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Adiabatic theory for the area-constrained Willmore flow

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Abstract

In this paper, we develop an adiabatic theory for the evolution of large closed surfaces under the area-constrained Willmore (ACW) flow in a three-dimensional asymptotically Schwarzschild manifold. We construct explicitly a map, defined on a certain four-dimensional manifold of barycenters, which characterizes key static and dynamical properties of the ACW flow. In particular, using this map, we find an explicit four-dimensional effective dynamics of barycenters, which serves as a uniform asymptotic approximation for the (infinite-dimensional) ACW flow, so long as the initial surface satisfies certain mild geometric constraints (which determine the validity interval). Conversely, given any prescribed flow of barycenters evolving according to this effective dynamics, we construct a family of surfaces evolving by the ACW flow, whose barycenters are uniformly close to the prescribed ones on a large time interval (whose size depends on the geometric constraints of initial configurations).

1 Introduction

Let \((M, g)\) be a 3-dimensional, complete, oriented Riemannian manifold with non-negative curvature. Consider the area-constrained Willmore (ACW) flow,

\[
\partial_t x^N = -W(x) - \lambda H(x).
\] (1.1)

Here, for \(t \geq 0, x = x_t : S \to M\) is a family of embeddings of spheres (with orientation compatible with that on \(M\)). \(\partial_t x^N\) denotes the normal velocity at \(x\), given by \(\partial_t x^N := g(\partial_t x, \nu)\), where \(\nu = \nu(x)\) is the unit normal vector to \(\Sigma\) at \(x\). \(H(x)\) denotes the mean curvature scalar at \(x\), and

\[
W(x) := \Delta H(x) + H(x)(\text{Ric}_M(\nu, \nu) + |\hat{A}|^2(x))
\]
is the Willmore operator, where $\hat{A}(x)$ denotes the traceless part of the second fundamental form. The number $\lambda = \lambda(t)$ is given by

$$
\lambda = \frac{1}{|\Sigma| H^2} \int_{\Sigma} \left( |\nabla H|^2 - H^2 \text{Ric}_M(\nu, \nu) - H^2 |\hat{A}|^2 \right) d\mu,
$$

where $\mu = \mu^g_{\Sigma}$ is the canonical measure on $\Sigma$, induced by the embedding $x$ and background metric $g$. This choice of $\lambda$ ensures that (1.1) is area-preserving, see Appendix B.

Mathematically, the ACW flow (1.1) can be viewed as the $L^2$-gradient flow of the Willmore energy

$$
W(\Sigma) = \frac{1}{4} \int_{\Sigma} H^2 d\mu,
$$

which is a well-defined $C^2$-functional in a suitable configuration space. Denote by $dW(x)$ the Fréchet derivative of $W$. Define the normal $L^2$-gradient $\nabla_N W(x) := dW(x)$ for every normal, area-preserving variation $\phi$ on the surface $\Sigma = x(\Sigma)$. Then $\nabla_N W(x)$ is given by the r.h.s. of (1.1). Hence, for a flow of immersions $x = x_t$, $t \geq 0$ in the configuration space of $W$, we can rewrite (1.1) as

$$
\partial_t x^N = \nabla_N W(x).
$$

See Section 2.2 for more details.

Physically, the ACW flow (1.1) is the gradient flow of the Hawking mass of embedded 2-spheres, defined in (2.9) below and first introduced in [19]. In mathematical biology, building upon Helfrich’s seminal work [20], the flow (1.1) and its variants are used to model the motion of cell membranes. More recently, (1.1) finds applications to nonlinear elasticity [13,14], among various other fields in applied science.

1.1 Main results

In this paper, we develop a rigorous adiabatic (or slow-motion approximation) theory for (1.1). To state our main result, denote by $H^k$ the Sobolev space of order $k$. Let

$$
X^k := H^k(S, M), \quad X^k_c := \{x \in X^k : |x(\Sigma)| = c\}.
$$

Here and below, for a surface $\Sigma := x(\Sigma) \subset M$, denote by

$$
|\Sigma| := \int_{\Sigma} d\mu
$$

the area of $\Sigma$. Following the nomenclature in [23], we call (the images of) static solutions to (1.1) surfaces of Willmore type.

Then we prove the following:

**Theorem 1 (Main).** Let $k \geq 4$, $c \gg 1$. Fix $R \gg 1$, $\delta \ll 1$ in Definition 1 of admissible surfaces.

Then there exists a map

$$
\Psi : M' := \mathbb{R}_{>R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3 \rightarrow X^k
$$

with the following property: Denote by $\zeta \equiv (r, z) \in M'$ and $\nu_\zeta \equiv \Psi(\zeta) \in X^k$. 

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1. (Critical point) The immersion \( v_\zeta \) parametrizes a surface of Willmore type if and only if \( \zeta \) is a critical point of the function \( W \circ \Psi : \mathbb{R}^4 \to \mathbb{R} \), restricted to the submanifold \( \{ \zeta \in M' : |v_\zeta(S)| = c \} \).

2. (Stability) Suppose \( v_\zeta \) parametrize an admissible surface of Willmore type. Then \( v_\zeta \) is uniformly stable with small area-preserving \( H^k \)-perturbation if \( \zeta \) is a strict local minimum of the function \( W \circ \Psi \) restricted to the submanifold \( \{ \zeta \in M' : |v_\zeta(S)| = c \} \).

3. (Effective dynamics) Let \( \Sigma^* = \Phi_\ast(S) \) be an admissible surface. Let \( \Sigma_t = \Phi_t(S), t \geq 0 \) be the global solution to (1.1) with initial configuration \( \Sigma|_{t=0} = \Sigma^* \) as in Theorem 3.

Then there exist \( \alpha > 0, T = O(R^{-\alpha}) \), and a path \( \zeta_t \in M', t \geq 0 \), such that for every \( t \geq T \), there holds

\[
\|v_{\zeta_t} - x_t\|_{X_k} = O(R^{-3}).
\]

Moreover, the path \( \zeta_t \equiv (r_t, z_t) \) evolves according to

\[
\begin{align*}
\dot{z} &= \frac{1}{4\pi} \nabla_z W \circ \Psi(r, z) + O(R^{-3}), \\
\dot{r} &= 4R^{-2} + O(R^{-3}).
\end{align*}
\]

In (1.3) the leading term is of the order \( O(R^{-2}) \).

4. Conversely, if \( \zeta_t \in M' \) is a flow evolving according to (1.3)-(1.4), then there exists a global solution \( x_t \) to (1.1) such that (1.2) holds for this choice of \( \zeta_t \) and every \( T \leq t \leq T + R \).

Remark 1. The map \( \Psi \) in Theorem 1 is defined by \( \Psi(r, z) = \theta(\Phi(r, z), r, z) \), where \( \theta, \Phi \) are given respectively in (3.2) and Definition 4. Following [8], we call \( \Phi \) (or equivalently \( \Psi \)) the Lyapunov-Schmidt map.

Remark 2. The content of Theorem 1 says that the static and dynamical properties of (2.4) are captured by the effective action \( W \circ \Psi \), on the 3-manifold given by the level set \( \{ \zeta \in M' : |v_\zeta(S)| = c \} \).

Remark 3. (1.2)-(1.4) constitute the crucial part of the theorem, which says that the infinite dimensional dynamical system (1.1) reduces uniformly in time to the finite system of ODEs, (1.3)-(1.4).

On the analytical ground, roughly speaking, Theorem 1 shows that the pull-back by the Lyapunov-Schmidt map \( \Psi \) to \( M' \) essentially reduces key static and dynamical properties of the infinite dimensional dynamical system (1.1) to finite dimensional ones.

For applications to various physical models, our results in this paper provide an explicit 4-dimensional dynamical system (1.3), whose behavior approximates that of (1.1) uniformly for all sufficiently large time, and captures key qualitative behaviours of (1.1). This can potentially reduce computational complexity for the complicated forth order PDE (1.1).

1 This means that if \( y_0 \) is another admissible surface that is \( H^k \)-close to \( v_\zeta \) in the topology given in Definition 3 then for every \( \epsilon > 0 \) there exists \( T > 0 \) such that \( \|v_t - y_t\|_{X_k} < \epsilon \) for all \( t \geq T \), where \( v_t, y_t \) are respectively the flows generated by \( v_\zeta, y_0 \) under (1.1).
1.2 Historical remarks

The problem we study here is motivated by a recent work [11] by M. Eichmair and T. Koerber, in which the authors study stationary solutions to (1.1) using Lyapunov-Schmidt reduction. Here we derive some dynamical analogues of the static existence results in [11], with, however, rather different focus. Indeed, the main point of our results is that we have 1. explicit information about the adiabatic parts of a flow of surfaces evolving according to (1.1), with 2. uniformly small errors in time, and 3. we can construct solution (1.1) with prescribe adiabatic behavior. See the precise statements of these points in Theorem 1.

The method of adiabatic approximation has a long history in classical field theory. See [26] for an excellent review in this context. Our work here is inspired by a series of papers by I.M. Sigal with several co-authors, in which the adiabatic theory is adapted to geometric problems [8, 17, 18, 25]. Other results along this line which we have referred to include [7, 9, 10, 15, 16, 21], which cover a range of static and dynamical problems of geometric equations using Lyapunov-Schmidt reduction.

Among the papers above, we single out a recent paper [8], in which the authors study a formally similar problem (namely, the volume-preserving mean curvature flow with initial configurations close to small geodesics spheres), from which we draw much inspiration. In particular, it appears that the notion of Lyapunov-Schmidt map is first mentioned in this paper.

It seems to us that our results is the first rigorous adiabatic theory for the area-constrained Willmore flow. We expect these results to be robust, in the sense that they can be easily adapted to problems related to (1.1), for instance, using the generalized Willmore energy developed in [12] (which covers, among others, the applications to biomembranes). In a separate paper, we will treat the abstract properties of the Lyapunov-Schmidt map defined in Appendix A.

2 Setup of the problem

2.1 Asymptotically Schwarzschild manifolds

A 3-dimensional complete Riemannian manifold \((M, g)\) is said to be \(C^k\)-close to Schwarzschild if the following holds:

1. \(M \setminus K\) is diffeomorphic to \(\mathbb{R}^3 \setminus B_1(0)\) for some compact subset \(K \subset M\).

2. The metric \(g\) splits as \(g_S + h\), where

\[
gs := \left(1 + \frac{m}{2|x|}\right)^4 \delta_{ij}
\]

is the Schwarzschild metric with ADM mass \(m > 0\), and \(h \in C^k\) is a small perturbation satisfying

\[
h_{ij} = h_{ji}, \quad \partial^\alpha h_{ij} \leq \eta |x|^{-(2+|\alpha|)} \quad (|\alpha| \leq k, \ |x| \gg 1), \tag{2.1}
\]

for some fixed small decay coefficient \(\eta \ll 1\). Here \(x \in \mathbb{R}^3\) denotes the coordinate in the asymptotic chart on \(M\).
Physically, for applications to general relativity, such manifold $M$ is a perturbation of the static Schwarzschild black hole $(\mathbb{R}^3 \setminus B_{m/2}(0), g_S)$.

To simplify notations, throughout the paper we normalize ADM mass to be $m = 2$. We assume that the ambient space $M$ is $C^k$-close to Schwarzschild for sufficiently large $k$, and that in (2.1), the decay coefficient $\eta \ll 1$. Thus in what follows we take 

$$(M, g) = (\mathbb{R}^3 \setminus B_1(0), g_S + h)$$

where $h$ is as in (2.1).

To use results in [11, 22, 23], we assume the scalar curvature $S_c$ on $M$ satisfies the following decay properties:

$$x^j \partial_{x^j} \left( |x|^2 S_c \right) = o(|x|^2), \quad (2.2)$$

$$S_c(x) - S_c(-x) = o(|x|^4). \quad (2.3)$$

The asymptotic flatness condition (2.2) is satisfied if $g$ is $C^k$-close to Schwarzschild with $k \geq 4$, and $S_c = o(|x|^4)$, in which case $\nabla S_c = o(|x|^{-5})$. (2.3) means the scalar curvature on $M$ is asymptotically even. Geometrically, condition (2.2) provides quantitative control for various estimates involving extrinsic geometric quantities. Condition (2.3) provides qualitative control of the effective action in Secs. 4, 5.

### 2.2 The geometric structure of (1.1)

In this subsection, we lay out the geometric structure of ACW flow (1.1). This structure is understood in the subsequent developments in Secs. 4, 5.

Let $c \gg 1, k \geq 4$ be given. Recall that in Section 1.1, we have defined the configuration spaces

$$X^k := H^k(S, M), \quad X^k_c := \{x \in X^k : |x(S)| = c\}, \quad (2.4)$$

where $|\Sigma| := \int_{\Sigma} d\mathcal{H}^2$ denotes the area of $\Sigma$ w.r.t. the embedding $x$ and background metric $g$. One can check easily that (1.1) is well-defined in $X^k_c$. The spaces in (2.4) are equipped with the $L^2$-inner product

$$\langle \phi, \phi' \rangle := \int_{\Sigma} \langle \phi, \phi' \rangle_{\text{Euclidean}} \quad (\phi, \phi' \in X^k). \quad (2.5)$$

Let $x \in X^k$ and write $\Sigma = x(S)$. The tangent spaces to $x$ at $X^k$ and $X^k_c$ are respectively given by

$$T_x X^k = X^k, \quad (2.6)$$

$$T_x X^k_c = \left\{ \phi \in T_x X^k : \int_{\Sigma} Hg(\phi, \nu) = 0 \right\}. \quad (2.7)$$

Here, (2.7) is due to the well-known first variation formula of the area functional. Notice that, slightly abusing notation, in (2.7) we view $\phi$ as a vector field over $\Sigma$. With (2.5), we have a formal Riemannian structure on the configuration spaces $X^k$ and $X^k_c$. 5
With this geometric structure of $X^k$, one can view the equation (1.1) as the $L^2$-gradient flow, restricted to $X^k_c$, of the Willmore energy

$$W(\Sigma) = \frac{1}{4} \int_\Sigma H^2 \, d\mu_\Sigma.$$  \hspace{1cm} (2.8)

Using Sobolev inequalities, one can show that for $k \geq 4$, the functional $W$ is well-defined and $C^2$ (in the sense of Fréchet derivatives) on $X^k_c$.

Let $dW(x) : T_x X^k_c \to T_x X^k_{c-4}$ be the Fréchet derivative of $W$ at an embedding $x$ in the class $X^k_c$. Define the normal $L^2$-gradient $\nabla N W(x) := dW(x)\phi$ for every normal, area-preserving variation $\phi$ on the surface $\Sigma = x(S)$. (This operator $\nabla N$ depends on $x$.) Then by the first variation formula of the Willmore energy (see e.g. [1, Sec. 3]), this $\nabla N W(x)$ is given by the r.h.s. of (1.1). This allows us to rewrite (1.1) as

$$\partial_t x^N = \nabla N W(x) \quad (x \in X^k_c).$$

Equivalently, (1.1) is the (negative) $L^2$-gradient flow of the Hawking mass,

$$m_{\text{Haw}}(\Sigma) := \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}} \left( 16\pi - \frac{1}{2} \int_\Sigma H^2 \, d\mu_\Sigma \right),$$  \hspace{1cm} (2.9)

in the sense that a flow of surfaces evolving according to (1.1) increases the mass $m_{\text{Haw}}$. For interests from physics related to this problem, especially in general relativity, see [24].

### 2.3 Preliminary results

Let $R \gg 1$ be given. Let $K \subset M$ be a fixed compact set as in Section 2.1. As explained in the last subsection, for asymptotically Schwarzschild manifold $M$, we can identify the $M \setminus K$ with its coordinate space $\mathbb{R}^3 \setminus B_R(0)$.

Let $\delta > 0$ be given.

**Definition 1** (Admissible surfaces). For a closed surface $\Sigma \subset M$, define the inner and outer radii $\rho(\Sigma)$, $\lambda(\Sigma)$ as

$$\rho(\Sigma) = \min_{x \in \Sigma} |x|,$$  \hspace{1cm} (2.10)

$$\lambda(\Sigma) := \sqrt{|\Sigma|/4\pi}.$$  \hspace{1cm} (2.11)

We say $\Sigma$ is admissible if the interior of $\Sigma$ contains the fixed compact set $K$, and

$$\rho(\Sigma) > R, \quad \left| \frac{\rho(\Sigma)}{\lambda(\Sigma)} - 1 \right| + \int_\Sigma |\tilde{A}|^2 < \delta.$$  \hspace{1cm} (2.12)

Here, recall, $\tilde{A}$ denotes the traceless part of the second fundamental form on $\Sigma$.

**Remark 4.** Geometrically, a surface $\Sigma$ is admissible if the origin lies sufficiently deep inside the interior of $\Sigma$ (this property is called *centering* in [11]), and at the same time the surface does not wiggle too much. It follows from the definition (2.11) that $\lambda(\Sigma) \leq \max_{x \in \Sigma} |x|$. Using the terminology in [11], every admissible surface $\Sigma$ satisfying (2.12) with $R, \delta^{-1} \gg 1$ is *on-center*. 

6
For the class of admissible surfaces, we have the following well-posedness result for (1.1):

**Theorem 2** ([22, Thm. 5.3]). Assume $M$ is $C^4$-close to Schwarzschild and satisfies (2.2)-(2.3). Then for $R \gg 1$, $\delta \ll 1$ and every admissible surface $\Sigma$, $\Sigma|_{t=0} = \Sigma_*$, there exists a global solution to (1.1) with initial configuration $\Sigma|_{t=0} = \Sigma_*$.

Recall that Stationary solutions to (1.1) are called surfaces of Willmore type. The existence and stability of such surfaces are studied in [11, 22].

**Theorem 3** ([23, Thm. 1], [22, Thm. 5.3]). Assume $M$ is $C^4$-close to Schwarzschild and satisfies (2.2)-(2.3). Then there exists a compact subset $K \subset M$ such that $M \setminus K$ is foliated by surfaces of Willmore type. Moreover, for $R \gg 1$, $\delta \ll 1$ and every admissible surface $\Sigma$, satisfying (2.12), the flow generated by $\Sigma$, under (1.1) converges smoothly to one of the leaves of this foliation.

### 2.4 Organization of the paper

We organize this paper as follows: In Section 2, we define the important map $\Phi$, which arises by reconceptualizing the Lyapunov-Schmidt reduction. This allows us to identify the adiabatic part of a flow evolving according to (1.1). This adiabatic part accounts for most of the (Willmore) energy change along the flow (modulo some uniformly small fluctuation), and is finite-dimensional. In Section 3, we discuss the static property of the effective action $W \circ \Psi$, and prove the first part of Theorem 1. A similar but different function defined on a domain in $\mathbb{R}^3$ is used in [11] (denoted by $G$ in that paper). In Section 4, we prove the remaining part of Theorem 1 by deriving the effective dynamics (1.3) of (1.1). Here we exploit the spectral property of certain linearized operators, in order to bound a Lyapunov-type functional that controls fluctuations.

**Notation**

Throughout the paper, the notation $A \lesssim B$ means that there is a constant $C > 0$ depends only on $c, k$ in (2.4) and $R, \delta$ in (2.12), such that $A \leq CB$. For two vectors $A, B$ in Banach spaces $X, Y$, the notation $A = O_Y(B)$ means $\|A\|_X \lesssim \|B\|_Y$. For a vector $A$ in some Sobolev space $H^k$, the notation $A = O_Y(B)$ means $\|A\|_{H^k} = O_Y(B)$.

### 3 The Lyapunov-Schmidt map

Let $k \geq 4, c \gg 1$. Let $K \subset M$, $R \gg 0$ to be determined, and let $M' := \mathbb{R}_{\geq R} \times B_1(0) \subset \mathbb{R} \times \mathbb{R}^3$. In this section we construct the map $\Psi : M' \to X^k$ as in Theorem 1.

#### 3.1 Graphs over sphere

Denote $H^k = H^k(S, \mathbb{R})$. This space is equipped with the $L^2$-inner product $\langle u, v \rangle = \int_S uv$. Define the configuration space

$$Y^k := H^k \times M'.$$  \hspace{1cm} (3.1)
Define a map
\[ \theta : (Y^k, r, z) \rightarrow (X^k, r(1 + \phi(v))v + z) \] (3.2)

Here \( v \in S \subset \mathbb{R}^3 \) is the spherical coordinate, and recall we identify the asymptotic part \((M \setminus K) \cong (\mathbb{R}^3 \setminus B_R(0))\). Define
\[ Y_c^k := \{(\phi, r, z) \in Y^k : \theta(\phi, r, z) = c\} \] (3.3)

This corresponds to the space of surfaces with fixed area, \( X_c^k \), as in [23].

For \( ||\phi||_{H^k} \ll 1 \), the map \( \theta(\phi, r, z) \) is a well-defined graph over the coordinate sphere \( \theta(0, r, z)(S) =: S_{r,z} \). Thus we can also identify \( \theta(\phi, r, z) \) as a function from \( S_{r,z} \subset M \rightarrow \mathbb{R} \). Note also that for sufficiently large \( c \gg 1 \) and every \( z \in B_1(0) \subset \mathbb{R}^3 \), there is a coordinate sphere with area \( c \) around \( z \). Thus the map \( \theta \) is surjective onto \( X_c^k \).

**Definition 2** (topology on graphs). We say two graphs \( \theta(\phi, r, z), \theta(\phi', r', z') \) are \( H^k \)-close if \( ||\phi - \phi'||_{H^k} + |r - r'| + |z - z'| \ll 1 \).

### 3.2 Lyapunov-Schmidt reduction

Denote \( \bar{W}(\phi, r, z), \Omega(\phi, r, z) \) the pullbacks of the r.h.s. of (1.1) and the Willmore energy (2.3) to \( Y^k \) through \( \theta \), respectively. Explicitly, we have
\[ \bar{W}(\phi, r, z) := -W(\theta(\phi, r, z)) - \lambda H(\theta(\phi, r, z)), \] (3.4)
\[ \Omega(\phi, r, z) := \mathcal{W}(\theta(\phi, r, z)). \] (3.5)

Since \( W \) is \( C^2 \) on \( X^k \) with \( k \geq 4 \) and \( \theta \) is smooth, the pullback energy \( \Omega \) is \( C^2 \) on \( Y^k \), \( k \geq 4 \). Using Sobolev inequalities, one can check that the partial Fréchet derivative \( \bar{W} \) is \( C^1 \) in \( \phi \) and smooth in \( r, z \). This \( \bar{W} \) is the \( L^2 \)-gradient of \( \Omega(\cdot, r, z) \) up to scaling, and satisfies the mapping property \( \bar{W} : Y^k \rightarrow H^{k-4} \).

**Remark 5.** Notice that (3.4)–(3.5) both implicitly depend on the background metric \( g \).

**Lemma 1.** The linearized operator \( L_{r,z} \) of \( \bar{W} \) at \((0, r, z)\) with background metric \( g \) is given by
\[ L_{r,z}^r := \partial_r \bar{W}(\phi, r, z)|_{\phi=0} \]
\[ = (\Delta^2 + 2r^{-2}\Delta + O(r^{-4}))\partial_r \theta(0, r, z) : H^k \rightarrow H^{k-4}. \] (3.6)

Here \( \Delta : X^k \rightarrow X^{k-2} \) denotes the Laplace-Beltrami operator on the coordinate sphere \( S_{r,z} \subset M \setminus K \), with center \( z \) and radius \( r \). The partial Fréchet derivative \( \partial_r \theta(0, r, z) : H^k \rightarrow X^k \) is given by \( \xi(v) \mapsto \xi(v)rv \).

Moreover, the operator \( L_{r,z} \) is self-adjoint on \( H^k \). The spectrum of \( L_{r,z} \) is purely discrete. The operator \( \partial_r \theta(0, r, z) \) is invertible and satisfies
\[ ||\partial_r \theta(0, r, z)||_{H^k \rightarrow X^k} = ||\partial_r \theta(0, r, z)^{-1}||_{X^k \rightarrow H^k} = r. \] (3.7)

**Proof.** The operator \( L_{r,z}^r \) is explicitly calculated in [23, Sec. 3]. The spectral properties of \( L_{r,z} \) are studied in [23, Sec. 7]. The mapping properties of \( \partial_r \theta \) is obvious. \( \square \)
Remark 6. The linearized operator (3.6) depends on (the curvature of) the background metric \( g \) on \( M \). In the special case when the ambient manifold \( M \) is flat, i.e. \( g = \delta_{ij} \), the linearized operator \( L^0_{r,z} \) has eigenvalue 0, and \( \ker L^0_{r,z} \) is spanned by the constant function \( y^0 \equiv 1 \), together with the spherical harmonics \( y^1, y^2, y^3 \). Thus, so long as \((M, g)\) is asymptotically flat and \( r \gg 1 \) in (3.6) (such as in our setting), one can view \( L^0_{r,z} \) as a perturbation of \( L^0_{r,z} \). This motivates the following definition.

Definition 3. Define \( P : H^k \to H^k \) to be the \( L^2 \)-orthogonal projection onto \( \text{span} \{ y^0, ..., y^4 \} = \ker L^0_{r,z} \). Define \( \bar{P} : H^k \to H^k \) be the complement of \( P \).

Let \( \mathcal{S} \) be the set of all smooth symmetric two tensors on \( M \). Define a map

\[
F : Y^k \times \mathcal{S} \to H^{k-4}, \quad (\phi, r, z, h) \mapsto \bar{P} \bar{W}(\phi, r, z),
\]

where \( \bar{W} \) is computed with background metric \( g = g_S + h \) (see Section 2.1).

Proposition 1. Assume the ambient manifold \((M, g)\) is \( C^k \)-closed to Schwarzschild.

1. For every \( z \in \mathbb{R}^3 \) with \( |z| < 1 \) and sufficiently large \( r \geq R \gg 1 \), there is a unique solution \( \phi = \phi_{r,z} \in \bar{PH}^k \) to the equation

\[
F(\phi, r, z, h) = 0,
\]

where \( F \) is defined in (3.9), and \( g = g_S + h \).

2. Moreover, the map \((r, z) \mapsto \phi_{r,z} \) is \( C^2 \), and satisfies the estimate

\[
\| \partial_r^m \partial_z^\alpha \phi_{r,z} \|_{H^k} \lesssim r^{-(2+2m)}
\]

for every \( m + |\alpha| \leq 2 \).

3. Moreover, the surface \( \theta(\phi_{r,z}, r, z) \) lies in the class of admissible surfaces in Definition 4.

Proof. 1. By the Implicit Function Theorem, it suffices to check that the map \( F \) defined in (3.9) satisfies the following properties:

1. \( F \) is \( C^1 \) in \( \phi \).
2. \( F(0, r, z, 0) = 0 \) for every \( r, z \).
3. \( \partial_\phi F(0, r, z, 0) = L^0_{r,z} \) is invertible on \( \bar{PH}^k \).

The first claim follows from the regularity of \( \bar{W} \) on \( Y^k \) and its smooth dependence on the background metric.

If the background metric is Schwarzschild, i.e. \( h = 0 \), then it is well-known that by conformal invariance the coordinate sphere \( \theta(0, r, z) \) is the global minimizer of the Willmore energy \( W \). Since \( \bar{W} = \partial_\phi \Omega \) (see (3.5)), the second claim follows.

The spectrum of \( L^0_{r,z} \) can be calculated explicitly. See for instance [11, Cor. 33]. In particular, 0 is an isolated eigenvalue with finite multiplicity.
By elementary spectral theory, this implies the restriction $L_{r,z}^0 := L_{r,z}^0|_{\bar{\mathcal{P}}}$ is invertible as a map from $PH^k \to PH^k$. Thus the third claim follows.

2. For the estimate (3.10), we expand

$$L_{r,z}^g = L_{r,z}^0 + V_{r,z},$$

where $V_{r,z}$ is defined by this expression. As we discuss in Remark [3] this $V_{r,z}$ is bounded from $H^k \to H^{k-4}$, and satisfies $\|V_{r,z}\|_{H^k \to H^{k-4}} = O(r^{-4})$. The restriction $L_{r,z}^0$ can be bounded from below by $Cr^{-2}$ for some $C > 0$ only depending on $k$. It follows that

$$\|(L_{r,z}^0)^{-1}V_{r,z}\|_{H^{k-4} \to H^k} = O(r^{-2}).$$

For sufficiently large $r$, this together with the expansion (3.11) implies that the restriction $L_{r,z}^g : PH^k \to PH^k$ is invertible, given explicitly as the Neumann series

$$(L_{r,z}^g)^{-1} = \sum_{n=0}^{\infty} (L_{r,z}^0)^{-1}(-V_{r,z}(L_{r,z}^0)^{-1})^n.$$ 

From here one can also read off the estimate

$$\|(L_{r,z}^g)^{-1}\|_{PH^k \to PH^k} = O(r^2).$$ (3.12)

Expand $F(\phi, r, z, h) = F(0, r, z, h) + L_{r,z}^g \phi + N_{r,z}(\phi)$, where the nonlinearity $N_{r,z}$ is defined by this expression. This $N_{r,z}$ is calculated explicitly in (C.4).

For every $\phi$ satisfying (3.9), we can rearrange to get

$$\phi = -(L_{r,z}^g)^{-1}(F(0, r, z, h) + N_{r,z}(\phi)).$$ (3.13)

In the r.h.s. we have $F(0, r, z, h) = O(r^{-4})$ by [11, Cor. 45]. Thus, for sufficiently small $\phi$, we have by (3.12)-(3.13) that $\|\phi\|_{H^k} = O(r^{-2})$.

We now claim for $\phi \in H^k$ and $m + |\alpha| \leq 2$, there hold

$$\|\partial_\nu^m \partial_{\zeta}^\alpha (L_{r,z}^g)^{-1}\phi\|_{H^{k-4}} \lesssim \|\phi\|_{H^{k-4}},$$ (3.14)

$$\|\partial_\nu^m \partial_{\zeta}^\alpha F(0, r, z, h)\|_{H^{k-4}} \lesssim r^{-4+|m|},$$ (3.15)

$$\|\partial_\nu^m \partial_{\zeta}^\alpha N_{r,z}(\phi)\|_{H^{k-4}} \lesssim \|\phi\|_{H^{k}}.$$ (3.16)

For (3.14), one uses the identity $\partial^\beta (L_{r,z}^g)^{-1} = -(L_{r,z}^g)^{-1}\partial^\beta L_{r,z}^g (L_{r,z}^g)^{-1}$, where $|\beta| \leq 2$ is a multi-index in both $r$ and $z$. This, together with the fact that $\partial^\beta L_{r,z}$ is uniformly bounded (see (C.5)), implies (3.14). The rest follows from the expansion in Proposition 4. Using (3.14)-(3.16) and differentiating both sides of (3.10), we conclude the estimates (3.10).

3. For sufficiently large $R$ and every $r \geq R$, we find using (3.10) with $m = 0, \alpha = 0$ that the surface $\theta(\phi_{r,z}, r, z)$ is $H^k$-close to the coordinate sphere $S_{r,z}$. This implies $\theta(\phi_{r,z}, r, z)$ is an admissible surface. 

From now on we write $\zeta = \zeta^\alpha, \alpha = 0, \ldots, 4$, for a point in $(r, z) \in M'$. Thus, $\zeta^0 = r$ and $\zeta^j = z^j$ for $j = 1, 2, 3$. 

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Definition 4 (The Lyapunov-Schmidt map $\Phi$). Let $K \subset M$ be the compact set as in Theorem 3. Let $R \gg 1$, $\delta \ll 1$ be given as in Theorem 2. Let $M' := R \times R \times B_1(0) \subset R \times R^3$.

Define the Lyapunov-Schmidt map $\Phi : M' \to H^k$ by $\zeta \mapsto \phi_\zeta$, where $\phi_\zeta$ is the solution to (3.9) given in Proposition 1.

Remark 7. This $\Phi$ is equivalent to the map $\Psi$ in Theorem 1, through the diffeomorphism $\Phi(\zeta) \mapsto \theta(\Phi(\zeta), \zeta)$.

In the next proposition, we describe the geometric structure induced by the map $\Phi$.

Proposition 2. The set $E := \{ \theta(\phi, \zeta) : \phi = \Phi(\zeta), \zeta \in M' \}$ forms an immersed $C^1$ submanifold in $X_k$. The tangent space $T_{\theta(\Phi(\zeta), \zeta)}E$ consists of vector fields over the surface $\theta(\Phi(\zeta), \zeta)(S)$. A basis of $T_{\theta(\Phi(\zeta), \zeta)}E$ is given by $\partial_{\zeta^\alpha}\theta(\Phi(\zeta), \zeta)$.

Remark 8. Using the projection constructed in Lemma 2, one can view this manifold $E$ as consisting of the adiabatic parts of low (Willmore) energy surfaces in $X^k$.

Proof. The manifold structure of $E$ follows from Definition 4, where $\Phi : M' \to E$ is a $C^1$ parametrization. We check the tangent space is non-degenerate. Compute

\[
\partial_{\zeta^\alpha}\theta(\Phi(\zeta), \zeta)(v) = (1 + \Phi(\zeta) + \zeta^0\partial_{\zeta^0}\Phi(\zeta))v, \\
\partial_{\zeta^j}\theta(\Phi(\zeta), \zeta)(v) = \zeta^0\partial_{\zeta^j}\Phi(\zeta)v + e^j,
\]

where $e^j$ is the $j$-th unit vector in $\mathbb{R}^3$. By the estimate (3.10), we find

\[
\langle \partial_{\zeta^\alpha}\theta(\Phi(\zeta), \zeta), \partial_{\zeta^\beta}\theta(\Phi(\zeta), \zeta) \rangle = 4\pi\delta_{\alpha\beta} + O(R^{-2}).
\]

This implies the claim if $R$ is sufficiently large.

In Appendix, we introduce the general concepts of the Lyapunov-Schmidt map, and relate it to our setting above.

3.3 Barycenter

In this subsection, we develop a new concept of barycenter for a certain class of closed surfaces in $X^k$.

Definition 5 (Barycenter). Let $x_*$ be an embedding of sphere that is $H^k$-close to the manifold $E \subset X^k$ constructed in Definition 4 w.r.t. the topology on graphs introduced in Definition 2. Then we can write $x_* = \theta(\Phi(\zeta) + \xi, \zeta)$ for some $\zeta \in M', \|\xi\|_{H^k} \ll 1$. (There can in general be many such choice of $\zeta$ and $\xi$.) Expand $x_*$ in $\xi$ around $\theta(\Phi(\zeta), \zeta)$ as

\[
x_* = \theta(\Phi(\zeta), \zeta) + \partial_\xi \theta(\Phi(\zeta), \zeta)\xi + O(\|\xi\|_{H^k}^2).
\]
Define $f_\alpha \in H^k$ as

$$f_\alpha(\zeta) = \partial_\zeta \theta(\Phi(\zeta), \zeta^N) = g(\partial_\zeta \theta(\Phi(\zeta), \zeta), \nu(\theta(\Phi(\zeta), \zeta)) \quad (\alpha = 0, \ldots, 3).$$ (3.20)

We say a point $\zeta_* \in M'$ is the barycenter of $x_*$ if $\zeta_*$ solves the following algebraic system:

$$\langle \xi, f_\alpha \rangle_{L^2} = 0 \quad (\alpha = 0, \ldots, 3),$$ (3.21)

where $\xi$ is defined by the relation $x_* = \theta(\Phi(\zeta_*) + \xi, \zeta_*)$.

**Remark 9.** The four vectors $f_\alpha$ span the tangent space at $\theta(\Phi(\zeta_*), \zeta_*)^N$ to $E^N \subset H^k$, where $E^N$ consists of the normal components of the elements in the manifold $E$ defined in Definition 4. Geometrically, the defining condition (3.21) for barycenter means that the Gâteaux derivative of the map $\theta(\cdot, \zeta_*)^N$ at $\Phi(\zeta_*)$ along $\xi$-direction is perpendicular to the tangent space $T_{\theta(\Phi(\zeta_*), \zeta_*)^N}E^N$. In terms of the expansion (3.19), this means the second term in the r.h.s. is $L^2$-orthogonal to the tangent space at the first term to $E$. In this sense, the choice of barycenter is optimal.

**Remark 10.** Our definition of barycenter differs from the classical one, given by averaging over $\Sigma$ w.r.t. Euclidean background metric, namely $|\Sigma|^{-1} \int_\Sigma x d\mu_{\Sigma}$. See [22] and the references therein. Our version of barycenter retains the key decay property as [22, Sec. 5]. Namely, the motion of barycenter is controlled by a differential inequality using a Lyapunov functional, defined in Section 5. Moreover, our definition allows us to retain explicit and uniform control of a flow evolving according to (1.1), as we show in Sec. 5.

In the next lemma, we define a nonlinear projection (or coordinate map) that associates barycenters to low energy configurations in $X^k$.

**Lemma 2** (nonlinear projection). There exists $\delta > 0$ such that on the space $U_\delta := \{ x = \theta(\Phi(\zeta) + \xi, \zeta) : \zeta \in M', \|\xi\|_{H^k} < \delta \},$ (3.22)

there exists a $C^1$ map $S : U_\delta \to M'$ such that $S(x)$ is the barycenter of $x$ as in Definition 5.

Moreover, we have uniform estimate on $S$ and its derivative.

**Remark 11.** Essentially, the existence of such projection depends on the non-degeneracy shown in Proposition 2. Later, we see that the barycenter $\zeta = S(x)$ determines the adiabatic (or slowly-varying) part of $x$.

We call the remainder $\xi$ that satisfies $x = \theta(\Phi(S(x)) + \xi, S(x))$ the fluctuation of $x$.

**Proof.** Define a map

$$\Gamma : M' \times U_\delta \subset \mathbb{R}^4 \times H^k \longrightarrow \mathbb{R}^4 \quad (\zeta, x) \longmapsto (\xi, f_\alpha)_{L^2},$$ (3.23)

where $\xi$ is defined by the relation $x = \theta(\Phi(\zeta) + \xi, \zeta)$. It suffices to find a map $S$ such that $\zeta = S(x)$ solves

$$\Gamma(\zeta, x) = 0.$$ (3.24)

By the Implicit Function Theorem, it suffices to check that the map $\Gamma$ satisfies the following properties:
1. $\Gamma$ is $C^1$ in $\zeta$.
2. $\Gamma(\zeta, \theta(\Phi(\zeta), \zeta)) = 0$.
3. The matrix $\nabla_\zeta \Gamma(\zeta, \theta(\Phi(\zeta), \zeta)) : \mathbb{R}^4 \to \mathbb{R}^4$ is invertible.

The first claim follows from the regularity of $\Phi$ as in Proposition [1]. The second claim is trivial because in this case $\zeta = 0$.

We now claim the rescaled matrix $A_{\alpha\beta} := \zeta^0 \partial_\zeta \Gamma(\beta|\theta(\Phi(\zeta), \zeta))$ is invertible, which implies the third claim above.

Using (3.17)-(3.18), the definition (3.20), and the assumption on the background metric $g_{ij} = \delta_{ij} + O((\zeta^0)^{-4})$, we compute

\[
 f_0 = (1 + O((\zeta^0)^{-1})) y^0 + \Phi(\zeta) + \zeta^0 \partial_\zeta \Phi(\zeta) + \Phi(\zeta) O((\zeta^0)^{-1}) + O((\zeta^0)^{-4}),
\]

(3.25)

\[
 f_j = (1 + O((\zeta^0)^{-1})) y^j + \zeta^0 \partial_\zeta \Phi(\zeta) + O((\zeta^0)^{-4}),
\]

(3.26)

where the vectors $y^\alpha$ span $\mathrm{ran} P$ as in Definition [3].

For $\xi = \xi(\zeta, x)$, we find

\[
 \xi(v) = \left\langle \frac{x(v) - \zeta}{\zeta^0}, v \right\rangle - 1 - \Phi(\zeta)(v),
\]

(3.27)

\[
 \partial_\zeta \xi(v) = -\left\langle \frac{\partial_\zeta x(v) - \zeta}{(\zeta^0)^2}, v \right\rangle - \partial_\zeta \Phi(\zeta),
\]

(3.28)

\[
 \partial_\xi \xi(v) = -\frac{\zeta^0}{\zeta^0} - \partial_\zeta \Phi(\zeta)(v).
\]

(3.29)

Now, since $x \in U_\delta$, we have $A_{\alpha\beta} = \zeta^0 (\partial_\zeta \xi, f_\beta) + \zeta^0 (\xi, \partial_\zeta f_\beta) = (\partial_\zeta \xi, f_\beta) + O(\zeta^0 \delta)$. By this, and the formula (3.25)-(3.29) above, we find that $A_{\alpha\beta} = O(1) h_{\alpha\beta} + O((\zeta^0)^{-2}) + O(\zeta^0 \delta)$. For sufficiently large $R \gg 1$, $\delta = o(R^{-1})$, and every $\zeta^0 \geq R$, we can conclude from here that $A_{\alpha\beta}$ is invertible. This proves the existence of the $C^1$ map $S$. The uniform estimates for $S$ and its Fréchet derivative are implicit in the arguments above.

\[
\square
\]

4 Effective action

In this section we prove Part 1 of Theorem [1]. We formulate this as follows:

**Theorem 4.** The embedding $\theta(\Phi(\zeta), \zeta)$ parametrizes a surface of Willmore type (i.e. static solution to (1.1)) if and only if $\zeta$ is a critical point of the function $G := \Omega(\Phi(\cdot), \cdot) : \mathbb{R}^4 \to \mathbb{R}$, where $\Omega$ is defined in (3.5).

Remark 12. Similar results are obtained in [11] Thms. 5, 8]. Using some expansion obtained in that paper, we can calculate $G$ explicitly as in (C.2).
Proof. For the forward direction, we use the chain rule to get
\[ \partial_{\zeta^\alpha} G(\zeta) = \langle \nabla^N W(\theta(\Phi(\zeta), \zeta)), f_\alpha \rangle. \] (4.1)
If \(\theta(\Phi(\zeta), \zeta)\) is a critical point of \(W\), i.e. the Fréchet derivative \(dW(\theta(\Phi(\zeta), \zeta)) = 0\), then the first factor of r.h.s. of (4.1) vanishes, and therefore \(\zeta\) is a critical point of \(G\).

Now, suppose \(\partial_{\zeta^\alpha} G(\zeta) = 0\). For the backward direction, it suffices to show the pullback \(\bar{W}(\Phi(\zeta), \zeta) = 0\).

Let \(Q_\zeta\) be the projection onto the tangent space at \(T_{\theta(\Phi(\zeta), \zeta)} N\), as in Remark 9. Explicitly, this map \(Q_\zeta\) is given by
\[ Q_\zeta \phi = V^{\alpha\beta} (f_\alpha, \phi)_{L^2} f_\beta \quad (\phi \in H^k), \] (4.2)
where the matrix \(V^{\alpha\beta} := (f_\alpha, f_\beta)_L^2\) is its inverse, and the \(f_\alpha\)'s are given in (3.25)-(3.26). Notice that this matrix \(V^{\alpha\beta}\) is indeed invertible because the tangent space \(T_{\theta(\Phi(\zeta), \zeta)} N\) is non-degenerate, c.f. Proposition 2.

From the definition (4.2) and the formula (3.25)-(3.26), we also get a uniform estimate
\[ \|Q_\zeta\| \lesssim 1. \] (4.3)

The claim now is that \(Q_\zeta : H^k \to H^k\) is uniformly close in operator norm to the projection \(P\) defined in Definition 3. Geometrically, this is because the manifold \(E_N\) is a uniformly small normal perturbation of ran \(P\). Analytically, this claim follows by comparing (3.25)-(3.26) to the basis \(y^\alpha\) of ran \(P\), and using the estimate (3.10) for the derivatives \(\partial_{\zeta^\alpha} \Phi\). As a result, we find
\[ \|Q_\zeta - P\|_{H^k \to H^k} = O(R^{-2}). \] (4.4)

Now, by the construction in Definition 4 we know \(PW(\Phi(\zeta), \zeta) = \bar{W}(\Phi(\zeta), \zeta)\). Thus we can write
\[ \bar{W}(\Phi(\zeta), \zeta) = PW(\Phi(\zeta), \zeta) = (P - Q_\zeta)\bar{W}(\Phi(\zeta), \zeta) + Q_\zeta \bar{W}(\Phi(\zeta), \zeta). \] (4.5)
From (4.1)-(4.2), one can see that \(\partial_{\zeta^\alpha} G = 0\) implies \(Q_\zeta \bar{W} = 0\). Thus by assumption, the last term in (4.5) vanishes. Using this fact and the estimate (4.3), we find
\[ \|\bar{W}(\Phi(\zeta), \zeta)\|_{H^k} \lesssim \|(P - Q_\zeta)\|_{H^k \to H^k} \|\bar{W}(\Phi(\zeta), \zeta)\|_{H^k} \lesssim R^{-1} \|\bar{W}(\Phi(\zeta), \zeta)\|_{H^k}. \]

For sufficiently large \(R\), this is impossible unless \(\bar{W}(\Phi(\zeta), \zeta) = 0\).

5 Effective dynamics

In this section we prove the rest of Theorem 1. We first derive the effective dynamics (1.2)-(1.4), and then use this to derive Parts 2 and 4 of Theorem 1.

Theorem 5. Let \(\Phi : M' \to H^k\) be the map defined in Definition 4. Let \(\Sigma_* = x_*(S)\) be an admissible surface. Let \(\Sigma_t = x_t(S)\) be the global solution to (1.1) with initial configuration \(\Sigma_{|t=0} = \Sigma_*\) as in Theorem 3.
Then there exist $\alpha > 0$, $T = O(R^{-\alpha})$, and a path $\zeta_t \in M'$, such that
\[
\|\Phi(\zeta_t) - x_t^N\|_{H^k} = O(R^{-3}) \quad (t \geq T).
\] (5.1)

Moreover, the path $\zeta_t$ evolves according to
\[
\dot{\zeta} = \frac{1}{4\pi} \nabla G(\zeta) + O(R^{-3}),
\] (5.2)
where the leading term in the r.h.s. is of the order $O(R^{-2})$.

Remark 13. The function $G$ is defined in Theorem 4 and calculated in Appendix C.

Proof. Step 1. Take some $0 < \delta \ll 1$ to be determined. To begin with, denote $T > 0$ the first time $u_\alpha$ enters the manifold $U_\delta$ defined in (3.22). By the stability result Theorem 3, this $T$ is finite, and there exists $\alpha > 0$ such that $T = O(\delta^{-\alpha})$.

Now we prove the claims (5.1)-(5.2), assuming the a priori estimate
\[
x_t \in U_\delta \text{ for all } t > T.
\] (5.3)

By Lemma 2, so long as (5.3) is satisfied, we can associate a path of barycenters $\zeta_t \in M'$ to the full flow $x_t$. Later on, we show this Ansatz is satisfied with $\delta = O(R^{-3})$ by proving an a priori estimate for the fluctuation of $x_t$ around its adiabatic part.

Write $x = \theta(\Phi(\xi) + \xi, \zeta)$, where $\zeta = S(x)$ is the barycenter as in Lemma 2 and $\xi$ is the fluctuation. Then we can rewrite the l.h.s. of (1.1) as
\[
\partial_t x^N = \partial_t \theta(\Phi + \xi, \zeta)^N \partial_t \xi + \partial_{\xi^\alpha} \theta(\Phi + \xi, \zeta)^N \xi^\alpha
\]
\[
= A(\xi) \partial_t \theta(\Phi, \xi)^N \partial_t \xi + B(\xi) \partial_{\xi^\alpha} \theta(\Phi, \xi)^N \xi^\alpha
\]
\[
= A(\xi)(\xi^0 + O_{H^k}(1)) \partial_t \xi + B(\xi) f_\alpha \xi^\alpha,
\] (5.4)
where we define
\[
A(\xi) := \frac{\partial_t \theta(\Phi, \xi)^N}{\partial_t \theta(\Phi, \xi)^N}, \quad B(\xi) := \frac{\partial_{\xi^\alpha} \theta(\Phi, \xi)^N}{\partial_{\xi^\alpha} \theta(\Phi, \xi)^N}.
\]

For simplicity, here and below we omit the dependence of $f_\alpha$ and $\Phi$ on $\zeta$.

The expansion (5.4) follows from the chain rule, (3.20), and the assumption on the background metric, $g_{ij} = \delta_{ij} + O((\xi^0)^{-1})$. From (3.2), one can see that the prefactors $A(\xi), B(\xi)$ satisfy
\[
\|A(\xi) - 1\|_{H^k} + \|B(\xi) - 1\|_{H^k} \lesssim \|\xi\|_{H^k}.
\] (5.5)

Let $Q = Q_{\xi(t)}$ be the projection onto the tangent space $T_{\theta(\Phi(\xi), \zeta)} E^N$, as given explicitly in (1.2). The space ran $Q$ is spanned by the four vectors $f_\alpha$, which we computed in (3.20). Applying $Q$ to (5.4), we find
\[
Q \partial_t x^N = (\xi^0 + O_{H^k}(1))Q \partial_t \xi + f_\alpha \xi^\alpha + O_{H^k}(\xi(|\dot{\xi}| + \partial_t \xi)).
\] (5.6)

By the definition of barycenter, (3.21), we know $Q \xi = 0$. Differentiating this, we find
\[
Q(\partial_t \xi) = -(\partial_t Q) \xi = -(\partial_{\xi^\alpha} Q \xi^\alpha) \xi.
\] (5.7)
Here, the partial Fréchet derivative $\partial_\zeta Q$ maps a real number to a linear operator from $H^k \to \text{ran } Q \subset H^k$. From the explicit formula (4.2) for $Q$ and the estimate 8.10, we get the uniform bound $\|\partial_\zeta Q\|_{L(H^k, H^k)} \lesssim (\zeta^0)^{-2}$. Thus plugging 5.7 to 5.10 gives

$$Q\partial_\zeta x^N = f_\alpha \zeta^\alpha + O_{H^s}(\zeta(|\zeta| + \partial_\xi)) + R^{-1}O_{H^s}(\xi|\zeta|)$$

(5.8)

Step 2. Next, expanding the r.h.s. of (5.1) at $\theta(\Phi(\zeta), \zeta)$, we find

$$W(x) + \lambda H(x) = \bar{W}(\Phi(\zeta) + \xi, \zeta) = W(\Phi(\zeta), \zeta) + L_\zeta \xi + N_\zeta(\zeta).$$

(5.9)

Here, as in Section 3, $\bar{W}$ is the partial Fréchet derivative $\partial_\xi \bar{W}$ evaluated at $\theta(\Phi(\zeta), \zeta)$. The nonlinear term $N_\zeta(\zeta)$ is defined by collecting the rest terms in this expansion.

Applying $Q$ to (5.9), we find

$$Q\bar{W}(\Phi(\zeta) + \xi, \zeta) = Q\bar{W}(\Phi(\zeta), \zeta) + QL_\zeta \xi + QN_\zeta(\zeta).$$

(5.10)

By the chain rule and the definitions 3.4, 4.2, the first term in (5.10) can be written as

$$Q\bar{W}(\Phi(\zeta), \zeta) = V^{\alpha\beta} \langle \nabla^N W(\theta(\Phi(\zeta), \zeta)), f_\alpha \rangle f_\beta.$$

(5.11)

By (4.1), this expression equals to $V^{\alpha\beta} \partial_\alpha G(\zeta) f_\beta$.

By the uniform estimates $\|Q\|_{H^s \to H^k} \lesssim 1$ and (C.11), we can bound the nonlinearity as

$$\|QN_\zeta(\zeta)\|_{H^{s-1}} = O(\|\xi\|^2_{H^s}).$$

(5.12)

Lastly, using the approximate zero modes property of the elements in $\text{ran } Q$, which we show in Lemma 3, one sees that the restriction of $L_\zeta$ is small on $\text{ran } Q$. In particular, this implies for $\xi, \xi' \in H^k$,

$$\langle QL_\zeta \xi, \xi' \rangle = \langle \xi, L_\zeta Q\xi' \rangle = O(R^{-2})\|\xi\|_{H^s} \|\xi'\|_{H^{s-1}}.$$  

Plugging $\xi' = QL_\zeta \xi$ into this expression, we find

$$\|QL_\zeta \xi\|_{H^{s-1}} = O(R^{-2})\|\xi\|_{H^s}.$$  

(5.13)

Recall the Ansatz 5.3, which implies $\|\xi\|_{H^s} < \delta \ll 1$. Collecting 5.7, 5.10-5.13, plugging these back to 5.4, 5.9, and rearranging, we find

$$\|f_\alpha \zeta^\alpha - V^{\alpha\beta} \partial_\alpha G(\zeta) f_\beta\|_{H^{s-1}} \lesssim \frac{R^{-2}Q(\delta) + O(\delta \|\zeta\|_{H^s})}{1 - O(\delta) - R^{-1}O(\delta)}.$$  

(5.14)

Consider the map

$$\psi : \mathbb{R}^4 \to \text{ran } Q \subset H^{k-4},$$

$$\eta^\alpha \mapsto V^{\alpha\beta} \eta^\alpha f_\beta.$$

This map is linear. Moreover, using the fact that the matrix $V^{\alpha\beta} = O(1)\delta_{\alpha\beta}$ is invertible, one can show $\psi$ is invertible on $\text{ran } Q$, and the operator norm of its inverse $\psi^{-1} : \text{ran } Q \subset H^{k-4} \to \mathbb{R}^4$ is of the order $O(1)$.

Now we rewrite $f_\alpha \zeta^\alpha = \psi(\eta^\alpha)$ with

$$\eta^\alpha = V_{\alpha\beta} \xi^\beta.$$  

(5.15)
Note that this choice of $\eta^a$ is unique. Then we can conclude from (5.14) and the discussion above that

$$|\eta^a - \partial_\zeta G(\zeta)| \lesssim \frac{R^{-2}O(\delta) + O(\delta||\partial_\zeta \xi||_{H^k})}{1 - O(\delta) - R^{-2}O(\delta)}.$$  \hfill (5.16)

Step 3. From (5.16), the relation (5.13), and the estimate $V_{\alpha\beta} = 4\pi\delta_{\alpha\beta} + O(R^{-2})$ which follows from (3.25)-(3.26) (see also Proposition 2), we see that (5.2) follows once we can prove an a priori estimate of the form

$$||\xi||_{H^k} + ||\partial_\zeta \xi||_{H^k} \lesssim ||\xi_T||_{H^k},$$  \hfill (5.17)

where $\xi_T$ is the fluctuation when the flow $x_t$ first enters the manifold $U_{\delta}$, as in Ansatz (5.3).

To this end, define

$$\Lambda(t) := \frac{1}{2} \langle \xi_t, L_\zeta \xi_t \rangle.$$  \hfill (5.18)

We show this is a Lyapunov-type functional along the flow of $\xi$.

Recall $\bar{Q} = 1 - Q$ is the complement of the (not necessarily orthogonal) projection $Q$.

Indeed, differentiating $\Lambda$, we find

$$\dot{\Lambda}(t) = \langle \partial_t \xi, L_\zeta \xi \rangle + \frac{1}{2} \langle (\partial_t L_\zeta) \xi, \xi \rangle.$$  \hfill (5.19)

The second term is bounded as

$$||\langle (\partial_t L_\zeta) \xi, \xi \rangle|| \leq ||(\partial_t L_\zeta) \xi||_{H^{k-1-\delta}}||\xi||_{L^2} = O(\bar{\xi}||\xi||_{H^k}),$$  \hfill (5.20)

where we used the estimate (C.10).

To bound the first term in the r.h.s. of (5.19), we isolate the dynamics of $\xi$ after equating (5.4) to (5.9). We find

$$\partial_t \xi = \frac{1}{\partial_\zeta \theta(\Phi, \zeta)^N A(\xi)} ( - \bar{W}(\Phi, \zeta) - L_\zeta \xi - N_\zeta(\xi) - B(\xi) f_\alpha \bar{\zeta}^\alpha).$$  \hfill (5.21)

Consider the r.h.s. of (5.20)-(5.21). Using (C.9)-(C.11) and the governing equation for barycenter, (5.13), we find for $R \gg 1$,

$$\partial_\zeta \theta(\Phi, \zeta)^N A(\xi) = O_H(\bar{\zeta}^N(1 + \xi) + O_{H^k}(1 + \xi))$$  \hfill (5.22)

$$||\langle \bar{W}(\Phi, \zeta), L_\zeta \xi \rangle|| \lesssim R^{-4}||\xi||_{L^2},$$  \hfill (5.23)

$$||N_\zeta(\xi), L_\zeta \xi|| \lesssim ||\xi||_{H^k},$$  \hfill (5.24)

$$\left| \langle f_\alpha \bar{\zeta}^\alpha, L_\zeta \xi \rangle \right| \lesssim R^{-2}(1 + ||\xi||_{H^k})||\xi||_{H^k}. $$  \hfill (5.25)

Plugging (5.20)-(5.25) back to (5.19), we find that so long as $||\xi||_{H^k} < 1/2$ and $R \gg 1$, there holds

$$\dot{\Lambda}(t) \leq C_1 R^{-3}||\xi||_{H^k}$$

$$- ||L_\zeta \xi||_{H^{k-4}}^2 + (C_2 R^{-3} + C_3 R^{-1}||\xi||_{H^k})||\xi||_{H^k}^2.$$  \hfill (5.26)

By the coercivity of $L_\zeta$ shown in Lemma 4, the first term in the second line of (5.26) can be bounded by $- ||L_\zeta \xi||_{H^k}^2 \leq - \beta ||\xi||_{H^k}^2$ for some $\beta > 0$ independent.
of $\zeta$. This, together with the upper bound in (D.5), implies that there exists some $\gamma > 0$ independent of $\zeta$, such that
\begin{equation}
\dot{\Lambda}(t) + \gamma \Lambda(t) \leq C_1 R^{-3} \||\xi\|_H^k + (C_2 R^{-3} + C_3 R^{-1}) \||\xi\|_H^k - \beta/2) \||\xi\|_H^k. \tag{5.27}
\end{equation}

Thus, so long as
\begin{equation}
C_2 R^{-3} + C_3 R^{-1} \||\xi\|_H^k - \beta/2 < 0, \tag{5.28}
\end{equation}
we can drop the last term in (5.27) to deduce that
\begin{equation}
\frac{d(\Lambda(t)e^{\gamma t})}{dt} \lesssim R^{-3} e^{\gamma t}. \tag{5.29}
\end{equation}

Integrating (5.29) on $[T, \infty)$ and using (1.1), we find
\begin{equation}
\||\xi_t\|_H^k \lesssim \Lambda(t) \lesssim e^{-\gamma t} \||\xi_t\|_H^k + R^{-3} M(t) \quad (M(t) := \sup_{T \leq t'} \||\xi_t\|_H^k), \tag{5.30}
\end{equation}
where $\xi_T$ is the fluctuation at $t = T$. Taking supremum of both sides of (5.30) in $t$, and then dividing by $M(t)$ (which is positive for all $t$ in the nontrivial case), we find
\begin{equation}
M(t) \lesssim e^{-\gamma t} \||\xi_t\|_H^k + R^{-3}. \tag{5.31}
\end{equation}
This, in particular, implies that the Ansatz (5.28) is satisfied as long as $R \gg 1$. Iterating this process, we get the estimate $M(t) \lesssim R^{-3}$. Plugging this back to the dynamics (5.14) and (5.21) for $\zeta$ and $\xi$ respectively, we find that the velocity of the fluctuation satisfies $\||\partial_t \xi\|_H^k \lesssim R^{-4}$. Thus for sufficiently large $R$, the claim (5.17) follows.

\[\square\]

**Corollary 1.** Let $\zeta_t \in M'$ be a flow evolving according to (5.2). Then there exist some $T > 0$ and a global solution $x_t$ to (1.1), such that (5.1) holds on the time interval $T \leq t \leq T + R$ with this choice of $\zeta_t$.

**Proof.** Let $T > 0$ be the first time $u_t$ enters the manifold $U_\delta$ defined in (5.22) (this $T$ is finite by Theorem 3). Consider the flow $x_t$, $t \geq T$ generated under (1.1) by $x_t = \Phi(\zeta_t)$, as in Theorem 2. Let $\zeta_t$, $t \geq T$ be the flow of barycenter as in (5.1). Then using (5.10) and (5.11), we can estimate
\begin{equation}
\||\Phi(\zeta_t) - x_t^N\|_H^k \leq \||\Phi(\zeta_t) - x_t^N\|_H^k + \||\Phi(\zeta_t) - \Phi(\tilde{\zeta}_t)\|_H^k
\lesssim R^{-2} \||\zeta_t - \tilde{\zeta}_t\|_H^k + R^{-3}
\lesssim R^{-2} \int_T^{T+R} \||\dot{\zeta}_t - \dot{\tilde{\zeta}}_t\|_H^k + R^{-3}
\lesssim R^{-3} \quad (T \leq t \leq T + R). \tag{5.32}
\end{equation}
In the last step we use the effective dynamics (5.2).

\[\square\]

In the following corollary, we display the $t$-dependence in subscripts.

**Corollary 2.** Suppose the embedding $\theta(\Phi(\zeta), \zeta)$ parametrizes a surface of Willmore type (i.e. static solution to (1.1)). Then $\theta(\Phi(\zeta), \zeta)$ is uniformly stable with small area-preserving $H^k$-perturbation if $\zeta$ is a strict local minimum of the function $G$ defined in Theorem 3.
Proof. Suppose $\zeta$ is a strict local minimum of $G$. Then every flow $\zeta_t$ starting at some $\zeta'$ near $\zeta$ under (5.2) converges to $\zeta$, i.e. $\zeta_t \to \zeta$ as $t \to \infty$. Now, for this $\zeta$, consider a perturbation $x' := \theta(\Phi(\zeta) + \xi, \zeta)$ with $\|\xi\|_{H^k} \ll 1$. By the regularity of the nonlinear projection $S$ in Lemma 2, this perturbation has barycenter $\zeta'$ close to $\zeta$. It follows that the flow of barycenters $\zeta_t$ associated to the flow of embeddings $x_t$ generated by $x'$ under (1.1) satisfies $|\zeta_t - \zeta| < \delta$ for any $\delta > 0$ and all sufficiently large $t > T$. If we choose $\delta \ll R^{-3}$, then we can conclude from (5.1) that $\|x_N^t - \Phi(\zeta)\|_{H^{k-4}} \lesssim R^{-3}$ for all large $t$.

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A The Abstract Lyapunov-Schmidt map

Motivated by the synonymous reduction procedure, in general we can define Lyapunov-Schmidt map as follows:

Definition 6. Let $X \subset Y$ be two Hilbert spaces. Let $U \subset X$ be an open set, and $u_0 \in U$. Let $P : X \to X$ be an orthogonal projection onto some subspace of $X$, and $\bar{P} := 1 - P$ be the complement of $P$.

The Lyapunov-Schmidt map $\Phi = \Phi_{P,u_0}$ is defined on the following set:

$$\left\{ F : U \subset X \to Y : F \text{ is } C^1, \bar{P}L\bar{P} : \bar{P}X \to \bar{P}Y \text{ has bounded inverse,} \right. $$

where $L := dF(u)|_{u = u_0} : X \to Y$ is the linearized operator of $F$ at $u_0$.

For such $F$, by Implicit Function Theorem, there exists an open neighbourhood $V \subset PX$ around $u = 0$ and a $C^1$ map $w : V \to \bar{P}X$ such that $\bar{PF}(u_0 + v + w(v)) = 0$ for every $v \in V$. This map $w$ parametrizes a manifold $E$ in $X$. This manifold $E$ is a normal perturbation of ran $P$.

Let $w'$ be the Fréchet derivative of $w$. (This is a map from $V$ to the linear operators on $X$.) Then the tangent space at $w = w(v)$ to $E$ is spanned by $w'(v)(V)$, and can be trivialized as a subspace in $X$, with dimension up to $\dim V$.
Now, the image $\Phi(F)$ of the Lyapunov-Schmidt map is given by

$$\Phi(F) : V \subset PX \rightarrow X, \quad v \mapsto Q_v F(u_0 + v + w(v)),$$

where $w = w(v)$ is as above, and $Q_v$ is the projection onto the tangent space $T_{w(v)}E$.

The notion above is first informally conceived in [8], in connection with the Feshbach-Schur map introduced in [2, 3]. Here, notice that the domain $V$ of the map $\Phi(F)$ depends on the spectrum of $L$ only. Essentially, the map $\Phi$ identifies a reduced space which can be either finite-dimensional or otherwise more tractable. The behavior of a map in the domain of $\Phi$ on this reduced space, which is locally isomorphic to $PX$, determines the behavior of this map in the vicinity of 0 in the full space $X$.

As such, one can view this map $\Phi$ in the context of the theory of infinite-dimensional invariant manifolds for semiflows in Banach spaces. See [4–6] and the references therein.

We will treat the analytical and geometric properties of the abstract Lyapunov-Schmidt map elsewhere.

In the context of this paper, the family of maps involved is $F(\cdot, r, z) : H^k \rightarrow H^{k-4}$, defined in Definition 3. These maps are parametrized by the manifold $M'$. The projection $P$ is defined as the orthogonal projection onto the subspace $\text{span}\{1, y_1, y_2, y_3\} \subset H^k$. The vector $u_0$ is the zero function in $H^k$, which, through $\theta$ given in (3.2), corresponds to a large coordinate sphere.

### B Area-preserving property of (1.1)

In this section, following [22, Sec. 2], we show that (1.1) is area-preserving.

Let $x = x_t : \mathbb{S} \rightarrow (M, g)$ be a family of embeddings, and $\Sigma := x(\mathbb{S})$. Suppose $x$ solves (1.1), namely

$$\partial_t x^N = -W(x) - \lambda H(x), \quad W(x) := \Delta H(x) + H(x)(\text{Ric}_M(\nu, \nu) + |\mathring{A}|^2(x)),$$

$$\lambda = \lambda(t) = \frac{1}{\int_{\Sigma} H^2 \, d\mu} \int_{\Sigma} \left( |\nabla H|^2 - H^2 \text{Ric}_M(\nu, \nu) - H^2 |\mathring{A}|^2 \right) \, d\mu,$$  \hspace{1cm} (B.1) \hspace{1cm} (B.2)

where $\mu = \mu^S$ is the canonical measure on $\Sigma$, induced by the embedding $x$ and background metric $g$.

If $\lambda$ is given by (B.2), then integration by part on the first term in the r.h.s. of (B.2) yields

$$\lambda(t) \int_{\Sigma_t} H^2 \, d\mu = - \int_{\Sigma} WH \, d\mu.$$  \hspace{1cm} (B.3)

On the other hand, for $\Sigma = \Sigma_t = x_t(\mathbb{S})$, the first variation formula for the area functional reads

$$\partial_t |\Sigma| = - \int_{\Sigma} H \partial_t x^N \, d\mu.$$  \hspace{1cm} (B.4)
Plugging the expression for $\partial_t x^N$ from (B.1) into (B.4), we find
\[
\partial_t |\Sigma| = \int_\Sigma (W + \lambda H) H \, d\mu
\]
\[= 0,
\]
where the last step follows from identity (B.3). This shows (1.1) is indeed area-preserving.

C Asymptotics of $W$

In this section, we record various expansions related to the Willmore energy $W$ at large coordinate spheres. Most of the results can be read off from [11, Sec. C], [23, Secs. 3, 7].

Fix some $R \gg 1$ and $0 < \delta \ll 1$. In what follows, we restore the parameter $r, z$ and assume $r \geq R, z \in \mathbb{R}^3, |z| < 1 - \delta$.

**Proposition 3.** For $\Omega$ given in (3.5), $G$ given in Theorem 4, there hold
\[
\Omega(0, r, z) = 4\pi - 16\pi r^{-1} + 2\pi r^{-2} \left( \frac{10 - 6|z|^2}{|z|^2 - 1}^2 + \frac{1}{|z|} \log \frac{1 + |z|}{1 - |z|} \right) + O(r^{-3}),
\]
\[= \Omega(0, r, z) + O(r^{-5}).
\]

**Proof.** (C.1) can be found in [11, Lem. 42]. To derive this expansion, one uses the assumption (2.3) that the scalar curvature $Sc$ is asymptotically even. (C.2) follows from the expansion
\[
G(r, z) = \Omega(0, r, z) + O(r^{-5}).
\]

The following results are used to derive the key estimate (3.10).

**Proposition 4.** Let $k \geq 4$. Let $\phi = \phi_{r, z} \in H^k$ be a family of functions with $\|\phi\|_{H^k} \ll 1$, and suppose the map $(r, z) \mapsto \phi_{r, z}$ is $C^2$. Let $\bar{W}$ be as in (3.4). Then for all $m + |\alpha| \leq 2$, there hold
\[
\bar{W}(\phi, r, z) = \bar{W}(0, r, z) + L_{r, z}\phi + N_{r, z}(\phi),
\]
\[
\|\partial^m_\mu \partial^{\alpha}_z \bar{W}(0, r, z)\|_{H^{k-4}} = O(r^{-(4+m)}),
\]
\[
\|\partial^m_\mu \partial^{\alpha}_z L_{r, z}\phi\|_{H^{k-4}} \lesssim \|\partial^m_\mu \partial^{\alpha}_z \phi\|_{H^{k}},
\]
\[
\|\partial^m_\mu \partial^{\alpha}_z N_{r, z}(\phi)\|_{H^{k-4}} \lesssim \|\partial^m_\mu \partial^{\alpha}_z \phi\|_{H^{k}}.
\]

**Proof.** The expansion (C.3) is valid by the regularity of $\bar{W}$ from $Y^k \to H^{k-4}$. (C.4) follows from the formula [11, Cor. 45]. Since $L_{r, z}$ is linear in $\phi$, (C.5) is
immediate. For (C.6), we calculate \(N_{r,z}(\phi)\) explicitly as

\[
N_{r,z}(\phi) = \Delta H(\theta(\phi, r, z)) + H(\theta(\phi, r, z)) \left( \text{Ric}_M(\nu, \nu) + |A|^2(\theta(\phi, r, z)) \right) \\
- \Lambda_{r,z}^2 \phi - \frac{1}{2} H_{r,z}^2 \Lambda_{r,z} \phi - \lambda \Lambda_{r,z} \phi \\
- 2H_{r,z}g(\bar{A}_{r,z}, \nabla^2 \phi) - 2H_{r,z} \text{Ric}_M(\nu, \nabla \phi) + 2\bar{A}_{r,z}(\nabla H_{r,z}, \nabla \phi) \\
- V_{r,z} \phi + O(r^{-4}),
\]

(C.7)

where \(\lambda\) is the Lagrange multiplier in (1.4), \(\Lambda = -\Delta - (|A|^2 + \text{Ric}_M(\nu, \nu))\) is the Jacobi operator, \(V\) is some smooth function depending on the sphere \(\theta(0, r, z)\) and the background metric only, and the subscript on \(r, z\) indicates evaluation at the sphere \(\theta(0, r, z)\). Formula (C.7) follows from [1, Sec. 7.2].

Now, inspecting each term in (C.7), one can see from this explicit expression that the nonlinearity \(N_{r,z}\) is a finite sum of the form \(N_{r,z}(D^4 \phi, g)\), where \(D^4 \phi\) denotes all derivatives of \(\phi\) up to order 4. Moreover, each term in this finite sum, together with its derivatives, is uniformly bounded by some \(C > 0\) independent of \(r, z\) as a map from \(H^k \to H^{k-4}\). This implies

\[
\|\partial^m \partial^2 \phi N_{r,z}(\phi)\|_{H^{k-4}} \lesssim \|\partial^m \partial^2 \phi\|_{H^k}^M
\]

for some \(M = M(m, \alpha) \geq 2\). For \(\|\phi\|_{H^k} \ll 1\), this implies (C.6).

By perturbing the estimates in Proposition 4, we get the following results:

**Proposition 5.** Let \(k \geq 4\). Let \(\phi = \phi_{r,z} \in H^k\) be a family of functions with \(\|\phi\|_{H^k} \ll 1\). Let \(\bar{W}\) be as in (3.4).

Suppose the map \((r, z) \to \phi_{r,z}\) is \(C^2\), and satisfies (3.10). Then, for every \(\|\xi\| \ll 1\), there hold

\[
\bar{W}(\phi + \xi, r, z) = \bar{W}(\phi, r, z) + \bar{L}_{r,z} \xi + \bar{N}_{r,z}(\xi),
\]

(C.8)
\[
\|\bar{W}(\phi, r, z)\|_{H^{k-4}} = O(r^{-4}),
\]

(C.9)
\[
\|\partial^m \partial^2 \bar{L}_{r,z} \xi\|_{H^{k-4}} \lesssim \|\partial^m \partial^2 \xi\|_{H^k} \quad (m + |\alpha| \leq 1),
\]

(C.10)
\[
\|\bar{N}_{r,z}(\xi)\|_{H^{k-4}} \lesssim \|\xi\|_{H^k}^2.
\]

(C.11)

**Proof.** The point is that, by the regularity of \(\Omega(\cdot, r, z)\) on \(H^k\) with \(k \geq 4\), the map \(\bar{W}(\cdot, r, z)\) together with its derivatives varies continuously as \(\phi\) varies. Thus, if \(\phi\) satisfies (3.10), then (C.9)–(C.11) follow from (C.4)–(C.6) with appropriate choice of \(r, \alpha\).

\(\square\)

### D Spectral properties of \(L_\zeta\)

Let \(k \geq 4\). Fix \(\zeta = (r, z^j) \in M' \subset \mathbb{R}_{\geq R} \times B_1(0)\) for sufficiently large \(R\). In this section, we prove various uniform estimates for the linearized operator \(L_\zeta\) of \(\bar{W}(\cdot, \zeta)\) at \(\Phi(\zeta)\). For more details, see [23 Sections 3, 7].

Recall \(P : H^k \to H^k\) is the \(L^2\)-orthogonal projection onto the span consisting of the constants and the three spherical coordinates. \(Q = Q_\zeta : H^k \to H^k\) is the (not necessarily orthogonal) projection onto the tangent space at \(\theta(\Phi(\zeta), \zeta)^N\) to \(E^N := \{\theta^N : \theta \in E\} \subset H^k\), where the manifold \(E\) is given in Definition 4.
Lemma 3 (approximate zero modes). For every \( \phi \in H^k \), there holds
\[
\|L_\xi Q\phi\|_{H^{k-4}} \lesssim O(R^{-2})\|\phi\|_{H^k}.
\]  
(D.1)

Proof. Recall that the functions \( f_\alpha \) are calculated in (3.25)-(3.26), and that the projection \( Q \) maps onto \( \text{span}\{f_\alpha\} \). Using these facts, we have shown \( \|Q - P\|_{H^k} \lesssim R^{-2} \) as in (4.13).

Denote by \( L^0 \) the linearized operator at the coordinate sphere \( S_\xi \) with Euclidean background metric, and by \( L^g \) the same with the metric \( g \) on \( M \). These two operators satisfy
\[
\|L^0_\xi \phi\|_{H^{k-4}} = \|L^0_\xi \phi\|_{H^{k-4}} + O(R^{-3})\|\phi\|_{H^k}.
\]  
(D.2)

See a discussion about this in Remark 6.

By the definition of \( P \) in Definition 3, \( L^0 \) vanishes on \( \text{ran} \ P \). It follows that
\[
\|L^0_\xi Q\phi\|_{H^{k-4}} \leq \|L^0_\xi (P - Q)\phi\|_{H^k} = O(R^{-2})\|\phi\|_{H^k}.
\]  
(D.3)

Since the functional \( W \) is \( C^2 \) on \( X^k \), we have by (3.10) that
\[
\|L_\xi - L^0_\xi\|_{H^{k-4}} \lesssim \|\xi_\alpha \partial_\xi \Phi\|_{H^k} = O(R^{-2}).
\]  
(D.4)

The claim follows by combining (D.2)-(D.3) with (D.4).

Lemma 4 (coercivity). Let \( k \geq 4, R \gg 1 \). There exist \( \alpha, \beta > 0 \) depending on \( k, R \) and the background metric \( g \) only, such that
\[
\alpha\|\xi\|_{H^k}^2 \leq \langle \xi, L_\xi \xi \rangle \leq \beta\|\xi\|_{H^k}^2 \quad (\xi \in \ker Q).
\]  
(D.5)

Proof. The upper bound follows from the fact that \( W \) is \( C^2 \) and the relation \( L_\xi = \partial_\xi W(\Phi(\xi), \xi) \). Thus it suffices to find the lower bound.

In [23, Thm. 10], it is shown that
\[
\langle \xi, L_\xi \xi \rangle \geq \alpha\|\xi\|_{H^2}^2 \quad (\xi \in \ker P),
\]  
(D.6)

where \( \alpha \) depends only on the lower bound \( R \) and the ambient metric \( g \). The claim now is
\[
\langle \xi, L_\xi \xi \rangle \geq \frac{\alpha}{4}\|\xi\|_{L^2}^2 \quad (\xi \in \ker Q),
\]  
(D.7)

provided \( R \) is sufficiently large. Notice that here we have one caveat, namely that \( Q \) is not necessarily orthogonal.

Put \( \bar{Q} = 1 - Q \). Then we can rewrite (D.7) as
\[
\langle L_\xi \xi, \xi \rangle = \langle L_\xi Q\xi, \xi \rangle + \langle L_\xi \bar{Q}\xi, \xi \rangle.
\]  
(D.8)

The first term is \( O(R^{-2})\|\xi\|_{H^k}^2 \) by the approximate zero mode property (D.1).

The second term further splits as \( \langle L_\xi Q\xi, \xi \rangle + \langle L_\xi \bar{Q}\xi, \xi \rangle = \langle L_\xi Q\xi, \xi \rangle + \langle Q\xi, L\xi Q\xi \rangle + O(R^{-2})\|\xi\|_{H^k}^2 \), again by (D.1). Thus it suffices to find a lower bound of the quadratic form \( \langle L_\xi Q\xi, \xi \rangle \geq (\alpha/2)\|\xi\|_{H^k}^2 \).

Consider the operator \( QLQ \). By the uniform closeness (4.3), we have \( \bar{Q}LQ = PLP + O_{H^{k-4}(R^{-2})} \). This and \( k \geq 4 \) imply that for sufficiently large \( R \), we have \( \|QLQ\|_{H^{k-4}} \geq \alpha/2 \) and therefore (D.7).
Thus (D.5) follows once we can find a uniform estimate \( \|\xi\|_{H^k} \lesssim \|\xi\|_{L^2} \). This can be done using the fact that \( \xi \) solves \( 5.21 \), which we can rewrite as a fourth order elliptic equation

\[
L_\xi \xi = -\bar{W}(\Phi, \xi) - N_\xi(\xi) + O(R^{-2}).
\]

By the regularity theory for elliptic operator of higher-order, this implies that there is some \( C > 0 \) depending on \( R \), the background metric \( g \), and the Sobolev order \( k \) only, such that \( \|\xi\|_{H^k} \leq C\|\xi\|_{L^2} \).


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