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Published in:
Indagationes Mathematicae

DOI:
10.1016/j.indag.2022.02.010

Publication date:
2022

Document version
Publisher's PDF, also known as Version of record

Document license:
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Citation for published version (APA):
Discrete series representations with non-tempered embedding

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Received 9 November 2021; received in revised form 25 February 2022; accepted 28 February 2022
Communicated by K.-H. Neeb

Abstract

We give an example of a semisimple symmetric space $G/H$ and an irreducible representation of $G$ which has multiplicity 1 in $L^2(G/H)$ and multiplicity 2 in $C^\infty(G/H)$.

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Keywords: Symmetric spaces; Gelfand pairs; Multiplicity

1. Introduction

Let $G$ be a real reductive group and $X = G/H$ an attached symmetric space. Let further $V$ be a Harish-Chandra module and $V^\infty$ its smooth Fréchet completion of moderate growth. We write $V^{-\infty} = (V^\infty)^\prime$ for the strong dual of $V^\infty$. We recall that $V$ is called $H$-spherical provided

$\text{Hom}_G(V^\infty, C^\infty(X)) \neq 0$.

By Frobenius reciprocity $\text{Hom}_G(V^\infty, C^\infty(X))$ is isomorphic to the space $(V^{-\infty})^H$ of $H$-invariants in $V^{-\infty}$. It follows from [2, Corollary 3.10] that the space $(V^{-\infty})^H$ is finite dimensional.

Inside $(V^{-\infty})^H$ there are several subspaces of interest. In particular we mention

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https://doi.org/10.1016/j.indag.2022.02.010
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\( (V^{-\infty})^H_{\text{disc}} \), the subspace of functionals whose generalized matrix coefficients lie in the Harish-Chandra Schwartz space \( \mathcal{C}(X) \subset L^2(X) \cap C^\infty(X) \),

\( (V^{-\infty})^H_{\text{temp}} \), the subspace of functionals whose generalized matrix coefficients are tempered, i.e., belong to \( L^{2+\epsilon}(X) \) for all \( \epsilon > 0 \).

These subspaces satisfy

\[
(V^{-\infty})^H_{\text{disc}} \subset (V^{-\infty})^H_{\text{temp}} \subset (V^{-\infty})^H.
\]  

(1.1)

It follows from the tempered embedding theorem [6, Théorème 2] and the construction of wave packets [3, Théorème 1] that the subspaces \((V^{-\infty})^H_{\text{temp}}\) serve as the multiplicity spaces in the Plancherel decomposition of \( L^2(X) \). The Plancherel decomposition of \( L^2(X) \) is in general a mixture of discrete and continuous parts. The most continuous part of \( L^2(X) \) is the closed invariant subspace of \( L^2(X) \) that decomposes into a direct integral over the largest continuous families of representations. For these representations the space \((V^{-\infty})^H\) has been explicitly determined for almost every representation \( V \) in the families. See [4]. In this case \((V^{-\infty})^H_{\text{disc}} = \{0\}\) if \( X \) is not compact and \((V^{-\infty})^H_{\text{temp}} = (V^{-\infty})^H\). The same phenomenon occurs more generally for real spherical spaces as is described in [11, Theorem C].

In general there is no a priori reason for any of the inclusions (1.1) to be an equality, even if \((V^{-\infty})^H_{\text{disc}} \neq \{0\}\). However, no example appears to be recorded in the literature.

The objective of this paper is to provide an example where

\[
0 \neq (V^{-\infty})^H_{\text{disc}} = (V^{-\infty})^H_{\text{temp}} \subsetneq (V^{-\infty})^H.
\]

To be more specific this happens for \( X \) the \( n \)-dimensional one-sheeted hyperboloid which is homogeneous for the connected Lorentzian group \( G = \text{SO}_0(n, 1) \).

Let us briefly introduce the standard notions.

### 1.1. Notation

Let \( n \geq 3 \) and let \( G = \text{SO}_0(n, 1) \) be the identity component of the special Lorentz group \( \text{SO}(n, 1) \). We denote by \( H \) the stabilizer of

\[
x_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}
\]

in \( G \). The entry in the lower right corner of any matrix in \( \text{SO}(n, 1) \) is non-zero, and \( \text{SO}_0(n, 1) \) consists of those matrices for which this entry is positive. From this fact we see that \( H \) is the connected subgroup

\[
H = \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}_0(n-1, 1) \end{pmatrix} \subset G.
\]

The group \( G \) acts transitively on the hyperboloid

\[
X := \{ x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1 \},
\]

and the homogeneous space

\[
X = G/H = \text{SO}_0(n, 1)/\text{SO}_0(n-1, 1)
\]

is a symmetric space. The corresponding involution \( \sigma \) of \( G \) is given by conjugation with the diagonal matrix \( \text{diag}(-1, 1, \ldots, 1) \). The subgroup \( G^\sigma \) of \( G \) of \( \sigma \)-fixed elements is the stabilizer of \( \mathbb{R}x_0 \). This subgroup has two components, one of which is \( H \). For our purpose it is important to use \( H \) rather than \( G^\sigma \). The pairs \( (G, H) \) and \( (G, G^\sigma) \) differ by the fact that \( (G, G^\sigma) \) is a
Gelfand pair, whereas \((G, H)\) is not. In fact it has been shown by van Dijk [7] that \(X\) is the only symmetric space of rank one, which is not obtained as the homogeneous space of a Gelfand pair.

The regular representation of \(G\) on \(C^\infty(X)\) decomposes as the direct sum

\[ C^\infty(X) = C^\infty_{\text{even}}(X) \oplus C^\infty_{\text{odd}}(X) \]

of the invariant subspaces of functions that are even or odd with respect to the \(G\)-equivariant symmetry \(x \mapsto -x\). The restriction of the regular representation to \(C^\infty_{\text{even}}(X)\) is isomorphic to the regular representation on \(C^\infty(G/G^\sigma)\), and the non-Gelfandness of \((G, H)\) is therefore caused by the presence of the odd functions.

### 1.2. Main results

The Lorentzian manifold \(X\) carries the \(G\)-invariant Laplace–Beltrami operator \(\Delta\), which is obtained as the radial part of

\[ \Box := -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2} \]

on \(\mathbb{R}^{n+1}\). That is, for \(f \in C^\infty(X)\) we define \(\Delta f \in C^\infty(X)\) as \((\Box \tilde{f})|_X\) where \(\tilde{f}\) is any extension of \(f\) to a function, homogeneous of degree 0, on some open neighborhood of \(X\) in \(\mathbb{R}^{n+1}\).

Let \(\rho := \frac{1}{2}(n - 1)\) and for each \(\lambda \in \mathbb{C}\) let \(E_\lambda(X)\) be the eigenspace

\[ E_\lambda(X) = \{ f \in C^\infty(X) \mid \Delta f = (\lambda^2 - \rho^2)f \}. \]

The Laplace–Beltrami operator is a scalar multiple of the Casimir element associated to the Lie group \(G\), and hence every irreducible subspace \(V\) of \(C^\infty(X)\) is contained in \(E_\lambda(X)\) for some \(\lambda \in \mathbb{C}\). Apart from the sign, the scalar \(\lambda\) is uniquely determined by the infinitesimal character of \(V\). Conversely, since \(X = G/H\) is rank one, \(\pm \lambda\) determines the infinitesimal character.

Let

\[ E^\text{even}_\lambda(X) = E_\lambda(X) \cap C^\infty_{\text{even}}(X), \quad E^\text{odd}_\lambda(X) = E_\lambda(X) \cap C^\infty_{\text{odd}}(X). \]

We can now state our main result.

The manifold \(X\) carries a \(G\)-invariant measure, which is unique up to scalar multiplication. We denote by \(L^2(X)\) the associated \(G\)-invariant space of square integrable functions.

The following theorem can be seen from the general theory of hyperbolic spaces over \(\mathbb{R}, \mathbb{C}\) and \(\mathbb{H}\) in [8]. See [15, Thm 6.1] and [16, Thm 6.4]. For the current simple situation it will be established along with Theorem 1.2.

**Theorem 1.1.** Let \(\lambda \in \mathbb{C}\) with \(\text{Re}\lambda > 0\). The intersections \(E^\text{even}_\lambda(X) \cap L^2(X)\) and \(E^\text{odd}_\lambda(X) \cap L^2(X)\) are either zero or irreducible. Moreover,

\[ E^\text{even}_\lambda(X) \cap L^2(X) \neq 0 \Leftrightarrow \lambda \in \rho + 1 + 2\mathbb{Z} \]

\[ E^\text{odd}_\lambda(X) \cap L^2(X) \neq 0 \Leftrightarrow \lambda \in \rho + 2\mathbb{Z}. \]

**Theorem 1.2.** For every \(0 < \lambda < \rho\) with \(\lambda \in \rho - \mathbb{N}\) the \(G\)-representations \(E^\text{even}_\lambda(X)\) and \(E^\text{odd}_\lambda(X)\) are irreducible and infinitesimally equivalent. Moreover in this case,
(1) if $\lambda - \rho$ is even then $\mathcal{E}_\lambda^{\text{odd}}(X)$ is contained in $L^2(X)$ and $\mathcal{E}_\lambda^{\text{even}}(X)$ is not contained in $C_{\text{temp}}^{\infty}(X)$,

(2) if $\lambda - \rho$ is odd then $\mathcal{E}_\lambda^{\text{even}}(X)$ is contained in $L^2(X)$ and $\mathcal{E}_\lambda^{\text{odd}}(X)$ is not contained in $C_{\text{temp}}^{\infty}(X)$.

For $n \geq 4$ we have $\rho > 1$ and it follows that there exists at least one discrete series representation for $X = G/H$ which has multiplicity 1 in $C_{\text{temp}}^{\infty}(X)$, but for which the underlying Harish-Chandra module has multiplicity 2 in $C^{\infty}(X)$. It is observed in [7] that the tempered multiplicity of the unitary principal series of $X$ is two, but the phenomenon described here of representations for which the tempered multiplicity differs from the smooth multiplicity, is new.

The complete Plancherel decomposition for $SO_0(n, 1)/SO_0(n - 1, 1)$ is given in [12]. However, this is not needed for the proof of the above theorems.

We assume $n \geq 3$ throughout, since $n = 2$ would require a separate treatment. Moreover, Theorem 1.2 is empty for $n = 2$, just as it is for $n = 3$.

2. Proof of the main results

The proof of the two main theorems is divided into several parts. We begin with the analysis on $K$-types.

2.1. $K$-Types

Let $K \subset G$ be the stabilizer of $e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1}$, then $K \simeq SO(n)$ is a maximal compact subgroup of $G$.

We are going to use the diffeomorphism $S^{n-1} \times \mathbb{R} \xrightarrow{\sim} X$ given by

$$(y, t) \mapsto (y_1 \cosh t, \ldots, y_n \cosh t, \sinh t) \in X$$

where $y = (y_1, \ldots, y_n) \in S^{n-1}$ and $t \in \mathbb{R}$. With the natural action of $SO(n)$ on $S^{n-1}$ the parameter dependence on $y$ is $K$-equivariant.

For each $j \in \mathbb{N}_0$ we denote by $\mathcal{H}_j \subset C^{\infty}(S^{n-1})$ the space of spherical harmonics of degree $j$. We recall that by definition $\mathcal{H}_j$ consists of the restrictions to $S^{n-1}$ of all harmonic polynomials on $\mathbb{R}^n$, homogeneous of degree $j$. Equivalently, $\mathcal{H}_j$ can be defined as the eigenspace

$$\mathcal{H}_j := \{ h \in C^{\infty}(S^{n-1}) | \Delta_K h = -j(j + n - 2)h \},$$

where $\Delta_K$ is the angular part of the $n$-dimensional Laplacian

$$\frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2}.$$ Each $\mathcal{H}_j$ is an irreducible $SO(n)$-invariant finite dimensional subspace of $C^{\infty}(S^{n-1})$, and the sum $\bigoplus_{j=0}^{\infty} \mathcal{H}_j$ of these subspaces is dense in $C^{\infty}(S^{n-1})$.

It follows that the space $C_K^{\infty}(X)$ of $K$-finite functions $f \in C^{\infty}(X)$ is spanned by all functions given in the coordinates $(y, t)$ by

$$f(y, t) = h(y)\varphi(t),$$

where $h \in C^{\infty}(S^{n-1})$ is a spherical harmonic, and $\varphi \in C^{\infty}(\mathbb{R})$. Thus

$$C_K^{\infty}(X) \simeq \bigoplus_{j=0}^{\infty} (\mathcal{H}_j \otimes C^{\infty}(\mathbb{R})).$$
Being homogeneous of degree $j$, the spherical harmonics $h \in \mathcal{H}_j$ satisfy $h(-y) = (-1)^j h(y)$ for $y \in S^{n-1}$. Therefore

$$C_{K}^{\infty}(X)_{\text{even}} \simeq \bigoplus_{j=0}^{\infty} (\mathcal{H}_j \otimes C_{\text{parity}(j)}^{\infty}(\mathbb{R})),$$

where parity$(j)$ denotes the parity even or odd of $j$. Likewise

$$C_{K}^{\infty}(X)_{\text{odd}} \simeq \bigoplus_{j=0}^{\infty} (\mathcal{H}_j \otimes C_{\text{parity}(j+1)}^{\infty}(\mathbb{R})).$$

2.2. Eigenspaces

With respect to the coordinates (2.1) on $X$ we have (see [14, p. 455])

$$\Delta = \frac{\partial^2}{\partial t^2} + 2 \rho \tanh t \frac{\partial}{\partial t} - \frac{1}{\cosh^2 t} \Delta_K.$$

The $K$-finite eigenfunctions for $\Delta$ belong to $C^\infty(X)$, and they can be determined as follows. Let

$$\mathcal{E}_{\lambda,K}(X) := \mathcal{E}_\lambda(X) \cap C_{K}^{\infty}(X),$$

be the Harish-Chandra module of $\mathcal{E}_\lambda(X)$, and for each $j \in \mathbb{N}_0$ let

$$\mathcal{E}_{\lambda,j}(X) := \mathcal{E}_\lambda(X) \cap (\mathcal{H}_j \otimes C^\infty(\mathbb{R})).$$

By separating the variables $y$ and $t$ we see that $\mathcal{E}_{\lambda,j}$ is spanned by the functions $f(y,t) = h(y)\varphi(t)$ for which $h \in \mathcal{H}_j$ and

$$\left(\frac{d^2}{dt^2} + 2 \rho \tanh t \frac{d}{dt} + \frac{j(j+n-2)}{\cosh^2 t}\right)\varphi = (\lambda^2 - \rho^2)\varphi. \tag{2.2}$$

This differential equation is invariant under sign change of $t$. The solution with $\varphi(0) = 1$ and $\varphi'(0) = 0$ is even, and the solution with $\varphi(0) = 0$ and $\varphi'(0) = 1$ is odd. Thus the solution space decomposes as the direct sum of the one-dimensional subspaces of even and odd solutions, and we have

$$\mathcal{E}_{\lambda,j}^{\text{even}}(X) = \bigoplus_{j=0}^{\infty} \mathcal{E}_{\lambda,j}^{\text{even}}(X), \quad \mathcal{E}_{\lambda,j}^{\text{odd}}(X) = \bigoplus_{j=0}^{\infty} \mathcal{E}_{\lambda,j}^{\text{odd}}(X)$$

where $\mathcal{E}_{\lambda,j}^{\text{even}}(X) \simeq \mathcal{E}_{\lambda,j}^{\text{odd}}(X) \simeq \mathcal{H}_j$ are equivalent irreducible $K$-types for each $j$.

2.3. Hypergeometric functions

In fact (2.2) can be transformed into a standard equation of special function theory. We first prepare for the anticipated asymptotic behavior of $\varphi$ by substituting $\Phi(t) = (\cosh t)^{\lambda+\rho} \varphi(t)$. This leads to the following equation for $\Phi(t)$

$$\Phi''(t) - 2\lambda \tanh t \Phi'(t) - ab (1 - \tanh^2 t) \Phi(t) = 0, \tag{2.3}$$

where $a = \lambda + \rho + j$ and $b = \lambda - \rho + 1 - j$.

Next we change variables. With

$$x = \frac{1}{2}(1 - \tanh t) = (1 + e^{2t})^{-1} \in (0, 1)$$

we replace the limits $t = \infty$ and $t = -\infty$ by $x = 0$ and $x = 1$, respectively. We write $\Phi(t) = F(x)$, so that

$$\varphi(t) = (\cosh t)^{-\lambda-\rho} F((1 + e^{2t})^{-1}).$$
Since \( \tanh t = 1 - 2x \) and \( 1 - \tanh^2 t = 4x(1-x) \) this gives
\[
(x')^2 F''(x) + x'' F'(x) - 2\lambda(1 - 2x) F'(x) x' - ab 4x(1-x) F(x) = 0,
\]
and since
\[
x' = -2x(1-x), \quad x'' = 4x(1-x)(1-2x),
\]
we arrive at the following equation for the function \( F(x) \)
\[
x(1-x) F''(x) + (\lambda + 1)(1-2x) F'(x) - ab F(x) = 0. \tag{2.4}
\]

Recall the hypergeometric function \( F(a, b; c; x) = \,_{2}F_{1}(a, b; c; x) \), which with the notation
\[
(a)_m := \prod_{k=0}^{m-1} (a + k)
\]
is defined by the Gauss series
\[
F(a, b; c; x) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{m! (c)_m} x^m
\]
for \( x \in \mathbb{C} \) with \( |x| < 1 \), for all \( a, b, c \in \mathbb{C} \) except \( c \in -\mathbb{N}_0 \). It solves Euler’s hypergeometric differential equation
\[
x(1-x) w'' + (c - (a + b + 1)x) w' - ab w = 0. \tag{2.5}
\]
The function \( F(a, b; c; x) \) is analytic at \( x = 0 \) with the value 1, and unless \( c \) is an integer it is the unique solution with this property.

With \( a = \lambda + \rho + j \) and \( b = \lambda - \rho + 1 - j \) as above we have \( a + b + 1 = 2(\lambda + 1) \). By comparing (2.4) and (2.5) we conclude that for each \( \lambda \notin -\mathbb{N} \) the function
\[
\varphi_{\lambda,j}(t) := (\cosh t)^{-\lambda-\rho} F(\lambda + \rho + j, \lambda - \rho + 1 - j; 1 + \lambda; (1 + e^{2t})^{-1})
\]
solves (2.2).

2.4. \( L^2 \)-Behavior

In the coordinates \((y, t)\) an invariant measure on \( X \) is given by
\[
\cosh^{n-1} t \, dt \, dy
\]
where \( dt \) and \( dy \) are invariant measures on \( \mathbb{R} \) and \( S^{n-1} \), respectively. Hence a function \( f(y, t) = h(y) \varphi(t) \) is square integrable if and only if
\[
\int_{\mathbb{R}} |\varphi(t)|^2 \cosh^{n-1} t \, dt < \infty.
\]

Let \( \tilde{\varphi}_{\lambda,j}(t) = \varphi_{\lambda,j}(-t) \). By symmetry this function also solves (2.2), and it belongs to \( L^2(\mathbb{R}, \cosh^{n-1} t \, dt) \) if and only if \( \varphi_{\lambda,j} \) does.

**Lemma 2.1.** Let \( \text{Re} \lambda > 0 \) and \( j \in \mathbb{N}_0 \).

1. If \( j \in \lambda - \rho + \mathbb{N} \) then \( \varphi_{\lambda,j} \in L^2(\mathbb{R}, \cosh^{n-1} t \, dt) \).
2. If \( j \notin \lambda - \rho + \mathbb{N} \) then \( \varphi_{\lambda,j} \) and \( \tilde{\varphi}_{\lambda,j} \) are linearly independent, and for a sufficiently small \( \epsilon > 0 \) no non-trivial linear combination belongs to \( L^{2+\epsilon}(\mathbb{R}, \cosh^{n-1} t \, dt) \).

**Proof.** It follows from the definition of \( \varphi_{\lambda,j}(t) \) that
\[
(\cosh t)^{\lambda+\rho} \varphi_{\lambda,j}(t) \to 1, \quad t \to \infty.
\]
Since $2\rho = n - 1$ this means $\varphi_{b,j}$ has the desired $L^2$-behavior in the positive direction for all $\Re \lambda > 0$. The only issue is with the negative direction, or equivalently, with $\tilde{\varphi}_{b,j}(t)$ for $t \to -\infty$.

We first consider (1). The assumption that $j \in \lambda - \rho + \mathbb{N}$ implies that $b = \lambda - \rho + 1 - j \in -\mathbb{N}_0$. The Gauss series for $F(a; b; c; x)$ terminates and defines a polynomial when $a$ or $b$ is a non-positive integer. In particular $F(a; b; c; x)$ is then a bounded function on $[0, 1]$. It then follows from the definition of $\varphi_{b,j}$ that

$$(\cosh t)^{\lambda + \rho} \varphi_{b,j}(t)$$

is bounded on $\mathbb{R}$, and hence $|\varphi_{b,j}(t)|^2(\cosh t)^{2\rho}$ is integrable. This proves (1).

Now consider (2). According to Gauss (see [1, Thm. 2.1.3]) we have

$$\lim_{x \to 1^-} (1 - x)^{a+b-c} F(a; b; c; x) = A := \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}$$

if $\Re(a + b - c) > 0$. We apply this to $i \to \infty$ in

$$(\cosh t)^{\lambda + \rho} \tilde{\varphi}_{b,j}(t) = F(\lambda + \rho + j, \lambda - \rho + 1 - j; 1 + \lambda; (1 + e^{-2t})^{-1}).$$

Here $a + b - c = \lambda$ and

$$A = \frac{\Gamma(1 + \lambda) \Gamma(\lambda)}{\Gamma(\lambda + \rho + j) \Gamma(\lambda - \rho + 1 - j)}.$$

Since $x = (1 + e^{-2t})^{-1}$ implies $1 - x = (1 + e^{2t})^{-1}$ it follows that

$$e^{-2lt}(\cosh t)^{\lambda + \rho} \tilde{\varphi}_{b,j}(t) \to A, \quad t \to \infty.$$}

In particular we note that $A \neq 0$ if $j \notin \lambda - \rho + \mathbb{N}$. Under this condition we see that no non-trivial linear combination of $\varphi_{b,j}$ and $\tilde{\varphi}_{b,j}$ exhibits $L^{2+\epsilon}$-behavior in both directions $\pm \infty$.

This proves (2) and concludes the proof of the lemma. \qed

2.5. Parity of $\varphi_{\lambda,j}$

We have seen that $\varphi_{b,j}$ and $\tilde{\varphi}_{b,j}$ are independent solutions to (2.2) when $j \notin \lambda - \rho + \mathbb{N}$. For $j \in \lambda - \rho + \mathbb{N}$ they are proportional, as the following lemma shows. For $\alpha, \beta > -1$ and $l \in \mathbb{N}_0$ let

$$P_l^{(\alpha,\beta)}(z) := \frac{(\alpha + 1)_l}{l!} F(l + \alpha + \beta + 1, -l; \alpha + 1; \frac{1}{2}(1 - z))$$

be the corresponding Jacobi polynomial (see [5, page 115]).

**Lemma 2.2.** Assume $j = \lambda - \rho + 1 + l \in \mathbb{N}_0$ where $l \in \mathbb{N}_0$. Then

1. $(\frac{\lambda + 1}{l})! \varphi_{\lambda,j}(t) = (\cosh t)^{-\lambda - \rho} P_l^{(\lambda,\lambda)}(\tanh t),$
2. $\varphi_{\lambda,j}(-t) = (-1)^l \varphi_{\lambda,j}(t),$

for all $t \in \mathbb{R}$.

Note that with the repeated indices $P_l^{(\lambda,\lambda)}$ is in fact a Gegenbauer polynomial.

**Proof.** By definition

$$\varphi_{\lambda,j}(t) = (\cosh t)^{-\lambda - \rho} F(2\lambda + 1 + l, -l; \lambda + 1; x)$$

where $x = (1 + e^{2t})^{-1} = \frac{1}{2}(1 - \tanh t)$. The List (1) follows immediately. Then (2) follows since a Gegenbauer polynomial is even or odd according to the parity of its degree. \qed
2.6. \( K \)-Types in \( L^2 \)

For \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda \geq 0 \) we define
\[
D_\lambda := \mathbb{N}_0 \cap (\lambda - \rho + \mathbb{N})
\]
if \( \lambda - \rho \in \mathbb{Z} \) and \( \lambda > 0 \), and by \( D_\lambda = \emptyset \) otherwise. Furthermore we let
\[
U_\lambda := \bigoplus_{j \in D_\lambda} (\mathcal{H}_j \otimes \varphi_{\lambda,j}).
\]

It follows from Lemma 2.2 and the fact that the parity of \( \mathcal{H}_j \) is \((-1)^j\) that
\[
U_\lambda \subset E_{\lambda,K}(\mathbb{X}) \text{ if } \lambda - \rho \text{ is odd, and } U_\lambda \subset E_{\lambda,K}^{\text{odd}}(\mathbb{X}) \text{ if } \lambda - \rho \text{ is even.}
\]

Lemma 2.3. For all \( \lambda \in \mathbb{C} \) with \( \text{Re}\lambda > 0 \) we have
\[
U_\lambda = E_{\lambda,K}(\mathbb{X}) \cap L^2(\mathbb{X}) = E_{\lambda,K}(\mathbb{X}) \cap C^\infty_{\text{temp}}(\mathbb{X}). \tag{2.6}
\]

Proof. Let \( \text{Re}\lambda > 0 \). When \( \lambda - \rho \notin \mathbb{Z} \) it follows immediately from Lemma 2.1(2) that
\[
E_{\lambda,j}(\mathbb{X}) \cap C^\infty_{\text{temp}}(\mathbb{X}) = \{0\}
\]
for each \( j \in \mathbb{N}_0 \), since \( U_\lambda = \{0\} \) in this case (2.6) follows. Assume from now on that \( \lambda - \rho \in \mathbb{Z} \).

It then follows from Lemma 2.1(1) that \( U_\lambda \subset L^2(\mathbb{X}) \).

To complete the proof we will find for each \( j \in \mathbb{N}_0 \) a second solution to (2.2), which is linearly independent from \( \varphi_{\lambda,j} \), and which is not tempered. There are two cases, depending on the parity of \( n \).

If \( n \) is even, then \( \rho \), and hence also \( \lambda \), is not an integer. In that case we already have a second solution at hand, namely \( \varphi_{-\lambda,j} \). Since
\[
(cosh(t))^{-\lambda+\rho} \varphi_{-\lambda,j}(t) \rightarrow 1, \quad t \rightarrow \infty,
\]
this function \( \varphi_{-\lambda,j} \) does not belong to any \( L^p(\mathbb{R}, \text{cosh}^{n-1} t \, dt) \) if \( \text{Re}\lambda \geq \rho \). When \( 0 < \text{Re}\lambda < \rho \) it belongs to \( L^{2+\epsilon} \) only for \( \epsilon > \frac{2 \text{Re}\lambda}{\rho - \text{Re}\lambda} \).

We now assume \( n \) is odd. Then \( \rho \) and \( \lambda \) are positive integers. We need to find a solution linearly independent from \( F(a,b;c;x) \) to the hypergeometric Eq. (2.5) with
\[
a = \lambda + \rho + j, \quad b = \lambda - \rho + 1 - j, \quad c = 1 + \lambda.
\]

By the method of Frobenius one finds (see [13, p. 5]) such a solution \( G(a,b,c;x) \). It has the form
\[
G(x) = x^{-\lambda} \sum_{v=0}^{\infty} a_v x^v + \log x \sum_{v=0}^{\infty} b_v x^v
\]
for some explicit power series with \( a_0 = 1 \) and \( b_0 \neq 0 \). The corresponding solution to (2.2) is
\[
(cosh(t))^{-\lambda-\rho} G(a,b,c; (1 + e^{2t})^{-1}).
\]

It behaves like \( (cosh(t))^{-\lambda-\rho}(1 + e^{2t})^{\lambda} \) as \( t \to \infty \) and as before it is not tempered with respect to the invariant measure. \( \square \)
2.7. Irreducibility

Let \( \lambda \in \rho + \mathbb{Z} \) and assume \( \lambda > 0 \). It follows from Lemma 2.3 that \( U_\lambda \) is \((g, K)\)-invariant. We will prove that it is an irreducible \((g, K)\)-module by using the infinitesimal element
\[
T = E_{n+1,1} + E_{1,n+1} \in g = \mathfrak{so}(n, 1),
\]
as a raising and lowering operator between the functions \( \varphi_{\lambda,j} \) which generate \( U_\lambda \) together with \( K \). For this we need to find the derivative of \( \varphi_{\lambda,j} \).

**Lemma 2.4.** Let \( j = \lambda - \rho + 1 + l \in \mathbb{N}_0 \) where \( l \in \mathbb{N}_0 \). There exist constants \( A_l, B_l \in \mathbb{R} \) such that
\[
\varphi'_{\lambda,j} = A_l \varphi_{\lambda,j+1} + B_l \varphi_{\lambda,j-1}.
\]
Both \( A_l \) and \( B_l \) are non-zero, except when \( l = 0 \) or \( j = 0 \), in which cases only \( A_l \) is non-zero.

**Proof.** Recall from Lemma 2.2
\[
\varphi_{\lambda,j}(t) = \frac{n}{(\lambda+1)l} (\cosh t)^{-\lambda-\rho} P_\lambda^{(\lambda,\lambda)}(\tanh t)
\]
for \( j = \lambda - \rho + 1 + l \). It follows that
\[
\varphi'_{\lambda,j}(t) = \frac{n}{(\lambda+1)l} (\cosh t)^{-\lambda-\rho} (-cosh(\lambda + \rho) x P_\lambda^{(\lambda,\lambda)}(x) + (1-x^2)(P_\lambda^{(\lambda,\lambda)})'(x))
\]
where \( x = \tanh t \).

We obtain from [9, (4.7)] that
\[
(1-x^2)(P_\lambda^{(\lambda,\lambda)})'(x) = (l+2\lambda+1)x P_\lambda^{(\lambda,\lambda)}(x) - \frac{(l+1)(l+2\lambda+1)}{l+\lambda+1} P_{l+1}^{(\lambda,\lambda)}(x).
\]

By [5, (5.5.5)] the polynomials \( P_\lambda^{(\lambda,\lambda)} \) satisfy a three term recurrence relation
\[
n(l+\lambda+1)(2l+2\lambda+1) x P_\lambda^{(\lambda,\lambda)}(x) = (l+\lambda)(l+\lambda+1) P_{l-1}^{(\lambda,\lambda)}(x) + (l+1)(l+2\lambda+1) P_{l+1}^{(\lambda,\lambda)}(x).
\]

With this relation we can eliminate \( x P_\lambda^{(\lambda,\lambda)}(x) \) and obtain \( \varphi'_{\lambda,j} \) as a linear combination of \( \varphi_{\lambda,j+1} \) and \( \varphi_{\lambda,j-1} \). The coefficients turn out to be
\[
A_l = -\frac{(\lambda + \rho + l)(2\lambda + l + 1)}{2\lambda + 2l + 1}, \quad B_l = \frac{l(\lambda - \rho + l + 1)}{2\lambda + 2l + 1}.
\]

All these coefficients are non-zero, except \( B_l \) when \( l = 0 \) or \( \lambda - \rho + l + 1 = 0 \). \( \square \)

Let \( M = K \cap H \) be the stabilizer in \( K \) of \( x_0 \), that is
\[
M = \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(n-1) \\ 0 & 1 \end{pmatrix} \subset \text{SO}(n, 1) = G.
\]
Then \( S^{n-1} \cong K/M \). Let \( h_j \in \mathcal{H}_j \) be the zonal spherical harmonic. This is the unique function in \( \mathcal{H}_j \) which is \( M \)-invariant and has the value 1 at the origin \((1, 0, \ldots, 0)\) of \( S^{n-1} \). Furthermore, let \( f_{\lambda,j} \in \mathcal{E}_\lambda(X) \) be defined by
\[
f_{\lambda,j}(y, t) = h_j(y) \varphi_{\lambda,j}(t).
\]
With the element $T$ from (2.7) the coordinates $(y, t)$ are determined from
\[ K/M \times \mathbb{R} \ni (kM, t) \mapsto k \exp(tT)x_0 \in X, \]
and since $T$ is centralized by $M$ the left derivative $L_T f_{\lambda,j}$ by $T$ is again $M$-invariant. It follows that for each $j \in \mathbb{N}_0$, the function $L_T f_{\lambda,j} \in \mathcal{E}_{\lambda,K}(X)$ is a linear combination of the same family of functions $f_{\lambda,j} \in \mathcal{E}_{\lambda,K}(X)$. Since $h_j(x_0) = 1$ for all $j$, the coefficients can be determined from the restriction to
\[
\{ (\cosh t, 0, \ldots, 0, \sinh t) \mid t \in \mathbb{R} \} \subset X,
\]
on which $L_T$ acts just by $\frac{d}{dt}$, and hence they are given by Lemma 2.4. It follows immediately that $U_\lambda$ has no non-trivial $(g, K)$-invariant subspaces.

2.8. Equivalence

Let $\lambda \in \rho + \mathbb{Z}$ and assume $0 < \lambda < \rho$.

**Lemma 2.5.** The $(g, K)$-modules $\mathcal{E}_{\lambda,K}^{even}(X)$ and $\mathcal{E}_{\lambda,K}^{odd}(X)$ are irreducible and equivalent.

**Proof.** The assumption on $\lambda$ implies that $D_\lambda = \mathbb{N}_0$, and hence $U_\lambda$ is equal to one of the two modules $\mathcal{E}_{\lambda,K}^{even}(X)$ and $\mathcal{E}_{\lambda,K}^{odd}(X)$, depending on the parity of $\lambda - \rho$. For simplicity of exposition, let us assume a specific parity, say even, of $\lambda - \rho$. Then $\mathcal{E}_{\lambda,K}^{odd}(X) = U_\lambda$ is irreducible as seen in Section 2.7.

By Kostant’s theorem [10, Thm. 8] an irreducible $(g, K)$-module, which contains the trivial $K$-type, is uniquely determined up to equivalence by its infinitesimal character. Hence $\mathcal{E}_{\lambda,K}^{odd}(X)$ is equivalent to the irreducible subquotient of $\mathcal{E}_{\lambda,K}^{even}(X)$ containing $\mathcal{E}_{\lambda,0}^{even}(X)$. Since $\mathcal{E}_{\lambda,K}^{odd}(X)$ and $\mathcal{E}_{\lambda,K}^{even}(X)$ contain the same $K$-types, all with multiplicity one, we conclude that this subquotient is equal to $\mathcal{E}_{\lambda,K}^{even}(X)$. The lemma is proved. \(\square\)

3. Conclusion

Assume $\text{Re} \lambda > 0$. Then $\mathcal{E}_{\lambda,K}(X) \cap L^2(X) = U_\lambda$ by Lemma 2.3. By definition $U_\lambda$ is non-zero if and only if $\lambda - \rho \in \mathbb{Z}$. In Section 2.6 we saw that it consists of even functions on $X$ when $\lambda - \rho$ is odd, and vice versa. Finally, irreducibility was seen in Section 2.7. Thus the proof of Theorem 1.1 is complete.

Assume $\lambda \in \rho + \mathbb{Z}$ and $0 < \lambda < \rho$. Then $\mathcal{E}_{\lambda,K}^{even}(X)$ and $\mathcal{E}_{\lambda,K}^{odd}(X)$ are irreducible and equivalent by Lemma 2.5. One of them equals $U_\lambda$ and belongs to $L^2(X)$, whereas we have seen in Lemma 2.3 that the other one is non-tempered. This proves Theorem 1.2.

References


