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Well-Separation and Hyperplane Transversals in High Dimensions

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Abstract

A family of $k$ point sets in $d$ dimensions is well-separated if the convex hulls of any two disjoint subfamilies can be separated by a hyperplane. Well-separation is a strong assumption that allows us to conclude that certain kinds of generalized ham-sandwich cuts for the point sets exist. But how hard is it to check if a given family of high-dimensional point sets has this property? Starting from this question, we study several algorithmic aspects of the existence of transversals and separations in high-dimensions.

First, we give an explicit proof that $k$ point sets are well-separated if and only if their convex hulls admit no $(k-2)$-transversal, i.e., if there exists no $(k-2)$-dimensional flat that intersects the convex hulls of all $k$ sets. It follows that the task of checking well-separation lies in the complexity class coNP. Next, we show that it is NP-hard to decide whether there is a hyperplane-transversal (that is, a $(d-1)$-transversal) of a family of $d+1$ line segments in $\mathbb{R}^d$, where $d$ is part of the input. As a consequence, it follows that the general problem of testing well-separation is coNP-complete. Furthermore, we show that finding a hyperplane that maximizes the number of intersected sets is NP-hard, but allows for an $\Omega \left( \frac{\log k}{\log \log k} \right)$-approximation algorithm that is polynomial in $d$ and $k$, when each set consists of a single point. When all point sets are finite, we show that checking whether there exists a $(k-2)$-transversal is in fact strongly NP-complete.

Finally, we take the viewpoint of parametrized complexity, using the dimension $d$ as a parameter: given $k$ convex sets in $\mathbb{R}^d$, checking whether there is a $(k-2)$-transversal is FPT with respect to $d$. On the other hand, for $k \geq d + 1$ finite point sets in $\mathbb{R}^d$, it turns out that checking whether there is a $(d-1)$-transversal is $W[1]$-hard with respect to $d$.

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1 Introduction

In the study of high-dimensional ham-sandwich cuts, the following notion has turned out to be fundamental: we call $k$ sets $S_1, \ldots, S_k$ in $\mathbb{R}^d$ are well-separated if for any proper index set $I \subset [k]$ (i.e., $I$ is neither empty nor all of $[k]$), the convex hulls of $[k]$ $S_I = \bigcup_{i \in I} S_i$ and of $S_{[k] \setminus I} = \bigcup_{i \notin I} S_i$ can be separated by a hyperplane. Since any two disjoint convex sets can be separated by a hyperplane [16], well-separation is equivalent to the fact that for any proper index set $I$, the convex hulls of $S_I$ and $S_{[k] \setminus I}$ do not intersect. A hyperplane $h$ is a transversal of $S_1, \ldots, S_k$ if we have $S_i \cap h \neq \emptyset$, for all $i \in [k]$. More generally, for $m \in \{0, \ldots, d-1\}$, an $m$-transversal of $S_1, \ldots, S_k$ is an $m$-flat (i.e., an $m$-dimensional affine subspace of $\mathbb{R}^d$) that intersects all the $S_i$. As we shall see below, it turns out that well-separation is intimately related to transversals: the sets $S_1, \ldots, S_k$ are well-separated if and only if there is no $(k-2)$-transversal of the convex hulls of $S_1, \ldots, S_k$.

In the past, transversals have been studied extensively, mostly from a combinatorial, but also from a computational perspective. Arguably the most well-known such theorem is Helly’s theorem [12], which states that for any finite family of convex sets in $\mathbb{R}^d$, it holds that if every $d+1$ of them have a point in common, then all of them do. In other words, Helly’s theorem gives a sufficient fingerprint condition for a family of convex sets to have a 0-transversal. In 1935, Vincensini asked whether such a statement holds for general $k$-transversals, that is, whether there is some number $m(k,d)$ such that if any $m(k,d)$ sets of a family have a $k$-transversal, then all of them do. This was disproved by Santaló, who showed that already the number $m(1,2)$ does not exist (cf. [13] for more details).

One reason why 0-transversals differ significantly from $k$-transversals with $k > 0$ is that the space of 0-transversals of a family of convex sets is itself a convex set. In contrast, for $k > 0$, the space of $k$-transversals can be very complicated, even for pairwise disjoint convex sets. Thus, in order to generalize Helly’s theorem to $k$-transversals with $k > 0$, additional assumptions become necessary. For example, Hadwiger’s Transversal Theorem [11] states that for any family $\mathcal{S}$ of compact and convex sets in the plane, it holds that if there exists a linear ordering on $\mathcal{S}$ such that any three sets can be transversed by a directed line in accordance with this ordering, then there is a line transversal for $\mathcal{S}$. This result has been extended to higher dimensions by Pollack and Wenger [18]. Note that to have a well-defined order in which a directed line intersects the sets, the sets should be pairwise disjoint. Now, well-separation is a way to extend this idea to transversals of higher dimensions: if no $k+1$ sets in a family $\mathcal{S}$ of convex sets have a $(k-1)$-transversal, then every $k$-transversal $H$ intersects the set $\mathcal{S}$ in a well-defined $k$-ordering, that is, for every way of choosing a $k$-tuple of points from the intersections of $H$ with $\mathcal{S}$, one point from each set, the orientation of the resulting simplices is the same (that is, they all have the same order type) [18]. Under well-separation, the space of transversals becomes simpler, in particular for hyperplane transversals: it is now a union of contractible sets [21]. Note that in $d$ dimensions, there can be no $d+2$ sets that are well-separated, due to Radon’s theorem which states that any set of $d+2$ points in $d$ dimensions can be partitioned into two sets whose convex hulls intersect. For more background on transversals, we refer the interested readers to the relevant surveys, e.g., [2,10,13].

---

1 Observe that for any $k \leq d$ sets in $\mathbb{R}^d$, there is always a $(k-1)$-transversal: choose one point from each set, and consider a $(k-1)$-flat that goes through these points. The $(k-1)$-flat is unique if the chosen points are in general position.
Thus, well-separation is a strong assumption on set-families, and it does not come as a surprise that for many problems it leads to stronger results and faster algorithms compared to the general case. One such example is obtained for Ham-Sandwich cuts, a well-studied notion that occurs in many places in discrete geometry and topology [16]. Given $d$ point sets $P_1, \ldots, P_d$ in $\mathbb{R}^d$, a Ham-Sandwich cut is a hyperplane that simultaneously bisects all point sets. While a Ham-Sandwich cut exists for any family of $d$ point sets [20], finding such a cut is PPA-complete when the dimension is not fixed [9], meaning that it is unlikely that there is an algorithm that runs in polynomial time in the dimension $d$. On the other hand, if $P_1, \ldots, P_d$ are well-separated, not only do there exist bisecting hyperplanes, but the Ham-Sandwich theorem can be generalized to hyperplanes cutting off arbitrary given fractions from each point set [5, 19]. Further, the problem of finding such a hyperplane lies in the complexity class UEOPL [8], a subclass of PPA that is believed to be much smaller than PPA.

From an algorithmic perspective, the main focus of the previous work has been efficient algorithms for finding line transversals in two and three dimensions, e.g., see [1, 4, 17]. To the authors’ knowledge, in higher dimensions only algorithms for hyperplane transversals have been studied, where the best known algorithm for deciding whether a set of $n$ polyhedra with $m$ edges has a hyperplane transversal runs in time $O(nm^{d-1})$ [3]. In particular, there is an exponential dependence on the dimension $d$, and there are no non-trivial algorithmic results for the case that the dimension is part of the input. This curse of dimensionality appears in many geometric problems. For several problems, it has been shown that there is probably no hope to get rid of the exponential dependence in the dimension. As a relevant example, Knauer, Tiwary, and Werner [14] showed the following: given $d$ point sets $S_1, \ldots, S_d$ in $\mathbb{R}^d$ and a point $p \in \mathbb{R}^d$, where $d$ is part of the input, it is $W[1]$-hard (and thus NP-hard) to decide whether there is there a Ham-sandwich cut for the sets that passes through $p$.

Our Results. First, we prove that a family of $k$ sets in $\mathbb{R}^d$ is well-separated if and only if the convex hulls of the sets have no $(k - 2)$-transversal. This fact seems to be known, but we could only find some references without proofs, and some proofs of only one direction, for similar definitions of well-separation [6, 7]. Therefore, for the sake of completeness, we present a short proof in Section 2. This immediately implies that testing well-separation is in coNP.

In [8], the authors ask for the complexity of determining whether a family of point sets is well-separated when $d$ is not fixed. We present several hardness results for finding $(k - 2)$-transversals in a family of $k$ sets in $\mathbb{R}^d$. We consider two cases: a) finite sets, and b) possibly infinite, but convex set.

**Theorem 1.** Given a set of $k > d$ point sets in $\mathbb{R}^d$, each with at most two points, it is NP-hard to check whether there is a $(d + 1)$-transversal, even in the special case $k = d + 1$.

Note that this decision problem is trivial for $k \leq d$, as the answer is always yes. The assumption $k = d + 1$ is of special interest to us since the transversals we are considering are hyperplanes in $\mathbb{R}^d$, as in the Ham-sandwich cuts problem. Moreover, it shows that the problem becomes NP-hard for the first non-trivial value of $k$. We extend Theorem 1 to show the following:

**Theorem 2.** Given a set of $k > d$ line segments in $\mathbb{R}^d$, it is NP-hard to check whether there is a $(d + 1)$-transversal, even in the special case $k = d + 1$.

Theorem 2 implies that testing well-separation is coNP-complete even for $d + 1$ segments in $\mathbb{R}^d$, answering the question from [8]. Further, we show the following result, with a stronger hardness than Theorem 1; however, we remove the additional constraint that $k = d + 1$. 
Theorem 3. Given a set of \( k \leq d + 1 \) point sets in \( \mathbb{R}^d \), each with most two points, it is strongly \( \text{NP} \)-hard to check whether there is a \((k - 2)\)-transversal.

Observe that for the problem of Theorem 3, we consider \((k - 2)\)-transversals instead of \((d - 1)\)-transversals. In this context, the problem becomes trivial for \( k \geq d + 2 \), because all sets lie in \( \mathbb{R}^d \). On the positive side, we can show the existence of the following approximation algorithm. This can be seen as the special case where each point set consists of a single point.

Theorem 4. Given a set \( P \) of \( k \) points in \( \mathbb{R}^d \), it is possible to compute in polynomial time in \( d \) and \( k \) a hyperplane that contains \( \Omega(\text{OPT} \log k \log \log k) \) points of \( P \), where \( \text{OPT} \) denotes the maximum number of points in \( P \) that a hyperplane can contain.

In Section 4, we study the problem through the lens of parametrized complexity. We show a significant difference between finite sets and convex sets.

Theorem 5. Checking whether a family of \( k \) convex sets in \( \mathbb{R}^d \) has a \((k - 2)\)-transversal (or equivalently, whether it is well-separated) is \( \text{FPT} \) with respect to \( d \).

Theorem 6. Checking whether a family of \( k \geq d + 1 \) finite point sets in \( \mathbb{R}^d \) has a \((d - 1)\)-transversal is \( \text{W[1]} \)-hard with respect to \( d \).

Observe that for finite point sets (and, more generally, for any non-convex sets), having no \((k - 2)\)-transversal does not a priori imply well-separation. The result of Theorem 6 bears a similarity with the following result, shown in [14]: given a point set \( S \subset \mathbb{R}^d \), it is possible to compute in polynomial time whether \( S \) is a \((k - 2)\)-transversal, yielding a proof that checking well-separation is in \( \text{coNP} \).

Lemma 7. Let \( S_1, \ldots, S_k \subset \mathbb{R}^d \) be \( k \) sets in \( d \) dimensions. Then \( S_1, \ldots, S_k \) are well-separated if and only if their convex hulls have no \((k - 2)\)-transversal.

Proof. In the following, we assume without loss of generality that the sets are convex, that is, the are equal to their convex hulls. Assume first that \( S_1, \ldots, S_k \) have a \((k - 2)\)-transversal \( h \). Consider the intersection of the sets with \( h \). This gives a collection of \( k \) sets \( S'_1, \ldots, S'_k \) in a \((k - 2)\)-dimensional space, thus by Radon’s theorem there is an index set \( I \subset [k] \) such that the convex hulls of \( S'_I \) and of \( S'_{[k] \setminus I} \) intersect. But then as well the convex hulls of \( S_I \) and of \( S_{[k] \setminus I} \) intersect, and thus \( S_1, \ldots, S_k \) are not well-separated.

2 Well-separation and transversals

Let us recall some definitions. Let \( S_1, \ldots, S_k \subset \mathbb{R}^d \) be \( k \) sets in \( d \) dimensions. An \( m \)-transversal of \( S_1, \ldots, S_k \) is an \( m \)-flat \( h \subset \mathbb{R}^d \) (that is, an affine subspace of dimension \( m \)) with \( h \cap S_i \neq \emptyset \) for \( i = 1, \ldots, k \). Transversals are intimately related to well-separation: the sets \( S_1, \ldots, S_k \subset \mathbb{R}^d \) are well-separated if and only if there is no \((k - 2)\)-transversal of their convex hulls. As mentioned in the introduction, this fact seems to be well known, but as we could not find a reference with all details for it, we give a short proof for the sake of completeness. In particular, a \((k - 2)\)-transversal of the convex hulls is a certificate that \( S_1, \ldots, S_k \) are not well-separated. For a given \((k - 2)\)-flat \( h \), it can be checked in polynomial time whether \( h \) is a \((k - 2)\)-transversal, yielding a proof that checking well-separation is in \( \text{coNP} \).

Lemma 7. Let \( S_1, \ldots, S_k \subset \mathbb{R}^d \) be \( k \) sets in \( d \) dimensions. Then \( S_1, \ldots, S_k \) are well-separated if and only if their convex hulls have no \((k - 2)\)-transversal.

Proof. In the following, we assume without loss of generality that the sets are convex, that is, they are equal to their convex hulls. Assume first that \( S_1, \ldots, S_k \) have a \((k - 2)\)-transversal \( h \). Consider the intersection of the sets with \( h \). This gives a collection of \( k \) sets \( S'_1, \ldots, S'_k \) in a \((k - 2)\)-dimensional space, thus by Radon’s theorem there is an index set \( I \subset [k] \) such that the convex hulls of \( S'_I \) and of \( S'_{[k] \setminus I} \) intersect. But then also the convex hulls of \( S_I \) and of \( S_{[k] \setminus I} \) intersect, and thus \( S_1, \ldots, S_k \) are not well-separated.
For the other direction, assume that \( S_1, \ldots, S_k \) are not well-separated, that is, there is an index set \( I \subset [k] \) such that the convex hulls of \( S_I \) and of \( S_{[k] \setminus I} \) intersect. Let \( p \) be a point in this intersection. The point \( p \) can be written as a convex combination of points in \( S_I \). Note that not only can we write it as a convex combination of points in \( S_I \), but we can even ensure that in this combination, we use at most one point of each \( S_i \). This is because the sets \( S_i \) are convex and so instead of taking two individual points we can take a convex combination of them. This means that in particular, there is a \((|I| - 1)\)-transversal \( h_I \) of \( S_I \) which contains \( p \). The same holds for \( S_{[k] \setminus I} \), giving a \((k - |I| - 1)\)-transversal \( h_{[k] \setminus I} \) of \( S_{[k] \setminus I} \) which contains \( p \). Then the affine hull of \( h_I \) and \( h_{[k] \setminus I} \) is a transversal of \( S_1, \ldots, S_k \) and has dimension at most \(|I| - 1 + k - |I| - 1 = k - 2 \).

\section{Hyperplane Transversals in High Dimensions}

Let \( S_1, \ldots, S_k \subset \mathbb{R}^d \) be \( k \) sets in \( d \) dimensions, where \( d \) is not fixed. Recall that a \textit{hyperplane transversal} of \( S_1, \ldots, S_k \) is a \((d - 1)\)-transversal. Note that we do not assume the sets to be convex. In particular, the sets can even be finite. We consider the decision problem \textsc{HypTrans}: Given sets \( S_1, \ldots, S_k \), decide if there is a hyperplane transversal for them. There are different variants of \textsc{HypTrans}, depending on what we require from the sets \( S_i \). We consider the finite case and the case of line segments. We also consider the optimisation formulation of \textsc{HypTrans}, that we name \textsc{MaxHyp}: Given the sets \( S_1, \ldots, S_k \), find a hyperplane that intersects as many of these sets as possible.

\subsection{Finite Case}

We begin with the case that all \( S_i \) are finite point sets. We provide an approximation algorithm for \textsc{MaxHyp} in the situation where every \( S_i \) contains a single point, for \( i = 1, \ldots, k \). Note that in this situation, \textsc{HypTrans} can be solved greedily. We also provide some hardness results for \textsc{HypTrans} even in the restricted setting where every \( S_i \) contains at most two points, for \( i = 1, \ldots, k \).

\subsubsection{Singleton sets}

We assume that every \( S_i \) contains a single point, for \( i = 1, \ldots, k \). We denote by \( P \) the point set that is the union of all \( S_i \). Let us denote by \( OPT \) the maximum number of points in \( P \) that a hyperplane may contain.

\begin{theorem}
It is possible to compute in polynomial time in \( d \) and \( k \) a hyperplane that contains \( \Omega\left(\frac{OPT \log k}{k \log \log k}\right) \) points in \( P \).
\end{theorem}

\begin{proof}
If \( k \leq d \), we just output a hyperplane that contains all points of \( P \). Otherwise, let \( f(k) = \log k / \log \log k \). If \( f(k) < d \), we pick \( d \) points from \( P \), and we output a hyperplane through these points. If \( f(k) \geq d \), we partition \( P \) into disjoint groups of size \( f(k) \). In each group, we compute all hyperplanes that go through some \( d \) points from the group. Among all hyperplanes for all groups, we output the hyperplane that contains the most points in \( P \). For each group, we have \( O(f(k)^d) = O(f(k)^{f(k)}) = O(k) \) hyperplanes to consider. Thus, the algorithm runs in polynomial time in \( d \) and \( k \).

We now analyze the approximation guarantee. If \( f(k) < d \), then we output a hyperplane with at least \( d > f(k) \geq f(k)OPT/k \) points, since \( OPT \leq k \). If \( f(k) \geq d \), we let \( h \) be an optimal hyperplane. If \( h \) contains at least \( d \) points in a single group, then we output an optimal solution. Otherwise, \( h \) contains less than \( d \) points in each group, so \( OPT \leq d(k/f(k)) \). This means that \( d \geq f(k)OPT/k \), and the claim follows from the fact that our solution contains at least \( d \) points.
\end{proof}
3.1.2 Sets of at most two points

Here, we restrict ourselves to the situation that every \( S_i \) contains at most two points, for \( i = 1, \ldots, k \). We prove that this version of HypTrans is NP-hard, with a reduction from SubsetSum. In SubsetSum, we are given \( n + 1 \) integers \( a_1, \ldots, a_n, b \in \mathbb{Z} \), and the goal is to decide whether there exists an index set \( I \subseteq \{1, \ldots, n\} \) with \( \sum_{i \in I} a_i = b \). It is well-known that SubsetSum is (weakly) NP-complete.

Given an input \( a_1, \ldots, a_n, b \in \mathbb{Z} \) for SubsetSum, we create an input \( S_1, \ldots, S_{n+2} \subseteq \mathbb{R}^{n+1} \) for HypTrans, as follows. Note that the number of sets and the dimension are differing by exactly one. First, we define \( 2n + 1 \) vectors \( u, v_1, \ldots, v_n, w_1, \ldots, w_n \in \mathbb{R}^{n+1} \), by setting

\[
\begin{align*}
   u(1) &= -b \quad \text{and} \quad u(j) = -1, & \text{for } j = 2, \ldots, n+1, \\
   v_i(1) &= a_i \quad \text{and} \quad v_i(j) = \delta_{i+1,j}, & \text{for } j = 2, \ldots, n+1, i = 1, \ldots, n, \text{ and} \\
   w_i(1) &= 0 \quad \text{and} \quad w_i(j) = \delta_{i+1,j}, & \text{for } j = 2, \ldots, n+1, i = 1, \ldots, n.
\end{align*}
\]

Here, for \( i, j \in \mathbb{Z} \),

\[
\delta_{i,j} = \begin{cases} 
   1, & \text{if } i = j, \\
   0, & \text{if } i \neq j,
\end{cases}
\]

denotes the Kronecker delta. Using these vectors, we define the input for HypTrans as \( S_1 = \{v_1, w_1\}, \ldots, S_n = \{v_n, w_n\}, S_{n+1} = \{u\}, \) and \( S_{n+2} = \{0\} \), where \( 0 \) is the origin of \( \mathbb{R}^{n+1} \).

\( \triangleright \) Claim 9. We have that \( a_1, \ldots, a_n, b \) is a yes-input for SubsetSum if and only if \( S_1, \ldots, S_{n+2} \) is a yes-input for HypTrans.

Proof. First, suppose that \( a_1, \ldots, a_n, b \) is a yes-input for SubsetSum, and let \( I \subseteq [n] \) be an index set with \( \sum_{i \in I} a_i = b \). Then, we define a point set \( x_1, \ldots, x_{n+2} \) with \( x_i \in S_i \) as follows:

for \( i = 1, \ldots, n \), if \( i \in I \), we set \( x_i = v_i \), and if \( i \notin I \), we set \( x_i = w_i \). Furthermore, we set \( x_{n+1} = u \) and \( x_{n+2} = 0 \). Then, the points \( x_1, \ldots, x_{n+2} \) lie on a common hyperplane. For this, it suffices to check that

\[
0 = \sum_{i=1}^{n+1} \frac{1}{n+1} x_i,
\]

which follows immediately from the definitions and the choice of the \( x_i \). Thus, there is a hyperplane transversal for \( S_1, \ldots, S_{n+2} \), as desired.

Conversely, suppose that \( S_1, \ldots, S_{n+2} \) is a yes-input for HypTrans. Thus, there is a choice \( x_i \in S_i \), for \( i = 1, \ldots, n+2 \), such that \( x_1, \ldots, x_{n+2} \), lie on a common hyperplane. Obviously, we have \( x_{n+1} = u \) and \( x_{n+2} = 0 \), so we can conclude that \( 0 \) is in the affine span of \( x_1, \ldots, x_n, u \) and can be written as

\[
0 = \sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} u,
\]

where \( \lambda_i \in \mathbb{R} \) with \( \sum_{i=1}^{n+1} \lambda_i = 1 \). Let \( I \subseteq [n] \) be the set of those indices \( i \) for which \( x_i = v_i \). By inspecting the coordinates and applying the definitions, we get the following system of equations:

\[
\sum_{i \in I} \lambda_i a_i = \lambda_{n+1} b, \quad \text{and} \quad \lambda_i = \lambda_{n+1}, \quad \text{for } i = 1, \ldots, n.
\]
From this, it now follows that $\lambda_1 = \cdots = \lambda_{n+1}$. Since $\sum_{i=1}^{n+1} \lambda_i = 1$, this implies that $\lambda_i = 1/(n + 1)$, for $i = 1, \ldots, n + 1$. Thus, the first equation implies that $a_1, \ldots, a_n, b$ is a yes-input for SUBSETSUM, with $I$ as the certifying index set.

### 3.1.3 A second reduction

Now, we prove that HypTrans is strongly NP-hard, by reducing from BinPacking. Our reduction will pass through two intermediate problems EqualBinPacking and FlatTrans. We start by defining all the involved problems.

In BinPacking, we are given a sequence $w_1, \ldots, w_n \in \mathbb{Z}_+$ of weights, a number $k$ of bins and a capacity $b \in \mathbb{Z}_+$. The goal is to decide whether there is a partition of $n$ items with weights $w_1, \ldots, w_n$ into $k$ bins such that in each bin the total weight of the items does not exceed the capacity $b$. It is known that BinPacking is strongly NP-hard. In EqualBinPacking, we are given the same input, but now the goal is to decide whether there exists a partition of the items into the bins such that in each bin the total weight of the items equals exactly the capacity. Note that BinPacking can easily be reduced to EqualBinPacking by adding the appropriate number of elements of weight 1, so EqualBinPacking is strongly NP-hard as well.

Finally, in FlatTrans, we are given $m$ sets $S_0, \ldots, S_{m-1} \in \mathbb{R}^d$, where $m$ and $d$ are both part of the input, and the goal is to decide whether there is an $(m - 2)$-transversal. In other words, the question is whether there exists an $(m - 2)$-dimensional affine subspace $h$ such that for all $i \in \{0, \ldots, m - 1\}$, we have $S_i \cap h \neq \emptyset$. Note that HypTrans with $k = d + 1$ is the same as FlatTrans with $m = d + 1$.

**Theorem 10.** FlatTrans is strongly NP-hard even when $S_0 = \{0\}$ and any other $S_i$ consists of at most two points.

**Proof.** We reduce from EqualBinPacking. Given an input $w_1, \ldots, w_n, k, b$ for to EqualBinPacking, we construct an instance of FlatTrans as follows: we set the dimension $d = k + n + kn$ and the number of sets $m = kn + 2$. For every pair $(i, j) \in [n] \times [k]$, define the vectors

$$v_{i,j}(x) := \begin{cases} w_i, & \text{if } x = j, \\ 1, & \text{if } x = k + i, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$u_{i,j}(x) := \begin{cases} 0, & \text{if } x = j, \\ 1, & \text{if } x = k + i, \\ 0, & \text{otherwise.} \end{cases}$$

Here, we denote by $x \in \{1, \ldots, n + k + kn\}$ the entries of the vector, e.g., the first entry of $v_{i,j}$ is denoted by $v_{i,j}(1)$. Furthermore, let $c$ be the vector whose entries are $-b$, for $1 \leq x \leq k$, and $-1$ everywhere else. Now set $S_0 = \{0\}$, and $S_l = \{v_{i,j}, u_{i,j}\}$, for $l = (i - 1)k + j$, $i = 1, \ldots, n$, $j = 1, \ldots, k$ (note that this choice of $l$ just gives that the order of the $l$’s corresponds to the lexicographic order of the $(i, j)$’s) and $S_{kn+1} = \{c\}$. All these vectors can be constructed in polynomial time.

We claim that there is a $kn$-transversal of the sets $S_0, \ldots, S_{kn+1}$, if and only if there is a valid solution for the EqualBinPacking instance.

Assume first that there is a solution for EqualBinPacking. For each $S_l$, $1 \leq l \leq kn$, $l = (i - 1)k + j$, choose $p_l = v_{i,j}$, if item $i$ is placed in bin $j$, and choose $p_l = u_{i,j}$, otherwise. Furthermore, set $p_0 = 0$, $p_{kn+1} = c$. We claim that there exist coefficients $\lambda_l$ such that

$$\sum_{l=1}^{kn+1} \lambda_l p_l = 0$$

(1)
and
\[ \sum_{i=1}^{kn+1} \lambda_i = kn + 1. \]
(2)
This implies the claim, because then 0 can be written as a non-trivial linear combination of the other points. Set \( \lambda_i := 1 \), for all \( l \). Then, (2) is certainly satisfied. Consider the \( x' \)th row of (1), where \( 1 \leq x \leq k \). By construction, and since we assumed a valid solution for the bin packing problem, this row evaluates to
\[ \left( \sum_{i \text{ item in bin \( x \)}} w_i \right) - b = 0. \]
Similarly, for \( k+1 \leq x \leq k+n \), the \( x' \)th row evaluates to \( 1 - 1 = 0 \), since each item is placed in exactly one bin. All remaining rows evaluate to \( 1 - 1 = 0 \), and thus (2) is also satisfied.

Assume now that there exist coefficients \( \lambda_i \) that satisfy (1) and (2) (which must be the case of 0 can be written as a non-trivial linear combination of the other points). From the \( x' \)th rows in (1) with \( x > k + n \), we get \( \lambda_i - \lambda_{kn+1} = 0 \), for \( 1 \leq l \leq kn \), and thus \( \lambda_1 = \cdots = \lambda_{kn+1} \). Together with (2), we thus get \( \lambda_i = 1 \), for all \( l \). Put item \( i \) into bin \( j \) if and only if \( p_l = v_i,j \) for \( l = (i-1)k+j \). Analogous to above we get from the \( x' \)th rows of (1), for \( k+1 \leq x \leq k+n \), that each item is placed into exactly one bin. Further, we get from the \( x' \)th rows of (1), for \( 1 \leq x \leq k \), that each bin is filled exactly to capacity. Thus, we have a valid solution for \textsc{EqualBinPacking}, as desired.

Now, there is only one reduction remaining:

\[ \textbf{Theorem 11.} \textsc{HypTrans} is strongly NP-hard even when \( S_0 = \{0\} \) and \( S_i \) consists of at most two points for all \( i = 1, \ldots, m-1 \). \]
\[ \textbf{Proof.} \] We reduce from \textsc{FlatTrans}. Let us assume that \( S_0 = \{0\} \) and let \( S_0, S_1, \ldots, S_{m-1} \subset \mathbb{R}^d \) be the sets in the instance of \textsc{FlatTrans}, and assume that \( m - 1 < d \). We construct sets in \( \mathbb{R}^{d+2} \) as follows: First, for each point \( p \) in some set \( S_i \) we define the point \( p' = (p, 0, 0) \) and place it in the set \( S'_i \). For \( m \leq i \leq d + 2 \), define \( S'_i \) as the set consisting only of the point \( s'_i = (0, \ldots, 0, 1, i) \). Additionally, let \( S'_0 := \{0\} \).

We claim that \( S_0, S_1, \ldots, S_{m-1} \subset \mathbb{R}^d \) have an \((m-2)\)-transversal, if and only if \( S'_0, S'_1, \ldots, S'_{d+2} \subset \mathbb{R}^{d+2} \) can be transversed by a hyperplane.

Assume first that \( S_0, S_1, \ldots, S_{m-1} \subset \mathbb{R}^d \) indeed have an \((m-2)\)-transversal, that is, there are points \( p_i \in S_i \) and parameters \( \lambda_i \) such that \( \sum_{i=1}^{m-1} \lambda_i p_i = 0 \) and \( \sum_{i=1}^{m-1} \lambda_i = 1 \). Choosing the corresponding points \( p'_i \) and setting \( \lambda'_i = \lambda_i \) for \( i \leq m-1 \) and \( \lambda'_i = 0 \) for \( i > m-1 \) we get \( \sum_{i=1}^{d+2} \lambda'_i p'_i = 0 \) and \( \sum_{i=1}^{d+2} \lambda'_i = 1 \), that is, \( S'_0, S'_1, \ldots, S'_{d+2} \subset \mathbb{R}^{d+2} \) can be transversed by a hyperplane.

Assume now that \( S'_0, S'_1, \ldots, S'_{d+2} \subset \mathbb{R}^{d+2} \) can be transversed by a hyperplane, that is, there are points \( p'_i \in S'_i \) and parameters \( \lambda'_i \), such that \( \sum_{i=1}^{d+2} \lambda'_i p'_i = 0 \) and \( \sum_{i=1}^{d+2} \lambda'_i = 1 \). The second to last row of the first equation evaluates to \( \sum_{i=m}^{d+2} \lambda'_i = 0 \), and we thus have \( \sum_{i=1}^{m-1} \lambda'_i = 1 \). Set \( p_i = p'_i \) and \( \lambda_i = \lambda'_i \). Then \( \sum_{i=1}^{m} \lambda_i = 1 \) by the observation above. Further, \( \sum_{i=1}^{m} \lambda_i p_i = 0 \) by the first \( m \) rows of the first equation. Thus, \( S_0, S_1, \ldots, S_{m-1} \subset \mathbb{R}^d \) can be transversed by a \((m-2)\)-flat.

\[ \textbf{3.2 Line segments} \]
In this section, we will show that deciding whether there is a hyperplane transversal for \( d \) line segments and the origin in \( \mathbb{R}^d \), where \( d \) is not fixed, is NP-hard.
We will reduce this to one of the previous cases shown, that is, to the restricted version of HypTran where the sets $S_i$ contain at most two points, see Section 3.1.2. This is done with the help of a gadget that enforces that every hyperplane transversal must use one of the two endpoints of a given line segment. The gadget is shown in Figure 1.

\[ \begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{gadget.png}
\caption{Every hyperplane transversal through $s_1$, $s_2$, $s_3$ must choose an endpoint of $s_1$ (and of $s_2$).}
\end{figure} \]

Given a collection of sets of size at most two, for each set we take the line segment formed by its points as $s_1$, the origin as point $s_3$, and we construct the corresponding new segment $s_2$ using the gadget presented in Figure 1. This gives a family $S$ of $2k$ line segments that all lie in a $k$-dimensional space. In order to prove our result, we need to lift our construction to $\mathbb{R}^{2k}$. Let $A_i, B_i$ in $\mathbb{R}^k$ denote the endpoints of the $i$'th original segment ($s_1$ in Figure 1) and let $G_i, H_i$ in $\mathbb{R}^k$ denote the endpoints of the $i$'th gadget segment ($s_2$ in Figure 1). Denote by $\epsilon_j$ the vector in $\mathbb{R}^k$ which is 0 everywhere except in the $j$'th entry, where it is $\varepsilon_j$. Further, we write $0_k$ for the zero vector in $\mathbb{R}^k$. We now lift the points $A_i, B_i, G_i, H_i$ to $\mathbb{R}^{2k}$ as follows:

\[ \begin{align*}
A_i' &:= \begin{pmatrix} A_i \\ 0_k \end{pmatrix}, B_i' := \begin{pmatrix} B_i \\ 0_k \end{pmatrix}, G_i' := \begin{pmatrix} G_i \\ \varepsilon_i \end{pmatrix}, H_i' := \begin{pmatrix} H_i \\ \varepsilon_i \end{pmatrix}.
\end{align*} \]

We denote the corresponding set of line segments $A_i'B_i'$ and $G_i'H_i'$ in $\mathbb{R}^{2k}$ by $S'$.

\[ \begin{lemma} 
S \subset \mathbb{R}^k \text{ has a hyperplane transversal if and only if } S' \subset \mathbb{R}^{2k} \text{ does.} 
\end{lemma} \]

\[ \begin{proof} 
We will prove this by explicitly computing affine combinations of points on the line segments that give us the required transversals. In this setting, $S \subset \mathbb{R}^k$ has a hyperplane transversal if and only if there are real numbers $\lambda_i, \gamma_i, \mu_j^{(i)}$, with $i \in [k], j \in \{0, \ldots, k\}$ and the following properties

\[ \begin{align*}
\sum_{i=1}^k \mu_0^{(i)} \left( \lambda_i A_i + (1 - \lambda_i) B_i \right) &= 0, \\
\sum_{i=1}^k \mu_0^{(i)} &= 1; 
\end{align*} \]

and for all $j \in \{1, \ldots, k\}$

\[ \begin{align*}
\sum_{i=1}^k \mu_j^{(i)} \left( \lambda_i A_i + (1 - \lambda_i) B_i \right) &= \gamma_j G_j + (1 - \gamma_j) H_j, \\
\sum_{i=1}^k \mu_j^{(i)} &= 1. 
\end{align*} \]

Here, the $\lambda_i$ and $\gamma_i$ fix points on the segments, and the $\mu_j^{(i)}$ write the origin (Equation (3)) and the points on the gadget segments (Equation (4)) as affine combinations of the points on the reduction segments.

Similarly, $S' \subset \mathbb{R}^{2k}$ has a hyperplane transversal if and only if there are real $l_i, g_i, m^{(i)}, n^{(i)}$, with $i \in [k]$ with the following property:

\[ \begin{align*}
\sum_{i=1}^k m^{(i)} (l_i A_i' + (1 - l_i) B_i') + \sum_{i=1}^k n^{(i)} (g_i G_i' + (1 - g_i) H_i') &= 0, \\
\sum_{i=1}^k m^{(i)} + n^{(i)} &= 1. 
\end{align*} \]
Here, the $l_i$ and $g_i$ fix points on the segments and the $n^{(i)}$ and $m^{(i)}$ write the origin as an affine combination of these points.

Assume first that $S \subset \mathbb{R}^k$ has a hyperplane transversal. Then Equation (5) can be satisfied by setting $l_i = \lambda_i, m^{(i)} := \mu_0^{(i)}, n^{(i)} := 0, g_i := 0$. Thus, if $S \subset \mathbb{R}^k$ has a hyperplane transversal then so does $S' \subset \mathbb{R}^{2k}$.

As for the other direction, assume that $S' \subset \mathbb{R}^{2k}$ has a hyperplane transversal. Note that the $(k+i)$’th row of Equation (5) reduces to $n^{(i)}\varepsilon = 0$, so in particular we must have $n^{(i)} = 0$ for every $i \in \{1, \ldots, k\}$. Thus, we may set $\lambda_i := l_i$ and $\mu_0^{(i)} := m^{(i)}$ and Equation (3) follows. As for Equation (4), fix some $j \in \{1, \ldots, k\}$ and note that by the construction of the gadget segments there exist real numbers $\alpha_j$ and $\beta_j$ such that $G_j = \alpha_j A_j$ and $H_j = \beta_j B_j$. Pick real numbers $\gamma_j$ and $x_j$ that satisfy the following two equations:

$$x_j \lambda_j = (1 + x_j) \gamma_j \alpha_j, \quad \text{and} \quad x_j (1 - \lambda_j) = (1 + x_j)(1 - \gamma_j) \beta_j,$$

It is straightforward to show that such numbers always exist, for the sake of readability we will not prove this here. Now, define $\mu_j^{(i)} := \frac{m^{(i)}}{1 + x_j}$ for $j \neq i$ and $\mu_j^{(i)} := \frac{m^{(i)} + x_j}{1 + x_j}$. Then

$$\sum_{i=1}^{k} \mu_j^{(i)} (\lambda_i A_i + (1 - \lambda_i) B_i) = \frac{1}{1 + x_j} \sum_{i=1}^{k} m^{(i)} (l_i A_i + (1 - l_i) B_i) + \frac{x_j}{1 + x_j} \left( l_i A_i + (1 - l_i) B_i \right).$$

By Equation 5, we have $\sum_{i=1}^{k} m^{(i)} (l_i A_i + (1 - l_i) B_i) = 0$ (recall that $n^{(i)} = 0$), thus we have

$$\sum_{i=1}^{k} \mu_j^{(i)} (\lambda_i A_i + (1 - \lambda_i) B_i) = \frac{1}{1 + x_j} \cdot x_j \left( l_i A_i + x_j (1 - l_i) B_i \right).$$

From our choice of $\gamma_j$ and $x_j$, we thus get

$$\frac{1}{1 + x_j} \cdot x_j (l_i A_i + x_j (1 - l_i) B_i) = \gamma_j \alpha_j A_j + (1 - \gamma_j) \beta_j B_j = \gamma_j G_j + (1 - \gamma_j) H_j,$$

which is what we want. Further, we have

$$\sum_{i=1}^{k} \mu_j^{(i)} = \frac{1}{1 + x_j} \left( \sum_{i=1}^{k} m^{(i)} + x_j \right) = \frac{1 + x_j}{1 + x_j} = 1,$$

so Equation (4) is indeed satisfied.

\section{Parametrized complexity}

\subsection{An FPT algorithm for $d$ sets}

Recall that our original motivation comes from determining whether $d$ sets in $\mathbb{R}^d$ are well-separated. Let us consider those $d$ sets, and let us denote by $n$ the total number of extreme vertices on their respective convex hulls (for general convex sets, this might be infinite, but we consider only the finite case). We say that $n$ is the \textit{convex hull complexity} of the set family. We assume that we are given the extreme points of the convex hull of every set and hence have a finite number of points for every set.

\textbf{Theorem 13.} Checking whether a family of $k$ sets in $\mathbb{R}^d$ with convex hull complexity $n$ is well-separated is FPT with parameter $d$. 

A row gadget forces the left value of each encoding gadget from the same row to be the same. Thus, there exists a constant \(d\) with value \(i\).

We use a framework similar to the one introduced by Marx [15]. The reduction is

\begin{proof}
\begin{theorem}
\end{proof}

\section{A \(W[1]\)-hardness proof}

\begin{proof}
\end{proof}

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from the set of gadgets to \([k']\). For each encoding gadget \(g\) in row \(\alpha\) and column \(\beta\), \(1 \leq \alpha, \beta \leq k\) we have a point set \(P^{\alpha,\beta}\) that contains \(O(n^2)\) points. First let us assume \(\alpha = \beta\). The point set \(P^{\alpha,\alpha}\) contains the points \(p^{\alpha,\alpha}_i\), for \(1 \leq i \leq n\), where the coordinates of \(p^{\alpha,\alpha}_i\) are: 
\[
p^{\alpha,\alpha}_i(x) = \delta f(g)_x + k^i \delta k_{\alpha} + \delta k_{\alpha} + x.
\]
Now let us assume that \(\alpha \neq \beta\). The point set \(P^{\alpha,\beta}\) contains the points \(p^{\alpha,\beta}_{i,j}\), for \(1 \leq i, j \leq n\) and \((i, j) \in E\), where the coordinates of \(p^{\alpha,\beta}_{i,j}\) are: 
\[
p^{\alpha,\beta}_{i,j}(x) = \delta f(g)_x + k^i \delta k_{\alpha} + \delta k_{\beta} + \beta x.
\]

Likewise if the left value of row \(\alpha\) is \(i\), we take the point \(p^{\alpha,\beta}_i\). Finally, if the right value of column \(\beta\) is \(j\), we take the point \(p^{\alpha,\beta}_j\). We denote those \(k'+1\) points by \(p_1, \ldots, p_{k'+1}\) and claim that they lie on a common hyperplane which contains \(0\). It suffices to show that 
\[
\sum_{1 \leq t \leq k'+1} \frac{1}{k'_{t} + 1} p_t = 0.
\]

Consider the first \(k'\) coordinates. Recall that \(f\) is a bijection between the set of gadgets and \([k']\) and recall that by definition, the points \(p_t\) have exactly one entry \(1\) in the first \(k'\) coordinates. Therefore in this sum, we have exactly one entry \(1\) from exactly one of the gadgets and exactly one entry \(-1\) from the point \(p_1\) in each of these coordinates. So it is clear that this equation is satisfied in the first \(k'\) coordinates. Now let us consider the coordinate \(k' + \alpha\), for some \(1 \leq \alpha \leq k\). As the encoding is valid, it implies that the left value in row \(\alpha\) of all encoding gadgets is the same. Let us denote by \(i\) this left value. We have indeed 
\[
\sum_{1 \leq t \leq k'+1} \frac{1}{k'_{t} + 1} p_t(k' + \alpha) = \frac{1}{k'+1} \left( \left( \sum_{1 \leq t \leq k} k'_{t} \right) - (k'+1) \right) = 0.
\]

Likewise if the coordinate is of the form \(k'' + \beta\) for some \(1 \leq \beta \leq k\), we argue using the fact that the right value of all encoding gadgets in column \(\beta\) is the same. This completes the first direction of our proof.

For the second direction, let us assume that there is a hyperplane \(h\) that contains at least one point from each point set. By assumption one of these points is \(0\), another is \(p_1\), and we denote the others by \(p_2, \ldots, p_{k'+1}\). This implies that we have \(0 = \lambda_1 p_1 + \sum_{2 \leq t \leq k'+1} \lambda_t p_t\), where \(\lambda_t \in \mathbb{R}\) and \(\sum_{1 \leq t \leq k'+1} \lambda_t = 1\). By looking at the \(k'\) first coordinates, we immediately obtain \(\lambda_i = \lambda_1 = \frac{1}{k'+1}\), for all \(2 \leq i \leq k'+1\). Let assume that in point set \(P^{\alpha,\beta}\) with \(1 \leq \alpha, \beta \leq k\), the point \(p^{\alpha,\beta}_{i,j}\) is contained in \(h\), for some \(1 \leq i, j \leq n\). Note that by definition, \((i, j)\) is an admissible tuple of the encoding gadget in row \(\alpha\) and column \(\beta\). We assign this tuple to this gadget, and do likewise with all other encoding gadgets. It remains to show
that the left value of all encoding gadgets in the same row is the same, and that the same holds with the right value of encoding gadgets from the same column. Let us consider row $\alpha$.

We consider the points contained in $h$ that belong to $P^{\alpha, \beta}$, for some $1 \leq \beta \leq k$. Let us denote by $Y$ the set of their $(k' + \alpha)$-th coordinate. Let $z$ be equal to $\max\{\log_k(y) \mid y \in Y\}$.

By assumption, we know that $\sum_{y \in Y} y = k^i$ for some $2 \leq i \leq n + 1$. This is because the coefficients $\lambda_\ell$ for these point sets are equal to the coefficient for the point in $P^{\alpha, \cdot}$ contained in $h$. As the elements in $Y$ are non-negative, we obtain $i \geq z + 1$. Assume for a contradiction that not all elements in $Y$ are equal. Then we have $\sum_{y \in Y} y < \sum_{y \in Y} k^z = k^{z+1} \leq k^i$. As this is not possible, we know that all elements in $Y$ are equal, which implies that the left value of all encoding gadgets in row $\alpha$ is the same. We can argue likewise for the columns.

\section{Conclusion and Open Problems}

We showed that the problem of testing well-separability of $k$ sets in $\mathbb{R}^d$ is hard. However, it may be that there exist some algorithms which solve the problem in a smarter way than simply testing the $2^k$ choices of index set. This question is still open.

It would be interesting to have some inapproximability results, or some better approximation algorithms, for the problem of finding a hyperplane that intersects as many points as possible in a point set $P$ in $\mathbb{R}^d$, where $d$ is not fixed.

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