Ellipticity and discrete series

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Abstract. We explain by elementary means why the existence of a discrete series representation of a real reductive group $G$ implies the existence of a compact Cartan subgroup of $G$. The presented approach has the potential to generalize to real spherical spaces.
1. Introduction

Let $G$ be a connected reductive algebraic group defined over $\mathbb{R}$ and $G := G(\mathbb{R})$ its group of real points. In this article we give an elementary proof that Harish-Chandra’s compact Cartan subgroup condition is necessary for $G$ to have discrete series. To explain the background, we first describe the problem in the more general context of real spherical spaces.

1.1. Real spherical spaces. Let $H \subset G$ be an algebraic subgroup defined over $\mathbb{R}$ and $H = H(\mathbb{R})$. A suitable framework for harmonic analysis on $Z := G/H$ is obtained by the request that $Z$ is real spherical, i.e., there exists an open orbit on $Z$ for the natural action of a minimal parabolic subgroup $P$ of $G$.

Our interest is to obtain a geometric criterion for the existence of discrete series on a unimodular real spherical space $Z$. We recall that by definition the discrete series for $Z$ consists of the irreducible subrepresentations of the regular representation of $G$ on $L^2(Z)$. The following condition for its existence was conjectured in [8, (1.2)]:

Conjecture 1.1. Let $Z$ be a unimodular real spherical space. A necessary and sufficient condition for the existence of a discrete series representation for $Z$ is that the interior of $(\mathfrak{h}^\perp)_{\text{ell}}$ in $\mathfrak{h}^\perp$ is non-empty.

Let us explain the notation. Let $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of $G$ and $H$. Then $\mathfrak{h}^\perp \cong (\mathfrak{g}/\mathfrak{h})^*$ is the cotangent space $T^*_z Z$ at the base point $z_0 = H \in Z$, and the index ‘ell’ stands for elliptic elements.

The sufficiency of the condition has been established in [3]. We recall the result:

Theorem 1.2. Let $Z$ be a unimodular real spherical space. If the interior of $(\mathfrak{h}^\perp)_{\text{ell}}$ in $\mathfrak{h}^\perp$ is non-empty, then there exist infinitely many representations in the discrete series for $Z$.

A central tool in the proof of this theorem is a property of the infinitesimal characters of discrete series representations for $Z$, derived in [8]. The same property is crucial for our approach to necessity. Some notation is needed in order to describe it.

Let $G = KAN$ be an Iwasawa decomposition for $G$ and $P = MAN$ the associated minimal parabolic subgroup, with $M = Z_K(A)$ the centralizer of $A$ in $K$. Let $t \subset \mathfrak{m}$ be a maximal torus. Then $\mathfrak{c} = \mathfrak{a} + \mathfrak{t}$ is a maximally split Cartan subalgebra of $G$, unique up to conjugation. With $\mathfrak{c}_R = \mathfrak{a} + i\mathfrak{t}$ we obtain a real form of $\mathfrak{c}_C$ which is characterized by the property that all roots $\gamma \in \Sigma_\mathfrak{c} = \Sigma(\mathfrak{g}_C, \mathfrak{c}_C) \subset \mathfrak{c}_C^*$ are real valued on $\mathfrak{c}_R$. Let $V$ be the Harish-Chandra module of a discrete series representation for $Z$, and let its infinitesimal character be denoted $\chi_V \in \text{Hom}_{\text{alg}}(Z(\mathfrak{g}), \mathbb{C})$. Using the Harish-Chandra isomorphism we identify $\chi_V$ with a $W_\mathfrak{c}$-orbit $[\Lambda_V] = W_\mathfrak{c} \cdot \Lambda_V \in \mathfrak{c}_C^*/W_\mathfrak{c}$, where $W_\mathfrak{c}$ is the big Weyl group, i.e. the Weyl group of the root system $\Sigma_\mathfrak{c}$ with respect to the Cartan subalgebra $\mathfrak{c}$.

The mentioned result of [8] asserts that there exists an explicit $W_\mathfrak{c}$-invariant rational lattice $\mathcal{L}$, such that

\begin{equation}
[\Lambda_V] \subset \mathcal{L} \subset \mathfrak{c}_R^*
\end{equation}
for all discrete series representations $V$ of $Z$. Let us emphasize in particular that the parameters $\Lambda_V$ of the discrete series are real, as the lattice $\mathcal{L}$ lies in the real form $c^*_R$.

The purpose of this article is to explore whether this property of the infinitesimal character can be used to establish the conjectured necessity of the condition. To be more precise, we show that this is the case for the group, regarded as a spherical space. We believe the approach has the potential to generalize to all real spherical spaces.

1.2. The group case. In the remainder of this article we consider the group case. The group $G$ is a real spherical space when looked upon as a geometric object under its both-sided symmetries of $G \times G$. Specialized to this case the conjecture is Harish-Chandra’s beautiful geometric criterion for the existence of discrete series representations for $G$, which results from his deep study of discrete series [4, 5].

**Theorem 1.3.** (Harish-Chandra, [5, Theorem 13]) A necessary and sufficient condition for $G$ to admit discrete series is that it has a compact Cartan subgroup.

As mentioned, we provide an elementary proof of the necessity, based on the property (1.1) for $G$. In the case at hand the proof of this property is also elementary, as explained in the introduction to [8].

Let us describe the argument. Let $\sigma$ be the conjugation on $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. We call an element $\Lambda \in c^*_C$ strongly regular provided that the stabilizer of $\Lambda$ in the extended Weyl group $W_{\text{ext}} := \langle W, -\sigma \rangle \subset \text{Aut}(\Sigma_c)$ is trivial. We show that the existence of a unitary representation with a strongly regular real infinitesimal character implies the existence of a compact Cartan subgroup, see Corollary 3.6. Knowing that infinitesimal characters of discrete series are real, the existence of a discrete series representation with strongly regular infinitesimal character therefore requires the existence of a compact Cartan subgroup. Finally, we complete the proof by using the Zuckerman translation principle [9] to produce from any representation of the discrete series a discrete series representation with strongly regular infinitesimal character, see Corollary 5.8. The tools used for this belong to general representation theory of Harish-Chandra modules. Beyond the characterization of square integrability in terms of the leading exponents of asymptotic expansions, the only property of discrete series used at this stage is the existence of the lattice $\mathcal{L}$ satisfying (1.1).

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2. Notation

Throughout this article we let $G$ be the open connected subgroup of $G(\mathbb{R})$ where $G$ is a connected reductive group defined over $\mathbb{R}$. We write $G_C$ for the connected group $G(\mathbb{C})$. As usual we denote the Lie algebra of $G$ by $\mathfrak{g}$ and keep this terminology for subgroups of $G$, i.e., if $H \subset G$ is a subgroup, then we denote by $\mathfrak{h}$ its Lie algebra. If $\mathfrak{h}$ is a Lie algebra, then we write $\mathfrak{h}_C$ for the complexification of $\mathfrak{h}$.

Fix a Cartan involution $\theta$ of $G$ and denote by $K = G^\theta$ the corresponding maximal compact subgroup. The Lie algebra automorphism of $\mathfrak{g}$ induced by $\theta$, and its
linear extension to $g_C$, will be denoted by $\theta$ as well. We write $g = t + s$ for the associated Cartan decomposition. We fix a maximal abelian subspace $a \subset s$ and write $A = \exp(a)$. Further we let $M = Z_K(A)$ and select with $t \subset m$ a maximal torus. We write $T$ for the Cartan subgroup $Z_M(t)$ of $M$.

We denote by $\sigma : g_C \to g_C$ the complex conjugation with respect to the real form $g$, and let $U := K \exp(i\sigma)$ be the $\theta$-stable maximal compact subgroup of $G_C$, which is obtained as the fixed point subgroup of the antilinear extension $\theta \circ \sigma$ of the Cartan involution $\theta$ to $G_C$.

We extend $a$ by $t$ to a Cartan subalgebra $c := a + t$ of $g$, and use the symbol $\sigma$ also for the restriction of $\sigma$ to $c_C$. We write $\Sigma_c = \Sigma(g_C, c_C)$ for the corresponding root system and $\Sigma_a := \Sigma_c|_a \setminus \{0\}$ for the corresponding restricted root system. Further we set $c_R := a + it$. Note that $\Sigma_c \subset c_R^*$, that $\sigma$ preserves $\Sigma_c$ and $c_R$ and that $\sigma|_{c_R} = -\theta|_{c_R}$. We write $C_C$ for the maximal torus of $G_C$ with Lie algebra $c_C$. As $G_C$ is a connected algebraic reductive group, the torus $C_C$ is connected. We further define $C := G \cap C_C$ and $C_U := C_C \cap U$. Note that $C = TA$ and $C_U = T \exp(i\sigma)$.

Let us denote by $W_c$ the Weyl group of the root system $\Sigma_c$ and likewise we denote by $W_a$ the Weyl group of the restricted root system $\Sigma_a$. With respect to $\Sigma_a$ we have the restricted root space decomposition

$$g = a \oplus m \oplus \bigoplus_{\alpha \in \Sigma_a} g^\alpha.$$ 

In the sequel we fix with $\Sigma_a^+ \subset \Sigma_a$ a positive system. We then let $\Sigma_c^+ \subset \Sigma_c$ be any positive system which is compatible with $\Sigma_a^+$, i.e., $\Sigma_a^+ = \Sigma_c^+|_a \setminus \{0\}$.

The positive system $\Sigma_a^+$ defines a maximal nilpotent subalgebra $n = \bigoplus_{\alpha \in \Sigma_a^+} g^\alpha$. Put $N = \exp n$ and note that $P = MAN \subset G$ defines a minimal parabolic subgroup of $G$. We write $\overline{P}$ and $\overline{n}$ for $\theta P$ and $\theta n$, respectively.

3. Reading of the existence of maximal compact Cartan subgroup from the infinitesimal character

As usual we write $Z(g)$ for the center of the universal enveloping algebra $U(g)$ of $g_C$. Recall that according to Harish-Chandra the characters $\chi$ of $Z(g)$ are parametrized by $c^*_C/W_c$ as follows. For any positive system $S$ of $\Sigma_c$ we set

$$u_S := \bigoplus_{\alpha \in S} g_{C,\alpha},$$

and write $\rho_S$ for half the trace of $\text{ad}(c)$ on $u_S$. Using the Poincaré-Birkhoff-Witt theorem we may decompose an element $Z \in Z(g)$ as

$$(3.1) \quad Z \in C_S + u_{-S} U(g) u_S$$

with $C_S \subset U(c)$, see the proof of [7, Lemma 8.17]. The element $[\Lambda] \in c^*_C/W_c$ parametrizing $\chi$ is then given by

$$(3.2) \quad \chi(Z) = (\Lambda - \rho_S)(C_S)$$

and does not depend on the choice of $S$.

Every irreducible Harish-Chandra module $V$ admits an infinitesimal character $\chi_V : Z(g) \to \mathbb{C}$ which then corresponds to a $W_c$-orbit

$$[\Lambda_V] := W_c \cdot \Lambda_V$$
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for some \( \Lambda_V \in \mathfrak{c}_c^* \). The following lemma is standard. For convenience we include its short proof.

**Lemma 3.1.** Let \( V \) be an irreducible Harish-Chandra module. The following hold.

1. \( [\Lambda_V] = [-\Lambda_V] \), where \( \tilde{V} \) is the contragredient of \( V \).
2. If \( V \) is unitarizable, then \( [\Lambda_V] = [-\sigma \Lambda_V] \).

**Proof.** Let \( Z \mapsto Z' \) denote the principal anti-automorphism of \( \mathcal{U}(g) \). Then \( \chi_{\tilde{V}}(Z) = \chi_V(Z') \) for \( Z \in \mathcal{Z}(g) \). Let \( S \) be any positive system of \( \Sigma_c \). Let \( C_S \in \mathcal{U}(c) \) be as in (3.1). By (3.2)

\[
\chi_{\tilde{V}}(Z) = (\Lambda_{\tilde{V}} - \rho_S)(C_S).
\]

As \( Z' \in C'_S + u_S \mathcal{U}(g) u_{-S} \), we have

\[
\chi_V(Z') = (\Lambda_V - \rho_S)(C'_S) = (-\Lambda_V - \rho_S)(C_S).
\]

This proves (1).

The conjugate representation \( \nabla \) of \( V \) has infinitesimal character \( [\Lambda_{\nabla}] = [\sigma(\Lambda_V)] \). If \( V \) is unitarizable, then the representation is isomorphic to its conjugate dual, hence assertion (2). \( \square \)

We recall that an element \( \lambda \in \mathfrak{c}_c^* \) is regular provided that the stabilizer of \( \lambda \) in \( W_c \) is trivial. Notice that the complex conjugation \( \sigma \) and \( -id \) induce automorphisms of \( \Sigma_c \), i.e., they determine elements of \( \text{Aut}(\Sigma_c) \). In particular \( -\sigma \in \text{Aut}(\Sigma_c) \). We define the extended Weyl group of \( W_c \) as the following subgroup of \( \text{Aut}(\Sigma_c) \):

\[
W_{c, \text{ext}} := \langle W_c, -\sigma \rangle_{\text{group}} \subset \text{Aut}(\Sigma_c).
\]

Furthermore \( \lambda \in \mathfrak{c}_c^* \) is called strongly regular in case the stabilizer in \( W_{c, \text{ext}} \) is trivial.

According to Harish-Chandra (see [5, Theorem 16]) the infinitesimal characters of representations of the discrete series \( V \) of \( G \) are real, i.e., \( \Lambda_V \in \mathfrak{c}_c^*/W_c \). A simplified proof of this fact was recently given in the more general context of real spherical spaces, see [8, Theorem 1.1].

**Proposition 3.2.** Assume that there exists a representation \( V \) of the discrete series for \( G \) with infinitesimal character \( [\Lambda] \in \mathfrak{c}_c^*/W_c \). Then the following assertions hold:

1. \( \Lambda \in \mathfrak{c}_c^* \) and there exists an element \( w \in W_c \) such that \( w \cdot \Lambda = -\sigma(\Lambda) \).
2. If in addition \( \Lambda \) is strongly regular, then there exists an element \( w \in W_c \) such that \( w = -\sigma \) on \( \mathfrak{c}_c^* \). In particular, \( -\sigma \mid_{\mathfrak{c}_c^*} \in W_c \subset \text{Aut}(\mathfrak{c}_c^*) \).

**Proof.** As mentioned above, \( \Lambda \in \mathfrak{c}_c^* \). Since representations of the discrete series are also unitarizable, Lemma 3.1 gives \( [-\sigma \Lambda] = [\Lambda] \). This shows the first assertion and the second is a consequence thereof. \( \square \)

We recall that \( W_c = N_{G_c}(\mathfrak{c}_c)/C \) and \( W_\alpha = N_K(\mathfrak{a})/M \). We denote by \( W^{\theta}_c \) the subgroup of \( W_c \) consisting of the elements which commute with \( \theta \), and recall the exact sequence

\[
(3.3) \quad 1 \to W_m \to W^{\theta}_c \to W_\alpha \to 1
\]
where $W_m$ is the Weyl group of the root system $\Sigma_m := \Sigma(m_C, t_C)$, which can be realized as $N_M(t)/T$.

**Lemma 3.3.** Let $\tau$ be an automorphism of $\mathfrak{g}_C$ and $J_C$ a Cartan subgroup of $G_C$. If $\tau$ acts trivially on $J_C$, then there exists a $t \in J_C$ so that $\tau = \text{Ad}(t)$.

**Proof.** Since $\tau$ acts trivially on $J_C$, it preserves all root spaces $\mathfrak{g}_C^\gamma$, $\gamma \in \Sigma$. Hence there exists for all $\gamma \in \Sigma$ numbers $c_\gamma \in \mathbb{C}$ such that $\tau|_{\mathfrak{g}_C^\gamma} = c_\gamma \cdot \text{id}_{\mathfrak{g}_C^\gamma}$. Let now $t \in J_C$ be such that $\text{Ad}(t)$ coincides with $\tau$ on all simple root spaces $\mathfrak{g}_C^\gamma$, $\gamma \in \Pi$. Now $\phi := \text{Ad}(t)^{-1} \circ \tau$ is an automorphism of $\mathfrak{g}_C$ which acts trivially on $\mathfrak{b}_C = J_C + \bigoplus_{\gamma \in \Sigma^+} \mathfrak{g}_C^\gamma$ and leaves all other $\mathfrak{g}_C^{-\gamma}$, $\gamma \in \Sigma^+$, invariant. In fact, $\phi$ acts trivially on all negative root spaces. To see this, let $\gamma \in \Sigma^+$ and $0 \neq E_\gamma \in \mathfrak{g}_C^\gamma$ and $0 \neq F_\gamma \in \mathfrak{g}_C^{-\gamma}$. Then $0 \neq [E_\gamma, F_\gamma] \in J_C$. As $\phi$ acts trivially on $J_C$ we have
\[ [E_\gamma, F_\gamma] = \phi[E_\gamma, F_\gamma] = [E_\gamma, \phi F_\gamma], \]
and hence $\phi F_\gamma = F_\gamma$. It follows that $\tau = \text{Ad}(t)$. \qed

**Proposition 3.4.** The following assertions are equivalent:

1. $-\sigma|_{c_R} \in W_e$.
2. $\theta|_{c_R} \in W_t$.
3. $\theta$ is an inner automorphism of $\mathfrak{g}_C$.
4. There exists a $g \in U$ such that $\theta = \text{Ad}(g)$ as an automorphism of $\mathfrak{g}_C$.

**Proof.** Since $-\sigma$ and $\theta$ coincide on $c_R$, the equivalence of (1) and (2) is clear.

Suppose now that (2) holds. Since $\theta|_a \in W_a$ there exists a $k \in N_K(a)$ so that $\theta|_a = \text{Ad}(k)|_a$. Since $N_K(a) \subseteq N_K(M)$, the restriction of $\text{Ad}(k)^{-1} \theta$ to $c_R$ defines an element of $W_t$ whose restriction to $a$ is trivial. In view of (3) $\text{Ad}(k)^{-1} \theta$ defines an element of $W_m$, and thus there exists an $m \in M$ so that $\text{Ad}(k)^{-1} \theta|_t = \text{Ad}(m)|_t$. Now $\text{Ad}(km)$ and $\theta$ coincide on $c_R$. Let $w = km$.

Let $\tau = \theta \circ \text{Ad}(w)$. Since $\tau$ is an automorphism of $\mathfrak{g}_C$ with $\tau|_{c_R} = \text{id}_{c_R}$, it follows from Lemma 3.3 that there exists a $t \in C_C$ so that $\tau = \text{Ad}(t)$. Since $\theta$ commutes with $\text{Ad}(w)$ (as $w$ in $K$) we have $\tau^2 \in C_K(\tau^2) \subseteq C_K$. Hence $\langle \tau \rangle = \langle \tau^2 \rangle \cup \langle \tau^2 \rangle$ is a relatively compact subgroup of $\text{Ad}(C_C)$. Consequently we see that $t$ can in fact be chosen in $C_U$. It follows $\theta = \text{Ad}(tw^{-1})$ with $g := tw^{-1} \in U$. This proves (4).

The implication of (3) from (1) is trivial.

Finally, if (3) holds, then there exists a $g \in G_C$ so that $\theta = \text{Ad}(g)$. Since $\theta$ preserves the Cartan subalgebra $c_R$, we have $g \in N_G(c_R)$. Therefore, $\theta|_{c_R} = \text{Ad}(g)|_{c_R} \in W_t$. This proves (2). \qed

The following statement can also be found in [1, Lemma 1.6].

**Corollary 3.5.** The Cartan involution $\theta$ is an inner automorphism of $\mathfrak{g}_C$ if and only if $\mathfrak{t} \subseteq \mathfrak{g}$ is a reductive subalgebra of maximal rank. In that case $\mathfrak{g}$ admits a compact Cartan subalgebra.

**Proof.** Assume that $\theta$ is an inner automorphism of $\mathfrak{g}_C$. By Proposition 3.4 there exists a $g \in U$ so that $\text{Ad}(g) = \theta$. As $g$ is semi-simple, the group $K_C := C^g_C$ is equal to $Z_{G_C}(g)$. The centralizer of a semi-simple element contains a maximal torus of $G_C$, and therefore, $\text{rank } K_C = \text{rank } G_C$. 

If \( \mathfrak{k} \) is reductive of maximal rank, then there exists a Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{g} \) in \( \mathfrak{k} \). The Cartan involution \( \theta \) acts trivially on \( \mathfrak{h} \). Now Lemma 3.3 is applicable to \( \tau = \theta \) and \( j_{\mathcal{C}} = h_{\mathcal{C}} \). It follows that \( \theta \) is inner. \( \square \)

**Corollary 3.6.** Suppose that there exists a representation of the discrete series for \( G \) with strongly regular infinitesimal character. Then \( G \) admits a compact Cartan subgroup.

**Proof.** The assertion follows from Propositions 3.2 and 3.4 and Corollary 3.5. \( \square \)

### 4. Power series expansion

In this section we summarize a few basic facts regarding the power series expansions of the matrix coefficients of an irreducible Harish-Chandra module \( V \). We denote the dual Harish-Chandra module of \( V \) with an \( \mathcal{W} \)-orbit \([\Lambda V] = \mathcal{W} \cdot \Lambda V \subset \mathfrak{c}^*\).

Let us denote by \( a^{++} \) the positive Weyl chamber in \( a \) with respect to \( \Sigma^+_\mathfrak{a} \) and denote by \( a^+ \) the closure of \( a^{++} \). Likewise we set \( A^{++} = \exp(a^{++}) \) and \( A^+ = \exp(a^+) \). As usual we denote by \( \rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+_\mathfrak{a}} (\dim \mathfrak{g}^\alpha) \alpha \in \mathfrak{a}^* \) the Weyl half sum.

Now given an irreducible Harish-Chandra module \( V \) each \( K \)-bi-finite matrix coefficient

\[
G \ni g \mapsto m_{v,\hat{v}}(g) := \langle \pi(g)v, \hat{v} \rangle
\]

for \( v \in V \) and \( \hat{v} \in \hat{V} \) admits a power series expansion on \( A^{++} \), see [7, Ch. VIII]. To be precise, we have

\[
m_{v,\hat{v}}(a) = \sum_{\xi \in [\Lambda V]|a-\mathbb{N}[\Sigma^+_\mathfrak{a}]} p_{v,\hat{v}}^{\xi}(\log a) a^{\xi-\rho} \quad (a \in A^{++}, v \in V, \hat{v} \in \hat{V})
\]

with unique polynomials \( p_{v,\hat{v}}^{\xi} \) on \( a \) which are of bounded degree and depend bilinearly on the pair \( v, \hat{v} \). In case \( V \) belongs to the discrete series only those elements \( \xi \) contribute for which \( \text{Re} \xi|_{a^+} \) is negative, i.e., \( \text{Re} \xi(X) < 0 \) for all \( X \in a^+ \backslash \{0\} \).

By definition, an element \( \xi \in [\Lambda V]|a-\mathbb{N}[\Sigma^+_\mathfrak{a}] \) is called an exponent of \( V \) if \( p_{v,\hat{v}}^{\xi} \neq 0 \) for some \( v, \hat{v} \). The maximal elements in the set of exponents with respect to the ordering given by \( \xi_1 \geq \xi_2 \) if \( \xi_1 - \xi_2 \in \mathbb{N}[\Sigma^+_\mathfrak{a}] \) are called the leading exponents. We denote by \( \mathcal{E}_V \subset \mathfrak{a}_\mathbb{C}^* \) the set of leading exponents and note that by [7] Theorem 8.33] we have \( \mathcal{E}_V \subseteq [\Lambda V]|a \). Then

\[
m_{v,\hat{v}}(a) = \sum_{\xi \in \mathcal{E}_V-\mathbb{N}[\Sigma^+_\mathfrak{a}]} p_{v,\hat{v}}^{\xi}(\log a) a^{\xi-\rho} \quad (a \in A^{++})
\]

The coefficients \( p_{v,\hat{v}}^{\lambda} \) for \( \lambda \in \mathcal{E}_V \) determine the principal asymptotics of the matrix coefficient in the sense that

\[
m_{v,\hat{v}}(a) = \sum_{\lambda \in \mathcal{E}_V} p_{v,\hat{v}}^{\lambda}(\log a) a^{\lambda-\rho} + \text{lower order terms} \quad (a \in A^{++})
\]
The condition that $V$ belongs to the discrete series can be read off by its set of leading exponents. Let
\[ C := (a^{+})^{*} := \{ \lambda \in a^{*} \mid \lambda(X) \geq 0, \ X \in a^{+} \} = \sum_{\alpha \in \Sigma^{+}} \mathbb{R}_{\geq 0} \alpha \]
be the dual Weyl chamber. By [7, Theorem 8.48] $V$ belongs to the discrete series if and only if it satisfies the condition
\[ (4.1) \quad \text{Re} \ E_{\nu} \subset -\text{int} \ C. \]

**Lemma 4.1.** Let $F = F_{\mu}$ be a finite dimensional representation of $G$ with highest weight $\mu$ with respect to $\Sigma^{+}_{c}$ and let $V$ be a Harish-Chandra module of the discrete series. The following are equivalent:
1. $\text{Re} \mu|_{a} + \text{Re} E_{\nu} \subset -\text{int} \ C$.
2. All matrix coefficients of $V \otimes F_{\mu}$ are contained in $L^{2}(G)$.

**Proof.** If $v \otimes f \in V \otimes F_{\mu}$ and $\tilde{v} \otimes \tilde{f} \in \tilde{V} \otimes F_{\mu}^{*}$, then
\[ m_{v \otimes f, \tilde{v} \otimes \tilde{f}} = m_{v, \tilde{v}} m_{f, \tilde{f}}. \]

The assertion (1) $\Rightarrow$ (2) now follows from (4.1) as $\text{spec}_{a} F_{\mu} \subset \mu|_{a} - N_{0} [\Sigma^{+}_{c}] \subset \mu|_{a} - C$. The other implication follows immediately from (4.2) with suitable choices of $f$ and $\tilde{f}$. \qed

5. APPLICATION OF THE TRANSLATION PRINCIPLE

For a Harish-Chandra module $V$ we denote by $H_{0}(\pi, V) = V/\pi V$ the finite dimensional $\pi$-homology of degree 0, and recall that the covariant functor $H_{0}(\pi, \cdot)$ is right exact. Notice that $H_{0}(\pi, V)$ is a module for $M_{\pi}$. By the Harish-Chandra homomorphism we have $Z(\mathfrak{m}) \cong \mathcal{U}(\mathfrak{t})^{W_{\mu}}$. Moreover we note $Z(\mathfrak{a} + \mathfrak{m}) = \mathcal{U}(\mathfrak{a}) \otimes Z(\mathfrak{m})$. Therefore we can consider the spectrum of a finite dimensional $Z(\mathfrak{a} + \mathfrak{m})$-module as a $W_{m}$-invariant subset of $C_{c}^{\ast}$. In addition we consider $\rho$ as a $W_{m}$-invariant element of $c_{C}^{\ast}$ by extending it trivially on $t$.

**Lemma 5.1.** Let $V$ be an irreducible Harish-Chandra module with infinitesimal character $[\Lambda]$. Then the following assertions hold:
1. $\text{spec}_{Z(\mathfrak{a} + \mathfrak{m})} H_{0}(\pi, V) \subset -\rho + [\Lambda]$,
2. $\text{spec}_{a} H_{0}(\pi, V) \subset -\rho + \text{Spec}_{V} - N_{0} [\Sigma^{+}_{c}]$.

**Proof.** For (1) see [3, Cor. 3.32]. For the inclusion in (2), let $\lambda \in \text{spec}_{a} H_{0}(\pi, V)$. Recall that it follows from Casselman’s version of Frobenius reciprocity that elements $\lambda \in \text{spec}_{a} H_{0}(\pi, V)$ correspond to embeddings of $V$ into a minimal principal series representation $\text{Ind}_{\mathbb{Z}}^{G}(\sigma \otimes (\lambda + \rho))$ (see [2] or [3, Theorem 4.9]). Without loss of generality, we may assume that $V \subset \text{Ind}_{\mathbb{Z}}^{G}(\sigma \otimes (\lambda + \rho))$. As in the derivation of [8, (1.4)] one sees that $\lambda + \rho$ occurs as an exponent of $V$, and hence is contained in $\text{Spec}_{V} - N_{0} [\Sigma^{+}_{a}]$. \qed

For the rest of this section we let $V$ be a Harish-Chandra module of the discrete series with infinitesimal character $[\Lambda] = W_{\zeta} \cdot \Lambda \in c_{C}^{\ast}/W_{\zeta}$. We set
\[ [\Lambda]^{+} := \{ \nu \in [\Lambda] \mid \text{Re} \nu|_{a} \in -\text{int} \ C \} = \{ \nu \in [\Lambda] \mid \text{Re} \nu|_{a^{+}\setminus\{0\}} < 0 \}. \]
Lemma 5.2. Let $V$ be a Harish-Chandra module of the discrete series with infinitesimal character $[\Lambda]$. Then
\begin{equation}
\text{spec}_{\mathcal{Z}(a+m)} H_0(\overline{\rho}, V) \subset -\rho + [\Lambda]^+.
\end{equation}

Proof. Immediate from Lemma 5.1 and (4.1). □

We pick the representative $\Lambda \in [\Lambda]$ such that $\lambda := \Lambda|_a \in E_V$. In view of [3], Theorem 1.1 and Remark 1.2(3), there exists an $N \in \mathbb{N}$, independent of the discrete series representation $V$, so that $N\Lambda$ is integral. We select such an $N$ and set $\mu_0 := N\Lambda$. Let $\mu$ be the unique dominant integral element in $W_\iota \cdot \mu_0$ and let $F_\mu$ be the corresponding finite dimensional representation of $G$ with highest weight $\mu \in \mathfrak{c}^*_R$.

We are interested in the $Z(\mathfrak{g})$-isotypical decomposition of $V \otimes F_\mu$. Let $\chi_{\Lambda+\mu_0} : Z(\mathfrak{g}) \to \mathbb{C}$ be the character corresponding to $[\Lambda + \mu_0]$. According to Zuckerman [3] Theorem 1.2 (1)] the element $[\Lambda + \mu_0]$ appears in $\text{spec}_{Z(\mathfrak{g})}(V \otimes F_\mu)$ and thus the corresponding isotypical component
\begin{equation}
W := \{ v \in V \otimes F_\mu \mid (\exists k \in \mathbb{N})(\forall z \in Z(\mathfrak{g}))(z - \chi_{\Lambda+\mu_0}(z))^k \cdot v = 0 \}
\end{equation}
is non-zero. Let $J \subset W$ be a maximal submodule and set $U := W/J$. Then $U$ is an irreducible Harish-Chandra module with infinitesimal character $[\Lambda_U] = [\Lambda + \mu_0] = [(N+1)\Lambda]$.

Lemma 5.3. For any finite dimensional representation $F$ we have
\[ \text{spec}_{\mathcal{Z}(a+m)} H_0(\overline{\rho}, V \otimes F) \subset -\rho + [\Lambda]^+ + \text{spec}_{\mathcal{Z}(a+m)} F. \]

Proof. Filter $F$ as $\overline{\rho}$-module as
\[ F_0 = \{0\} \subsetneq F_1 \subsetneq \ldots \subsetneq F_n = F \]
such that $F_k/F_{k-1}$ is an irreducible $\overline{\rho}$-module for each $1 \leq k \leq n$. In particular, each $F_k/F_{k-1}$ is a trivial $\overline{\rho}$-module and thus $H_0(\overline{\rho}, V \otimes F_k/F_{k-1}) = H_0(\overline{\rho}, V) \otimes F_k/F_{k-1}$ as $MA$-modules.

We apply now $H_0$ to the exact sequence of $MA$-modules
\[ 0 \to V \otimes F_{k-1} \to V \otimes F_k \to V \otimes F_k/F_{k-1} \to 0. \]
and obtain the right exact sequence
\[ H_0(\overline{\rho}, V \otimes F_k) \to H_0(\overline{\rho}, V \otimes F_k/F_{k-1}) \to H_0(\overline{\rho}, V) \otimes F_k/F_{k-1} \to 0. \]
This implies
\[ \text{spec}_{\mathcal{Z}(a+m)} H_0(\overline{\rho}, V \otimes F_k) \subset \text{spec}_{\mathcal{Z}(a+m)} H_0(\overline{\rho}, V \otimes F_k/F_{k-1}) \cup \text{spec}_{\mathcal{Z}(a+m)} H_0(\overline{\rho}, V) \otimes F \]
and the assertion follows by induction on $k$ and (5.1). □

Lemma 5.4. Let $\mu \in \mathfrak{c}^*_R$ be dominant and integral and let $F_\mu$ be the highest weight representation with highest weight $\mu$. Let $\mu_0 \in W_\iota \cdot \mu$ and let $\Lambda \in \mathbb{R}_{+}\mu_0$. Further, let $\nu \in [\Lambda]$, $\sigma \in \text{spec}_c F_\mu$ and $w \in W_\iota$. If $w(\Lambda + \mu_0) = \nu + \sigma$, then $w\Lambda = \nu$ and $w\mu_0 = \sigma$.

Proof. Let $r > 0$ be so that $\mu_0 = r\Lambda$. We have $\sigma \in \text{spec}_c F_\mu \subset \text{conv}(W_\iota \cdot \mu_0)$. In particular, $\|\sigma\| \leq \|\mu_0\|$. Moreover, $\|\nu\| = \|\Lambda\|$. The Cauchy-Schwarz inequality applied to $\nu$ and $\sigma$ then gives that $\sigma = r\nu$. It follows that $\sigma = w\mu_0$ and $\nu = w\Lambda$. □
For a Harish-Chandra module $U$ and infinitesimal character $[\Lambda_U]$ we define a subset $[\Lambda_U]_E \subset [\Lambda_U]$ by

$$[\Lambda_U]_E := \{ \Upsilon \in [\Lambda_U] \mid \Upsilon|_a \in E_U \}.$$ 

**Proposition 5.5.** For $U = W/J$ as defined after (5.2) one has $[\Lambda_U]_E \subset [\Lambda + \mu_0]^+$. In particular, $U$ is square integrable.

**Proof.** First recall that $W \subset V \otimes F_\mu$ is a direct summand as it is a generalized $Z(\mathfrak{g})$-eigenspace. Thus $H_0(\overline{\pi}, W) \subset H_0(\overline{\pi}, V \otimes F_\mu)$ as $MA$-module and therefore

$$\text{spec}_{Z(a+m)} H_0(\overline{\pi}, W) \subset -\rho + [\Lambda]^+ + \text{spec}_{Z(a+m)} F_\mu$$

by Lemma 5.3. Now $U = W/J$ is a quotient of $W$ and thus the natural map $H_0(\overline{\pi}, W) \rightarrow H_0(\overline{\pi}, U)$ is surjective. We conclude that

$$\text{spec}_{Z(a+m)} H_0(\overline{\pi}, U) \subset -\rho + [\Lambda_U]$$

by Lemma 5.3. Now $U = W/J$ is a quotient of $W$ and thus the natural map $H_0(\overline{\pi}, W) \rightarrow H_0(\overline{\pi}, U)$ is surjective. We conclude that

$$\text{spec}_{Z(a+m)} H_0(\overline{\pi}, U) \subset -\rho + [\Lambda_U]$$

by Lemma 5.3. Now $U = W/J$ is a quotient of $W$ and thus the natural map $H_0(\overline{\pi}, W) \rightarrow H_0(\overline{\pi}, U)$ is surjective. We conclude that

$$\text{spec}_{Z(a+m)} H_0(\overline{\pi}, U) \subset -\rho + [\Lambda_U]$$

On the other hand we have $\text{spec}_{Z(a+m)} H_0(\overline{\pi}, U) \subset -\rho + [\Lambda_U]$ by Lemma 5.1(1). Comparing this with (5.3) and applying Lemma 5.4 yields

$$\text{spec}_{Z(a+m)} H_0(\overline{\pi}, U) \subset -\rho + [\Lambda_U]$$

Finally, from (4.1) we deduce that $U$ is square integrable. \qed

Repeated application of Proposition 5.5 yields:

**Corollary 5.6.** There exists a $N \in \mathbb{N}$ such that if $V$ is a representation of the discrete series with infinitesimal character $[\Lambda]$, then for every $k \in \mathbb{N}$ there exists a representation $U$ of the discrete series with infinitesimal character $[(kN + 1)\Lambda]$ and $E_U \subset [(kN + 1)\Lambda]^+_a$.

**Corollary 5.7.** Suppose that there exists a representation of the discrete series. Then there exists a representation of the discrete series with strongly regular infinitesimal character.

**Proof.** Let $V$ be a representation of the discrete series with infinitesimal character $[\Lambda]$ such that $\lambda = [\Lambda]|_a \in E_V$. By Corollary 5.6 there exists a discrete series representation $V_k$ for every $k \in \mathbb{N}$ with infinitesimal character $[(kN + 1)\Lambda]$ and $E_k := E_{V_k} \subset [(kN + 1)\Lambda]^+_a$. Since $[\Lambda]^+_a \subset -\text{int} \mathcal{C}$, we have

$$\lim_{k \rightarrow \infty} \text{dist}(E_k, -\partial \mathcal{C}) \geq \lim_{k \rightarrow \infty} (kN + 1) \text{dist}([\Lambda]^+_a, -\partial \mathcal{C}) = \infty.$$ 

It follows that for any $\mu \in \mathfrak{c}_R^*$ there exists a $k$ such that

$$E_k + \text{conv} \left( W_r \cdot \mu|_a \right) \subset -\text{int} \mathcal{C}.$$ 

In view of Lemma 4.1 this implies that for every $m \in \mathbb{N}$ and any choice of fundamental representations $F_{\mu_1}, \ldots, F_{\mu_m}$ there exists a $n \in \mathbb{N}$ so that for every $k \in \mathbb{N}$ with $k \geq n$ all matrix coefficients of the representation

$$V_k \otimes F_{\mu_1} \otimes \ldots \otimes F_{\mu_m}$$

are contained in $L^2(G)$. Let $\hat{\Lambda} \in [\Lambda]$ be the dominant element with respect to $\Sigma^+_\mathfrak{g}$. In view of [9 Theorem 1.2(1)] the representation (5.4) contains a subrepresentation with infinitesimal character $[(kN + 1)\hat{\Lambda} + \mu_1 + \ldots + \mu_m]$. 


The proof will be finished by showing that \((kN+1)\Lambda + \mu_1 + \ldots + \mu_m\) is strongly regular for a suitable choice of \(\mu_1, \ldots, \mu_m\) and for all \(k\) sufficiently large. The strongly regular elements comprise the complement of a finite union of proper subspaces of \(c^*_k\). We first choose \(m\) and \(\mu_1, \ldots, \mu_m\) such that \(\mu := \mu_1 + \cdots + \mu_m\) is outside of those subspaces which contain \(\Lambda\). Then so is \((kN+1)\Lambda + \mu\) for any \(k\). Clearly each remaining subspace can contain \((kN+1)\Lambda + \mu\) for at most one value of \(k\).

\[\square\]

**Corollary 5.8** (Harish-Chandra). *If a real reductive group \(G\) admits a representation of the discrete series, then there exists a compact Cartan subalgebra.*

*Proof.* Combine Corollary 5.7 with Corollary 3.6

\[\square\]

**References**


