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Projective representation theory for compact quantum groups and the quantum Baum-Connes assembly map

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Abstract. We study the theory of projective representations for a compact quantum group $G$, i.e. actions of $G$ on $B(H)$ for some Hilbert space $H$. We show that any such projective representation is inner, and is hence induced by an $\Omega$-twisted representation for some unitary measurable 2-cocycle $\Omega$ on $G$. We show that a projective representation is continuous, i.e. restricts to an action on the compact operators $\mathcal{K}(H)$, if and only if the associated 2-cocycle is regular, and that this condition is automatically satisfied if $G$ is of Kac type. This allows in particular to characterise the torsion of projective type of $\hat{G}$ in terms of the projective representation theory of $G$. For a given regular unitary 2-cocycle $\Omega$, we then study $\Omega$-twisted actions on $C^*$-algebras. We define deformed crossed products with respect to $\Omega$, obtaining a twisted version of the Baaj-Skandalis duality and a quantum version of the Packer-Raeburn’s trick. As an application, we provide a twisted version of the Green-Julg isomorphism and obtain the quantum Baum-Connes assembly map for permutation torsion-free discrete quantum groups.

Keywords. Assembly map, Baum-Connes conjecture, cleftness, 2-cocycle, compact objects, crossed products, Galois co-objects, quantum groups, projective representations, regularity, torsion, triangulated categories, twisting.
1. Introduction

The Baum-Connes conjecture has been formulated in 1982 by P. Baum and A. Connes. We still do not know any counterexample to the original conjecture, but it is known that the one with coefficients is false. For this reason we refer to the Baum-Connes conjecture with coefficients as the Baum-Connes property. The goal of the conjecture is to understand the link between two operator $K$-groups of different nature that would establish a strong connection
between geometry and topology in a more abstract and general index-theory context. More precisely, if $G$ is a (second countable) locally compact group and $A$ is a (separable) $G$-$C^*$-algebra, then the Baum-Connes property for $G$ with coefficients in $A$ claims that the assembly map $\mu_A^G : \hat{K}_a^\text{top}(G; A) \to \hat{K}_a(G \rtimes A)$ is an isomorphism, where $\hat{K}_a^\text{top}(G; A)$ is the equivariant $K$-homology with compact support of $G$ with coefficients in $A$ and $K_a(G \rtimes A)$ is the $K$-theory of the reduced crossed product $G \rtimes A$. This property has been proved for a large class of groups; let us mention the remarkable work of N. Higson and G. Kasparov [21] about groups with Haagerup property and the one of V. Lafforgue [23] about hyperbolic groups.

The equivariant $K$-homology with compact support $\hat{K}_a^\text{top}(G; A)$ is the geometrical object obtained from the classifying space of proper actions of $G$, thus it is, a priori, easier to calculate than the group $K_a(G \rtimes A)$, which is the one of analytical nature and less flexible in its structure. Nevertheless, sometimes the group $\hat{K}_a^\text{top}(G; A)$ creates non-trivial troubles. This is why R. Meyer and R. Nest provide in 2006 a new formulation of the Baum-Connes property in a well-suited category framework, using the language of triangulated categories and derived functors [27]. More precisely, if $\mathcal{K}$ is the $G$-equivariant Kasparov category and $F(A) := K_a(G \rtimes A)$ is the homological functor $\mathcal{K} \to \mathbf{Ab}$ defining the right-hand side of the Baum-Connes assembly map, they show that the assembly map $\mu_A^G$ is equivalent to the natural transformation $\eta_A^G : LF(A) \to F(A)$, where $LF$ is the localisation of the functor $F$ with respect to an appropriated complementary pair of (localizing) subcategories $(\mathcal{L}_G, \mathcal{N}_G)$. Here $\mathcal{L}_G$ is the subcategory of $\mathcal{K}$ of compactly induced $G$-$C^*$-algebras, and $\mathcal{N}_G$ is the subcategory of $\mathcal{K}$ of compactly contractible $G$-$C^*$-algebras. We say that $G$ satisfies the strong Baum-Connes property if $\mathcal{L}_G = \mathcal{K}^G$, which corresponds, in usual terms, to the existence of a $\gamma$-element that equals $1_\mathbb{C}$. This approach yields as well a characterization of the Baum-Connes property only in terms of compact subgroups, $K$-theory and crossed products. This reformulation allows in particular to avoid any geometrical construction, and thus to replace $G$ by a locally compact quantum group $\hat{G}$. For instance, this approach has already been implemented by R. Meyer and R. Nest [28] by proving that duals of compact connected\(^1\) groups satisfy the strong Baum-Connes property. Also, for a genuine discrete quantum group, $\hat{G}$, the strong Baum-Connes property has been studied, leading to explicit $K$-theory computations of the $C^*$-algebra $C(\hat{G})$ in remarkable examples: [44, 45, 18]. A major problem when studying the quantum counterpart of the Baum-Connes property in this setting is the torsion structure of such a $\hat{G}$. Indeed, if $G$ is a discrete group, its torsion phenomena are completely described in terms of the finite subgroups of $G$ and encoded in the localizing subcategory $\mathcal{L}_G$ using the Meyer-Nest reformulation. More precisely, induction and restriction functors provide a pair of adjoint functors allowing to apply the general Meyer-Nest machinery to define the complementary pair encoding the Baum-Connes property. The notion of torsion for a genuine discrete quantum group, $\hat{G}$, has been introduced first by R. Meyer and R. Nest (see [28] and [26]) in terms of ergodic actions of $G$, and re-interpreted later by Y. Arano and K. De Commer in terms of fusion rings and module $C^*$-categories [1]. Following the Meyer-Nest approach, we need an analogous complementary pair of localizing subcategories, $(\mathcal{L}_{\hat{G}}, \mathcal{N}_{\hat{G}})$, where $\mathcal{L}_{\hat{G}}$ must encode the torsion phenomena of $\hat{G}$. In this case, the induction-restriction approach is no

\(^1\)In an upcoming paper, the second author (together with P. Fima) has extended this result by removing the connectedness assumption.
longer valid since finite discrete quantum groups do not exhaust the torsion phenomena for $\hat{G}$. Thus, in order to apply the Meyer-Nest machinery to define $(\mathcal{L}_G, \mathcal{M}_G)$, it was already foreseen by the second author in [24, Section 5.4] that we should define, for each torsion action $(T, \delta)$ of $G$, a triangulated category $\mathcal{J}_\delta$ and a triangulated functor $F_\delta : \mathcal{K}\mathcal{K}\hat{G} \to \mathcal{J}_\delta$ such that the functor $F := \left( F_\delta \right)_{\delta \in \text{Tor}(\hat{G})} : \mathcal{K}\mathcal{K}\hat{G} \to \prod_{\delta \in \text{Tor}(\hat{G})} \mathcal{J}_\delta$ admits a (possibly partially defined) adjoint functor $F^\dagger$. In this case, the Meyer-Nest machinery allows to construct projective objects in $\mathcal{K}\mathcal{K}\hat{G}$ with respect to the homological ideal $\ker_{\text{Hom}}(F)$, which are precisely the objects in the image of $F^\dagger$. Accordingly, $\langle F^\dagger(\text{Obj}(\prod_{\delta \in \text{Tor}(\hat{G})} \mathcal{J}_\delta)) \rangle$ together with $\ker_{\text{Obj}}(F)$ form a complementary pair in $\mathcal{K}\mathcal{K}\hat{G}$.

A candidate for the pair $(\mathcal{L}_G, \mathcal{M}_G)$ was proposed in [28] and [45], but it had been an open question since the work of Meyer-Nest to prove whether it is complementary in $\mathcal{K}\mathcal{K}\hat{G}$. This has prevented from having a definition of a quantum assembly map whenever $\hat{G}$ is not torsion-free. Thus in the related works up to the present authors proved directly the abstract condition $\mathcal{L}_G = \mathcal{K}\mathcal{K}\hat{G}$, where $\mathcal{L}_G$ is the natural candidate for a quantum Baum-Connes property formulation.

Recently, Y. Arano and A. Skalski [2] have observed that the candidates for $\mathcal{L}_G$ and $\mathcal{M}_G$ form indeed a complementary pair of subcategories in $\mathcal{K}\mathcal{K}\hat{G}$, which allows to define a quantum assembly map for every discrete quantum group $\hat{G}$ (torsion-free or not). In this paper, we will revisit this result from a different perspective in the case when there is only projective torsion, i.e. any finite dimensional C*-algebra carrying an ergodic action of $G$ is simple. More precisely, when constructing the associated quantum assembly map, our proof of the adjointness between $j_{G,T}$ and $\tau_{\gamma}$ differs from the one in [2]. In the course of these investigations, we have resolved open questions about cleftness properties of compact quantum groups, as we explain below.

Assume that $G := G$ is a classical compact group. A continuous action of $G \to \mathbb{K}(H)$, for some Hilbert space $H$ is usually referred as a projective representation of $G$ on $H$. Such a $\delta$ is always of the form $\delta_g(T) = \pi(g)T \pi(g)^*$ for all $g \in G$ and $T \in \mathbb{K}(H)$, where $\pi : G \to \mathbb{U}(H)$ is a measurable map and $\omega : G \times G \to S^1$ is a measurable 2-cocycle such that $\pi(x)\pi(y) = \omega(x,y)\pi(xy)$, for all $x, y \in G$. Such a map $\pi$ is called a $\omega$-representation of $G$ on $H$.

A projective representation theory for compact quantum groups was introduced already in [14] by the first author in terms of the so called Galois co-objects for $G$, which are regarded as generalized 2-cocycle functions on $G$. These are in bijective correspondence with the obstructions for actions $G \to \mathbb{B}(H)$ to be inner. Particular instances of Galois co-objects are those defined in terms of (measurable) 2-cocycles $\Omega$ on $G$, extending the classical setting described above. Such Galois co-objects are called von Neumann cleft. We show in this article that Galois co-objects for compact quantum groups are automatically cleft, so that we can restrict to projective representations defined through measurable 2-cocycles. See Section 3.1. For the sake of completeness of the present article, we have included an explicit development of the corresponding $\Omega$-representation theory for $G$ in Section 3.2.

An action of $G$ on $\mathbb{B}(H)$ does not automatically restrict to a continuous action on the compact operators $\mathbb{K}(H)$. If this is the case, we call the projective representation continuous, and the associated 2-cocycle of finite type. In general, not all 2-cocycles are of finite type. We show
however that if $G$ is of Kac type, then all projective representations are continuous, and all 2-cocycles are of finite type. See Section 3.3.

From the above, we obtain that torsion of projective type of $G$ is given by (ergodic) actions of $G$ implemented by (irreducible) continuous projective representations of $G$ with respect to (measurable) 2-cocycles. Given such a 2-cocycle $\Omega$ on $G$ we introduce the notion of twisted action of $G$ with respect to $\Omega$ and define a twisted crossed product for which we obtain a twisted version of the Baaj-Skandalis duality. This generalizes previous results by J. C. Quigg \cite{36} in the case of classical locally compact groups. As a consequence, we obtain a quantum version of the Packer-Raeburn’s trick \cite{35}. It is important to mention that a similar approach was already presented in a more general framework in \cite{32} (see also \cite{42} for the von Neumann algebra setting), but the setting of compact quantum groups allows for some alternate perspectives. Note that even in the compact quantum group setting, one needs to address the delicate question about regularity of a 2-cocycle. We show that regularity is equivalent to $\Omega$ being of finite type. See Section 4.

As an application, we provide a quantum assembly map for discrete quantum groups with projective torsion. More precisely, let $\mathcal{D}_G$ be the localizing subcategory of $\mathcal{K}\mathcal{K}^G$ generated by objects of the form $C \otimes T$, where $C \in \text{Obj}(\mathcal{K}\mathcal{K})$ is a $C^*$-algebra and $(T, \delta) \in \text{Obj}(\mathcal{K}\mathcal{K}^G)$ is a projective torsion action of $G$. We show that the pair $(\mathcal{D}_G, \mathcal{D}_G^\perp)$ is complementary in $\mathcal{K}\mathcal{K}^G$. To do so we generalize the Green-Julg isomorphism in the following way. We define a “twisted” descent map $F_\delta := j_{G,T} : \mathcal{K}\mathcal{K}^G \to \mathcal{K}\mathcal{K}$ for each projective torsion action $(T, \delta)$. We show that it is adjoint to the functor $\tau_T : \mathcal{K}\mathcal{K} \to \mathcal{K}\mathcal{K}^G$ given by making the tensor product by $T$ on the right. As byproduct we obtain that the projective torsion actions of $G$ are compact objects in $\mathcal{K}\mathcal{K}^G$. This is a first step towards spectra computations in the quantum Kasparov category in the realm of tensor triangular geometry \cite{6, 16}. See Section 5.

## 2. Preliminaries

### 2.1. Conventions and notations

Let us fix the notations and the conventions that we use throughout the whole article.

Whenever $\mathcal{C}$ denotes a category, we shall assume that $\mathcal{C}$ is essentially small, so morphisms $\text{Hom}_\mathcal{C}(\cdot, \cdot)$ form sets. Given a category $\mathcal{C}$, we denote by $\mathcal{C}^{\text{op}}$ its opposite category. We say that $\mathcal{C}$ is countable additive if it is additive and it admits countable direct sums. If $F$ is an additive functor on an additive category, it is, by definition, compatible with finite direct sums. Whenever we require $F$ to be compatible with infinite (countable) direct sums, it will be explicitly indicated.

We denote by $\mathcal{Ab}$ the abelian category of abelian groups, by $\mathcal{Ab}^\mathbb{Z}/2$ the abelian category of $\mathbb{Z}/2$-graded groups of $\mathcal{Ab}$.

If $E$ is a $\mathbb{C}$-vector space and $S$ is a subset of vectors of $E$, then we write $\text{span} S$ for the corresponding $\mathbb{C}$-vector subspace generated by $S$. If $(E, ||\cdot||)$ is a normed $\mathbb{C}$-vector space and $F \subset E$ is a vector subspace, we write $[F] := \overline{F^{||\cdot||}}$ for the $||\cdot||$-closure of $F$ in $E$. We then also write $\overline{\text{span}} S = [\text{span} S]$ for $S \subset E$. 
Let $H$ be a Hilbert space. We denote by $\mathcal{B}(H)$ (resp. $\mathcal{K}(H)$) the space of all linear bounded (resp. compact) operators on $H$. If $\mathcal{S}$ is a subset of $\mathcal{B}(H)$, then we write $\text{span}^\sigma$-weak $\mathcal{S}$ for the closure of the linear subspace generated by $\mathcal{S}$ with respect to the $\sigma$-weak topology. We denote by $\mathcal{B}(H)_\ast$ the space of normal functionals on $\mathcal{B}(H)$, and for $\xi, \eta \in H$ we denote by $\omega_{\xi, \eta} \in \mathcal{B}(H)_\ast$ the linear form defined by $\omega_{\xi, \eta}(T) := \langle \xi, T(\eta) \rangle$, for all $T \in \mathcal{B}(H)$. If $V \in \mathcal{B}(H \otimes H)$ is a unitary operator, we put $\mathcal{C}(V) := \{[(id \otimes \eta)(\Sigma V) \mid \eta \in \mathcal{B}(H)_\ast]\}$. Observe that $(id \otimes \eta)(\Sigma V) = (\eta \otimes id)\circ \text{Ad}_\Sigma(\Sigma V) = (\eta \otimes id)(V \Sigma)$, for all $\eta \in \mathcal{B}(H)_\ast$. Also, we clearly have $\mathcal{C}(V) = \text{span}\{(id \otimes \omega_{\xi, \eta})(\Sigma V) \mid \xi, \eta \in H\}$.

If $A$ is a $C^*$-algebra and $H$ a Hilbert $A$-module, we denote by $\mathcal{L}_A(H)$ (resp. $\mathcal{K}_A(H)$) the space of all (resp. compact) adjointable operators on $H$. Hilbert $A$-modules are considered to be right $A$-modules, so that the corresponding inner products are considered to be conjugate-linear on the left and linear on the right. Given a Hilbert $A$-module $H$ and $\xi, \eta \in H$ we denote by $\theta_{\xi, \eta} \in \mathcal{L}_A(H)$ the rank one operator defined by $\theta_{\xi, \eta}(\zeta) := \langle \xi, \zeta \rangle \eta$, for all $\zeta \in H$. Then $\mathcal{K}_A(H) = \text{span}\{\theta_{\xi, \eta} \mid \xi, \eta \in H\}$.

All our $C^*$-algebras (except for obvious exceptions such as multiplier $C^*$-algebras and von Neumann algebras) are supposed to be separable and all our Hilbert modules are supposed to be countably generated. If $A$ is a $C^*$-algebra and $\mathcal{S}$ is a subset of elements in $A$, we write $C^*<\mathcal{S}> := C^*<\mathcal{S} \cup \mathcal{S}^*>$ for the corresponding $C^*$-subalgebra of $A$ generated by $\mathcal{S}$, that is, the intersection of all $C^*$-subalgebras of $A$ containing $\mathcal{S}$. In this case, the elements of $\mathcal{S}$ are called generators of $C^*<\mathcal{S}>$.

The symbol $\otimes$ stands for the minimal tensor product of $C^*$-algebras and the exterior tensor product of Hilbert modules depending on the context. The symbol $\otimes_{\text{max}}$ stands for the maximal tensor product of $C^*$-algebras. The symbol $\overline{\otimes}$ stands for the tensor product of von Neumann algebras. In any of the previous cases, the elementary tensors in the corresponding tensor product are denoted simply by $\otimes$ and the context will distinguish the specific situation. If $H$ is a Hilbert $A$-module and $(K, \pi)$ is a Hilbert $(A, B)$-bimodule, the interior tensor product of $H$ and $K$ with respect to $\pi$ is denoted by $H \otimes_K K$. If $A$ and $B$ are two $C^*$-algebras, $\Sigma : A \otimes B \longrightarrow B \otimes A$ denotes the flip map. The symbol $\Sigma$ is used as well for the suspension functor in the framework of triangulated categories. The context will distinguish the specific situation. We use systematically the leg numbering, so if $H$ is a Hilbert space then $X_{12} = X \otimes 1 \in \mathcal{B}(H^{\otimes 3})$ for $X \in \mathcal{B}(H^{\otimes 2})$, etc.

If $S, A$ are $C^*$-algebras, we denote by $M(A) = \mathcal{L}_A(A)$ the multiplier algebra of $A$ and we put $\widetilde{M}(A \otimes S) := \{x \in M(A \otimes S) \mid x(id_A \otimes S) \subset A \otimes S \text{ and } (id_A \otimes S)x \subset A \otimes S\}$, which contains $M(A) \otimes S$. If $H$ is a Hilbert $A$-module, we put $M(H) := \mathcal{L}_A(A, H)$, which contains canonically $H \cong K_A(A, H)$. We put $\widetilde{M}(H \otimes S) := \{X \in M(H \otimes S) \mid X(id_A \otimes S) \subset H \otimes S \text{ and } (id_H \otimes S)X \subset H \otimes S\}$, which contains $H \otimes M(S)$.

If $T := \mathcal{B}(H)$ is a type $I$-factor, we denote by $\text{Tr}$ the usual trace on $T$. If $\varphi$ is any state on $T$, we denote by $\varrho \in T$ the density matrix (i.e. the positive matrix with trace 1) such that $\varphi = \text{Tr}(\varrho \cdot)$. Given a state $\varphi$ on $T$, we denote by $(L^2(T), \lambda_T, \Lambda_T, \xi_T)$ the corresponding GNS construction, but we drop the notation $\lambda_T$ when it is clear from the context. If $T^{op}$ denotes the opposite von Neumann algebra of $T$, then the modular properties for $\varphi$ yield a $*$-representation $\rho_T$.
of $T^{\text{op}}$ on $L^2(T)$ defined by the formula $\rho_T(s^{\text{op}})(t\xi_T) := t^{1/2}s \xi_T$ for all $s, t \in T$ of finite rank with respect to an eigenbasis of $T$. We consider the anti $*$-homomorphism $(\cdot)^\circ : T \to B(L^2(T))$ defined by $s^\circ := \rho_T(s^{\text{op}}) = J_T s^* J_T$ for all $s \in T$, where $J_T t \xi_T := t^{1/2} t^* \rho^{-1/2} \xi_T$ is the modular conjugation on $L^2(T)$. We then have $T^\circ = T$. In the following, we will also identify $j : B(H)^{\text{op}} \cong B(\mathcal{T})$ through the $*$-isomorphism $T \to \mathcal{T}$.

2.2. Compact/Discrete Quantum Groups

In this section we recall elementary and fundamental facts concerning compact quantum groups and their corresponding duality theory. We refer to the books [31, 40] or to the original papers [46, 5] for more details.

2.2.1 Definition. A compact quantum group $\mathbb{G}$ is the data $(C(\mathbb{G}), \Delta)$ where $C(\mathbb{G})$ is a unital $C^*$-algebra and $\Delta : C(\mathbb{G}) \to C(\mathbb{G}) \otimes C(\mathbb{G})$ is a unital $*$-homomorphism such that:

i) $\Delta$ is co-associative meaning that $(\text{id} \otimes \Delta)\Delta = (\Delta \otimes \text{id})\Delta$ and

ii) $\Delta$ satisfies the cancellation property meaning that $[\Delta(C(\mathbb{G}))(C(\mathbb{G}) \otimes 1)] = C(\mathbb{G}) \otimes C(\mathbb{G}) = [\Delta(C(\mathbb{G}))(1 \otimes C(\mathbb{G}))]$.

A compact quantum group has a unique Haar state $h_\mathbb{G}$ such that $(h_\mathbb{G} \otimes \text{id})\Delta(x) = h_\mathbb{G}(x)1_{C(\mathbb{G})} = (\text{id} \otimes h_\mathbb{G})\Delta(x)$ for all $x \in C(\mathbb{G})$. We will make the standing assumption that $h_\mathbb{G}$ is faithful, so we only work with the reduced form $C(\mathbb{G})$ of a compact quantum group.

The GNS construction corresponding to $h_\mathbb{G}$ is denoted by $(L^2(\mathbb{G}), \lambda, \xi_\mathbb{G})$. We also write $\lambda(x) = \lambda(x)|_{C(\mathbb{G})}$ for $x \in C(\mathbb{G})$. We adopt the standard convention for the inner product on $L^2(\mathbb{G})$, which means that $\langle \lambda(x), \lambda(y) \rangle := h_\mathbb{G}(x^* y)$ for all $x, y \in C(\mathbb{G})$. We suppress the notation $\lambda$ in computations so that we simply write $x\lambda(y) = \lambda(xy)$ for all $x, y \in C(\mathbb{G})$.

2.2.2 Theorem-Definition (Regular representation). Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group.

i) There exists a unique unitary operator $V_\mathbb{G} \in M(\mathcal{K}(L^2(\mathbb{G})) \otimes C(\mathbb{G}))$ such that $V_\mathbb{G}(\lambda(x) \otimes \xi) = \Delta(x)(\xi_\mathbb{G} \otimes \xi)$, for all $x \in C(\mathbb{G})$ and $\xi \in L^2(\mathbb{G})$.

ii) For all $x \in C(\mathbb{G})$ we have $\Delta = \Delta_{V_\mathbb{G}}$ where $\Delta_{V_\mathbb{G}}(x) = V_\mathbb{G}(x \otimes 1)V_\mathbb{G}^*$. 

iii) The following identity holds: $(\text{id} \otimes \Delta)(V_\mathbb{G}) = (V_\mathbb{G})_{12}(V_\mathbb{G})_{13}$.

iv) The following pentagonal equation holds: $(V_\mathbb{G})_{12}(V_\mathbb{G})_{13}(V_\mathbb{G})_{23} = (V_\mathbb{G})_{23}(V_\mathbb{G})_{12}$. So $V_\mathbb{G}$ is a multiplicative unitary on $L^2(\mathbb{G})$ in the sense of Baaj-Skandalis [5].

v) We have that $C(\mathbb{G}) = S_{V_\mathbb{G}} := \{\eta \otimes \text{id}(V_\mathbb{G}) \mid \eta \in B(L^2(\mathbb{G}))^*\}$.

The unitary $V_\mathbb{G}$ is called right regular representation of $\mathbb{G}$ on $L^2(\mathbb{G})$ or fundamental unitary of $\mathbb{G}$.
2.2.3 Remark. In a similar way, we can define the left regular representation of $G$: there exists a unique multiplicative unitary $W_G \in M(C(\mathbb{G}) \otimes K(L^2(\mathbb{G})))$ such that $(W_G)^*(\xi \otimes \Lambda(x)) = \Delta(x)(\xi \otimes \xi_G)$, for all $x \in C(\mathbb{G})$ and $\xi \in L^2(\mathbb{G})$. For all $x \in C(\mathbb{G})$ we have $\Delta(x) = W_G^*(1 \otimes x)W_G$ and the following identity holds: $(\Delta \otimes \text{id})(W_G) = (W_G)_{13}(W_G)_{23}$.

The coproduct on $C(\mathbb{G})$ can be extended to $L^\infty(\mathbb{G}) = C(\mathbb{G})^\sigma$ using Theorem-Definition 2.2.2.(ii), obtaining the normal map $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$. The Haar state extends uniquely to a normal faithful state on $L^\infty(\mathbb{G})$, and we denote by $J_G$ the associated modular conjugation on $L^2(\mathbb{G})$.

Conversely, if $L^\infty(\mathbb{G})$ is a von Neumann algebra with a coassociative normal $*$-homomorphism $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\mathbb{G})$ and admitting an invariant normal faithful state $h_\mathbb{G}$, then $(L^\infty(\mathbb{G}), \Delta)$ arises from a (reduced) compact quantum group $G$ in a unique way.

2.2.4 Definition. A unitary representation of $G$ on a Hilbert space $H = H_u$ is a unitary element in $M(K(H) \otimes C(\mathbb{G}))$ with $(\text{id} \otimes \Delta)(u) = u_{12}u_{13}$. For $u, v$ unitary representations, we denote $\text{Hom}_G(u, v) = \{T : H_u \to H_v \mid T \text{ bounded and } (T \otimes 1)v = v(T \otimes 1)\}$. One calls $u$ irreducible if $\text{End}_G(u) = C(id_{H_u})$.

Any irreducible unitary representation $u$ has finite dimensional $H$, so then $u \in \mathcal{B}(H) \otimes C(\mathbb{G})$. The set of all unitary equivalence classes of irreducible unitary representations of $G$ is denoted by $\text{Irr}(G)$. If $x \in \text{Irr}(G)$ is such a class, we write $u^x \in \mathcal{B}(H_x) \otimes C(\mathbb{G})$ for a representative of $x$ and $H_x$ for the finite dimensional Hilbert space on which $u^x$ acts. We write $\text{dim}(x) := n_x$ for the dimension of $H_x$. The trivial representation of $G$ is denoted by $\epsilon$, and we put $u^\epsilon = 1_{C(\mathbb{G})}$. Given $x, y \in \text{Irr}(G)$, the tensor product of $x$ and $y$ is denoted by $x \otimes y$. Given $x \in \text{Irr}(G)$, there exists a unique class $\mathfrak{F}$ of irreducible unitary representations of $G$ such that $\text{Hom}_G(\epsilon, u^x \otimes u^\mathfrak{F}) \neq 0 \neq \text{Hom}_G(\epsilon, u^\mathfrak{F} \otimes u^x)$. It is called contragredient or conjugate representation of $x$.

The linear span of matrix coefficients of all finite dimensional unitary representations of $G$ is denoted by $\text{Pol}(\mathbb{G})$. It is a $*$-Hopf algebra by restriction of the co-multiplication $\Delta$, and we denote its co-unit by $\varepsilon$ and its antipode by $S$. Let $I_0$ be the anti-linear involutive map $\Lambda(\text{Pol}(\mathbb{G})) \to L^2(\mathbb{G})$ defined by $\Lambda(x) \mapsto \Lambda(S(x)^*)$ for $x \in \text{Pol}(\mathbb{G})$. Then $I_0$ is closeable, and we denote $I = \hat{J}_G[I]$ for the polar decomposition of its closure. The map $R(x) = \hat{J}_G x^* \hat{J}_G$, for all $x \in C(\mathbb{G})$, is a well-defined anti-multiplicative and anti-co-multiplicative map on $C(\mathbb{G})$ preserving $\text{Pol}(\mathbb{G})$, called unitary antipode.

2.2.5 Theorem-Definition (Discrete quantum group). Let $G = (C(\mathbb{G}), \Delta)$ be a compact quantum group. We switch between the following notations for the same space $c_0(\hat{G}) = C^*_{\mathbb{G}}(\mathbb{G}) = \tilde{S}_{V_G} := \{[(\text{id} \otimes \eta)(V_G) \mid \eta \in \mathcal{B}(L^2(\mathbb{G}))_+]\} \subset \mathcal{B}(L^2(\mathbb{G}))$. Then $c_0(\hat{G})$ is a $C^*$-algebra, and we denote also the identity map by:

$$\hat{\lambda} : c_0(\hat{G}) \to \mathcal{B}(L^2(\mathbb{G})).$$

Furthermore, we have the following:

i) The formula $\hat{\Delta}(x) = \hat{\Delta}^{\text{cop}}(x) := \Sigma V_G^*(1 \otimes x)V_G \Sigma$ defines a non-degenerate $*$-homomorphism $c_0(\hat{G}) \to \hat{M}(c_0(\hat{G}) \otimes c_0(\hat{G}))$ such that the pair $(\hat{G} = (c_0(\hat{G}), \hat{\Delta}))$ is a locally compact quantum group. One calls $\hat{G}$ the (Pontryagin) dual discrete quantum group of $G$. 

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Moreover, we denote \( U = c_0(\hat{G}) \otimes C(\mathbb{G}) \).

iii) We have \( V_G \in M(c_0(\hat{G}) \otimes C(\mathbb{G})) \).

iv) We have \( \left[ (\eta \otimes id)(W^*_G) \mid \eta \in \mathcal{B}(L^2(\mathbb{G})), \right] = \hat{J}_Gc_0(\hat{G})\hat{J}_G \subset \mathcal{B}(L^2(\mathbb{G})). \)

We denote \( l^\infty(\hat{G}) \) for the \( \sigma \)-weak closure of \( c_0(\hat{G}) \). It is a von Neumann algebra with coproduct \( \hat{\Delta} \) given by extending the formula in item i) above. It has a left, resp. right invariant normal, semifinite faithful weight \( \hat{h}_L \), resp. \( \hat{h}_R \). We can identify \( L^2(\mathbb{G}) \) with the standard space of \( l^\infty(\hat{G}) \) in such a way that \( \hat{J}_G \) becomes the associated modular conjugation. We further have inside \( c_0(\hat{G}) \) the dense 2-sided ideal:

\[
c_{00}(\hat{G}) \cong \bigoplus_{x \in \text{irr}(\hat{G})} \mathcal{B}(H_x),
\]

contained in the set of integrable elements for \( \hat{h}_L \) and \( \hat{h}_R \).

2.2.6 Theorem-Definition (Kac system associated to \( \mathbb{G} \)). Let \( \mathbb{G} = (C(\mathbb{G}), \Delta) \) be a compact quantum group. Then \( U_G = J_G\hat{J}_G = \hat{J}_GJ_G \in \mathcal{B}(L^2(\mathbb{G})) \) is a symmetry, and we call the pair \( (V_G, U_G) \) the standard Kac system associated to \( \mathbb{G} \). We then denote

\[
\rho(a) = U_G\lambda(a)U_G, \quad \hat{\rho}(x) = U_G\hat{\lambda}(x)U_G, \quad a \in C(\mathbb{G}), x \in c_0(\hat{G}).
\]

Moreover, we have \( W_G = \tilde{V}_G \) where

\[
\tilde{V}_G := \Sigma(U_G \otimes 1)V_G(U_G \otimes 1)\Sigma \equiv (U_G)_2(V_G)_2(U_G)_2 \in M(C(\mathbb{G}) \otimes \hat{\rho}(c_0(\mathbb{G}))),
\]

and \( V_G, \tilde{V}_G \) together with

\[
\tilde{V}_G := \Sigma(1 \otimes U_G)V_G(1 \otimes U_G)\Sigma \equiv (U_G)_1(V_G)_2(U_G)_1 \in M(\rho(C(\mathbb{G})) \otimes c_0(\mathbb{G})),
\]

\[
\tilde{V}_G = (U_G \otimes U_G)V_G(U_G \otimes U_G) \in M(\rho(C(\mathbb{G})) \otimes \hat{\rho}(c_0(\mathbb{G}))).
\]

i) are multiplicative on \( L^2(\mathbb{G}) \) in the sense of Baaj-Skandalis,

ii) are regular, meaning that \( \mathcal{K}(L^2(\mathbb{G})) = C(V_G) = C(\tilde{V}_G) = C(\tilde{V}_G), \)

iii) satisfy the following identity in \( V \): \( (\Sigma(1 \otimes U_G)V)^3 = \text{id} \).

Moreover, the following properties hold:

i) \( (V_G)_3(V_G)_2(V_G)_1 (V_G)_2(V_G)_1 = (V_G)_2(V_G)_1 \) and \( (\tilde{V}_G)_3(\tilde{V}_G)_2(\tilde{V}_G)_1 = (\tilde{V}_G)_2(\tilde{V}_G)_1 \).

ii) \( (c_0(\hat{G}), \hat{\Delta}^{cop}) = (S_{\tilde{V}_G}, \Delta_{\tilde{V}_G}) \); in particular, \( \hat{\Delta}^{cop}(x) = \tilde{V}_G(x \otimes 1)\tilde{V}_G* \), for all \( x \in c_0(\hat{G}) \).

iii) \( (C(\mathbb{G}), \Delta) = (S_{\tilde{V}_G}, \Delta_{\tilde{V}_G}) \); in particular, \( \Delta(a) = \tilde{V}_G^*(1 \otimes a)\tilde{V}_G, \) for all \( a \in C(\mathbb{G}) \).
iv) \((V_G)_{12}(\tilde{V}_G)_{23} = (\tilde{V}_G)_{23}(V_G)_{12}\); in particular, \(\tilde{V}_G(a \otimes 1)V_G^* = a \otimes 1\), for all \(a \in C(G)\).

v) \((V_G)_{23}(\tilde{V}_G)_{12} = (\tilde{V}_G)_{12}(V_G)_{23}\); in particular, \(V_G(1 \otimes x)V_G^* = 1 \otimes x\), for all \(x \in c_0(\hat{G})\).

vi) \((\tilde{V}_G)_{12}(\tilde{V}_G)_{23} = (\tilde{V}_G)_{23}(\tilde{V}_G)_{12}\); in particular, \(V_G(x \otimes 1)V_G^* = x \otimes 1\), for all \(x \in U_Gc_0(\hat{G})U_G\).

vii) \((U_GC(G)U_G), \Delta_{U_G} = (\hat{S}_{V_G}, \hat{\Delta}_{V_G})\), where \(\Delta_{U_G}(U_GaU_G) := \text{Ad}_{U_G \otimes U_G} (\Delta(a))\), for all \(a \in C(G)\); in particular, \(V_G(1 \otimes a)V_G^* = 1 \otimes a\), for all \(a \in U_GC(G)U_G\).

viii) \(\text{span}\{C(G) \cdot c_0(\hat{G})\} = \mathcal{K}(L^2(G))\).

We refer to [40] or [5] for more details about these computations.

2.3. Actions of compact and discrete quantum groups

In this section we recall elementary notions and results concerning actions of quantum groups.

2.3.1 Definition. Let \(G = (C(G), \Delta)\) be a compact quantum group and \(A\) a \(C^*\)-algebra. A right (continuous) action of \(G\) on \(A\) (or a right co-action of \(C(G)\) on \(A\)) is a non-degenerate \(*\)-homomorphism \(\delta : A \rightarrow A \otimes C(G)\) such that:

i) \(\delta\) intertwines the co-multiplication, meaning that \((\delta \otimes \text{id}_{C(G)}) \circ \delta = (\text{id}_A \otimes \Delta) \circ \delta\) and

ii) \(\delta\) satisfies the density condition \([\delta(A)(1 \otimes C(G))] = A \otimes C(G)\).

We write \(A \triangleright \bowtie G\). We say that \((A, \delta)\) is a right \(G\)-\(C^*\)-algebra if moreover \(\delta\) is injective.

If \(M\) is a von Neumann algebra, then a right (measurable) action of \(G\) on \(M\) is a normal unital \(*\)-homomorphism \(\delta : M \rightarrow M \otimes L^\infty(G)\) intertwining the co-multiplication (the density condition being superfluous in this case).

2.3.2 Example. The co-multiplication of any compact quantum group \(G\) defines an action of \(G\) on its defining \(C^*\)-algebra. This action is called the regular action of \(G\).

Similarly, we can define a left action of \(G\) on \(A\) (or a left co-action of \(C(G)\) on \(A\)) as a non-degenerate \(*\)-homomorphism \(\delta : A \rightarrow C(G) \otimes A\) satisfying the analogous properties of the preceding definition. In the present article, an action of a compact quantum group \(G\) is supposed to be a right one unless the contrary is explicitly indicated. Hence, we refer to such actions simply as \(\text{actions of } G\). Observe however that if \((A, \delta)\) is a right \(G\)-\(C^*\)-algebra, then \((A^{op}, \overline{\delta})\) is a left \(G\)-\(C^*\)-algebra where \(A^{op}\) denotes the opposite \(C^*\)-algebra of \(A\) and \(\overline{\delta} : A^{op} \rightarrow C(G) \otimes A^{op}\) is defined by:

\[ \overline{\delta} := (R \otimes \text{id}) \circ \Sigma \circ \delta, \]  

(2.1)

where \(R\) denotes the unitary antipode of \(G\).
2.3.3 Definition. If \((A, \delta)\) is a \(\mathbb{G}\text{-}C^*\)-algebra, we denote \(A^\delta = \{a \in A \mid \delta(a) = a \otimes 1\}\). We call \((A, \delta)\) ergodic if \(A\) is unital and \(A^\delta = \mathbb{C}1_A\).

In general, we denote \(E_\delta : A \to A^\delta\) for the \(\delta\)-invariant conditional expectation given by \(E_\delta(a) = (id_A \otimes h_\mathbb{G})\delta(a)\), for all \(a \in A\).

2.3.4 Remark. Recall that we assume \(\mathbb{G}\) to be a reduced compact quantum group, so \(E_\delta\) is automatically faithful. Moreover, if \((u_i)_{i \in \mathbb{N}}\) is an approximate unit for \(A\), then by non-degeneracy of \(\delta\) we have that \((\delta(u_i))_{i \in \mathbb{N}}\) is an approximate unit for \(A \otimes C(\mathbb{G})\). Thanks to the continuity of \(id \otimes h_\mathbb{G}\), the operators \(v_i := (id \otimes h_\mathbb{G})(\delta(v_i))\) form an approximate unit for \(A\) inside \(A^\delta\), and we have \([AA^\delta] = [AA^\delta] = A\). In particular, if \(A\) acts non-degenerately on a Hilbert space \(H\), then also \(A^\delta\) acts non-degenerately on \(H\).

Given a \(\mathbb{G}\text{-}C^*\)-algebra \(A\), we can equip \(A\) with the pre-Hilbert \(A^\delta\)-module structure given by \(\langle a, b \rangle_{E_\delta} := E_\delta(a^*b)\), for all \(a, b \in A\). We denote by \(L^2(A, E_\delta)\) the completion of \(A\) with respect to the inner product \(\langle \cdot, \cdot \rangle_{E_\delta}\). When \(\delta\) is ergodic, we have \(E_\delta(a) = \varphi_\delta(a)1_A\) for \(\varphi_\delta\) a (unique) \(\delta\)-invariant state on \(A\). We then write \(L^2(A) = L^2(A, \varphi)\) for the Hilbert space completion of \(A\) with respect to the inner product \(\langle a, b \rangle_\varphi := \varphi(a^*b)\), for all \(a, b \in A\).

The notion of action of \(\mathbb{G}\) can be defined also for Hilbert modules.

2.3.5 Definition. Let \(\mathbb{G} = (C(\mathbb{G}), \Delta)\) be a compact quantum group and \((A, \delta)\) a \(\mathbb{G}\text{-}C^*\)-algebra. Let \(E\) be a Hilbert \(A\)-module. A right action of \(\mathbb{G}\) on \(E\) (or a right co-action of \(C(\mathbb{G})\) on \(E\)) is a linear map \(\delta_E : E \to E \otimes C(\mathbb{G})\) such that:

\begin{align*}
\text{i)} \quad & \delta_E(\xi \cdot a) = \delta_E(\xi)\delta(a) \text{ for all } \xi \in E, a \in A; \\
\text{ii)} \quad & \delta(\langle \xi, \eta \rangle) = \langle \delta_E(\xi), \delta_E(\eta) \rangle \text{ for all } \xi, \eta \in E; \\
\text{iii)} \quad & \delta_E \text{ intertwines the co-multiplication meaning that } (\delta_E \otimes id_{C(\mathbb{G})}) \circ \delta_E = (id_A \otimes \Delta) \circ \delta_E; \\
\text{iv)} \quad & \text{The density conditions } [\delta_E(E)(1 \otimes C(\mathbb{G}))] = [(1 \otimes C(\mathbb{G}))\delta_E(E)] = E \otimes C(\mathbb{G}) \text{ are satisfied.}
\end{align*}

We write \(E^{\delta_E} \mathbb{G}\). We say that \((E, \delta_E)\) is a right \(\mathbb{G}\)-equivariant Hilbert \(A\)-module if moreover \(\delta_E\) is injective.

If \((E, \delta_E)\) is a \(\mathbb{G}\)-equivariant Hilbert \(A\)-module as above, then \(\mathcal{K}_A(E)\) is a \(\mathbb{G}\text{-}C^*\)-algebra with action \(\delta_{\mathcal{K}_A(E)}(\delta_E)\) defined by \(\delta_{\mathcal{K}_A(E)}(\delta_E)(\theta_{\xi, \eta}) = \delta_E(\xi)\delta_E(\eta)^* \in \mathcal{K}_A(E) \otimes C(\mathbb{G})\), for all \(\xi, \eta \in E\) where \(\theta_{\xi, \eta}\) denotes the corresponding rank one operator in \(E\). By abuse of notation, we still denote by \(\delta_{\mathcal{K}_A(E)}\) the extension of this homomorphism to \(\mathcal{L}_A(E) = M(\mathcal{K}_A(E)) \to M(\mathcal{K}_A(E) \otimes C(\mathbb{G}))\). The latter is however not in general an action of \(\mathbb{G}\) on \(\mathcal{L}_A(E)\).

Recall further that giving an action \(\delta_E\) is equivalent to giving a unitary operator \(V_E \in \mathcal{L}_{A \otimes C(\mathbb{G})}(E \otimes (A \otimes C(\mathbb{G})), E \otimes (A \otimes C(\mathbb{G})))\) such that \(\delta_E(\xi) = V_E \circ T_\xi\) for all \(\xi \in E\) where \(T_\xi \in \mathcal{L}_{A \otimes C(\mathbb{G})}(A \otimes C(\mathbb{G}), E \otimes (A \otimes C(\mathbb{G})))\) is such that \(T_\xi(x) = \xi \otimes x\), for all \(x \in A \otimes C(\mathbb{G})\). One calls \(V_E\) the admissible operator for \((E, \delta_E)\). Moreover, we have \(\delta_{\mathcal{K}_A(E)} = Ad_{V_E}\). We refer to [4] for more details.

Next, we recall the following useful result (recall the notations from Definition 2.3.3).
2.3.6 Proposition. Let $\mathbb{G}$ be a compact quantum group. Let $(A, \delta)$ be a unital $\mathbb{G}$-$C^*$-algebra. If $\delta$ is ergodic, then there exists a unitary representation $V_A \in M(\mathcal{K}(L^2(A)) \otimes C(\mathbb{G}))$ of $\mathbb{G}$ such that $\delta(a) = V_A(a \otimes 1)V_A^*$, for all $a \in A$.

Proof. Consider the map $A \otimes C(\mathbb{G}) \xrightarrow{V_A} A \otimes C(\mathbb{G})$ such that $a \otimes x \to \delta(a)(1_A \otimes x)$. By $\delta$-invariance of $\phi_\delta$, this map is isometric with respect to the natural pre-Hilbert $C(\mathbb{G})$-module structure on $A \otimes C(\mathbb{G})$. Moreover, since $\delta$ is an action of $\mathbb{G}$ on $A$, we know that $[\delta(A)(1 \otimes C(\mathbb{G}))] = A \otimes C(\mathbb{G})$, that is, $V_A$ has dense range. Accordingly, $V_A$ extends to a unitary operator in $M(\mathcal{K}(L^2(A)) \otimes C(\mathbb{G}))$, which we still denote by $V_A$.

The relation $\delta(a)V_A = V_A(a \otimes 1)$, for all $a \in A$ is obvious. The coaction property for $\delta$ straightforwardly leads to $(V_A)_{12}(V_A)_{13}(V_G)_{23} = (V_G)_{23}(V_A)_{12}$, so $(id \otimes \Delta)V_A = (V_A)_{12}(V_A)_{13}$ and $V_A$ is a representation of $\mathbb{G}$ on $L^2(A)$ (see [9] for more details).

2.3.7 Remark. A similar result can be obtained when $\delta$ is not ergodic by considering instead the Hilbert $A^\delta$-module $L^2(A, E_\delta)$. One also has a corresponding result in the von Neumann algebraic setting: if $M \to M\overline{\otimes}L^2(\mathbb{G})$ on a von Neumann algebra (say with separable predual), we can find a $\mathbb{G}$-invariant state on $M$ leading to a unitary $V_M : L^2(M) \otimes L^2(\mathbb{G}) \to L^2(M) \otimes L^2(\mathbb{G})$ as above. This map is independent of the chosen state [41].

We also recall the notion of action for discrete quantum groups.

2.3.8 Definition. Let $\mathbb{G}$ be a compact quantum group and $A$ a $C^*$-algebra. A right action of $\hat{\mathbb{G}}$ on $A$ (or a right co-action of $c_0(\hat{\mathbb{G}})$ on $A$) is a non-degenerate $*$-homomorphism $\delta : A \to \hat{\mathcal{M}}(A \otimes c_0(\hat{\mathbb{G}}))$ such that:

i) $\delta$ intertwines the co-multiplication meaning that $(\delta \otimes id)\delta = (id \otimes \hat{\Delta})\delta$ and

ii) $\delta$ satisfies the cancellation property meaning that $[\delta(A)(1 \otimes c_0(\hat{\mathbb{G}}))] = A \otimes c_0(\hat{\mathbb{G}})$.

We say that $(A, \delta)$ is a right $\hat{\mathbb{G}}$-$C^*$-algebra if moreover $\delta$ is injective.

Again, one has the analogous notion of a left action of $\hat{\mathbb{G}}$. In the following, an action of a discrete quantum group $\hat{\mathbb{G}}$ is supposed to be a right one unless the contrary is explicitly indicated.

2.4. Torsion phenomena for discrete quantum groups

In this section we recall elementary notions and results concerning the notion of torsion-freeness for discrete quantum groups. It was initially introduced by R. Meyer and R. Nest and it can be characterized as in Theorem-Definition 2.4.4 below (see [28, 26] and [45] for more details).

2.4.1 Definition. Let $\mathbb{G} = (C(\mathbb{G}), \Delta)$ be a compact quantum group and let $(A, \delta)$ be a right $\mathbb{G}$-$C^*$-algebra with $A$ finite dimensional and $\delta$ ergodic. Then we say that $(A, \delta)$ is a torsion action of $\mathbb{G}$. The set of all non-trivial torsion actions of $\mathbb{G}$ is denoted by $\text{Tor}(\hat{\mathbb{G}})$. 

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2.4.2 Remark. If $\widehat{G}$ is a classical discrete group $\Gamma$, then $\text{Tor}(\Gamma)$ detects the torsion in $\Gamma$, hence the notational use in general of the dual discrete quantum group.

2.4.3 Examples. 1. The trivial action $(C, \text{trv.})$ is of course a torsion action of any compact quantum group $G$.

2. If $\widehat{H} < \widehat{G}$ is a discrete quantum subgroup of $\widehat{G}$, we have by definition an inclusion of $C^*$-algebras $C(\widehat{H}) \subseteq C(\widehat{G})$ intertwining the corresponding co-multiplications. Therefore, if $(B, \beta)$ is a $\mathbb{H}$-$C^*$-algebra, we can obviously extend $\beta$ (by composing with $\iota$) into an action of $G$ on $B$, which is denoted by $\tilde{\beta}$. We denote by $\text{Ind}^G_{\mathbb{H}}(B, \beta)$ the same $C^*$-algebra $B$ but equipped with the composition $\tilde{\beta} := (id_B \otimes \iota) \circ \beta$ as an action of $G$. Observe that if $(B, \beta)$ is a torsion action of $\mathbb{H}$, then $\text{Ind}^G_{\mathbb{H}}(B, \beta)$ is a torsion action of $G$.

3. Let $\widehat{G}$ be a discrete quantum group that has a non-trivial finite discrete quantum subgroup, say $\widehat{H} < \widehat{G}$. Then $(C(\mathbb{H}), \Delta_{\mathbb{H}})$ defines a non-trivial torsion action of $G$.

4. If $u \in \mathcal{B}(H_u) \otimes C(G)$ is a unitary representation of $G$ on a finite dimensional Hilbert space $H_u$, then it defines an action of $G$ on $\mathcal{B}(H_u)$ given by:

$$Ad_u : \mathcal{B}(H_u) \rightarrow \mathcal{B}(H_u) \otimes C(G), \quad T \mapsto Ad_u(T) := u(T \otimes 1_{C(G)})u^*.$$  

It is clear that $\mathcal{B}(H)^{Ad_u} = \text{End}_G(u)$. Hence, the pair $(\mathcal{B}(H_u), Ad_u)$ is a torsion action of $G$ if and only if $u$ is irreducible.

5. Consider the rotation group $SO(3)$. Recall that $SO(3) \cong SU(2)/\mathbb{Z}_2$, where $\mathbb{Z}_2 \cong \mathbb{Z}(SU(2))$ is the center of $SU(2)$. Then the conjugation action of $SU(2)$ on $M_2(\mathbb{C})$ descends to an action $\delta$ of $SO(3)$ on $M_2(\mathbb{C})$. Similar considerations can be made for $SO_q(3)$ with $q \in (-1, 1) \setminus \{0\}$ (see [38] for more details).

The following characterisation of torsion-freeness for discrete quantum groups is well-known. A full proof can be found in [24, Theorem 1.6.1.4].

2.4.4 Theorem-Definition. Let $G$ be a compact quantum group. We say that $\widehat{G}$ is torsion-free if one of the following equivalent conditions hold:

i) Any torsion action of $G$ is $G$-equivariantly Morita equivalent to the trivial $G$-$C^*$-algebra $\mathbb{C}$.

ii) Every finite dimensional $G$-$C^*$-algebra is $G$-equivariantly isomorphic to a direct sum of $G$-$C^*$-algebras which are $G$-equivariantly Morita equivalent to the trivial $G$-$C^*$-algebra $\mathbb{C}$.

iii) Let $(A, \delta)$ be a finite dimensional $G$-$C^*$-algebra.

a) If $\delta$ is ergodic, then $A$ is simple. In other words, there are no non-simple ergodic finite dimensional $G$-$C^*$-algebras. In this case, we say that $\widehat{G}$ is permutation torsion-free; otherwise we say that $\widehat{G}$ has torsion of permutation type.
b) If $A$ is simple, then there exists a finite dimensional unitary representation $(H_u, u)$ of $\mathbb{G}$ such that $A \cong \mathcal{K}(H_u)$ as $\mathbb{G}$-$C^*$-algebras. In this case, we say that $\hat{\mathbb{G}}$ is projective torsion-free; otherwise we say that $\hat{\mathbb{G}}$ has torsion of projective type.

2.4.5 Example. The $SO(3)$-$C^*$-algebra $(\mathcal{M}_2(\mathbb{C}), \delta)$ introduced in Examples 2.4.3 is a torsion action of $SO(3)$ of projective type. Indeed, notice that $(\mathcal{M}_2(\mathbb{C}), \delta)$ is not $SO(3)$-equivariantly Morita equivalent to $\mathbb{C}$, as there are no irreducible 2-dimensional $SO(3)$-representations to implement this equivalence. Hence, $SO(3)$ is not torsion-free. Moreover, $(\mathcal{M}_2(\mathbb{C}), \delta)$ is the only, up to equivariant Morita equivalence, non-trivial torsion action of $SO(3)$. Similar considerations can be made for $SO_q(3)$ with $q \in (-1, 1) \setminus \{0\}$; namely $SO_q(3)$ is not torsion-free and $SO_q(3)$ has only one, up to equivariant Morita equivalence, non-trivial torsion action, which is of projective type (see for instance [45]).

2.5. Triangulated categories

In this section we recall elementary notions and results concerning triangulated categories. We refer to [30, 22, 29] or [27] for more details.

2.5.1 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category. A triangulated subcategory of $\mathcal{T}$ is an additive full subcategory $\mathcal{S}$ of $\mathcal{T}$ such that: i) every object of $\mathcal{T}$ isomorphic to an object in $\mathcal{S}$ is an object of $\mathcal{S}$, ii) $\Sigma(\mathcal{S}) = \mathcal{S}$ and iii) if $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma(X)$ is a distinguished triangle in $\mathcal{T}$ with $X, Y \in \text{Obj}(\mathcal{S})$, then $Z \in \text{Obj}(\mathcal{S})$.

2.5.2 Definition. Let $(\mathcal{T}, \Sigma, \Delta_\Sigma)$ be a triangulated category.

- A thick subcategory of $\mathcal{T}$ is a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ such that if $X, Y \in \text{Obj}(\mathcal{T})$ are such that $X \oplus Y \in \text{Obj}(\mathcal{S})$, then $X, Y \in \text{Obj}(\mathcal{S})$.

- A localizing subcategory of $\mathcal{T}$ is a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ such that every countable direct sum of objects of $\mathcal{S}$ is an object of $\mathcal{S}$.

- If $\mathcal{S} \subset \text{Obj}(\mathcal{T})$ is any class of objects in $\mathcal{T}$, we denote by $\langle \mathcal{S} \rangle$ the smallest triangulated subcategory of $\mathcal{T}$ such that: i) the objects of $\mathcal{S}$ are in $\langle \mathcal{S} \rangle$, ii) every countable direct sum of objects of $\mathcal{S}$ is an object of $\langle \mathcal{S} \rangle$ and iii) the subcategory $\langle \mathcal{S} \rangle$ is thick. In this case, we say that $\langle \mathcal{S} \rangle$ is the localizing subcategory of $\mathcal{T}$ generated by $\mathcal{S}$.

2.5.3 Remarks. 1. One can show that any localizing subcategory is automatically a thick subcategory (see Proposition 1.6.8 in [30]). Notice that [30] works in a more general framework. Namely, a localization subcategory is defined with respect to a given cardinal. We restrict our attention to the countable cardinal $\aleph_0$ as indicated in Section 2.1 and recall that we assume that all our additive categories admit countable direct sums.

2. With these definitions and the preceding remark, we observe that $\langle \mathcal{S} \rangle$ is the localizing subcategory generated by $\mathcal{S}$ meaning that $\langle \mathcal{S} \rangle$ is the smallest triangulated category containing the objects of $\mathcal{S}$ and stable with respect to countable direct sums.
2.5.4 Definition. Let \((\mathcal{I}, \Sigma, \Delta\Sigma)\) be a triangulated category. An ideal \(J\) in \(\mathcal{I}\) is a family of subgroups \(\{J(X, Y)\}_{X,Y \in \text{Obj}(\mathcal{I})} \subset \text{Hom}_{\mathcal{I}}(X, Y)\) such that \(\text{Hom}_{\mathcal{I}}(Z, W) \circ J(Y, Z) \circ \text{Hom}_{\mathcal{I}}(X, Y) \subset \text{Hom}_{\mathcal{I}}(X, W)\), for all \(X, Y, Z, W \in \text{Obj}(\mathcal{I})\).

We say that \(J\) is additive if \(J\) is compatible with countable direct sums meaning that \(J(\bigoplus X_i, Y) \cong \prod J(X_i, Y)\) through the canonical map \(\text{Hom}_{\mathcal{I}}(\bigoplus X_i, Y) \longrightarrow \prod \text{Hom}_{\mathcal{I}}(X_i, Y)\), for all \(X_i, Y \in \text{Obj}(\mathcal{I})\).

2.5.5 Definition. Let \((\mathcal{I}, \Sigma, \Delta\Sigma)\) and \((\mathcal{I}', \Sigma', \Delta\Sigma')\) be two triangulated categories. Let \(F : \mathcal{I} \longrightarrow \mathcal{I}'\) be an additive functor. We say that \(F\) is a triangulated functor if: i) \(F\) is stable meaning that \(F \circ \Sigma \cong \Sigma' \circ F\) and ii) \(F\) transforms distinguished triangles into distinguished triangles.

2.5.6 Definition. Let \((\mathcal{I}, \Sigma, \Delta\Sigma)\) be a triangulated category and \(\mathcal{A}\) an abelian category. Let \(F : \mathcal{I} \longrightarrow \mathcal{A}\) be a covariant (resp. contravariant) additive functor. We say that \(F\) is a homological (resp. cohomological) functor if for every distinguished triangle \(X \overset{u}{\longrightarrow} Y \overset{v}{\longrightarrow} Z \overset{w}{\longrightarrow} \Sigma(X)\) in \(\mathcal{I}\), the sequence \(F(X) \overset{F(u)}{\longrightarrow} F(Y) \overset{F(v)}{\longrightarrow} F(Z)\) (resp. \(F(Z) \overset{F(u)}{\longrightarrow} F(Y) \overset{F(v)}{\longrightarrow} F(X)\)) is exact in \(\mathcal{A}\).

2.5.7 Example. Given a triangulated functor \(F : \mathcal{I} \longrightarrow \mathcal{I}'\) (or a stable homological functor \(F : \mathcal{I} \longrightarrow \mathcal{A}\), where \(\mathcal{A}\) is a stable additive category) we have a natural additive full subcategory of \(\mathcal{I}\) defined by:

\[
\ker_{\text{Obj}}(F) := \{X \in \text{Obj}(\mathcal{I}) \mid F(X) \not\cong 0\}.
\]

It is straightforward to show that \(\ker_{\text{Obj}}(F)\) is a thick subcategory of \(\mathcal{I}\). Moreover, \(\ker_{\text{Obj}}(F)\) is a localizing subcategory whenever \(F\) is compatible with countable direct sums. Analogously, we have an obvious ideal in \(\mathcal{I}\) defined by:

\[
\ker_{\text{Hom}}(F)(X, Y) := \{f \in \text{Hom}_{\mathcal{I}}(X, Y) \mid F(f) \not\cong 0\},
\]

for all \(X, Y \in \text{Obj}(\mathcal{I})\). It is additive whenever \(F\) is compatible with countable direct sums.

We say that an ideal \(J\) in \(\mathcal{I}\) is homological if there exists a stable homological functor \(F : \mathcal{I} \longrightarrow \mathcal{A}\) such that \(J = \ker_{\text{Hom}}(F)\). Freyd’s theorem (see Remark 2.21 in [29] for the details) assures that a homological ideal can be realized as \(\ker_{\text{Hom}}(F)\) with \(F\) triangulated.

2.5.8 Definition. Let \((\mathcal{I}, \Sigma, \Delta\Sigma)\) be a triangulated category and \(\mathcal{S}\) a thick subcategory of \(\mathcal{I}\). The right orthogonal complement of \(\mathcal{S}\) is the thick subcategory of \(\mathcal{I}\) defined by \(\mathcal{S}^+ := \{X \in \text{Obj}(\mathcal{I}) \mid \text{Hom}_{\mathcal{I}}(S, X) = 0 \forall S \in \text{Obj}(\mathcal{S})\}\). The left orthogonal complement of \(\mathcal{S}\) is the thick subcategory of \(\mathcal{I}\) defined by \(\mathcal{S}^- := \{X \in \text{Obj}(\mathcal{I}) \mid \text{Hom}_{\mathcal{I}}(S, X) = 0 \forall S \in \text{Obj}(\mathcal{S})\}\).

2.5.9 Remark. If \(\mathcal{S}\) is a localizing subcategory of \(\mathcal{I}\), so are \(\mathcal{S}^+\) and \(\mathcal{S}^-\).

2.5.10 Definition. Let \((\mathcal{I}, \Sigma, \Delta\Sigma)\) be a triangulated category and \(\mathcal{L}, \mathcal{N}\) two thick subcategories of \(\mathcal{I}\). We say that \((\mathcal{L}, \mathcal{N})\) is a complementary pair of thick subcategories if: i) \(\mathcal{L} \subset \mathcal{N}^+\) and ii) for every object \(X \in \text{Obj}(\mathcal{I})\) there exists a distinguished triangle of the form \(L \overset{u}{\longrightarrow} X \overset{v}{\longrightarrow} N \overset{w}{\longrightarrow} \Sigma(L)\), for some \(L \in \text{Obj}(\mathcal{L})\) and \(N \in \text{Obj}(\mathcal{N})\). Such a distinguished triangle is called \((\mathcal{L}, \mathcal{N})\)-triangle associated to \(X\) in the sequel.
The following are fundamental properties of complementary pairs (see for instance \([27]\) for a proof).

2.5.11 Theorem (Bousfield localization). Let \((\mathcal{T}, \Sigma, \Delta_{\Sigma})\) be a triangulated category and \((\mathcal{L}, \mathcal{N})\) a complementary pair of thick subcategories.

i) \(\mathcal{L} = \mathcal{N}^\perp\) and \(\mathcal{N} = \mathcal{L}^\perp\).

ii) Given an object \(X \in \text{Obj}(\mathcal{T})\), the associated \((\mathcal{L}, \mathcal{N})\)-triangle \(L \xrightarrow{u} X \xrightarrow{v} N \xrightarrow{w} \Sigma(L)\) is unique up to canonical isomorphism and depends functorially on \(X\). In particular, its entries define functors \(L : \mathcal{T} \to \mathcal{L}\) and \(N : \mathcal{T} \to \mathcal{N}\), which are unique up to natural isomorphism.

iii) The functors \(L, N : \mathcal{T} \to \mathcal{T}\) are triangulated.

2.5.12 Note. In the situation of the previous theorem, let \(L(X) \xrightarrow{u} X \xrightarrow{v} N(X) \xrightarrow{w} \Sigma(L(X))\) be the \((\mathcal{L}, \mathcal{N})\)-distinguished triangle associated to an object \(X \in \text{Obj}(\mathcal{T})\). The homomorphism \(u\) is called Dirac homomorphism for \(X\) (compare with Note 5.1.2). Given a functor \(F : \mathcal{T} \to \mathcal{C}\), where \(\mathcal{C}\) is some category, we define the localization of \(F\) with respect to \((\mathcal{L}, \mathcal{N})\) to be the functor \(\mathbb{L}F := F \circ L\) and \(\mathbb{r}_X := F(u)\), for all \(X \in \text{Obj}(\mathcal{T})\). The latter yields a natural transformation \(\eta : \mathbb{L}F \to F\).

2.5.13 Theorem (Complementary pairs from \(\mathcal{F}\)-projective objects, \([26]\)). Let \((\mathcal{T}, \Sigma, \Delta_{\Sigma})\) be a triangulated category. Let \(F : \mathcal{T} \to \mathcal{T}'\) be a triangulated functor compatible with countable direct sums. We put \(\mathcal{J} := \ker_{\text{Hom}}(F)\).

i) If \(\mathcal{J}\) has enough \(\mathcal{F}\)-projective objects, then \((\langle p_{\mathcal{J}} \rangle, \ker_{\text{Obj}}(F))\) is a complementary pair of localizing subcategories in \(\mathcal{T}\), where \(p_{\mathcal{J}}\) denotes the class of \(\mathcal{F}\)-projective objects in \(\mathcal{T}\).

ii) If \(F\) admits a left adjoint \(F^\leftarrow : \mathcal{T}' \to \mathcal{T}\), \(\mathcal{T}\) has enough \(\mathcal{F}\)-projective objects and consequently \(\langle F^\leftarrow(\text{Obj}(\mathcal{T}'))\rangle, \ker_{\text{Obj}}(F)\) is a complementary pair of localizing subcategories in \(\mathcal{T}\).

2.5.14 Definition. Let \((\mathcal{T}, \Sigma, \Delta_{\Sigma})\) be a triangulated category. An object \(X \in \text{Obj}(\mathcal{T})\) is called compact if the functor \(\text{Hom}_\mathcal{T}(X, \cdot)\) commutes with countable direct sums. The full (thick) subcategory of compact objects in \(\mathcal{T}\) is denoted by \(\mathcal{T}^c\). We say that \(\mathcal{T}\) is compactly generated if there is a countable set of compact objects \(S \subset \text{Obj}(\mathcal{T}^c)\) generating \(\mathcal{T}\), that is, such that \(\mathcal{T} = \langle S \rangle\).

The following is a fundamental result in the theory of triangulated categories (see for instance \([30]\) or \([27]\) for a proof). Notice that for our purpose we restrict ourselves to work with the countable cardinality \(\aleph_0\).

2.5.15 Theorem (Brown representability). Let \((\mathcal{T}, \Sigma, \Delta_{\Sigma})\) be a triangulated category. Let \(S\) be a countable set of compact objects in \(\mathcal{T}\) such that \(\mathcal{T} = \langle S \rangle\). Let \(F : \mathcal{T} \to \text{Ab}\) be an additive, contravariant functor. Then \(F\) is representable if and only if: i) \(F\) is cohomological and ii) \(F\) is compatible with countable direct sums.
2.5.16 Corollary. Let \((\mathcal{F}, \Sigma, \Delta\Sigma)\) be a triangulated category. If \(S\) is a countable set of compact objects in \(\mathcal{F}\), then \(\langle S \rangle, \langle S \rangle^\perp\) is a complementary pair of localizing subcategories in \(\mathcal{F}\).

Proof. First of all, it is clear that both \(\langle S \rangle\) and \(\langle S \rangle^\perp\) are localizing subcategories in \(\mathcal{F}\) and we have \(\langle S \rangle \subset \langle S \rangle^\perp\). It remains to construct the \((\langle S \rangle, \langle S \rangle^\perp)\)-triangle. Let \(X\) be an object of \(\mathcal{F}\) and put \(\mathcal{L} := \langle S \rangle\). Consider the restriction of the functor \(\text{Hom}_\mathcal{F}(\cdot, X)\) to the subcategory \(\mathcal{L} \subset \mathcal{F}\). It is an additive, contravariant, cohomological functor which is compatible with countable direct sums. By Brown representability theorem, \(\text{Hom}_\mathcal{F}(\cdot, X)\) is representable. This means that there exists an object \(L_X \in \text{Obj}(\mathcal{L})\) such that \(\text{Hom}_\mathcal{F}(Y, X) \cong \text{Hom}_\mathcal{F}(Y, L_X)\), naturally for all \(Y \in \text{Obj}(\mathcal{L})\). In particular, taking \(Y := L_X\), we deduce that there exists a homomorphism \(\varepsilon_X : L_X \to X\) in \(\mathcal{F}\) (the image of \(\text{id}_{L_X}\) by the representability isomorphism). Next, we complete this homomorphism into a distinguished triangle in \(\mathcal{F}\), say \(L_X \xrightarrow{\varepsilon_X} X \to C_{\varepsilon_X} \to \Sigma(L_X)\), where \(C_{\varepsilon_X}\) is the cone of \(\varepsilon_X\). Given any object \(Y \in \text{Obj}(\mathcal{F})\), consider the long exact sequence induced by applying the homological functor \(\text{Hom}_\mathcal{F}(Y, \cdot)\) to this distinguished triangle:

\[
\cdots \to \text{Hom}_\mathcal{F}(Y, L_X) \xrightarrow{(\varepsilon_X)_*} \text{Hom}_\mathcal{F}(Y, X) \to \text{Hom}_\mathcal{F}(Y, C_{\varepsilon_X}) \to \cdots
\]

In particular, if \(Y \in \text{Obj}(\mathcal{L})\), then we remark that \((\varepsilon_X)_*\) is the isomorphism of the representability theorem above because \((\varepsilon_X)_*\) is, by construction, the counit of the adjunction between the inclusion functor \(\mathcal{L} \to \mathcal{F}\) and the functor \(\mathcal{F} \to \mathcal{L}\) given by \(X \mapsto L_X\). In conclusion, \(\text{Hom}_\mathcal{F}(Y, C_{\varepsilon_X}) \cong (0)\), for all \(Y \in \text{Obj}(\mathcal{L})\), which means that \(C_{\varepsilon_X} \in \mathcal{L}^\perp\).

2.5.17 Corollary. Let \((\mathcal{F}, \Sigma, \Delta\Sigma)\) be a triangulated category. \(\mathcal{F}\) is compactly generated if and only if there exists a countable set of compact objects \(S \subset \text{Obj}(\mathcal{F}^c)\) such that for all \(X \in \text{Obj}(\mathcal{F})\), \(\text{Hom}_\mathcal{F}(\Sigma^n(S), X) = (0)\) for all \(n \in \mathbb{Z}\) and all \(S \in S\) implies \(X \cong 0\).

Proof. Assume that \(\mathcal{F}\) is compactly generated, that is, there exists a countable set of compact objects \(S \subset \text{Obj}(\mathcal{F}^c)\) such that \(\mathcal{F} = \langle S \rangle\). Given an object \(X \in \text{Obj}(\mathcal{F})\), consider \(X^\perp := \{Y \in \text{Obj}(\mathcal{F}) \mid \text{Hom}_\mathcal{F}(\Sigma^n(Y), X) = (0), \forall n \in \mathbb{Z}\}\), which defines a localizing subcategory of \(\mathcal{F}\). Next, if \(S \subset X^\perp\), then \(X^\perp\) must contain \(\langle S \rangle = \mathcal{F}\). In other words, we have \(X \in \mathcal{F} \cap X^\perp\), which implies \(X \cong 0\). Conversely, assume that for all \(X \in \text{Obj}(\mathcal{F})\), \(\text{Hom}_\mathcal{F}(\Sigma^n(S), X) = (0)\) for all \(n \in \mathbb{Z}\) and all \(S \in S\) implies \(X \cong 0\). Given \(X \in \text{Obj}(\mathcal{F})\), we have already seen in the previous corollary that \(\text{Hom}_\mathcal{F}(Y, C_{\varepsilon_X}) \cong (0)\), for all \(Y \in \text{Obj}(\langle S \rangle)\). Hence, by hypothesis we obtain that \(C_{\varepsilon_X} \cong 0\), which implies that \(X \cong L_X \in \langle S \rangle\). Hence \(\mathcal{F} = \langle S \rangle\).

3. Projective representation theory for compact quantum groups

In this section, we develop the theory of projective representations for compact quantum groups based on the notion of (measurable) 2-cocycle. We obtain a projective representation theory analogous to the one for classical compact groups. Namely, given a unitary 2-cocycle, we construct the associated projective regular representation containing all irreducible \(\Omega\)-twisted representations and reaching thus a twisted version of the Peter-Weyl theorem. The
content of this section concerns a particular case of the more general framework developed in [14] by the first author, but we give more attention here to the associated $C^*$-algebraic theory.

3.1. Projective representations of compact quantum groups

3.1.1 Definition. Let $G$ be a compact quantum group. A measurable left projective representation of $G$ consists of a Hilbert space $H$ and a (measurable) right coaction $\delta : \mathcal{B}(H) \to \mathcal{B}(H) \otimes L^\infty(G)$. A continuous left projective representation of $G$ consists of a Hilbert space $H$ and a (continuous) right coaction $\delta : \mathcal{K}(H) \to \mathcal{K}(H) \otimes C(G)$.

Note that any continuous coaction $\delta : \mathcal{K}(H) \to \mathcal{K}(H) \otimes C(G)$ extends uniquely to a (normal) coaction $\delta : \mathcal{B}(H) = M(\mathcal{K}(H)) \to \mathcal{B}(H) \otimes L^\infty(G)$. Hence a continuous projective representation can be seen as a special type of measurable projective representation. On the other hand, any measurable left projective representation $\delta$ on a finite dimensional Hilbert space $H$ is automatically continuous: We can endow $\mathcal{B}(H)$ with a Hilbert space structure for which $\delta$ becomes a finite dimensional unitary representation, hence its matrix coefficients lie in $Pol(G) \subset C(G)$. On the other hand, it is not true that a general measurable left projective representation is automatically continuous, as we will comment on later.

3.1.2 Remark. There is also an obvious notion of right projective representation. Identifying $j : \mathcal{B}(H)^{op} \cong \mathcal{B}(\overline{H})$ via $j(x) = \overline{x}$, there is a natural correspondence between left and right measurable/continuous projective representations by $\delta \leftrightarrow \overline{\delta} := \Sigma(j \otimes R)\delta j^{-1}$, so we consider $\overline{\delta}$ as a right continuous projective representation of $G$ on $\overline{H}$. More directly, one can also view a left projective representation of $G$ as a right projective representation of $G^{op}$.

Recall from the introduction that any continuous action of a classical compact group $G$ on $\mathcal{K}(H)$, for some Hilbert space $H$, is implemented by an $\omega$-representation of $G$ on $H$, where $\omega$ is a measurable 2-cocycle on $G$. The same in fact holds for measurable actions of $G$ on $\mathcal{B}(H)$. The main goal of this section will be to show that these statements are still true for compact quantum groups. This will in particular justify the terminology projective torsion action (recall Theorem 2.4.4).

3.1.3 Definition. Let $\delta$ be a measurable left projective representation. We say that $\delta$ is cleft if there exists a unitary $u \in \mathcal{B}(H) \otimes L^\infty(G)$ such that $\delta(a) = u(a \otimes 1)u^*$. Similarly, we say that a measurable projective representation $\delta$ is cleft if there exists a unitary $u \in \mathcal{B}(H) \otimes L^\infty(G)$ such that $\delta(a) = \Sigma(u^*(a \otimes 1)u)$. Clearly $\delta$ is cleft if and only if $\overline{\delta}$ is cleft. We call $u$ an implementing unitary.

We will show in Theorem 3.1.15 that all measurable projective representations are cleft. We start with the following elementary well-known result.

3.1.4 Lemma. Let $M$ be a von Neumann algebra and $k \in \mathbb{N}$. If $\{e_{i,j}\}_{i,j=1,...,k}$ and $\{f_{i,j}\}_{i,j=1,...,k}$ are the matrix units of two copies of $\mathcal{M}_k(\mathbb{C})$ inside $M$, then there exists a unitary $U$ in $M$ such that $Ue_{i,j}U^* = f_{i,j}$, for all $i,j = 1,...,k$. Moreover, $U$ is unique up to a unitary in $\{e_{i,j}\}_{i,j=1,...,k} \cap M$. 

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Proof. Note that $e_{11}$ and $f_{11}$ have central support 1 and $k[e_{11}] = [1] = k[f_{11}]$ in $K_0(M)$. Hence $e_{11}$ and $f_{11}$ are Murray-von Neumann equivalent by a partial isometry $u$. Then $U = \sum_s f_{s1}ue_{1s}$ is the sought-after unitary. The stated uniqueness of $U$ is clear.

3.1.5 Theorem. Let $G$ be a compact quantum group. Then any finite dimensional projective representation is cleft.

Proof. Let $\delta : B(H) \to B(H) \otimes L^\infty(G)$ be a right coaction with $H$ finite dimensional. Applying Lemma 3.1.4 with respect to the matrix units $\delta(e_{ij})$ and $e_{ij} \otimes 1$ provides a unitary $u \in B(H) \otimes L^\infty(G)$ implementing the coaction $\delta$, hence $\delta$ is cleft.

To extend this result to arbitrary projective representations, we first take a further look at implementing unitaries. Note that if $\delta : B(H) \to B(H) \otimes L^\infty(G)$ is a measurable projective representation implemented by $u \in B(H) \otimes L^\infty(G)$, then

$$(\delta \otimes id)(a) = (\delta \otimes id) \left( u(a \otimes 1)u^* \right) = u_{12} \left( u(a \otimes 1)u^* \right)_{13} u_{12} = u_{12} u_{13} (a \otimes 1 \otimes 1) u_{13} u_{12}^*,$$

while

$$(id \otimes \Delta)(a) = (id \otimes \Delta) \left( u(a \otimes 1)u^* \right) = (V_G)_{23} \left( u(a \otimes 1)u^* \right)_{12} (V_G)^*_2,$$

$$= (V_G)_{23} u_{12} (a \otimes 1 \otimes 1) u_{12}^* (V_G)^*_2.$$

Hence by the coaction property $(\delta \otimes id) \delta = (id \otimes \Delta) \delta$, we obtain $u_{12} u_{13} (a \otimes 1 \otimes 1) u_{13} u_{12}^* = (V_G)_{23} u_{12} (a \otimes 1 \otimes 1) u_{12}^* (V_G)^*_2$, for all $a \in B(H)$, so there exists a unitary $X \in B(L^2(G)) \otimes L^\infty(G)$ such that $(V_G)_{23} u_{12} = u_{12} u_{13} X_{23}$. This relation allows to write the following: $(id \otimes \Delta)(u) = (V_G)_{23} u_{12} (V_G)^*_2 = u_{12} u_{13} X_{23} (V_G)^*_2$.

Since the left hand side is an element of $B(H) \otimes L^\infty(G) \otimes L^\infty(G)$ and $u$ is a unitary, then $	ilde{\Omega} := u_{13} u_{12} \Delta_{23}(u_{12}) \in C \otimes L^\infty(G) \otimes L^\infty(G)$ and $\tilde{\Omega} = 1 \otimes \Omega$ with $\Omega \in L^\infty(G) \otimes L^\infty(G)$ unitary. Moreover, we have $\Delta_{23}(u_{12}) \cdot \Omega^*_2 = u_{12} u_{13}$. Applying this equation to the identity $(u_{12} u_{13} u_{14}) u_{12} = u_{12} (u_{13} u_{14})$, we obtain that $\Omega^*$ satisfies the 2-cocycle relation:

$$(\Delta \otimes id)(\Omega^*) \Omega^*_{12} = (id \otimes \Delta)(\Omega^*) \Omega^*_{23}.$$

Let us formalize this in the following definition.

3.1.6 Definition. Let $G$ be a compact quantum group. A (measurable, unitary) 2-cocycle on $G$ is a unitary element $\Omega \in L^\infty(G) \otimes L^\infty(G)$ such that:

$$(\Omega \otimes 1)(\Delta \otimes id)(\Omega) = (1 \otimes \Omega)(id \otimes \Delta)(\Omega).$$

Two 2-cocycles $\Omega$ and $\Omega'$ on $G$ are said to be coboundary equivalent if there exists a unitary $X \in L^\infty(G)$ such that $\Omega' = (X^* \otimes X^*) \Omega \Delta(X)$.

3.1.7 Note. If one replaces $L^\infty(G)$ by $C(G)$ or $Pol(G)$, then we define analogously a continuous or algebraic 2-cocycle on $G$, respectively.
Given a 2-cocycle $\Omega$ on $G$ we can define the following linear maps

$$
\Omega \Delta : L^\infty(G) \rightarrow L^\infty(G) \otimes L^\infty(G), \quad \Delta \Omega^* : L^\infty(G) \rightarrow L^\infty(G) \otimes L^\infty(G)
$$

$$
x \mapsto \Omega \Delta (x) := \Omega \cdot \Delta (x) \quad x \mapsto \Delta \Omega^*(x) := \Delta (x) \cdot \Omega^*.
$$

We call $\Omega \Delta$ and $\Delta \Omega^*$ the right/left twisted pseudo co-multiplication on $G$ with respect to $\Omega$ or the right/left $\Omega$-pseudo co-multiplication on $G$; respectively. Observe that both $\Omega \Delta$ and $\Delta \Omega^*$ are linear maps satisfying the following identities:

1. $\Omega \Delta(xy) = \Omega \Delta(x) \Delta(y)$ and $\Delta \Omega^*(xy) = \Delta(x) \Delta \Omega^*(y)$, for all $x, y \in L^\infty(G)$;
2. $\Omega \Delta(x)^* \Delta(y) = \Delta(x^* y)$ and $\Delta \Omega^*(x) \Delta \Omega^*(y)^* = \Delta(xy^*)$, for all $x, y \in L^\infty(G)$;
3. $(\Omega \Delta \otimes \text{id}) \Omega \Delta = (\text{id} \otimes \Omega \Delta) \Omega \Delta$ and $(\Delta \Omega^* \otimes \text{id}) \Delta \Omega^* = (\text{id} \otimes \Delta \Omega^*) \Delta \Omega^*$; and
4. $\text{S}_{\text{pair}}^\text{weak} \{ \Omega \Delta(x)(y \otimes z) \mid x, y, z \in L^\infty(G) \} = L^\infty(G) \otimes L^\infty(G)$ and $\text{S}_{\text{pair}}^\text{weak} \{ (y \otimes z) \Delta \Omega^*(x) \mid x, y, z \in L^\infty(G) \} = L^\infty(G) \otimes L^\infty(G)$.

These identities are obtained after a straightforward computation. Hence, $\Omega \Delta$ and $\Delta \Omega^*$ are in particular von Neumann algebraic analogues of module coalgebras, and form concrete instances of the notion of Galois co-object introduced in [12].

**3.1.8 Definition.** Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle on $G$. A (measurable) unitary $\Omega$-representation of $G$ on a Hilbert space $H$ is a unitary element $u \in \mathcal{B}(H) \otimes L^\infty(G)$ such that $(\text{id} \otimes \Omega \Delta)(u) = u_{12}u_{13}$. A (measurable) unitary $\Omega^*$-representation on $H$ is a unitary element $u \in \mathcal{B}(H) \overline{\otimes} L^\infty(G)$ satisfying $(i\otimes \Delta \Omega^*) (u) = u_{12}u_{13}$.

The elements of the form $u_{\xi, \zeta} := (\omega_{\xi, \zeta} \otimes 1)(u) \in L^\infty(G)$ with $\xi, \zeta \in H$ are called matrix coefficients of $u$. In particular, if we fix a basis $\{ \xi_i \}_{i=1, \ldots, \dim(H)}$ in $H$, we write $u_{ij} := u_{\xi_i, \xi_j}$. Then for an $\Omega$-representation $u$ we obtain the usual corepresentation identities $\Omega \Delta(u_{ij}) = \sum_{k=1}^{\dim(H)} u_{ik} \otimes u_{kj}$, for all $i, j = 1, \ldots, n_x$, where the sum converges in (say) the strong operator topology. The same conclusion holds for $\Omega^*$-representations.

Note that if the $u_i$ are measurable unitary $\Omega$-representations on Hilbert spaces $H_i$, then clearly $u = \bigoplus_i u_i$ is a measurable unitary $\Omega$-representation on $H = \bigoplus_i H_i$, called the direct sum $\Omega$-representation.

Summarizing the discussion following Theorem 3.1.5, we obtain the following result.

**3.1.9 Proposition.** Let $\delta : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \otimes L^\infty(G)$ be a cleft measurable projective representation. Then there exists a 2-cocycle $\Omega \in L^\infty(G) \otimes L^\infty(G)$ and a unitary $\Omega^*$-representation $u \in \mathcal{B}(H) \otimes L^\infty(G)$ such that $\delta(a) = u(a \otimes 1)u^*$.

Conversely, if $u$ is a unitary $\Omega^*$-representation, we obtain a measurable right coaction:

$$
\delta_u : \mathcal{B}(H) \rightarrow \mathcal{B}(H) \otimes L^\infty(G), \quad \delta_u(a) = u(a \otimes 1)u^*, \quad a \in \mathcal{B}(H), \quad (3.1)
$$

where the coaction property follows immediately from the $\Omega^*$-representation property of $u$. Similarly, any unitary $\Omega$-representation $u$ provides a measurable left coaction:

$$
\delta_u : \mathcal{B}(H) \rightarrow L^\infty(G) \overline{\otimes} \mathcal{B}(H), \quad \delta_u(a) = \Sigma(a^*(a \otimes 1)u), \quad a \in \mathcal{B}(H). \quad (3.2)
$$
3.1.10 Note. If \( v \) is another implementing unitary for \( \delta \), we see that \( v^* u(a \otimes 1) u^* v = a \otimes 1 \) for all \( a \in \mathcal{B}(H) \), hence \( v = u(1 \otimes X) \) for some unitary \( X \in L^\infty(G) \). If \( v \) has associated 2-cocycle \( \Omega' \), it then follows from the projective representation identities for \( v \) and \( u \) that \( \Omega' = (X^* \otimes X^*) \Omega \Delta(X) \), so \( \Omega \) and \( \Omega' \) are cohomologous.

3.1.11 Lemma. Let \( \delta : \mathcal{B}(H) \to \mathcal{B}(H) \otimes L^\infty(G) \) be a measurable projective representation on \( H \). Assume that \( p \in \mathcal{B}(H) \) is a coinvariant projection, and let \( \delta_p : \mathcal{B}(pH) \to \mathcal{B}(pH) \otimes L^\infty(G) \) be the restriction of \( \delta \). Then \( \delta \) is cleft if and only if \( \delta_p \) is cleft. Moreover, if \( \delta_p \) is implemented by an \( \Omega^* \)-representation \( v \) for some 2-cocycle \( \Omega \), then \( \delta \) can be implemented by some \( \Omega^* \)-representation \( u \) such that \( u(p \otimes 1) = (p \otimes 1) u = v \).

Proof. Assume that \( \delta \) is cleft, with implementing unitary \( u \). Then as \( \delta(p) = p \otimes 1 \), it follows immediately that \( u \) commutes with \( p \), hence \( v = (p \otimes 1) u = u(p \otimes 1) \) is a unitary in \( \mathcal{B}(pH) \otimes L^\infty(G) \) implementing \( \delta_p \). This shows that \( \delta_p \) is cleft.

Conversely, assume that \( \delta_p \) is cleft. Then \( \mathcal{B}(H) \) contains a minimal projection \( e \) such that \( \delta(e) \) and \( e \otimes 1 \) are unitarily equivalent in \( \mathcal{B}(H) \otimes M \). The same reasoning as in Theorem 3.1.5 then shows that \( \delta \) is cleft.

Assume now that \( v \) is a unitary in \( \mathcal{B}(pH) \otimes L^\infty(G) \) implementing \( \delta_p \) and \( \tilde{u} \in \mathcal{B}(H) \otimes L^\infty(G) \) a unitary implementing \( \delta \). Assume that \( \Omega \) is the 2-cocycle associated to \( v \), and let \( \hat{\Omega} \) be the 2-cocycle associated to \( \tilde{u} \). Then it is easily seen that \( (p \otimes 1) \tilde{u} \) is a \( \hat{\Omega}^* \)-representation implementing \( \delta_p \). Hence \( \Omega \) and \( \hat{\Omega} \) are cohomologous, say \( \hat{\Omega} = (X^* \otimes X^*) \Omega \Delta(X) \) where \( \tilde{u}(p \otimes 1) = v(1 \otimes X) \).

Hence \( u = \tilde{u}(1 \otimes X^*) \) is an \( \Omega^* \)-representation implementing \( \delta \) with \( u(1 \otimes p) = v \).

We can now generalize Theorem 3.1.5 to the infinite dimensional setting for continuous projective representations. Although we will later show that this holds in general, independently of this result, it seems worthwhile to present the more direct argument for this case.

3.1.12 Theorem. Let \( G \) be a compact quantum group. Then any continuous projective representation is cleft.

Proof. Assume that \( \delta : \mathcal{K}(H) \to \mathcal{K}(H) \otimes C(G) \) is a continuous projective representation. Recall from Remark 2.3.4 that \( \mathcal{K}(H)^\delta \) acts non-degenerately on \( H \). Hence, as \( \mathcal{K}(H)^\delta \) is a separable \( C^* \)-algebra of compact operators, we can find an ascending sequence \( p_i \) of finite rank projections in \( \mathcal{K}(H)^\delta \) converging strongly to 1. By Theorem 3.1.5 and Lemma 3.1.11, we can find a 2-cocycle \( \Omega \in L^\infty(G) \otimes L^\infty(G) \) and \( \Omega^* \)-representations \( u_i \in \mathcal{K}(p_i H) \otimes L^\infty(G) \) such that \( u_i \) implements \( \delta_{p_i} \) and such that \( u_i(p_j \otimes 1) = u_j \) for \( j \leq i \). Then clearly the \( u_i \) converge \( \sigma \)-strongly to a unitary \( u \in \mathcal{B}(H) \otimes L^\infty(G) \), and \( u \) is a unitary \( \Omega^* \)-representation implementing \( \delta \).

Let now \( \delta : \mathcal{B}(H) \to \mathcal{B}(H) \otimes L^\infty(G) \) be a measurable projective representation of \( G \), and consider in this setting the averaging operator:

\[
E_\delta : \mathcal{B}(H) \to \mathcal{B}(H), \quad x \mapsto (id \otimes h_G) \delta(x).
\]
Then $E_\theta$ is a normal conditional expectation on $\mathcal{B}(H)^\theta$, and it is well-known that then necessarily $\mathcal{B}(H)^\theta$ is a (possibly infinite) direct sum of type I factors. In particular, $\mathcal{B}(H)^\theta$ contains a minimal projection $p$, and $\delta_q$ is then an irreducible projective representation, meaning $\mathcal{B}(pH)^\theta = \mathbb{C}p$. This leads to the following:

3.1.13 Corollary. If $G$ is of Kac type, then all measurable projective representations of $G$ are continuous, hence cleft.

Proof. Let $\delta : \mathcal{B}(H) \to \mathcal{B}(H) \bar{\otimes} L^\infty(G)$ be a measurable projective representation of $G$. If $p_i$ are a maximal set of orthogonal minimal projections in $\mathcal{B}(H)^\delta$, then each $\delta_{p_i}$ is an irreducible projective representation. By [14, Corollary 5.2], it follows that each $p_i H$ is finite dimensional, and in particular each $\delta_{p_i}$ is continuous. Since $\mathcal{K}(H)$ is the directed union of all $\mathcal{K}(qH)$ with $q$ a finite sum of $p_i$, it follows that $\delta$ is continuous, and in particular cleft.

Recall now that any compact quantum group allows a maximal compact quantum subgroup of Kac type [37, Appendix A.1]. We will slightly modify this construction as follows.

3.1.14 Lemma. Let $G$ be a compact quantum group. Let $p$ be the maximal properly infinite projection of $M = L^\infty(G)$, and put $q = 1 - p$. Then $qM$ defines a von Neumann algebraic compact quantum group of Kac type with coproduct $\Delta_q(x) = \Delta(x)(q \otimes q)$.

We call $qM$ the normal Kac quotient of $M$.

Proof. Note that $pM = \{x \in M \mid \tau(x^* x) = 0 \text{ for all normal tracial states } \tau\}$. Since the convolution product of two normal tracial states is still a normal tracial state, it follows that $\Delta$ descends to a coproduct $\Delta_q$ on $qM = M/pM$. We are to show that $qM$ has an invariant normal tracial state.

Note first that $p$ is invariant under the scaling group and the unitary antipode. It follows that $pM \cap Pol(G)$ is preserved under the antipode, hence $Pol(\mathbb{H}) := qPol(G) \cong Pol(G)/Pol(G) \cap pM$ is a Hopf $*$-algebra with respect to $\Delta_q$. As it spanned by matrix coefficients of unitary corepresentations, it defines indeed a compact quantum group $\mathbb{H}$. As $Pol(\mathbb{H})$ has a separating family of tracial states by construction, it follows that $\mathbb{H}$ must be of Kac type.

Write $\pi : Pol(G) \to Pol(\mathbb{H})$ for the natural quotient map, and consider $N = L^\infty(\mathbb{H})$. As $\mathbb{H}$ is a compact quantum subgroup of $G$, we have a normal coaction $\alpha : M \to M \bar{\otimes} N$ restricting to $(id \otimes \pi)\Delta$ on $Pol(G)$. As the Haar state of $\mathbb{H}$ is tracial, $\alpha$ descends to a normal coaction $\alpha_q : qM \to qM \bar{\otimes} N$. Moreover, we have

$$\Delta_q \otimes id)\alpha_q = (id \otimes \alpha_q)\Delta_q.$$  \hfill (3.3)

Let $E : qM \to qM$ be given by $E(x) = (id \otimes h_{\mathbb{H}})\alpha_q(x)$, for all $x \in qM$, where $h_{\mathbb{H}} \in N_*$ is the Haar state for $\mathbb{H}$. Since $E$ is normal and $E(x) = h_{\mathbb{H}}(x)1$ for $x \in Pol(\mathbb{H})$, we have by $\sigma$-weak density of $Pol(\mathbb{H})$ in $qM$ that there exists a normal state $h_{qM}$ on $qM$ with $E(x) = h_{qM}(x)1$, for all $x \in qM$. From (3.3), it then easily follows that $h_{qM}$ is left invariant, i.e. $(id \otimes h_{qM})(\Delta_q(x)) = h_{qM}(x)$, for all $x \in qM$. As the unitary antipode of $M$ descends to $qM$, we also have that $qM$ has a right invariant normal state, hence $(qM, \Delta_q)$ defines a compact quantum group in its own right. It is then clear that $(qM, \Delta_q)$ in fact equals $L^\infty(\mathbb{H})$, and in particular defines a compact quantum group of Kac type. \hfill ■
We are now ready to prove the main theorem of this section.

3.1.15 Theorem. Let $\mathbb{G}$ be a compact quantum group. Then all projective representations of $\mathbb{G}$ are cleft.

Proof. Let $\delta : \mathcal{B}(H) \to \mathcal{B}(H) \otimes L^\infty(\mathbb{G})$ be a measurable projective representation of $\mathbb{G}$. To show that $\delta$ is cleft, we may by the discussion before Corollary 3.1.13 assume that $\delta$ is ergodic. As we are assuming that $C(\mathbb{G})$ is separable, it then follows in particular that $H$ is separable. We can then moreover find a unique normal state $\Phi$ on $\mathcal{B}(H)$ such that $\Phi(x)1 = E_\delta(x)$ for all $x \in \mathcal{B}(H)$. We necessarily have that $\Phi \leq \text{Tr}$, with $\text{Tr}$ the usual trace of $\mathcal{B}(H)$.

Let $e$ be a minimal projection in $\mathcal{B}(H)$. It is sufficient to show that $\delta(e)$ and $e \otimes 1$ are unitarily equivalent in $\mathcal{B}(H) \otimes M$, as we can then proceed as in Theorem 3.1.5.

Let $p$ be the (central) maximal properly infinite projection of $L^\infty(\mathbb{G})$, and put $q = 1 - p$. We are to show that $\delta(e)(1 \otimes p) \sim e \otimes p$ and $\delta(e)(1 \otimes q) \sim e \otimes q$, where $\sim$ denotes Murray-von Neumann equivalence.

To show that $\delta(e)(1 \otimes p) \sim e \otimes p$, let us show first that $\delta(e)(1 \otimes p)$ is properly infinite. Assume this were not the case. Then there exists a non-zero semi-finite normal weight $\tau$ on $pL^\infty(\mathbb{G})$ with $(\text{Tr} \otimes \tau)(\delta(e)(1 \otimes p)) < \infty$. But the left hand side is larger than $(\Phi \otimes \tau)(\delta(e)(1 \otimes p)) = \Phi(e)\tau(p)$, which is infinite since $p$ is maximally properly infinite and $\Phi$ is faithful. This contradiction shows that $\delta(e)(1 \otimes p)$ is necessarily properly infinite. Since $\delta(e)(1 \otimes w) \neq 0$ for any non-zero $w$, again using faithfulness of $\Phi$, it follows that the properly infinite projections $\delta(e)(1 \otimes p)$ and $e \otimes p$ have the same central support $1 \otimes p$, and hence $\delta(e)(1 \otimes p) \sim e \otimes p$.

To show that also $\delta(e)(1 \otimes q) \sim e \otimes q$, we note that $x \mapsto \delta(x)(1 \otimes q)$ defines a projective representation of the Kac type compact quantum group $(qL^\infty(\mathbb{G}), \Delta_q)$. By Corollary 3.1.13 we have that this projective representation is necessarily cleft, which implies that $\delta(e)(1 \otimes q) \sim e \otimes q$. $\blacksquare$

3.2. Measurable $\Omega$-representations

We recall some of the results of [14], where by Theorem 3.1.15 we can restrict to the cleft case.

3.2.1 Definition. Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a 2-cocycle on $\mathbb{G}$. Let $(u, H_u)$ be a measurable unitary $\Omega$-representation of $\mathbb{G}$. A (closed) subspace $E \subset H$ is called $u$-invariant if $(p_E \otimes 1)u(p_E \otimes 1) = u(p_E \otimes 1)$, where $p_E$ denotes the orthogonal projection from $H$ onto $E$. We say that $u$ is irreducible if the only $u$-invariant subspaces are (0) and $H_u$, and we say that $u$ is indecomposable if $H$ cannot be written as a direct sum of two non-zero $u$-invariant subspaces.

If $(v, H_v)$ is another unitary $\Omega$-representation, an intertwiner between $u$ and $v$ is a linear bounded operator $T : H_u \to H_v$ such that $(T \otimes 1)u = v(T \otimes 1)$. The space of all intertwiners between $u$ and $v$ is denoted by $\text{Hom}_G(u, v)$. We write $\text{End}_G(u)$ if $u = v$.

We say that $(u, H_u)$ and $(v, H_v)$ are (unitary) equivalent if $\text{Hom}_G(u, v)$ contains a unitary operator.
We denote by $\text{Irr}(G, \Omega)$ and by $\text{Irr}(G, \Omega^*)$ the set of all unitary equivalence classes of irreducible unitary $\Omega$-representations of $G$ and the one of irreducible unitary $\Omega^*$-representations of $G$, respectively.

If $x \in \text{Irr}(G, \Omega)$ is such a class, we denote by $u^x \in \mathcal{B}(H_x)\hat{\otimes}L^\infty(G)$ a representative of $x$, where $H_x$ denotes the Hilbert space on which $u^x$ acts. We put $n_x := \dim(H_x)$, where possibly $n_x = \infty$. Notice that these notations are similar to the ones used for ordinary representations of $G$, so that the context will explain in which situation the notations are used.

For further use, let us note here that if $\Omega$ is a 2-cocycle for $G$, then $\Omega^*$ is a 2-cocycle for $G^{op} = (C(G)^{op}, \Delta)$. Hence all arguments valid for $\Omega$-representations are also valid for $\Omega^*$-representations.

Another way to link up $\Omega$-representations with $\Omega^*$-representations is given by the following result.

3.2.3 Lemma. Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle on $G$. Let $u \in \mathcal{B}(H)\hat{\otimes}L^\infty(G)$ be a (measurable) unitary $\Omega^*$-representation on $H$, and put $\delta = \delta_u : \mathcal{B}(H) \to \mathcal{B}(H)\hat{\otimes}L^\infty(G)$ as in (3.1). With $T = \mathcal{B}(H)$, the unitary operator $V_T \in \mathcal{B}(L^2(T)) \otimes C(G)$ implementing the action $\delta$ (recall Remark 2.3.7) can then be written as $V_T = u_{13}u_{23}^* \in T \otimes T^{op} \otimes C(G)$, where $u^x$ is an $\Omega$-representation implementing $\delta$ in the sense of (3.2).

Proof. We have by construction that $\delta(t) = u(t \otimes 1)u^*$, for all $t \in T$. On the other hand, we have by Remark 2.3.7 that there exists a canonical unitary operator $V_T \in \mathcal{B}(L^2(T)) \otimes C(G)$ implementing $\delta$, that is, $\delta(t) = V_T(t \otimes 1)V_T^*$, for all $t \in T$. Hence, for all $t \in T$ we write, upon identifying $T \otimes T^{op} \cong \mathcal{B}(L^2(T))$:

$$V_T^* u_{13} (t \otimes 1 \otimes 1) = V_T^* \delta(t)_{13} u_{13} = (t \otimes 1 \otimes 1) V_T^* u_{13},$$

which shows that there exists a unitary operator $u^x \in T^{op} \otimes C(G)$ such that $V_T = u_{13}u_{23}^*$. As $V_T$ is a corepresentation, it is easily seen that $u^x$ must necessarily be an $\Omega$-representation.

Finally, we have by [41, Proposition 3.7.3] that $(J_T \otimes J_G) V_T(J_T \otimes J_G) = V_T^\ast$. Since $J_G$ implements the unitary antipode $R$, and since for $x \in T^{op}$ we have $J_T(1 \otimes x)J_T = x^* \otimes 1$ by definition, it then follows that for $x \in T^{op}$ we have $1 \otimes (u^x)^*(x \otimes 1)u^x = V_T^\ast(1 \otimes x \otimes 1)V_T = \overline{\delta(x)}_{32}$. 

3.2.4 Corollary. There exists a canonical element $X_\Omega \in L^\infty(G)$ such that

$$u^x = (j \otimes R)(u)(1 \otimes X_\Omega^x)$$

for all $\Omega^*$-representations. Moreover, $X_\Omega$ is then a coboundary between $\Omega$ and the 2-cocycle $\widetilde{\Omega} = (R \otimes R)\Omega_2$, so $\widetilde{\Omega} = (X_\Omega^\ast \otimes X_\Omega^\ast)\Omega \Delta(X_\Omega)$.

We obtain in particular a one-to-one correspondence between unitary $\Omega$-representations and unitary $\Omega^*$-representations by the map $u \mapsto u^x = (j \otimes R)(u)(1 \otimes X_\Omega^x)$.
Note that the fact that $\Omega$ and $\tilde{\Omega}$ are canonically coboundary equivalent holds in the general context of locally compact quantum groups, see [13, Proposition 6.3.(iii)], but we can give in our setting an easier, more direct proof. It can be shown that the coboundary element obtained here indeed coincides with the one from [13, Proposition 6.3.(iii)], but we refrain from showing this explicitly.

**Proof.** If $u$ is an $\Omega^*$-representation, it is easily seen that $u^\circ$ and $(j \otimes R)(u)$ both implement the same left coaction on $\mathcal{B}(H)^{op}$, hence by Note 3.1.10 we have $u^\circ = (j \otimes R)(u)(1 \otimes X_u^*)$ for some unitary $X_u$. It is also easily seen that $\delta_{J \otimes R}(u)$ is cleft with associated 2-cocycle $\tilde{\Omega}$, showing that $X_u$ is a coboundary between $\Omega$ and $\tilde{\Omega}$.

It remains to show that $X_u$ is independent of $u$. But by construction, it is easily shown that $(u \oplus v)^\circ = u^\circ \oplus v^\circ$. It then follows that $X_u = X_{u \oplus v} = X_v$ for any two $\Omega^*$-representations $u,v$.

**3.2.5 Lemma** (Twisted Schur’s lemma). If $(u, H_u)$ and $(v, H_v)$ are two irreducible unitary $\Omega$-representations (resp. $\Omega^*$-representations) of $\mathbb{G}$, then either $u$ is not unitary equivalent to $v$ and $\text{Hom}_\mathbb{G}(u,v) = (0)$; or $u$ is unitary equivalent to $v$ and $\text{Hom}_\mathbb{G}(u,v)$ is a 1-dimensional subspace of $\mathcal{B}(H_u, H_v)$. In particular, $\text{End}_\mathbb{G}(u) = \mathbb{C}$, and $u$ is irreducible if and only if $u$ is indecomposable.

**Proof.** Let us prove this for unitary $\Omega^*$-representations, the result for $\Omega$-representations then follows by considering $(C(\mathbb{G})^{op}, \Delta)$.

Let $p_E$ be a unitary $\Omega^*$-representation, and let $E \subset H_u$ be an invariant closed subspace. Let $p_E$ be the projection onto $E$. Let $\omega$ be a faithful normal state on $\mathcal{B}(H)$. By possibly replacing $\omega$ by $(\omega \otimes h_G)\delta_u$, we may assume that $(\omega \otimes id)\delta_u(x) = \omega(x)1$. The operator $V_{\delta_u}$ on $L^2(\mathcal{B}(H), \omega) \otimes L^2(\mathbb{G})$ sending $\Lambda_\omega(x) \otimes \Lambda(a)$ to $(\Lambda_\omega \otimes \Lambda)(\delta_u(x)(1 \otimes a))$ is a unitary corepresentation implementing $\delta_u$ by $\delta_u(x) = V_{\delta_u}(x \otimes 1)V_{\delta_u}^*$. Then the invariance of $E$ gives that $(p_E \otimes 1)\delta_u(p_E) = \delta_u(p_E)$, which implies $(p_E \otimes 1)V_{\delta_u}(p_E \otimes 1) = V_{\delta_u}(p_E \otimes 1)$. But it is well-known that this implies $p_E \otimes 1$ commutes with $V_{\delta_u}$. In particular, $\delta_u(p_E) = p_E \otimes 1$, so $p_E \otimes 1$ commutes with $u$.

It is now immediate that $u$ is indecomposable if and only if it is irreducible, and that for $u, v$ irreducible one has $\text{Hom}_\mathbb{G}(u,v)$ at either 0 or one-dimensional, the latter case occurring if $u$ and $v$ are unitarily equivalent.

**3.2.6 Lemma.** Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a 2-cocycle on $\mathbb{G}$. Let $(u, H_u)$ and $(v, H_v)$ be two measurable unitary $\Omega$-representations of $\mathbb{G}$.

i) If $T : H_u \longrightarrow H_v$ is a linear bounded operator, then the bounded linear operator $S := (id \otimes h_G)(v^*(T \otimes 1)u)$ lies in $\text{Hom}_\mathbb{G}(u,v)$. It is called average intertwiner with respect to $T$.

ii) Every $\Omega$-representation $u$ of $\mathbb{G}$ decomposes into a direct sum of irreducible unitary $\Omega$-representations.
Proof. (i) Assume that \( T \in \mathcal{B}(H_u, H_v) \). Clearly the linear operator \( S = (id \otimes h_G)(v^* (T \otimes 1)u) \) is a well-defined bounded operator. By definition of \( \Omega \)-representation, we have

\[
(id \otimes \Omega \Delta)(u) = u_{12}u_{13} \text{ and } (id \otimes \Omega \Delta)(v) = v_{12}v_{13},
\]

which, using the definition of \( \Omega \Delta \), can be written as:

\[
(id \otimes \Delta)(u) = (1 \otimes \Omega^*)(u_{12}u_{13}) \text{ and } (id \otimes \Delta)(v) = (1 \otimes \Omega^*)(v_{12}v_{13}).
\]

Apply the unital *-homomorphism \((id \otimes \Delta)\) to \( v^*(T \otimes 1)u \):

\[
(id \otimes \Delta)(v^*(T \otimes 1)u) = (id \otimes \Delta)(v^*)(T \otimes 1)(id \otimes \Delta)(u)
\]

\[
= v_{13}v_{12}^*(1 \otimes \Omega)(T \otimes 1 \otimes 1)(1 \otimes \Omega^*)u_{12}u_{13}
\]

\[
= v_{13}v_{12}^*(T \otimes 1 \otimes 1)u_{12}u_{13}.
\]

Next, the \( G \)-invariance of the Haar state of \( G \) yields that \((id \otimes h_G \otimes id)((id \otimes \Delta)(v^*(T \otimes 1)u)) = S \otimes 1\). Also, we have:

\[
(id \otimes h_G \otimes id)(v_{13}v_{12}^*(T \otimes 1 \otimes 1)u_{12}u_{13})
\]

\[
= v_{12}^*(id \otimes h_G)(v^*(T \otimes 1)u)_1 u_{12}
\]

\[
= v^*(S \otimes 1)u
\]

and the conclusion follows.

(ii) If \( u \) is an \( \Omega \)-representation, then the averaging operator \( \mathcal{B}(H) \to \text{End}_G(u) = \mathcal{B}(H)^G_u \) sending \( T \) to \((id \otimes h_G)(u^*(T \otimes 1)u)\) is a normal conditional expectation onto \( \text{End}_G(u) \), as already observed, hence \( \text{End}_G(u) \) is a direct sum of type I-factors, proving that \( u \) is a direct sum of irreducible \( \Omega \)-representations.

\[\blacksquare\]

Again, a similar statement holds for \( \Omega^* \)-representations.

3.2.7 Remark. Let \( \delta : \mathcal{B}(H) \to \mathcal{B}(H) \otimes L^\infty(G) \) be a measurable projective representation on \( H \) with implementing unitary \( u \). By the argument used in point ii) of Lemma 3.2.6, one has a one-to-one correspondence between the set of all \( \delta \)-invariant projections in \( \mathcal{B}(H) \) and the \( u \)-invariant subspace of \( H \). Accordingly, \( \delta \) is ergodic if and only if \( u \) is irreducible. In particular, \( \delta \) is a torsion action of projective type (recall Theorem-Definition 2.4.4) if and only if \( u \) is finite dimensional and irreducible.

To any 2-cocycle \( \Omega \) one can associate a canonical \( \Omega \)-representation.

3.2.8 Theorem-Definition (Projective regular representation). Let \( G \) be a compact quantum group and \( \Omega \) a 2-cocycle on \( G \). Defining \( V^\Omega = \Omega \mathcal{V}_G \), the following properties hold:

i) For all \( x \in L^\infty(G) \) and \( \xi \in L^2(G) \) we have \( V^\Omega(\Lambda(x) \otimes \xi) = \Omega \Delta(x)(\xi \otimes \xi) \).
ii) For all \( x \in L^\infty(G) \) we have \( \omega \Delta(x) = V^\Omega(x \otimes 1)V^\Omega_* \).

iii) The following identity holds: \( (id \otimes \Delta)(V^\Omega) = V^\Omega_{12}V^\Omega_{13} \), so \( V^\Omega \in \mathcal{B}(L^2(G)) \otimes L^\infty(G) \) is a unitary \( \Omega \)-representation.

iv) The following pentagonal equation holds: \( V^\Omega_{12}V^\Omega_{13}(V^\Omega)_23 = V^\Omega_{23}V^\Omega_{12} \).

The unitary \( V^\Omega \) is called right projective regular representation of \( G \) on \( L^2(G) \) with respect to \( \Omega \) or simply right \( \Omega \)-regular representation of \( G \) on \( L^2(G) \).

3.2.9 Remark. Similarly, defining \( W^\Omega = W_G \Omega^* \), we have that \( (W^\Omega)^*(x \otimes \Delta(x)) = \omega \Delta(x)(x \otimes x) \), for all \( x \in L^\infty(G) \) and \( \xi \in L^2(G) \). For all \( x \in L^\infty(G) \) we have \( \omega \Delta(x) = (W^\Omega)^*(1 \otimes x)W_G \) and the pentagonal equation: \( (W^\Omega)_{12}W^\Omega_{13}W^\Omega_{23} = W^\Omega_{23}W^\Omega_{12} \). Moreover, the following identity holds: \( \Delta(\Omega^* \otimes id)(W^\Omega) = W^\Omega_{13}W^\Omega_{23} \), so \( \Sigma W^\Omega \Sigma \) is an \( \Omega^* \)-projective representation.

The unitary \( W^\Omega \) is called left projective regular representation of \( G \) on \( L^2(G) \) with respect to \( \Omega \) or simply left \( \Omega^* \)-regular representation of \( G \) on \( L^2(G) \).

The following lemma follows from direct computations by using the relations from Theorem 2.2.6.

3.2.10 Lemma. Let \( \Omega \) be a 2-cocycle for \( G \). Given the canonical Kac system, \( (V_G, U_G) \), associated to \( G \), the following identities hold:

i) \( (V^\Omega)_12(V^\Omega)_{13}(V^\Omega)_23 = (V^\Omega)_{13}(V^\Omega)_23 \).

ii) \( (V^\Omega)_{12}(V^\Omega)_{23} = (V^\Omega)_{23}(V^\Omega)_{12} \).

3.2.11 Lemma. Let \( G \) be a compact quantum group and \( \Omega \) a 2-cocycle on \( G \). We have:

\[
L^\infty(G) \cong \overline{\text{span}}^\text{\-weakly} \{ (\eta \otimes id)(V^\Omega) \mid \eta \in \mathcal{B}(L^2(G))^* \}
= \overline{\text{span}}^\text{\-weakly} \{ (id \otimes \eta)((W^\Omega)^*) \mid \eta \in \mathcal{B}(L^2(G))^* \}.
\]

Proof. Let us show the first identification. The second one follows analogously. Given \( x, y \in L^\infty(G) \) consider the coordinate linear functional \( \omega_{\xi, \eta} \in \mathcal{B}(L^2(G))^* \) and write the following:

\[
\langle (\omega_{\xi, \eta} \otimes id)(V^\Omega)(\xi), \xi' \rangle = \langle V^\Omega(x \xi \otimes \xi, y \xi \otimes \xi') \rangle = \langle \eta \Delta(x)(\xi \otimes \xi), y \xi \otimes \xi' \rangle = \langle (y^* \otimes 1)\Delta(x)(\xi \otimes \xi), \xi \otimes \xi' \rangle = \langle (h_G \otimes id)((y^* \otimes 1)\Delta(x))\xi, \xi' \rangle,
\]

for all \( \xi, \xi' \in L^2(G) \). Hence, \( (\omega_{\xi, \eta} \otimes id)(V^\Omega) = (h_G \otimes id)((y^* \otimes 1)\Delta(x)) \), for all \( x, y \in L^\infty(G) \).

It is enough to show that the linear span of these elements is \( \sigma \)-weakly dense in \( L^\infty(G) \).

As \( h_G \) is a faithful normal state on \( L^\infty(G) \), it is in fact sufficient to show that the linear span of elements of the form \( (h_G \otimes id)((y^* \otimes 1)\Delta(x)) \) is \( \sigma \)-weakly dense in \( L^\infty(G) \). But by the cancellation property of \( \Delta \) we find immediately that \( \omega \Delta(L^\infty(G))(L^\infty(G) \otimes 1) \) is \( \sigma \)-weakly dense in \( L^\infty(G) \otimes L^\infty(G) \), which yields the conclusion. \( \blacksquare \)
A standard argument by combining the previous lemma and Lemma 3.2.6 yields the following Peter-Weyl theorem.

3.2.12 Theorem (Twisted Peter-Weyl theorem I). Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a 2-cocycle. The right projective regular representation $(V^\Omega, L^2(\mathbb{G}))$ contains all irreducible $\Omega$-representations of $\mathbb{G}$ in its direct sum decomposition.

Following [14] we have a twisted version of the Schur’s orthogonality relations. This theorem follows straightforwardly by applying Lemma 3.2.6.i) with respect to rank one operators.

3.2.13 Theorem (Twisted Schur’s orthogonality relations). Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a 2-cocycle on $\mathbb{G}$. Let $\{u^x\}_{x \in \text{Irr}(\mathbb{G}, \Omega)}$ be a complete set of mutually inequivalent, irreducible unitary $\Omega$-representations, with fixed bases for the associated Hilbert spaces $H_x$. For each $x \in \text{Irr}(\mathbb{G}, \Omega)$, let $\{\xi^x_i\}_{i=1,\ldots,n_x}$ be a complete set of mutually inequivalent irreducible unitary $\Omega$-representations, with fixed bases for the associated Hilbert spaces $H_x$. For each $x,y \in \text{Irr}(\mathbb{G}, \Omega)$, $i,j = 1,\ldots,n_x$ and $k,l = 1,\ldots,n_y$.

The matrix $F^x$ is nothing but the density matrix of the $\delta_x$-invariant state $\varphi_x$ with $\varphi_x(T)1t = (h_\mathbb{G} \otimes \text{id})\delta(T)$ for $T \in \mathcal{B}(H)$, so $\varphi_x = T_T(F^x - )$.

Given $x \in \text{Irr}(\mathbb{G}, \Omega)$ and the corresponding positive operator $F^x \in \mathcal{B}(H_x)$ from the previous theorem, we fix an orthonormal basis of $H_x$, $\{\xi^x_i\}_{i=1,\ldots,n_x}$, that diagonalises $F^x$. If $F^x_j \in \mathbb{R}^+$ denotes the eigenvalue of $F^x$ for the eigenvector $\xi^x_j$, for every $j = 1,\ldots,n_x$, then the orthogonality relations become $h_\mathbb{G}((u^x_i)^*u^x_j) = \delta_{xy}\delta_{kl}F^x_{ik}$. Following these notations, we obtain as an immediate corollary of the previous two theorems the following decomposition for $L^2(\mathbb{G})$.

3.2.14 Theorem (Twisted Peter-Weyl theorem II). Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a 2-cocycle on $\mathbb{G}$. We have a unitary transformation $L^2(\mathbb{G}) \cong \bigoplus_{x \in \text{Irr}(\mathbb{G}, \Omega)} H_x \otimes \Pi_x$ such that $\Lambda(u^x_{ij}) \mapsto \sqrt{F^x_{ij}} \otimes \xi^x_i$, for all $j = 1,\ldots,n_x$, $x \in \text{Irr}(\mathbb{G}, \Omega)$.

3.3. Continuous $\Omega$-representations and 2-cocycles of finite type

We now consider a special type of measurable 2-cocycles.

3.3.1 Definition. We say that a 2-cocycle $\Omega$ on $\mathbb{G}$ is of finite type if there exists a finite dimensional $\Omega$-representation.

Not all 2-cocycles $\Omega$ are of finite type, see e.g. [14, Section 8] for an example of a 2-cocycle which is not of finite type. These types of 2-cocycles will however be sufficient for our needs. The following lemma shows that being of finite type is an ambidextrous notion.

3.3.2 Lemma. A 2-cocycle $\Omega$ is finite type if and only if there exists a finite dimensional $\Omega^*$-representation.
Proof. This follows immediately from Corollary 3.2.4.

Recall the notation introduced in (3.1) and (3.2).

3.3.3 Definition. Let $\Omega \in L^\infty(G)\widehat{\otimes}L^\infty(G)$ be a measurable 2-cocycle. We say that a measurable $\Omega$-representation (resp. $\Omega^*$-representation) $u$ is continuous if $\delta_u$ is a continuous right (resp. left) projective representation.

Note that this notion is strictly weaker than demanding that $u \in M(\mathcal{K}(H)\otimes C(G))$, which is a too strong condition in practice. Note also that any $\Omega$-representation on a finite dimensional Hilbert space is automatically continuous.

We will show that continuous $\Omega$-representations can only exist if $\Omega$ is of finite type, and that then all $\Omega$-representations are continuous.

3.3.4 Theorem (Twisted Maschke’s theorem). Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle on $G$. Let $(u,H_u)$ and $(v,H_v)$ be two continuous unitary $\Omega$-representations of $G$.

i) If $T : H_u \rightarrow H_v$ is a linear compact operator, then the average intertwiner $S$ with respect to $T$ is again compact.

ii) The $C^*$-algebra $D_u = \mathcal{K}(H_u)\delta_u$ acts non-degenerately on $H_u$, that is, $[D_u H_u] = H_u$.

iii) If $u$ is irreducible, then $u$ is finite dimensional.

iv) Every continuous left $\Omega$-representation of $G$ decomposes into a direct sum of finite dimensional irreducible unitary $\Omega$-representations.

Proof. i) Assume that $u$ is a continuous $\Omega$-representation. If then $T \in \mathcal{K}(H_u)$, we have that the average $S = (id \otimes h_G)(u^*(T \otimes 1)u) = (id \otimes h_G)\delta_u(T) \in \mathcal{K}(H)$. This proves the first point when $u = v$.

ii) This is just a special case of the observation in Remark 2.3.4.

iii) Since $\mathcal{K}(H)^\delta$ is necessarily non-trivial when $H$ is infinite dimensional, it follows that $u$ can only be irreducible when $H$ is finite dimensional. A general continuous $\Omega$-representation $u$ must then be a direct sum of finite dimensional (and hence continuous) $\Omega$-representations. It then also follows immediately that if $u, v$ are continuous $\Omega$-representations, an average intertwiner with respect to an operator in $\mathcal{K}(H_u, H_v)$ remains in this space, proving the first point in general.

iv) The last point follows from iii) and Lemma 3.2.6.iii).

Note that the previous lemma shows in particular that the class of continuous $\Omega$-representations is stable under direct sums, which is not immediately obvious. By Remark 3.2.7, we moreover see that $\hat{G}$ is projective torsion-free if and only if all 2-cocycles of finite type on $G$ are cohomologous to the trivial one.
3.3.5 Theorem. Let $\mathcal{G}$ be a compact quantum group, and let $\Omega$ be a 2-cocycle. Then the following are equivalent:

i) There exists a continuous $\Omega$-representation.

ii) $\Omega$ is of finite type.

iii) All irreducible $\Omega$-representations are finite dimensional.

Moreover, if one (hence any) of these conditions hold, then all $\Omega$-representations of $\mathcal{G}$ are continuous.

Proof. The implication i) $\Rightarrow$ ii) follows from Theorem 3.3.4.iv). The implication ii) $\Rightarrow$ iii) follows from [14, Proposition 4.3]. The implication iii) $\Rightarrow$ i) is trivial, and the final statement follow from Lemma 3.2.6.iii). 

3.3.6 Corollary. Assume that $\Omega \in C(\mathcal{G}) \otimes C(\mathcal{G})$ is a continuous 2-cocycle. Then $\Omega$ is of finite type.

Proof. In this case we have that the right regular projective representation $V^\Omega \in M(K(L^2(\mathcal{G})) \otimes C(\mathcal{G}))$, so a fortiori the associated projective representation is continuous. By the previous theorem, this forces $\Omega$ to be of finite type.

Assume now again that $\Omega$ is a general measurable 2-cocycle. Then clearly

$$\Omega \Delta_{\Omega^*} : L^\infty(\mathcal{G}) \to L^\infty(\mathcal{G}) \otimes L^\infty(\mathcal{G}), \quad x \mapsto \Omega \Delta(x) \Omega^*$$

defines a coassociative coproduct on $L^\infty(\mathcal{G})$. It follows from [13] that $(L^\infty(\mathcal{G}), \Omega \Delta_{\Omega^*})$ is again a locally compact quantum group. It need not necessarily be compact, as the example in [11] shows. However, one has the following theorem as a particular case of Proposition 4.3.2 in [14].

3.3.7 Theorem. Let $\mathcal{G}$ be a compact quantum group and let $\Omega$ be a 2-cocycle. Then the couple $(L^\infty(\mathcal{G}), \Omega \Delta_{\Omega^*})$ defines a compact quantum group if and only if $\Omega$ is of finite type.

We will use the notation $(L^\infty(\mathcal{G}_\Omega), \Delta_{\mathcal{G}_\Omega}) = (L^\infty(\mathcal{G}), \Omega \Delta_{\Omega^*})$. We denote $C(\mathcal{G}_\Omega) \subseteq L^\infty(\mathcal{G}_\Omega)$ for the associated reduced C*-algebra, and $Pol(\mathcal{G}_\Omega)$ for the polynomial Hopf *-subalgebra.

By standard von Neumann algebra theory [39, Theorem IX.1.14], there is a canonical identification $L^2(\mathcal{G}_\Omega) \cong L^2(\mathcal{G})$ intertwining in particular the modular conjugations $J_{\mathcal{G}_\Omega}$ and $J_{\mathcal{G}}$ and the left action of $L^\infty(\mathcal{G})$. In the following, we will then simply identify $L^2(\mathcal{G}_\Omega) \cong L^2(\mathcal{G})$.

3.3.8 Remark. An interesting situation where $\mathcal{G}_\Omega$ is always compact is when $\mathcal{G}$ is of Kac type (see [14, Proposition 5.1]). More precisely, by [14, Proposition 4.3] it follows that any 2-cocycle of a compact quantum group of Kac type is of finite type.
In [14, Proposition 4.3], the language of Galois co-objects is used (in the measurable setting), of which \((L^\infty(G), \Omega \Delta)\) is a particular case, see Example 1.20 in [14]. Although we are able to avoid this more abstract theory in the measurable setting, it is necessary to use this formalism in the C*-algebraic and algebraic setting, due to the fact that most 2-cocycles are in practice not cohomologous to continuous or algebraic ones, even when of finite type, and that in general one can expect \(C(G_\Omega) \neq C(G)\) inside \(L^\infty(G)\).

Let us provide now some more information on the relation between \(Pol(G)\) and \(Pol(G_\Omega)\).

3.3.9 Definition. Let \(\Omega\) be a 2-cocycle of finite type. We denote by \(Pol(G, \Omega) \subset L^\infty(G)\) the linear span of matrix coefficients of all irreducible unitary \(\Omega\)-representations of \(G\), and by \(C(G, \Omega)\) its normclosure.

By Lemma 3.2.11 and Theorem 3.3.4.iv), it follows that \(Pol(G, \Omega)\) is \(\sigma\)-weakly dense in \(L^\infty(G)\).

Contrary to the ordinary case when \(\Omega = 1 \otimes 1\), \(Pol(G, \Omega)\) is not a Hopf \(*\)-algebra. More precisely, it is a coalgebra but not an algebra. However, \(Pol(G, \Omega)\) will be a \(Pol(G_\Omega)\)-\(Pol(G)\)-bimodule (and in fact bimodule coalgebra) such that:

\[
Pol(G, \Omega)^* Pol(G, \Omega) = Pol(G), \quad Pol(G, \Omega) Pol(G, \Omega)^* = Pol(G_\Omega).
\]

Indeed, if \(u\) is a finite dimensional unitary \(\Omega\)-representation and \(v\) a finite dimensional unitary \(G\)-representation, then \(u_{13} v_{23}\) is a finite dimensional unitary \(\Omega\)-representation, showing that \(Pol(G, \Omega)\) is a right \(Pol(G)\)-module. We obtain then for example the equality \(Pol(G, \Omega)^* Pol(G, \Omega) = Pol(G)\) as clearly the left hand side is a \(\sigma\)-weakly dense \(*\)-subbialgebra of \(L^\infty(G)\). The analogous properties for \(C(G, \Omega)\) then follow immediately.

From Theorem 3.2.13, we also deduce the following.

3.3.10 Corollary. Let \(G\) be a compact quantum group and \(\Omega\) a 2-cocycle of finite type on \(G\). The set of matrix coefficients of all representatives of irreducible unitary \(\Omega\)-representations of \(G\) (with respect to fixed bases) form a basis for \(Pol(G, \Omega)\).

4. Twisted Baaj-Skandalis duality

We study the construction of a twisted crossed product by a compact quantum group based on the notion of twisted dynamical system. Twisted \(C^*\)-algebras associated to classical locally compact groups and, more generally, twisted crossed products with respect to a 2-cocycle have been studied in the literature by several hands (standard references are [34, 7]). The more general framework of locally compact quantum groups has been addressed for instance in [47, 32], see also the work of S. Vaes and L. Vainerman in the von Neumann algebraic setting [42].

In this paper, we focus on the case of compact quantum groups. First, we relate the regularity of a 2-cocycle as defined in [32] to our notion of being of finite type. Then we study twisted crossed products coming from projective representations. We end by considering a twisted version of the Takesaki-Takai duality and the Baaj-Skandalis duality.

The contents of this section have been initially inspired by [36].

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4.1. Twisted group C*-algebras and regularity

4.1.1 Definition. Let \( G \) be a compact quantum group and \( \Omega \) a measurable 2-cocycle on \( G \). The twisted reduced C*-algebra of \( G \) with respect to \( \Omega \) is the C*-algebra defined by:

\[
C_r^*(G, \Omega) = c_0(\hat{G}, \Omega) := C^*\langle (id \otimes \eta)(V^\Omega) \mid \eta \in \mathcal{B}(L^2(\hat{G})) \rangle \subset \mathcal{B}(L^2(\hat{G})).
\]

The following result can be found in [8, Lemma 4.9] when \( G \) is discrete, but is valid for general regular locally compact quantum groups as was already remarked in [32]. As we are using a slightly different setup (on which we will comment after the lemma) we include a proof for \( G \) compact, following a different path.

4.1.2 Lemma. We have an equality \( C_r^*(G, \Omega) = \{(id \otimes \eta)(V^\Omega) \mid \eta \in \mathcal{B}(L^2(\hat{G}))\} \). Moreover, \( C_r^*(G, \Omega) \) acts non-degenerately on \( L^2(\hat{G}) \).

Proof. By a direct computation, we have \((id \otimes \eta)(V^\Omega)\Lambda(x) = \Lambda((id \otimes \eta)(\Omega \Delta(x)))\) for \( x \in L^\infty(\hat{G}) \) and \( \eta \in L^\infty(\hat{G})^* \). By Lemma 3.2.11 and the fact that \( V^\Omega \in \mathcal{B}(L^2(\hat{G})) \otimes L^\infty(\hat{G}) \), we have that \( \{(id \otimes \eta)((V^\Omega) \mid \eta \in \mathcal{B}(L^2(\hat{G})))\} = \{(id \otimes \omega_{\Lambda(\Omega \Delta(x)))})((V^\Omega) \mid \eta \in \mathcal{B}(L^2(\hat{G}))\} \). But a direct computation using the twisted orthogonality relations in Lemma 3.2.5 shows that with respect to the basis in Theorem 3.2.14 we have that

\[
(F^\eta)^{-1}(id \otimes \omega_{\Lambda(\Omega \Delta(x)))})((V^\Omega)(\xi_x \otimes \xi_y) = \delta_{x,y} \delta_{s,t} \xi_x \otimes \xi_y.
\]

Hence \( \{(id \otimes \eta)(V^\Omega) \mid \eta \in \mathcal{B}(L^2(\hat{G}))\} \) forms a C*-algebra acting non-degenerately on \( L^2(\hat{G}) \), and we moreover obtain the following corollary.

4.1.3 Corollary (Twisted Peter-Weyl theorem III). Let \( G \) be a compact quantum group and \( \Omega \) a 2-cocycle. Then we have a C*-isomorphism \( C_r^*(G, \Omega) \cong \bigoplus_{\eta \in \text{Irr}(\hat{G}, \Omega)} \text{K}(\mathcal{H}) \).

Denoting \( L^\infty(\hat{G}, \Omega) = L(\hat{G}, \Omega) = C_r^*(G, \Omega)^* \), we then also have that \( L^\infty(\hat{G}, \Omega) \cong \bigoplus_{\eta \in \text{Irr}(\hat{G}, \Omega)} \mathcal{B}(\mathcal{H}) \).

4.1.4 Remark. It follows from Corollary 3.2.4 that there is a one-to-one correspondence between projective representations of \( G \) and \( G_{\Omega} \), where we assume that \( \Omega \) is of finite type and hence \( G_{\Omega} \) compact. First of all, \( \Theta := \Omega^* \) is then easily seen to be a 2-cocycle of finite type for \( G_{\Omega} \) with \( (G_{\Omega})_{\Omega^*} = G \). If then \( u \) is a unitary \( \Omega \)-representation of \( G \), it is also a unitary \( \Theta^* \)-representation of \( G_{\Omega} \), leading to the left projective \( G_{\Omega} \)-representation \( \delta_u(a) = u(a \otimes 1)u^* \) for \( a \in \mathcal{B}(H_u) \). By Corollary 3.2.4, we obtain a one-to-one correspondence between \( \Omega \)-representations of \( G_{\Omega} \) and \( \Omega^* \)-representations of \( G \).

4.1.5 Remark. We can also relate the regular representations of \( G \) and \( G_{\Omega} \). Indeed, since \( L^\infty(G) = L^\infty(G_{\Omega}) \), we can canonically identify \( L^2(G_{\Omega}) \) and \( L^2(G) \), and by [13, Proposition 5.4] the right regular representation of the twisted quantum group is given by \( \Omega^* V_{G_{\Omega}} = (J \otimes \hat{J}X_{\Omega^*})(\Omega V_{G})^*(J \otimes \hat{J}) \). To see that we can use here the same \( \Omega \) as before, it is sufficient to calculate that \( (\Omega^* V_{G_{\Omega}})^* = \Omega V_{G} \) (after identifying \( L^2(G) \) with \( L^2(\hat{G}) \) by \( J \)), that is that \( V_B(L^2(G)) = \Omega_{13}(V_{G_{\Omega}})_{13} \Omega_{23}(V_{G})_{23} \), where \( B(L^2(\hat{G})) \) carries the coaction \( Ad_{\Omega^*}V_{G_{\Omega}} \). This can be
verified using the techniques of [13]. Alternatively, one can also follow more directly the proof of Proposition 4.1.5 of [14].

In any case, from the above we immediately get that $C^\ast_r(\mathbb{G}_\Omega, \Omega^\ast) = J C^\ast_r(\mathbb{G}, \Omega) J$. Analogously, we have $\Omega^\ast W^\ast_{\mathbb{G}} = (X_0 \hat{\otimes} J) W^\ast_{\mathbb{G}} \Omega^\ast (\hat{\otimes} J)$. 

4.1.6 Remark. In [32, Theorem 2.1], a different twisted group $C^\ast$-algebra is introduced, which we will write $C^\ast_r(\Omega, \mathbb{G}) := [(\omega \otimes \text{id})(W\Omega^\ast)]$. Indeed, by a similar computation to (4.1.2), we obtain that:

$$(F^\ast(F^\ast)^{-1})^{1/2}(\omega_{\lambda(u\omega)_x}, \xi \otimes \text{id})((W\Omega^\ast)^\ast) \xi \otimes \overline{\xi},$$

so that also $C^\ast_r(\Omega, \mathbb{G})$ forms a $C^\ast$-algebra, isomorphic to the direct sum $\bigoplus_{\xi \in \text{irr}(\mathbb{G}, \Omega)} \mathcal{K}(H_\xi)$, whose $\sigma$-weak closure equals the commutant $l^2((\hat{\mathbb{G}}, \Omega)')$. In fact, using that $V = (\hat{\mathcal{J}} \otimes \mathcal{J}) W_{21}(\hat{\mathcal{J}} \otimes \hat{\mathcal{J}})$, we see that $\hat{\mathcal{J}} C^\ast_r(\Omega, \mathbb{G}) \hat{\mathcal{J}} = C^\ast_r(\hat{\mathcal{G}}, \Omega)$, with $\hat{\mathcal{G}}$ as in Corollary 3.2.4. It hence follows (cf. [32, Proposition 3.12]) that $C^\ast_r(\Omega, \mathbb{G}) = X_0 \hat{\otimes} C^\ast_r(\mathbb{G}, \Omega) \hat{\otimes} X^\ast_{\Omega}$. 

4.1.7 Example. Let us briefly relate the above construction to the classical setting. Let $G$ be a classical compact group. Let $\omega : G \times G \to S^1$ be a 2-cocycle on $G$, that is, $\omega(x, y) \omega(xy, z) = \omega(x, yz) \omega(y, z)$ for all $x, y, z \in G$. By passing to a cohomologous 2-cocycle, we may without loss of generality assume that $\omega$ is normalized, so $\omega(e, x) = 1 = \omega(x, e)$ for all $x \in G$. If $\rho$ denotes the right regular representation of $G$, then the right $\omega$-regular representation of $G$ on $L^2(G)$ is the map:

$$\rho^\omega : G \to \mathcal{B}(L^2(G)), \quad g \mapsto \rho^\omega_g, \quad \rho^\omega_g(f)(x) := \omega(x, g) \rho_g(f)(x) = \omega(x, g) f(xg),$$

for all $f \in L^2(G)$ and $x \in G$. A direct computation shows that $\rho^\omega$ is a $\omega$-representation of $G$ on $L^2(G)$. The corresponding Peter-Weyl theory can be obtained in this context (for instance, see [10] for more details). The twisted reduced $C^\ast$-algebra of $G$ with respect to $\omega$ is defined as the $C^\ast$-algebra $C^\ast_r(G, \omega) := \{ \int_G f(g) \rho^\omega(g) dg \mid f \in C(G) \}$. 

Let us now relate the notion of ‘being of finite type’ to the regularity of a 2-cocycle. This will be essential for the twisted Takesaki-Takai duality in the next section. For what follows, we recall that $\mathcal{C}(V) := \{ (\eta \otimes \text{id})(\Sigma V) \mid \eta \in \mathcal{B}(H_\ast) \}$ for any $V \in \mathcal{B}(H \otimes H)$. 

4.1.8 Definition. A 2-cocycle $\Omega$ is called regular if $\mathcal{C}(V^\Omega) = \mathcal{K}(L^2(\mathbb{G}))$. 

4.1.9 Theorem. Let $\mathbb{G}$ be a compact quantum group and $\Omega$ a (measurable) 2-cocycle on $\mathbb{G}$.

i) The set $\mathcal{C}(V^\Omega) \mathcal{C}(V^\Omega)$ is linearly dense in $\mathcal{C}(V^\Omega)$. In particular, $\mathcal{C}(V^\Omega)$ is an algebra. Moreover, $\mathcal{C}(V^\Omega)$ acts non-degenerately on $L^2(\mathbb{G})$, so $[\mathcal{C}(V^\Omega)L^2(\mathbb{G})] = L^2(\mathbb{G})$.

ii) The 2-cocycle $\Omega$ is regular if and only if $\Omega$ is of finite type.

iii) We have $\overline{\text{span}}\{U_G C(\mathbb{G}) U_G \cdot C^\ast_r(\mathbb{G}, \Omega)\} = \mathcal{K}(L^2(\mathbb{G}))$ if and only if $\Omega$ is of finite type.
Proof. i) Applying the definition of \( \mathcal{C}(V^\Omega) \) together with the pentagonal equation satisfied by \( V^\Omega \) from Theorem 3.2.8, we write the following:

\[
\overline{\text{span}}\{\mathcal{C}(V^\Omega)\mathcal{C}(V^\Omega)\} = \overline{\text{span}}\{(id \otimes \eta)(\Sigma V^\Omega) \cdot (id \otimes \eta')(\Sigma V^\Omega) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(id \otimes \eta)(\Sigma V^\Omega) \cdot (\eta' \otimes id)(V^\Omega \Sigma) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(\eta' \otimes id \otimes \eta)(\Sigma_{23} V_{12}^{\Omega} V_{13}^{\Omega} \Sigma_{12}) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(\eta' \otimes id \otimes \eta)(\Sigma_{23} V_{12}^{\Omega} V_{13}^{\Omega} \Sigma_{12}) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(id \otimes \eta \otimes \eta')(\Sigma_{13} V_{12}^{\Omega} V_{13}^{\Omega} \Sigma_{13}) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(id \otimes \eta \otimes \eta')(\Sigma_{13} V_{12}^{\Omega} V_{13}^{\Omega} \Sigma_{13}) \mid \eta, \eta' \in \mathcal{B}(L^2(G))^*\} \\
= \overline{\text{span}}\{(id \otimes \eta)(\Sigma V^\Omega) \mid \eta \in \mathcal{B}(L^2(G))^*\} \\
= \mathcal{C}(V^\Omega).
\]

To see that \( \mathcal{C}(V^\Omega) \) acts non-degenerately, take \( \zeta \in (\mathcal{C}(V^\Omega)L^2(G))^* \). For every \( \xi, \xi', x \in L^2(G) \) we have \( 0 = \langle \zeta, (id \otimes \omega_{\xi, \xi'})(\Sigma V^\Omega)(x) \rangle = \langle \zeta \otimes \xi', \Sigma V^\Omega(x \otimes \xi') \rangle \). Since \( \Sigma V^\Omega \) is a unitary in \( \mathcal{B}(L^2(G) \otimes L^2(G)) \) so surjective, the above equality implies \( \langle \zeta \otimes \xi', \Sigma V^\Omega(x \otimes \xi') \rangle = 0 \).

ii) By Theorem 3.2.13, the set \( \{(id \otimes \omega_{(u_{y_i}, \Lambda(a))})(\Sigma V^\Omega) \mid y \in \text{Irr}(G, \Omega), k, l = 1, \ldots, n_y, a \in \text{Pol}(G)\} \) is dense in \( \mathcal{C}(V^\Omega) \). Given \( u_{y_j} \in \text{Pol}(G, \Omega) \) we compute with the help of the twisted orthogonality relations from Theorem 3.2.13 that for \( \zeta \in L^2(G) \):

\[
\langle \zeta, (id \otimes \omega_{(u_{y_l}, \Lambda(a))})(\Sigma V^\Omega)\Lambda(u_{y_j}) \rangle = \langle \Lambda(u_{y_j}^*) \otimes \zeta, V^\Omega(\Lambda(u_{y_j}) \otimes \Lambda(a)) \rangle \\
= \langle \Lambda(u_{y_j}^*) \otimes \zeta, \Delta_{\Omega}(u_{y_j}) \otimes \Lambda(a) \rangle \\
= \langle \Lambda(u_{y_j}^*) \otimes \zeta, (\sum_{r=1}^{n_x} u_{y_{i_r}}^* \otimes u_{y_{j_r}}^*)(\xi_{\Omega} \otimes \Lambda(a)) \rangle \\
= \sum_{r=1}^{n_x} h_{\xi_{\Omega}}((u_{y_j}^*)^* u_{y_{i_r}}^*) (\zeta, \lambda(u_{y_{i_r}}^*) \Lambda(a)) \\
= F_k^x \delta_{x,y} \delta_{k,i} (\zeta, \lambda(u_{y_j}^*) \Lambda(a)),
\]

so with respect to the orthonormal basis of Theorem 3.2.14 we have:

\[
(F_k^x)^{-1}(id \otimes \omega_{(u_{y_l}, \Lambda(a))})(\Sigma V^\Omega)(F_k^x)^{-1/2} \Lambda(u_{y_j}) = \delta_{x,y} \delta_{k,i} (F_k^x)^{-1/2} \lambda(u_{y_j}^*) \Lambda(a).
\] (4.3)

Assume now that \( \Omega \) is regular. Taking \( a = 1 \), we see from the above that \( (id \otimes \omega_{(u_{y_l}, \xi_{\Omega})})(\Sigma V^\Omega) \) is of the form \( T_{rs}^x \otimes 1 \) for some non-zero operator \( T_{rs}^x \) with respect to the model \( L^2(G) \cong \oplus_{x \in \text{Irr}(G, \Omega)} H_x \otimes \mathcal{T}_x \). As this operator needs to be compact, we see that all \( n_x \) need to be finite, hence \( \Omega \) is of finite type.

Conversely, if \( \Omega \) is of finite type, we see from the above computation (and the fact that the operators \( \Lambda(b) \mapsto \Lambda(ba) \) are bounded for \( a \in \text{Pol}(G) \)) that \( \mathcal{C}(V^\Omega) \subset \mathcal{K}(L^2(G)) \).

To see that this is an equality, it is sufficient to show that the commutant \( \mathcal{C}(V^\Omega)' = \mathcal{C}(V^\Omega) \) of \( \mathcal{C}(V^\Omega) \).
Now if $x \in \mathcal{G}(V^\Omega)'$, then in particular $x$ commutes with all $(id \otimes \omega_{\lambda(\omega_{\eta}^e)})\Sigma V^\Omega$, and it follows from the above computation and (4.1) that $x \in l^2(\hat{G}, \Omega)$. On the other hand, the computation above also shows that then $x \in \rho(L^2(\hat{G}))' = L^\infty(\hat{G})$, hence $x \in l^2(\hat{G}, \Omega) \cap L^\infty(\hat{G})$. But by Remark 4.1.6, this implies $(W^\Omega)^*(1 \otimes x)W^\Omega = 1 \otimes x$, hence $\Omega_{\Delta_{\Omega^*}}(x) = 1 \otimes x$ and so $x \in \mathbb{C}$.

iii) By (4.1) we see that $C^*_r(\hat{G}, \Omega)$ is formed by compact operators if and only if $\Omega$ is of finite type. Hence $\overline{span}\{U_G C(\hat{G})U_G \cdot C^*_r(\hat{G}, \Omega)\} \subseteq \mathcal{K}(L^2(\hat{G}))$ if and only if $\Omega$ is of finite type. To see that this is an equality if $\Omega$ is of finite type, we can follow as similar strategy as in ii). Alternatively, conjugating with $\hat{J}$ and taking into account Remark 4.1.6, we see that we have an inclusion $C^*_r(\Omega, G)JC(\hat{G})J \subseteq \mathcal{K}(L^2(\hat{G}))$, and this must be an equality by the discussion following [32, Definition 2.9]. Conjugating back with $\hat{J}$, we see that $\overline{span}\{U_G C(\hat{G})U_G \cdot C^*_r(\hat{G}, \Omega)\} = \mathcal{K}(L^2(\hat{G}))$.

4.1.10 Remarks. i) As follows from the end of the above proof, our notion of regularity indeed coincides with the notion of regularity of a 2-cocycle as introduced in [32, Definition 2.9].

ii) Up to unitary conjugation, our operator $V^\Omega$ also coincides with the operator $\Sigma(V^\Omega)'\Sigma$ with $V^\Omega_{21}$ as it appears in [3, Proposition 2.44]. It hence follows from that result that $\Omega$ is regular if and only if $G\Omega$ is regular, and hence that $G\Omega$ is never regular if $\Omega$ is not of finite type. It is unclear at the moment if in general $G\Omega$ is semi-regular (this holds in all known cases). By [3, Proposition 2.44], this is equivalent to $\mathcal{K}(L^2(\hat{G})) \subseteq \mathcal{G}(V^\Omega)$.

The following proposition is a straightforward adaptation of [32, Proposition 2.3] to our setting.

4.1.11 Proposition. Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle. Then the twisted reduced $C^*$-algebra $C^*_r(\hat{G}, \Omega)$ is a left $\hat{G}$-$C^*$-algebra with action $\hat{G} \to C^*_r(\hat{G}, \Omega)$ defined by:

$$\alpha^\Omega(x) = \Sigma(V^\Omega)'(1 \otimes x)V^\Omega\Sigma,$$

for all $x \in C^*_r(\hat{G}, \Omega)$.

Proof. Given $x \in C^*_r(\hat{G}, \Omega)$, assume without loss of generality that $x := (id \otimes \eta)(V^\Omega)$ for some $\eta \in \mathcal{B}(L^2(\hat{G}))$. Then, with the help of the pentagonal equation satisfied by $V^\Omega$ (see Theorem-Definition 3.2.8), we have $\alpha^\Omega(x) = (id \otimes id \otimes \eta)V^\Omega_{23}(V_G)_{13} \in \hat{M}(c_0(\hat{G}) \otimes C^*_r(\hat{G}, \Omega))$, which shows that $\alpha^\Omega$ is well-defined as a (injective) *-homomorphism $C^*_r(\hat{G}, \Omega) \to \hat{M}(c_0(\hat{G}) \otimes C^*_r(\hat{G}, \Omega))$. Next, we are going to show that $\alpha^\Omega$ defines an action of $\hat{G}$ on $C^*_r(\hat{G}, \Omega)$. On the one hand, since the elements of the form $x = (id \otimes \eta)(V^\Omega) \in C^*_r(\hat{G}, \Omega)$ with $\eta \in \mathcal{C}(\hat{G})$, are dense in $C^*_r(\hat{G}, \Omega)$, then the previous computation shows also that the subspace $\alpha^\Omega(C^*_r(\hat{G}, \Omega))c_0(\hat{G}) \otimes 1$ is dense in $c_0(\hat{G}) \otimes C^*_r(\hat{G}, \Omega)$. On the other hand, applying $id \otimes \alpha^\Omega$ and $\hat{\Delta} \otimes id$ to the above expression, we obtain $(id \otimes \alpha^\Omega)\alpha^\Omega(x) = (id \otimes id \otimes id \otimes \eta)V^\Omega_{23}(V_G)_{24}(V_{G})_{14} = (\hat{\Delta} \otimes id)\alpha^\Omega(x)$ by a direct computation.

4.1.12 Remark. By the formula $\beta^\Omega(x) := (V^\Omega)^*(1 \otimes x)V^\Omega$ for $x \in C^*_r(\hat{G}, \Omega)$, we can also view $(C^*_r(\hat{G}, \Omega), \beta^\Omega)$ as a (right) $\hat{G}$-cop-$C^*$-algebra.
4.2. Twisted crossed products

In this section, we consider twisted crossed products, cf. again [32]. We start however from the twisted side, and work our way back to the original compact quantum group. As to spare the reader a battle with conventions, we spell out some of the details particular to our setting.

4.2.1 Definition. A (measurable) right twisted dynamical system is the data \((G, A, \delta, \Omega)\) where \(G\) is a compact quantum group, \(\Omega\) is a 2-cocycle of finite type on \(G\), and \(\delta : A \to A \otimes C(G_\Omega)\) is right action of \(G_\Omega\).

We write \(A \overset{\delta, \Omega}{\longrightarrow} G\), and say that \(\delta\) is a twisted action of \(G\) on \(A\) with respect to \(\Omega\) or simply that \(\delta\) is an \(\Omega\)-action of \(G\) on \(A\). We say that \((A, \delta)\) is a right \(\Omega\)-G-C*-algebra if moreover \(\delta\) is injective.

4.2.2 Definition. Let \((G, A, \delta, \Omega)\) be a twisted dynamical system. The twisted reduced crossed product of \(A\) by \(G\) with respect to \((\delta, \Omega)\), denoted by \(A \rtimes_{r,(\delta,\Omega)} G\), is the \(C^*\)-algebra defined by:

\[
A \rtimes_{r,(\delta,\Omega)} G := C^*\langle \langle (id \otimes \lambda)\delta(A)(1 \otimes C^*_r(G, \Omega)) \rangle \rangle \subset \mathcal{L}_A(A \otimes L^2(G)).
\]

4.2.3 Note. To lighten the notation, we will omit the representation \(\lambda\) appearing in the definition of \(A \rtimes_{r,(\delta,\Omega)} G\).

4.2.4 Lemma. We have \(A \rtimes_{r,(\delta,\Omega)} G = \text{span} \{\delta(A)(1 \otimes C^*_r(G, \Omega))\}\).

Proof. It is enough to show that

\[
[(1 \otimes C^*_r(G, \Omega))\delta(A)] = [\delta(A)(1 \otimes C^*_r(G, \Omega))]
\]

But, using the implementation of \(\Omega \Delta = \Omega \cdot \Delta\) in terms of \(V_G\) and \(V^\Omega\), the compatibility of \(\delta\) with \(\Delta\) as a twisted action of \(G\) on \(A\) yields that \((1 \otimes (id \otimes \eta)\delta(\alpha)) = (id \otimes id \otimes \eta)((\delta \otimes id)\delta(\alpha)) = \lim \sum a_i(1 \otimes (id \otimes \eta \cdot u_i)\delta(\alpha))\), for all \(a \in A\) and all \(\eta \in \mathcal{B}(L^2(G))_\ast\), where \(\delta(\alpha) = \lim \sum a_i \otimes u_i\). This proves (4.4).

As a consequence, the maps \(A \to \mathcal{L}_A(A \otimes L^2(G))\) and \(C^*_r(G,\Omega) \to \mathcal{L}_A(A \otimes L^2(G))\) given by \(a \to \delta(\alpha)\) and \(x \to 1 \otimes x\), send \(A\) and \(C^*_r(G,\Omega)\) respectively onto non-degenerate \(C^\ast\)-subalgebras of \(M(\times_{r,(\delta,\Omega)} G)\).

4.2.5 Example. We note that both Definition 4.2.1 and Definition 4.2.2 are natural dual versions of the classical framework. Let \(G\) be a classical compact group and \(A\) a unital \(C^\ast\)-algebra. Given a map \(\omega : G \times G \to U(A)\), a \(\omega\)-action of \(G\) on \(A\) is a map \(\alpha : G \to Aut(A)\) such that \(\alpha_{g_1} \circ \alpha_{g_2} = Ad_{\omega(g_1,g_2)} \circ \alpha_{g_1 g_2}\) for all \(g_1, g_2 \in G\); \(\omega(x,y)\omega(xy, z) = \omega(x,y,z)\omega(x,y)\omega(x,y)\omega(x,y)\omega(x,yz)\) for all \(x, y, z \in G\) and \(\omega(x,e) = 1 = \omega(e, x)\) for all \(x \in G\). Consider the vector space of continuous functions on \(G\) with values in \(A\) equipped with the usual point-wise operations, \(C(G, A)\). We define the twisted convolution product on \(C(G, A)\) with respect to \(\omega\) by:

\[
(f \ast_{\omega} g)(x) := \int_G f(y)\omega(y^{-1}x)\omega(y,y^{-1}x)dy,
\]

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for all \( f, g \in C(G, A) \) and \( x \in G \). We define the twisted involution on \( C(G, A) \) with respect to \( \omega \) by:

\[
f^\ast(x) := \omega(x, x^{-1})\alpha_x(f(x^{-1})^\ast),
\]

for all \( f \in C(G, A) \) and \( x \in G \). Straightforward computations show that \( C(G, A) \) is a \( * \)-algebra with the product and involution above. Next, by applying standard arguments (analogous to the untwisted case) we find that \( (L^2(G) \otimes H_0, \pi, \lambda^\omega) \) is a faithful covariant \( \omega \)-representation of \((A, \alpha)\), where \( \pi : A \to B(L^2(G) \otimes H_0) \) is such that \( \pi(a)(f \otimes \xi)(x) := \left(f \otimes \pi_0(\alpha_{x^{-1}}(a))(\xi)(x)\right) \), for all \( a \in A \), \( f \in L^2(G), \xi \in H_0, x \in G \); and \( \pi_0, H_0 \) is any faithful representation of \( A \). Thus we define the reduced twisted crossed product by \( A \rtimes_r \alpha \in \mathcal{G} := (\pi, \lambda^\omega)(C(G, A))^{||\phi||_{B(L^2(G, H_0))}} \) and one shows that this definition does not depend on the faithful representation \( \pi_0 \). Alternatively, we have \( G \rtimes_r \alpha \in \mathcal{G} := \text{span}(\alpha(A)(1 \otimes C_r(\mathcal{G}, \omega)))) \), where the action \( G \rtimes_r \alpha \) is viewed as a map \( \alpha : A \to M(A \otimes C(G)) \hookrightarrow \mathcal{L}_A(A \otimes L^2(G)) \).

Specially interesting is the case when \( \omega \) is scalar valued (that is, \( \omega \) takes its values on \( S^1 \simeq S^1 \cdot 1 ) \), then \( \omega \) is a usual (normalized) 2-cocycle on \( G \) and \( \alpha \) a group homomorphism. In this case, we observe that these constructions yield the usual \( C^* \)-algebras: if \( \omega = 1 \), we have \( A \rtimes_r \alpha \in \mathcal{G} := \text{span}(\alpha(A)(1 \otimes C^*_r(\mathcal{G}, \omega)))) \); if \( \alpha \) is trivial, we have \( A \rtimes_r \alpha \in \mathcal{G} := A \otimes C^*_r(\mathcal{G}, \omega) \).

**4.2.6 Proposition-Definition.** Let \((\mathcal{G}, A, \delta, \Omega)\) be a twisted dynamical system. The twisted reduced crossed product, \( A \rtimes_r \delta \mathcal{G} \), is a \( \mathcal{G}^{\text{cop}} \)-\( C^* \)-algebra with action \( A \rtimes_r \delta \mathcal{G} \rtimes \mathcal{G}^{\text{cop}} \) such that:

\[
\hat{\delta}(\delta(a)(1 \otimes x)) = (\delta(a) \otimes 1)(1 \otimes (V^\Omega)^*(1 \otimes x)V^\Omega),
\]

for all \( a \in A \) and all \( x \in C^*_r(\mathcal{G}, \Omega) \). The action \( A \rtimes_r \delta \mathcal{G} \rtimes \mathcal{G}^{\text{cop}} \) is called twisted dual action of \((\delta, \Omega)\) or \( \Omega \)-dual action of \( \delta \).

**Proof.** Let us consider the unitary \( \hat{\mathcal{V}}_\mathcal{G} \) as in Theorem-Definition 2.2.6. We are going to show that \( \hat{\delta} \) can be written as a conjugation by \( 1 \otimes \hat{\mathcal{V}}_\mathcal{G} \). Given \( x \in C^*_r(\mathcal{G}, \Omega) \), assume without loss of generality that \( x := (id \otimes \eta)(V^\Omega) \) for some \( \eta \in C(\mathcal{G}) \). On the one hand, using the pentagonal equation satisfied by \( V^\Omega \) (see Theorem-Definition 3.2.8) and Lemma 3.2.10, a direct computation shows that \( (V^\Omega)^*(1 \otimes x)V^\Omega = \hat{\mathcal{V}}_\mathcal{G}(x \otimes 1)\hat{\mathcal{V}}_\mathcal{G}^* \). On the other hand, Theorem-Definition 2.2.6 guarantees that \( \hat{\mathcal{V}}_\mathcal{G}(y \otimes 1)\hat{\mathcal{V}}_\mathcal{G}^* = y \otimes 1 \), for all \( y \in C(\mathcal{G})^\ast = L^2(\mathcal{G}) \supseteq C(\mathcal{G}) \). Combining these two expressions, it is easy to see that \( (\delta(a) \otimes 1)(1 \otimes (V^\Omega)^*(1 \otimes x)V^\Omega) = (1 \otimes \hat{\mathcal{V}}_\mathcal{G})(\delta(a)(1 \otimes x) \otimes 1)(1 \otimes \hat{\mathcal{V}}_\mathcal{G}) \), for all \( a \in A \) and all \( x \in C^*_r(\mathcal{G}, \Omega) \). In other words, these expressions show that the formula of the statement defines a (injective) \( * \)-homomorphism \( A \rtimes_r \mathcal{G} \to \hat{M}(A \rtimes_r \mathcal{G} \otimes c_0(\mathcal{G})) \) given precisely by \( \hat{\delta}(z) = (1 \otimes \hat{\mathcal{V}}_\mathcal{G})(z \otimes 1)(1 \otimes \hat{\mathcal{V}}_\mathcal{G}) \), for all \( z \in A \rtimes_r \mathcal{G} \). It remains to show that \( \hat{\delta} \) defines an action of \( \hat{\mathcal{G}} \) on \( A \rtimes_r \mathcal{G} \). On the one
hand, the density condition for $\hat{\delta}$ is obtained as follows:

$$
\left[\hat{\delta}(A \rtimes_{r,(\delta,\Omega)} G)(1 \otimes c_0(\hat{G}))\right] = \left[(\delta(A) \otimes 1)(1 \otimes (V^\Omega)^* (1 \otimes C^*_p(G, \Omega))V^\Omega)(1 \otimes 1 \otimes c_0(\hat{G}))\right]
$$

$$
\overset{(*)}{=} \left[(\delta(A) \otimes 1)(1 \otimes C^*_p(G, \Omega) \otimes c_0(\hat{G}))\right]
$$

$$
= (A \rtimes_{r,(\delta,\Omega)} G) \otimes c_0(\hat{G}),
$$

where in $(*)$ we use Remark 4.1.12. On the other hand, the compatibility of $\hat{\delta}$ with $\hat{\Delta}_{\text{cop}}$ is obtained by a direct computation using again Remark 4.1.12.  

To end this section, let us introduce the following nomenclature for a special type of quantum dynamical systems.

**4.2.7 Definition.** Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle of finite type on $G$. Let $H$ be a Hilbert space. A right twisted dynamical system $(G, K(H), \delta, \Omega)$ is called $\Omega$-inner if there exists an $\Omega$-representation $u \in \mathcal{B}(H) \hat{\otimes} L^\infty(G)$ such that $\delta(a) = \delta_\omega(a) = u(a \otimes 1)u^*$, for all $a \in K(H)$. In this case, the data $(G, K(H), \delta, \Omega, u)$ is called a right twisted inner dynamical system or right $\Omega$-inner dynamical system.

So a right twisted dynamical system $(G, K(H), \delta, \Omega)$ is nothing but a projective left $G_{\Omega}$-representation induced from an $\Omega$-representation of $G$.

It is well-known that an inner action is exterior equivalent to the trivial one, so that the corresponding crossed products are isomorphic (see for instance [19] for more details). The following proposition shows that a similar phenomenon occurs in the quantum group setting when the action is $\Omega$-inner.

**4.2.8 Proposition.** Let $G$ be a compact quantum group and $\Omega$ a 2-cocycle of finite type on $G$. Let $H$ be a Hilbert space. If $(G, K(H), \delta, \Omega, v)$ is a right $\Omega$-inner dynamical system, then

$$
K(H) \rtimes_{r,(\delta,\Omega)} G \cong K(H) \otimes c_0(\hat{G}).
$$

**Proof.** We can represent $K(H) \rtimes_{r,(\delta,\Omega)} G$ as the normclosure of $(1 \otimes C^*_p(G, \Omega))\delta_\omega(K(H))$ on $H \otimes L^2(G)$. It is then sufficient to show that $u^*[((1 \otimes C^*_p(G, \Omega))\delta_\omega(K(H)))]u = K(H) \otimes c_0(\hat{G})$. But since $u^*_{23}(V^\Omega)_{23}u_{12} = u_{13}(V_G)_{23}$ as $u$ is an $\Omega$-representation, we have that $u^*[((1 \otimes C^*_p(G, \Omega))\delta_\omega(K(H)))] = [((id \otimes id \otimes \omega)(u_{13}(x \otimes V_G)) | x \in K(H), \omega \in \mathcal{B}(L^2(G)))_\omega]$. As $u$ is the direct sum of finite-dimensional $\Omega$-representations, it follows that this last set equals $[((id \otimes id \otimes \omega)(x \otimes V_G)) | x \in K(H), \omega \in \mathcal{B}(L^2(G)))_\omega] = K(H) \otimes c_0(\hat{G})$.  

**4.2.9 Remark.** Note that if $v$ is a unitary $\Omega^*$-representation, we also have an ordinary right action $K(H) \overset{\delta}{\leftarrow} G$ by putting $\delta(a) = \delta_\omega(a) = v(a \otimes 1)v^*$, for all $a \in \mathcal{B}(H)$. We then say that the action $K(H) \overset{\delta}{\leftarrow} G$ is $\Omega$-inner. An analogous computation as above yields that in this case we have for the untwisted crossed product that $K(H) \rtimes_{r,\Omega} G \cong K(H) \otimes C^*_p(G, \Omega)$. Analogously, from an $\Omega$-representation $u$ of $G$, one could consider an ordinary left action
\[ G \xrightarrow{\delta} \mathcal{K}(H) \] by putting \( \delta(a) = \Sigma(u^*(a \otimes 1)u) \), for all \( a \in B(H) \). In this case, the (untwisted) crossed product is defined as \( G \rtimes \mathcal{K}(H) = C^*\langle \lambda(c_{0}(G) \otimes 1)(\rho \otimes \text{id})\delta(A) \rangle \) where we recall that \( \rho(x) = U_G \lambda(x)U_G \) for \( x \in C(G) \). Then we now have, upon using the regular representation \( W_G = \hat{V}_G \) that \( G \rtimes \mathcal{K}(H) \cong \text{Ad}(u_{21} U_G, 1) \) where we use the notation of Remark 4.1.6.

### 4.3. Twisted Takesaki-Takai duality, twisted descent map and twisted Baaj-Skandalis duality

#### 4.3.1 Definition

Let \((G, A, \delta, \Omega)\) be a twisted dynamical system. The double reduced crossed product of \( A \) by \( \hat{G}^{cop} \) with respect to \( \delta \), denoted by \( (A \rtimes G) \rtimes \hat{G}^{cop} \), is the \( C^* \)-algebra defined by:

\[
(A \rtimes G) \rtimes \hat{G}^{cop} := C^*\langle \delta_U(A \rtimes G)(1 \otimes C(G)) \rangle \subset \mathcal{L}_A((A \otimes L^2(G)) \otimes L^2(G)),
\]

where \( \delta_U(x) = (1 \otimes U_G)\hat{\delta}(x)(1 \otimes U_G) \) for \( x \in A \rtimes G \).

#### 4.3.2 Remark

Observe that the crossed product \( (A \rtimes G) \rtimes \hat{G}^{cop} \) defined above is the usual one as, say, given in [5, Section 7] (we do not consider any deformation in this definition, contrary to Definition 4.2.2). In particular, we have automatically that \( (A \rtimes G) \rtimes \hat{G}^{cop} = \overline{\text{span}}\{\delta_U(A \rtimes G)(1 \otimes C(G))\} \).

Since \( (A \rtimes G) \rtimes \hat{G}^{cop} \) defined above is a usual crossed product, we can define the corresponding dual action \( \hat{\delta} \). Hence, the following proposition follows from the standard theory of crossed products.

#### 4.3.3 Proposition-Definition

Let \((G, A, \delta, \Omega)\) be a twisted dynamical system. The double reduced crossed product \( (A \rtimes G) \rtimes \hat{G}^{cop} \) is a \( G \)-\( C^* \)-algebra with action \( (A \rtimes G) \rtimes \hat{G}^{cop} \overset{\hat{\delta}}{\longrightarrow} G \) such that:

\[
\hat{\delta}(\delta_U(z)(1 \otimes y)) = (\hat{\delta}_U(z) \otimes 1)(1 \otimes V_G(y \otimes 1)V_G^*),
\]

for all \( z \in A \rtimes G \) and all \( y \in C(G) \). The action \( (A \rtimes G) \rtimes \hat{G}^{cop} \overset{\hat{\delta}}{\longrightarrow} G \) is called twisted double dual action of \( (\delta, \Omega) \) or \( \Omega \)-double dual action of \( \delta \).

The following theorem is a special case of [32, Theorem 3.6] by means of [32, Proposition 3.5], see also [42, Section 1] in the von Neumann algebraic setting (where no regularity assumption is needed).
4.3.4 Theorem (Twisted Takesaki-Takai duality). Let \((\mathbb{G}, A, \delta, \Omega)\) be a twisted dynamical system with \(\Omega\) of finite type.

i) There is a canonical isomorphism of \(C^*\)-algebras, \(A \otimes \mathcal{K}(L^2(\mathbb{G})) \cong (A \rtimes_{r,(\delta,\Omega)} \mathbb{G}) \rtimes \hat{\mathbb{G}}^{\text{cop}},\) given by the map \(x \mapsto (U_\delta)_3(\Omega^*)_{23}(\delta \otimes \text{id})(x)(\Omega)_3(\Omega)\).

ii) Under the above isomorphism, the twisted double dual action \(\hat{\delta}\) of \((\delta, \Omega)\) is conjugate to the action \(A \otimes \mathcal{K}(L^2(\mathbb{G})) \overset{\hat{\delta}}{\to} \mathbb{G}\) defined by \(\hat{\delta} := \text{Ad}_{(W_\Omega)_3} \circ \delta_{13}\), where \(\delta_{13}\) denotes the amplified twisted action of \(\mathbb{G}\) on \(A \otimes \mathcal{K}(L^2(\mathbb{G}))\) such that:

\[\delta_{13}(a \otimes T) = (1 \otimes \Sigma)(\delta(a) \otimes \text{id})(1 \otimes \Sigma)(1 \otimes T \otimes 1) \in A \otimes \mathcal{K}(L^2(\mathbb{G})) \otimes \mathcal{C}(\mathbb{G}_{\Omega}),\]

for all \(a \in A\) and \(T \in \mathcal{K}(L^2(\mathbb{G}))\).

**Proof.** By Theorem 4.1.9, we have \(\mathcal{K}(L^2(\mathbb{G})) = [C^*_r(\mathbb{G}) U_\Omega C(\mathbb{G}) U_\Omega]\). Since \(\delta\) is continuous, we can hence write \(A \otimes \mathcal{K}(L^2(\mathbb{G})) = [\delta(A)(1 \otimes \mathcal{K}(L^2(\mathbb{G})))] = [\delta(A)(1 \otimes C^*_r(\mathbb{G}, \Omega) U_\Omega C(\mathbb{G}) U_\Omega)]\), and the first item then follows by a straightforward computation. The second item then follows from Remark 4.1.6 together with the fact that \(W_\mathbb{G}(1 \otimes U_y U)(W_\mathbb{G})^* = (1 \otimes U) \Delta^{op}(y)(1 \otimes U)\), which follows from the identity \(W = V_\mathbb{G}\) in Theorem-Definition 2.2.6. ■

As a corollary of the twisted Takesaki-Takai duality established in Theorem 4.3.4 we obtain the following generalization of the well known Packer-Raeburn’s untwisting trick or Packer-Raeburn’s stabilisation trick [35] to compact quantum groups.

4.3.5 Proposition (Quantum Packer-Raeburn’s untwisting trick). Let \((\mathbb{G}, A, \delta, \Omega)\) be a twisted dynamical system with \(\Omega\) of finite type. Then

\[(A \rtimes_{r,(\delta,\Omega)} \mathbb{G}) \otimes \mathcal{K}(L^2(\mathbb{G})) \cong (A \otimes \mathcal{K}(L^2(\mathbb{G})) \rtimes \mathbb{G}).\]

**Proof.** Put \(B := A \rtimes_{r,(\delta,\Omega)} \mathbb{G}\), which is a \(\hat{\mathbb{G}}^{\text{cop}}\)-\(C^*\)-algebra with action \((B \rtimes \hat{\mathbb{G}}^{\text{cop}} \rtimes \mathbb{G}\). Using the usual version of Takesaki-Takai duality for quantum groups, we can write \(B \otimes \mathcal{K}(L^2(\mathbb{G})) \cong (B \rtimes \hat{\mathbb{G}}^{\text{cop}} \rtimes \mathbb{G}\). Next, using the twisted version of Takesaki-Takai duality from Theorem 4.3.4 we have the following:

\[(B \rtimes \hat{\mathbb{G}}^{\text{cop}} \rtimes \mathbb{G}) = ((A \rtimes_{r,(\delta,\Omega)} \mathbb{G}) \rtimes \hat{\mathbb{G}}^{\text{cop}} \rtimes \mathbb{G}) \cong (A \otimes \mathcal{K}(L^2(\mathbb{G})) \rtimes \mathbb{G}).\]

This trick helps to establish a twisted version of the well known Baaj-Skandalis duality [5, 43].

4.3.6 Note. “Takesaki-Takai duality” for quantum groups is also referred as “Baaj-Skandalis duality” in the literature. We prefer to reserve the latter terminology for such duality at the level of KK-theory.
4.3.7 Theorem (Twisted Baaj-Skandalis duality). Let \((G, A, \delta, \Omega)\) and \((\mathcal{G}, B, \vartheta, \Omega)\) be two twisted dynamical systems with respect to a given 2-cocycle \(\Omega\) of finite type on \(G\). Then there exists a canonical group isomorphism:

\[
J_G^\Omega : KK^{G_\Omega}(A, B) \xrightarrow{\sim} KK^{G_{\mathcal{G}}} (A \rtimes_r \Omega, B \rtimes_r \mathcal{G}, \Omega),
\]

which is compatible with the Kasparov product, that is, if \((G, C, \nu, \Omega)\) is another twisted dynamical system with respect to \(\Omega\), then we have:

\[
J_G^\Omega(\mathcal{X}) \otimes C_\mathcal{G} \cong J_G^\Omega(\mathcal{X}) \otimes C_\mathcal{G} \otimes J_G^\Omega(\mathcal{Y}) \quad \text{and} \quad J_G^\Omega(1_A) = 1_A \rtimes_r \mathcal{G},
\]

for all \(\mathcal{X} \in KK^{G_\Omega}(A, C)\) and \(\mathcal{Y} \in KK^{G_\Omega}(C, B)\). Moreover, we have \(J_G^\Omega = \mathcal{O}_G \circ J_G^\Omega\), where \(\mathcal{O}_G\) is the obvious forgetful functor.

Proof. Given the twisted dynamical system \((G, A, \delta, \Omega)\), put \(A_1 := A \rtimes \Omega\). Thus \((A_1, \mu)\) is an object in \(\mathcal{K}_{\mathcal{K}}^{G_\Omega}\). It is known that \(G_\Omega\) and \(\hat{G}\) are monoidally co-Morita equivalent in the sense of \([12], [15]\). The corresponding co-linking quantum groupoid takes the form 

\[
C^r(G_\Omega) \oplus C^r(\mathcal{G}, \Omega) \oplus C^r(G, \Omega) \oplus C^r(G). \quad \text{Next, following the notations from } [3, \text{Section 2.4}], \text{we consider the exterior comultiplication } \hat{\Delta}_{ext} = (\hat{\Delta}_{\Omega})_\Omega^2 : C^r(G_\Omega) \rightarrow \hat{\mathcal{M}}(C^r(G_\Omega, \Omega^*) \otimes C^r(G, \Omega)). \quad \text{Following } [3, \text{Proposition 4.1}], \text{we consider the } C^*-\text{algebra:}
\]

\[
A_2 := \text{span}\{(id \otimes id \otimes \mu)(\eta A_1 \otimes \Delta_{ext}^\mu) \delta(a') \mid a' \in A_1, \eta \in B(L^2(G))_*\}
= \text{span}\{(id \otimes id \otimes \mu)(\delta(a) \otimes 1)(1 \otimes \Delta_{ext}^\mu(x)) \mid a' \in A, x \in C^r(G_\Omega), \eta \in B(L^2(G))_*\}.
\]

Since \(\{id \otimes \mu) \Delta_{ext}^\mu(x) \mid x \in C^r(G_\Omega), \eta \in B(L^2(G))_*\}\) is norm-dense in \(C^r(G_\Omega)\), we have by construction that \(A_2 = A \rtimes_r \mathcal{G}\). Now, \(A_2\) is an object in \(\mathcal{K}_{\mathcal{K}}^{G_\Omega}\) with \(\hat{\mathcal{G}}^\mu\)-action given by Proposition 4.2.6, which we still denote by \(\hat{\mu}\).

In other words, the equivalence of triangulated categories \(\mathcal{K}_{\mathcal{K}}^{G_\Omega} \cong \mathcal{K}_{\mathcal{K}}^{\hat{\mathcal{G}}^\mu}\) from \([3, \text{Section 4.5}]\) sends \(A \rtimes \Omega\) to \(A \rtimes_r \mathcal{G}\). An application of the equivariant Morita equivalence of \([3, \text{Section 4.4}]\) yields that \(A \rtimes \mathcal{K}(L^2(G)) \rtimes \mathcal{G} \cong A \rtimes \hat{\mathcal{G}}^\mu\rtimes \mathcal{G}\) in \(\mathcal{K}_{\mathcal{K}}^{G_\Omega}\) (here \(\hat{\mu}\) is defined by Takesaki-Takai duality for \(\mathcal{G}\)) is sent to \(A \rtimes \mathcal{K}(L^2(G)) \rtimes \mathcal{G} \cong A \rtimes \hat{\mathcal{G}}^\mu\rtimes \mathcal{G}\) in \(\mathcal{K}_{\mathcal{K}}^{G_\Omega}\) (here \(\hat{\mu}\) is defined through the twisted Takesaki-Takai from Theorem 4.3.4) through \(J_{\mathcal{G}_\Omega}^\mu, \hat{\mathcal{G}}^\mu\).
Therefore, the twisted Baaj-Skandalis is obtained as follows:

\[
KK^{G_\alpha}(A, B) \simeq KK^{G_\alpha}(A \otimes K(L^2(G)), B \otimes K(L^2(G)))
\]

\[
\simeq KK^{G_\alpha}( (A \otimes K(L^2(G)) \rtimes _{r,\delta} G, (B \otimes K(L^2(G)) \rtimes _{r,\delta} G)
\]

\[
\simeq KK^{G_\alpha}(A \rtimes _{r, \alpha, \delta} \Omega G, B \rtimes _{r, \alpha, \delta} \Omega G)
\]

As a consequence, we obtain a twisted version of the well-known descent map for quantum groups [43].

4.3.8 Corollary (Twisted descent map). Let \((G, A, \delta, \Omega)\) and \((G, B, \vartheta, \Omega)\) be two twisted dynamical systems with respect to a given 2-cocycle \(\Omega\) of finite type on \(G\). Then there exists a canonical group homomorphism:

\[
j_\Omega^G : KK^{G_\alpha}(A, B) \longrightarrow KK(A \rtimes _{r, \alpha, \delta} \Omega G, B \rtimes _{r, \alpha, \delta} \Omega G)
\]

called twisted descent map (with respect to \(G\)). Moreover, \(j_\Omega^G\) is compatible with the Kasparov product, that is, if \((G, C, \nu, \Omega)\) is another twisted dynamical system with respect to \(\Omega\), then we have:

\[
j_\Omega^G(\mathcal{A} \otimes \mathcal{Y}) = j_\Omega^G(\mathcal{A}) \otimes _{r,(\nu, \Omega)} \mathcal{G} j_\Omega^G(\mathcal{Y}) \quad \text{and} \quad j_\Omega^G(1_A) = 1_A \rtimes _{r,(\delta, \Omega)} \mathcal{G},
\]

for all \(\mathcal{A} \in KK^{G_\alpha}(A, C)\) and \(\mathcal{Y} \in KK^{G_\alpha}(C, B)\).

Proof. Similarly as in Theorem 4.3.7 we have \(KK^{G_\alpha}(A, B) \simeq KK^{G_{\alpha, \delta}}( (A \otimes K(L^2(G)) \rtimes _{r,\delta} G, (B \otimes K(L^2(G)) \rtimes _{r,\delta} G)
\]

Applying Baaj-Skandalis duality for \(G\), the latter is isomorphic to \(KK^G( (A \otimes K(L^2(G)), B \otimes K(L^2(G)) \rtimes _{r,\delta} G)
\]

Applying the ordinary descent map \(j_G\), we get \(KK( (A \otimes K(L^2(G)) \rtimes _{r,\delta} G, (B \otimes K(L^2(G)) \rtimes _{r,\delta} G), which is isomorphic to \(KK(A \rtimes _{r, \alpha, \delta} \Omega G, B \rtimes _{r, \alpha, \delta} \Omega G)\) thanks to the untwisting trick from Proposition 4.3.5. The map \(j_\Omega^G\) of the statement is obtained as result of these compositions.

5. APPLICATION: QUANTUM ASSEMBLY MAP FOR PERMUTATION TORSION-FREE DISCRETE QUANTUM GROUPS

5.1. The Baum-Connes property for discrete quantum groups

We collect here the main category framework for the formulation of the Baum-Connes property for discrete quantum groups. We refer to [27] or [22] for a complete presentation of the subject.
Let $\hat{G}$ be a discrete quantum group and consider the corresponding equivariant Kasparov category, $\mathcal{K}_G^G$, with canonical suspension functor denoted by $\Sigma$. $\mathcal{K}_G^G$ is a triangulated category whose distinguished triangles are given by mapping cone triangles (see [27] for more details). The word homomorphism (resp. isomorphism) will mean homomorphism (resp. isomorphism) in the corresponding Kasparov category; it will be a true homomorphism (resp. isomorphism) between $\hat{G}$-C*-algebras or any Kasparov triple between $\hat{G}$-C*-algebras (resp. any equivariant KK-equivalence between $\hat{G}$-C*-algebras). Analogously, we can consider the equivariant Kasparov category $\mathcal{K}_L^L$.

Assume for the moment that $\hat{G}$ is torsion-free. In that case, consider the usual complementary pair of localizing subcategories in $\mathcal{K}_G^G$, $(\mathcal{L}_G, \mathcal{N}_G)$. Denote by $(L, N)$ the canonical triangulated functors associated to this complementary pair. More precisely we have that $\mathcal{L}_G$ is defined as the localizing subcategory of $\mathcal{K}_G^G$ generated by the objects of the form $\text{Ind}_G^C(C) = C \otimes c_0(\hat{G})$ with $C$ any C*-algebra in the Kasparov category $\mathcal{K}_G^G$ and $\mathcal{N}_G$ is defined as the localizing subcategory of objects which are isomorphic to $0$ in $\mathcal{K}_G^G$: $\mathcal{L}_G := \langle \{\text{Ind}_G^C(C) = C \otimes c_0(\hat{G}) \mid C \in \text{Obj}(\mathcal{K}_G^G)\} \rangle$ and $\mathcal{N}_G = \{A \in \text{Obj}(\mathcal{K}_G^G) \mid \text{Res}_G^A(A) = 0\}$.

If $\hat{G}$ is not torsion-free, then a technical property lacked in the literature in order to define a suitable complementary pair. The natural candidate used in the related works (see for instance [28] and [45]) is given by the following localizing subcategories of $\mathcal{K}_G^G$:

$$\mathcal{L}_G := \langle \{C \otimes G \otimes T \mid C \in \text{Obj}(\mathcal{K}_G^G), T \in \text{Tor}(\hat{G})\} \rangle,$$

$$\mathcal{N}_G := \mathcal{L}_G^\perp = \{A \in \text{Obj}(\mathcal{K}_G^G) \mid KR^G(L, \text{Res}_G^A(A)) = 0, \forall L \in \text{Obj}(\mathcal{L}_G)\}.$$  

5.1.1 Remark. We put $\hat{\mathcal{L}}_G := \langle \{T \otimes C \mid C \in \text{Obj}(\mathcal{K}_G^G), T \in \text{Tor}(\hat{G})\} \rangle$, so that we have $G \otimes \hat{\mathcal{L}}_G = \mathcal{L}_G$ by definition. Similarly we put $\hat{\mathcal{N}}_G := \hat{G} \otimes \mathcal{N}_G$.

Recently, Y. Arano and A. Skalski [2] have showed that these two subcategories form indeed a complementary pair of localizing subcategories in $\mathcal{K}_G^G$. As an application of our cleanness property obtained in Theorem 3.1.5, we are going to prove in Section 5.3 that $(\mathcal{L}_G, \mathcal{N}_G)$ is indeed complementary when $\mathcal{L}_G$ is defined only in terms of torsion actions of $G$ of projective type. It consists in generalizing the Green-Julg isomorphism.

5.1.2 Note. The following nomenclature is useful. Given $A \in \text{Obj}(\mathcal{K}_G^G)$ consider a $(\mathcal{L}_G, \mathcal{N}_G)$-triangle associated to $A$, say $\Sigma N(A) \rightarrow L(A) \rightarrow A \rightarrow N(A)$. We know that such triangles are distinguished and unique up to isomorphism (recall Theorem 2.5.11). The homomorphism $D : L(A) \rightarrow A$ is called Dirac homomorphism for $A$. In particular, we consider the Dirac homomorphism for $\mathbb{C}$ (as trivial $\hat{G}$-C*-algebra), $D_\mathbb{C} : L(\mathbb{C}) \rightarrow \mathbb{C}$. We refer to $D_\mathbb{C}$ simply as Dirac homomorphism.

5.2. Two-sided crossed products

In this section we give some technical tools necessary for the next section. In order to define an assembly map for discrete quantum groups, we need to define a suitable pair of adjoint
functors on $\mathcal{H}$ taking into account the torsion phenomena of $\hat{G}$. For classical discrete groups, the torsion phenomena are completely described in terms of finite subgroups. Hence, the induction and restriction functors will provide such an adjunction. In the quantum case, the torsion is described in terms of $G$-C*-algebras, so the induction-restriction approach is no longer valid since finite discrete quantum groups do not exhaust the torsion phenomena for $\hat{G}$. Moreover, the torsion objects in the Kasparov category are also $G$-C*-algebras. In this sense, we need a construction encoding both the induction process from the classical setting and the diagonal action with respect to $G$. We call it a two-sided crossed product and it is already used in [2], see also [33, Section 2.6].

5.2.1 Definition. Let $G$ be a compact quantum group. If $(B, \beta)$ is a right $G$-C*-algebra and $(A, \alpha)$ is a left $G$-C*-algebra, then the two-sided crossed product of $B$ and $A$ by $G$, denoted by $B \times G \ltimes A$, is the C*-algebra defined by:

$$B \times G \ltimes A := C^*(\langle (id \otimes \lambda)(B) \otimes 1)(1 \otimes \lambda(c_0(\hat{G}))) \otimes 1)(1 \otimes (\rho \otimes id)(\alpha(A))) \rangle \subset \mathcal{L}_{B \otimes A}(B \otimes L^2(G) \otimes A).$$

5.2.2 Remark. First, to lighten the notations we will omit the representations $\lambda, \hat{\lambda}$ and $\rho$ in the definition of $B \times G \ltimes A$, and note that then $\rho(x) = U_G x U_G$ for $x \in C(G)$. We also write $\alpha_U(x) = (U_G \otimes id)\alpha(x)(U_G \otimes id)$ for $x \in A$. Next, it is easy to show that $B \times G \ltimes A = \text{span}\{((\beta(B) \otimes 1)(1 \otimes c_0(\hat{G})) \otimes 1)(1 \otimes (r \otimes id)(\alpha(A)))\}$, cf. Lemma 4.2.4. From now on we will use these two descriptions of $B \times G \ltimes A$ interchangeably. As a consequence, we see that the maps $B \rightarrow \mathcal{L}_{B \otimes A}(B \otimes L^2(G) \otimes A)$, $A \rightarrow \mathcal{L}_{B \otimes A}(B \otimes L^2(G) \otimes A)$ and $c_0(\hat{G}) \rightarrow \mathcal{L}_{B \otimes A}(B \otimes L^2(G) \otimes A)$ given by $b \mapsto \beta(b) \otimes 1$, $a \mapsto 1 \otimes \alpha_U(a)$ and $x \mapsto 1 \otimes \lambda(x) \otimes 1$ respectively, send $B$, $A$ and $c_0(\hat{G})$ respectively onto non-degenerate $C^*$-subalgebras of $M(B \times G \ltimes A)$.

As for usual crossed products, we can show that the two-sided crossed product construction is functorial. More precisely, we have the following:

5.2.3 Proposition. Let $G$ be a compact quantum group. Let $(B, \beta)$, $(B', \beta')$ be right $G$-C*-algebras and $(A, \alpha)$, $(A', \alpha')$ left $G$-C*-algebras.

i) If $\phi : B \rightarrow M(B')$ is a non-degenerate $G$-equivariant $*$-homomorphism, then there exists a non-degenerate $*$-homomorphism:

$$\phi \times id \times id : B \times G \ltimes A \rightarrow M(B' \times G \ltimes A)$$

such that $\phi \times id \times id((\beta(b) \otimes 1)(1 \otimes x \otimes 1)(1 \otimes \alpha_U(a))) = (\beta'(\phi(b)) \otimes 1)(1 \otimes x \otimes 1)(1 \otimes \alpha_U(a))$, for all $b \in B$, $a \in A$ and $x \in c_0(\hat{G})$.

ii) If $\psi : A \rightarrow M(A')$ is a non-degenerate $G$-equivariant $*$-homomorphism, then there exists a non-degenerate $*$-homomorphism:

$$id \times id \times \psi : B \times G \ltimes A \rightarrow M(B \times G \ltimes A')$$

such that $id \times id \times \psi((\beta(b) \otimes 1)(1 \otimes x \otimes 1)(1 \otimes \alpha_U(a))) = (\beta(b) \otimes 1)(1 \otimes x \otimes 1)(1 \otimes \alpha_U'(\psi(a)))$, for all $b \in B$, $a \in A$ and $x \in c_0(\hat{G})$. 44
If a torsion action of projective type of \( \mathbb{G} \) is involved in a two-sided crossed product, then we can give an alternative description of the latter, which is useful for our purpose. Recall first from Lemma 3.2.3 that, given a torsion action of projective type \( \delta \) on \( T = \mathcal{B}(H) \) with implementing \( \Omega^* \)-representation \( u \) for some \( \Omega \), we can construct the \( \Omega \)-representation \( u^\circ \) on \( \mathcal{F} \) with associated coaction \( \delta^\circ \) on \( T^{op} \cong \mathcal{B}(\mathcal{F}) \). By Remark 4.1.4 we can view \( u^\circ \) as a \( \Omega = (\Omega^*)^* \)-representation of \( \mathbb{G}_\Omega \), and correspondingly \( \delta^\circ \) as a right action of \( \mathbb{G}_\Omega \) on \( T^{op} \).

### 5.2.4 Proposition

Let \( \mathbb{G} \) be a compact quantum group. Let \( (T, \delta) \) be a torsion action of projective type of \( \mathbb{G} \). Let \( u \) be an \( \Omega^* \)-representation of \( \mathbb{G} \) implementing \( \delta \) for some \( 2 \)-cocycle \( \Omega \) (necessarily of finite type). Let \( (B, \beta) \) be a \( \mathbb{G} \)-\( C^* \)-algebra. Then \( B \rtimes \mathbb{G} \), \( B \rtimes \mathbb{G} \rtimes \mathbb{G}_\Omega \), and \( B \rtimes \mathbb{G} \rtimes \mathbb{G}_\Omega \) are isomorphic as \( \mathbb{G} \)-algebras, where \( \tilde{\beta} := Ad_{u^\circ_{23}} \circ \beta_{13} \).

We use here that the twisted quantum group \( \mathbb{G}_\Omega \) is again a compact quantum group by Theorem 3.3.7.

**Proof.** First of all, it is straightforward to check that \( \tilde{\beta} := Ad_{u^\circ_{23}} \circ \beta_{13} \) is an action of \( \mathbb{G}_\Omega \) on \( B \otimes T^{op} \). For given \( b \in B \) and \( x \in T^{op} \) we have:

\[
(id \otimes \Omega \Delta \delta^\circ)(b \otimes x) = (id \otimes \Omega \Delta \delta^\circ)(u^\circ_{23}(b_{(0)} \otimes x \otimes b_{(1)})(u^\circ_{23})^*)
\]

\[
= \Omega_{34} \Delta_3(u^\circ_{23}(b_{(0)} \otimes x \otimes \Delta(b_{(1)})) \Delta_3(u^\circ_{23})^* \Omega_{34}
\]

\[
\overset{(*)}{=} u^\circ_{23} u^\circ_{24}(b_{(0)} \otimes x \otimes \Delta(b_{(1)})) u^\circ_{24} (u^\circ_{23})^*
\]

\[
\overset{(**)}{=} u^\circ_{23} u^\circ_{24}(b_{(0)} \otimes x \otimes b_{(0)} \otimes b_{(1)} \otimes b_{(1)})(u^\circ_{24})^* (u^\circ_{23})^*
\]

\[
= (\tilde{\beta} \otimes id)(u^\circ_{23}(b_{(0)} \otimes x \otimes b_{(1)})(u^\circ_{23})^*) = (\tilde{\beta} \otimes id)(b \otimes x),
\]

where in \((*)\) we used the fact that \( u^\circ \) is a \( \Omega \)-representation of \( \mathbb{G} \) and in \((**)\) we use the fact that \( \beta \) is a \( \mathbb{G} \)-action on \( B \).

Next, recall that \( u^\circ \) implements \( \delta \). Then on the one hand, by Remark 4.2.9 we have \( \mathbb{G} \rtimes T^{op} \cong C^*_r(\mathbb{G}_\Omega, \Omega) \otimes T^{op} \) (here the identification is given by conjugating with \( u^\circ_{21}(U_{\mathbb{G}} \otimes 1) \)). On the other hand, by Remark 4.1.4 we view \( u^\circ \) as a \( \Theta^* := (\Omega^*)^* \)-representation of \( \mathbb{G}_\Omega \) so that, by applying again Remark 4.2.9, we obtain that \( T^{op} \rtimes \mathbb{G}_\Omega \cong T^{op} \otimes C^*_r(\mathbb{G}_\Omega, \Theta) = T^{op} \otimes C^*_r(\mathbb{G}_\Omega, \Omega^*) \) (here the identification is given by conjugating with \( (u^\circ)^* \)). Recall from Remark 4.1.5 that \( C^*_r(\mathbb{G}_\Omega, \Omega^*) = JC^*_r(\mathbb{G}, \Omega)J \). Therefore, by Remark 4.1.6 we see that \( T^{op} \rtimes \mathbb{G}_\Omega \) is \(*\)-isomorphic to \( \mathbb{G} \rtimes T^{op} \) by introducing an extra conjugation with \( X_{\mathbb{G}} U_{\mathbb{G}} \) (and a flip map).

This allows to conclude the isomorphism of the statement. Indeed, since \( U_{\mathbb{G}} X_{\mathbb{G}} U_{\mathbb{G}} \in L^\infty(\mathbb{G})' \),
we compute

\[
(B \otimes T^{op}) \otimes \mathbb{G} = \text{span}\{ (id \otimes id \otimes \lambda) \tilde{\beta}(B \otimes T^{op})(1 \otimes 1 \otimes c_0(\mathcal{G}_\Omega)) \}
\]

\[
= \text{span}\{ (id \otimes id \otimes \lambda) u_{23}^\circ (B(0) \otimes T^{op} \otimes B(1))(u_{23}^\circ)^*(1 \otimes 1 \otimes c_0(\mathcal{G}_\Omega)) \}
\]

\[
\cong \text{span}\{ (id \otimes id \otimes \lambda) (B(0) \otimes T^{op} \otimes B(1))(u_{23}^\circ)^*(1 \otimes 1 \otimes c_0(\mathcal{G}_\Omega)) \}
\]

\[
= \text{span}\{ (id \otimes id \otimes \lambda)(B(0) \otimes 1 \otimes B(1))(1 \otimes T^{op} \otimes 1)(1 \otimes (u^\circ)^*(1 \otimes c_0(\mathcal{G}_\Omega))u^\circ) \}
\]

\[
= \text{span}\{ (id \otimes id \otimes \lambda)(B(0) \otimes 1 \otimes B(1))(1 \otimes T^{op} \otimes C^*_r(\mathbb{G}_\Omega, \Omega^*)) \}
\]

\[
\cong \text{span}\{ (id \otimes id \otimes \lambda)(B(0) \otimes 1 \otimes B(1))(1 \otimes T^{op} \otimes JC^*_r(\mathbb{G}, \Omega)J) \}
\]

\[
\cong \text{span}\{ (id \otimes \lambda \otimes id)(B(0) \otimes B(1) \otimes 1)(1 \otimes U_G C^*_r(\Omega, \mathbb{G})U_G \otimes T^{op}) \}
\]

\[
\cong \text{span}\{ (id \otimes \lambda \otimes id)(B(0) \otimes B(1) \otimes 1)(1 \otimes c_0(\mathcal{G}_\Omega) \otimes 1)(1 \otimes (\rho \otimes 1)(1 \otimes (u^\circ)^*(1 \otimes T^{op})u^\circ \Sigma) \}
\]

\[
= \text{span}\{ \{(id \otimes \lambda)(B(0) \otimes 1)(1 \otimes c_0(\mathcal{G}_\Omega) \otimes 1)(1 \otimes (\rho \otimes 1)(\tilde{T}(T^{op})) \} = B \otimes \mathbb{G} \otimes T^{op},
\]

where in (*) and (**) we have used the identifications $T^{op} \cong \mathbb{G}_\Omega \cong T^{op} \otimes C^*_r(\mathbb{G}_\Omega, \Omega^*)$ and $\mathbb{G} \otimes T^{op} \cong C^*_r(\Omega, \mathbb{G}) \otimes T^{op}$ explained above, respectively.  

The two-sided crossed product construction can also be defined for Hilbert modules in a similar way as we do for usual crossed products.

**5.2.5 Definition.** Let $\mathbb{G}$ be a compact quantum group. Let $(B, \beta)$ be a right $\mathbb{G}$-$C^*$-algebra and $(A, \alpha)$ a left $\mathbb{G}$-$C^*$-algebra. If $(E, \delta_E)$ is a $\mathbb{G}$-equivariant Hilbert $B$-module, we define the two-sided crossed product of $E$ and $A$ by $\mathbb{G}$, denoted by $E \rtimes \mathbb{G} \ltimes A$, as the following Hilbert $B \rtimes \mathbb{G} \ltimes A$-module $E \rtimes \mathbb{G} \ltimes A := E \otimes B \rtimes \mathbb{G} \ltimes A$.

As for the usual crossed products, the embeddings of $E \cong \mathcal{K}_B(B, E)$ and $B \cong \mathcal{K}_B(B)$ into $\mathcal{K}_B(B \oplus E) \cong B \rtimes \mathbb{G} \ltimes A$ induce an embedding of $E \rtimes \mathbb{G} \ltimes A$ into $\mathcal{K}_B(B \oplus E) \cong \mathbb{G} \ltimes A$. In this way we have the following (see for instance [43, Lemme 5.2] for a proof):

**5.2.6 Proposition.** Let $\mathbb{G}$ be a compact quantum group. Let $(B, \beta)$ be a right $\mathbb{G}$-$C^*$-algebra and $(A, \alpha)$ a left $\mathbb{G}$-$C^*$-algebra. If $(E, \delta_E)$ is a $\mathbb{G}$-equivariant Hilbert $B$-module, then $\mathcal{K}_B(E) \rtimes \mathbb{G} \ltimes A \cong \mathcal{K}_B \rtimes \mathbb{G} \ltimes A (E \rtimes \mathbb{G} \ltimes A)$.

Following similar arguments as for usual crossed products (see for instance [43, Proposition 5.3] for more details), it is easy to show that Definition 5.2.5 above passes also at the level of Kasparov triples. More precisely, we have the following:

**5.2.7 Proposition.** Let $\mathbb{G}$ be a compact quantum group. Let $(B, \beta), (B', \beta')$ be right $\mathbb{G}$-$C^*$-algebras and $(A, \alpha)$ a left $\mathbb{G}$-$C^*$-algebra. If $((E, \delta_E), \pi, F)$ is a $\mathbb{G}$-equivariant Kasparov
(B', B)-module in KK^G(B', B), then the triple \((E \times \mathbb{G} \times A, \pi \times id \times id, F \otimes id)\) defines a Kasparov \((B' \times \mathbb{G} \times A, B \times \mathbb{G} \times A)\)-module in KK'^G(B' \times \mathbb{G} \times A, B \times \mathbb{G} \times A).

Finally, as for usual crossed products, the two-sided crossed product functor intertwines the suspension of \(\mathbb{G}\)-C*-algebras and transforms mapping cone triangles into mapping cone triangles. In other words, the functor \((\cdot) \times \mathbb{G} \times A\) preserves semi-split extensions i.e. extensions of \(\mathbb{G}\)-equivariant C*-algebras that split through a \(\mathbb{G}\)-equivariant completely positive contractive linear section, see for instance [25]; and the class of all triangles in \(\mathcal{X}\mathcal{X}^G\) isomorphic to mapping cone triangles is the same as the class of all triangles in \(\mathcal{X}\mathcal{X}^G\) isomorphic to extension triangles (see for instance [24, Lemma 1.2.3.7]). In conclusion, we have obtained that, for a fixed left \(\mathbb{G}\)-C*-algebra \((A_0, \alpha_0)\), the association \((B, \beta) \mapsto B \times \mathbb{G} \times A_0\), for all right \(\mathbb{G}\)-C*-algebra \((B, \beta)\) defines a triangulated functor \(j_{G,A_0} := (\cdot) \times \mathbb{G} \times A_0 : \mathcal{X}\mathcal{X}^G \rightarrow \mathcal{X}\mathcal{X}^G\).

5.3. Twisted Green-Julg isomorphism

First of all, let us recall briefly the Green-Julg isomorphism for compact quantum groups (see [43] for more details). If \(C\) is a C*-algebra equipped with the trivial action of \(\mathbb{G}\), then we have that \(\Psi : KK^G(C, B) \xrightarrow{r,\beta} KK(C, B \rtimes \mathbb{G})\), for all \(\mathbb{G}\)-C*-algebra \((B, \beta)\). Since \(C\) is equipped with the trivial action of \(\mathbb{G}\), then \(C \times \mathbb{G} \simeq C \otimes c_0(\mathbb{G})\) and we have a natural *-homomorphism:

\[
\phi_C : C \rightarrow C \otimes c_0(\mathbb{G}), \quad c \mapsto \phi(c) := c \otimes p_0,
\]

where \(p_0 := (id \otimes h_\mathbb{G})(V_\mathbb{G}) \in c_0(\mathbb{G})\) is the canonical projection onto the subspace of invariant vectors of \((V_\mathbb{G}, L^2(\mathbb{G}))\). In this way we obtain a Kasparov triple \([\phi_C] \in KK(C, C \otimes c_0(\mathbb{G}))\). The Green-Julg isomorphism is given precisely by \(\Psi(\mathcal{X}) := \phi_C^*(j_C(\mathcal{X})) = [\phi_C] \otimes j_C(\mathcal{X}), \quad \mathcal{X} \in KK^G(C, B)\), for all \(\mathcal{X} \in KK^G(C, B)\).

It is also possible to give an explicit expression of its inverse. Given any C*-algebra \(C\) in \(\mathcal{X}\mathcal{X}^G\), we denote by \(\tau(C)\) the same C*-algebra \(C\) equipped with the trivial action of \(\mathbb{G}\) and so we regard it as an object in \(\mathcal{X}\mathcal{X}^G\). In this way, we define the Kasparov triple \(\mathcal{E}_B := [(B \otimes L^2(\mathbb{G}), \pi_r, 0)] \in KK^G(\tau(B \rtimes \mathbb{G}), B)\), where \(\pi_r\) denotes the canonical representation of \(B \rtimes \mathbb{G}\) in \(B \otimes L^2(\mathbb{G})\). The action of \(\mathcal{G}\) on \(B \otimes L^2(\mathbb{G})\) is defined as the tensor product action of \(\beta\) with the action of \(\mathbb{G}\) on \(L^2(\mathbb{G})\) induced by the unitary \(\Sigma V_\mathbb{G} = (U_\mathbb{G} \otimes 1)V_\mathbb{G}(U_\mathbb{G} \otimes 1)\) (see [43] for the precise definitions). Then we have \(\Psi^{-1}(\mathcal{Y}) = \tau(\mathcal{Y}) \otimes \mathcal{E}_B, \quad \mathcal{Y} \in KK(C, B \rtimes \mathbb{G})\). In other words, the Green-Julg isomorphism can be rephrased by saying that the functors \(\mathcal{X}\mathcal{X}^G \xrightarrow{j_C} \mathcal{X}\mathcal{X}^G\) and \(\mathcal{X}\mathcal{X}^G \xrightarrow{r} \mathcal{X}\mathcal{X}^G\) are adjoint: \(\tau\) is a left adjoint of \(j_C\). Precisely, the unit of the adjunction is given by \(\eta_C := [\phi_C]\) and the counit by \(\varepsilon_B := \mathcal{E}_B, \quad \mathcal{X} \in Obj(\mathcal{X}\mathcal{X}^G)\) and \(B \in Obj(\mathcal{X}\mathcal{X}^G)\).

The goal of this section is to generalise these constructions when \(C\) is replaced by an object of the form \(C \otimes T \in \mathcal{L}_\mathbb{G}\), where \((T, \delta)\) is a torsion action of \(\mathbb{G}\) of projective type. Recall
that a torsion action of projective type of $G$, $(T, \delta)$, means simply that $T = M_k(\mathbb{C})$ for some $k \in \mathbb{N}$ and that $\delta$ is ergodic such that $T$ is not $G$-Morita equivalent to $\mathbb{C}$. We fix a state $\varphi_T = \text{Tr}(\varrho \cdot)$ on $T$ (recall Section 2.1). Recall as well that, by virtue of Theorem 3.1.5, $\delta$ is implemented by a $\Omega^*$-representation of $G$, say $\omega$, for some (measurable) 2-cocycle $\Omega$ on $G$. The 2-cocycle $\Omega$ is necessarily of finite type (recall Definition 3.3.1). Hence $G_\Omega$ is again a compact quantum group by Theorem 3.3.7. Following Equation (2.1), we denote by $(T^\text{op}, \tilde{\delta})$ the corresponding opposite twisted dynamical system. In this case, that a torsion action of projective type of $G$ sense of (3.2)) by a $\Omega$-representation of $G$ of $\tilde{G}$ from the usual Green-Julg isomorphism applied to $G$. Let us consider the following triangulated functors:

\[ j_{G,T} : \mathcal{K} \mathcal{K}^G \rightarrow \mathcal{K} \mathcal{K}, \quad (B, \beta) \mapsto j_{G,T}(B, \beta) := B \rtimes G \rtimes T^\text{op}, \]

\[ \tau_T : \mathcal{K} \mathcal{K} \rightarrow \mathcal{K} \mathcal{K}^G, \quad C \mapsto \tau_T(C) := (C \otimes T, \text{id} \otimes \delta). \]

We are going to show that $\tau_T$ is a left adjoint of $j_{G,T}$ for every torsion action of projective type $(T, \delta)$ of $G$. To do so we start by showing an appropriate equivalence of triangulated categories between $\mathcal{K} \mathcal{K}^G$ and $\mathcal{K} \mathcal{K}^{G_\Omega}$. Then the adjunction between $\tau_T$ and $j_{G,T}$ will result from the usual Green-Julg isomorphism applied to $G_\Omega$.

Let us consider the following triangulated functors:

\[ \Pi_{T^\text{op}} : \mathcal{K} \mathcal{K}^G \rightarrow \mathcal{K} \mathcal{K}^{G_\Omega}, \quad (B, \beta) \mapsto \Pi_{T^\text{op}}(B, \beta) := (B \otimes T^\text{op}, \tilde{\beta} := \text{Ad}_{u_{23}} \circ \beta_{13}), \]

\[ \Pi_T : \mathcal{K} \mathcal{K}^{G_\Omega} \rightarrow \mathcal{K} \mathcal{K}^G, \quad (C, \gamma) \mapsto \Pi_T(C, \gamma) := (C \otimes T, \tilde{\gamma} := \text{Ad}_{u_{23}} \circ \gamma_{13}). \]

First of all, observe that these functors are well defined. On the one hand, given $(B, \beta) \in \text{Obj}(\mathcal{K} \mathcal{K}^G)$, we have proved in Proposition 5.2.4 that $\beta := \text{Ad}_{u_{23}} \circ \beta_{13}$ is an action of $G_\Omega$ on $B \otimes T^\text{op}$. So $\Pi_{T^\text{op}}(B, \beta) \in \text{Obj}(\mathcal{K} \mathcal{K}^{G_\Omega})$. A similar computation yields that if $(C, \gamma) \in \text{Obj}(\mathcal{K} \mathcal{K}^{G_\Omega})$, then $\gamma := \text{Ad}_{u_{23}} \circ \gamma_{13}$ is an action of $G$ on $C \otimes T$. So $\Pi_T(C, \gamma) \in \text{Obj}(\mathcal{K} \mathcal{K}^G)$.

On the other hand, given two objects $(B_1, \beta_1), (B_2, \beta_2) \in \text{Obj}(\mathcal{K} \mathcal{K}^G)$ and a Kasparov triple $X \in KK^G(B_1, B_2)$, then $\Pi_{T^\text{op}}(X)$ is given by the right exterior tensor product of Kasparov triples with respect to $T^\text{op}$ i.e. $\Pi_{T^\text{op}}(X) = X \otimes T^\text{op} \in KK^G_1(B_1 \otimes T^\text{op}, B_2 \otimes T^\text{op})$ (if $X$ is represented by the $G$-equivariant Hilbert $B_2$-module $E$ with action $\delta_E$, then $\Pi_{T^\text{op}}(X)$ is represented by the Hilbert $B_2 \otimes T^\text{op}$-module $E \otimes T^\text{op}$ with action of $G_\Omega$ given by $\text{Ad}_{u_{23}} \circ (\delta_E)_{13}$). Similarly, $\Pi_T(Y) = Y \otimes T$, for all $Y \in KK^{G_\Omega}(C_1, C_2)$ with $(C_1, \gamma_1), (C_2, \gamma_2) \in \text{Obj}(\mathcal{K} \mathcal{K}^{G_\Omega})$. Clearly, both $\Pi_{T^\text{op}}$ and $\Pi_T$ intertwine the suspensions of each category. Moreover, they transform mapping cone triangles into mapping cone triangles. This is true by the following general fact: if $\phi : A \rightarrow B$ is a homomorphism between $C^*$-algebras and $D$ is any other $C^*$-algebra, then we have that $C_\phi \otimes D \cong C_{\phi \otimes id}$ induced by the canonical identification $C_0((0,1], B) \otimes D \cong C_0((0,1], B \otimes D), f \otimes d \mapsto \left(t \mapsto f(t) \otimes d\right)$. If in addition $\phi$ is a $G$-equivariant homomorphism between the $G$-$C^*$-algebras $(A, a)$ and $(B, \beta)$, then $C_\phi$ is a $G$-$C^*$-algebra with action $\gamma((a, h)) := (a(a), \beta \circ h)$, for all $(a, h) \in C_\phi$. In this way, it is straightforward to check that given a $G$-$C^*$-algebra $(B_1, \beta_1)$ and $(B_2, \beta_2)$, say $\phi : B_1 \rightarrow B_2$, then the isomorphism $C_\phi \otimes T^\text{op} \cong C_{\phi \otimes id}$ is $G_\Omega$-equivariant. As a consequence, the functor $\Pi_{T^\text{op}}$ preserves mapping cone triangles; and similarly for $\Pi_T$. In conclusion, both $\Pi_{T^\text{op}}$ and $\Pi_T$ are well defined triangulated functors.
5.3.1 Lemma. Following the previous notations, the pair of functors $(\Pi_{T^{op}}, \Pi_T)$ defines an equivalence of triangulated categories between $\mathcal{K}^{G}$ and $\mathcal{K}^{G_{\Omega}}$.

Proof. It only remains to show that $\Pi_T \circ \Pi_{T^{op}} \cong \text{id}_{\mathcal{K}^{G}}$ and $\Pi_{T^{op}} \circ \Pi_T \cong \text{id}_{\mathcal{K}^{G_{\Omega}}}$. On the one hand, given an object $(B, \beta) \in \text{Obj}(\mathcal{K}^{G})$, we have $\Pi_T(\Pi_{T^{op}}(B, \beta)) = B \otimes T^{op} \otimes T \overset{id}{\cong} B \otimes T \otimes T^{op}$ equipped with the $G$-action $\tilde{\beta} = \text{Ad}_{u_{24}} \circ \text{Ad}_{u_{34}} \circ \beta_{14}$. By identifying $T \otimes T^{op} \cong B(L^2(T))$ along with the $G$-action $\text{Ad}_{u_T}$, where $V_T = u_{13}u_{23}$ as in Lemma 3.2.3, we obtain that $\Pi_T(\Pi_{T^{op}}(B, \beta))$ is $G$-equivariantly Morita equivalent to $(B, \beta)$; and this identification is natural. So $\Pi_T \circ \Pi_{T^{op}} \cong \text{id}_{\mathcal{K}^{G}}$. On the other hand, given an object $(C, \gamma) \in \text{Obj}(\mathcal{K}^{G_{\Omega}})$, we have $\Pi_{T^{op}}(\Pi_T(C, \gamma)) = C \otimes T \otimes T^{op} \overset{id}{\cong} B \otimes T^{op} \otimes T$ equipped with the $G$-action $\tilde{\gamma} = \text{Ad}_{u_{24}} \circ \text{Ad}_{u_{34}} \circ \gamma_{14}$. By identifying $T^{op} \otimes T = (T \otimes T^{op})^{op} \cong B(L^2(T))^{op} \cong B(L^2(T^{op})) = K(L^2(T^{op}))$ along with the $G_{\Omega}$-action $\text{Ad}_{V_{T^{op}}}$, where we define the (ordinary) $G_{\Omega}$-representation $V_{T^{op}} = u_{13}u_{23}$, we obtain that $\Pi_{T^{op}}(\Pi_T(C, \gamma))$ is $G_{\Omega}$-equivariantly Morita equivalent to $(C, \gamma)$; and this identification is natural. So $\Pi_{T^{op}} \circ \Pi_T \cong \text{id}_{\mathcal{K}^{G_{\Omega}}}$.

5.3.2 Theorem (Twisted Green-Julg isomorphism). Let $G$ be a compact quantum group. Let $(T, \delta)$ be a torsion action of projective type of $G$. Let $u$ be an $\Omega^*$-representation of $G$ implementing $\delta$ for some 2-cocycle $\Omega$ (necessarily of finite type). Then $\tau_T : \mathcal{K}^{G} \longrightarrow \mathcal{K}^{G_{\Omega}}$ is a left adjoint of $f_{G,T} : \mathcal{K}^{G_{\Omega}} \longrightarrow \mathcal{K}^{G}$ as triangulated functors. More precisely,

$$\Psi_T : KK^{G}(C \otimes T, B) \xrightarrow{\sim} KK(C, B \rtimes G \ltimes T^{op}),$$

for all $C \in \text{Obj}(\mathcal{K}^{G})$ and $(B, \beta) \in \text{Obj}(\mathcal{K}^{G})$.

Proof. Since the 2-cocycle $\Omega$ is necessarily of finite type, the twisted quantum group $G_{\Omega}$ is compact by Theorem 3.3.7. Given a $C^*$-algebra $C \in \text{Obj}(\mathcal{K}^{G})$ and a $G$-$C^*$-algebra $(B, \beta) \in \text{Obj}(\mathcal{K}^{G})$, the previous lemma allows to write the following:

$$KK^{G}(C \otimes T, B) \cong KK^{G_{\Omega}}(C \otimes T \otimes T^{op}, B \otimes T^{op}) \cong KK^{G_{\Omega}}(C, B \otimes T^{op}).$$

Next, by applying the usual Green-Julg isomorphism we obtain that: $KK^{G_{\Omega}}(C, B \otimes T^{op}) \overset{\Psi}{\cong} KK(C, (B \otimes T^{op}) \rtimes G_{\Omega})$. To conclude, we observe that $(B \otimes T^{op}) \rtimes G_{\Omega} \cong B \rtimes G \ltimes T^{op}$ by virtue of Proposition 5.2.4. Therefore $\Psi_T = \Psi \circ \Pi_{T^{op}}$.

5.4. Compact objects in $\mathcal{K}^{G}$

First of all, as a corollary of the twisted Green-Julg isomorphism established in Theorem 5.3.2 we obtain a finiteness presentation property type for the projective torsion objects within the Kasparov category $\mathcal{K}^{G}$.

5.4.1 Theorem. Let $G$ be a compact quantum group. Let $(T, \delta) \in \text{Obj}(\mathcal{K}^{G})$ be a torsion action of projective type of $G$. Then $(T, \delta)$ is a compact object in $\mathcal{K}^{G}$, that is, the functor $KK^{G}(T, -)$ is compatible with countable direct sums.
5.5.1 Theorem. Let \( \{(B_n, \beta_n)\}_{n \in \mathbb{N}} \) be a countable family of \( \mathbb{G}\)-\( C^*\)-algebras. By virtue of the twisted Green-Julg isomorphism of Theorem 5.3.2 with \( C := \mathbb{C} \) we have

\[
KK^G(T, \bigoplus_{n \in \mathbb{N}} B_n) \cong KK^G(\mathbb{C}, (\bigoplus_{n \in \mathbb{N}} B_n) \times_{r,\beta} \mathbb{G} \times \mathbb{T}^{op}) \cong K_0\left( (\bigoplus_{n \in \mathbb{N}} B_n) \times_{r,\beta} \mathbb{G} \times \mathbb{T}^{op} \right)
\]

\[
= K_0 \circ (\cdot) \times_{r,\beta} \mathbb{G} \times \mathbb{T}^{op} \left( \bigoplus_{n \in \mathbb{N}} B_n^{op} \right).
\]

To conclude it is enough to note that the \( K_0 \) functor and the two-sided crossed product functor \( (\cdot) \times_{r,\beta} \mathbb{G} \times \mathbb{T}^{op} \) are compatible with countable direct sums.

\[\blacksquare\]

5.4.2 Remark. Thanks to the more general result in [2], any torsion object in \( \mathcal{X} \mathcal{H}^G \) (not necessarily of projective type) is a compact object in \( \mathcal{X} \mathcal{H}^G \). This is because an analogous isomorphism as the one from Theorem 5.3.2 is obtained in [2] for any torsion action of \( \mathbb{G} \) (not necessarily of projective type) allowing to apply the same argument as above.

We believe that this property will be useful to study the equivariant Kasparov category \( \mathcal{X} \mathcal{H}^G \) from a geometrical and topological perspective according to works by I. Dell’Ambrogio and his collaborators (see for example [16], [17]). For instance, the above theorem (and remark) yields that the subcategory \( \mathcal{P}_{D(G)} := \langle \{T \mid T \in \mathrm{Tor}(\mathcal{D}(\mathbb{G}))\} \rangle \) is a compactly generated tensor triangular subcategory of \( \mathcal{X} \mathcal{H}^D(G) \) when \( \mathbb{G} \) is finite (note that we need to consider the Drinfeld double construction to provide a tensor structure on the Kasparov category). In particular, \( (\mathcal{P}_{D(G)}, \mathcal{P}_{\mathbb{G}}^{-1}) \) is a complementary pair of localizing subcategories in \( \mathcal{X} \mathcal{H}^D(G) \) (recall Corollary 2.5.16). Hence it will be interesting to compute its spectrum, \( \mathrm{Spc}(\mathcal{P}_{D(G)}) \), in the sense of Balmer [6] and to make a connection with the Baum-Connes property for \( \hat{\mathbb{G}} \).

5.5. The quantum assembly map

5.5.1 Theorem. Let \( \hat{\mathbb{G}} \) be a compact quantum group. If \( \hat{\mathbb{G}} \) is permutation torsion-free, then the subcategories \( \mathcal{L}_{\hat{\mathbb{G}}} \) and \( \mathcal{N}_{\hat{\mathbb{G}}} \) form a complementary pair of localizing subcategories in \( \mathcal{X} \mathcal{H}^\hat{\mathbb{G}} \).

Proof. Since \( \hat{\mathbb{G}} \) is permutation torsion-free, any torsion action of \( \mathbb{G} \) is of projective type. Moreover, by virtue of Theorem 3.1.5 (and Remark 3.2.7), the projective torsion of \( \hat{\mathbb{G}} \) is parametrised by the family of all (measurable) 2-cocycles on \( \mathbb{G} \), which is a countable one. We put the following triangulated functor \( F := (j_{\mathcal{C}, T})_{T \in \mathrm{Tor}(\hat{\mathbb{G}})} : \mathcal{X} \mathcal{H}^G \rightarrow \prod_{T \in \mathrm{Tor}(\hat{\mathbb{G}})} \mathcal{X} \mathcal{H}^G \) defined in the obvious way. Then we put \( \mathcal{C}C_{\hat{\mathbb{G}}} := \ker_{\mathrm{Obj}}(F) \) and \( J_{\hat{\mathbb{G}}} := \ker_{\mathrm{Hom}}(F) \). By virtue of Theorem 5.3.2 the triangulated functor \( F^\perp := \bigoplus_{T \in \mathrm{Tor}(\hat{\mathbb{G}})} \prod_{T \in \mathrm{Tor}(\hat{\mathbb{G}})} \mathcal{X} \mathcal{H} \rightarrow \mathcal{X} \mathcal{H}^{\hat{\mathbb{G}}} \) is the left adjoint of \( F \) (note that the equivariant KK-theory is additive on the first variable). Consequently, \( \mathcal{X} \mathcal{H}^{\hat{\mathbb{G}}} \) has enough \( J \)-projective objects and so the pair \( (\mathcal{P}_{\mathcal{J}}, \mathcal{C}C_{\hat{\mathbb{G}}}) \) is complementary in \( \mathcal{X} \mathcal{H}^{\hat{\mathbb{G}}} \) with \( \langle \mathcal{P}_{\mathcal{J}} \rangle = \langle F^\perp(\mathrm{Obj}(\prod_{T \in \mathrm{Tor}(\hat{\mathbb{G}})} \mathcal{X} \mathcal{H})) \rangle \) (recall Theorem 2.5.13). Note that, by using again the twisted Green-Julg isomorphism, we have \( \mathcal{C}C_{\hat{\mathbb{G}}} = \mathcal{N}_{\hat{\mathbb{G}}} \). To conclude, recall that \( \mathcal{L}_{\hat{\mathbb{G}}} \) is, by definition, the minimal localising subcategory containing the objects of the form
$C \otimes T$ with $T \in \text{Tor} (\hat{\mathbb{G}})$ and $C \in \text{Obj}(\mathscr{KH})$. So we have $\langle F^- (\text{Obj}(\prod_{T \in \text{Tor}(\hat{\mathbb{G}})} \mathscr{KH})) \rangle \subset \hat{\mathcal{L}}_{\hat{\mathbb{G}}}$ and by minimality this inclusion must be an equality. By applying Baaj-Skandalis duality, we obtain the complementary pair as in the statement.

We are now ready to apply the general Meyer-Nest’s machinery to define a quantum assembly map for permutation torsion-free discrete quantum groups and thus a quantum version of the (strong) Baum-Connes property for this class of discrete quantum groups. More precisely, consider the following homological functor:

$$F : \mathscr{KH}^{\hat{\mathbb{G}}} \rightarrow \mathcal{M}_{\hat{\mathbb{G}}}, \quad (A, \delta) \mapsto F(A) := K_*(\hat{\mathbb{G}} \rtimes \Delta).$$

The quantum assembly map for $\hat{\mathbb{G}}$ is given by the natural transformation $\eta^{\hat{\mathbb{G}}} : LF \rightarrow F$ (recall Note 2.5.12).

5.5.2 Definition. Let $\hat{\mathbb{G}}$ be a discrete quantum group. We say that $\hat{\mathbb{G}}$ satisfies the quantum Baum-Connes property (with coefficients) if the natural transformation $\eta^{\hat{\mathbb{G}}} : LF \rightarrow F$ is a natural equivalence. We say that $\hat{\mathbb{G}}$ satisfies the strong Baum-Connes property if $\mathscr{KH}^{\hat{\mathbb{G}}} = \mathcal{L}_{\hat{\mathbb{G}}}$.

5.5.3 Remark. If $\hat{\mathbb{G}} := \Gamma$ is a classical discrete group, then the Meyer-Nest approach to the Baum-Connes conjecture leads to consider the following complementary pair of localizing subcategories in $\mathscr{KH}^\Gamma$:

$$\mathcal{L}_\Gamma = \langle \{ A \in \text{Obj}(\mathscr{KH}^\Gamma) \mid A \cong \text{Ind}^\Gamma_\Lambda (B) \text{ with } \Lambda \in \mathcal{F}, B \in \text{Obj}(\mathscr{KH}^\Lambda) \rangle, $$

$$\mathcal{N}_\Gamma = \{ A \in \text{Obj}(\mathscr{KH}^\Gamma) \mid \text{Res}^\Gamma_\Lambda (A) \cong 0 \ \forall \Lambda \in \mathcal{F} \} = \bigcap_{\Lambda \in \mathcal{F}} \ker_{\text{Obj}}(\text{Res}^\Gamma_\Lambda),$$

where $\mathcal{F}$ denotes the family of all finite subgroups of $\Gamma$. Observe that every generator of $\mathcal{L}_\Gamma$ is a proper $\Gamma$-$C^*$-algebra over $\Gamma/\Lambda$. Conversely, it is well-known that any proper $\Gamma$-$C^*$-algebra is built up out of induced $C^*$-algebras from finite subgroups, so they belong to $\mathcal{L}_\Gamma$ (see [20] for more details). However, one open question is to decide whether every object of $\mathcal{L}_\Gamma$ is a proper $\Gamma$-$C^*$-algebra in $\mathscr{KH}^\Gamma$. It is well-known that any torsion action of $\hat{\Gamma}$ is of the form $C^*_r(\Lambda, \omega)$ for some finite subgroup $\Lambda \subseteq \Gamma$ and some 2-cocycle $\omega \in H^2(\Lambda, S^1)$. In addition, it can be shown that $\hat{\Gamma} \rtimes C^*_r(\Lambda, \omega) \in \text{Obj}(\mathscr{KH}^\Gamma)$ is a proper $\Gamma$-$C^*$-algebra over $\Gamma/\Lambda$, so it is an object in $\mathcal{L}_\Gamma$. Therefore, the previous discussion yields that $\mathcal{L}_\Gamma$ contains the subcategory $\mathcal{L}_\Gamma^\Gamma$, where $\Gamma := (C^*_r(\Gamma), \Delta \Gamma)$ as defined previously in the more general setting of quantum groups. In other words, the condition $\mathscr{KH}^{\hat{\mathbb{G}}} = \mathcal{L}_{\hat{\mathbb{G}}}$ is stronger than the condition $\mathscr{KH}^{\hat{\mathbb{G}}} = \mathcal{L}_\Gamma$. The latter translates into the usual strong Baum-Connes conjecture for $\Gamma$, that is, the existence of a $\gamma$-element that equals $1_{\hat{\mathbb{G}}}$ (see [27, Section 8] for a proof). For this reason, it is convenient to refer to the property $\mathscr{KH}^{\hat{\mathbb{G}}} = \mathcal{L}_{\hat{\mathbb{G}}}$ as $\mathcal{L}_{\hat{\mathbb{G}}}$-strong Baum-Connes property instead of simply strong Baum-Connes property.
References


