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ABSTRACT
In this paper, we develop an adiabatic theory for the evolution of large closed surfaces under the area-constrained Willmore (ACW) flow in a three-dimensional asymptotically Schwarzschild manifold. We explicitly construct a map, defined on a certain four-dimensional manifold of barycenters, which characterizes key static and dynamical properties of the ACW flow. In particular, using this map, we find an explicit four-dimensional effective dynamics of barycenters, which serves as a uniform asymptotic approximation for the (infinite-dimensional) ACW flow, so long as the initial surface satisfies certain mild geometric constraints (which determine the validity interval). Conversely, given any prescribed flow of barycenters evolving according to this effective dynamics, we construct a family of surfaces evolving by the ACW flow, whose barycenters are uniformly close to the prescribed ones on a large time interval (whose size depends on the geometric constraints of initial configurations).

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I. INTRODUCTION
Let \((M, g)\) be a three-dimensional, complete, oriented Riemannian manifold with non-negative curvature. Consider the area-constrained Willmore (ACW) flow,

\[ \partial_t x^N = -W(x) - \lambda H(x). \]  

Here, for \(t \geq 0, x = x_t : S \to M\) is a family of embeddings of spheres (with orientation compatible with that on \(M\)). \(\partial_t x^N\) denotes the normal velocity at \(x\), given by \(\partial_t x^N := g(\partial_t x, \nu)\), where \(\nu = \nu(x)\) is the unit normal vector to \(\Sigma\) at \(x\). \(H(x)\) denotes the mean curvature scalar at \(x\), and

\[ W(x) := \Delta H(x) + H(x)(\text{Ric}_M(\nu, \nu) + |\tilde{A}|^2(x)) \]

is the Willmore operator, where \(\tilde{A}(x)\) denotes the traceless part of the second fundamental form. The number \(\lambda = \lambda(t)\) is given by

\[ \lambda = \frac{1}{\int_{\Sigma} H^2} \int_{\Sigma} \left( |\nabla H|^2 - H^2 \text{Ric}_M(\nu, \nu) - H^2 |\tilde{A}|^2 \right) \, d\mu, \]

where \(\mu = \mu^g_\Sigma\) is the canonical measure on \(\Sigma\), induced by the embedding \(x\) and background metric \(g\). This choice of \(\lambda\) ensures that (1.1) is area-preserving; see Appendix B.

Mathematically, the ACW flow (1.1) can be viewed as the \(L^2\)-gradient flow of the Willmore energy,
which is a well-defined $C^2$-functional in a suitable configuration space. Denote by $dW(x)$ the Fréchet derivative of $W$. Define the normal $L^2$-gradient $\nabla^N W(x, \phi) := dW(x)\phi$ for every normal, area-preserving variation $\phi$ on the surface $\Sigma = x(\bar{\Sigma})$. Then, $\nabla^N W(x)$ is given by the rhs of (1.1). Hence, for a flow of immersions $x = x_t, t \geq 0$ in the configuration space of $W$, we can rewrite (1.1) as

$$\partial_t x \nabla^N W(x).$$

See Sec. II B for more details.

Physically, the ACW flow (1.1) is the gradient flow of the Hawking mass of embedded 2-spheres, defined in (2.9) and first introduced in Ref. 1. In mathematical biology, building upon Helfrich’s seminal work, the flow (1.1) and its variants are used to model the motion of cell membranes. More recently, (1.1) finds applications to nonlinear elasticity, among various other fields in applied science.

A. Main results

In this paper, we develop a rigorous adiabatic (or slow-motion approximation) theory for (1.1). To state our main result, denote by $H^k$ the Sobolev space of order $k$. Let

$$X^k := H^k(\Sigma, M), \quad X^k_c := \{x \in X^k : |x(\bar{\Sigma})| = c\}.$$ 

Here and below, for a surface $\Sigma := x(S) \subset M$, denote by

$$|\Sigma| := \int_{\Sigma} d\mu$$

the area of $\Sigma$. Following the nomenclature in Ref. 5, we call (the images of) static solutions to (1.1) surfaces of Willmore type.

Then, we prove the following.

**Theorem 1** (main). Let $k \geq 4, c > 1$. Fix $R \gg 1, \delta \ll 1$ in Definition 1 of admissible surfaces.

Then, there exists a map

$$\Psi : M' := \mathbb{R}^2 \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3 \to X^k$$

with the following property: Denote by $\xi \equiv (r, z) \in M'$ and $v_{\xi} \equiv \Psi(\xi) \in X^k$.

1. **Critical point:** The immersion $v_{\xi}$ parameterizes a surface of Willmore type if and only if $\xi$ is a critical point of the function $W \circ \Psi : \mathbb{R}^4 \to \mathbb{R}$, restricted to the submanifold $\{\xi \in M' : |v_{\xi}(\bar{\Sigma})| = c\}$.

2. **Stability:** Suppose that $v_{\xi}$ parameterize an admissible surface of Willmore type. Then, $v_{\xi}$ is uniformly stable with small area-preserving $H^k$-perturbation if $\xi$ is a strict local minimum of the function $W \circ \Psi$, restricted to the submanifold $\{\xi \in M' : |v_{\xi}(\bar{\Sigma})| = c\}$.

3. **Effective dynamics:** Let $\Sigma_\ast = x_\ast(\bar{\Sigma})$ be an admissible surface. Let $\Sigma_t = x_t(\bar{\Sigma})$, $t \geq 0$ be the global solution to (1.1) with initial configuration $\Sigma_{t=0} = \Sigma_\ast$ as in Theorem 2.

Then, there exist $\alpha > 0$, $T = O(R^{-\alpha})$, and a path $\zeta_t \in M'$, $t \geq 0$ such that for every $t \geq T$, the following holds:

$$\|v_{\zeta_t} - x_t\|_{X^k} = O(R^{-3}). \quad (1.2)$$

Moreover, the path $\zeta_t \equiv (r_t, z_t)$ evolves according to

$$\dot{z} = \frac{1}{4\pi} \nabla_z W \circ \Psi(r, z) + O(R^{-3}), \quad (1.3)$$

$$\dot{r} = 4R^{-2} + O(R^{-3}). \quad (1.4)$$

In (1.3), the leading term is of the order $O(R^{-3})$. 

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4. Conversely, if \( \xi \in M' \) is a flow evolving according to (1.3) and (1.4), then there exists a global solution \( x_t \) to (1.1) such that (1.2) holds for this choice of \( \xi \), and every \( T \leq t \leq T + R \).

Remark 1. The map \( \Psi \) in Theorem 1 is defined by \( \Psi(r, z) = \theta(\Phi(r, z), r, z) \), where \( \theta, \Phi \) are given, respectively, in (3.2) and Definition 4. Following Ref. 6, we call \( \Phi \) (or equivalently \( \Psi \)) the Lyapunov–Schmidt map.

Remark 2. The content of Theorem 1 says that the static and dynamical properties of (2.4) are captured by the effective action \( W \circ \Psi \) on the 3-manifold given by the level set \( \{ \xi \in M' : |\nu_{\xi}(S)| = c \} \).

Remark 3. (1.2)–(1.4) constitute the crucial part of the theorem, which says that the infinite-dimensional dynamical system (1.1) reduces uniformly in time to the finite system of ODEs, (1.3) and (1.4).

On the analytical ground, roughly speaking, Theorem 1 shows that the pullback by the Lyapunov–Schmidt map \( \Psi \) to \( M' \) essentially reduces key static and dynamical properties of the infinite-dimensional dynamical system (1.1) to the finite-dimensional ones.

For applications to various physical models, our results in this paper provide an explicit four-dimensional dynamical system, (1.3), whose behavior approximates that of (1.1) uniformly for all sufficiently large time and captures key qualitative behaviors of (1.1). This can potentially reduce the computational complexity for the complicated fourth order partial differential equations (PDE) (1.1).

B. Historical remarks

The problem we study here is motivated by a recent work by Eichmair and Koerber, in which the authors studied stationary solutions to (1.1) using Lyapunov–Schmidt reduction. Here, we derive some dynamical analogs of the static existence results in Ref. 7, with, however, rather different focus. Indeed, the main point of our results is that we have (1) explicit information about the adiabatic parts of a flow of surfaces evolving according to (1.1), with (2) uniformly small errors in time and (3) we can construct solution (1.1) with prescribe adiabatic behavior. See the precise statements of these points in Theorem 1.

The method of adiabatic approximation has a long history in classical field theory. See Ref. 8 for an excellent review in this context. Our work here is inspired by a series of papers by Sigal et al., in which the adiabatic theory is adapted to geometric problems.\(^{6,9–11}\) Other results along this line that we have referred to include Refs. 12–17, which cover a range of static and dynamical problems of geometric equations using Lyapunov–Schmidt reduction.

Among the papers above, we single out a recent paper, in which the authors studied a formally similar problem (namely, the volume-preserving mean curvature flow with initial configurations close to small geodesic spheres), from which we draw much inspiration. In particular, it appears that the notion of the Lyapunov–Schmidt map is first mentioned in this paper.

It seems to us that our results are the first rigorous adiabatic theory for the area-constrained Willmore flow. We expect these results to be robust in the sense that they can be easily adapted to problems related to (1.1), for instance, using the generalized Willmore energy developed in Ref. 18 (which covers, among others, the applications to biomembranes). In a separate paper, we will treat the abstract properties of the Lyapunov–Schmidt map defined in Appendix A.

II. SETUP OF THE PROBLEM

A. Asymptotically Schwarzschild manifolds

A three-dimensional complete Riemannian manifold \((M, g)\) is said to be \(C^4\)-close to Schwarzschild if the following holds:

1. \( M \setminus K \) is diffeomorphic to \( \mathbb{R}^3 \setminus B_1(0) \) for some compact subset \( K \subset M \).
2. The metric \( g \) splits as \( g = g_s + h \), where

\[
g_s := \left( 1 + \frac{m}{2|x|} \right)^4 \delta_{ij}
\]

is the Schwarzschild metric with Arnowitt-Deser-Misner (ADM) mass \( m > 0 \) and \( h \in C^4 \) is a small perturbation satisfying

\[
h_{ij} = h_{ji}, \quad \partial^a h_{ij} \leq \eta |x|^{-2+|a|} \quad (|a| \leq k, |x| \gg 1)
\]  

(2.1)

for some fixed small decay coefficient \( \eta \ll 1 \). Here, \( x \in \mathbb{R}^3 \) denotes the coordinate in the asymptotic chart on \( M \).

Physically, for applications to general relativity, such manifold \( M \) is a perturbation of the static Schwarzschild black hole \((\mathbb{R}^3 \setminus \overline{B_m}^3(0), g_s)\).
To simplify notations, throughout this paper, we normalize ADM mass to be \( m = 2 \). We assume that the ambient space \( M \) is \( C^k \)-close to Schwarzschild for sufficiently large \( k \) and that in (2.1), the decay coefficient \( \eta \ll 1 \). Thus, in what follows, we take

\[
(M, g) = (\mathbb{R}^3 \setminus B_1(0), g_3 + h),
\]

where \( h \) is as in (2.1).

To use results in Refs. 5, 7, and 19, we assume that the scalar curvature \( Sc \) on \( M \) satisfies the following decay properties:

\[
\begin{align*}
x^i \partial_x (|x|^2 Sc) &= o(|x|^4), \\
Sc(x) - Sc(-x) &= o(|x|^5).
\end{align*}
\]

The asymptotic flatness condition (2.2) is satisfied if \( g \) is \( C^k \)-close to Schwarzschild with \( k \geq 4 \) and \( Sc = o(|x|^4) \), in which case \( \nabla Sc = o(|x|^{-5}) \).

(2.3) means that the scalar curvature on \( M \) is asymptotically even. Geometrically, condition (2.2) provides quantitative control for various estimates involving extrinsic geometric quantities. Condition (2.3) provides qualitative control of the effective action in Sec. IV and Appendix C.

B. The geometric structure of (1.1)

In this subsection, we lay out the geometric structure of ACW flow (1.1). This structure is understood in the subsequent developments in Secs. IV and V.

Let \( c \gg 1, k \geq 4 \) be given. Recall that in Sec. I A, we have defined the configuration spaces

\[
X^k := H^k(S, M), \quad X^k_c := \{ x \in X^k : |x(S)| = c \}, \tag{2.4}
\]

where \( |\Sigma| := \int_S d\mu_g^S \) denotes the area of \( \Sigma \) with respect to the embedding \( x \) and background metric \( g \). One can check easily that (1.1) is well-defined in \( X^k_c \). The spaces in (2.4) are equipped with the \( L^2 \)-inner product

\[
\langle \phi, \phi' \rangle := \int_\Sigma \langle \phi, \phi' \rangle_{\text{Euclidean}} \quad (\phi, \phi' \in X^k).
\]

Let \( x \in X^k \), and write \( \Sigma = x(S) \). The tangent spaces to \( x \) at \( X^k \) and \( X^k_c \) are, respectively, given by

\[
\begin{align*}
T_x X^k &= X^k, \\
T_x X^k_c &= \{ \phi \in T_x X^k : \int_\Sigma Hg(\phi, v) = 0 \}.
\end{align*}
\]

Here, (2.7) is due to the well-known first variation formula of the area functional. Note that, slightly abusing notation, in (2.7), we view \( \phi \) as a vector field over \( \Sigma \). With (2.5), we have a formal Riemannian structure on the configuration spaces \( X^k \) and \( X^k_c \).

With this geometric structure of \( X^k \), one can view Eq. (1.1) as the \( L^2 \)-gradient flow, restricted to \( X^k_c \), of the Willmore energy

\[
\mathcal{W}(\Sigma) = \frac{1}{4} \int_\Sigma H^2 d\mu_g^S. \tag{2.8}
\]

Using Sobolev inequalities, one can show that for \( k \geq 4 \), the functional \( \mathcal{W} \) is well-defined and \( C^2 \) (in the sense of Fréchet derivatives) on \( X^k_c \).

Let \( d\mathcal{W}(x) : T_x X^k \rightarrow T_x X^{k+4} \) be the Fréchet derivative of \( \mathcal{W} \) at an embedding \( x \) in the class \( X^k_c \). Define the normal \( L^2 \)-gradient \( \nabla^N \mathcal{W}(x) \phi := d\mathcal{W}(x)\phi \) for every normal, area-preserving variation \( \phi \) on the surface \( \Sigma = x(S) \). (This operator \( \nabla^N \) depends on \( x \).) Then, by the first variation formula of the Willmore energy (see, e.g., Ref. 20, Sec. 3), this \( \nabla^N \mathcal{W}(x) \) is given by the rhs of (1.1). This allows us to rewrite (1.1) as

\[
\partial_t x^N = \nabla^N \mathcal{W}(x) \quad (x \in X^k_c).
\]

Equivalently, (1.1) is the (negative) \( L^2 \)-gradient flow of the Hawking mass,
\[ m_{\text{law}}(\Sigma) := \left| \frac{\Sigma}{\sqrt{\lambda}} \right|^2 \frac{1}{16\pi} \int_{\Sigma} \left( \frac{1}{2} \int_{\Sigma} H^2 \cdot d\mu_S \right), \]  

(2.9)

in the sense that a flow of surfaces evolving according to (1.1) increases the mass \( m_{\text{law}} \). For interest from physics related to this problem, especially in general relativity, see Ref. 21.

**C. Preliminary results**

Let \( R \gg 1 \) be given. Let \( K \subset M \) be a fixed compact set as in Sec. II A. As explained in Sec. II A, for asymptotically Schwarzschild manifold \( M \), we can identify \( M \setminus K \) with its coordinate space \( \mathbb{R}^3 \setminus \overline{B}_R(0) \).

Let \( \delta > 0 \) be given.

**Definition 1** (admissible surfaces). For a closed surface \( \Sigma \subset M \), define the inner and outer radii \( \rho(\Sigma), \lambda(\Sigma) \) as

\[ \rho(\Sigma) = \min_{x \in \Sigma} |x|, \quad \lambda(\Sigma) := \sqrt{\rho(\Sigma)/4\pi}. \]  

(2.10)

(2.11)

We say that \( \Sigma \) is admissible if the interior of \( \Sigma \) contains the fixed compact set \( K \), and

\[ \rho(\Sigma) > R, \quad \left| \frac{\rho(\Sigma)}{\lambda(\Sigma)} - 1 \right| + \int_\Sigma |A|^2 < \delta. \]  

(2.12)

Here, recall that \( \dot{\Lambda} \) denotes the traceless part of the second fundamental form on \( \Sigma \).

**Remark 4.** Geometrically, a surface \( \Sigma \) is admissible if the origin lies sufficiently deep inside the interior of \( \Sigma \) (this property is called *centering* in Ref. 7), and at the same time, the surface does not wiggle too much. It follows from the definition (2.11) that \( \lambda(\Sigma) \leq \max_{x \in \Sigma} |x| \).

Using the terminology in Ref. 7, every admissible surface \( \Sigma \) satisfying (2.12) with \( R, \delta^{-1} \gg 1 \) is *on-center*.

For the class of admissible surfaces, we have the following well-posedness result for (1.1).

**Theorem 2** (Ref. 19, Theorem 5.3). Assume that \( M \) is \( C^4 \)-close to Schwarzschild and satisfies (2.2) and (2.3). Then, for \( R \gg 1, \delta \ll 1 \), and every admissible surface \( \Sigma \) satisfying (2.12), there exists a global solution to (1.1) with initial configuration \( \Sigma_{t=0} = \Sigma \).

Recall that stationary solutions to (1.1) are called surfaces of Willmore type. The existence and stability of such surfaces are studied in Refs. 7 and 19.

**Theorem 3** (Ref. 5, Theorem 1 and Ref. 19, Theorem 5.3). Assume that \( M \) is \( C^4 \)-close to Schwarzschild and satisfies (2.2) and (2.3). Then, there exists a compact subset \( K \subset M \) such that \( M \setminus K \) is foliated by surfaces of Willmore type.

Moreover, for \( R \gg 1, \delta \ll 1 \), and every admissible surface \( \Sigma \) satisfying (2.12), the flow generated by \( \Sigma \) under (1.1) converges smoothly to one of the leaves of this foliation.

**D. Organization of this paper**

We organize this paper as follows: In Sec. II, we define the important map \( \Phi \), which arises by reconceptualizing the Lyapunov-Schmidt reduction. This allows us to identify the adiabatic part of a flow evolving according to (1.1). This adiabatic part accounts for most of the (Willmore) energy change along the flow (modulo some uniformly small fluctuation) and is finite-dimensional. In Sec. III, we discuss the static property of the effective action \( \mathcal{W} \circ \Psi \) and prove the first part of Theorem 1. A similar but different function defined on a domain in \( \mathbb{R}^3 \) is used in Ref. 7 (denoted by \( G \) in that paper). In Sec. IV, we prove the remaining part of Theorem 1 by deriving the effective dynamics (1.3) of (1.1). Here, we exploit the spectral property of certain linearized operators in order to bound a Lyapunov-type functional that controls fluctuations.

**1. Notation**

Throughout this paper, the notation \( A \leq B \) means that there is a constant \( C > 0 \) that depends only on \( c, k \) in (2.4) and \( R, \delta \) in (2.12) such that \( A \leq CB \). For two vectors \( A, B \) in Banach spaces \( X, Y \), the notation \( A = O_Y(B) \) means \( \|A\| \leq \|B\|_Y \). For a vector \( A \) in some Sobolev space \( H^k \), the notation \( A = O_Y(B) \) means \( \|A\|_{H^k} = O_Y(B) \).
III. THE LYAPUNOV–SCHMIDT MAP

Let \( k \geq 4, c \gg 1 \). Let \( K \subset M, R \gg 0 \) to be determined, and let \( M' := \mathbb{R}_{\geq0} \times B_{1}(0) \subset \mathbb{R} \times \mathbb{R}^{3} \). In this section, we construct the map \( \Psi : M' \to X^{k} \) as in Theorem 1.

A. Graphs over sphere

Denote \( H^{k} = H^{k}(\mathbb{S} \times \mathbb{R}) \). This space is equipped with the \( L^{2} \)-inner product \( \langle u, v \rangle = \int_{\mathbb{S}} uv \). Define the configuration space

\[
Y^{k} := H^{k} \times M'.
\]  

Define a map

\[
\theta : Y^{k} \longrightarrow X^{k} \\
(\phi, r, z) \longmapsto (r(1 + \phi(v))v + z).
\]

Here, \( v \in \mathbb{S} \subset \mathbb{R}^{3} \) is the spherical coordinate, and recall that we identify the asymptotic part \( (M\setminus K) \cong (\mathbb{R}^{3}\backslash B_{0}(0)) \). Define

\[
Y^{k}_{c} := \left\{ (\phi, r, z) \in Y^{k} : \theta(\phi, r, z) = c \right\}.
\]

This corresponds to the space of surfaces with fixed area, \( X^{k}_{c} \), as in (2.4).

For \( |\phi|_{H^{k}} \ll 1 \), the map \( \theta(\phi, r, z) \) is a well-defined graph over the coordinate sphere \( \theta(0, r, z)(\mathbb{S}) = S_{r,z} \). Thus, we can also identify \( \theta(\phi, r, z) \) as a function from \( S_{r,z} \subset M \to \mathbb{R} \). Note also that for sufficiently large \( c \gg 1 \) and every \( z \in B_{1}(0) \subset \mathbb{R}^{3} \), there is a coordinate sphere with area \( c \) around \( z \). Thus, the map \( \theta \) is surjective onto \( X^{k}_{c} \).

Definition 2 (topology on graphs). We say that two graphs \( \theta(\phi, r, z), \theta(\phi', r', z') \) are \( H^{k} \)-close if \( \|\phi - \phi'\|_{H^{k}} + |r - r'| + |z - z'| \ll 1 \).

B. Lyapunov–Schmidt reduction

Denote \( \bar{W}(\phi, r, z) \), \( \Omega(\phi, r, z) \) the pullbacks of the rhs of (1.1) and the Willmore energy (2.8) to \( Y^{k} \) through \( \theta \), respectively. Explicitly, we have

\[
\bar{W}(\phi, r, z) := -W(\theta(\phi, r, z)) - \lambda H(\theta(\phi, r, z)), \quad \Omega(\phi, r, z) := \mathcal{W}(\theta(\phi, r, z)).
\]

Since \( \mathcal{W} \) is \( C^{1} \) on \( X^{k} \) with \( k \geq 4 \) and \( \theta \) is smooth, the pullback energy \( \Omega \) is \( C^{1} \) on \( Y^{k}, k \geq 4 \). Using Sobolev inequalities, one can check that the partial Fréchet derivative \( \bar{W} \) is \( C^{1} \) in \( \phi \) and smooth in \( r, z \). This \( \bar{W} \) is the \( L^{2} \)-gradient of \( \Omega(\cdot, r, z) \) up to scaling and satisfies the mapping property \( \bar{W} : Y^{k} \to H^{k-4} \).

Remark 5. Note that (3.4) and (3.5) implicitly depend on the background metric \( g \).

Lemma 1. The linearized operator \( L_{M} \) of \( \bar{W} \) at \( (0, r, z) \) with background metric \( g \) is given by

\[
L^{2}_{M} := \partial_{\phi} \bar{W}(\phi, r, z)|_{\phi=0} \quad (\Delta^{2} + 2r^{-2}\Delta + O(r^{-4}))\partial_{\phi} \theta(0, r, z) : H^{k} \to H^{k-4}.
\]

Here, \( \Delta : X^{k} \to X^{k-2} \) denotes the Laplace–Beltrami operator on the coordinate sphere \( S_{r,z} \subset M\setminus K \), with center \( z \) and radius \( r \). The partial Fréchet derivative \( \partial_{\phi} \theta(0, r, z) : H^{k} \to X^{k} \) is given by \( \xi(v) \mapsto \xi(v)v \).

Moreover, the operator \( L_{M} \) is self-adjoint on \( H^{k} \). The spectrum of \( L_{M} \) is purely discrete. The operator \( \partial_{\phi} \theta(0, r, z) \) is invertible and satisfies

\[
\| \partial_{\phi} \theta(0, r, z) \|_{H^{k-2} \to X^{k}} \geq \| \partial_{\phi} \theta(0, r, z)^{-1} \|_{X^{k} \to H^{k}} = r.
\]
Proof. The operator $L^F_{t,z}$ is explicitly calculated in Ref. 5, Sec. 3. The spectral properties of $L_{t,z}$ are studied in Ref. 5, Sec. 7. The mapping properties of $\partial_y \theta$ are obvious.

Remark 6. The linearized operator (3.6) depends on (the curvature of) the background metric $g$ on $M$. In the special case when the ambient manifold $M$ is flat, i.e., $g = \delta_{ij}$, the linearized operator $L^0_{t,z}$ has eigenvalue 0 and ker $L^0_{t,z}$ is spanned by the constant function $y^0 \equiv 1$, together with the spherical harmonics $y^1, y^2, y^3$. Thus, so long as $(M, g)$ is asymptotically flat and $r > 1$ in (3.6) (such as in our setting), one can view $L^0_{t,z}$ as a perturbation of $L^0_{t,z}$. This motivates the following definition.

Definition 3. Define $P : H^k \to H^k$ to be the $L^2$-orthogonal projection onto span$\{y^0, \ldots, y^4\} = \ker L^0_{t,z}$. Define $\overline{P} := 1 - P : H^k \to H^k$ to be the complement of $P$.

Let $S$ be the set of all smooth symmetric two tensors on $M$. Define a map

$$F : \mathbb{Y}^k \times S \to H^{k-4}$$

$$(\phi, r, z, h) \mapsto \overline{P}W(\phi, r, z),$$

where $W$ is computed with background metric $g = g_S + h$ (see Sec. II A).

Proposition 1. Assume that the ambient manifold $(M, g)$ is $C^4$-closed to Schwarzschild.

1. For every $z \in \mathbb{R}^3$ with $|z| < 1$ and sufficiently large $r \geq R > 1$, there is a unique solution $\phi = \phi_{t,z} \in \overline{PH}^k$ to the following equation:

$$F(\phi, r, z, h) = 0,$$

where $F$ is defined in (3.8), and $g = g_S + h$.

2. Moreover, the map $(r, z) \mapsto \phi_{t,z}$ is $C^2$ and satisfies the estimate

$$\|\partial^m_t \partial^p_z \phi_{t,z}\|_{H^k} \lesssim r^{-(2+2m)}$$

for every $m + |a| \leq 2$.

3. Moreover, the surface $\theta(\phi_{t,z}, r, z)$ lies in the class of admissible surfaces in Definition 1.

Proof 1. By the implicit function theorem, it suffices to check that the map $F$ defined in (3.9) satisfies the following properties:

1. $F$ is $C^1$ in $\phi$.
2. $F(0, r, z, 0) = 0$ for every $r, z$.
3. $\partial_F F(0, r, z, 0) = L^0_{t,z}$ is invertible on $\overline{PH}^k$.

The first claim follows from the regularity of $W$ on $Y^k$ and its smooth dependence on the background metric.

If the background metric is Schwarzschild, i.e., $h = 0$, then it is well-known that by conformal invariance, the coordinate sphere $\theta(0, r, z)$ is the global minimizer of the Willmore energy $W$. Since $W = \partial_r \Omega$, the second claim follows.

The spectrum of $L^0_{t,z}$ can be calculated explicitly. See, for instance, Ref. 7, Corollary 33. In particular, 0 is an isolated eigenvalue with finite multiplicity. By elementary spectral theory, this implies that the restriction $L^0_{t,z} := L^0_{t,z}|_{PH}^k$ is invertible as a map from $\overline{PH}^k \to \overline{PH}^k$. Thus, the third claim follows.

2. For the estimate (3.10), we expand

$$L^F_{t,z} = L^0_{t,z} + V_{t,z},$$

where $V_{t,z}$ is defined by this expression. As we discuss in Remark 6, this $V_{t,z}$ is bounded from $H^k \to H^{k-4}$ and satisfies $\|V_{t,z}\|_{H^k \to H^{k-4}} = O(r^{-4})$. The restriction $L^0_{t,z}$ can be bounded from below by $Cr^{-2}$ for some $C > 0$ only depending on $k$. It follows that

$$\|(L^0_{t,z})^{-1}V_{t,z}\|_{H^{k-4} \to H^k} = O(r^{-2}).$$

For sufficiently large $r$, this together with the expansion (3.11) implies that the restriction $L^F_{t,z} : \overline{PH}^k \to \overline{PH}^{k-4}$ is invertible, given explicitly as the Neumann series.
\[ (L_{rz})^{-1} = \sum_{n=0}^{\infty} (L_{rz}^n)^{-1} (-V_{rz}(L_{rz})^{-1})^n. \]

From here, one can also read off the estimate
\[ \| (L_{rz})^{-1} \|_{\mathcal{M}^0 \to \mathcal{M}^0} = O(r^2). \] (3.12)

Expand \( F(\phi, r, z, h) = F(0, r, z, h) + \tilde{L}_{rz} \phi + N_{rz}(\phi) \), where the nonlinearity \( N_{rz} \) is defined by this expression. This \( N_{rz} \) is calculated explicitly in (C7). For every \( \phi \) satisfying (3.5), we can rearrange to get
\[ \phi = -(L_{rz})^{-1} (F(0, r, z, h) + N_{rz}(\phi)). \] (3.13)

In the rhs, we have \( F(0, r, z, h) = O(r^{-4}) \) by Ref. 7, Corollary 45. Thus, for sufficiently small \( \phi \), we have by (3.12) and (3.13) that \( \| \phi \|_{\mathcal{M}^0} = O(r^{-5}) \).

We now claim for \( \phi \in H^k \) and \( m + |a| \leq 2 \) that the following holds:
\[ |\partial^m \tilde{L} \phi|_{\mathcal{M}^0} \leq |\phi|_{\mathcal{M}^{0+m}}, \] (3.14)
\[ |\partial^m \tilde{L} F(0, r, z, h) \phi|_{\mathcal{M}^0} \leq r^{-4+m}, \] (3.15)
\[ |\partial^m \tilde{L} N_{rz}(\phi) \phi|_{\mathcal{M}^0} \leq |\phi|^2_{\mathcal{M}^0}. \] (3.16)

For (3.14), one uses the identity \( \partial^\delta (L_{rz})^{-1} = -(L_{rz})^{-1} \partial^\delta (L_{rz}(L_{rz})^{-1}) \), where \( |\delta| \leq 2 \) is a multi-index in both \( r \) and \( z \). This, together with the fact that \( \partial^\delta L_{rz} \) is uniformly bounded [see (C5)], implies (3.14). The rest follows from the expansion in Proposition 4. Using (3.14)–(3.16) and differentiating both sides of (3.13), we conclude the estimates (3.10).

3. For sufficiently large \( R \) and every \( r \geq R \), we find using (3.10) with \( m = 0, a = 0 \) that the surface \( \theta(\phi_{r.z}, r, z) \) is \( H^k \)-close to the coordinate sphere \( S_{r.z} \). This implies that \( \theta(\phi_{r.z}, r, z) \) is an admissible surface. \( \square \)

From now on, we write \( \zeta = \zeta^a, a = 0, \ldots, 4, \) for a point in \( (r, z) \in M' \). Thus, \( \zeta^0 = r \) and \( \zeta^1 = z' \) for \( j = 1, 2, 3 \).

**Definition 4** (the Lyapunov–Schmidt map \( \Phi \)). Let \( K \subset M \) be the compact set as in Theorem 3. Let \( R \gg 1, \delta \ll 1 \) be given as in Theorem 2. Let \( M' := \mathbb{R}_{z>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3 \).

Define the Lyapunov–Schmidt map \( \Phi : M' \to H^k \) by \( \zeta \mapsto \Phi(\zeta) \), where \( \Phi(\zeta) \) is the solution to (3.9) given in Proposition 1.

**Remark 7.** This \( \Phi \) is equivalent to the map \( \Psi \) in Theorem 1, through the diffeomorphism \( \Phi(\zeta) \mapsto \theta(\Phi(\zeta), \zeta) \).

In the next proposition, we describe the geometric structure induced by the map \( \Phi \).

**Proposition 2.** The set
\[ E := \{ \theta(\phi, \zeta) : \phi = \Phi(\zeta), \zeta \in M' \} \]
forms an immersed \( C^1 \) submanifold in \( X^4 \). The tangent space \( T_{\theta(\Phi(\zeta))} E \) consists of vector fields over the surface \( \theta(\Phi(\zeta), \zeta) \). A basis of \( T_{\theta(\Phi(\zeta))} E \) is given by \( \partial_{\zeta} \theta(\Phi(\zeta), \zeta) \).

**Remark 8.** Using the projection constructed in Lemma 2, one can view this manifold \( E \) as consisting of the adiabatic parts of low (Willmore) energy surfaces in \( X^4 \).

**Proof.** The manifold structure of \( E \) follows from Definition 4, where \( \Phi : M' \to E \) is a \( C^1 \) parameterization. We check that the tangent space is non-degenerate. Compute
\[ \partial_{\zeta} \theta(\Phi(\zeta), \zeta)(v) = (1 + \Phi(\zeta) + \zeta^0 \partial_{\zeta} \Phi(\zeta)) v, \] (3.17)
\[ \partial_{\zeta} \theta(\Phi(\zeta), \zeta)(v) = \zeta^0 \partial_{\zeta} \Phi(\zeta) v + e^j, \] (3.18)
where $e^j$ is the $j$th unit vector in $\mathbb{R}^3$. By the estimate (3.10), we find

$$\{\partial_\xi, \theta(\Phi(\zeta), \zeta), \partial_\zeta, \theta(\Phi(\zeta), \zeta)\} = 4\pi \delta_{\xi\zeta} + O(R^{-2}).$$

This implies the claim if $R$ is sufficiently large. \hfill $\Box$

In Appendix A, we introduce the general concepts of the Lyapunov–Schmidt map and relate it to our setting above.

### C. Barycenter

In this subsection, we develop a new concept of barycenter for a certain class of closed surfaces in $X^k$.

**Definition 5** (barycenter). Let $x_*$ be an embedding of a sphere that is $H^k$-close to the manifold $E \subset X^k$ constructed in Definition 4 with respect to the topology on graphs introduced in Definition 2. Then, we can write $x_* = \theta(\Phi(\zeta) + \xi, \zeta)$ for some $\zeta \in M'$, $\|\zeta\|_{H^k} \ll 1$. (There can, in general, be many such choices of $\zeta$ and $\xi$.) Expand $x_*$ in $\zeta$ around $\theta(\Phi(\zeta), \zeta)$ as

$$x_* = \theta(\Phi(\zeta), \zeta) + \partial_\zeta \theta(\Phi(\zeta), \zeta) \xi + O(\|\xi\|^2_{H^k}). \tag{3.19}$$

Define $f_* \in H^k$ as

$$f_*(\xi)(v) = \partial_\zeta \theta(\Phi(\zeta), \zeta)(v)^N = g(\partial_\zeta \theta(\Phi(\zeta), \zeta), v(\theta(\Phi(\zeta), \zeta)) \quad (\alpha = 0, \ldots, 3). \tag{3.20}$$

We say that a point $\zeta_* \in M'$ is the barycenter of $x_*$ if $\zeta_*$ satisfies the following algebraic system:

$$\langle (\xi, f_* \zeta)_{L^2} = 0 \quad (\alpha = 0, \ldots, 3), \tag{3.21}$$

where $\xi$ is defined by the relation $x_* = \theta(\Phi(\zeta_*) + \xi, \zeta_*)$.

**Remark 9.** The four vectors $f_*$ span the tangent space at $\theta(\Phi(\zeta_*), \zeta)^N$ to $E^N \subset H^k$, where $E^N$ consists of the normal components of the elements in the manifold $E$ defined in Definition 4. Geometrically, the defining condition (3.21) for barycenter means that the Gâteaux derivative of the map $\theta(\cdot, \zeta_*)^N$ at $\Phi(\zeta_*)$ along the $\xi$-direction is perpendicular to the tangent space $T_{\theta(\Phi(\zeta_*), \zeta_*)^N} E^N$. In terms of the expansion (3.19), this means that the second term in the rhs is $L^2$-orthogonal to the tangent space at the first term to $E$. In this sense, the choice of barycenter is optimal.

**Remark 10.** Our definition of the barycenter differs from the classical one, given by averaging over $\Sigma$ with respect to the Euclidean background metric, namely, $\int_{\Sigma} f_\Sigma d\mu_{g_\Sigma}$. See Ref. 19 and the references therein. Our version of the barycenter retains the key decay property as Ref. 19, Sec. 5. Namely, the motion of the barycenter is controlled by a differential inequality using a Lyapunov functional, defined in Sec. V.

Moreover, our definition allows us to retain explicit and uniform control of a flow evolving according to (1.1), as in Sec. V.

In the next lemma, we define a nonlinear projection (or coordinate map) that associates barycenters with low energy configurations in $X^k$.

**Lemma 2** (nonlinear projection). There exists $\delta > 0$ such that on the space

$$U_\delta := \{x = \theta(\Phi(\zeta) + \xi, \zeta) : \zeta \in M', \|\zeta\|_{H^k} < \delta\}, \tag{3.22}$$

there exists a $C^3$ map $S : U_\delta \to M'$ such that $S(x)$ is the barycenter of $x$ as in Definition 5. Moreover, we have a uniform estimate on $S$ and its derivative.

**Remark 11.** Essentially, the existence of such a projection depends on the non-degeneracy shown in Proposition 2. Later, we see that the barycenter $\zeta = S(x)$ determines the adiabatic (or slowly varying) part of $x$.

We call the remainder $\xi$ that satisfies $x = \theta(\Phi(S(x)) + \xi, S(x))$ the fluctuation of $x$. 

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IV. EFFECTIVE ACTION

In this section, we prove part 1 of Theorem 1. We formulate this as follows.

Theorem 4. The embedding \( \theta(\Phi(\zeta)) \) parameterizes a surface of Willmore type [i.e., static solution to (1.1)] if and only if \( \zeta \) is a critical point of the function \( G := \Omega(\Phi(\cdot), \cdot) : \mathbb{R}^4 \to \mathbb{R} \), where \( \Omega \) is defined in (3.5).

Remark 12. Similar results are obtained in Ref. 7, Theorems 5 and 8. Using some expansion obtained in that paper, we can calculate \( G \) explicitly as in (C2).

Proof. For the forward direction, we use the chain rule to get
\[ \partial_{\zeta} G(\zeta) = \left\langle \nabla^{N} W(\theta(\Phi(\zeta), \zeta)), f_{\alpha} \right\rangle. \] (4.1)

If \( \theta(\Phi(\zeta), \zeta) \) is a critical point of \( W \), i.e., the Fréchet derivative \( dW(\theta(\Phi(\zeta), \zeta)) = 0 \), then the first factor of rhs of (4.1) vanishes, and therefore, \( \zeta \) is a critical point of \( G \).

Now, suppose that \( \partial_{\zeta} G(\zeta) = 0 \). For the backward direction, it suffices to show the pullback \( \tilde{W}(\Phi(\zeta), \bar{\zeta}) = 0 \).

Let \( Q_{t} \) be the projection onto the tangent space at \( T_{\tilde{\theta}(\Phi(\zeta), \bar{\zeta})} E^{N} \), as in Remark 9. Explicitly, this map \( Q_{t} \) is given by

\[ Q_{t}\psi = V^{\alpha\beta}(f_{\alpha}, \phi)_{t} f_{\beta} \quad (\phi \in H^{1}), \] (4.2)

where the matrix \( V_{\alpha\beta} := \{ f_{\alpha}, f_{\beta} \} \), the matrix \( V^{\alpha\beta} \) is its inverse, and \( f_{\alpha}'s \) are given in (3.25) and (3.26). Note that this matrix \( V_{\alpha\beta} \) is, indeed, invertible because the tangent space \( T_{\tilde{\theta}(\Phi(\zeta), \bar{\zeta})} E^{N} \) is non-degenerate; cf. Proposition 2. From definition (4.2) and formula (3.25) and (3.26), we also get a uniform estimate \( \| Q_{t} \| \leq 1 \).

The claim now is that \( Q_{t} : H^{k} \rightarrow H^{k} \) is uniformly close in operator norm to the projection \( P \) defined in Definition 3. Geometrically, this is because the manifold \( E^{N} \) is a uniformly small normal perturbation of ran \( P \). Analytically, this claim follows by comparing (3.25) and (3.26) to the basis \( y^{\alpha} \) of ran \( P \) and using the estimate (3.10) for the derivatives \( \partial_{\zeta} \Phi \). As a result, we find

\[ \| Q_{t} - P \|_{H^{k} \rightarrow H^{k}} = O(R^{-2}). \] (4.3)

Now, by the construction in Definition 4, we know that \( P\tilde{W}(\Phi(\zeta), \bar{\zeta}) = \tilde{W}(\Phi(\zeta), \bar{\zeta}) \). Thus, we can write

\[ \tilde{W}(\Phi(\zeta), \bar{\zeta}) = P\tilde{W}(\Phi(\zeta), \bar{\zeta}) = (P - Q_{t})\tilde{W}(\Phi(\zeta), \bar{\zeta}) + Q_{t}\tilde{W}(\Phi(\zeta), \bar{\zeta}). \] (4.4)

From (4.1) and (4.2), one can see that \( \partial_{\zeta} G = 0 \) implies \( Q_{t}\tilde{W} = 0 \). Thus, by assumption, the last term in (4.4) vanishes. Using this fact and the estimate (4.3), we find

\[ \| \tilde{W}(\Phi(\zeta), \bar{\zeta}) \|_{H^{k}} \leq \| (P - Q_{t}) \|_{H^{k} \rightarrow H^{k}} \| \tilde{W}(\Phi(\zeta), \bar{\zeta}) \|_{H^{k}} \]
\[ \leq R^{-2} \| \tilde{W}(\Phi(\zeta), \bar{\zeta}) \|_{H^{k}}. \]

For sufficiently large \( R \), this is impossible unless \( \tilde{W}(\Phi(\zeta), \bar{\zeta}) = 0 \).

\[ \square \]

V. EFFECTIVE DYNAMICS

In this section, we prove the rest of Theorem 1. We first derive the effective dynamics (1.2)–(1.4) and then use this to derive parts 2 and 4 of Theorem 1.

**Theorem 5.** Let \( \Phi : M' \rightarrow H^{k} \) be the map defined in Definition 4. Let \( \Sigma_{*} = x_{*}(\mathcal{S}) \) be an admissible surface. Let \( \Sigma_{t} = x_{t}(\mathcal{S}) \) be the global solution to (1.1) with initial configuration \( \Sigma_{|t=0} = \Sigma_{*} \) as in Theorem 2.

Then, there exist \( \alpha > 0 \), \( T = O(R^{-\alpha}) \), and a path \( \zeta_{t} \in M' \) such that

\[ \| \Phi(\zeta_{t}) - x_{t} \|_{H^{k}} = O(R^{-1}) \quad (t \geq T). \] (5.1)

Moreover, the path \( \zeta_{t} \) evolves according to

\[ \dot{\zeta} = \frac{1}{4\pi} \nabla G(\zeta) + O(R^{-3}), \] (5.2)

where the leading term in the rhs is of the order \( O(R^{-2}) \).

**Remark 13.** The function \( G \) is defined in Theorem 4 and calculated in Appendix C.

**Proof.** Step 1. Take some \( 0 < \delta < 1 \) to be determined. To begin with, denote \( T > 0 \) the first time \( u_{t} \) enters the manifold \( U_{\delta} \) defined in (3.22). By the stability result, Theorem 3, this \( T \) is finite, and there exists \( \alpha > 0 \) such that \( T = O(\delta^{-\alpha}) \).

Now, we prove claims (5.1) and (5.2), assuming the a priori estimate

\[ x_{t} \in U_{\delta} \text{ for all } t > T. \] (5.3)
By Lemma 2, so long as \((5.3)\) is satisfied, we can associate a path of barycenters \(\zeta_t \in M^r\) with the full flow \(x_t\). Later on, we show that this ansatz is satisfied with \(\delta = O(R^{-1})\) by proving an \(a\ priori\) estimate for the fluctuation of \(x_t\) around its adiabatic part.

Write \(x = \theta(\Phi(\zeta) + \xi, \zeta)\), where \(\zeta = S(x)\) is the barycenter as in Lemma 2, and \(\xi\) is the fluctuation. Then, we can rewrite the lhs of \((1.1)\) as

\[
\partial_t x^N = \partial_t \theta(\Phi + \xi, \zeta) = \partial_t \theta(\Phi, \zeta) N \partial_t \xi + \partial_t \theta(\Phi, \zeta) N \xi_0 = A(\xi) \partial_t \theta(\Phi, \zeta) N \partial_t \xi + B(\Phi, \theta(\Phi, \zeta) N \xi_0 = A(\xi)(\xi^0 + O_{I^R}(1)) \partial_t \xi + B(\xi) f_\alpha \xi_0,
\]

(5.4)

where we define

\[
A(\xi) := \frac{\partial_t \theta(\Phi + \xi, \zeta) N}{\partial_t \theta(\Phi, \zeta) N}, \quad B(\xi) := \frac{\partial_t \theta(\Phi + \xi, \zeta) N}{\partial_t \theta(\Phi, \zeta) N}.
\]

For simplicity, here and below, we omit the dependence of \(f_\alpha\) and \(\Phi\) on \(\zeta\).

Expansion \((5.4)\) follows from the chain rule, \((3.20)\), and the assumption on the background metric, \(g_{ij} = \delta_{ij} + O((\zeta_0)^{-1})\). From \((3.2)\), one can see that the prefactors \(A(\xi), B(\xi)\) satisfy

\[
|A(\xi)| L^1 + |B(\xi)| L^1 \leq \|\xi\|_{I^R}.
\]

(5.5)

Let \(Q = Q_{(\xi)}\) be the projection onto the tangent space \(T_{\theta(\Phi(\zeta), \xi)} Y^N\), as given explicitly in \((4.2)\). The space \(\text{ran}Q\) is spanned by the four vectors \(f_\alpha\), which we computed in \((3.25)\) and \((3.26)\). Applying \(Q\) to \((5.4)\), we find

\[
Q \partial_t x^N = (\xi^0 + O_{I^R}(1)) Q \partial_t \xi + f_\alpha \xi_0 + O_{I^R}(\xi(\xi^1 + \partial_t \xi)).
\]

(5.6)

By the definition of the barycenter, \((3.21)\), we know that \(Q \xi = 0\). Differentiating this, we find

\[
Q(\partial_t \xi) = -(\partial_t Q) \xi = -(\partial_t Q \xi) \xi.
\]

(5.7)

Here, the partial Fréchet derivative \(\partial_t Q\) maps a real number to a linear operator from \(H^k \rightarrow \text{ran}Q \subset H^k\). From the explicit formula \((4.2)\) for \(Q\) and the estimate \((3.10)\), we get the uniform bound \(\|\partial_t Q\|_{L^1(I^R, H^k)} \leq \|\xi^0\|^{-2}\). Thus, plugging \((5.7)\) to \((5.6)\) gives

\[
Q \partial_t x^N = f_\alpha \xi_0 + O_{I^R}(\xi(\xi^1 + \partial_t \xi)) + R^{-1} O_{I^R}(\xi(\xi^1).
\]

(5.8)

Step 2. Next, expanding the rhs of \((1.1)\) at \(\theta(\Phi(\zeta), \zeta)\), we find

\[
W(x) + \lambda H(x) = \tilde{W}(\Phi(\zeta) + \xi, \zeta) + \tilde{W}(\Phi(\zeta), \zeta) + L_\xi \zeta + N_\zeta(\zeta).
\]

(5.9)

Here, as in Sec. III, \(\tilde{W} : Y^k \rightarrow H^{k-4}\) is the pullback of \(W + \lambda H\), and \(L_\xi\) is the partial Fréchet derivative \(\partial_\xi \tilde{W}\) evaluated at \(\theta(\Phi(\zeta), \zeta)\). The nonlinear term \(N_\zeta(\zeta)\) is defined by collecting the rest terms in this expansion.

Applying \(Q\) to \((5.9)\), we find

\[
Q \tilde{W}(\Phi(\zeta) + \xi, \zeta) = Q \tilde{W}(\Phi(\zeta), \zeta) + Q L_\xi \zeta + Q N_\zeta(\zeta).
\]

(5.10)

By the chain rule and the definitions \((3.4)\) \((4.2)\), the first term in \((5.10)\) can be written as

\[
Q \tilde{W}(\Phi(\zeta), \zeta) = V^{\Phi}\{V^N(\theta(\Phi(\zeta), \zeta)), f_\alpha\} f_\beta.
\]

(5.11)

By \((4.1)\), this expression equals to \(V^{\Phi}\partial_\zeta \tilde{G}(\zeta) f_\beta\). By the uniform estimates \(\|Q\|_{H^k \rightarrow H^k} \leq 1\) and \((C1)\), we can bound the nonlinearity as
Finally, using the approximate zero mode property of the elements in ran Q, which we show in Lemma 3, one sees that the restriction of $L_{\xi}$ is small on ran Q. In particular, this implies for $\xi, \xi' \in H^4$,

$$\langle Q L_{\xi} \xi', \xi \rangle = \langle \xi, L_{\xi} Q \xi' \rangle = O(R^{-2}) \| \xi \|_{H^2} \| \xi' \|_{H^4}.$$  

Plugging $\xi' = Q L_{\xi} \xi$ into this expression, we find

$$\| Q L_{\xi} \xi \|_{H^4} = O(R^{-2}) \| \xi \|_{H^4}. \tag{5.13}$$

Recall the ansatz (5.3), which implies that $\| \xi \|_{H^4} < \delta \ll 1$. Collecting (5.8), (5.10)–(5.13), plugging these back to (5.4), (5.9), and rearranging, we find

$$\left\| f_\alpha \xi - V^\alpha \partial_x G(\xi) f_\beta \right\|_{H^4} \leq \frac{R^{-2} O(\delta) + O(\delta \| \xi \|_{H^4})}{1 - O(\delta) - R^{-1} O(\delta)}. \tag{5.14}$$

Consider the map

$$\psi : \mathbb{R}^4 \longrightarrow \text{ran} Q \subset H^{k-4},$$

$$\eta^\alpha \longrightarrow V^\alpha \eta^\beta f_\beta.$$ \hspace{1cm} \tag{5.15}

This map is linear. Moreover, using the fact that the matrix $V^\alpha \eta_\beta = O(1) \delta_{\alpha \beta}$ is invertible, one can show that $\psi$ is invertible on ran $Q$, and the operator norm of its inverse $\psi^{-1} : \text{ran} Q \subset H^{k-4} \rightarrow \mathbb{R}^4$ is of the order $O(1)$.

Now, we rewrite $f_\alpha \xi^\alpha = \psi(\eta^\alpha)$ with

$$\eta^\alpha = V^\alpha \eta^\beta.$$ \hspace{1cm} \tag{5.16}

Note that this choice of $\eta^\alpha$ is unique. Then, we can conclude from (5.14) and the discussion above that

$$\left| \eta^\alpha - \partial_x G(\xi) \right| \leq \frac{R^{-2} O(\delta) + O(\delta \| \xi \|_{H^4})}{1 - O(\delta) - R^{-1} O(\delta)}. \tag{5.17}$$

Step 3. From (5.16), the relation (5.15), and the estimate $V^\alpha \eta_\beta = O(R^{-2})$, which follows from (3.25) and (3.26) (see also Proposition 2), we see that (5.2) follows once we can prove an a priori estimate of the form

$$\| \xi \|_{H^4} + \| \partial_x \xi \|_{H^4} \leq \| \xi_T \|_{H^4}, \tag{5.18}$$

where $\xi_T$ is the fluctuation when the flow $x_T$ first enters the manifold $U_{\eta}$, as in ansatz (5.3).

To this end, define

$$\Lambda(x) := \frac{1}{2} \left\{ \xi, L_{\xi} \xi \right\}. \tag{5.19}$$

We show that this is a Lyapunov-type functional along the flow of $\xi$.

Recall that $Q = 1 - Q$ is the complement of the (not necessarily orthogonal) projection $Q$. Indeed, differentiating $\Lambda$, we find

$$\dot{\Lambda}(x) = \left( \partial_x \xi, L_{\xi} \xi \right) + \frac{1}{2} \left( \left( \partial_x L_{\xi} \xi, \xi \right). \tag{5.20}$$

The second term is bounded as

$$\left| \left( \partial_x L_{\xi} \xi, \xi \right) \right| \leq \left| \left( \partial_x L_{\xi} \xi \right) \xi \right|_{H^{-1}} \| \xi \|_{L^2} = O(\| \xi \|_{H^2}^2), \tag{5.21}$$

where we used the estimate (C10).

To bound the first term in the rhs of (5.19), we isolate the dynamics of $\xi$ after equating (5.4)–(5.9). We find
\[
\frac{\partial \xi}{\partial t} = \frac{1}{\partial_\theta (\Phi, \xi) N A(\xi)} (-\dot{W}(\Phi, \xi) - L_\xi \xi - N_\xi (\xi) - B(\xi) f_a \xi^a). \tag{5.21}
\]

Consider the rhs of (5.20) and (5.21). Using (C9)–(C11) and the governing equation for barycenter, (5.14), we find for \( R \gg 1, \)

\[
\partial_\theta (\Phi, \xi) N A(\xi) = O_{\mu} (\xi^0 (1 + \xi)) + O_{\mu} (1 + \xi), \tag{5.22}
\]

\[
\left[ W(\Phi, \xi), L_\xi \xi \right] \leq R^{-4} \| \xi \|_{H^2}, \tag{5.23}
\]

\[
\left[ N_\xi (\xi), L_\xi \xi \right] \leq R^{-3} \| \xi \|_{H^2}, \tag{5.24}
\]

\[
\left( f_\alpha \xi^\alpha, L_\xi \xi \right) \leq R^{-2} \left( 1 + \| \xi \|_{H^2} \right) \| \xi \|_{H^2}. \tag{5.25}
\]

Plugging (5.20)–(5.25) back to (5.19), we find that so long as \( \| \xi \|_{H^2} < 1/2 \) and \( R \gg 1 \), the following holds:

\[
\dot{\Lambda}(t) \leq C_1 R^{-3} \| \xi \|_{H^2} \tag{5.26}
\]

\[
- \\left[ L_\xi \xi \right] \| \xi \|_{H^2} + \left( C_2 R^{-3} + C_3 R^{-2} \| \xi \|_{H^2} \right) \| \xi \|_{H^2}.
\]

By the coercivity of \( L_\xi \) shown in Lemma 4, the first term in the second line of (5.26) can be bounded by \(- \| L_\xi \xi \|_{H^2} \leq -\beta \| \xi \|_{H^2} \) for some \( \beta > 0 \) independent of \( \xi \). This, together with the upper bound in (D5), implies that there exists some \( \gamma > 0 \) independent of \( \xi \) such that

\[
\dot{\Lambda}(t) + \gamma \Lambda(t) \leq C_1 R^{-3} \| \xi \|_{H^2} + \left( C_2 R^{-3} + C_3 R^{-2} \| \xi \|_{H^2} - \beta/2 \right) \| \xi \|_{H^2}. \tag{5.27}
\]

Thus, so long as

\[
C_2 R^{-3} + C_3 R^{-2} \| \xi \|_{H^2} - \beta/2 < 0, \tag{5.28}
\]

we can drop the last term in (5.27) to deduce that

\[
\frac{d(\Lambda(t) e^{\beta t})}{dt} \leq R^{-3} e^{\beta t}. \tag{5.29}
\]

Integrating (5.29) on \([T, \infty)\) and using (D5), we find

\[
\| \xi_T \|_{H^2} \leq \Lambda(t) \leq e^{\beta T} \| \xi_T \|_{H^2} + R^{-3} M(t), \quad \left( M(t) := \sup_{\tau \leq T} \| \xi_\tau \|_{H^2} \right), \tag{5.30}
\]

where \( \xi_T \) is the fluctuation at \( t = T \). Taking supremum of both sides of (5.30) in \( t \) and then dividing by \( M(t) \) (which is positive for all \( t \) in the nontrivial case), we find

\[
M(t) \leq e^{\beta t} \| \xi_T \| + R^{-3}. \tag{5.31}
\]

This, in particular, implies that the ansatz (5.28) is satisfied as long as \( R \gg 1 \). Iterating this process, we get the estimate \( M(t) \leq R^{-3} \). Plugging this back to the dynamics (5.14) and (5.21) for \( \xi \) and \( \dot{\xi} \), respectively, we find that the velocity of the fluctuation satisfies \( \| \partial_\theta \xi \|_{H^2} \leq R^{-4} \). Thus, for sufficiently large \( R \), the claim (5.17) follows.

\[
\textbf{Corollary 1.}\text{ Let } \xi \in M' \text{ be a flow evolving according to (5.2). Then, there exist some } T > 0 \text{ and a global solution } x_t \text{ to } (1.1) \text{ such that (5.1) holds on the time interval } T \leq t \leq T + R \text{ with this choice of } \xi.
\]

\[
\textbf{Proof.}\text{ Let } T > 0 \text{ be the first time } u_t \text{ enters the manifold } U_A \text{ defined in (3.22) (thisis finite by Theorem 3). Consider the flow } x_t, t \geq T \text{ generated under (1.1) by } x_\ast = \Phi(\xi_t), \text{ as in Theorem 2. Let } \xi_t, t \geq T \text{ be the flow of barycenter as in (5.1). Then, using (3.10) and (5.1), we can estimate}
\]

\[
\| \Phi(\xi_t) - x_t^N \|_{H^2} \leq \| \Phi(\xi_t) - x_t^N \|_{H^2} + \| \Phi(\xi_t) - \Phi(\xi_t) \|_{H^2}
\]

\[
\leq R^{-2} \| \xi_t - \xi \| + R^{-3}
\]

\[
\leq R^{-2} \int_T^{T + R} \| \xi_t - \xi \| + R^{-3}
\]

\[
\leq R^{-3} \text{ (} T \leq t \leq T + R\text{).}
\]
In the last step, we use the effective dynamics (5.2).

In the following corollary, we display the $t$-dependence in subscripts.

**Corollary 2.** Suppose that the embedding $\theta(\Phi(\zeta), \zeta)$ parameterizes a surface of Willmore type [i.e., static solution to (1.1)]. Then, $\theta(\Phi(\zeta), \zeta)$ is uniformly stable with small area-preserving $H^k$-perturbation if $\zeta$ is a strict local minimum of the function $G$ defined in Theorem 4.

**Proof.** Suppose that $\zeta$ is a strict local minimum of $G$. Then, every flow $\zeta_t$ starting at some $\zeta'$ near $\zeta$ under (5.2) converges to $\zeta$, i.e., $\zeta_t \to \zeta$ as $t \to \infty$. Now, for this $\zeta$, consider a perturbation $x^\prime := \theta(\Phi(\zeta) + \zeta, \zeta)$ with $|x^\prime|_{H^k} \ll 1$. By the regularity of the nonlinear projection $S$ in Lemma 2, this perturbation has barycenter $\zeta'$ close to $\zeta$. It follows that the flow of barycenters $\zeta_t$ associated with the flow of embeddings $x$, generated by $x^\prime$ under (1.1) satisfies $|\zeta_t - \zeta| < \delta$ for any $\delta > 0$ and all sufficiently large $t > T$. If we choose $\delta \ll R^{-3}$, then we can conclude from (5.1) that $|\zeta^N_t - \Phi(\zeta)|_{H^k-1} \ll R^{-3}$ for all large $t$. \hfill $\square$

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**AUTHOR DECLARATIONS**

**Conflict of Interest**

The author has no conflicts to disclose.

**DATA AVAILABILITY**

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

**APPENDIX A: THE ABSTRACT LYAPUNOV–SCHMIDT MAP**

Motivated by the synonymous reduction procedure, in general, we can define the Lyapunov–Schmidt map as follows.

**Definition 6.** Let $X \subset Y$ be two Hilbert spaces. Let $U \subset X$ be an open set and $u_0 \in U$. Let $P : X \to X$ be an orthogonal projection onto some subspace of $X$ and $P^\perp := 1 - P$ be the complement of $P$.

The Lyapunov–Schmidt map $\Phi = \Phi_{P,u_0}$ is defined on the following set:

\[
\{ F : U \subset X \to Y : F \text{ is } C^1, \quad P L P : P X \to P Y \text{ has bounded inverse,} \\
\text{where } L := dF(u)|_{u=u_0} : X \to Y \text{ is} \\
\text{the linearized operator of } F \text{ at } u_0. \}
\]

For such $F$, by the implicit function theorem, there exists an open neighborhood $V \subset PX$ around $v = 0$ and a $C^1$ map $w : V \to PX$ such that $P(F(u_0 + v + w(v))) = 0$ for every $v \in V$. This map $w$ parameterizes a manifold $E$ in $X$. This manifold $E$ is a normal perturbation of ran $P$.

Let $w'$ be the Fréchet derivative of $w$. (This is a map from $V$ to the linear operators on $X$.) Then, the tangent space at $w = w(v)$ to $E$ is spanned by $w'(v)(V)$ and can be trivialized as a subspace in $X$, with the dimension up to dim $V$.

Now, the image $\Phi(F)$ of the Lyapunov–Schmidt map is given by

\[
\Phi(F) : \quad V \subset PX \quad \longrightarrow \quad X \\
\quad v \quad \longmapsto \quad Q_v F(u_0 + v + w(v)),
\]

where $w = w(v)$ is as above, and $Q_v$ is the projection onto the tangent space $T_{w(v)}E$.

The notion above is first informally conceived in Ref. 6, in connection with the Feshbach–Schur map introduced in Ref. 22 and 23. Here, note that the domain $V$ of the map $\Phi(F)$ depends on the spectrum of $L$ only.

Essentially, the map $\Phi$ identifies a reduced space, which can be either finite-dimensional or otherwise more tractable. The behavior of a map in the domain of $\Phi$ on this reduced space, which is locally isomorphic to $PX$, determines the behavior of this map in the vicinity of $0$ in the full space $X$. 

As such, one can view this map $\Phi$ in the context of the theory of infinite-dimensional invariant manifolds for semiflows in Banach spaces. See Refs. 24–26 and the references therein.

We will treat the analytical and geometric properties of the abstract Lyapunov–Schmidt map elsewhere.

In the context of this paper, the family of maps involved is $F(\cdot, r, z) : H^k \to H^{k-4}$, defined in Definition 3. These maps are parameterized by the manifold $M'$. The projection $P$ is defined as the orthogonal projection onto the subspace span$\{1, y^1, y^2, y^3\} \subset H^k$. The vector $u_0$ is the zero function in $H^k$, which, through $\theta$ given in (3.2), corresponds to a large coordinate sphere.

APPENDIX B: AREA-RESERVING PROPERTY OF (1.1)

In this section, following Ref. 19, Sec. 2, we show that (1.1) is area-preserving.

Let $x = x_t : \mathbb{S} \to (M, g)$ be a family of embeddings and $\Sigma := x(\mathbb{S})$. Suppose that $x$ solves (1.1), namely,

$$
\partial_t x^\Sigma = -W(x) - \lambda H(x), \quad W(x) := \Delta H(x) + H(x) (\text{Ric}_M(v,v) + |A|^2(x)),
$$

where $\mu = \mu_{\Sigma}$ is the canonical measure on $\Sigma$, induced by the embedding $x$ and background metric $g$.

If $\lambda$ is given by (B2), then integration by part on the first term in the rhs of (B2) yields

$$
\lambda(t) \int_{\Sigma_t} H^2 \, d\mu = -\int_{\Sigma} WH \, d\mu.
$$

On the other hand, for $\Sigma = \Sigma_t = x_t(\mathbb{S})$, the first variation formula for the area functional reads

$$
\partial_t |\Sigma| = -\int_{\Sigma} H \partial_t x^\Sigma \, d\mu.
$$

Plugging the expression for $\partial_t x^\Sigma$ from (B1) into (B4), we find

$$
\partial_t |\Sigma| = \int_{\Sigma} (W + \lambda H)H \, d\mu = 0,
$$

where the last step follows from identity (B3). This shows that (1.1) is, indeed, area-preserving.

APPENDIX C: ASYMPTOTICS OF $\mathcal{W}$

In this section, we record various expansions related to the Willmore energy $\mathcal{W}$ at large coordinate spheres. Most of the results can be read off from Ref. 7, Sec. C and Ref. 5, Secs. 3 and 7.

Fix some $R \gg 1$ and $0 < \delta \ll 1$. In what follows, we restore the parameter $r, z$ and assume that $r \geq R, z \in \mathbb{R}^3, |z| < 1 - \delta$.

Proposition 3. For $\Omega$ given in (3.5) and $G$ given in Theorem 4, the following holds:

$$
\Omega(0, r, z) = 4\pi - 16\pi r^{-1} + 2\pi r^{-2} \left( \frac{10}{2} - 6|z|^2 + 3 \frac{1}{|z|^2} \log \frac{1 + |z|}{1 - |z|} \right) + O(r^{-3}),
$$

(C1)

$$
G(r, z) = \Omega(0, r, z) + O(r^{-5}).
$$

(C2)

Proof. (C1) can be found in Ref. 7, Lemma 42. To derive this expansion, one uses the assumption (2.3) that the scalar curvature $S_c$ is asymptotically even. (C2) follows from the expansion

$$
G(r, z) = \Omega(0, r, z) + \{\hat{W}(0, r, z), \Phi(r, z)\} + O(\|\Phi(r, z)\|_{L^p}),
$$

the expansion for $\hat{W}(0, r, z)$, and the estimate (3.10). \qed
The following results are used to derive the key estimate (3.10).

Proposition 4. Let \( k \geq 4 \). Let \( \phi = \phi_{r,z} \in H^{k} \) be a family of functions with \( \| \phi \|_{H^{k}} \ll 1 \), and suppose that the map \( (r,z) \mapsto \phi_{r,z} \) is \( C^{2} \). Let \( \tilde{W} \) be as in (3.4). Then, for all \( m + |\alpha| \leq 2 \), the following holds:

\[
\begin{align*}
\tilde{W}(\phi, r, z) &= \tilde{W}(0, r, z) + L_{r,z} \phi + N_{r,z}(\phi), \\
\| \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} \tilde{W}(0, r, z) \|_{H^{m-4}} &= O(r^{-(4+m)}), \\
| \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} L_{r,z} \phi |_{H^{m-4}} &\leq | \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} \phi |_{H^{m}}, \\
| \partial_{r}^{m} \partial_{z}^{\alpha} N_{r,z}(\phi) |_{H^{m-4}} &\leq | \partial_{r}^{m} \partial_{z}^{\alpha} \phi |^{2}_{H^{m}}.
\end{align*}
\]

Proof. The expansion (C3) is valid by the regularity of \( \tilde{W} \) from \( Y^{k} \to H^{k-4} \). (C4) follows from the formula Ref. 7, Corollary 45. Since \( L_{r,z} \) is linear in \( \phi \), (C5) is immediate. For (C6), we calculate \( N_{r,z}(\phi) \) explicitly as

\[
N_{r,z}(\phi) = \Delta H(\theta(r, z)) + H(\theta(r, z)) \left( \text{Ric}_{\bar{M}}(v, v) + |\bar{A}|^{2}(\theta(r, z)) \right) - \Lambda_{r}^{2} \phi - \frac{1}{2} H_{r}^{2} \Lambda_{r} \phi - \lambda \Lambda_{r} \phi
\]

where \( \lambda \) is the Lagrange multiplier in (1.1), \( \Lambda = -\Delta - (|\bar{A}|^{2} + \text{Ric}_{\bar{M}}(v, v)) \) is the Jacobian operator, \( V \) is some smooth function depending on the sphere \( \theta(0, r, z) \) and the background metric only, and the subscript on \( r, z \) indicates evaluation at the sphere \( \theta(0, r, z) \). Formula (C7) follows from Ref. 20, Sec. 7.2.

Now, inspecting each term in (C7), one can see from this explicit expression that the nonlinearity \( N_{r,z} \) is a finite sum of the form \( N_{r,z} = N_{r,z}(D^{4}\phi, g) \), where \( D^{4}\phi \) denotes all derivatives of \( \phi \) up to order 4. Moreover, each term in this finite sum, together with its derivatives, is uniformly bounded by some \( C > 0 \) independent of \( r, z \) as a map from \( H^{k} \to H^{k-4} \). This implies

\[
\| \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} N_{r,z}(\phi) \|_{H^{m-4}} \leq \| \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} \phi \|^{M}_{H^{m}}
\]

for some \( M = M(m, \alpha) \geq 2 \). For \( \| \phi \|_{H^{k}} \ll 1 \), this implies (C6). \( \square \)

By perturbing the estimates in Proposition 4, we get the following results.

Proposition 5. Let \( k \geq 4 \). Let \( \phi = \phi_{r,z} \in H^{k} \) be a family of functions with \( \| \phi \|_{H^{k}} \ll 1 \). Let \( \tilde{W} \) be as in (3.4). Suppose that the map \( (r, z) \mapsto \phi_{r,z} \) is \( C^{1} \) and satisfies (3.10). Then, for every \( |\xi| \ll 1 \), the following holds:

\[
\begin{align*}
\tilde{W}(\phi + \xi, r, z) &= \tilde{W}(\phi, r, z) + L_{r,z} \xi + N_{r,z}(\xi), \\
\| \tilde{W}(\phi, r, z) \|_{H^{m-4}} &= O(r^{-4}), \\
| \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} L_{r,z} \xi |_{H^{m-4}} &\leq | \partial_{\nu}^{m} \partial_{\alpha}^{\alpha} \xi |_{H^{m}} \quad (m + |\alpha| \leq 1), \\
| N_{r,z}(\epsilon) \|_{H^{m-4}} &\leq | \xi |^{2}_{H^{m}}.
\end{align*}
\]

Proof. The point is that, by the regularity of \( \Omega(\cdot, r, z) \) on \( H^{k} \) with \( k \geq 4 \), the map \( \tilde{W}(\cdot, r, z) \) together with its derivatives varies continuously as \( \phi \) varies. Thus, if \( \phi \) satisfies (3.10), then (C9)–(C11) follow from (C4)–(C6) with an appropriate choice of \( r, \alpha \). \( \square \)

APPENDIX D: SPECTRAL PROPERTIES OF \( L_{\zeta}^{i} \)

Let \( k \geq 4 \). Fix \( \zeta = (\zeta, \zeta') \in M' \subset \mathbb{R}_{\geq} \times B_{l}(0) \) for sufficiently large \( R \). In this section, we prove various uniform estimates for the linearized operator \( L_{\zeta} \) of \( \tilde{W}(\cdot, \zeta) \) at \( \Phi(\zeta) \). For more details, see Ref. 5, Secs. 3 and 7.

Recall that \( P : H^{k} \to H^{k} \) is the \( L^{2} \)-orthogonal projection onto the span consisting of the constants and the three spherical coordinates.

\[
Q = Q_{\mathbb{R}} : H^{k} \to H^{k} : \text{(the (not necessarily orthogonal) projection onto the tangent space at } \Phi(\zeta), \zeta' \text{ to}} E^{N} = \{ \Phi^{N} : \Phi \in E \} \subset H^{k}, \text{where the manifold } E \text{ is given in Definition 4.}
\]
Lemma 3 (approximate zero modes). For every \( \phi \in H^k \), the following holds:
\[
|L_\xi Q\phi|_{H^{k-1}} \leq O(R^{-2}) \|\phi\|_{H^k}.
\] (D1)

\textbf{Proof.} Recall that the functions \( f_a \) are calculated in (3.25) and (3.26) and that the projection \( Q \) maps onto \( \text{span}\{f_a\} \). Using these facts, we have shown \( |Q - P|_{H^k} \lesssim R^{-2} \) as in (4.3).

Denote by \( L^0 \) the linearized operator at the coordinate sphere \( S_c \) with the Euclidean background metric and by \( L^R \) the same with the metric \( g \) on \( M \). These two operators satisfy
\[
\|L^0_\xi \phi\|_{H^{k-1}} = \|L^R_\xi \phi\|_{H^{k-1}} + O(R^{-2}) \|\phi\|_{H^k}.
\] (D2)

See a discussion about this in Remark 6.

By the definition of \( P \) in Definition 3, \( L^0 \) vanishes on \( \text{ran} P \). It follows that
\[
\|L^0_\xi Q\phi\|_{H^k} \leq \|L^0_\xi\|_{H^{k-1}, H^{k-1}}(|P - Q|\phi)_{H^k} = O(R^{-2}) \|\phi\|_{H^k}.
\] (D3)

Since the functional \( W \) is \( C^2 \) on \( X^k \), we have by (3.10) that
\[
\|L_\xi - L^0_\xi\|_{H^{k-1}, H^{k-1}} \leq \|\xi^a \partial_\alpha \Phi\|_{H^k} = O(R^{-2}).
\] (D4)

The claim follows by combining (D2) and (D3) with (D4).

\[\square\]

Lemma 4 (coercivity). Let \( k \geq 4, R \gg 1 \). There exist \( \alpha, \beta > 0 \) depending on \( k, R \), and the background metric \( g \) only such that
\[
\alpha \|\xi\|^2_{H^k} \leq \langle \xi, L_\xi \xi \rangle \leq \beta \|\xi\|^2_{H^k} \quad (\xi \in \text{ker} Q).
\] (D5)

\textbf{Proof.} The upper bound follows from the fact that \( \tilde{W} \) is \( C^2 \) and the relation \( L_\xi = \partial_\xi \tilde{W}(\Phi(\xi), \zeta) \). Thus, it suffices to find the lower bound. In Ref. 5, Theorem 10, it is shown that
\[
\langle \xi, L_\xi \xi \rangle \geq \alpha \|\xi\|^2_{L^2} \quad (\xi \in \text{ker} P),
\] (D6)

where \( \alpha \) depends only on the lower bound \( R \) and the ambient metric \( g \). The claim now is
\[
\langle \xi, L_\xi \xi \rangle \geq \frac{\alpha}{4} \|\xi\|^2_{L^2} \quad (\xi \in \text{ker} Q),
\] (D7)

provided that \( R \) is sufficiently large. Note that here we have one caveat, namely, that \( Q \) is not necessarily orthogonal.

Put \( Q = 1 - Q \). Then, we can rewrite (D7) as
\[
\langle L_\xi \xi, \xi \rangle = \langle L_\xi Q\xi, \xi \rangle + \langle L_\xi Q\xi, \xi \rangle.
\] (D8)

The first term is \( O(R^{-2})\|\xi\|^2_{H^k} \) by the approximate zero mode property (D1). The second term further splits as \( \langle L_\xi Q\xi, Q\xi \rangle + \langle L_\xi Q\xi, Q\xi \rangle = \langle L_\xi Q\xi, Q\xi \rangle + \langle L_\xi Q\xi, Q\xi \rangle + O(R^{-2})\|\xi\|^2_{H^k} \), again by (D1). Thus, it suffices to find a lower bound of the quadratic form \( \langle L_\xi Q\xi, Q\xi \rangle \geq (\alpha/2)\|\xi\|^2_{H^k} \).

Consider the operator \( QLQ \). By the uniform closeness (4.3), we have \( QLQ = P \phi + O_{H^{k-1}, H^{k-1}}(R^{-2}) \). Thus, and \( k \geq 4 \) imply that for sufficiently large \( R \), we have \( \|QLQ\|_{H^{k-1}, H^{k-1}} \geq 4/2 \) and, therefore, (D7).

Thus, (D5) follows once we can find a uniform estimate \( \|\xi\|_{H^k} \lesssim \|\xi\|_{L^2} \). This can be done using the fact that \( \xi \) solves (5.21), which we can be rewritten as a fourth order elliptic equation as follows:
\[
L_\xi \xi = -\tilde{W}(\Phi(\xi)) + N_\xi(\xi) + O(R^{-2}).
\]

By the regularity theory for an elliptic operator of higher-order, this implies that there is some \( C > 0 \) depending on \( R \), the background metric \( g \), and the Sobolev order \( k \) only such that \( \|\xi\|_{H^k} \leq C\|\xi\|_{L^2} \).
This means that if $y_0$ is another admissible surface that is $H^k$-close to $v_{\zeta}$ in the topology given in Definition 3, then for every $\epsilon > 0$, there exists $T > 0$ such that $\|v_t - y_t\|_{H^k} < \epsilon$ for all $t \geq T$, where $v_t, y_t$ are, respectively, the flows generated by $v_{\zeta}, y_0$ under (1.1).