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Discrete series representations with non-tempered embedding

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Abstract

We give an example of a semisimple symmetric space $G/H$ and an irreducible representation of $G$ which has multiplicity 1 in $L^2(G/H)$ and multiplicity 2 in $C^\infty(G/H)$.

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1. Introduction

Let $G$ be a real reductive group and $X = G/H$ an attached symmetric space. Let further $V$ be a Harish-Chandra module and $V^\infty$ its smooth Fréchet completion of moderate growth. We write $V^{-\infty} = (V^\infty)'$ for the strong dual of $V^\infty$. We recall that $V$ is called $H$-spherical provided

\[ \text{Hom}_G(V^\infty, C^\infty(X)) \neq 0. \]

By Frobenius reciprocity $\text{Hom}_G(V^\infty, C^\infty(X))$ is isomorphic to the space $(V^{-\infty})^H$ of $H$-invariants in $V^{-\infty}$. It follows from [2, Corollary 3.10] that the space $(V^{-\infty})^H$ is finite dimensional.

Inside $(V^{-\infty})^H$ there are several subspaces of interest. In particular we mention

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• \((V^{-\infty})^H_{\text{disc}}\), the subspace of functionals whose generalized matrix coefficients lie in the Harish-Chandra Schwartz space \(\mathcal{C}(X) \subset L^2(X) \cap C^\infty(X)\),

• \((V^{-\infty})^H_{\text{temp}}\), the subspace of functionals whose generalized matrix coefficients are tempered, i.e., belong to \(L^{2+\varepsilon}(X)\) for all \(\varepsilon > 0\).

These subspaces satisfy
\[(V^{-\infty})^H_{\text{disc}} \subset (V^{-\infty})^H_{\text{temp}} \subset (V^{-\infty})^H.\]  

It follows from the tempered embedding theorem [6, Théorème 2] and the construction of wave packets [3, Théorème 1] that the subspaces \((V^{-\infty})^H_{\text{temp}}\) serve as the multiplicity spaces in the Plancherel decomposition of \(L^2(X)\). The Plancherel decomposition of \(L^2(X)\) is in general a mixture of discrete and continuous parts. The most continuous part of \(L^2(X)\) is the closed invariant subspace of \(L^2(X)\) that decomposes into a direct integral over the largest continuous families of representations. For these representations the space \((V^{-\infty})^H\) has been explicitly determined for almost every representation \(V\) in the families. See [4]. In this case \((V^{-\infty})^H_{\text{disc}} = \{0\}\) if \(X\) is not compact and \((V^{-\infty})^H_{\text{temp}} = (V^{-\infty})^H\). The same phenomenon occurs more generally for real spherical spaces as is described in [11, Theorem C].

In general there is no a priori reason for any of the inclusions (1.1) to be an equality, even if \((V^{-\infty})^H_{\text{disc}} \neq \{0\}\). However, no example appears to be recorded in the literature.

The objective of this paper is to provide an example where
\[0 \neq (V^{-\infty})^H_{\text{disc}} = (V^{-\infty})^H_{\text{temp}} \subsetneq (V^{-\infty})^H.\]

To be more specific this happens for \(X\) the \(n\)-dimensional one-sheeted hyperboloid which is homogeneous for the connected Lorentzian group \(G = \text{SO}_0(n, 1)\).

Let us briefly introduce the standard notions.

1.1. Notation

Let \(n \geq 3\) and let \(G = \text{SO}_0(n, 1)\) be the identity component of the special Lorentz group \(\text{SO}(n, 1)\). We denote by \(H\) the stabilizer of
\[x_0 := (1, 0, \ldots, 0) \in \mathbb{R}^{n+1}\]
in \(G\). The entry in the lower right corner of any matrix in \(\text{SO}(n, 1)\) is non-zero, and \(\text{SO}_0(n, 1)\) consists of those matrices for which this entry is positive. From this fact we see that \(H\) is the connected subgroup
\[H = \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}_0(n-1, 1) \end{pmatrix} \subset G.\]

The group \(G\) acts transitively on the hyperboloid
\[X := \{x \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 = 1\},\]
and the homogeneous space
\[X = G/H = \text{SO}_0(n, 1)/\text{SO}_0(n-1, 1)\]
is a symmetric space. The corresponding involution \(\sigma\) of \(G\) is given by conjugation with the diagonal matrix \(\text{diag}(-1, 1, \ldots, 1)\). The subgroup \(G^\sigma\) of \(\sigma\)-fixed elements is the stabilizer of \(\mathbb{R} x_0\). This subgroup has two components, one of which is \(H\). For our purpose it is important to use \(H\) rather than \(G^\sigma\). The pairs \((G, H)\) and \((G, G^\sigma)\) differ by the fact that \((G, G^\sigma)\) is a...
Gelfand pair, whereas \((G, H)\) is not. In fact it has been shown by van Dijk [7] that \(X\) is the only symmetric space of rank one, which is not obtained as the homogeneous space of a Gelfand pair.

The regular representation of \(G\) on \(C^\infty(X)\) decomposes as the direct sum
\[
C^\infty(X) = C^\infty_{\text{even}}(X) \oplus C^\infty_{\text{odd}}(X)
\]
of the invariant subspaces of functions that are even or odd with respect to the \(G\)-equivariant symmetry \(x \mapsto -x\). The restriction of the regular representation to \(C^\infty_{\text{even}}(X)\) is isomorphic to the regular representation on \(C^\infty(G/G^\sigma)\), and the non-Gelfandness of \((G, H)\) is therefore caused by the presence of the odd functions.

1.2. Main results

The Lorentzian manifold \(X\) carries the \(G\)-invariant Laplace–Beltrami operator \(\Delta\), which is obtained as the radial part of
\[
\square := -\frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} + \frac{\partial^2}{\partial x_{n+1}^2}
\]
on \(\mathbb{R}^{n+1}\). That is, for \(f \in C^\infty(X)\) we define \(\Delta f \in C^\infty(X)\) as \((\square \tilde{f})|_X\) where \(\tilde{f}\) is any extension of \(f\) to a function, homogeneous of degree 0, on some open neighborhood of \(X\) in \(\mathbb{R}^{n+1}\).

Let
\[
\rho := \frac{1}{2}(n - 1)
\]
and for each \(\lambda \in \mathbb{C}\) let \(\mathcal{E}_\lambda(X)\) be the eigenspace
\[
\mathcal{E}_\lambda(X) = \{ f \in C^\infty(X) \mid \Delta f = (\lambda^2 - \rho^2)f \}.
\]
The Laplace–Beltrami operator is a scalar multiple of the Casimir element associated to the Lie group \(G\), and hence every irreducible subspace \(\mathcal{V}\) of \(C^\infty(X)\) is contained in \(\mathcal{E}_\lambda(X)\) for some \(\lambda \in \mathbb{C}\). Apart from the sign, the scalar \(\lambda\) is uniquely determined by the infinitesimal character of \(\mathcal{V}\). Conversely, since \(X = G/H\) is rank one, \(\pm \lambda\) determines the infinitesimal character.

Let
\[
\mathcal{E}^\text{even}_\lambda(X) = \mathcal{E}_\lambda(X) \cap C^\infty_{\text{even}}(X), \quad \mathcal{E}^\text{odd}_\lambda(X) = \mathcal{E}_\lambda(X) \cap C^\infty_{\text{odd}}(X).
\]
We can now state our main result.

The manifold \(X\) carries a \(G\)-invariant measure, which is unique up to scalar multiplication. We denote by \(L^2(X)\) the associated \(G\)-invariant space of square integrable functions.

The following theorem can be seen from the general theory of hyperbolic spaces over \(\mathbb{R}, \mathbb{C}\) and \(\mathbb{H}\) in [8]. See [15, Thm 6.1] and [16, Thm 6.4]. For the current simple situation it will be established along with Theorem 1.2.

**Theorem 1.1.** Let \(\lambda \in \mathbb{C}\) with \(\text{Re}\lambda > 0\). The intersections \(\mathcal{E}^\text{even}_\lambda(X) \cap L^2(X)\) and \(\mathcal{E}^\text{odd}_\lambda(X) \cap L^2(X)\) are either zero or irreducible. Moreover,
\[
\mathcal{E}^\text{even}_\lambda(X) \cap L^2(X) \neq 0 \iff \lambda \in \rho + 1 + \mathbb{Z}
\]
\[
\mathcal{E}^\text{odd}_\lambda(X) \cap L^2(X) \neq 0 \iff \lambda \in \rho + \mathbb{Z}.
\]

**Theorem 1.2.** For every \(0 < \lambda < \rho\) with \(\lambda \in \rho - \mathbb{N}\) the \(G\)-representations \(\mathcal{E}^\text{even}_\lambda(X)\) and \(\mathcal{E}^\text{odd}_\lambda(X)\) are irreducible and infinitesimally equivalent. Moreover in this case,
where the eigenspace \( R \) consists of harmonic polynomials on \( X \) of degree \( j \)

\[ \oplus \]

The proof of the two main theorems is divided into several parts. We begin with the analysis on \( K \)-types.

### 2.1. \( K \)-Types

Let \( K \subset G \) be the stabilizer of \( e_{n+1} = (0, \ldots, 0, 1) \in \mathbb{R}^{n+1} \), then \( K \simeq \text{SO}(n) \) is a maximal compact subgroup of \( G \).

We are going to use the diffeomorphism \( S^{n-1} \times \mathbb{R} \sim X \) given by

\[ (y, t) \mapsto (y_1 \cosh t, \ldots, y_n \cosh t, \sinh t) \in X \]  \hspace{1cm} (2.1)

where \( y = (y_1, \ldots, y_n) \in S^{n-1} \) and \( t \in \mathbb{R} \). With the natural action of \( \text{SO}(n) \) on \( S^{n-1} \) the parameter dependence on \( y \) is \( K \)-equivariant.

For each \( j \in \mathbb{N}_0 \) we denote by \( \mathcal{H}_j \subset C^\infty(S^{n-1}) \) the space of spherical harmonics of degree \( j \). We recall that by definition \( \mathcal{H}_j \) consists of the restrictions to \( S^{n-1} \) of all harmonic polynomials on \( \mathbb{R}^n \), homogeneous of degree \( j \). Equivalently, \( \mathcal{H}_j \) can be defined as the eigenspace

\[ \mathcal{H}_j \coloneqq \{ h \in C^\infty(S^{n-1}) \mid \Delta_K h = -j(j + n - 2)h \} \]

where \( \Delta_K \) is the angular part of the \( n \)-dimensional Laplacian

\[ \frac{\partial^2}{\partial y_1^2} + \cdots + \frac{\partial^2}{\partial y_n^2} \]

Each \( \mathcal{H}_j \) is an irreducible \( \text{SO}(n) \)-invariant finite dimensional subspace of \( C^\infty(S^{n-1}) \), and the sum \( \bigoplus_{j=0}^\infty \mathcal{H}_j \) of these subspaces is dense in \( C^\infty(S^{n-1}) \).

It follows that the space \( C^\infty_K(X) \) of \( K \)-finite functions \( f \in C^\infty(X) \) is spanned by all functions given in the coordinates \((y, t)\) by

\[ f(y, t) = h(y) \varphi(t) \]

where \( h \in C^\infty(S^{n-1}) \) is a spherical harmonic, and \( \varphi \in C^\infty(\mathbb{R}) \). Thus

\[ C^\infty_K(X) \simeq \bigoplus_{j=0}^\infty (\mathcal{H}_j \otimes C^\infty(\mathbb{R})) \]
Being homogeneous of degree $j$, the spherical harmonics $h \in \mathcal{H}_j$ satisfy $h(-y) = (-1)^j h(y)$ for $y \in S^{n-1}$. Therefore
\[ C_K^\infty(X)_{\text{even}} \simeq \bigoplus_{j=0}^\infty (\mathcal{H}_j \otimes C^\infty_{\text{parity}(j)}(\mathbb{R})) \]
where parity$(j)$ denotes the parity even or odd of $j$. Likewise
\[ C_K^\infty(X)_{\text{odd}} \simeq \bigoplus_{j=0}^\infty (\mathcal{H}_j \otimes C^\infty_{\text{parity}(j+1)}(\mathbb{R})). \]

2.2. Eigenspaces

With respect to the coordinates (2.1) on $X$ we have (see [14, p. 455])
\[ \Delta = \frac{\partial^2}{\partial r^2} + 2\rho \tanh \frac{t}{r} - \frac{1}{\cosh^2 t} \Delta_K. \]
The $K$-finite eigenfunctions for $\Delta$ belong to $C^\infty(X)$, and they can be determined as follows. Let
\[ \mathcal{E}_{\lambda,K} := \mathcal{E}_\lambda \cap C_K^\infty(X), \]
be the Harish-Chandra module of $\mathcal{E}_\lambda(X)$, and for each $j \in \mathbb{N}_0$ let
\[ \mathcal{E}_{\lambda,j} := \mathcal{E}_\lambda \cap (\mathcal{H}_j \otimes C^\infty(\mathbb{R})). \]
By separating the variables $y$ and $t$ we see that $\mathcal{E}_{\lambda,j}$ is spanned by the functions $f(y,t) = h(y)\varphi(t)$ for which $h \in \mathcal{H}_j$ and
\[ \left( \frac{d^2}{dt^2} + 2\rho \tanh \frac{t}{r} \frac{d}{dt} + \frac{j(j+n-2)}{\cosh^2 t} \right) \varphi = (\lambda^2 - \rho^2)\varphi. \] (2.2)
This differential equation is invariant under sign change of $t$. The solution with $\varphi(0) = 1$ and $\varphi'(0) = 0$ is even, and the solution with $\varphi(0) = 0$ and $\varphi'(0) = 1$ is odd. Thus the solution space decomposes as the direct sum of the one-dimensional subspaces of even and odd solutions, and we have
\[ \mathcal{E}_{\lambda,K}^{\text{even}}(X) = \bigoplus_{j=0}^\infty \mathcal{E}_{\lambda,j}^{\text{even}}(X), \quad \mathcal{E}_{\lambda,K}^{\text{odd}}(X) = \bigoplus_{j=0}^\infty \mathcal{E}_{\lambda,j}^{\text{odd}}(X) \]
where $\mathcal{E}_{\lambda,j}^{\text{even}}(X) \simeq \mathcal{E}_{\lambda,j}^{\text{odd}}(X) \simeq \mathcal{H}_j$ are equivalent irreducible $K$-types for each $j$.

2.3. Hypergeometric functions

In fact (2.2) can be transformed into a standard equation of special function theory. We first prepare for the anticipated asymptotic behavior of $\varphi$ by substituting $\Phi(t) = (\cosh t)^{\lambda+\rho} \varphi(t)$. This leads to the following equation for $\Phi(t)$
\[ \Phi''(t) - 2\lambda \tanh t \Phi'(t) - ab(1 - \tanh^2 t) \Phi(t) = 0, \] (2.3)
where $a = \lambda + \rho + j$ and $b = \lambda - \rho + 1 - j$.

Next we change variables. With
\[ x = \frac{1}{2}(1 - \tanh t) = (1 + e^{2t})^{-1} \in (0, 1) \]
we replace the limits $t = \infty$ and $t = -\infty$ by $x = 0$ and $x = 1$, respectively. We write $\Phi(t) = F(x)$, so that
\[ \varphi(t) = (\cosh t)^{-\lambda-\rho} F((1 + e^{2t})^{-1}). \]
Since \( \tanh t = 1 - 2x \) and \( 1 - \tanh^2 t = 4x(1 - x) \) this gives
\[
(x')^2 F''(x) + x'' F'(x) - 2\lambda (1 - 2x) F'(x)x' - ab 4x(1 - x)F(x) = 0,
\]
and since
\[
x' = -2x(1 - x), \quad x'' = 4x(1 - x)(1 - 2x),
\]
we arrive at the following equation for the function \( F(x) \)
\[
x(1 - x)F''(x) + (\lambda + 1)(1 - 2x)F'(x) - ab F(x) = 0. \tag{2.4}
\]
Recall the hypergeometric function \( F(a, b; c; x) = \sum_{m=0}^{\infty} (a)_m (b)_m (c)_m x^m / m! (c)_m \). 

The function \( F(a, b; c; x) \) is analytic at \( x = 0 \) with the value 1, and unless \( c \) is an integer it is the unique solution with this property.

With \( a = \lambda + \rho + j \) and \( b = \lambda - \rho + 1 - j \) as above we have \( a + b + 1 = 2(\lambda + 1) \). By comparing (2.4) and (2.5) we conclude that for each \( \lambda \), \( \lambda \notin \mathbb{N} \) the function
\[
\varphi_{\lambda, j}(t) := (\cosh t)^{\lambda - \rho} F(\lambda + \rho + j, \lambda - \rho + 1 - j; 1 + \lambda; (1 + e^{2t})^{-1})
\]
solves (2.2).

**2.4. \( L^2 \)-Behavior**

In the coordinates \((y, t)\) an invariant measure on \( X \) is given by
\[
\cosh^{n-1} dt \, dy
\]
where \( dt \) and \( dy \) are invariant measures on \( \mathbb{R} \) and \( S^{n-1} \), respectively. Hence a function \( f(y, t) = h(y)\varphi(t) \) is square integrable if and only if
\[
\int_{\mathbb{R}} |\varphi(t)|^2 \cosh^{n-1} dt < \infty.
\]
Let \( \tilde{\varphi}_{\lambda, j}(t) = \varphi_{\lambda, j}(-t) \). By symmetry this function also solves (2.2), and it belongs to \( L^2(\mathbb{R}, \cosh^{n-1} dt) \) if and only if \( \varphi_{\lambda, j} \) does.

**Lemma 2.1.** Let \( \text{Re} \lambda > 0 \) and \( j \in \mathbb{N} \).

1. If \( j \in \lambda - \rho + \mathbb{N} \) then \( \varphi_{\lambda, j} \in L^2(\mathbb{R}, \cosh^{n-1} dt) \).
2. If \( j \notin \lambda - \rho + \mathbb{N} \) then \( \varphi_{\lambda, j} \) and \( \tilde{\varphi}_{\lambda, j} \) are linearly independent, and for a sufficiently small \( \epsilon > 0 \) no non-trivial linear combination belongs to \( L^{2+\epsilon}(\mathbb{R}, \cosh^{n-1} dt) \).

**Proof.** It follows from the definition of \( \varphi_{\lambda, j}(t) \) that
\[
(\cosh t)^{\lambda + \rho} \varphi_{\lambda, j}(t) \to 1, \quad t \to \infty.
\]
Since $2\rho = n - 1$ this means $\varphi_{\lambda, j}$ has the desired $L^2$-behavior in the positive direction for all $\Re \lambda > 0$. The only issue is with the negative direction, or equivalently, with $\tilde{\varphi}_{\lambda, j}(t)$ for $t \to \infty$.

We first consider (1). The assumption that $j \in \lambda - \rho + \mathbb{N}$ implies that $b = \lambda - \rho + 1 - j \in -\mathbb{N}_0$. The Gauss series for $F(a, b; c; x)$ terminates and defines a polynomial when $a$ or $b$ is a non-positive integer. In particular $F(a, b; c; x)$ is then a bounded function on $[0, 1]$. It then follows from the definition of $\varphi_{\lambda, j}$ that

$$(\cosh t)^{\lambda + \rho} \varphi_{\lambda, j}(t)$$

is bounded on $\mathbb{R}$, and hence $|\varphi_{\lambda, j}(t)|^2(\cosh t)^{2\rho}$ is integrable. This proves (1).

Now consider (2). According to Gauss (see [1, Thm. 2.1.3]) we have

$$\lim_{x \to 1^-} (1 - x)^{a+b-c} F(a, b; c; x) = A = \frac{\Gamma(c) \Gamma(a + b - c)}{\Gamma(a) \Gamma(b)}$$

if $\Re(a + b - c) > 0$. We apply this to $t \to \infty$ in

$$(\cosh t)^{\lambda + \rho} \tilde{\varphi}_{\lambda, j}(t) = F(\lambda + \rho + j, \lambda - \rho + 1 - j; 1 + \lambda; (1 + e^{-2t})^{-1}).$$

Here $a + b - c = \lambda$ and

$$A = \frac{\Gamma(1 + \lambda) \Gamma(\lambda)}{\Gamma(\lambda + \rho + j) \Gamma(\lambda - \rho + 1 - j)}.$$

Since $x = (1 + e^{-2t})^{-1}$ implies $1 - x = (1 + e^{2t})^{-1}$ it follows that

$$e^{-2\lambda t}(\cosh t)^{\lambda + \rho} \tilde{\varphi}_{\lambda, j}(t) \to A, \quad t \to \infty.$$  

In particular we note that $A \neq 0$ if $j \notin \lambda - \rho + \mathbb{N}$. Under this condition we see that no non-trivial linear combination of $\varphi_{\lambda, j}$ and $\tilde{\varphi}_{\lambda, j}$ exhibits $L^{2+\varepsilon}$-behavior in both directions $\pm \infty$.

This proves (2) and concludes the proof of the lemma.  

2.5. Parity of $\varphi_{\lambda, j}$

We have seen that $\varphi_{\lambda, j}$ and $\tilde{\varphi}_{\lambda, j}$ are independent solutions to (2.2) when $j \notin \lambda - \rho + \mathbb{N}$. For $j \in \lambda - \rho + \mathbb{N}$ they are proportional, as the following lemma shows. For $\alpha, \beta > -1$ and $l \in \mathbb{N}_0$ let

$$P_l^{(\alpha, \beta)}(z) := \frac{(\alpha + 1)_l}{l!} F(l + \alpha + \beta + 1, -l; \alpha + 1; \frac{1}{2}(1 - z))$$

be the corresponding Jacobi polynomial (see [5, page 115]).

**Lemma 2.2.** Assume $j = \lambda - \rho + 1 + l \in \mathbb{N}_0$ where $l \in \mathbb{N}_0$. Then

1. $(\frac{(\lambda + 1)_l}{l!}) \varphi_{\lambda, j}(t) = (\cosh t)^{-\lambda - \rho} P_l^{(\lambda, \lambda)}(\tanh t),$
2. $\varphi_{\lambda, j}(-t) = (-1)^l \varphi_{\lambda, j}(t),$ 

for all $t \in \mathbb{R}$.

Note that with the repeated indices $P_l^{(\lambda, \lambda)}$ is in fact a Gegenbauer polynomial.

**Proof.** By definition

$$\varphi_{\lambda, j}(t) = (\cosh t)^{-\lambda - \rho} F(2\lambda + 1 + l, -l; \lambda + 1; x)$$

where $x = (1 + e^{2t})^{-1} = \frac{1}{2}(1 - \tanh t)$. The List (1) follows immediately. Then (2) follows since a Gegenbauer polynomial is even or odd according to the parity of its degree.  

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2.6. $K$-Types in $L^2$

For $\lambda \in \mathbb{C}$ with $\text{Re} \lambda \geq 0$ we define
\[ D_\lambda := \mathbb{N}_0 \cap (\lambda - \rho + \mathbb{N}) \]
if $\lambda - \rho \in \mathbb{Z}$ and $\lambda > 0$, and by $D_\lambda = \emptyset$ otherwise. Furthermore we let
\[ U_\lambda := \bigoplus_{j \in D_\lambda} (H_j \otimes \phi_{\lambda, j}). \]

It follows from Lemma 2.2 and the fact that the parity of $H_j$ is $(-1)^j$ that $U_\lambda \subset \mathcal{E}_{\lambda, K}^\text{even}(X)$ if $\lambda - \rho$ is odd, and $U_\lambda \subset \mathcal{E}_{\lambda, K}^\text{odd}(X)$ if $\lambda - \rho$ is even.

**Lemma 2.3.** For all $\lambda \in \mathbb{C}$ with $\text{Re} \lambda > 0$ we have
\[ U_\lambda = \mathcal{E}_{\lambda, K}(X) \cap L^2(X) = \mathcal{E}_{\lambda, K}(X) \cap C^\infty_{\text{temp}}(X). \tag{2.6} \]

**Proof.** Let $\text{Re} \lambda > 0$. When $\lambda - \rho \not\in \mathbb{Z}$ it follows immediately from Lemma 2.1(2) that $\mathcal{E}_{\lambda, j}(X) \cap C^\infty_{\text{temp}}(X) = \{0\}$ for each $j \in \mathbb{N}_0$. Since $U_\lambda = \{0\}$ in this case (2.6) follows. Assume from now on that $\lambda - \rho \in \mathbb{Z}$. It then follows from Lemma 2.1(1) that $U_\lambda \subset L^2(X)$.

To complete the proof we will find for each $j \in \mathbb{N}_0$ a second solution to (2.2), which is linearly independent from $\phi_{\lambda, j}$, and which is not tempered. There are two cases, depending on the parity of $n$.

If $n$ is even, then $\rho$, and hence also $\lambda$, is not an integer. In that case we already have a second solution at hand, namely $\phi_{-\lambda, j}$. Since
\[(\cosh t)^{-\lambda - \rho} \phi_{-\lambda, j}(t) \to 1, \quad t \to \infty,
\]
this function $\phi_{-\lambda, j}$ does not belong to any $L^p(\mathbb{R}, \cosh^{n-1} t \, dt)$ if $\text{Re} \lambda \geq \rho$. When $0 < \text{Re} \lambda < \rho$ it belongs to $L^{2+\epsilon}$ only for $\epsilon > \frac{2 \text{Re} \lambda}{\rho - \text{Re} \lambda}$.

We now assume $n$ is odd. Then $\rho$ and $\lambda$ are positive integers. We need to find a solution linearly independent from $F(a, b; c; x)$ to the hypergeometric Eq. (2.5) with
\[ a = \lambda + \rho + j, \quad b = \lambda - \rho + 1 - j, \quad c = 1 + \lambda. \]

By the method of Frobenius one finds (see [13, p. 5]) such a solution $G(a, b, c; x)$. It has the form
\[ G(x) = x^{-\lambda} \sum_{v=0}^{\infty} a_v x^v + \log x \sum_{v=0}^{\infty} b_v x^v \]
for some explicit power series with $a_0 = 1$ and $b_0 \neq 0$. The corresponding solution to (2.2) is
\[(\cosh t)^{-\lambda - \rho} G(a, b, c; (1 + e^{2t})^{-1}). \]
It behaves like $(\cosh t)^{-\lambda - \rho}(1 + e^{2t})^\lambda$ as $t \to \infty$ and as before it is not tempered with respect to the invariant measure. $\square$
2.7. Irreducibility

Let \( \lambda \in \rho + \mathbb{Z} \) and assume \( \lambda > 0 \). It follows from Lemma 2.3 that \( U_\lambda \) is \((g, K)\)-invariant. We will prove that it is an irreducible \((g, K)\)-module by using the infinitesimal element

\[
T = E_{n+1,1} + E_{1,n+1} \in g = \mathfrak{so}(n, 1),
\]

(2.7)
as a raising and lowering operator between the functions \( \varphi_{\lambda, j} \) which generate \( U_\lambda \) together with \( K \). For this we need to find the derivative of \( \varphi_{\lambda, j} \).

**Lemma 2.4.** Let \( j = \lambda - \rho + 1 + l \in \mathbb{N}_0 \) where \( l \in \mathbb{N}_0 \). There exist constants \( A_l, B_l \in \mathbb{R} \) such that

\[
\varphi'_{\lambda, j} = A_l \varphi_{\lambda, j+1} + B_l \varphi_{\lambda, j-1}.
\]

Both \( A_l \) and \( B_l \) are non-zero, except when \( l = 0 \) or \( j = 0 \), in which cases only \( A_l \) is non-zero.

**Proof.** Recall from Lemma 2.2

\[
\varphi_{\lambda, j}(t) = \frac{n}{(\lambda + 1)_j} (\cosh t)^{-\lambda-\rho} P^{(\lambda, l)}_l(tanh t)
\]

for \( j = \lambda - \rho + 1 + l \). It follows that

\[
\varphi'_{\lambda, j}(t) = \frac{n}{(\lambda + 1)_j} (\cosh t)^{-\lambda-\rho} \left(- (\lambda + \rho)x P^{(\lambda, l)}_l(x) + (1 - x^2)(P^{(\lambda, l)}_l)'(x) \right)
\]

where \( x = \tanh t \).

We obtain from [9, (4.7)] that

\[
(1 - x^2)(P^{(\lambda, l)}_l)'(x) = (l + 2\lambda + 1)x P^{(\lambda, l)}_l(x) - \frac{l(l+1)(l+2\lambda+1)}{l+\lambda+1} P^{(\lambda, l)}_{l+1}(x).
\]

By [5, (5.5.5)] the polynomials \( P^{(\lambda, l)}_l \) satisfy a three term recurrence relation

\[
n(l + \lambda + 1)(2l + 2\lambda + 1) x P^{(\lambda, l)}_l(x) = (l + \lambda)(l + \lambda + 1) P^{(\lambda, l)}_{l-1}(x) + (l + 1)(l + 2\lambda + 1) P^{(\lambda, l)}_{l+1}(x).
\]

With this relation we can eliminate \( x P^{(\lambda, l)}_l(x) \) and obtain \( \varphi'_{\lambda, j} \) as a linear combination of \( \varphi_{\lambda, j+1} \) and \( \varphi_{\lambda, j-1} \). The coefficients turn out to be

\[
A_l = \frac{(\lambda + \rho + l)(2\lambda + l + 1)}{2\lambda + 2l + 1}, \quad B_l = \frac{l(l - \rho + l + 1)}{2\lambda + 2l + 1}.
\]

All these coefficients are non-zero, except \( B_l \) when \( l = 0 \) or \( \lambda - \rho + l + 1 = 0 \). \( \square \)

Let \( M = K \cap H \) be the stabilizer in \( K \) of \( x_0 \), that is

\[
M = \begin{pmatrix} 1 & 0 \\ 0 & \text{SO}(n-1) \end{pmatrix} \subset \text{SO}(n, 1) = G.
\]

Then \( S^{n-1} \cong K/M \). Let \( h_j \in \mathcal{H}_j \) be the zonal spherical harmonic. This is the unique function in \( \mathcal{H}_j \) which is \( M \)-invariant and has the value 1 at the origin \((1, 0, \ldots, 0)\) of \( S^{n-1} \). Furthermore, let \( f_{\lambda, j} \in \mathcal{E}_\lambda(X) \) be defined by

\[
f_{\lambda, j}(y, t) = h_j(y) \varphi_{\lambda, j}(t).
\]
With the element $T$ from (2.7) the coordinates $(y, t)$ are determined from

$$K/M \times \mathbb{R} \ni (kM, t) \mapsto k \exp(tT)x_0 \in X,$$

and since $T$ is centralized by $M$ the left derivative $L_T f_{\lambda, j}$ by $T$ is again $M$-invariant. It follows that for each $j \in \mathbb{N}_0$, the function $L_T f_{\lambda, j} \in \mathcal{E}_{\lambda, K}(X)$ is a linear combination of the same family of functions $f_{\lambda, j}$ in $\mathcal{E}_{\lambda, K}(X)$. Since $h_j(x_0) = 1$ for all $j$, the coefficients can be determined from the restriction to

$$(\cosh t, 0, \ldots, 0, \sinh t) \mid t \in \mathbb{R} \subset X,$$

on which $L_T$ acts just by $\frac{d}{dt}$, and hence they are given by Lemma 2.4. It follows immediately that $U_{\lambda}$ has no non-trivial $(g, K)$-invariant subspaces.

2.8. Equivalence

Let $\lambda \in \rho + \mathbb{Z}$ and assume $0 < \lambda < \rho$.

Lemma 2.5. The $(g, K)$-modules $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$ and $\mathcal{E}^{\text{odd}}_{\lambda, K}(X)$ are irreducible and equivalent.

Proof. The assumption on $\lambda$ implies that $D_{\lambda} = \mathbb{N}_0$, and hence $U_{\lambda}$ is equal to one of the two modules $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$ and $\mathcal{E}^{\text{odd}}_{\lambda, K}(X)$, depending on the parity of $\lambda - \rho$. For simplicity of exposition, let us assume a specific parity, say even, of $\lambda - \rho$. Then $\mathcal{E}^{\text{odd}}_{\lambda, K}(X) = U_{\lambda}$ is irreducible as seen in Section 2.7.

By Kostant’s theorem [10, Thm. 8] an irreducible $(g, K)$-module, which contains the trivial $K$-type, is uniquely determined up to equivalence by its infinitesimal character. Hence $\mathcal{E}^{\text{odd}}_{\lambda, K}(X)$ is equivalent to the irreducible subquotient of $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$ containing $\mathcal{E}^{\text{even}}_{\lambda, 0}(X)$. Since $\mathcal{E}^{\text{odd}}_{\lambda, K}(X)$ and $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$ contain the same $K$-types, all with multiplicity one, we conclude that this subquotient is equal to $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$. The lemma is proved.

3. Conclusion

Assume Re $\lambda > 0$. Then $\mathcal{E}_{\lambda, K}(X) \cap L^2(X) = U_{\lambda}$ by Lemma 2.3. By definition $U_{\lambda}$ is non-zero if and only if $\lambda - \rho \in \mathbb{Z}$. In Section 2.6 we saw that it consists of even functions on $X$ when $\lambda - \rho$ is odd, and vice versa. Finally, irreducibility was seen in Section 2.7. Thus the proof of Theorem 1.1 is complete.

Assume $\lambda \in \rho + \mathbb{Z}$ and $0 < \lambda < \rho$. Then $\mathcal{E}^{\text{even}}_{\lambda, K}(X)$ and $\mathcal{E}^{\text{odd}}_{\lambda, K}(X)$ are irreducible and equivalent by Lemma 2.5. One of them equals $U_{\lambda}$ and belongs to $L^2(X)$, whereas we have seen in Lemma 2.3 that the other one is non-tempered. This proves Theorem 1.2.

References


