+_X(-)$ is a strong classification functor for a class of $C^*$-algebras with exactly one proper nontrivial ideal containing $C^*$-algebras associated to finite graphs. The results of this section will be used in the next section to show that $K^+_X(-)$ together with the appropriate scale is a complete isomorphism invariant for $C^*$-algebras associated to graphs. Moreover, in a forthcoming paper, we use these results to solve the following extension problem: If $\mathfrak{A}$ fits into the following exact sequence

$$0 \to C^*(E) \otimes K \to \mathfrak{A} \to C^*(G) \to 0,$$

where $C^*(E)$ and $C^*(G)$ are simple $C^*$-algebras, then when is $\mathfrak{A} \cong C^*(F)$ for some graph $F$?

**Theorem 4.1. (Existence Theorem)** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)_Y$ is a positive isomorphism for all $Y \in \mathcal{L}C(X_2)$. Suppose $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$ is a full extension, $\mathfrak{A}_i[2]$ is a stable $C^*$-algebra, $\mathfrak{A}_i$ is a nuclear $C^*$-algebra with real rank zero, and either

(i) $\mathfrak{A}_i[2]$ is a purely infinite simple $C^*$-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra;

(ii) $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple $C^*$-algebra.

Then there exists an $X_2$-equivariant homomorphism $\phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$, and $\phi[2]$ and $\phi[1]$ are injective, where $\{e_{ij}\}$ is a system of matrix units for $K$.

Proof. Set $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_2; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$. Note that by Lemma 3.10 and Theorem 3.8 of [11], $\mathfrak{A}_i[2] \otimes K$ satisfies the corona factorization property (see [21] for the definition of the corona factorization property). Since $\mathfrak{A}_i[k]$ is an AF-algebra or an Kirchberg algebra, $\mathfrak{A}_i[k]$ has the stable weak cancellation. By Lemma 3.15 of [15], $\mathfrak{A}_i$ has stable weak cancellation. Let $\xi_i$ be the extension

$$0 \to \mathfrak{A}_i[2] \otimes K \to \mathfrak{A}_i \otimes K \to \mathfrak{A}_i[1] \otimes K \to 0.$$
By Corollary 3.24 of [15], $e_i$ is a full extension since $\mathfrak{A}_i[1]$ has cancellation of projections (in the AF case) and $\mathfrak{A}_i[1]$ is properly infinite (in the purely infinite case).

Case (i): $\mathfrak{A}_i[2]$ is a purely infinite simple C*-algebra and $\mathfrak{A}_i[1]$ is an AF-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) \times \{r_{X_2}^{(2)}\}$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$, we have that $r_{X_2}^{(1)}(y)$ is invertible in $KK([1] \otimes k, \mathfrak{A}_2[2] \otimes k)$ and $\Gamma(r_{X_2}^{(1)}(y)) = \Gamma(x_{\{1\}})$ is a positive isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_1 : [1] \otimes k \to \mathfrak{A}_2[1] \otimes k$ such that $KK(\psi_1) = r_{X_2}^{(1)}(y)$. Since $y$ is invertible in $KK(X_2, \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK([1] \otimes k, \mathfrak{A}_2[2] \otimes k)$. Thus, by Kirchberg-Phillips classification (see [20] and [29]), there exists an isomorphism $\psi_0 : [2] \otimes k \to \mathfrak{A}_2[2] \otimes k$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$. By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in M(\mathfrak{A}[2] \otimes k)$ such that $\psi = (Ad(u) \circ \psi_0, Ad(u) \circ \psi_0)$ is an X₂-equivariant isomorphism from $\mathfrak{A}_1 \otimes k$ to $\mathfrak{A}_2 \otimes k$, where $\psi_0 : M([1] \otimes k) \to M([1] \otimes k)$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi_{\{1\}}) = r_{X_2}^{(1)}(y)$ for $k = 1, 2$.

Note that

$$0 \to i_{X_2}^{(1)}((\mathfrak{A}_1 \otimes k)[2]) \xrightarrow{\lambda_2} \mathfrak{A}_1 \otimes k \xrightarrow{\beta_1} i_{X_2}^{(1)}((\mathfrak{A}_1 \otimes k)[1]) \to 0$$

is a semi-split extension of C*-algebras over $X_2$ (see Definition 3.5 of [27]). Set

$$\mathfrak{J}_i = i_{X_2}^{(i)}((\mathfrak{A}_i \otimes k)[2]) \quad \text{and} \quad \mathfrak{B}_i = i_{X_2}^{(1)}((\mathfrak{A}_i \otimes k)[1]).$$

By Theorem 3.6 of [27] (see also Korollar 3.4.6 of [4]),

$$KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{J}_2) \xrightarrow{(\lambda_2)_*} KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{B}_2 \otimes k)$$

and

$$KK(X_2; \mathfrak{B}_1 \otimes k, \mathfrak{J}_2) \xrightarrow{(\beta_2)_*} KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{B}_2)$$

is exact. By Proposition 3.12 of [27], $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{B}_2)$ and $KK([1] \otimes k, \mathfrak{A}_2[1] \otimes k)$ are naturally isomorphic. Hence, there exists $z \in KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{J}_2)$ such that $y - KK(X_2; \psi) = z \times KK(X_2; \lambda_2)$ since $KK(\psi_{\{1\}}) = r_{X_2}^{(1)}(y)$.

By Proposition 3.13 of [27], $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{J}_2)$ and $KK([1] \otimes k, \mathfrak{A}_2 \otimes k)[2]$ are isomorphic. By Theorem 8.3.3 of [36] (see also Hauptsatz 4.2 of [20]), there exists a *-homomorphism $\eta : \mathfrak{A}_1 \otimes k \to \mathfrak{A}_2 \otimes k[2]$ such that $KK(\eta) = \pi$, where $\pi$ is the image of $z$ under the isomorphism $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{J}_2) \cong KK([1] \otimes k, \mathfrak{A}_2 \otimes k)[2]$. Note that $\eta$ induces an X₂-equivariant homomorphism $\eta : \mathfrak{A}_1 \otimes k \to \mathfrak{J}_2$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\lambda_2 \circ \eta)$, where the sum is the Cuntz sum in $M(\mathfrak{B}_2 \otimes k)$. Then $\phi : \mathfrak{A}_1 \otimes k \to \mathfrak{B}_2 \otimes k$ is an X₂-equivariant homomorphism such that $KK(X_2; \phi) = y$. Since $\psi_{\{2\}}$ and $\psi_{\{1\}}$ are injective homomorphisms, $\phi_{\{2\}}$ and $\phi_{\{1\}}$ are injective homomorphisms.

Case (ii): $\mathfrak{A}_i[2]$ is an AF-algebra and $\mathfrak{A}_i[1]$ is a purely infinite simple C*-algebra. By Theorem 3.3 of [14], $r_{X_2}^{(1)}(y) \times \{r_{X_2}^{(2)}\}$ is invertible in $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$. Since $y$ is invertible in $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$, we have that $r_{X_2}^{(2)}(y)$ is invertible in $KK([1] \otimes k, \mathfrak{A}_2[2] \otimes k)$ and $\Gamma(r_{X_2}^{(2)}(y)) = \Gamma(x_{\{2\}})$ is an order isomorphism. Thus, by Elliott’s classification [19], there exists an isomorphism $\psi_0 : [2] \otimes k \to \mathfrak{A}_2[2] \otimes k$ such that $KK(\psi_0) = r_{X_2}^{(2)}(y)$. Since $y$ is invertible in $KK(X_2; \mathfrak{A}_1 \otimes k, \mathfrak{A}_2 \otimes k)$, we have that $r_{X_2}^{(1)}(y)$ is invertible in
there exists an isomorphism $\psi : \mathfrak{A}_1[1] \otimes \mathbb{K} \to \mathfrak{A}_2[1] \otimes \mathbb{K}$. Thus, by Kirchberg-Phillips classification (see [20] and [29]), there exists a unitary $u \in M(\mathfrak{A}_2[2] \otimes \mathbb{K})$ such that $KK(\psi) = r_{X_2}^\varepsilon(y)$. By Lemma 4.5 of [14] and its proof, there exists a unitary $u \in M(\mathfrak{A}_2[2] \otimes \mathbb{K})$ such that $\psi = (Ad(u) \circ \psi_0, Ad(u) \circ \psi_0, \psi_1)$ is an $X_2$-equivalent isomorphism from $\mathfrak{A}_1 \otimes \mathbb{K}$ to $\mathfrak{A}_2 \otimes \mathbb{K}$, where $\psi_0 : M(\mathfrak{A}_1[2] \otimes \mathbb{K}) \to M(\mathfrak{A}_1[2] \otimes \mathbb{K})$ is the unique isomorphism extending $\psi_0$. Note that $KK(\psi) = r_{X_2}^\varepsilon(y)$ for $k = 1, 2$.

Note that

$$0 \to i_{\{2\}}(\mathfrak{A}_1 \otimes \mathbb{K}[2]) \xrightarrow{\lambda} \mathfrak{A}_1 \otimes \mathbb{K} \xrightarrow{\beta} i_{\{1\}}(\mathfrak{A}_1 \otimes \mathbb{K}[1]) \to 0$$

is a semi-split extension of $C^*$-algebras over $X_2$ (see Definition 3.5 of [27]). Set

$$\mathcal{J}_i = i_{\{2\}}(\mathfrak{A}_1 \otimes \mathbb{K}[2]) \quad \text{and} \quad \mathcal{B}_i = i_{\{1\}}(\mathfrak{A}_1 \otimes \mathbb{K}[1]).$$

By Theorem 3.6 of [27] (see also Korollar 3.4.6 [4]), $KK(\mathcal{J}_i, \mathcal{B}_i, \mathfrak{A}_1 \otimes \mathbb{K})$ is exact. By Proposition 3.12 of [27], $KK(\mathcal{J}_i, \mathfrak{A}_1 \otimes \mathbb{K})$ and $KK(\mathfrak{A}_1[2] \otimes \mathbb{K}, \mathfrak{A}_2[2] \otimes \mathbb{K})$ are naturally isomorphic. Hence, there exists $z \in KK(\mathcal{J}_i, \mathcal{B}_i, \mathfrak{A}_1 \otimes \mathbb{K})$ such that $y - KK(X; \psi) = KK(\mathcal{J}_i, \mathcal{B}_i, \mathfrak{A}_1 \otimes \mathbb{K})$ and $KK((\mathfrak{A}_1 \otimes \mathbb{K})[1], \mathfrak{A}_2 \otimes \mathbb{K})$ are isomorphic. Therefore, by Theorem 8.3.3 of [36], there exists a homomorphism $\eta : (\mathfrak{A}_1 \otimes \mathbb{K})[1] \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\eta) = z$, where $z$ is the image of $z$ under the isomorphism $KK(X; \mathcal{J}_i, \mathcal{B}_i, \mathfrak{A}_1 \otimes \mathbb{K})$. The existence of the homomorphism uses the fact that $\mathfrak{A}_2 \otimes \mathbb{K}$ is a proper infinite $C^*$-algebra which follows from Proposition 3.21 and Theorem 3.22 of [15]. Note that $\eta$ induces an $X_2$-equivariant homomorphism $\eta : \mathcal{J}_i \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \eta) = z$.

Set $\phi = \psi + (\eta \circ \beta_1)$, where the sum is the Cuntz sum in $M(\mathfrak{A}_2 \otimes \mathbb{K})$. Then $\phi$ is an $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = z$. Since $\psi[2]$ and $\psi[1]$ are injective homomorphisms, $\phi[2]$ and $\phi[1]$ are injective homomorphisms.

4.1. Strong classification of extensions of AF-algebras by purely infinite $C^*$-algebras.

**Definition 4.2.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable $C^*$-algebras over $X$. Two $X$-equivariant homomorphisms $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are said to be approximately unitarily equivalent if there exists a sequence of unitaries $\{u_n\}_{n=1}^\infty$ in $M(\mathfrak{B})$ such that

$$\lim_{n \to \infty} \|u_n \phi(a) u_n^* - \psi(a)\| = 0$$

for all $a \in \mathfrak{A}$.

We now recall the definition of $KL(\mathfrak{A}, \mathfrak{B})$ from [33].

**Definition 4.3.** Let $\mathfrak{A}$ be a separable, nuclear $C^*$-algebra in $\mathcal{N}$ and let $\mathfrak{B}$ be a $\sigma$-unital $C^*$-algebra. Let

$$\text{Ext}^1_\mathcal{L}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B})) = \text{Ext}^1_\mathcal{L}(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Ext}^1_\mathcal{L}(K_1(\mathfrak{A}), K_0(\mathfrak{B})).$$

Since $\mathfrak{A}$ is in $\mathcal{N}$, by [37], $\text{Ext}^1_\mathcal{L}(K_*(\mathfrak{A}), K_{*+1}(\mathfrak{B}))$ can be identified as a sub-group of the group $KK(\mathfrak{A}, \mathfrak{B})$. 
For abelian groups, $G$ and $H$, let $\text{Pext}_2^1(G, H)$ be the subgroup of $\text{Ext}_2^1(G, H)$ of all pure extensions of $G$ by $H$. Set
\[
\text{Pext}_2^1(K_*(\mathfrak{A}), K_{*-1}(\mathfrak{B})) = \text{Pext}_2^1(K_0(\mathfrak{A}), K_1(\mathfrak{B})) \oplus \text{Pext}_2^1(K_1(\mathfrak{A}), K_0(\mathfrak{B})).
\]
Define $KL(\mathfrak{A}, \mathfrak{B})$ as the quotient
\[
KL(\mathfrak{A}, \mathfrak{B}) = KK(\mathfrak{A}, \mathfrak{B})/\text{Pext}_2^1(K_*(\mathfrak{A}), K_{*-1}(\mathfrak{B})).
\]
Rørdam in [33] proved that if $\phi, \psi : \mathfrak{A} \to \mathfrak{B}$ are approximately unitarily equivalent, then $KL(\phi) = KL(\psi)$.

**Notation 4.4.** Let $x \in KK(\mathfrak{A}, \mathfrak{B})$. Then the element $x + \text{Pext}_2^1(K_*(\mathfrak{A}), K_{*-1}(\mathfrak{B}))$ in $KL(\mathfrak{A}, \mathfrak{B})$ will be denoted by $KL(x)$.

A nuclear, purely infinite, separable, simple $C^*$-algebra will be called a *Kirchberg algebra*.

**Theorem 4.5. (Uniqueness Theorem 1)** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be separable, nuclear, $C^*$-algebras over $X_2$ such that $\mathfrak{A}_1$ has real rank zero, $\mathfrak{A}_1$ is stable, $\mathfrak{A}_1[2]$ is a Kirchberg algebra in $N$, $\mathfrak{A}_1[1]$ is an AF-algebra, and $\mathfrak{A}_1[2]$ is an essential ideal of $\mathfrak{A}_1$. Suppose $\phi, \psi : \mathfrak{A}_1 \to \mathfrak{A}_2$ be $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; \psi)$, and $\phi_{(2)}$, $\phi_{(1)}$, $\psi_{(2)}$, and $\psi_{(1)}$ are injective homomorphisms. Then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Proof.** Since $\mathfrak{A}_1[1]$ is an AF algebra, every finitely generated subgroup of $K_0(\mathfrak{A}_1[1])$ is torsion free (hence free) and every finitely generated subgroup of $K_1(\mathfrak{A}_1[1])$ is zero. Thus, $\text{Pext}_2^1(K_*(\mathfrak{A}_1[1]), K_{*-1}(Q(\mathfrak{A}_1[2]))) = \text{Ext}_2^1(K_*(\mathfrak{A}_1[1]), K_{*-1}(Q(\mathfrak{A}_1[2])))$ which implies that $KL(\mathfrak{A}_1[1], Q(\mathfrak{A}_1[2])) \cong \text{Hom}(K_*(\mathfrak{A}_1[1]), K_{*-1}(Q(\mathfrak{A}_1[2])))$.

Let $\varepsilon_i$ denote the extension $0 \to \mathfrak{A}_i[2] \to \mathfrak{A}_i \to \mathfrak{A}_i[1] \to 0$. Since $\mathfrak{A}_i$ has real rank zero and $K_1(\mathfrak{A}_i[1]) = 0$, we have that $K_j(\varepsilon_i) = 0$, where $\varepsilon_i$ is the Busby invariant of $\varepsilon_i$. Hence, $\varepsilon_i = 0$ in $KL(\mathfrak{A}_i[1], Q(\mathfrak{A}_i[2]))$. By Corollary 6.7 of [24], $\varepsilon_i$ is quasi-diagonal. Thus, there exists an approximate identity of $\mathfrak{A}_i[2]$ consisting of projections $\{e_k\}_{k \in \mathbb{N}}$ such that
\[
\lim_{n \to \infty} \|e_k x - xe_k\| = 0
\]
for all $x \in \mathfrak{A}_i$.

Since $\mathfrak{A}_1[1]$ is an AF-algebra and $\mathfrak{A}_1$ has real rank zero, as in the proof of Lemma 9.8 of [10], there exists a sequence of finite dimensional sub-$C^*$-algebras $\{\mathfrak{B}_k\}_{k=1}^\infty$ of $\mathfrak{A}_1$ such that $\mathfrak{B}_k \cap \mathfrak{A}_1[2] = \{0\}$ and for each $x \in \mathfrak{A}_1$, there exist $y_1 \in \bigcup_{k=1}^\infty \mathfrak{B}_k$ and $y_2 \in \mathfrak{A}_1[2]$ such that $x = y_1 + y_2$.

Let $\varepsilon > 0$ and $\mathcal{F}$ be a finite subset of $\mathfrak{A}_1$. Note that we may assume $\mathcal{F}$ is the union of the generators of $\mathfrak{B}_m$, for some $m \in \mathbb{N}$ and $\mathcal{G}$, for some finite subset $\mathcal{G}$ of $\mathfrak{A}_1[2]$ . Since $\mathfrak{B}_m$ is a finite dimensional $C^*$-algebra,
\[
\lim_{k \to \infty} \|e_k x - xe_k\| = 0
\]
for all $x \in \mathfrak{A}_1$, and $\{e_k\}_{k \in \mathbb{N}}$ is an approximate identity for $\mathfrak{A}_1[2]$ consisting of projections, there exist $k \in \mathbb{N}$, a finite dimensional sub-$C^*$-algebra $\mathfrak{D}$ of $\mathfrak{A}_1$ with $\mathfrak{D} \subseteq (1_{M(\mathfrak{A}_1)} - e_k)\mathfrak{A}_1(1_{M(\mathfrak{A}_1)} - e_k)$ and $\mathfrak{D} \cap \mathfrak{A}_1[2] = \{0\}$, and there exists a finite subset $\mathcal{H}$ of $e_k \mathfrak{A}_1[2] e_k$ such that for all $x \in \mathcal{F}$, there exist $y_1 \in \mathfrak{D}$ and $y_2 \in \mathcal{H}$
\[
\|x - (y_1 + y_2)\| < \varepsilon / 3.
\]
Set \( \mathfrak{D} = \bigoplus_{\ell=1}^{s} M_{n_{\ell}} \) and let \( \{ f_{ij}^{\ell} \}_{i,j=1}^{n_{\ell}} \) be a system of matrix units for \( M_{n_{\ell}} \). Let \( \mathcal{I}_{\ell} \) be the ideal in \( \mathfrak{A}_{i} \) generated by \( f_{11}^{\ell} \). Since \( \mathfrak{A}_{i}[2] \) is simple and \( \mathfrak{A}_{i}[2] \) is an essential ideal of \( \mathfrak{A}_{i} \), we have that \( \mathfrak{A}_{i}[2] \subseteq \mathcal{I}_{i} \) for all nonzero ideal \( \mathcal{I} \) of \( \mathfrak{A}_{i} \). Thus, \( \mathfrak{A}_{i}[2] \subseteq \mathcal{I}_{\ell} \) since \( \mathfrak{D} \cap \mathfrak{A}_{i}[2] = 0 \).

Let \( \mathcal{I}_{\ell}^{\psi} \) be the ideal in \( \mathfrak{A}_{2} \) generated by \( \phi(f_{11}^{\ell}) \) and let \( \mathcal{I}_{\ell}^{\psi} \) be the ideal in \( \mathfrak{A}_{2} \) generated by \( \psi(f_{11}^{\ell}) \). Since \( \phi \) and \( \psi \) are \( X_{2} \)-equivariant homomorphisms and since \( \phi(1) \) and \( \psi(1) \) are injective homomorphisms, we have that \( \phi(f_{11}^{\ell}) \not\in \mathfrak{A}_{2}[2] \) and \( \psi(f_{11}^{\ell}) \not\in \mathfrak{A}_{2}[2] \). Therefore, \( \mathfrak{A}_{2}[2] \subseteq \mathcal{I}_{\ell}^{\psi} \) and \( \mathfrak{A}_{2}[2] \subseteq \mathcal{I}_{\ell}^{\phi} \). Since \( K_{0}(\phi(1)) = K_{0}(\psi(1)) \) and since \( \mathfrak{A}_{2}[1] \) is an AF-algebra, we have that \( \phi(1)(\mathcal{I}_{11}^{\phi}) \) is Murray-von Neumann equivalent to \( \psi(1)(\mathcal{I}_{11}^{\psi}) \), where \( \mathcal{I}_{11}^{\phi} \) is the image of \( f_{11}^{\ell} \) in \( \mathfrak{A}_{1}[1] \). Thus, they generate the same ideal in \( \mathfrak{A}_{2}[1] \). Since \( \mathfrak{A}_{2}[2] \subseteq \mathcal{I}_{\ell}^{\phi} \) and \( \mathfrak{A}_{2}[2] \subseteq \mathcal{I}_{\ell}^{\psi} \) and since \( \psi(1)(\mathcal{I}_{11}^{\psi}) \) and \( \phi(1)(\mathcal{I}_{11}^{\phi}) \) generate the same ideal in \( \mathfrak{A}_{2}[1] \), we have that \( \mathcal{I} = \mathcal{I}_{\ell}^{\psi} = \mathcal{I}_{\ell}^{\phi} \).

Note that the following diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & K_{0}(\mathfrak{A}_{2}[2]) \\
\downarrow & & \downarrow K_{0}(\mathcal{I}) \\
0 & \longrightarrow & K_{0}(\mathfrak{A}_{2}[2])
\end{array}
\bigg| \quad \begin{array}{ccc}
& & \bigg| \quad 0 \\
& & K_{0}(\mathfrak{A}_{2}[2]) & \longrightarrow & K_{0}(\mathfrak{A}_{2}) & \longrightarrow & K_{0}(\mathfrak{A}_{2}[2]) \\
& & K_{0}(\mathcal{I}) & \bigg| & K_{0}(\mathfrak{A}_{2}) & \bigg| & K_{0}(\mathfrak{A}_{2}[2])
\end{array}
\]

is commutative, the rows are exact, and \( \iota \) and \( \tau \) are the canonical embeddings. Since \( \mathfrak{A}_{2}[1] \) is an AF-algebra, \( K_{0}(\mathcal{I}) \) is injective. A diagram chase shows that \( K_{0}(\iota) \) is injective. Since \( KK(X_{2}; \phi) = KK(X_{2}; \psi) \), we have that \( [\phi(f_{11}^{\ell})] = [\psi(f_{11}^{\ell})] \) in \( K_{0}(\mathfrak{A}_{2}) \). Since \( \phi(f_{11}^{\ell}) \) and \( \psi(f_{11}^{\ell}) \) are elements of \( \mathcal{I} \) and \( K_{0}(\iota) \) is injective, we have that \( [\phi(f_{11}^{\ell})] = [\psi(f_{11}^{\ell})] \) in \( K_{0}(\mathfrak{A}_{2}) \). Since \( \mathfrak{A}_{i}[1] \) is an AF-algebra and \( \mathfrak{A}_{i}[2] \) is a Kirchberg algebra, they both have stable weak cancellation. By Lemma 3.15 of [15], \( \mathfrak{A}_{i} \) has stable weak cancellation. Thus, \( \phi(f_{11}^{\ell}) \) is Murray-von Neumann equivalent to \( \psi(f_{11}^{\ell}) \). Hence, there exists \( v_{\ell} \in \mathfrak{A}_{2} \) such that \( v_{\ell}^{*}v_{\ell} = \phi(f_{11}^{\ell}) \) and \( v_{\ell}v_{\ell}^{*} = \psi(f_{11}^{\ell}) \).

Set

\[
u_{1} = \sum_{\ell=1}^{s} \sum_{i=1}^{n_{\ell}} \psi(f_{11}^{\ell})v_{\ell}\phi(f_{11}^{\ell})
\]

Then, \( u_{1} \) is a partial isometry in \( \mathfrak{A}_{1} \) such that \( u_{1}^{*}u_{1} = \phi(1_{\mathfrak{D}}) \), \( u_{1}u_{1}^{*} = \psi(1_{\mathfrak{D}}) \), and \( u_{1}\phi(x)u_{1}^{*} = \psi(x) \) for all \( x \in \mathfrak{D} \).

Let \( \beta : e_{k}\mathfrak{A}_{1}[2]e_{k} \to \mathfrak{A}_{1}[2] \) be the usual embedding. Note that \( KK(\phi_{2}\circ\beta) = KK(\psi_{2}\circ\beta) \) and \( \phi_{2} \circ \beta, \psi_{2} \circ \beta \) are monomorphisms. Therefore, by Theorem 6.7 of [23], there exists a partial isometry \( u_{2} \in \mathfrak{A}_{2}[2] \) such that \( u_{2}^{*}u_{2} = \phi(e_{k}) \), \( u_{2}u_{2}^{*} = \psi(e_{k}) \), and

\[
\|u_{2}\phi(x)u_{2}^{*} - \psi(x)\| < \frac{\epsilon}{3}
\]

for all \( x \in \mathcal{H} \).

Since \( \mathfrak{A}_{2} \) is stable, there exists \( u_{3} \in \mathcal{M}(\mathfrak{A}_{2}) \) such that \( u_{3}^{*}u_{3} = 1_{\mathcal{M}(\mathfrak{A}_{2})} - (u_{1} + u_{2})^{*}(u_{1} + u_{2}) \) and \( u_{3}u_{3}^{*} = 1_{\mathcal{M}(\mathfrak{A}_{2})} - (u_{1} + u_{2})(u_{1} + u_{2})^{*} \). Set \( u = u_{1} + u_{2} + u_{3} \in \mathcal{M}(\mathfrak{A}_{2}) \). Then \( u \) is a unitary in \( \mathcal{M}(\mathfrak{A}_{2}) \).
Let $x \in \mathcal{F}$. Choose $y_1 \in \mathcal{D}$ and $y_2 \in \mathcal{H} \subseteq \epsilon_k \mathfrak{A}_1[n]e_k$ such that $\|x - (y_1 + y_2)\| < \frac{\epsilon}{3}$. Then

$$\|u \phi(x) u^* - \psi(x)\| \leq \|u \phi(x) u^* - u \phi(y_1 + y_2) u^*\| + \|u_1 \phi(y_1) u_1^* + u_2 \phi(y_2) u_2^* - \psi(y_1) - \psi(y_2)\| + \|\psi(y_1 + y_2) - \psi(x)\| < \epsilon.$$

We have just shown that for each $\epsilon > 0$ and for each finite subset $\mathcal{F}$ of $\mathfrak{A}_1$, there exists a unitary $u \in \mathcal{M}(\mathfrak{A}_2)$ such that $\|u \phi(x) u^* - \psi(x)\| < \epsilon$ for all $x \in \mathcal{F}$. Since $\mathfrak{A}_1$ is a separable $C^*$-algebra, we have that $\phi$ is approximately unitarily equivalent to $\psi$. □

**Lemma 4.6.** Let $\mathfrak{A}$ be a separable $C^*$-algebra over a finite topological space $X$. Let $u$ be unitary in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$. Then $K_X(\text{Ad}(u)|_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$.

**Proof.** Since $\mathfrak{A} \otimes \mathbb{K}$ is stable, we have that there exists a norm continuous path of unitaries $\{u_t\}$ in $\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})$ such that $u_0 = u$ and $u_1 = 1_{\mathcal{M}(\mathfrak{A} \otimes \mathbb{K})}$. It follows that $K_X(\text{Ad}(u)|_{\mathfrak{A} \otimes \mathbb{K}}) = \text{id}_{K_X(\mathfrak{A})}$.

**Theorem 4.7.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)\mathcal{Y}$ is an order isomorphism for all $Y \subseteq \mathcal{L}(X_2)$. Suppose $\mathfrak{A}_1[2]$ is a Kirchberg algebra, $\mathfrak{A}_1[1]$ is an AF-algebra, $\mathfrak{A}_1$ has real rank zero, and $\mathfrak{A}_2[2]$ is an essential ideal of $\mathfrak{A}_1$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KL(\phi) = KL(g_{X_2}^1(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_n; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$.

**Proof.** Since $\mathfrak{A}_1[2]$ is a purely infinite simple $C^*$-algebra, $\mathfrak{A}_1[2]$ is either unital or stable. Since $\mathfrak{A}_1[2]$ is an essential ideal of $\mathfrak{A}_1$, $\mathfrak{A}_1[2]$ is non-unital else $\mathfrak{A}_1[2]$ is isomorphic to a direct summand of $\mathfrak{A}_1$ which would contradict the essential assumption. Therefore, $\mathfrak{A}_1[2]$ is stable. Moreover, $Q(\mathfrak{A}_1[2])$ is simple which implies that $0 \to \mathfrak{A}_1[2] \to \mathfrak{A}_1 \to \mathfrak{A}_1[1] \to 0$ is a full extension. Since $\mathfrak{A}_1[2]$ and $\mathfrak{A}_1[1]$ are nuclear $C^*$-algebras, $\mathfrak{A}_1$ is a nuclear $C^*$-algebra.

Let $z \in KK(X_2; \mathfrak{A}_2 \otimes \mathbb{K}, \mathfrak{A}_1 \otimes \mathbb{K})$ such that $y \times z = [\text{id}_{\mathfrak{A}_1 \otimes \mathbb{K}}]$ and $y \times z = [\text{id}_{\mathfrak{A}_2 \otimes \mathbb{K}}]$. By Theorem 4.1. there exists an $X_2$-equivariant homomorphism $\psi_1 : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(X_2; \psi_1) = x$, and $(\psi_1)_1(1)$ and $(\psi_1)_1(1)$ are injective homomorphisms. By Theorem 4.1. there exists an $X_2$-equivariant homomorphism $\psi_2 : \mathfrak{A}_2 \otimes \mathbb{K} \to \mathfrak{A}_1 \otimes \mathbb{K}$ such that $KK(X_2; \psi_2) = y$, and $(\psi_2)_1(1)$ and $(\psi_2)_1(1)$ are injective homomorphisms. Using Theorem 4.5 and a typical approximate intertwining argument, there exists an isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $\phi$ and $\psi_1$ are approximately unitarily equivalent.

Let $\tau_2 : \mathfrak{A}_2 \to \mathfrak{A}_2[1]$ be the canonical quotient map. Then $\tau_2 \circ \phi|_{\mathfrak{A}_2[2]}$ is either zero or injective since $\mathfrak{A}_2[2]$ is simple. Since $\mathfrak{A}_2[2]$ is purely infinite and $\mathfrak{A}_2[1]$ is an AF-algebra, we must have that $\tau_2 \circ \phi|_{\mathfrak{A}_2[2]} = 0$. Thus, $\phi$ is an $X_2$-equivariant homomorphism. Similarly, $\phi^{-1}$ is an $X_2$-equivariant homomorphism. Hence, $\phi$ is an $X_2$-equivariant isomorphism. By construction, $KL(\phi) = KL(\psi_1) = KL(g_{X_2}^1(y))$. By Lemma 4.4, $K_{X_2}(\phi) = K_{X_2}(x)$. □

**Corollary 4.8.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be in $\mathcal{B}(X_2)$ and let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible element such that $\Gamma(x)\mathcal{Y}$ is an order isomorphism for all $Y \subseteq \mathcal{L}(X_2)$. Suppose $\mathfrak{A}_1[2]$ is a Kirchberg algebra, $\mathfrak{A}_1[1]$ is an AF-algebra, $\mathfrak{A}_1$ has real rank zero, $\mathfrak{A}_2[2]$ is an essential ideal of $\mathfrak{A}_1$, and $K_i(\mathfrak{A}[Y])$ and $K_i(\mathfrak{B}[Y])$ are finitely generated for all $Y \subseteq \mathcal{L}(X_2)$. Then there exists an $X_2$-equivariant isomorphism $\phi : \mathfrak{A}_1 \otimes \mathbb{K} \to \mathfrak{A}_2 \otimes \mathbb{K}$ such that $KK(\phi) = KK(g_{X_2}^1(y))$ and $K_{X_2}(\phi) = K_{X_2}(y)$, where $y = KK(X_2; \text{id}_{\mathfrak{A}_1} \otimes e_{11})^{-1} \times x \times KK(X_n; \text{id}_{\mathfrak{A}_2} \otimes e_{11})$.
Proof. This follows from Theorem 4.7 and the fact that if $G$ is finitely generated, then $\text{Pext}_2^1(G, H) = 0$. □

4.2. Strong classification of extensions of purely infinite by $\mathcal{K}$. We recall the following from [1] p. 341. Let $\psi : \mathfrak{A} \to B(\mathcal{H})$ be a representation of $\mathfrak{A}$. Let $\mathcal{H}_e$ denote the subspace of $\mathcal{H}$ spanned by the ranges of all compact operators in $\psi(\mathfrak{A})$. Since $\psi(\mathfrak{A}) \cap \mathcal{K}$ is an ideal of $\psi(\mathfrak{A})$, we have that $\mathcal{H}_e$ reduces $\pi(\mathfrak{A})$, and so the decomposition $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_e^\perp$ induces a decomposition of $\psi$ into sub-representations $\psi = \psi_e \oplus \psi'$. The summand $\psi_e$, considered as a representation of $\mathfrak{A}$ on $\mathcal{H}_e$, will be called the essential part of $\psi$ and $\mathcal{H}_e$ is called the essential subspace for $\psi$.

Let $\mathfrak{B}$ be a tight $C^*$-algebra over $X_2$. Consider the essential extension

$$\epsilon_{\mathfrak{B}} : 0 \to \mathfrak{B}[1] \to \mathfrak{B} \to \mathfrak{B}[1] \to 0.$$  

If $\tau_{\mathfrak{B}} : \mathfrak{B}[1] \to Q(\mathfrak{B}[2])$ is the Busby invariant of $\epsilon$, then there exists an injective homomorphism $\sigma_{\mathfrak{B}} : \mathfrak{B} \to \mathcal{M}(\mathfrak{B}[2])$ such that the diagram

$$\begin{array}{ccc}
0 & \to & \mathfrak{B}[2] \\
\pi_{\mathfrak{B}} & & \pi_{\mathfrak{B}} \\
\tau_{\mathfrak{B}} & & \tau_{\mathfrak{B}} \\
0 & \to & \mathfrak{B}[1] \\
\pi_{\mathfrak{B}} & & \pi_{\mathfrak{B}} \\
\tau_{\mathfrak{B}} & & \tau_{\mathfrak{B}} \\
0 & \to & \mathcal{M}(\mathfrak{B}[2]) \\
\pi_{\mathfrak{B}} & & \pi_{\mathfrak{B}} \\
\tau_{\mathfrak{B}} & & \tau_{\mathfrak{B}} \\
0 & \to & Q(\mathfrak{B}[2]) \\
\pi_{\mathfrak{B}} & & \pi_{\mathfrak{B}} \\
\tau_{\mathfrak{B}} & & \tau_{\mathfrak{B}} \\
0 & \to & 0
\end{array}$$

If $\mathfrak{B}[2] \cong \mathcal{K}$, let $\eta_{\mathfrak{B}} : \mathcal{M}(\mathfrak{B}[2]) \to B(\ell^2)$ be the isomorphism extending the isomorphism $\mathfrak{B}[2] \cong \mathcal{K}$ and let $\eta_{\mathfrak{B}} : Q(\mathfrak{B}[2]) \to B(\ell^2)/\mathcal{K}$ be the induced isomorphism.

Lemma 4.9. Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}[2] \cong \mathfrak{B}[2] \cong \mathcal{K}$. Let $\psi_1, \psi_2 : \mathfrak{A} \to \mathfrak{B}$ be two, full $X_2$-equivariant homomorphisms such that $K_0(\psi_1(\mathfrak{A}[2])) = K_0(\psi_2(\mathfrak{A}[2]))$ and $\eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1$ is a non-degenerate representation of $\mathfrak{A}$. Then there exists a sequence of unitaries $\{U_n\}_{n=1}^{\infty}$ in $\mathcal{M}(\mathfrak{B}[2])$ such that

$$U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a) \in \mathfrak{B}[2]$$

for all $a \in \mathfrak{A}$ and for all $n \in \mathbb{N}$, and

$$\lim_{n \to \infty} \| U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a)U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a) \| = 0$$

for all $a \in \mathfrak{A}$.

Proof. We argue as in the proof of Lemma 2.8 of [22]. Set $\sigma_i = \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_i$. By assumption, $\sigma_i : \mathfrak{A} \to B(\ell^2)$ is a non-degenerated representation of $\mathfrak{A}$. We claim that there exists a sequence of unitaries $\{V_n\}_{n=1}^{\infty}$ in $B(\ell^2)$ such that $V_n \sigma_1(a) V_n^* - \sigma_2(a) \in \mathcal{K}$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to \infty} \| V_n \sigma_1(a) V_n^* - \sigma_2(a) \| = 0$$

for all $a \in \mathfrak{A}$. This will be a consequence of Theorem 5(iii) of [1].

Let $\rho : \mathfrak{A} \to B(\ell^2)$ be the unique irreducible faithful representation defined by the isomorphism $\mathfrak{A}[2] \cong \mathcal{K}$. Since $\psi_i, \sigma_{\mathfrak{B}}, \eta_{\mathfrak{B}}$ are injective homomorphisms, $\sigma_i$ is injective. Therefore, $\ker(\sigma_1) = \ker(\sigma_2) = \{0\}$. Let $\pi : B(\ell^2) \to B(\ell^2)/\mathcal{K}$ be the natural projection. Note that

$$\pi \circ \sigma_1 = \pi \circ \eta_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1 = \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \sigma_{\mathfrak{B}} \circ \psi_1 = \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ \psi_1 = \pi_{\mathfrak{B}} \circ \pi_{\mathfrak{B}} \circ (\psi_1)(1) \circ \pi_{\mathfrak{A}}.$$
It now follows that \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) = \mathfrak{A}[2] \) since \( \pi_{\mathfrak{B}}, \tau_{\mathfrak{B}}, \) and \( \{\psi_i\}_{\{1\}} \) are injective homomorphisms.

Let \( H_1 \) be the essential subspace of \( \sigma_1 \). Since \( \sigma_1(\mathfrak{A}[2]) \subseteq \mathbb{K} \) and for each \( x \notin \mathfrak{A}[2] \), we have that \( \sigma_1(x) \notin \mathbb{K} \), we have that \( H_1 = \sigma_1(\mathfrak{A}[2])^{\ell^2} \). Similarly, we have that \( H_2 = \sigma_2(\mathfrak{A}[2])^{\ell^2} \), where \( H_2 \) is the essential subspace of \( \sigma_2 \). Let \( e \) be a minimal projection of \( \mathfrak{A}[2] \cong \mathbb{K} \). Suppose \( \sigma_1(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_1(-)|_{H_1} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \). Since \( K_0((\psi_1)(2)) = K_0((\psi_2)(2)) \), we have that \( \sigma_1(e) \) is Murray-von Neumann equivalent to \( \sigma_2(e) \). Hence, \( \sigma_2(e) \) has rank \( k \). Standard representation theory now implies that \( \sigma_2(-)|_{H_2} \) is unitarily equivalent to the direct sum of \( k \) copies of \( \rho \).

The above paragraph imply that \( \sigma_2(-)|_{H_2} \) and \( \sigma_1(-)|_{H_1} \) are unitarily equivalent. Since \( \ker(\sigma_1) = \ker(\sigma_2) \) and \( \ker(\pi \circ \sigma_1) = \ker(\pi \circ \sigma_2) \) by Theorem 5(iii) of \([1]\), there exists a sequence of unitaries \( \{V_n\}_{n=1}^{\infty} \) in \( B(\ell^2) \) such that \( V_n \sigma_1(a) V_n^* - \sigma_2(a) \in \mathbb{K} \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and

\[
\lim_{n \to \infty} \|V_n \sigma_1(a) V_n^* - \sigma_2(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

Set \( U_n = \eta^{-1}(V_n) \). Then \( \{U_n\}_{n=1}^{\infty} \) is a sequence of unitaries in \( \mathcal{M}(\mathfrak{B}[2]) \) such that \( U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a) U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a) \in \mathfrak{B}[2] \) for all \( n \in \mathbb{N} \) and for all \( a \in \mathfrak{A} \), and

\[
\lim_{n \to \infty} \|U_n(\sigma_{\mathfrak{B}} \circ \psi_1)(a) U_n^* - (\sigma_{\mathfrak{B}} \circ \psi_2)(a)\| = 0
\]

for all \( a \in \mathfrak{A} \).

**Definition 4.10.** A \( C^* \)-algebra \( \mathfrak{A} \) is called *weakly semiprojective* if we can always solving the *-homomorphism lifting problem

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\phi} & \prod_{n=1}^{\infty} \mathfrak{B}_n \\
\downarrow{\rho_N} & & \downarrow{(b_N, b_{N+1}, \ldots)} \\
\prod_{n=1}^{\infty} \mathfrak{B}_n / \bigoplus_{n=1}^{\infty} \mathfrak{B}_n & & [0, \ldots, 0, b_N, b_{N+1}, \ldots]
\end{array}
\]

and \( \mathfrak{A} \) is called *semiprojective* if we can always solve the lifting problem

\[
\begin{array}{ccc}
\mathfrak{A} & \xrightarrow{\phi} & \mathfrak{B} / \bigcup_{n=1}^{\infty} \mathfrak{I}_n \\
\downarrow{\rho_N} & & (\mathfrak{I}_1 \subseteq \mathfrak{I}_2 \subseteq \cdots \subseteq \mathfrak{B})
\end{array}
\]

**Lemma 4.11.** Let \( \mathfrak{A}_0 \) be a unital, separable, nuclear, tight \( C^* \)-algebra over \( X_2 \) such that \( \mathfrak{A}_0[2] \cong \mathbb{K} \) and \( \mathfrak{A}_0 \) has the stable weak cancellation property. Set \( \mathfrak{A} = \mathfrak{A}_0 \otimes \mathbb{K} \). Suppose \( \beta : \mathfrak{A} \rightarrow \mathfrak{A} \) is a full \( X_2 \)-equivariant homomorphism such that \( K_{X_2}(\beta) = K_{X_2}(\text{id}_{\mathfrak{A}}) \) and \( \beta_{\{1\}} = \text{id}_{\mathfrak{A}[1]} \). Then there exists a sequence of contractive, completely positive, linear maps \( \{\alpha_n : \mathfrak{A} \rightarrow \mathfrak{A}\}_{n=1}^{\infty} \) such that

1. \( \alpha_n \mid e_{n, \mathfrak{A}_0} \) is a homomorphism for all \( n \in \mathbb{N} \) and
2. for all \( a \in \mathfrak{A} \),

\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(a) - a\| = 0
\]
where \( e_n = \sum_{k=1}^{n} 1_{\mathbb{A}_0} \otimes e_{kk} \) and \( \{ e_{ij} \}_{i,j} \) is a system of matrix units for \( \mathbb{K} \). If, in addition, \( \mathbb{A} \) is assumed to be weakly semiprojective, then \( \alpha_n \) can be chosen to be a homomorphism for all \( n \in \mathbb{N} \).

**Proof.** Since \( \beta \) is a full \( X_2 \)-equivariant homomorphism and the ideal in \( \mathbb{A} \) generated by \( e_n \) is \( \mathbb{A} \), we have that the ideal in \( \mathbb{A} \) generated by \( (e_n) \) is \( \mathbb{A} \). Since \( K_{X_2}(\beta) = K_{X_2}([1]) \), we have that \( [\beta(e_n)] = [e_n] \) in \( K_{0}(\mathbb{A}) \). It now follows that \( \beta(e_n) \) and \( e_n \) are Murray-von Neumann equivalent since \( \mathbb{A}_0 \) has the stable weak cancellation property. Since \( \mathbb{A} \) is stable, there exists a unitary \( v_n \) in the unitization of \( \mathbb{A} \) such that \( v_n \beta(e_n) v_n^{\ast} = e_n \).

Fix \( n \in \mathbb{N} \). Let \( \epsilon \) be the extension \( 0 \to e_n \mathbb{A}[2] e_n \to e_n \mathbb{A} e_n \to \tau_{\mathbb{A}}[1] \mathbb{A} e_n \to 0 \). By Lemma 1.5 of [16], \( \epsilon \) is a full extension. Therefore, \( \sigma_{\epsilon}(e_n) \) is Murray-von Neumann equivalent to \( 1_{\mathcal{M}(\mathbb{A}[2])} \). Hence, \( e_n \mathbb{A}[2] e_n \cong \mathbb{A}[2] \cong \mathbb{K} \). Set \( \mathbb{A}_n = e_n \mathbb{A} e_n \) and define \( \beta_n : \mathbb{A}_n \to \mathbb{A}_n \) by \( \beta_n(x) = \text{Ad}(v_n) \circ \beta(x) \). Then \( \beta_n \) is a unital, full \( X_2 \)-equivariant homomorphism. Since \( \eta_{\mathbb{A}_n} \circ \sigma_{\epsilon_n} \circ \beta_n \) is a unital representation of \( \mathbb{A}_n \), the closed subspace of \( \ell^2 \) generated by \( \{ (\eta_{\mathbb{A}_n} \circ \epsilon_n \circ \beta_n)(x)\} : x \in \mathbb{A}_n, \xi \in \ell^2 \} \) is \( \ell^2 \). Therefore, \( \eta_{\mathbb{A}_n} \circ \sigma_{\epsilon_n} \circ \beta_n \) is a non-degenerate representation of \( \mathbb{A}_n \).

Since \( K_{X_2}(\beta) = K_{X_2}([1]) \) and the \( X_2 \)-equivariant embedding of \( \mathbb{A}_n \) as a sub-algebra of \( \mathbb{A} \) induces an isomorphism in ideal related \( K \)-theory, we have that \( K_{X_2}(\beta_n) = K_{X_2}([1]) \).

By Lemma 4.9 there exists a sequence of unitaries \( W_{k,n} \in \mathcal{M}(\mathbb{A}_n[2]) \) such that

\[
(\text{Ad}(W_{k,n}) \circ \sigma_{\epsilon_n} \circ \beta_n)(x) - \sigma_{\epsilon_n}(x) \in \mathbb{A}_n[2]
\]

for all \( x \in \mathbb{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \|(\text{Ad}(W_{k,n}) \circ \sigma_{\epsilon_n} \circ \beta_n)(x) - \sigma_{\epsilon_n}(x)\| = 0
\]

for all \( x \in \mathbb{A}_n \).

Note that \( \mathcal{M}(\mathbb{A}_n[2]) \cong \sigma_{\epsilon}(e_n) \mathcal{M}(\mathbb{A}[2]) \sigma_{\epsilon}(e_n) \) with an isomorphism mapping \( \mathbb{A}_n[2] \) onto \( e_n \mathbb{A}[2] e_n \). Thus, we get a partial isometry \( \tilde{W}_{k,n} \in \mathcal{M}(\mathbb{A}[2]) \) such that \( \tilde{W}_{k,n}^{\ast} \tilde{W}_{k,n} = \tilde{W}_{k,n} \tilde{W}_{k,n}^{\ast} = \sigma_{\epsilon}(e_n) \) and

\[
(\text{Ad}(\tilde{W}_{k,n}) \circ \sigma_{\epsilon} \circ \text{Ad}(v_n) \circ \beta)(x) - \sigma_{\epsilon}(x) \in \mathbb{A}[2]
\]

for all \( x \in \mathbb{A}_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \|(\text{Ad}(\tilde{W}_{k,n}) \circ \sigma_{\epsilon} \circ \text{Ad}(v_n) \circ \beta)(x) - \sigma_{\epsilon}(x)\| = 0
\]

for all \( x \in \mathbb{A}_n \).

Set \( V_{k,n} = (\tilde{W}_{k,n} + 1_{\mathcal{M}(\mathbb{A}[2])} - \sigma_{\epsilon}(e_n)) \sigma_{\epsilon}(e_n) \). Then \( V_{k,n} \) is a unitary in \( \mathcal{M}(\mathbb{A}[2]) \) such that

\[
(\text{Ad}(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta)(x) - \sigma_{\epsilon}(x) \in \mathbb{A}[2]
\]

for all \( x \in e_n \mathbb{A} e_n \) and for all \( k \in \mathbb{N} \), and

\[
\lim_{k \to \infty} \|(\text{Ad}(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta)(x) - \sigma_{\epsilon}(x)\| = 0
\]

for all \( x \in e_n \mathbb{A} e_n \). A consequence of the first part is that \( (\text{Ad}(V_{k,n}) \circ \sigma_{\epsilon} \circ \beta)(x) \in \sigma_{\epsilon}(e_n \mathbb{A} e_n) + \mathbb{A}[2] \) for all \( x \in e_n \mathbb{A} e_n \). Since \( \beta_{[1]} = \text{id}_{\mathbb{A}[2]} \), we have that \( x - \beta(x) \in \mathbb{A}[2] \) for all \( x \in e_n \mathbb{A} e_n \). Therefore,

\[
\text{Ad}(V_{k,n})(\sigma_{\epsilon}(x)) = \text{Ad}(V_{k,n}) \circ \sigma_{\epsilon}(x - \beta(x)) + \text{Ad}(V_{k,n}) \circ \beta(x) \in \sigma_{\epsilon}(e_n \mathbb{A} e_n) + \mathbb{A}[2]
\]

Thus, \( \alpha_{k,n} = \sigma_{\epsilon}^{-1} \circ (\text{Ad}(V_{k,n}) \circ \sigma_{\epsilon} \circ \text{Ad}(v_n)) | e_n \mathbb{A} e_n \) is a homomorphism from \( e_n \mathbb{A} e_n \) to \( \mathbb{A} \).
Since
\[
\lim_{k \to \infty} \|(\text{Ad}(V_{k,n}) \circ \sigma_t \circ \beta)(x) - \sigma_t(x)\| = 0
\]
for all \( x \in e_n \mathfrak{A}_n \) and \( e_n \mathfrak{A}_n \subseteq e_{n+1} \mathfrak{A}_{n+1} \), there exists a strictly increasing sequence \( \{k(n)\}_{n=1}^\infty \) of positive integers such that
\[
\lim_{n \to \infty} \|\alpha_{k(n),n} \circ \beta(x) - x\| = 0
\]
for all \( x \in \bigcup_{n=1}^\infty e_n \mathfrak{A}_n \). Let \( \alpha_n \) be a completely, contractive, positive linear extension of \( \alpha_{k(n),n} \). Since \( \bigcup_{n=1}^\infty e_n \mathfrak{A}_n \) is dense in \( \mathfrak{A} \), we have that
\[
\lim_{n \to \infty} \|\alpha_n \circ \beta(x) - x\| = 0
\]
for all \( x \in \mathfrak{A} \). We have just proved the first part of the lemma.

We now show that \( \mathfrak{A} \) is weakly semiprojective. Suppose \( \mathfrak{A} \) is weakly semiprojective. Let \( \epsilon > 0 \) and \( \mathcal{F} \) be a finite subset of \( \mathfrak{A} \). By Theorem 2.4 of [23] (see also Definition 2.1 and Theorem 2.3 of [25], and Theorem 19.1.3 of [26]), there exist a \( \delta > 0 \) and a finite subset \( \mathcal{G} \) of \( \mathfrak{A} \) such that for any \( C^* \)-algebra \( \mathfrak{B} \) and any contractive, completely positive, linear map \( L : \mathfrak{A} \to \mathfrak{B} \) such that
\[
\|L(ab) - L(a)L(b)\| < \delta
\]
for all \( a, b \in \mathcal{G} \), there exists a homomorphism \( h : \mathfrak{A} \to \mathfrak{B} \) such that
\[
\|h(x) - L(x)\| < \frac{\epsilon}{2}
\]
for all \( x \in \beta(\mathcal{F}) \).

Without loss of generality, we may assume that \( \epsilon < 1 \) and \( \delta < 1 \). Set
\[
M = 1 + \max \left( \{|a| : a \in \mathcal{G} \} \cup \{|x| : x \in \mathcal{F} \} \right)
\]
Since \( e_n \mathfrak{A}_n \subseteq e_{n+1} \mathfrak{A}_{n+1} \) and \( \bigcup_{n=1}^\infty e_n \mathfrak{A}_n \) is dense in \( \mathfrak{A} \), there exist \( n \in \mathbb{N} \) and a finite subset \( \mathcal{H} \subseteq e_n \mathfrak{A}_n \) such that for each \( a \in \mathcal{G} \), there exists \( y \in \mathcal{H} \) such that \( \|a - y\| < \frac{\delta}{4M} \) and
\[
\|\alpha_n \circ \beta(x) - x\| < \frac{\epsilon}{2}
\]
for all \( x \in \mathcal{F} \). Let \( a, b \in \mathcal{G} \). Choose \( x, y \in \mathcal{H} \subseteq e_n \mathfrak{A}_n \) such that \( \|a - x\| < \frac{\delta}{4M} \) and \( \|b - y\| < \frac{\delta}{4M} \). Note that \( |x| \leq 1 + |a| \leq M \) and \( |y| \leq 1 + |b| \leq M \). Then
\[
\|\alpha_n(ab) - \alpha_n(a)\alpha_n(b)\| = \|\alpha_n(ab - xb + xb - xy) + \alpha_n(xy) - \alpha_n(a)\alpha_n(b)\|
\leq \|\alpha_n(ab - xb + xb - xy)\| + \|\alpha_n(xy) - \alpha_n(a)\alpha_n(b)\|
\leq 2M\|a - x\| + 2M\|b - y\|
< 4M\frac{\delta}{4M} = \delta.
\]
By the choice of \( \delta \) and \( \mathcal{G} \), there exists a homomorphism \( \psi : \mathfrak{A} \to \mathfrak{A} \) such that
\[
\|\psi(t) - \alpha_n(t)\| < \frac{\epsilon}{2}
\]
for all $t \in \beta(\mathcal{F})$. Let $x \in \mathcal{F}$. Then

$$\|\psi \circ \beta(x) - x\| \leq \|\psi(\beta(x)) - \alpha_n(\beta(x))\| + \|\alpha_n(\beta(x)) - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$  

We have just shown that for every $\epsilon > 0$ and for every finite subset $\mathcal{F}$ of $\mathcal{A}$, there exists a homomorphism $\psi : \mathcal{A} \rightarrow \mathcal{A}$ such that

$$\|\psi \circ \beta(x) - x\| < \epsilon$$

for all $x \in \mathcal{F}$. Consequently, there exists a sequence of endomorphisms $\{\psi_n : \mathcal{A} \rightarrow \mathcal{A}\}_{n=1}^{\infty}$ such that

$$\lim_{n \rightarrow \infty} \|\psi_n \circ \beta(x) - x\| = 0$$

for all $x \in \mathcal{A}$ since $\mathcal{A}$ is separable. \hfill \Box

To prove a uniqueness theorem involving tight $C^*$-algebras $\mathcal{A}$ over $X_2$, we require that $\mathcal{A}[1]$ belongs to a class of $C^*$-algebras whose injective homomorphisms between two objects in this class are classified by $KK$.

**Definition 4.12.** We will be interested in classes $\mathcal{C}$ of separable, nuclear, simple $C^*$-algebras satisfying the following property that if $\mathcal{A}, \mathcal{B} \in \mathcal{C}$ and $\phi, \psi : \mathcal{A} \otimes K \rightarrow \mathcal{B} \otimes K$ are two injective homomorphisms such that $KK(\phi) = KK(\psi)$, then $\phi$ and $\psi$ are approximately unitarily equivalent.

**Remark 4.13.**

(1) By Theorem 4.1.3 of [29] if $\mathcal{C}$ is the class of Kirchberg algebras, then $\mathcal{C}$ satisfies the property in Definition 4.12.

(2) Let $\mathcal{C}$ be the class of unital, separable, nuclear, simple tracially AF $C^*$-algebras in $\mathcal{N}$. Then $\mathcal{C}$ satisfies the property in Definition 4.12.

**Theorem 4.14.** (Uniqueness Theorem 2) Let $\mathcal{C}$ be a class of $C^*$-algebras satisfying the property in Definition 4.12 and let $\mathcal{A}$ be a unital, separable, nuclear, tight $C^*$-algebra over $X_2$ such that $\mathcal{A}[2] \cong K$ and $\mathcal{A}[1] \in \mathcal{C}$. Suppose $\mathcal{A} \otimes K$ is semiprojective and $\mathcal{A}$ has the stable weak cancellation property. Let $\phi : \mathcal{A} \otimes K \rightarrow \mathcal{A} \otimes K$ be a full $X_2$-equivariant homomorphism such that $KK(X_2; \phi) = KK(X_2; id_{\mathcal{A} \otimes K})$. Then there exists a sequence of full $X_2$-equivariant endomorphisms $\{\alpha_n : \mathcal{A} \otimes K \rightarrow \mathcal{A} \otimes K\}_{n=1}^{\infty}$ such that $KK(X_2; \alpha_n) = KK(X_2; id_{\mathcal{A} \otimes K})$ and

$$\lim_{n \rightarrow \infty} \|(\alpha_n \circ \phi)(x) - x\| = 0$$

for all $x \in \mathcal{A} \otimes K$.

**Proof.** Set $\mathcal{B} = \mathcal{A} \otimes K$. Note that $\mathcal{B}$ is a tight $C^*$-algebra over $X_2$ with $\mathcal{B}[2] \cong K$. Throughout the proof, $\pi : \mathcal{B} \rightarrow \mathcal{B}[1]$ will denote the canonical projection. Note that $KK(\phi_{[1]}) = KK(id_{\mathcal{B}[1]})$ since $KK(X_2; \phi) = KK(X_2; id_{\mathcal{B}})$. Since $\mathcal{A}[1] \in \mathcal{C}$, there exists a sequence of unitaries $\{z_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathcal{B}[1])$ such that

$$\lim_{k \rightarrow \infty} \|z_k \phi_{[1]}(\pi(b)) z_k^* - \pi(b)\| = 0$$

for all $b \in \mathcal{B}$. Using the fact that $\phi$ is an $X_2$-equivariant homomorphism, we have that $\pi \circ \phi = \phi_{[1]} \circ \pi$, and hence

$$\lim_{k \rightarrow \infty} \|z_k (\pi \circ \phi(b)) z_k^* - \pi(b)\| = 0$$
for all $b \in \mathcal{B}$.

Let $\pi : \mathcal{M}(\mathcal{B}) \to \mathcal{M}(\mathcal{B}[1])$ be the surjective homomorphism induced by $\pi$. Since $\mathcal{B}$ is stable, by Corollary 2.3 of [35], we have that $\mathcal{B}[1]$ is stable. Thus, the unitary group of $\mathcal{M}(\mathcal{B}[1])$ is path-connected, which implies that every unitary in $\mathcal{M}(\mathcal{B}[1])$ lifts to a unitary in $\mathcal{M}(\mathcal{B})$. Hence, there exists a sequence of unitaries $\{w_k\}_{k=1}^{\infty}$ in $\mathcal{M}(\mathcal{B})$ such that $\pi(w_k) = z_k$.

Since $\mathcal{B}$ is semiprojective, by Proposition 2.2 of [7] (see [20]), there exists a sequence of homomorphisms $\{\beta_\ell : \mathcal{B} \to \mathcal{B}\}_{\ell=1}^{\infty}$ and a strictly increasing sequence $\{k(\ell)\}_{\ell=1}^{\infty}$ of positive integers such that $\pi \circ \beta_\ell = \pi$ and

$$\lim_{\ell \to \infty} \|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| = 0$$

for all $b \in \mathcal{B}$.

By Remark 2.5 there exists $N_1 \in \mathbb{N}$ such that $\beta_\ell$ is a full $X_2$-equivariant homomorphism for all $\ell \geq N_1$. By Proposition 2.3 of [7], we may choose $N_2 \geq N_1$ such that for all $\ell \geq N_2$, we have that $\beta_\ell$ and $\text{Ad}(w_{k(\ell)}) \circ \phi$ is homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \beta_\ell) = KK(X_2; \text{Ad}(w_{k(\ell)}) \circ \phi) = KK(X_2; \phi) = KK(X_2; \text{id}_\mathcal{B})$.

Let $\ell \geq N_2$. Note that $(\beta_\ell)^{(1)} = \text{id}_\mathcal{B}[1]$ since $\pi \circ \beta_\ell = \pi$. Since $\mathcal{B}$ is semiprojective, by Corollary 3.6 of [6] (also see Chapter 19 of [26]), $\mathcal{B}$ is weakly semiprojective. Hence, by Lemma 4.11 there exists a sequence of homomorphisms $\{\alpha_{m,\ell} : \mathcal{B} \to \mathcal{B}\}_{m=1}^{\infty}$ such that

$$\lim_{m \to \infty} \|\alpha_{m,\ell} \circ \beta_\ell(x) - x\| = 0$$

for all $x \in \mathcal{B}$. Since $\beta_\ell$ and $\text{id}_\mathcal{B}$ are full $X_2$-equivariant homomorphisms, by Remark 2.5 there exists $N_3$ such that, for all $m \geq N_3$, we have that $\alpha_{m,\ell}$ is a full $X_2$-equivariant homomorphism. Moreover, by Proposition 2.3 of [7], we can choose $N_3 \geq N_2$ such that $\alpha_{m,\ell} \circ \beta_\ell$ and $\text{id}_\mathcal{B}$ are homotopic. It follows from Theorem 5.5 of [8] that $KK(X_2; \alpha_{m,\ell} \circ \beta_\ell) = KK(X_2; \text{id}_\mathcal{B})$ for all $m \geq N_3$. Consequently, $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B})$ for all $m \geq N_3$ since $KK(X_2; \beta_\ell) = KK(X_2; \text{id}_\mathcal{B})$.

Let $\mathcal{F}$ be a finite subset of $\mathcal{B}$ and $\epsilon > 0$. Then there exists $\ell \geq N_2$ such that

$$\|\text{Ad}(w_{k(\ell)}) \circ \phi(b) - \beta_\ell(b)\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Moreover, there exists $m \geq N_3$ such that

$$\|\alpha_{m,\ell} \circ \beta_\ell(b) - b\| < \frac{\epsilon}{2}$$

for all $b \in \mathcal{F}$. Set $\alpha_1 = \text{Ad}(w_{k(\ell)})|_{\mathcal{B}}$ and $\alpha = \alpha_{m,\ell} \circ \alpha_1$. Since $w_{k(\ell)}$ is a unitary in $\mathcal{M}(\mathcal{B})$, we have that $\alpha_1$ is an automorphism of $\mathcal{B}$ and $KK(X_2; \alpha_1) = KK(X_2; \text{id}_\mathcal{B})$. Therefore, $\alpha$ is a full $X_2$-equivariant homomorphism. Since $\ell \geq N_2$ and $m \geq N_3$, we have that $KK(X_2; \alpha_{m,\ell}) = KK(X_2; \text{id}_\mathcal{B})$. Therefore, $KK(X_2; \alpha) = KK(X_2; \text{id}_\mathcal{B})$. Let $b \in \mathcal{F}$. Then

$$\|\alpha \circ \phi(b) - b\| = \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - b\|$$

$$\leq \|\alpha_{m,\ell} \circ \text{Ad}(w_{k(\ell)}) \circ \phi(b) - \alpha_{m,\ell} \circ \beta_\ell(b)\| + \|\alpha_{m,\ell} \circ \beta_\ell(b) - b\|$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$
for all \( b \in \mathcal{B} \). Since \( \mathcal{B} \) is a separable \( C^* \)-algebra, there exists a sequence of full \( X_2 \)-equivariant homomorphisms \( \{ \alpha_n : \mathcal{B} \to \mathbb{B} \}_{n=1}^{\infty} \) such that \( KK(X_2; \alpha_n) = KK(X_2; \text{id}_{\mathbb{B}}) \) and

\[
\lim_{n \to \infty} \| \alpha_n \circ \phi(b) - b \| = 0
\]

for all \( b \in \mathcal{B} \).

\[ \Box \]

**Theorem 4.15.** Let \( C \) be a class of \( C^* \)-algebras satisfying the property in Definition 4.12 and let \( \mathfrak{A}_1 \) and \( \mathfrak{A}_2 \) be unital, separable, nuclear, tight \( C^* \)-algebras over \( X_2 \) such that \( \mathfrak{A}_i[2] \cong K \) and \( \mathfrak{A}_i[1] \in C \). Suppose \( \mathfrak{A}_i \otimes K \) is semi-projective and \( \mathfrak{A}_i \) has the stable weak cancellation property. If there exist full \( X_2 \)-equivariant homomorphisms, \( \phi : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) and \( \psi : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K \), such that \( KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \) and \( KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K}) \), then for any finite subset \( \mathcal{F} \) and \( \epsilon > 0 \), there exists an isomorphism \( \gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) such that \( KK(X_2; \gamma) = KK(\phi) \) and

\[
\| \gamma(x) - \phi(x) \| < \epsilon
\]

for all \( x \in \mathcal{F} \).

**Proof.** Let \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) be a sequence of finite subsets of \( \mathfrak{A}_1 \otimes K \) such that \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_1 \otimes K \) and let \( \{ \mathcal{F}_n \}_{n=1}^{\infty} \) be a sequence of finite subsets of \( \mathfrak{A}_2 \otimes K \) such that \( \mathcal{F}_n \subseteq \mathcal{F}_{n+1} \) and \( \bigcup_{n=1}^{\infty} \mathcal{F}_n \) is dense in \( \mathfrak{A}_2 \otimes K \).

Let \( \epsilon > 0 \) and \( \mathcal{F} \) be a finite subset of \( \mathfrak{A}_1 \). Set \( \mathcal{F}_1 = \mathcal{F} \cup \mathcal{F}_1 \) and choose \( m_1 \in \mathbb{N} \) such that \( \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon \). By Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \alpha_1 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_1 \otimes K \) such that \( KK(X_2; \alpha_1) = KK(X_2; \text{id}_{\mathfrak{A}_1 \otimes K}) \) and

\[
\| \alpha_1 \circ \psi \circ \phi(a) - a \| < \frac{1}{2m_1 + 1}
\]

for all \( a \in \mathcal{F}_1 \). Set \( \phi_1 = \phi \) and \( \psi_1 = \alpha_1 \circ \psi \). Then \( KK(X_2; \psi_1) = KK(X_2; \phi) \) and \( \| \psi_1 \circ \phi_1(a) - a \| < \frac{1}{2m_1 + 1} \) for all \( a \in \mathcal{F}_1 \).

Set \( \mathcal{G}_1 = \mathcal{G}_1 \cup \phi_1(\mathcal{F}_1) \). Note that \( KK(X_2; \phi \circ \psi_1) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \). Hence, by Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \beta_1 : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_2 \otimes K \) such that \( KK(X_2; \beta_1) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \) and

\[
\| \beta_1 \circ \phi \circ \psi_1(x) - x \| < \frac{1}{2m_1 + 1}
\]

for all \( x \in \mathcal{G}_1 \). Set \( \phi_2 = \beta_1 \circ \phi \). Then \( KK(X_2; \phi_2) = KK(X_2; \phi) \) and

\[
\| \phi_2 \circ \psi_1(x) - x \| < \frac{1}{2m_1 + 1}
\]

for all \( x \in \mathcal{G}_1 \). Note that for all \( x \in \mathcal{F}_1 \), then

\[
\| \phi(x) - \phi_2(x) \| \leq \| \phi_1(x) - \phi_2 \circ \psi_1(\phi_1(x)) \| + \| \phi_2 \circ \psi_1(\phi_1(x)) - \phi_2(x) \|
\]

\[
< \frac{1}{2m_1 + 1} + \| \psi_1 \circ \phi_1(x) - x \| < \frac{1}{2m_1}.
\]

Set \( \mathcal{F}_2 = \mathcal{F}_2 \cup \phi_2(\mathcal{G}_1) \). Note that \( KK(X_2; \psi \circ \phi_2) = KK(X_2; \psi \circ \phi) = KK(X_2; \text{id}_{\mathfrak{A}_2 \otimes K}) \). Hence, by Theorem 4.14 there exists a full \( X_2 \)-equivariant homomorphism \( \alpha_2 : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K \) such that

\[
\| \alpha_2 \circ \psi \circ \phi_2(a) - a \| < \frac{1}{2m_2 + 1}
\]

for all \( a \in \mathcal{F}_2 \). Set \( \phi_3 = \alpha_2 \circ \psi \circ \phi_2 \) and \( \psi_3 = \alpha_2 \circ \psi \circ \psi_1 \). Then \( KK(X_2; \phi_3) = KK(X_2; \phi) \) and \( KK(X_2; \psi_3) = KK(X_2; \psi) \) and

\[
\| \phi_3 \circ \phi_3(x) - x \| < \frac{1}{2m_2 + 1}
\]

for all \( x \in \mathcal{F}_2 \). Note that for all \( x \in \mathcal{F}_2 \), then

\[
\| \phi(x) - \phi_3(x) \| \leq \| \phi_2(x) - \phi_3 \circ \phi_2(\phi_2(x)) \| + \| \phi_3 \circ \phi_2(\phi_2(x)) - \phi_3(x) \|
\]

\[
< \frac{1}{2m_2 + 1} + \| \phi_2 \circ \psi_2(\phi_2(x)) - \phi_2(x) \| < \frac{1}{2m_2}.
\]
$\mathfrak{A} \otimes K$ such that $KK(X_2; \alpha_2) = KK(X_2; \text{id}_{\mathfrak{A} \otimes K})$ and

$$\|\alpha_2 \circ \psi \circ \phi_2(a) - a\| < \frac{1}{2m_1 + 2}$$

for all $a \in \mathcal{F}_2$. Set $\psi_2 = \alpha_2 \circ \psi$. Then $KK(X_2; \psi_2) = KK(X_2; \psi)$ and

$$\|\psi_2 \circ \phi_2(a) - a\| < \frac{1}{2m_1 + 2}$$

for all $x \in \mathcal{F}_2$.

Set $\mathcal{G}_2 = \mathcal{F}_2 \cup \phi_2(\mathcal{F}_2)$. Note that $KK(X_2; \phi \circ \psi_2) = KK(X_2; \phi \circ \psi) = KK(X_2; \text{id}_{\mathfrak{A} \otimes K})$.

Hence, by Theorem 4.14 there exists a full $X_2$-equivariant homomorphism $\beta_2 : \mathfrak{A} \otimes K \to \mathfrak{A} \otimes K$ such that $KK(X_2; \beta_2) = KK(X_2; \text{id}_{\mathfrak{A} \otimes K})$ and

$$\|\beta_2 \circ \phi \circ \psi_2(x) - x\| < \frac{1}{2m_1 + 2}$$

for all $x \in \mathcal{G}_2$. Set $\phi_3 = \beta_2 \circ \phi$. Then $KK(X_2; \phi_3) = KK(X_2; \phi)$ and

$$\|\phi_3 \circ \psi_2(x) - x\| < \frac{1}{2m_1 + 2}$$

for all $x \in \mathcal{G}_2$. Note that for all $x \in \mathcal{F}_2$, we have that

$$\|\phi_2(x) - \phi_3(x)\| \leq \|\phi_2(x) - \phi_3 \circ \psi_2(\phi_2(x))\| + \|\phi_3 \circ \psi_2(\phi_2(x)) - \phi_3(x)\|$$

$$< \frac{1}{2m_1 + 2} + \|\psi_2(\phi_2(x)) - x\| < \frac{1}{2m_1 + 1}.$$  

Continuing this process, we have constructed a sequence $\{\mathcal{F}_n\}_{n=1}^{\infty}$ of finite subsets of $\mathfrak{A} \otimes K$, a sequence $\{\mathcal{G}_n\}_{n=1}^{\infty}$ of finite subsets of $\mathfrak{A}_2 \otimes K$, a sequence of full $X_2$-equivariant homomorphisms $\{\phi_n : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K\}_{n=1}^{\infty}$, and a sequence of full $X_2$-equivariant homomorphisms $\{\psi_n : \mathfrak{A}_2 \otimes K \to \mathfrak{A}_1 \otimes K\}_{n=1}^{\infty}$ such that

1. $KK(X_2; \phi_n) = KK(X_2; \phi)$ for all $n \in \mathbb{N}$ and $\phi_1 = \phi$;
2. $KK(X_2; \psi_n) = KK(X_2; \psi)$ for all $n \in \mathbb{N}$;
3. $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $\mathcal{F}_n \subseteq \mathcal{F}_n$;
4. $\mathcal{G}_n \subseteq \mathcal{G}_{n+1}$ and $\mathcal{G}_n \subseteq \mathcal{G}_n$;
5. for each $x \in \mathcal{F}_n$ and for each $x \in \mathcal{G}_n$

$$\|\psi_n \circ \phi_n(x) - x\| < \frac{1}{2m_1 + n} \quad \text{and} \quad \|\phi_{n+1} \circ \psi_n(x) - x\| < \frac{1}{2m_1 + n}$$

6. for each $x \in \mathcal{F}_n$, 

$$\|\phi_n(x) - \phi_{n+1}(x)\| < \frac{1}{2m_1 + n}$$

Since $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes K$ and $\mathcal{F}_n \subseteq \mathcal{F}_n$, we have that $\bigcup_{n=1}^{\infty} \mathcal{F}_n$ is dense in $\mathfrak{A}_1 \otimes K$. Similarly, $\bigcup_{n=1}^{\infty} \mathcal{G}_n$ is dense in $\mathfrak{A}_2 \otimes K$. Therefore, there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that

$$\|\gamma(a) - \phi_n(a)\| < \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k}$$
for all $a \in \mathcal{A}_n$. Since $\mathcal{F} \subseteq \mathcal{F}_1$, we have that
\[
\|\phi(x) - \gamma(x)\| = \|\phi_1(x) - \gamma(x)\| < \sum_{k=m_1}^{\infty} \frac{1}{2^k} < \epsilon.
\]
Since
\[
\lim_{n \to \infty} \sum_{k=m_1+n-1}^{\infty} \frac{1}{2^k} = 0,
\]
we have that
\[
\lim_{n \to \infty} \|\gamma(a) - \phi_n(a)\| = 0
\]
for all $a \in \mathcal{A}_1 \otimes \mathcal{K}$. Since $\mathcal{A}_1 \otimes \mathcal{K}$ is semiprojective, by Proposition 2.3 of [7], there exists $\gamma_{\mathcal{A}_1,n} \otimes \mathcal{K}$ such that $\gamma$ and $\phi_{\mathcal{A}_1,n}$ are homotopic. Hence, by Theorem 5.5 of [8], $KK(X_2;\gamma) = KK(X_2;\phi_{\mathcal{A}_1,n})$.

4.3. Unital Classification. We now combine the above results with the Meta-theorem of Section 3 (see Theorem 3.3) to get a strong classification for a class of unital $C^*$-algebras which includes all unital graph $C^*$-algebras with exactly one non-trivial ideal.

**Corollary 4.16.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital, tight $C^*$-algebras over $X_n$ such that $\mathcal{A}_i$ has real rank zero, $\mathcal{A}_i[n]$ is a Kirchberg algebra in $\mathcal{N}$, and $\mathcal{A}_i[1,n-1]$ is an AF-algebra. Let $x \in KK(X_2;\mathcal{A}_1,\mathcal{A}_2)$ be an invertible such that $K_{X_n}(x)|_{Y}$ is an order isomorphism for each $Y \in \mathcal{L}(\mathcal{C}(X_n))$ and $K_{X_n}(X_2)(\mathcal{A}_1[n]) = [1_{\mathcal{A}_1}]$ in $K_0(\mathcal{A}_2)$. Then there exists an isomorphism $\phi : \mathcal{A}_1 \to \mathcal{B}$ such that $K_{X_n}(\phi) = K_{X_n}(x)$.

**Proof.** Since $\mathcal{A}_i[1]$ and $\mathcal{A}_i[2]$ are separable and nuclear, we have that $\mathcal{A}_i$ is separable and nuclear. Since $\mathcal{A}_i[1,n-1]$ is an AF-algebra and $\mathcal{A}_i[n]$ is a Kirchberg algebra, they both have the stable weak cancellation property. By Lemma 3.15 of [15], $\mathcal{A}_i$ has stable weak cancellation property. By Lemma 4.6 for each tight $C^*$-algebra $\mathcal{A}$ over $X_n$, we have that $K_{X_n}(\text{Ad}(u)|_{\mathcal{A}})$ for each unitary $u \in \mathcal{M}(\mathcal{A})$. A computation shows that $K_{X_n}(-)$ satisfies (1), (2), and (3) of Theorem 3.3 since $K_{\mathcal{A}_1,n}(-)$ does. The corollary now follows from Theorem 3.3 and Theorem 4.7.

**Corollary 4.17.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathcal{A}_i[2] \cong \mathcal{K}$ and $\mathcal{A}_i[1]$ is a Kirchberg algebra in $\mathcal{N}$. Let $x \in KK(X_2;\mathcal{A}_1,\mathcal{A}_2)$ be an invertible such that $K_{X_2}(x)|_{Y}$ is an order isomorphism for each $Y \in \mathcal{L}(\mathcal{C}(X_2))$ and $K_{X_2}(X_2)(\mathcal{A}_1[1]) = [1_{\mathcal{A}_1}]$ in $K_0(\mathcal{A}_2)$. If $\mathcal{A}_i \otimes \mathcal{K}$ is semiprojective, then there exists an isomorphism $\gamma : \mathcal{A}_1 \otimes \mathcal{K} \to \mathcal{A}_2 \otimes \mathcal{K}$ such that $KK(X_2;\gamma) = x$.

**Proof.** Since $\mathcal{A}_i[1]$ and $\mathcal{A}_i[2]$ are separable and nuclear, we have that $\mathcal{A}_i$ is separable and nuclear. Since $\mathcal{A}_i[2]$ and $\mathcal{A}_i[1]$ have real rank zero and $K_1(\mathcal{A}_i[2]) = 0$, we have that $\mathcal{A}$ has real rank zero. Since $\mathcal{A}_i[2]$ is an AF-algebra and $\mathcal{A}_i[1]$ is a Kirchberg algebra, they both have the stable weak cancellation property. Therefore, by Lemma 3.15 of [15], $\mathcal{A}$ has the stable weak cancellation property.

By Lemma 1.5 of [16], the extension $0 \to \mathcal{A}_i[2] \to \mathcal{A}_i \to \mathcal{A}_i[1] \to 0$ is full, and hence by Proposition 1.6 of [16], $0 \to \mathcal{A}_i[2] \otimes \mathcal{K} \to \mathcal{A}_i \otimes \mathcal{K} \to \mathcal{A}_i[1] \otimes \mathcal{K} \to 0$ is full. The corollary now follows from Theorem 4.1(ii), Theorem 4.15 and Theorem 3.3.
It is an open question to determine if every unital, separable, nuclear, tight $C^*$-algebra $\mathfrak{A}$ over $X_2$ whose unique proper nontrivial ideal is isomorphic to $K$ and quotient is a Kirchberg algebra in $\mathcal{N}$ with finitely generated $K$-theory is semiprojective. The following results show that under some $K$-theoretical conditions, $\mathfrak{A}$ is semiprojective.

**Lemma 4.18.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$ with $C^*(E)[1]$ being purely infinite. Then $C^*(E)$ and $C^*(E) \otimes K$ are semiprojective.

**Proof.** The fact that $C^*(E)$ is semiprojective follows from the results of [12]. By Proposition 6.4 of [18], $C^*(E)[2]$ is stable. Since $C^*(E)$ is a unital $C^*$-algebra, by Lemma 1.5 of [16], the extension $\varepsilon : 0 \to C^*(E)[2] \to C^*(E) \to C^*(E)[1] \to 0$ is a full extension. By Proposition 3.21 and Corollary 3.22 of [15], $C^*(E)$ is properly infinite. Therefore, by Theorem 4.1 of [3], $C^*(E) \otimes K$ is semiprojective. □

**Proposition 4.19.** Let $\mathfrak{A}$ be unital, separable, nuclear, tight $C^*$-algebras over $X_2$. If $\mathfrak{A}[2] \cong K$ and $\mathfrak{A}[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}[1])) \leq \text{rank}(K_0(\mathfrak{A}[1]))$, $K_1(\mathfrak{A}[1])$ is free, and the $K$-groups of $\mathfrak{A}[i]$ are finitely generated, then $\mathfrak{A}$ and $\mathfrak{A} \otimes K$ are semiprojective. Consequently, $\mathfrak{A}$ semiprojective.

**Proof.** By Lemma 1.5 of [16], $\varepsilon : 0 \to \mathfrak{A}[2] \to \mathfrak{A} \to \mathfrak{A}[1] \to 0$ is a full extension. By Corollary 3.22 of [15], $K_0(\mathfrak{A})_+ = K_0(\mathfrak{A})$. By Theorem 6.4 of [11], there exists a graph $E$ with finitely many vertices such that $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$ such that $C^*(E)$ is a tight $C^*$-algebra over $X_2$. Since $E$ has finitely many vertices, $C^*(E)$ is unital. Since $K^+_{X_2}(\mathfrak{A}) \cong K^+_{X_2}(C^*(E))$, we have that $C^*(E)[1]$ is a Kirchberg algebra. By Theorem 3.9 of [16], we have that $\mathfrak{A} \otimes K \cong C^*(E) \otimes K$. By Lemma 4.18, $C^*(E)$ and $C^*(E) \otimes K$ are semiprojective. Hence, by Proposition 2.7 of [3], $\mathfrak{A}$ and $\mathfrak{A} \otimes K$ are semiprojective. □

**Corollary 4.20.** Let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be unital, tight $C^*$-algebras over $X_2$ such that $\mathfrak{A}_1[2] \cong K$ and $\mathfrak{A}_1[1]$ is a Kirchberg algebra in $\mathcal{N}$ such that $\text{rank}(K_1(\mathfrak{A}_1[1])) \leq \text{rank}(K_0(\mathfrak{A}_1[1]))$, $K_1(\mathfrak{A}_1[1])$ is free, and the $K$-groups of $\mathfrak{A}_1$ are finitely generated. Let $x \in KK(X_2; \mathfrak{A}_1, \mathfrak{A}_2)$ be an invertible such that $K_{X_2}(x)_Y$ is an order isomorphism for each $Y \in \mathbb{L}(X_2)$ and $K_{X_2}(x)_{\mathfrak{A}_2}(\mathfrak{I}_{\mathfrak{A}_2}) = \mathfrak{I}_{\mathfrak{A}_2}$ in $K_0(\mathfrak{A}_2)$. Then there exists an isomorphism $\gamma : \mathfrak{A}_1 \otimes K \to \mathfrak{A}_2 \otimes K$ such that $KK(X_2; \gamma) = x$.

**Proof.** This follows from Proposition 4.19 and Corollary 4.17 □

5. Applications

Let $E$ be a graph satisfying Condition (K) (in particular, if $C^*(E)$ has finitely many ideals, then $E$ satisfies Condition (K)). Let $\mathfrak{J}_1, \mathfrak{J}_2$ be ideals of $C^*(E)$ such that $\mathfrak{J}_1 \subseteq \mathfrak{J}_2$ and $\mathfrak{J}_2/\mathfrak{J}_1$ is simple. Then by Theorem 5.1 of [38] and Corollary 3.5 of [2], $\mathfrak{J}_2/\mathfrak{J}_1$ is a simple graph $C^*$-algebra. Hence, $\mathfrak{J}_2/\mathfrak{J}_1$ is either a Kirchberg algebra or an AF algebra.

5.1. Classification of graph $C^*$-algebras with exactly one ideal.

**Lemma 5.1.** Let $E$ be a graph with finitely many vertices such that $C^*(E)$ is a simple AF-algebra. Then $C^*(E) \otimes K \cong K$. Consequently, if $F$ is a graph with finitely many vertices such that $C^*(F)$ is a tight $C^*$-algebra over $X_2$ and $C^*(F)[2]$ is an AF-algebra, then $C^*(F)[2] \cong K$. 

Proof. We claim that $E$ is a finite graph. By Corollary 2.13 and Corollary 2.15 of [9], $E$ has no cycles, and for every vertex $v_0$ that emits infinitely many edges and for each vertex $v$, there exists a path from $v$ to $v_0$. Since $E$ has no cycles, we have that every vertex of $E$ emits only finitely many edges. Hence, $E$ is a finite graph. By Proposition 1.18 of [30], $C^*(E) \cong M_n$.

We now prove the second statement. First note that $C^*(F)[2]$ is a simple AF-algebra. Since $C^*(F)[2]$ is stably isomorphic to a subgraph of $E$, $C^*(F)[2] \otimes K \cong C^*(E)$ for some graph $E$ with finitely many vertices. Since $C^*(E)$ is a simple AF-algebra, we have that $C^*(E) \otimes K \cong K$. Hence, $C^*(F)[2] \otimes K \cong K$ which implies that $C^*(F)[2] \cong M_n$ or $C^*(F)[2] \cong K$. Since $C^*(F)[2]$ is a non-unital $C^*$-algebra ($C^*(E)$ is a tight $C^*$-algebra over $X_2$), we have that $C^*(F)[2] \cong K$. 

**Definition 5.2.** For a $C^*$-algebra $\mathfrak{A}$, set

$$\Sigma \mathfrak{A} = \{ x \in K_0(\mathfrak{A}) : x = [p] \text{ for some projection } p \text{ in } \mathfrak{A} \}.$$ 

Let $\mathfrak{B}$ be a $C^*$-algebra. An order isomorphism $\alpha : K_0(\mathfrak{A}) \rightarrow K_0(\mathfrak{B})$ is scale preserving if one of the following holds:

1. $\mathfrak{A}$ is unital if and only if $\mathfrak{B}$ unital and $\alpha([1_\mathfrak{A}]) = [1_\mathfrak{B}]$.
2. $\mathfrak{A}$ is non-unital if and only if $\mathfrak{B}$ is non-unital and $\alpha(\Sigma \mathfrak{A}) = \Sigma \mathfrak{B}$.

**Theorem 5.3.** Let $E_1$ and $E_2$ be graphs with finitely many vertices and $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. If $\alpha : K_{X_2}^+(C^*(E_1)) \rightarrow K_{X_2}^+(C^*(E_2))$ is an isomorphism such that $\alpha Y$ is scale preserving for all $Y \in \mathbb{LC}(X_2)$, then there exists an isomorphism $\phi : C^*(E_1) \rightarrow C^*(E_2)$ such that $K_{X_2}^+(\phi) = \alpha$.

**Proof.** Since $E_i$ has finitely many vertices, $C^*(E_1)$ and $C^*(E_2)$ are unital $C^*$-algebras.

Case 1: Suppose $C^*(E_1)$ is an AF-algebra. Then $C^*(E_2)$ is an AF-algebra. Hence, the result follows from Elliott’s classification of AF-algebras [19].

Case 2: Suppose $C^*(E_1)$ is not an AF-algebra. Then $C^*(E_2)$ is not an AF-algebra.

Subcase 2.1: Suppose $C^*(E_1)[1]$ is an AF-algebra. Then $C^*(E_2)[1]$ is an AF-algebra. By Corollary 1.10 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \rightarrow C^*(E_2)$ such that $K_{X_2}^+(\phi) = \alpha$.

Subcase 2.2: Suppose $C^*(E_1)[1]$ is a Kirchberg algebra. Then $C^*(E_2)[1]$ is a Kirchberg algebra. Since $C^*(E_1)$ is not an AF-algebra, either $C^*(E_1)[2]$ is Kirchberg algebra or an AF-algebra.

Suppose $C^*(E_1)[2]$ is a Kirchberg algebra. By Theorem 2.4 of [32], there exists an isomorphism $\phi : C^*(E_1) \rightarrow C^*(E_2)$ such that $K_{X_2}^+(\phi) = \alpha$. Suppose $C^*(E_1)[2]$ is an AF-algebra. Then, by Lemma 5.1 $C^*(E_1)[2] \cong K$. By Corollary 4.20 and Corollary 2.11 there exists an isomorphism $\phi : C^*(E_1) \rightarrow C^*(E_2)$ such that $K_{X_2}^+(\phi) = \alpha$. □

The following theorem completes the classification of graph $C^*$-algebras with exactly one non-trivial ideal.

**Corollary 5.4.** Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_2$. Then $C^*(E_1) \cong C^*(E_2)$ if and only if there exists an isomorphism $\alpha : K_{X_2}^+(C^*(E_1)) \rightarrow K_{X_2}^+(C^*(E_2))$ such that $\alpha Y$ is a scale preserving isomorphism for all $Y \in \mathbb{LC}(X_2)$. 

Proof. The only case that is not covered by Theorem 4.9 of [15] is the case that $C^*(E_i)$ is unital. The unital case follows from Theorem 5.3 because of Theorem 3.3.

5.2. Classification of graph $C^*$-algebras with more than one ideal. For a tight $C^*$-algebra $\mathfrak{A}$ over $X_n$, the finite and infinite simple sub-quotients of $\mathfrak{A}$ are separated if there exists $U \in \mathfrak{O}(X_n)$ such that either

(1) $\mathfrak{A}(U)$ is an AF-algebra and $\mathfrak{A}(X_n \setminus U) \otimes \mathcal{O}_\infty \cong \mathfrak{A}(X_n \setminus U)$ or

(2) $\mathfrak{A}(X_n \setminus U)$ is an AF-algebra and $\mathfrak{A}(U) \otimes \mathcal{O}_\infty \cong \mathfrak{A}(U)$.

In [14], the authors proved that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are graph $C^*$-algebras that are tight $C^*$-algebras over $X_n$ such that the finite and infinite simple sub-quotients are separated, then $\mathfrak{A}_1 \otimes \mathcal{K} \cong \mathfrak{A}_2 \otimes \mathcal{K}$ if and only if $K^+_X(\mathfrak{A}_1) \cong K^+_X(\mathfrak{A}_2)$. We will show in this section that under mild $K$-theoretical conditions, we may remove the separated condition for the case $n = 3$.

Lemma 5.5. Let $E$ be a graph such that $C^*(E)$ is a tight $C^*$-algebra over $X_n$.

(i) If $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra, then

$$\varepsilon_1 : 0 \to C^*(E)[2, n] \otimes \mathcal{K} \to C^*(E) \otimes \mathcal{K} \to C^*(E)[1] \otimes \mathcal{K} \to 0$$

is a full extension.

(ii) If $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite, then

$$\varepsilon_2 : 0 \to C^*(E)[k, n] \otimes \mathcal{K} \to C^*(E) \otimes \mathcal{K} \to C^*(E)[1, k - 1] \otimes \mathcal{K} \to 0$$

is a full extension.

Proof. Suppose $C^*(E)[n]$ and $C^*(E)[1]$ are purely infinite and $C^*(E)[2, n - 1]$ is an AF-algebra. Note that $C^*(E)[1, n - 1]/C^*(E)[2, n - 1] \cong C^*(E)[1]$ and $C^*(E)[2, n - 1]$ is the largest ideal of $C^*(E)[1, n - 1]$ which is an AF-algebra. Since $C^*(E)[1, n - 1]$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[2, n - 1] \otimes \mathcal{K} \to C^*(E)[1, n - 1] \otimes \mathcal{K} \to C^*(E)[1] \otimes \mathcal{K} \to 0$$

is a full extension. Since $C^*(E)[n] \otimes \mathcal{K}$ is a purely infinite simple $C^*$-algebra, we have that

$$0 \to C^*(E)[n] \otimes \mathcal{K} \to C^*(E)[2, n] \otimes \mathcal{K} \to C^*(E)[2, n - 1] \otimes \mathcal{K} \to 0$$

is a full extension. Hence, by Proposition 3.2 of [17], $\varepsilon_1$ is a full extension.

Suppose $C^*(E)[k, n]$ and $C^*(E)[1, k - 2]$ are AF-algebras and $C^*(E)[k - 1]$ is purely infinite. Note that $C^*(E)[k, n]$ is the largest ideal of $C^*(E)[k - 1, n]$ such that $C^*(E)[k, n]$ is an AF-algebra and $C^*(E)[k - 1, n]/C^*(E)[k, n] \cong C^*(E)[k - 1]$ is purely infinite. Since $C^*(E)[k - 1, n] \otimes \mathcal{K}$ is isomorphic to a graph $C^*$-algebra, by Proposition 3.10 of [18],

$$0 \to C^*(E)[k, n] \otimes \mathcal{K} \to C^*(E)[k - 1, n] \otimes \mathcal{K} \to C^*(E)[k - 1] \otimes \mathcal{K} \to 0$$

is a full extension. By Proposition 5.4 of [14], $\varepsilon_2$ is a full extension.

Theorem 5.6. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_n$. Suppose

(i) $C^*(E_1)[n]$ and $C^*(E_1)[1]$ are purely infinite;

(ii) $C^*(E_2)[2, n - 1]$ is an AF-algebra; and
(iii) $KK^1(C^*(E_1)[1], C^*(E_2)[2, n]) = KL^1(C^*(E_1)[1], C^*(E_2)[2, n])$.

Then $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$ if and only if $K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K)$.

Proof. Let $\xi_i$ be the extension

$$0 \to C^*(E_i)[2, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0.$$

By Lemma 5.5(i), $\xi_i$ is a full extension. Suppose $\alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K)$.

Lift $\alpha$ to an invertible element $x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K)$. Note that $r_{X_n}^{[2, n]}(x)$ is invertible in $KK([2, n]; C^*(E_1)[2, n] \otimes K, C^*(E_2)[2, n] \otimes K)$ and $r_{X_n}^{[1]}(x)$ is invertible in $KK(C^*(E_1)[1] \otimes K, C^*(E_2)[1] \otimes K)$. By Theorem 4.7 there exists an isomorphism $\phi_0 : C^*(E_1)[2, n] \otimes K \to C^*(E_2)[2, n] \otimes K$ such that $KL(\phi_0) = z$, where $z$ is the invertible element of $KL(C^*(E_1)[2, n] \otimes K, C^*(E_2)[2, n] \otimes K)$ induced by $r_{X_n}^{[2, n]}(x)$. By the Kirchberg-Phillips classification (27) and (29), there exists an isomorphism $\phi_2 : C^*(E_1)[1] \otimes K \to C^*(E_2)[1] \otimes K$ such that $KK(\phi_2) = r_{X_n}^{[1]}(x).

Consider $C^*(E_i)$ as a $C^*$-algebra over $X_2$ by setting $C^*(E_i)[2] = C^*(E_i)[2, n]$ and $C^*(E_i)[1, 2] = C^*(E_i)$. Let $y$ be the invertible element of $KK(X_2, C^*(E_1), C^*(E_2))$ induced by $x$. Note that $r_{X_2}^{[1]}(y) = r_{X_2}^{[1]}(x) = KK(\phi_2)$ and $KL(r_{X_2}^{[2]}(y)) = z = KL(\phi_0)$ in $KL(C^*(E_1)[2, n], C^*(E_2)[2, n])$.

By Theorem 3.7 of [14],

$$r_{X_2}^{[1]}(y) \times [r_{X_2}^{[2]}(y)] = [r_{X_2}^{[1]}(y)] \times r_{X_2}^{[2]}(y)$$

in $KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$, where $\xi_1$ is the extension

$$0 \to C^*(E_i)[2, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0.$$

Thus,

$$KL(\phi_2) \times [r_{X_2}^{[2]}(y)] = [r_{X_2}^{[1]}(y)] \times KL(\phi_0)$$

in $KL^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$. Since $KL^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K) = KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$,

$$KK(\phi_2) \times [r_{X_2}^{[2]}(y)] = [r_{X_2}^{[1]}(y)] \times KK(\phi_0)$$

in $KK^1(C^*(E_1)[1] \otimes K, C^*(E_2)[2, n] \otimes K)$. By Lemma 4.5 of [14], $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$. □

Theorem 5.7. Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_n$.

Suppose

(i) $C^*(E_i)[k, n]$ and $C^*(E_i)[1, k - 2]$ are AF-algebras;

(ii) $C^*(E_i)[k - 1]$ is purely infinite; and

(iii) $KK^1(C^*(E_i)[k, k - 1], C^*(E_i)[k, n]) = KL^1(C^*(E_i)[k, k - 1], C^*(E_i)[k, n])$.

Then $C^*(E_1) \otimes K \cong C^*(E_2) \otimes K$ if and only if $K^+_{X_n}(C^*(E_1) \otimes K) \cong K^+_{X_n}(C^*(E_2) \otimes K)$.

Proof. Let $\xi_i$ be the extension $0 \to C^*(E_i)[k, n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1] \otimes K \to 0$.

By Lemma 5.5(ii), $\xi_i$ is a full extension. Suppose $\alpha : K^+_{X_n}(C^*(E_1) \otimes K) \to K^+_{X_n}(C^*(E_2) \otimes K)$.

Lift $\alpha$ to an invertible element $x \in KK(X_n; C^*(E_1) \otimes K, C^*(E_2) \otimes K)$. Note that $r_{X_n}^{[k, n]}(x)$ is invertible in $KK([k, n]; C^*(E_1)[k, n] \otimes K, C^*(E_2)[k, n] \otimes K)$ and $r_{X_n}^{[1, k - 1]}(x)$ is invertible in $KK(C^*(E_1)[1, k - 1], C^*(E_2)[1, k - 1])$. By Theorem 4.7 there exists an isomorphism
\[ \phi_2 : C^*(E_1)[1,k-1] \otimes K \to C^*(E_2)[1,k-1] \otimes K \text{ such that } KL(\phi_2) = z_2, \text{ where } z_2 \text{ is the invertible element in } KL(C^*(E_1)[1,k-1], C^*(E_2)[1,k-1]) \text{ induced by } r_{X_2}^{[1,k-1]}(x). \]  

By Elliott’s classification [10], there exists an isomorphism \( \phi_0 : C^*(E_1)[k,n] \otimes K \to C^*(E_2)[k,n] \otimes K \) such that \( KK(\phi_0) = z_0 \), where \( z_0 \) is the invertible element in \( KK(C^*(E_1)[k,n] \otimes K, C^*(E_2)[k,n] \otimes K) \) induced by \( r_{X_2}^{[k,n]}(x) \).

Consider \( C^*(E_i) \) as a C*-algebra over \( X_2 \) by setting \( C^*(E_i)[2] = C^*(E_i)[k,n] \) and \( C^*(E_i)[1,2] = C^*(E_i) \). Let \( y \) be the invertible element in \( KK(X_2, C^*(E_1), C^*(E_2)) \) induced by \( x \). Note that \( KL(r_{X_2}^{[1]}(y)) = z_2 = KL(\phi_2) \) and \( r_{X_2}^{[2]}(y) = z_0 = KK(\phi_0) \). By Theorem 3.7 of [14],

\[
\tau_{\xi_2} \times [\tau_{\xi_2}] = [\tau_{\xi_2}] \times r_{X_2}^{[1]}(y) \text{ in } KK(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K), \text{ where } \xi_2 \text{ is the extension}
\]

\[ 0 \to C^*(E_i)[k,n] \otimes K \to C^*(E_i) \otimes K \to C^*(E_i)[1,k-1] \otimes K \to 0. \]

Thus,

\[ KL(\phi_2) \times [\tau_{\xi_2}] = [\tau_{\xi_2}] \times KL(\phi_0) \]

in \( KL(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \). Since \( KL(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) = KK(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K), \)

\[ KK(\phi_2) \times [\tau_{\xi_2}] = [\tau_{\xi_2}] \times KK(\phi_0) \]

in \( KK(C^*(E_1)[1,k-1] \otimes K, C^*(E_2)[k,n] \otimes K) \). By Lemma 4.5 of [14], \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K. \)

**Theorem 5.8.** Let \( E_1 \) and \( E_2 \) be graphs such that \( C^*(E_i) \) is a tight C*-algebra over \( X_3 \). Suppose \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is purely infinite and \( K_0(C^*(E_1)[1,2]) \) is the direct sum of cyclic groups if \( C^*(E_1)[1] \) is an AF-algebra. Then \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K \) if and only if \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)). \)

**Proof.** The “only if” direction is clear. Suppose \( K_{X_3}^+(C^*(E_1)) \cong K_{X_3}^+(C^*(E_2)). \) Suppose \( C^*(E_1)[1] \) is purely infinite. Then \( K_0(C^*(E_1)[1]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_1^3(K_0(C^*(E_1)[1]), K_0(C^*(E_2)[2])) = 0. \) Since \( K_1(C^*(E_1)[1]) \) is a free group, \( \text{Pext}_1^3(K_1(C^*(E_1)[1]), K_1(C^*(E_2)[2])) = 0. \) Hence,

\[ KK^1(C^*(E_1)[1], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,3]). \]

Suppose \( C^*(E_1)[1] \) is an AF-algebra. Then \( K_0(C^*(E_1)[1,2]) \) is the direct sum of cyclic groups. Thus, \( \text{Pext}_1^3(K_0(C^*(E_1)[1,2]), K_0(C^*(E_2)[2,3])) = 0. \) Since \( K_1(C^*(E_1)[1,2]) \) is a free group, \( \text{Pext}_1^3(K_1(C^*(E_1)[1,2]), K_1(C^*(E_2)[2,3])) = 0. \) Therefore,

\[ KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3]). \]

**Case 1:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are separated. Hence, by Theorem 6.9 of [14], \( C^*(E_1) \otimes K \cong C^*(E_2) \otimes K. \)

**Case 2:** Suppose the finite and infinite simple sub-quotients of \( C^*(E_1) \) are not separated. Then the finite and infinite simple sub-quotients of \( C^*(E_2) \) are not separated.
Subcase 2.1: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are purely infinite and $C^*(E_1)[2]$ is an AF-algebra. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are purely infinite and $C^*(E_2)[2]$ is an AF-algebra. Then by the above paragraph we have that $KK^1(C^*(E_1)[1], C^*(E_2)[2,3]) = KL^1(C^*(E_1)[1], C^*(E_2)[2,3])$. Hence, by Theorem 5.6, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

Subcase 2.2: Suppose $C^*(E_1)[3]$ and $C^*(E_1)[1]$ are AF-algebras and $C^*(E_1)[2]$ is purely infinite. Then $C^*(E_2)[3]$ and $C^*(E_2)[1]$ are AF-algebras and $C^*(E_2)[2]$ is purely infinite. Then by the above paragraph we have that $KK^1(C^*(E_1)[1,2], C^*(E_2)[3]) = KL^1(C^*(E_1)[1,2], C^*(E_2)[3])$.

Hence, by Theorem 5.7, $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$.

\begin{corollary}
Let $E_1$ and $E_2$ be graphs such that $C^*(E_i)$ is a tight $C^*$-algebra over $X_3$. Suppose that $K_0(C^*(E_i))$ is finitely generated. Then $C^*(E_1) \otimes \mathbb{K} \cong C^*(E_2) \otimes \mathbb{K}$ if and only if $K^+_X(C^*(E_1)) \cong K^+_X(C^*(E_2))$.
\end{corollary}

\begin{proof}
Since $C^*(E_1)$ is real rank zero, the canonical projection $\pi : C^*(E_1) \rightarrow C^*(E_1)[1]$ induces a surjective homomorphism $\pi : K_0(C^*(E_1)) \rightarrow K_0(C^*(E_1)[1])$. Hence, $K_0(C^*(E_1)[1])$ is finitely generated since $K_0(C^*(E_1))$ is finitely generated. The corollary now follows from Theorem 5.8.
\end{proof}

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